Stalk and sheafification Functors

Jinwen XU
Sorbonne Université
21, rue de l'école de médecine 75006 Paris

Penglin LI
The University of Sheffield
Western Bank, Sheffield, S10 2TN

August 2023

Contents

1 Yoneda Lemma and adjoint functors

Recall (Locally small categories)

A locally small category is a category whose Hom-sets are (small) sets, i.e. for any objects $A, B \in \mathrm{obj}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a (small) set.

1.1 Lemma (Yoneda Lemma)

Let F be a functor from a locally small category $\mathcal{C}$ to **Set**. Then for each object $A \in \mathcal{C}$, we have isomorphism

$$\mathrm{Nat}(h_A, F) \simeq F(A)$$

where $h_A := \mathrm{Hom}(A, \square)$ and $\mathrm{Nat}(h_A, F) := \mathrm{Hom}(\mathrm{Hom}(A, \square), F)$ is the set of all natural transformations from functor $\mathrm{Hom}(A, \square)$ to functor F.

Proof. For each natural transformation $\phi : h_A \to F$ in $\mathrm{Nat}(h_A, F)$, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{id}_A \arrow[rrr, mapsto, dashed] \arrow[ddd, mapsto, dashed] & & & \phi_A(\mathrm{id}_A) \arrow[ddd, mapsto, dashed] \\
& \mathrm{Hom}(A, A) \arrow[r, "\phi_A"] \arrow[d, "\mathrm{Hom}(A, f)"'] & F(A) \arrow[d, "F(f)"] & \\
& \mathrm{Hom}(A, B) \arrow[r, "\phi_B"] & F(B) & \\
f \arrow[rrr, mapsto, dashed] & & & \phi_B(f) = F(f)\phi_A(\mathrm{id}_A)
\end{tikzcd}$$

for any morphism $f : A \to B$ according to the naturality of ϕ. Then the component of the natural transformation ϕ on each object B could be described by

$$\begin{aligned} \phi_B : \mathrm{Hom}(A, B) &\longrightarrow F(B) \\ f &\longmapsto F(f)\phi_A(\mathrm{id}_A) \end{aligned}$$

which means that ϕ is uniquely determined by the object $\phi_A(\mathrm{id}_A) \in \mathrm{obj}(F(A))$, since both $F(f)$ and $\phi_A(\mathrm{id}_A)$ are independent of B. Hence, we naturally have well-defined maps:

$$\alpha : \mathrm{Nat}(h_A, F) \to F(A) \quad , \quad \phi \mapsto \phi_A(\mathrm{id}_A)$$

and

$$\beta : F(A) \to \mathrm{Nat}(h_A, F) \quad , \quad x \mapsto \phi^x$$

where the natural transformation ϕ^x is defined by components:

$$\phi^x_B : \mathrm{Hom}(A,B) \to F(B) \quad , \quad f \mapsto F(f)x$$

Thus, we get the desired isomorphism $\mathrm{Nat}(h_A, F) \simeq F(A)$ by $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$. ■

1.2 **Construction** (Yoneda Embedding)

Let $\mathcal{C}$ be a locally small category, there is an embedding (functor)

$$\begin{aligned} h_\bullet : \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathbf{Set}^{\mathcal{C}} \\ A &\longmapsto \mathrm{Hom}(A, \square) = h_A \end{aligned}$$

(where $\mathbf{Set}^{\mathcal{C}} = [\mathcal{C}, \mathbf{Set}]$ is the category of all covariant functors from $\mathcal{C}$ to **Set**), and for a morphism $f : B \to A$ in category $\mathcal{C}$ (which is the same morphism $f : A \to B$ in $\mathcal{C}^{\mathrm{op}}$), its induced morphism is naturally given by $h_\bullet(f) = \mathrm{Hom}(f, \square) : \mathrm{Hom}(A, \square) \to \mathrm{Hom}(B, \square)$, where $\mathrm{Hom}(f, \square)$ is a natural transformation by

$$\begin{tikzcd} \mathrm{Hom}(A,C) \arrow[r, "\mathrm{Hom}(f,C)"] \arrow[d, "\mathrm{Hom}(A,g)"'] & \mathrm{Hom}(B,C) \arrow[d, "\mathrm{Hom}(B,g)"] & C \arrow[d, "g"] \\ \mathrm{Hom}(A,C') \arrow[r, "\mathrm{Hom}(f,C')"'] & \mathrm{Hom}(B,C') & C' \end{tikzcd} ,$$

the components $\mathrm{Hom}(f,C) : \mathrm{Hom}(A,C) \to \mathrm{Hom}(B,C), u \mapsto u \circ f$ and $\mathrm{Hom}(A,g) : \mathrm{Hom}(A,C) \to \mathrm{Hom}(A,C'), u \mapsto u \circ f$.

According to the Yoneda Lemma (Lemma 1.1), for any objects A, B in category $\mathcal{C}$ (and $\mathcal{C}^{\mathrm{op}}$), we have

$$\mathrm{Nat}(h_A, h_B) = \mathrm{Nat}(\mathrm{Hom}(A, \square), \mathrm{Hom}(B, \square)) \simeq \mathrm{Hom}_{\mathcal{C}}(B, A) = \mathrm{Hom}_{\mathcal{C}^{op}}(A, B),$$

(if we recognize the functor F in Lemma 1.1 as the functor $\mathrm{Hom}(B, \square)$). **Hence, the embedding functor $h_\bullet : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$ is fully faithfull.**

1.3 **Claim** (Fully faithful functor preserves isomorphism)

If a functor $F : \mathcal{C} \to \mathcal{D}$ is fully faithful, and $F(c) \simeq F(c')$ in $\mathcal{D}$, we could conclude that $c \simeq c'$ in $\mathcal{C}$.

Recall (Adjoint Functors)

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and a pair of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "G"'] & \mathcal{D} \arrow[l, shift right, "F"'] \end{tikzcd}$$

are called a pair of adjoint functors (F left adjoint, G right adjoint) if

$$\mathrm{Hom}_{\mathcal{C}}(F(d), c) \simeq \mathrm{Hom}_{\mathcal{D}}(d, G(c))$$

for all objects $c \in \mathrm{Obj}(C), d \in \mathrm{Obj}(D)$, and this family of bijections is natural in c and d.

1.4 ***Remark*** (Naturality)

The family of bijections $\mathrm{Hom}_{\mathcal{C}}(F(d), c) \simeq \mathrm{Hom}_{\mathcal{D}}(d, G(c))$ is natural in c and d means that for any morphism $f : c_2 \to c_1$ and $g : d_2 \to d_1$, the following diagram is commutative

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(F(d_1), c_1) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(F(g),f)"'] & \mathrm{Hom}_{\mathcal{D}}(d_1, G(c_1)) \arrow[d, "\mathrm{Hom}_{\mathcal{D}}(g,G(f))"] \\ \mathrm{Hom}_{\mathcal{C}}(F(d_2), c_2) \arrow[r, "\simeq"'] & \mathrm{Hom}_{\mathcal{D}}(d_2, G(c_2)) \end{tikzcd} .$$

In one word, functors F and G are a pair of adjoint functors if there is a natural isomorphism between bifunctors

$$\mathsf{Hom}_{\mathcal{C}}(F(\square),\square)\simeq \mathsf{Hom}_{\mathcal{D}}(\square,G(\square)).$$

1.5 **Theorem** (Uniqueness of left (right) adjoint functor)

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, if both functors $F:\mathcal{D}\to\mathcal{C}$ and $F':\mathcal{D}\to\mathcal{C}$ are left adjoint functors to $G:\mathcal{C}\to\mathcal{D}$, then we have natural isomorphism $F\simeq F'$, *i.e.*left adjoint functor is unique up to natural isomorphism. The situation for right adjoint functors is same.

Proof. According to the definition of adjoint functors, if both functors F and F' are left adjoint functors of G, which means there are natural isomorphisms

$$\mathsf{Hom}_{\mathcal{C}}(F(\square),\square)\simeq \mathsf{Hom}_{\mathcal{D}}(\square,G(\square))\simeq \mathsf{Hom}_{\mathcal{C}}(F'(\square),\square),$$

so $\mathsf{Hom}_{\mathcal{C}}(F(\square),\square)\simeq \mathsf{Hom}_{\mathcal{C}}(F'(\square),\square)$. Thus, we have natural isomorphisms $\alpha_d:\mathsf{Hom}_{\mathcal{C}}(F(d),\square)\to \mathsf{Hom}_{\mathcal{C}}(F'(d),\square)$ and $\alpha'_d:\mathsf{Hom}_{\mathcal{C}}(F'(d),\square)\to\mathsf{Hom}_{\mathcal{C}}(F(d),\square)$ for each $d\in\mathsf{Obj}(\mathcal{D})$, and satisfied naturality on d

$$\begin{tikzcd}
\alpha_{d_1}:\mathsf{Hom}_{\mathcal{C}}(F(d_1),\square) \arrow[r, "\simeq"] \arrow[d, "\mathsf{Hom}_{\mathcal{C}}(F(f),\square)"'] & \mathsf{Hom}_{\mathcal{C}}(F'(d_1),\square) \arrow[d, "\mathsf{Hom}_{\mathcal{C}}(F'(f),\square)"] & d_2 \arrow[d, "f"'] \\
\alpha_{d_2}:\mathsf{Hom}_{\mathcal{C}}(F(d_2),\square) \arrow[r, "\simeq"] & \mathsf{Hom}_{\mathcal{C}}(F'(d_2),\square) & d_1
\end{tikzcd}$$

for each morphism $f:d_2\to d_1$ in $\mathcal{D}$ (same in α'_d). Then according to Yoneda embedding $h_\bullet:\mathcal{D}^{\mathsf{op}}\to\mathbf{Set}^{\mathcal{D}}$ (or $h_\bullet:\mathcal{D}\to[\mathcal{D}^{\mathsf{op}},\mathbf{Set}]$), we have corresponded objects $h_\bullet^{-1}\mathsf{Hom}_{\mathcal{C}}(F(d),\square)=F(d)$ and $h_\bullet^{-1}\mathsf{Hom}_{\mathcal{C}}(F'(d),\square)=F'(d)$, corresponded morphisms

$$h_\bullet^{-1}\alpha_d:F(d)\xrightarrow{\simeq}F'(d),$$

(since fully faithful $h_\bullet$ preserves isomorphism by 1.3) and

$$h_\bullet^{-1}\mathsf{Hom}_{\mathcal{C}}(F(f),\square)=F(f), h_\bullet^{-1}\mathsf{Hom}_{\mathcal{C}}(F'(f),\square)=F'(f):F(d_1)\to F(d_2).$$

Since functor $h_\bullet$ is fully faithful, it preserves commutative diagrams

$$\begin{tikzcd}
h_\bullet^{-1}\alpha_{d_1}:F(d_1) \arrow[r, "\simeq"] \arrow[d, "F(f)"'] & F'(d_1) \arrow[d, "F'(f)"] & d_1 \arrow[d, "f"'] \\
h_\bullet^{-1}\alpha_{d_2}:F(d_2) \arrow[r, "\simeq"] & F'(d_2) & d_2
\end{tikzcd},$$

which means $h_\bullet^{-1}\alpha:F\xrightarrow{\simeq}F'$ is a natural isomorphism (the inverse transformation is $h_\bullet^{-1}\alpha':F'\xrightarrow{\simeq}F$), thus we have $F\simeq F'$. ∎

1.6 **Claim** (Hom-functor is continuous)

The Hom-functor $\mathsf{Hom}_{\mathcal{C}}(c,\square):\mathcal{C}\to\mathbf{Set}$ is continuous for fixed object $c\in\mathsf{Obj}(\mathcal{C})$, *i.e.*, it preserves all limits

$$\mathsf{Hom}_{\mathcal{C}}(c,\varprojlim x_i)\simeq\varprojlim\mathsf{Hom}(c,x_i)$$

where $x:I\to\mathcal{C}$ is a diagram exists limit called a limit diagram. The dual Hom-functor $\mathsf{Hom}_{\mathcal{C}}(\square,c):\mathcal{C}\to\mathbf{Set}$ is cocontinuous, *i.e.*, it preserves all colimits $\mathsf{Hom}_{\mathcal{C}}(\varinjlim x_i,c)\simeq\varinjlim\mathsf{Hom}_{\mathcal{C}}(x_i,c)$.

More exactly, Hom-functor $\mathsf{Hom}_{\mathcal{C}}(c,\square):\mathcal{C}\to\mathbf{Set}$ sends limit diagram $x:I\to\mathcal{C}, i\mapsto x_i$ to the corresponded limit diagram $\mathsf{Hom}\,\mathcal{C}(c,x):I\to\mathbf{Set}, i\mapsto\mathsf{Hom}_{\mathcal{C}}(c,x_i)$ in $\mathbf{Set}$, which means the isomorphism $\mathsf{Hom}_{\mathcal{C}}(c,\varprojlim x_i)\simeq\varprojlim\mathsf{Hom}(c,x_i)$ should exactly be 'equation' $\mathsf{Hom}_{\mathcal{C}}(c,\varprojlim x_i)=\varprojlim\mathsf{Hom}(c,x_i)$. Hence, we could conclude that the isomorphism is also natural in c, by

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c_1, \varprojlim x_i) \arrow[r, equal] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(f,\square)"'] & \varprojlim \mathrm{Hom}_{\mathcal{C}}(c_1, x_i) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(f,\square)"] & c_2 \arrow[d, "f"'] \\
\mathrm{Hom}_{\mathcal{C}}(c_2, \varprojlim x_i) \arrow[r, equal] & \varprojlim \mathrm{Hom}_{\mathcal{C}}(c_2, x_i) & c_1
\end{tikzcd}.$$

1.7 Proposition (Adjoint functors are (co)continuous)

Every right adjoint functor (it has a left adjoint functor) is continuous, *i.e.*, it preserves all limit. Every left adjoint functor (it has a right adjoint functor) is cocontinuous, *i.e.*, it preserves all colimit.

Proof. Let functors $\mathcal{C} \xleftarrow{F} \mathcal{D}, \mathcal{C} \xrightarrow{G} \mathcal{D}$ be a pair of adjoint functors. According to the definition of adjoint functors and the claim of Hom –functors 1.6, we have a series of isomorphismes which are also natural in d

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(d, G(\varprojlim c_i)) &\simeq \mathrm{Hom}_{\mathcal{C}}(F(d), \varprojlim c_i) \\
&\simeq \varprojlim \mathrm{Hom}_{\mathcal{C}}(F(d), c_i) \\
&\simeq \varprojlim \mathrm{Hom}_{\mathcal{D}}(d, G(c_i)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}}(d, \varprojlim G(c_i))
\end{aligned}$$

which means that there is a natural isomorphism in the functor category $\mathbf{Set}^{\mathcal{D}^{op}}$ $\mathrm{Hom}_{\mathcal{D}}(\square, G(\varprojlim c_i)) \simeq \mathrm{Hom}_{\mathcal{D}}(\square, \varprojlim G(c_i))$. Then according to Yoneda embedding Construction 1.2, we get isomorphism $G(\varprojlim c_i) \simeq \varprojlim G(c_i)$ in category $\mathcal{D}$, which implies right adjoint functor G is continuous. We could prove its dual case by same way. ∎

2 Pushforward and pullbcak functors

2.1 Definition (Pushforward)

Let $f : X \to Y$ be a continuous map of topological spaces. The pushforward functor is defined by

$$f_* : \mathsf{PSh}(X) \to \mathsf{PSh}(Y), \quad \mathscr{F} \mapsto f_*\mathscr{F}$$

where the presheaf $f_*\mathscr{F}$ is defined by $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$ for each open set V in Y.

2.2 *Remark*

The presheaf $f_*\mathscr{F}$ is well-defined, since for any open sets $V_1 \subset V_2 \subset Y$, the restriction map $f_*\rho_{V_2V_1} : f_*\mathscr{F}(V_2) \to f_*\mathscr{F}(V_1)$ is given by the commutative diagram

$$\begin{tikzcd}
f_*\mathscr{F}(V_2) \arrow[r, equal] \arrow[d, dashed, "f_*\rho_{V_2V_1}"'] & \mathscr{F}(f^{-1}(V_2)) \arrow[d, "\rho_{f^{-1}(V_2)f^{-1}(V_1)}"] \\
f_*\mathscr{F}(V_1) \arrow[r, equal] & \mathscr{F}(f^{-1}(V_1))
\end{tikzcd}$$

where the restriction map $\rho_{f^{-1}(V_2)f^{-1}(V_1)} : \mathscr{F}(f^{-1}(V_2)) \to \mathscr{F}(f^{-1}(V_1))$ is well-defined in presheaf $\mathscr{F}$, since $f^{-1}(V_1) \subset f^{-1}(V_2)$ by $V_1 \subset V_2$ and both $f^{-1}(V_1), f^{-1}(V_2)$ are open in X by $f : X \to Y$ continuous.

2.3 Lemma (Colimits of different diagrams)

Let $M : \mathcal{I} \to \mathcal{C}$ and $N : \mathcal{J} \to \mathcal{C}$ be two diagrams whose limits exist, and $H : \mathcal{I} \to \mathcal{J}$ be a functor, there is a new diagram $N \circ H : \mathcal{I} \to \mathcal{C}$ with index category $\mathcal{I}$. Suppose $t : M \to N \circ H$ is a natural transformation of functors, then there is a unique morphism of colimits $\theta : \mathrm{colim}_{\mathcal{I}} M \to \mathrm{colim}_{\mathcal{J}} N$ such that the following diagram commute for each object i in $\mathcal{I}$

$$\begin{array}{ccc} M_i & \xrightarrow{\phi_i^M} & \operatorname{colim}_{\mathcal{I}} M \\ {\scriptstyle t_i}\downarrow & & \downarrow{\scriptstyle \theta} \\ N_{H(i)} & \xrightarrow[\phi_{H(i)}^N]{} & \operatorname{colim}_{\mathcal{J}} N \end{array}$$

where ϕ_i^M is the canonical morphism in the colimit system $(\operatorname{colim}_{\mathcal{I}} M, \phi_i^M)$, so is $\phi_{H(i)}^N$.

2.4 **Corollary** (Colimit of subsystem)

Let diagram $M : \mathcal{I} \to \mathcal{C}$ be a subsystem of $N : \mathcal{J} \to \mathcal{C}$, *i.e.*, the index category $\mathcal{I}$ is a subcategory of $\mathcal{J}$, and there is a natural isomorphism (equation) $M = N \circ \tau, M_i = N_{\tau(i)} = N_i$ for all object i in subcategory $\mathcal{I}$, where $\tau : \mathcal{I} \to \mathcal{J}$ is the 'inclusion' functor of index categories. Then according to Lemma 2.3 there is a unique morphism $\theta : \operatorname{colim}_{\mathcal{I}} M \to \operatorname{colim}_{\mathcal{J}} N$ of colimits such that the following diagram commute

$$\begin{array}{ccc} M_i & \xrightarrow{\phi_i^M} & \operatorname{colim}_{\mathcal{I}} M \\ & {\scriptstyle \phi_i^N}\searrow & \downarrow{\scriptstyle \theta} \\ & & \operatorname{colim}_{\mathcal{J}} N \end{array}$$

where ϕ_i^M is the canonical morphism in the colimit system of M and ϕ_i^N is the canonical morphism in the colimit system of N.

2.5 **Proposition** (Existance of left adjoint with pushforward)

Any pushforward functor $f_* : \mathsf{PSh}(X) \to \mathsf{PSh}(Y)$ has a left adjoint functor, *i.e.*, it is a right adjoint functor. More precisely, for a continuous map $f : X \to Y$, there exists a functor $f_P : \mathsf{PSh}(Y) \to \mathsf{PSh}(X)$ which is left adjoint to f_*, and for each presheaf $\mathscr{G}$ over Y, the presheaf $f_P\mathscr{G}$ over X is defined by

$$f_p\mathscr{G}(U) = \operatorname*{colim}_{f(U)\subset V} \mathscr{G}(V)$$

for each open set U in X (where the colimit is over the collection of all open neighborhoods V of $f(U)$ via restrictions ρ in Y).

2.6 ***Remark*** (Sheaf $f_p\mathscr{G}$ is well-defined)

The sheaf $f_p\mathscr{G}$ is well-defined, since for any subset $U' \subset U$, its restriction map is defined by the morphism

$$\theta : f_p\mathscr{G}(U) = \operatorname*{colim}_{f(U)\subset V} \mathscr{G}(V) \to f_p\mathscr{G}(U') = \operatorname*{colim}_{f(U')\subset V} \mathscr{G}(V)$$

in corollary 2.4 (where $\{\mathscr{G}(V)|f(U) \subset V\}$ is a subsystem of $\{\mathscr{G}(V)|f(U') \subset V\}$ by $f(U') \subset f(U)$).

Proof. We just need to verify the correspondence $\operatorname{Hom}_{\mathsf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F}) \simeq \operatorname{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f^*\mathscr{F})$ for any objects $\mathscr{F}$ in $\mathsf{PSh}(X)$ and $\mathscr{G}$ in $\mathsf{PSh}(Y)$, and also natural in $\mathscr{F}, \mathscr{G}$.

Firstly, we construct the map $\operatorname{Hom}_{\mathsf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F}) \to \operatorname{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f^*\mathscr{F})$ by the following way: Since $f(f^{-1}(U)) \subset U$, the open set U is a neighborhood of $f(f^{-1}(U))$, which means $\mathscr{G}(U)$ is in the colimit system $\{\mathscr{G}(V)|f(f^{-1}(U)) \subset V\}$. So according to the definition of colimit, there is a canonical morphism to the colimit of this system

$$\phi_U : \mathscr{G}(U) \to \operatorname*{colim}_{f(f^{-1}(U))\subset V} \mathscr{G}(V) = f_* \operatorname*{colim}_{f(U)\subset V} \mathscr{G}(V) = f_*f_p\mathscr{G}(U).$$

This is compatible with restriction maps, because for subset $W \subset U$, there is another canonical morphism $\phi'_U : \mathscr{G}(U) \to \operatorname{colim}_{f(f^{-1}(W))}$ in the colimit system $\{\mathscr{G}(V)|f(f^{-1}(W)) \subset V\}$ (since $f(f^{-1}(W)) \subset U$, $\mathscr{G}(U)$ belongs to this system), which is compatible with restriction maps $\phi'_U = \phi_W \circ \rho_{UW}$ in the system, as well as $\phi'_U = \theta \circ \phi_U$ according to proposition 2.4 ($\{\mathscr{G}(V)|f(f^{-1}(U)) \subset V\}$ is a subsystem of $\{\mathscr{G}(V)|f(f^{-1}(W)) \subset V\}$).

Thus, the following diagram commute

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\phi_U"] \arrow[d, "\rho_{UW}"'] \arrow[dr, dashed, "\phi'_U"] & \mathrm{colim}_{f(f^{-1}(U))\subset V}\mathscr{G}(V) = f_*f_p\mathscr{G}(U) \arrow[d, "\theta"] \\
\mathscr{G}(W) \arrow[r, "\phi_W"'] & \mathrm{colim}_{f(f^{-1}(W))\subset V}\mathscr{G}(V) = f_*f_p\mathscr{G}(W)
\end{tikzcd},$$

the compatibility with restriction maps implies there is a canonical morphism of presheaves:

$$\phi_{\mathscr{G}} : \mathscr{G} \to f_*f_p\mathscr{G}$$

defined by such canonical maps ϕ_U.

Then let $\varphi : f_p(\mathscr{G}) \to \mathscr{F}$ be a morphism of presheaves on X, there is an induced morphism $f_*\varphi : f_*f_p(\mathscr{G}) \to f_*\mathscr{F}$ under the pushforward functor, then the corresponding map in $\mathsf{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f^*\mathscr{F})$ is given by $f_*\varphi \circ \phi_{\mathscr{G}} : \mathscr{G} \xrightarrow{\phi_{\mathscr{G}}} f_*f_p\mathscr{G} \xrightarrow{f_*\varphi} f_*\mathscr{F}$, so we get the map

$$\begin{aligned}
\alpha : \mathsf{Hom}_{\mathsf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F}) &\longrightarrow \mathsf{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f^*\mathscr{F}) \\
\varphi &\longmapsto f_*\varphi \circ \phi_{\mathscr{G}}.
\end{aligned}$$

Secondly, we construct the inverse map $\mathsf{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f^*\mathscr{F}) \to \mathsf{Hom}_{\mathsf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F})$. Let $U \subset X$ be an open set, for every open neighborhood $f(U) \subset V$, there is a restriction map $\rho^{\mathscr{F}}_{f^{-1}(V),U} : f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$, since $U \subset f^{-1}(V)$. And this restriction map $\rho^{\mathscr{F}}_{f^{-1}(V),U}$ is obviously compatible with restriction maps $\rho^{\mathscr{F}}_{f^{-1}(V'),f^{-1}(V)} : \mathscr{F}(f^{-1}(V')) \to \mathscr{F}(f^{-1}(V))$ for all open sets $V \subset V'$ in Y. So $\mathscr{F}(U)$ is an object with morphisms $\{\rho^{\mathscr{F}}_{f^{-1}(V),U}\}_{f(U)\subset V}$, which is compatible with the colimit system $\{\mathscr{F}(f^{-1}(V)) = f_*\mathscr{F}(V) | f(U) \subset V\}$,

$$\begin{tikzcd}
\cdots\ \mathscr{F}(f^{-1}(V')) \arrow[rr, "\rho^{f_*\mathscr{F}}_{V',V}=\rho_{f^{-1}(V'),f^{-1}(V)}"] \arrow[dr, "\phi_{V'}"] \arrow[ddr, bend right, "\rho^{\mathscr{F}}_{f^{-1}(V'),U}"'] & & \mathscr{F}(f^{-1}(V))\ \cdots \arrow[dl, "\phi_V"'] \arrow[ddl, bend left, "\rho^{\mathscr{F}}_{f^{-1}(V),U}"] \\
& f_pf_*(U) = \underset{f(U)\subset V}{\mathrm{colim}}\,\mathscr{F}(f^{-1}(V)) \arrow[d, dashed, "u"] & \\
& \mathscr{F}(U) &
\end{tikzcd},$$

according to the definition of colimit, there is a unique morphism

$$u : f_pf_*(U) = \underset{f(U)\subset V}{\mathrm{colim}}\,\mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$$

such that the diagram upside commute, *i.e.*, $\rho^{\mathscr{F}}_{f^{-1}(V),U} = u \circ \phi_V$ for each $f(U) \subset V$. This morphism is compatible with restriction maps, since for any open sets $U' \subset U$ in X, the following diagram commute

$$\begin{tikzcd}
f_pf_*(U) = \underset{f(U)\subset V}{\mathrm{colim}}\,\mathscr{F}(f^{-1}(V)) \arrow[rr, "\theta"] \arrow[dd, "u"] & & f_pf_*(U') = \underset{f(U')\subset V}{\mathrm{colim}}\,\mathscr{F}(f^{-1}(V)) \arrow[dd, "u'"] \\
& \cdots\mathscr{F}(f^{-1}(V))\cdots \arrow[ul, dashed, "\phi_V"] \arrow[ur, dashed, "\phi'_V"'] \arrow[dl, dashed, "\rho_{f^{-1}(V),U}"'] \arrow[dr, dashed, "\rho_{f^{-1}(V),U'}"] & \\
\mathscr{F}(U) \arrow[rr, "\rho^{\mathscr{F}}_{U,U'}"] & & \mathscr{F}(U')
\end{tikzcd}.$$

This is because for each object $\mathscr{F}(f^{-1}(V))$ in the colimit system $\{\mathscr{F}(f^{-1}(V)) | f(U) \subset V\}$, the four triple diagrams surrounding $\mathscr{F}(f^{-1}(V))$ are commutative in the diagram upside, and they are also compatible with restriction maps in the system (according to the definition of colimit and lemma2.3), so we have the following commutative diagram between the colimit system $\{\mathscr{F}(f^{-1}(V))|f(U) \subset V\}$ and $\mathscr{F}(U)$

$$\begin{tikzcd}
\mathscr{F}(f^{-1}(V)) \arrow[r, "\phi_V"] \arrow[rr, dashed, bend left, "u'\circ\theta\circ\phi_V"] \arrow[rr, dashed, bend right, "\rho^{\mathscr{F}}_{U,U'}\circ u\circ\phi_V"'] & \operatorname*{colim}_{f(U)\subset V} \mathscr{F}(f^{-1}(V)) \arrow[r, shift left, "u'\circ\theta"] \arrow[r, shift right, "\rho^{\mathscr{F}}_{U,U'}\circ u"'] & \mathscr{F}(U)
\end{tikzcd}$$

where $u' \circ \theta \circ \phi_V = \rho^{\mathscr{F}}_{U,U'} \circ u \circ \phi_V$, then according to uniqueness in the universal property of colimit, we have $u' \circ \theta = \rho^{\mathscr{F}}_{U,U'} \circ u$. Hence, we get a morphism of presheaves

$$u_{\mathscr{F}} : f_p f_* \mathscr{F} \to \mathscr{F}$$

defined by the morphism $u : f_p f_*(U) = \operatorname{colim}_{f(U)\subset V} \mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$ for each open set $U \subset X$.

Then let $\psi : \mathscr{G} \to f^*\mathscr{F}$ be a map of presheaves on Y, there is an induced morphism $f_p\psi : f_p\mathscr{G} \to f_p f_* \mathscr{F}$ under the functor f_p, then the corresponding morphism of ψ in $\mathsf{Hom}_{\mathsf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F})$ is given by $u_{\mathscr{F}} \circ f_p\psi : f_p\mathscr{G} \xrightarrow{f_p\psi} f_p f_* \mathscr{F} \xrightarrow{u_{\mathscr{F}}} \mathscr{F}$. Hence, we get a map

$$\beta : \mathsf{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f^*\mathscr{F}) \longrightarrow \mathsf{Hom}_{\mathsf{PSh}(X)}(f_p\mathscr{G}, \mathscr{F})$$
$$\psi \longmapsto u_{\mathscr{F}} \circ f_p\psi.$$

Finally, we just need to verify that the map β is an inverse map of α, this could be immediately verified follow $\beta \circ \alpha(\varphi)(f_p\mathscr{G})(U) = u_{\mathscr{F}} \circ f_p(f_*\varphi \circ \phi_{\mathscr{G}})\mathscr{G}(U) = \varphi(f_p\mathscr{G})(U)$ for every open set $U \subset Y$, and $\alpha \circ \beta(\psi)(\mathscr{G})(U) = f_*(u_{\mathscr{F}} \circ f_p\psi) \circ \phi_{\mathscr{G}}(f_p\mathscr{G})(U) = \psi\mathscr{G}(U)$ for every open set $U \subset X$. And the naturality is automatically from the details in previous proof. Hence, the functor f_p we have already constructed is left adjoint to the pushforward functor f_*.

Reference to stacks 6.21 Continuous maps and sheaves. ■

2.7 Definition (Pullback)

Let $f : X \to Y$ be a continuous map of topological spaces and its pushforward is $f_* : \mathsf{PSh}(X) \to \mathsf{PSh}(Y)$ as we have defined previously, then the pullback functor $f_p : \mathsf{PSh}(Y) \to \mathsf{PSh}(X)$ of continuous map f is defined by the left adjoint functor of the pushforward f_*.

2.8 *Remark*

The pullback $f_p : \mathsf{PSh}(Y) \to \mathsf{PSh}(X)$ of a continuous map $f : X \to Y$ is existed and unique up to natural isomorphism, since the pullback could be exactly expressed by

$$f_p\mathscr{G}(U) = \operatorname*{colim}_{f(U)\subset V} \mathscr{G}(V)$$

for any presheaf $\mathscr{G}$ on Y, according to proposition 2.5, and its is unique up to natural isomorphism according to the property of left adjoint functors Theorem 1.5.

3 Stalk and Sheafification

3.1 Definition (Stalk (functors))

Let $\mathscr{F}$ be a presheaf of sets (abelian groups) over topological space X, for each point $P \in X$, we define the stalk $\mathscr{F}_P$ of $\mathscr{F}$ at point P to be the direct limit

$$\mathscr{F}_P = \varinjlim \mathscr{F}(U)$$

of the direct system $\langle \mathscr{F}(U), \rho \rangle$ for all open sets $P \in U$ in X (where the morphisms in this direct system is just restrictions ρ).

More precisely, according to the definition of direct limit

$$\mathscr{F}_P = \varinjlim \mathscr{F}(U) = \bigsqcup_{P\in U} \mathscr{F}(U)/\sim = \{\langle U, s\rangle : s \in \mathscr{F}(U)\}_{P\in U}/\sim$$

where the equivalence '$\sim$' means elements $\langle U, s\rangle = \langle V, t\rangle$ iff there is an open neighborhood $P \in W \subset U \cap V$ such that $s|_W = t|_W$ (via restriction maps $\rho : \mathscr{F}(U) \to \mathscr{F}(W)$ and $\rho : \mathscr{F}(V) \to \mathscr{F}(W)$).

3.2 *Remark* (Functoriality)

In the version of category theory, stalk could be seen as a functor

$$\square_P : \mathsf{PSh}(X) \to \mathsf{Set}, \quad \mathscr{F} \mapsto \mathscr{F}_P$$

or $\square_P : \mathsf{PSh}_{\mathsf{Ab}}(X) \to \mathsf{Ab}$ for the category of presheaves of abelian groups over X and $\square_P : \mathsf{Sh}_{\mathsf{Ab}}(X) \to \mathsf{Ab}$ for the category of sheaves of abelian groups over X. And the induced morphism of a morphism of (pre)sheaves $\alpha : \mathscr{F} \to \mathscr{G}$ is naturally defined by

$$\alpha_P : \mathscr{F}_P \to \mathscr{G}_P, \quad \langle U, s\rangle \mapsto \langle U, \alpha(s)\rangle.$$

3.3 *Remark* (Stalk as a (pre)sheaf)

Any set (albelian group) G could be recognized as a trivial (pre)sheaf $\mathscr{G}$ on a space X, where $\mathscr{G}(U) \equiv G$ for all open set $U \subset X$, and restriction maps $\rho_{UV} = \mathrm{id}_G$ for any open sets $V \subset U$. So the stalk $\mathscr{F}_P$ of a (pre)sheaf $\mathscr{F}$ on X could also be recognized as a (pre)sheaf on X.

3.4 Proposition (Stalk is a pullback functor)

Let $i_x : \{x\} \to X$ be an inclusion map, its pullback functor is the stalk functor $\square_x : \mathscr{F} \mapsto \mathscr{F}_x$ of presheaves on X at point $x \in X$.

Proof. According to the definition of pullback functor (Definition 2.7) and Remark 2.8, the pullback functor of the inclusion map i_x could be expressed by

$$(i_x)_p \mathscr{F}(\{x\}) = \operatorname*{colim}_{i_x(\{x\})\subset V} \mathscr{F}(V) = \operatorname*{colim}_{x\in V} \mathscr{F}(V)$$

for any presheaf $\mathscr{F}$ on X (where the only one open set in $\{x\}$ is itself). Then by the definition of stalk functors 3.1, we have the equation

$$(i_x)_p \mathscr{F}(\{x\}) = \operatorname*{colim}_{x\in V} \mathscr{F}(V) = \varinjlim \mathscr{F}(V) = \mathscr{F}_x$$

where the middle equation is because the direct limit of the direct system $\langle \mathscr{F}(V), \rho\rangle_{x\in V}$ for all open neighborhoods containing point x, via restriction maps ρ is equal to the colimit of the same system $\langle \mathscr{F}(V), \rho\rangle_{x\in V}$. Also it is easy to verify that for any morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$, the induced morphisms $(i_x)_p(\varphi) = \varphi_x$ are equal, since they are colimits of a same system. Hence, we could say the stalk functor $\square_x$ is the pullback functor of the inclusion map i_x. ∎

3.5 *Remark* (Skyscraper)

The pushforward $i_x^*(\mathscr{F})$ of a presheaf $\mathscr{F}$ under the inclusion map $i_x : \{x\} \to X$ is called the Skyscraper of $\mathscr{F}$.

3.6 Construction (Sheafification)

Let $\mathscr{F}$ be a presheaf, its sheafification is a sheaf $\mathscr{F}^+$ with a morphism (of presheaves) $\theta : \mathscr{F} \to \mathscr{F}^+$ such that the universal property:

For any sheaf $\mathscr{G}$ and morphism (of presheaves) $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism of sheaves $\phi : \mathscr{F}^+ \to \mathscr{G}$ such that the following diagram commute

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathscr{G} \\ \mathscr{F}^+ \arrow[ur, dashed, "\phi"'] & \end{tikzcd},$$

i.e., $\varphi = \phi \circ \theta$ in morphisms of presheaves. Such pair $(\mathscr{F}^+, \theta)$ is also called the sheaf associated to presheaf $\mathscr{F}$.

3.7 ***Remark*** (Existance and Uniqueness)

- The sheafification of a presheaf always exists, it could be constructed by the following way: For any open set $U \subset X$, we define $\mathscr{F}^+(U)$ to be the set of functions $s : U \to \bigcup_{P \in U} \mathscr{F}_P$ (union of the stalks of points in U) which satisfy following two conditions:
  (1) For each point $P \in U$, the value of the function $s(P) \in \mathscr{F}_P$.
  (This condition aims to ensure $\mathscr{F}^+(U)$ to be an abelian group. For any functions $s_1, s_2 \in \mathscr{F}^+(U)$, the operation could be defined by $s_1 * s_2(P) = s_1(P) \cdot s_2(P)$ since $s_1(P), s_2(P)$ are in the same abelian group $\mathscr{F}_P$ by this condition).
  Notice that the restriction map $\rho^+_{UV} : \mathscr{F}^+_U \to \mathscr{F}^+_V$ for sets $V \subset U$ could be naturally defined by $\rho^+_{UV}(s) = s|_V : V \to \bigcup_{P \in V} \mathscr{F}_P, P \mapsto s(P)$ (the restriction of function s on V).
  (2) For each point $P \in U$, there is a neighborhood $P \in V \subset U$, and an element $t \in \mathscr{F}(U)$ such that for each point $Q \in V$, the value of function $s(Q) = t_Q$ (where t_Q is the germ of t in $\mathscr{F}_Q$).
  (This condition aims to ensure the universal property of sheafification).

  As well as, there is a natural morphism $\theta : \mathscr{F} \to \mathscr{F}^+$, which is described by

$$\begin{aligned} \theta(U) : \mathscr{F}(U) &\longrightarrow \mathscr{F}^+(U) \\ t &\longmapsto s_t \end{aligned}$$

  where the function $s_t : U \to \bigcup_{P \in U} \mathscr{F}_P, Q \mapsto t_Q$ (t_Q is the germ of t at point Q). Hence, we have already constructed the sheafification $(\mathscr{F}^+, \theta)$ of $\mathscr{F}$. The sheaf $\mathscr{F}^+$ is well-defined, since for an open cover $\{V_i\}$ of U:
  If there is an element $s \in \mathscr{F}^+(U)$ such that $s|_{V_i} = 0$ for all i, then we have $s(P) = 0$ for all points $P \in U$, so we got $s = 0$.
  If we have functions $s_i \in \mathscr{F}^+(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for any i, j, which means values $s_i(P) = s_j(P)$ for all points $P \in V_i \cap V_j$, then there is a function

$$\begin{aligned} s : U &\longrightarrow \bigcup_{P \in U} \mathscr{F}_P \\ Q &\longmapsto s_i(Q) \quad , \quad Q \in V_i \end{aligned}$$

  in $\mathscr{F}^+(U)$ (the compatibility of s is provided by the assumption), such that $s_i = s|_{V_i}$.
  The sheafification $\mathscr{F}^+$ we constructed satisfies universal property, since for any sheaf $\mathscr{G}$ and a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of presheaves, if there is a morphism $u : \mathscr{F}^+ \to \mathscr{G}$ of sheaves such that $u \circ \theta = \varphi$, then for each open set U, we have the following diagram commute

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\theta(U)"'] & \mathscr{G}(U) \\ \mathscr{F}^+(U) \arrow[ur, dashed, "u(U)"'] & \end{tikzcd} \qquad \begin{tikzcd} t \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(U)(t) \\ s_t \arrow[ur, dashed, mapsto, "u(U)"'] & \end{tikzcd} ,$$

  where t is an element in $\mathscr{F}(U)$. We will show that $u(U)(s_t) = \varphi(U)(t)$ uniquely determines the morphism u. According to the condition (2) in our construction, for any element (function) $s \in \mathscr{F}^+(U)$, each point $P \in U$ has an open neighborhood V_P such that $s|_{V_P} = s_{t_P}|_{V_P}$ for some element $t_P \in \mathscr{F}(U)$. So there is an open cover $\{V_P\}_{P \in U}$ of U, and for each V_P, we have

$$u(U)(s)|_{V_P} = u(U)(s|_{V_P}) = u(U)(s_{t_P}|_{V_P}) = u(U)(s_{t_P})|_{V_P} = \varphi(U)(t_P)|_{V_P}$$

  by the morphism of presheaves and $u(U)(s_t) = \varphi(U)(t)$. Now, we get a set of elements $\varphi(U)(t_P)|_{V_P} \in \mathscr{G}(V_P)$ with the open cover $\{V_P\}_{P \in U}$ of U, and they are obviously compatible since restrictions $s_{t_P}|_{V_P} = s|_{V_P} \in \mathscr{F}^+(V_P)$ are compatible on U. Because $\mathscr{G}$ is a sheaf, there is a unique element $u(U)(s) \in \mathscr{G}(U)$ such that $u(U)(s)|_{V_P} = \varphi(U)(t_P)|_{V_P}$ for each V_P. Hence, the morphism $u(U)$ is uniquely determined.

- The sheafification $(\mathscr{F}^+, \theta)$ of a presheaf $\mathscr{F}$ is unique up to unique isomorphism, since if another pair $(\mathscr{G}, \gamma)$ is also a sheafification of $\mathscr{F}$, then there is a unique pair of morphisms $\phi : \mathscr{F}^+ \to \mathscr{G}$ and $\varphi : \mathscr{G} \to \mathscr{F}^+$ such that the following diagram commute

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\gamma"] \arrow[d, "\theta"'] & \mathscr{G} \arrow[dl, dashed, shift left, "\varphi"] \\ \mathscr{F}^+ \arrow[ur, dashed, "\phi"] & \end{tikzcd},$$

so we have $\gamma = \phi \circ \theta$ and $\theta = \varphi \circ \gamma$, they can deduce that $\theta = (\varphi \circ \phi) \circ \theta$ and $\gamma = (\phi \circ \varphi) \circ \gamma$, however $\mathsf{id}_{\mathscr{F}^+}$ is the unique morphisms such that $\theta = \mathsf{id}_{\mathscr{F}^+} \circ \theta$ by the universal property of θ (definition of sheafification)

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\theta"] \arrow[d, "\theta"'] & \mathscr{F}^+ \\ \mathscr{F}^+ \arrow[ur, dashed, "\mathsf{id}_{\mathscr{F}^+} = \varphi\circ\gamma"'] & \end{tikzcd},$$

so we have $\varphi \circ \gamma = \mathsf{id}_{\mathscr{F}^+}$, and so is $\phi \circ \varphi = \mathsf{id}_{\mathscr{G}}$. Hence, we got an isomorphism $\mathscr{F}^+ \simeq \mathscr{G}$ and the isomorphism is unique.
A natural corollary is if presheaves $\mathscr{F} \simeq \mathscr{G}$ are isomorphic, then their sheafification $\mathscr{F}^+ \simeq \mathscr{G}^+$ (isomorphism of sheaves).

3.8 ***Remark*** (Sheafification as a functor)

The sheafification could be recognized as a functor $\mathsf{Sh} : \mathsf{PSh}(X) \to \mathsf{Sh}(X), \mathscr{F} \mapsto \mathscr{F}^+$, and for a presheaf morphism $\varphi : \mathscr{F} \to \mathscr{G}$, its induced sheaf morphism is naturally defined by $\varphi^+(U) : \mathscr{F}^+(U) \to \mathscr{G}^+(U), s \mapsto \varphi^+(U)(s)$, and the function

$$\begin{aligned} \varphi^+(U)(s) : U &\longrightarrow \bigcup_{P \in U} \mathscr{G}_P \\ Q &\longmapsto \varphi_Q(s(Q)) \end{aligned}$$

where $\varphi_Q : \mathscr{F}_P \to \mathscr{G}_P$ is the induced morphism under stalk functor as we have mentioned in remark 3.2

3.9 ***Remark*** (The Induced morphism and universal property)

The unique map ϕ in sheafification (construction 3.6) could be recognized as the induced map φ^+ of the morphism φ under sheafification, since we have the following commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathscr{G} \\ \mathscr{F}^+ \arrow[r, "\varphi^+"'] \arrow[ur, dashed, "\phi"] & \mathscr{G}^+ \arrow[u, "\simeq"'] \end{tikzcd},$$

where the isomorphism $\mathscr{G}^+(U) \xrightarrow{\simeq} \mathscr{G}(U), s_t \longmapsto t$, since $\mathscr{G}$ is a sheaf, we have $\mathscr{G} \simeq \mathscr{G}^+$. It is easy to verify that

$$t \overset{\theta}{\longmapsto} s_t \overset{\varphi^+}{\longmapsto} s_{\varphi(t)} \overset{\simeq}{\longmapsto} \varphi(t)$$

for each element $t \in \mathscr{F}(U)$, and each open set U. So the unique morphism ϕ in universal property is the composition of the isomorphism and induced map φ^+, which means it could also just be recognized as the induced map φ^+.

3.10 **Definition** (Inclusion Functor)

Let $\mathscr{F}$ be a sheaf on space X, then it could also be recognized as a presheaf (forgetting the two additional condition of sheaves). So there is an 'inclusion functor' $i : \mathsf{Sh}(X) \to \mathsf{PSh}(X)$, which preserves all elements and morphisms in $\mathsf{Sh}(X)$.

3.11 **Theorem**

Consider the category of abelian presheaves (sheaves) on X. The Stalk $\square_x$ and Skyscraper i_x^* are adjoint functors, sheafification Sh and inclusion functor i are adjoint functors as follow:

$$\begin{tikzcd} \mathbf{Ab} \arrow[r, bend right, "i_x^*"'] & \mathsf{Sh}(X) \arrow[l, bend right, "\square_x"'] \arrow[r, bend right, "i"'] & \mathsf{PSh}(X) \arrow[l, bend right, "\mathsf{Sh}"'] \end{tikzcd}.$$

Proof.
- Stalk $\square_x$ is a left adjoint functor of Skyscraper i_x^*, since stalk $\square_x$ is a pullback functor of the inclusion map $i : \{x\} \to X$ by proposition 3.4, and the Skyscraper i_x^* is the pushforward functor of the same inclusion map i.
- To prove sheafification Sh is a left adjoint functor of the inclusion i, it is sufficiently to prove $\mathsf{Hom}_{\mathsf{Sh}(X)}(\mathsf{Sh}(\mathscr{G}),\mathscr{F}) \simeq \mathsf{Hom}_{\mathsf{PSh}(X)}(\mathscr{G}, i(\mathscr{F}))$ for any sheaf $\mathscr{F} \in \mathsf{Sh}(X)$ and presheaf $\mathscr{G} \in \mathsf{PSh}(X)$, *i.e.*,

$$\mathsf{Hom}_{\mathsf{Sh}(X)}(\mathscr{G}^+,\mathscr{F}) \simeq \mathsf{Hom}_{\mathsf{PSh}(X)}(\mathscr{G},\mathscr{F}).$$

Let $\phi : \mathscr{G}^+ \to \mathscr{F}$ be a morphism of sheaves, then there is a natural presheaf morphism $\phi \circ \theta : \mathscr{G} \to \mathscr{F}$ by sheafification, so we have a map

$$\begin{aligned} \alpha : \mathsf{Hom}_{\mathsf{Sh}(X)}(\mathscr{G}^+,\mathscr{F}) &\longrightarrow \mathsf{Hom}_{\mathsf{PSh}(X)}(\mathscr{G},\mathscr{F}) \\ \phi &\longmapsto \phi \circ \theta \end{aligned}$$

On the other side, for each presheaf morphism $\varphi : \mathscr{G} \to \mathscr{F}$, according to sheafification, there is a unique morphism u_φ such that the following diagram commute

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathscr{F} \\ \mathscr{G}^+ \arrow[ur, dashed, "u_\varphi"'] & \end{tikzcd}$$

, *i.e.*, $u_\varphi \circ \theta = \varphi$. So we could construct another map

$$\begin{aligned} \alpha : \mathsf{Hom}_{\mathsf{PSh}(X)}(\mathscr{G},\mathscr{F}) &\longrightarrow \mathsf{Hom}_{\mathsf{Sh}(X)}(\mathscr{G}^+,\mathscr{F}) \\ \varphi &\longmapsto u_\varphi \end{aligned}$$

Then we immediately get $\beta \circ \alpha(\varphi) = u_\varphi \circ \theta = \varphi$, and $\alpha \circ \beta(\phi) = u_{\phi\circ\theta} = \phi$ by the uniqueness. Hence, we get the isomorphism $\mathsf{Hom}_{\mathsf{Sh}(X)}(\mathscr{G}^+,\mathscr{F}) \simeq \mathsf{Hom}_{\mathsf{PSh}(X)}(\mathscr{G},\mathscr{F})$.

■

3.12 ***Remark***

In general, according to Proposition 3.4, Stalk and Skyscraper are adjoint functors between category of sets the category of presheaves on X,

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend right, "i_x^*"'] & \mathsf{PSh}(X) \arrow[l, bend right, "\square_x"'] \end{tikzcd}.$$

3.13 **Corollary** (Sheafification preserves stalks)

Let $\mathscr{F}$ be a presheaf on space X, and $\mathscr{F}^+$ is its sheafification, then their stalks

$$\mathscr{F}_x = \mathscr{F}_x^+$$

for each point $x \in X$.

Proof. By Theorem 3.11, sheafification Sh is a left adjoint functor, so $\mathsf{Sh} : \mathsf{PSh}(X) \to \mathsf{Sh}(X)$ is cocontinuous (preserves colimits) according to Proposition 1.7. Since for a presheaf $\mathscr{F}$ on X, its stalk $\mathscr{F}_x = \varinjlim \mathscr{F}(U) = \operatorname{colim}\langle \mathscr{F}(U), \rho\rangle_{x\in U}$, we have

$$\begin{aligned}\mathscr{F}_x^+ &= \operatorname{colim}\langle \mathscr{F}^+(U), \rho^+\rangle_{x\in U} = \operatorname{colim}\langle \mathsf{Sh}(\mathscr{F})(U), \mathsf{Sh}(\rho)\rangle_{x\in U}\\ &= \mathsf{Sh}(\operatorname{colim}\langle \mathscr{F}(U), \rho\rangle_{x\in U})\\ &= \mathsf{Sh}(\mathscr{F}_x) = \mathscr{F}_x\end{aligned}$$

for each point $x \in X$. ■

3.14 ***Remark*** (Another version of this result)

For this proof, we will use the fact that for a point $P \in X$, the stalk functor $-_P : \mathbf{Ab}_{\mathrm{pre}}(X) \to \mathbf{Ab}$ defined by $\mathscr{F} \mapsto \mathscr{F}_P$ has a right adjoint, given by $i_{P*} : \mathbf{Ab} \to \mathbf{Ab}_{\mathrm{pre}}(X)$. This is defined where $i_{P*}(G)$ is the presheaf whose sections over U are G if $P \in U$, or 0 if $P \notin U$; and the restriction maps are id_G if $P \in V \subseteq U$, or the zero map otherwise. Thus, we have $\mathsf{Hom}(\mathscr{F}_P, G) \simeq \mathsf{Hom}(\mathscr{F}, i_{P*}(G))$.

However, we can check that $i_{P*}(G)$ is actually a sheaf. Therefore, by the universal property of the sheafification, we get a canonical isomorphism $\mathsf{Hom}(\mathscr{F}, i_{P*}(G)) \simeq \mathsf{Hom}(\mathscr{F}^+, i_{P*}(G))$. And now applying the adjunction again in reverse, we see that $\mathsf{Hom}(\mathscr{F}^+, i_{P*}(G)) \simeq \mathsf{Hom}(\mathscr{F}_P^+, G)$.

Furthermore, we can see that these isomorphisms are functorial in G, so we get isomorphisms of functors $\mathbf{Ab}_{\mathrm{pre}}(X) \to \mathbf{Set}$:

$$\mathrm{Hom}(\mathscr{F}_P, -) \simeq \mathrm{Hom}(\mathscr{F}, i_{P*}(-)) \simeq \mathrm{Hom}(\mathscr{F}^+, i_{P*}(-)) \simeq \mathrm{Hom}(\mathscr{F}_P^+, -).$$

Therefore, applying Yoneda's Lemma, we get a canonical isomorphism $\mathscr{F}_P \simeq \mathscr{F}_P^+$.