OpenRank Convergence Provement

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For convenience we will give the definition and convergence proof of OpenRank for heterogeneous information network with more than one type of node but only one type of edge, and then give a proof that OpenRank for heterogeneous information network with more than one type of edge is equivalent to the former one.

I. Problem Definition

With a certain directed weighted graph $G = (V, E)$, V is the node set and E is the edge set. Its adjacency matrix W consists with element w_{ij} which is the weight of the edge from node v_i to node v_j. For the node set V has an initial nodes' weight vector $\boldsymbol{v}^{(0)}$. OpenRank will give a function

$$f = V \to \mathcal{R}$$

to map each node in G to a real number.

II. Algorithm

The proposition of OpenRank is quite similar to PageRank, which is node should be ranked higher if it is linked by higher-ranked nodes. So the basic rule is that a node's weight is determined by the weights of all the nodes linked to it:

$$v_i = \sum^{|V|} v_j$$

For weighted network, the influence of each v_j should be determined by the weight of edges, so we can use edge weight to modify the formula:

$$v_i = \sum^{|V|} w_{ji} v_j$$

But for the iterative algorithm, we need a normalization method to make sure the algorithm converge. The basic linear normalization method is introduced to modify the formula which is using weighted out-degree of v_j:

$$v_i = \sum^{|V|} \frac{w_{ji}}{d_{oj}} v_j$$

And for many real network, nodes usually have some prior information or features which can be presented as the initial weight nodes, so assuming for node v_i, the hyper-parameter a_i represents how much weight v_i relies on the network structure, then $1 - a_i$ is how much weight v_i relies on its own initial weight, given matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$$

And the element in matrix $\boldsymbol{S}$ is:

$$s_{ij} = \frac{w_{ji}}{d_{oj}}$$

Then the iterative algorithm for k-th iteration is:

$$\boldsymbol{v}^{(k)} = \boldsymbol{A}\boldsymbol{S}\boldsymbol{v}^{(k-1)} + (\boldsymbol{E} - \boldsymbol{A})\boldsymbol{v}^{(0)}$$

So $\boldsymbol{v} = \lim_{k\to\infty} \boldsymbol{v}^{(k)}$ is the final weight vector for all the nodes.

III. Proof of Convergence

The above algorithm performs a method to map each node in a network to a real number for any weighted directed heterogeneous information network with only one type of edge. But we need to make sure the algorithm can converge while step k grows. Since

$$\boldsymbol{v}^{(k)} = (\boldsymbol{A}\boldsymbol{S})^k \boldsymbol{v}^{(0)} + \sum_{t=0}^{k-1} (\boldsymbol{A}\boldsymbol{S})^t (\boldsymbol{E} - \boldsymbol{A})\boldsymbol{v}^{(0)}$$

The convergence of the algorithm relies on the convergence of $\lim_{k\to\infty} (\boldsymbol{A}\boldsymbol{S})^k$ and $\sum_{k=0}^{\infty} (\boldsymbol{A}\boldsymbol{S})^k$.

When $\boldsymbol{A} = \boldsymbol{E}$, $\boldsymbol{A}\boldsymbol{S} = \boldsymbol{S}$ and $\boldsymbol{E} - \boldsymbol{A} = \boldsymbol{0}$, then the convergence of OpenRank is equivalent to the convergence of $\lim_{k\to\infty} (\boldsymbol{A}\boldsymbol{S})^k$. In this situation, nodes' finial values only depends on the network, so it is equivalent to the original PageRank. Then if $\boldsymbol{S}$ is a primitive stochastic matrix, $\boldsymbol{v}^{(k)}$ will converge to the stationary probability vector of the corresponding stochastic process.

When $\boldsymbol{A} \neq \boldsymbol{E}$, $\lim_{k\to\infty} (\boldsymbol{A}\boldsymbol{S})^k$ and $\sum_{k=0}^{\infty} (\boldsymbol{A}\boldsymbol{S})^k$ should both converge to make sure the process converges. According to the convergence theorem of Neumann series, $\lim_{k\to\infty} (\boldsymbol{A}\boldsymbol{S})^k$ converges if and only if the spectral radius of matrix $\boldsymbol{A}\boldsymbol{S}$ should meet $\rho(\boldsymbol{A}\boldsymbol{S}) < 1$, then the convergence result will be $(\boldsymbol{E} - \boldsymbol{A}\boldsymbol{S})^{-1}$. Specially, if all nodes partially depends on its own initial weight, which is for all elements in diagonal matrix $\boldsymbol{A}$, $a_{ii} \in [0, 1)$. Since the spectral radius is the infimum of all norms of the matrix, if we consider 1-norm of matrix $\boldsymbol{AS}$, $\rho(\boldsymbol{AS}) \leq ||\boldsymbol{AS}||_1 \leq ||\boldsymbol{A}||_1||\boldsymbol{S}||_1 = max(a_{ii}) \times 1 < 1$, so OpenRank converges.

In summary, OpenRank converges for all directed weighted heterogeneous information network with more than one type of node and only one type of edge if every node in the network partially depends on its own initial weight, and the result is

$$\begin{aligned}
\boldsymbol{v} &= \lim_{k\to\infty} \boldsymbol{v}^{(k)} \\
&= \lim_{k\to\infty} [\boldsymbol{ASv}^{(k-1)} + (\boldsymbol{E}-\boldsymbol{A})\boldsymbol{v}^{(0)}] \\
&= \lim_{k\to\infty} [(\boldsymbol{AS})^k\boldsymbol{v}^{(0)} + \sum_{t=0}^{k-1}(\boldsymbol{AS})^t(\boldsymbol{E}-\boldsymbol{A})\boldsymbol{v}^{(0)}] \\
&= (\boldsymbol{E}-\boldsymbol{AS})^{-1}(\boldsymbol{E}-\boldsymbol{A})\boldsymbol{v}^{(0)}
\end{aligned}$$

IV. Extend to General Heterogeneous Information Network

Assume we have a heterogeneous information network with only one type of node and n types of edge, and for node i, it will transfer its value through type k edge e_k by ratio a_k, which requires $\sum_{k=1}^{n} a_k = 1$.

Specially, consider value transfer from node v_i to v_j, assuming w_{ijk} is the normalized weight of type k edge e_{ijk} which means $\sum_j w_{ijk} = 1$. Then the value transfer from v_i to v_j is $\sum_{k=1}^{n} v_i w_{ijk} a_k = v_i \sum_{n=k}^{n} w_{ijk} a_k$.

So if we use a single edge e_{ij} with normalized weight $w_{ij} = \sum_{n=k}^{n} w_{ijk} a_k$ to replace all the edges e_{ijk}, the value transformation is equivalent.

That is, in a heterogeneous information network containing k types of edges, for any node pair v_i and v_j, using a single edge e_{ij} with edge weight $w_{ij} = \sum_{n=k}^{n} w_{ijk} a_k$ to replace all the heterogeneous edges between v_i and v_j, the result of OpenRank is the same as that of the original network.

Similarly, it can be extended to heterogeneous node heterogeneous networks, that is, OpenRank is applicable to any heterogeneous information network.