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Metapopulation Differential Co-Evolution of Trading Strategies in a Model Financial Market

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***Abstract*—This paper reports results from experiments using differential evolution (DE) in a high-fidelity simulation model of a contemporary financial market in which various traders are each simultaneously trying to adapt their own trading strategy to be as profitable as possible, given the distribution of strategies currently deployed at that time by other traders in the market. In our model, each trader maintains its own private local population of trading strategies, and uses DE to adaptively improve its strategies over time. Because all traders are simultaneously trying to adapt their strategies, and because the profitability of any one strategy at time t can only be determined in reference to all other strategies also active in the market at time t, the system is *co-evolutionary* rather than simply evolutionary. Furthermore, the existence of multiple separate DE populations in the system (i.e., one local DE population for each trader) means that technically this is a co-evolutionary *metapopulation* system. Using DE in a co-evolutionary metapopulation context requires extension of the usual DE approaches used in less challenging applications, chief of which is the introduction of a mechanism to detect and actively prevent convergence within each local DE population. Results are presented which demonstrate that when all traders are using this nonconvergent DE, the overall economic efficiency (i.e., the sum of profitability over all traders) of the market is greatly higher than a baseline established when all traders were using a simple stochastic hill-climbing strategy optimizer instead of DE. Source-code for the experiments described in this paper has been released on GitHub as open-source, freely available for other researchers to use to replicate and extend the results presented here.**

***Index Terms*—Zero-Intelligence Traders; Financial Markets; Automated Trading; Co-Evolution; Differential Evolution; Metapopulation Dynamics.**

I. Introduction

Many present-day financial markets involve very high proportions of adaptive automated trading systems, each continuously adjusting its trading strategy to try to be as profitable as it can be, given the current distribution of other trading strategies present in the market that it finds itself competing against. But each of those competing strategies is also simultaneously engaged in the same form of continual adjustment to try to maintain its profitability, and hence contemporary financial markets are manifestly *co-evolutionary* systems. Understanding the dynamics of today's financial markets is an important problem, given the obvious concerns that arise over whether such markets are stable or unstable, and whether they economically efficient or inefficient; but it is also a fiendishly difficult problem, given the large number of nonlinearly interacting constituent elements within such market systems. As in the study of other complex adaptive systems, researchers seeking to explore or explain contemporary markets typically either analyse and model fine-grained data from real-world systems (which, in the context of financial exchanges, is an approach known as the study of *market microstructure*: see, e.g., [1]–[4]); or they conduct series of laboratory experiments on different types of simplified situations under carefully controlled conditions (an approach known as *experimental economics*: see, e.g., [5]–[8]); or increasingly they turn to accurate computer-simulation models of markets and the populations of traders that inhabit them, using agent-based modelling (ABM) techniques (an approach known as *agent-based computational economics*, or ACE: see, e.g., [9]–[11]). This paper reports on the introduction of differential evolution (DE: [12]–[14]) into a long-established public-domain open-source agent-based model of a contemporary financial exchange: to do that, traditional DE methods needed to be modified, but the resulting huge increase in economic efficiency in the markets indicates that DE is a promising method for use in such models, and potentially also in corresponding deployments in real-world markets.

The open-source financial exchange ABM used here is *BSE* [15], the Python source-code for which has been freely available on GitHub since 2012 [16]. BSE is a high-fidelity simulation of a modern financial exchange centred on a detailed implementation of an exchange's *matching engine* (which matches bid orders from buyers with compatible ask orders from sellers, joining them to create transactions) and the exchange's data-structure known variously as the *limit order book* or *ladder*, which is a record of all bid orders and ask orders currently active at the exchange but not yet matched with a counterparty, i.e. not yet transacted. BSE simulates the market at sub-second time-resolution, and can support simulations involving very large numbers of traders. Each trader in BSE is an instance of a particular type of automated trading system – some of which are adaptive, changing their trading behavior in response to market conditions; and others of which are nonadaptive, issuing their bids and/or asks at time and prices set by their internal algorithm that does not alter over time. In this paper I introduce to BSE a new type of adaptive trader, using DE to continuously improve its trading strategy, which (for reasons that are explained in Section II) is known by the acronym PRDE (for *Parameterized-Response Differential Evolution*; pronounced *"purdy"*), and show that PRDE traders are more profitable than their predecessors.

Section II explains the background to this work in more detail. Section III describes how DE was used within each PRDE trader, the problem that arose involving convergence within a trader's local DE population, and the remedy that was introduced to address this. After that, Section IV presents results from our markets populated by PRDE traders, and the PRDE markets are shown to be much more economically efficient when DE is used, in comparison to a baseline established in earlier work where traders used stochastic-hill-cilmbing instead of DE. Section V then concludes this paper with a discussion of various potential paths for further work.

II. Background

A. Automated Trading in Financial Markets

Formally instituted financial markets originated in the 1680s with the founding of what became the Amsterdam Stock Exchange, and for the next three centuries such markets involved human traders engaging with one another, interacting initially face-to-face and latterly via telephones and then via computer terminals, the traders bargaining and haggling, buyers quoting their bid prices and sellers quoting their ask prices, each trying to get a deal at a price that satisfies their needs. However in the past twenty years, at the point of execution in most major financial markets, very many human traders have been replaced by machines which do exactly the same job, but which can process super-human quantities of data, and can react to changes in the market at super-human speeds. The days of markets populated mainly by human traders are over.

These automated trading systems, known various as *algorithmic traders*, *robot traders*, and in their speediest form *high-frequency traders*, typically involve mathematical models that seek to exploit high-order statistical regularities in the time-series of financial data: see, e.g., [17]; although the development has been met with dismay by several seasoned market professionals, as documented in [18]–[20], and there is clear evidence that modern-day market dynamics can be dramatically different from their relatively stable pre-automation regimes, a point illustrated most starkly by the now-infamous "Flash Crash" of 6 May 2011, when more than US$800Bn was wiped from the value of US capital markets in roughly 15 minutes of frenzied and largely machine-to-machine trading (see e.g. [21], [22]), and scarily similar events in various other markets since then. Understanding the dynamics of modern automated markets is a problem given the high rates of innovation, with the continuous introduction of new types of robot trader and of existing robots being modified and improved, as law-makers and regulators struggle to keep up.

B. Models of Markets and Traders

Working with data from real financial exchanges poses a number of major difficulties: the data streams can be truly huge, and need to be treated with high degrees of confidentiality; it is often impossible or wholly impracticable to run controlled tests on real exchanges; and any trader running a profitable robot-trading operation will most likely keep the details of the algorithms involved as a very closely guarded commercial secret.

Faced with these obstacles, researchers have sought alternative methods of understanding highly automated markets. Research in experimental economics, a field founded and grown by the economist Vernon Smith (for which he was awarded the 2002 Nobel Prize in Economics) typically involves having a small number (tens, rather than hundreds) of human subjects interact with one another via an electronic market platform, with the experimenters carefully controlling the supply and demand in the experimental market, and/or the flows of information within the market, and/or the opportunity sets (i.e., space of possible actions) of the participants in those markets. Smith published his seminal paper on this approach in 1962 [23], and for the next 30 years all the focus in experimental economics was on studying the dynamics of experimental markets populated by human traders. This anthropocentric focus was interrupted in 1993, when Gode & Sunder [24] introduced some startling results from experimental markets populated by minimally simple algorithmic trading systems, referred to as *zero intelligence* (ZI) traders.

Gode & Sunder's 1993 study is simple to describe. First, they set up an unremarkable experimental economics study, with human subjects buying and selling in the kind of market mechanism (technically: a *continuous double auction*, or CDA) that is the basis of most the world's financial exchanges, and they recorded and analysed the time-series of transaction prices in this market, to establish some baseline statistics for how humans perform in this particular market. Next, they replaced the human traders with robot traders that simply quoted random prices for their bids or asks, a type of trader that Gode & Sunder named ZIU (*Zero Intelligence Unconstrained*); the time-series of transactions for markets populated by ZIU looked much like random noise, with no discernible convergence to any kind of sensible stable price. Finally, they modified the ZIU traders, introducing a simple constraint that when the robot trader generated a random price for a bid or an ask, that price should not be one that would lead to a loss-making transaction – these traders were named ZIC (*Zero Intelligence Constrained*) and they gave surprisingly human-like market dynamics: in Gode & Sunder's paper, the common measure of the efficiency of a market, a metric known as *allocative efficiency*, was essentially the same for ZIC-populated markets as it was for the human-populated markets. Gode & Sunder's work can fairly be described as the sparking point of the robot trader revolution.

Gode & Sunder's groundbreaking ZIC trader was entirely non-adaptive: each ZIC trader relied on generating its random quote-prices from a specified uniform distribution, bounded above by the maximum purchase price assigned to a specific buyer, or below by the minimum sale-price assigned to a specific seller. But Gode & Sunder's results prompted others to explore the feasibility of adding in simple adaptivity or machine learning to such ZI traders, extending the ZI approach with the smallest amount of additional machinery necessary and sufficient to give better trading performance, an approach more accurately described as *minimal-intelligence* (MI) trading. Two notable developments in this line of research was the minimal machine-learning-based robot trader called ZIP (Zero Intelligence Plus) described in [25]; and the elegantly simple probabilistic-model-building method reported by Gjerstad & Dickhaut in [26], an algorithmic trading strategy now widely referred to simply as "GD". These two robot trader strategies were catapulted into prominence with the publication of a paper at IJCAI 2001 [27] by a team of researchers at IBM's main T. J. Watson Research Labs, who demonstrated for the first time that both ZIP and GD could repeatedly outperform human traders in experimental economics studies that mixed robots and human traders in the same markets – that is, these two robot-trader strategies were consistently more profitable than human traders. This was a result that generated worldwide media coverage, was subsequently replicated by other researchers [28]–[31], and which poured fuel on the flame lit by Gode & Sunder: the rise of robot trading in financial markets accelerated rapidly in the years following IBM's announcement of the supremacy of machine traders, which the IBM authors described as heralding the birth of a new business opportunity worth billions of dollars annually.

In the two decades since the IBM study was published, a fair number of further minimal-intelligence algorithmic trading strategies have been proposed, by researchers in industry and in academia, with each new strategy proposed as an improvement in one respect or another on those that had been published previously: for a summary of this sequence of developments in trading strategies, see [32]; for reviews of the surprising utility of ZI-style traders as models of real markets, see [33]–[35]. However for the purposes of this paper, we need only to discuss two recently-introduced minimal-intelligence trading strategies, PRZI and PRSH, which are each direct descendants of Gode & Sunder's ZIC:

- PRZI (*Parameterized-Response Zero Intelligence*; pronounced "prezzy": see [32]) traders are a nonadaptive generalisation of ZIC traders: whereas each individual ZIC trader uses a fixed uniform distribution as the probability mass function (PMF) to randomly generate its quote-prices, instead each PRZI trader has a strategy parameter value $s \in [-1.0, +1.0] \in \mathbb{R}$ which determines the PMF envelope for that trader: when a PRZI trader has $s = 0$, its PMF is identical to that of a ZIC trader (i.e., uniform); but as $s \to \pm 1.0$ the shape of the trader's quote-price generator PMF smoothly distorts to be more biased toward quoting either "urgent" or "relaxed" prices, such that when a PRZI trader has $s = +1.0$ it is maximally urgent, and will generate quote-prices from a PMF envelope very heavily weighted towards the least profitable price for that trader (which is hence most likely to attract a willing counterparty, all other things being equal), and when a PRZI trader has $s = -1.0$, it is maximally relaxed, using a quote-price-generation PMF that is very heavily biased toward prices that generate the most profit for that trader, but which are hence less likely to attract a counterparty. Thus in summary any PRZI trader has a particular strategy value s and that determines the PMF for its stochastic generation of quote-prices, but PRZI is not defined to adapt is strategy over time.
- PRSH (*PRZI Stochastic Hillclimber*; pronounced "pursh": see [36]) is a minimally simple instance of an adaptive PRZI trader, i.e. one that dynamically alters its trading strategy value s, attempting to always be increasing profitability. Each PRSH trader uses a simple stochastic hillclimber to optimise its s-value, in a manner that is similar to a k-armed bandit (see, e.g., [37]–[40]): that is, a PRSH trader maintains a private local population of k strategy-values; it loops forever on a process where in each loop it evaluates each of the k strategies in turn and then identifies which is currently the most profitable – the most profitable strategy is referred to as the *elite* strategy. On each iteration of the loop, once all k strategies have been evaluated and the elite identified, a new population of k individuals (i.e., the next *generation* of the population) is created by copying in the elite from the previous generation, and then creating $k-1$ "mutants" of the elite strategy, typically by adding a small amount of Gaussian noise to the elite s-value.

This paper presents first results from experiments where the entire market is populated by a successor to PRSH, named PRDE (for *PRZI with Differential Evolution*; pronounced "purdy"), in which the simple stochastic hiill-climber of PRSH is replaced by a differential evolution system: whereas each PRSH trader can be thought of as operating a very primitive k-armed bandit algorithm, instead each PRDE trader operates its own private DE system with a specific population-size (traditionally denoted by NP in the DE literature). Initial results from co-evolving markets populated entirely with PRSH traders were reported in [36] and the methods and results are summarised below in Section II-C – these results provide a set of baseline measures against which any changes resulting from the switch to PRDE can be evaluated. Results from co-evolutionary markets populated by PRDE, using the same experiment methods as for PRSH, are then presented in Section III.

C. Co-evolution with Stochastic Hillclimbing (PRSH)

Reference [36] reports on a set of experiments in which BSE is used to simulate a financial market for a single abstract tradeable commodity in which the number of active traders (denoted by N_T) is 60, split equally into 30 buyers and 30 sellers, and where each trader is running PRSH with k=4. All buyers were instructed to pay no more than \$140 per unit when purchasing, and all sellers were instructed to sell for no less than \$60 per unit: this style of supply and demand schedule gives what economists refer to as perfect elasticity of supply and of demand, and has been used before in many experimental economics studies (see, e.g., [41]); it means that in principle every seller can find a buyer (and *vice versa*) who is able to be a counterparty to a transaction – i.e., no traders are given extra-marginal limit prices that would prevent them from finding a counterparty. When two traders enter into a transaction, both the buyer's assignment of cash and the seller's assignment of units of stock to sell are depleted to render them inactive, and they each then wait a random period of a few (≈ 5) seconds before they are re-assigned fresh cash or stock, and re-enter the market as active traders.

BSE simulates continuous time using a discrete time-slicing approach using a temporal step-size of $\Delta_t = 1/N_T$, i.e. 0.0167sec for N_T=60, so that each trader interacts with the market at least once per second; and the experiments reported in [36] were run for very long durations, typically simulating sequences of hundreds of days of continuous round-the-clock 24×7 trading: although the traders are interacting with one another on sub-second timescales, the co-evolutionary dynamics play out over much longer time periods. The simple stochastic hillclimber in PRSH evaluated each of the k strategies for 2 hours of simulated trading, an evaluation period denoted here by Δ_E = 7200sec, so with k=4 it took 8 simulated hours of trading for one trader to evaluate all of its strategies (i.e., all of the arms on its k-armed bandit), and then it would identify which of the strategies was the elite, and then copy the elite and $k-1$ mutants of the elite into its set of strategies for the next iteration of the evaluation loop. Running on a 2022-model Apple Mac Mini with M1 silicon, the simulation of BSE with 60 PRSH traders ran at roughly 120× real-time, so simulating 100 days of trading took roughly 20 hours of wall-clock CPU time.

Evaluating a strategy s at time t involved measuring the profitability of s over a time-period from $t-\Delta_E$ to t, and then dividing that profitability by Δ_E to give a measure referred to as *profit per second* (PPS) for s. The profit on any one transaction was given by the absolute difference between the transaction price agreed by the buyer and seller, and those traders' individual limit prices. Because in these experiments all buyers had the same limit price of \$140, and all sellers had the same limit price of \$60, if a transaction took place at a price of \$90 then the buyer's profit on that transaction was \$50 and the seller's profit was \$30. An individual PRSH trader's strategy-value affects the probability mass function (the envelope of the distribution) for its stochastically-generated quote prices, so altering a trader's s-value can in principle increase the profitability of individual trades, at the cost of increasing the expected wait-time before a trade takes place at that price; or conversely it can increase the rate at which trades take place, but decrease the expected profitability of each such trade.

As illustration of typical results from one such experiment with PRSH, Figure 1 shows results from a 300-day experiment, plotting the total PPS generated by the 30 buyers (denoted by π_B), the total PPS generated by the 30 sellers (denoted by π_S), and the total PPS extracted from these two sides of the market (i.e., by all buyers and sellers combined, denoted by π_T, s.t. $\pi_T = \pi_B + \pi_S$). As can be seen, there is an initial adaptive transient over days 0 to ≈25, in which the π_T value rises from ≈50 at $t = 0$ to ≈90 by Day 25. After that period in which the system's random initial conditions are manifestly improved upon, the system then settles to an ongoing dynamic in which π_T is relatively constant, but in which there are temporally protracted periods where one side is consistently being more profitable than the other (i.e. either $\pi_B > \pi_S$ or $\pi_S > \pi_B$ continuously for some period of time): the roughly-constant π_T is to be expected, given the zero-sum nature of the PPS calculation (for π_B to go up, π_S must go down, and *vice versa*); but the temporal coherence, the fact that either the buyers or the sellers are consistently the more profitable group for long sequences lasting multiple days (i.e., periods of strong autocorrelation in PPS) is an indication that the underlying co-evolutionary dynamics involve surprisingly long-term transients: long periods where one side or the other holds the upper hand, is more profitable, until the other side co-evolves a response that improves its profitability; the tables are then turned, and the pressure is now on the previously more-profitable side to evolve to regain the upper hand.

Fig. 1. Plot of profitability data from one 300-day experiment in a market populated entirely by PRSH traders (i.e., using stochastic hill-climbing as their strategy-optimizer). Horizontal axis is time, measured in days; vertical axis is simple moving average of profit per second (PPS) over the preceding 7 days. The line labelled *B-profit* shows the total PPS generated by the sub-population of buyers (denoted by $\pi_B(t)$ in the text); the line labelled *S-profit* shows the total PPS generated by the sub-population of sellers (i.e., $\pi_S(t)$), and the line labelled *Sum* shows the total PPS extracted by all traders (i.e., $\pi_T(t) : \pi_T(t) = \pi_B(t) + \pi_S(t)$). See text for further discussion.

To illustrate the co-evolutionary dynamics at the level of individual trader's strategies, Figure 2 shows a heatmap of the elite s-values for the 30 buyers in the market over the same 300-day experiment for which PPS values were shown in Figure 1, and Figure 3 shows the corresponding heatmap for the elite s-values for the 30 sellers in that same experiment. In this experiment the initial (t=0) strategy values were assigned randomly from a uniform distribution over the range $[-1.0, +1.0]$: after the 25-day opening adaptive transient

Fig. 2. Heat-map of individual strategy-values for the population of 30 PRSH buyers in the experiment for which profitability values were plotted in Figure 1. Horizontal axis is time measured in days; vertical axis is simple moving average of strategy value over past 7 days. The space is pixelated into 40 bins on the vertical axis (i.e., bin-size of 0.05) and 300 bins (i.e., one per day) on the vertical: intensity of pixel-shading ('"heat") increases with the sample probability density – i.e., each bin is displayed as its raw count divided by the total number of counts and the bin width. There is a dominant mode at $s \approx -1.0$ and a second more diffuse mode meandering around the range $[+0.35, +0.75]$; see text for further discussion.

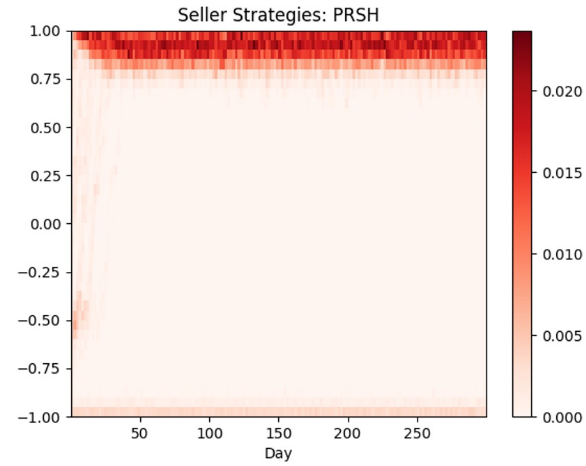

Fig. 3. Heat-map of individual strategy-values for the population of 30 PRSH sellers in the experiment for which profitability values were plotted in Figure 1; format is the same as for Figure 2. There is a major broad mode in the range $s \in [+0.8, +1.0]$ and a much weaker but more focused minor mode at $s \approx -1.0$; see text for further discussion.

when π_T is climbing to its steady-state value, thereafter there is clear multi-modality in the distribution of strategy values, and even after hundreds of days of trading there is no sign of the system settling to a fixed distribution. This multimodality is a very common outcome for this type of experiment with PRSH traders: [36] shows further sets of independent and identically distributed (IID) individual-experiment results from this type PRSH-market, each of which results in multimodal distributions of strategies; here, multimodality in the norm.

Note that, if ever the population did converge to a static set of strategies, using the language of dynamical systems analysis we might say that it was settling to a fixed-point attractor in the 30-dimensional space of possible buyer strategy-sets. That is: at any time t each buyer b has its own single elite strategy-value $s_b(t) \in [-1.0, +1.0] \in \mathbb{R}$ and hence the distribution of elite strategies in the population of 30 buyers can be conceived of as a vector $\vec{V}_B(t)$ in a 30-dimensional buyer-strategy phase-space, denoted by PS_B, s.t. $PS_B \in [-1.0, +1.0]^{30} \in \mathbb{R}^{30}$; and over time the co-evolving population of buyers traces a trajectory through PS_B as their individual s_b values alter: if ever the population converged on a fixed set of 30 strategies which remained unchanged thereafter (i.e., to a set of 30 *evolutionary stable strategies*: see e.g. [42]) then the trajectory would halt at that point in PS_B.

Effectively visualising trajectories in such high-dimensional phase-spaces is nontrivial, but thankfully in recent decades physics researchers have developed a set of conceptually intuitive visualisation tools and techniques that have proven to be usefully effective: the primary visualisation tool of choice here is the *recurrence plot* (RP), a 2-D matrix of cells/pixels that (in the simplest formulation of RPs) are either shaded or unshaded, with a shaded cell denoting that the system at some time t_1 is at a point in its phase space that is sufficiently close to a point in the phase space that it previously visited at an earlier time $t_0 : t_0 < t_1$ that the state of the system at t_0 is said to have *recurred* at t_1. RPs will often show particular distributions or macro-scale patternings of shaded and unshaded cells that are immediately obvious to the human eye, and that are informative of the system's dynamics; these can be quantified by the application of straightforward image-processing algorithms to the RP, a technique known as *recurrence quantification analysis* (RQA): for further details of RP and RQA, see the Appendix of this paper. Figure 4 shows a RP for the co-evolving set of 60 trader strategies during the experiment whose profitability was illustrated in Figure 1: the caption to Figure 4 explains the salient features of that particular RP.

The description in this section has introduced the experiment design/setup and the three modes of visualization that will be used in Section IV to show the improvements arising when the traders populating the market are switched from PRSH to PRDE; i.e., when differential evolution is introduced as the adaptive mechanism within the co-evolutionary market.

III. Co-evolution with Differential Evolution

Continuing in the spirit of minimalism that pervades research in markets populated by ZI/MI traders, the implementation of DE as the adaptation mechanism for PRZI traders (i.e., extending PRZI to give PRDE) is deliberately bare-bones. Each PRDE trader maintains its own private local population of potential strategy-values, of population-size $NP \geq 4$, which for trader i can be denoted by $s_{i,1}, s_{i,2}, \ldots, s_{i,NP}$. Because PRZI traders involve only a single real scalar value to specify their bargaining behavior, each individual in the

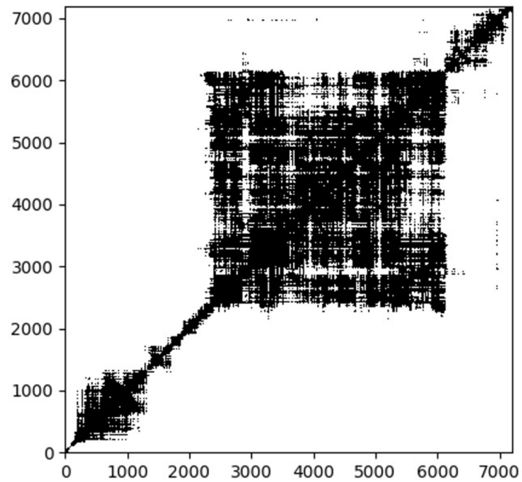

Fig. 4. Recurrence plot (RP) for the co-evolutionary trajectory through strategy phase-space of the population of PRSH traders in the 300-day experiment that was illustrated in Figure 1: scale on both axes is time measured in hours (i.e.: $24 \times 300 = 7200$). The threshold distance for recurrence (denoted by ϵ: see Appendix) is 2.5% of the maximum possible distance in phase-space. The *trapping time* for this plot is 18.01 hours: i.e., on the average, when the population is at any one point in the 60-dimensional phase-space, it will remain within ϵ of that point for roughly 18 hours – this is evidenced by the thickness of the diagonal Line of Identity (LOI) running from bottom left to top right. Also obvious from visual inspection are the multiple roughly-square shaded regions occurring along the LOI: e.g. running from $t{\approx}200$ to $t{\approx}1300$; from $\approx$1300 to $\approx$1600; from $\approx$2500 to $\approx$6100; and so on. These square shaded zones on the RP are sustained periods during which the trajectory is rarely more than ϵ distant from previously-visited points in that square's zone of the strategy phase-space – that is, the squares signify periods where relatively little changes in terms of the distribution of strategies within the market, and might reasonably be described as periods of co-evolutionary equilibria. These are punctuated by the transits between equilibria, indicated by those sections of the LOI that are relatively thin.

Fig. 5. Plot of profitability data from one 300-day experiment in a market populated entirely by PRDE traders (i.e., using differential evolution as their strategy-optimizer). Format is the same as for Figure 1.

DE population is a single value, a 1-D vector, and so the conventional DE notion of crossover (i.e., switching between two parents when selecting alleles, one allele per dimension of the genomes) is not relevant here: in PRDE, each new genome is constructed entirely from the operations on the base vector. As currently configured, PRDE applies the basic "vanilla" DE/rand/1 where, once a particular strategy $s_{i,x}$ has been evaluated, three other distinct s-values are chosen at random from the population: $s_{i,a}$, $s_{i,b}$, and $s_{i,c}$ where $x{\neq}a{\neq}b{\neq}c$, and then a new candidate strategy $s_{i,y}$ is created s.t. $s_{i,y} = \max(\min(s_{i,a}{+}F_i(s_{i,b}{-}s_{i,c}), +1), -1)$ where F_i is the trader's differential weight coefficient, (in the experiments reported here, $F_i = 0.8; \forall i$) and where the min amd max functions are introduced to keep the candidate strategy within $[-1.0, +1.0]$. Then the fitness of $s_{i,y}$ is evaluated and if it is better than that of $s_{i,x}$ then $s_{i,y}$ replaces $s_{i,x}$, otherwise it is discarded; and then the next strategy $s_{i,x+1}$ is evaluated.

In early trials with PRDE it became clear that convergence within any one trader's population of candidate strategies was a problem: in a highly-converged population, the length of the (one-dimensional) difference vector $s_{i,b} - s_{i,c}$ can tend very close to zero, and so each $s_{i,y}$ is then only very marginally different from the $s_{i,x}$ it is being compared to, and any signal in the comparison between $s_{i,x}$ and $s_{i,y}$ is swamped by noise arising from the inherent nondeterminism of the market environment in which the evaluation takes place. To address this, a simple vector-perturbation mechanism was added into PRDE, such that if at any time the maximum distance between any pair of vectors in trader i's population of candidate strategies is less than some threshold distance, then one of the strategies is chosen at random, and it is randomised to a new value generated as a draw from a uniform distribution over the entire range of allowable s-values, i.e. $[-1.0, +1.0] \in \mathbb{R}$

To enable a like-with-like comparison, in the experiments reported here $NP{=}4$ was used in each PRDE trader, as a direct correlate of the $k{=}4$ used in generating the PRSH results that PRDE will be compared against; however future work will explore the effects of using larger values of NP.

IV. Results

Figure 5 shows a PPS plot from a typical experiment for a market populated by PRDE traders.

Comparing this to Figure 1 we see that the total profit extracted (i.e., $\pi_T(t)$) by PRDE traders is roughly double that of PRSH traders; and Figure 6 shows summary data from multiple PRSH and PRDE experiments, to demonstrate that the results in Figures 1 and 5 are indeed typical: markets populated by PRDE traders are consistently $\approx$100% more profitable, more economically efficient,[1] than those populated by PRSH. Given that the only difference between PRDE and PRSH is the use of DE in PRDE *versus* stochastic hill-climbing in PRSH, this increase in performance can safely be attributed to DE.

[1]The usual measure of economic efficiency in experimental economics is calculated by dividing the actual total profit extracted from the market, by the total profit that would be expected if all transactions had instead taken place at that market"s *competitive equilibrium price* conventionally denoted by P_0. Therefore the total-PPS measure π_T used here can be thought of as a measure of efficiency, absent the division by some constant P_0.

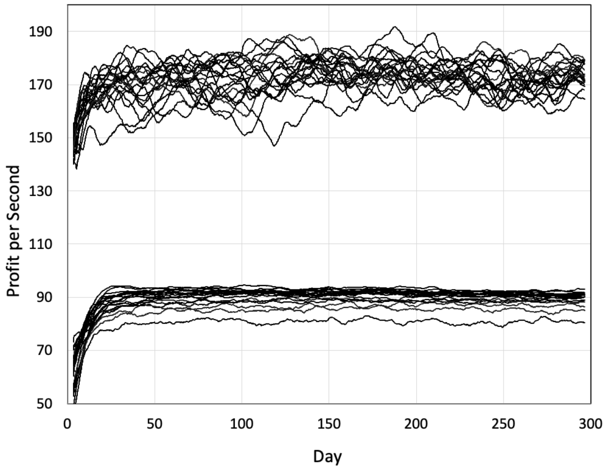

Fig. 6. Comparison of overall profitability of markets populated by stochastic hill-climber (PRSH) vs. those populated by differential evolution (PRDE) trader-agents. Horizontal axis is time measured in days of continuous trading; vertical axis is sum of total profit-per-second (PPS) measures of all traders in the market. The data plotted here come from 20 IID experiments each involving 300 days of continuous round-the-clock trading, in which the market is populated with 60 PRSH traders, and 20 IID experiments again of 300 days but where the market is instead populated with 60 PRDE traders. The only difference between the two sets of experiments is whether the traders are using stochastic hill-climbing or differential evolution. The PPS values of the PRSH-trader markets form the obvious tightly-clustered set of traces that converge to values around 90, while the PPS values of the PRDE-trader markets all form the looser cluster around 170-180. That is, the switch to DE roughly doubles the overall profitability/efficiency of the market.

Fig. 7. Heat-map of individual strategy-values for the population of 30 PRDE buyers in the experiment for which profitability values were plotted in Figure 5; format is the same as for Figure 2. See text for discussion.

Heat-maps of the individual strategies in the populations of PRDE buyers and sellers from the experiment illustrated in Figure 5 are illustrated in figures 7 and 8, respectively: as is evident from visual inspection, the multimodality is much less pronounced in the PRDE buyer population (Figure 7) than it was in the PRSH buyers (Figure 2); and it is seemingly eliminated in the PRDE sellers (Figure 8).

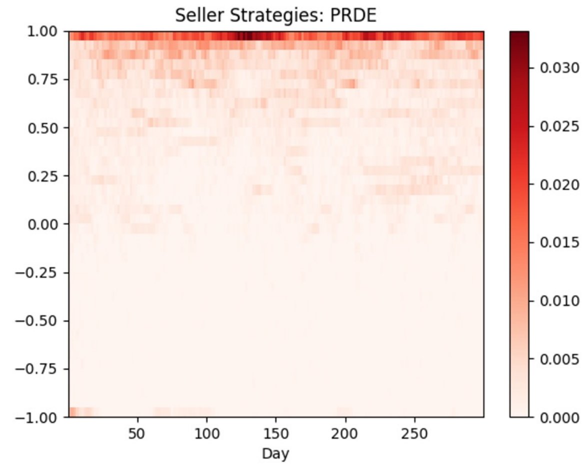

Fig. 8. Heat-map of individual strategy-values for the population of 30 PRDE sellers in the experiment for which profitability values were plotted in Figure 5; format is the same as for Figure 2. See text for discussion.

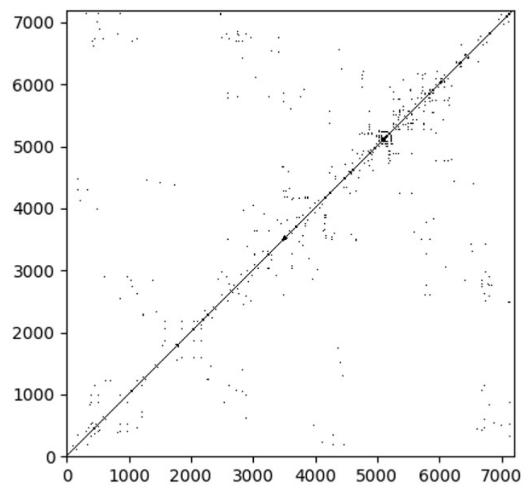

Fig. 9. Recurrence plot for the PRDE trader population whose strategy-value time-series was illustrated in figures 7 and 8: format is as for Figure 4; see text for further discussion. The trapping time for this plot is 2.37 hours, which is reflected in the LOI which is markedly thinner than that in the PRSH RP of Figure 4. Also glaringly obvious in comparison to Figure 4 is the lack of any sustained equilibrium periods (i.e., there are no big shaded squares on this RP) – instead, this very sparse RP indicates a co-evolutionary dynamic that rarely revisits previously-sampled areas of the strategy phase-space, with any recurrences lasting for only brief periods of time.

Finally, Figure 9 shows the recurrence plot (RP) for the PRDE experiment whose data was plotted in Figures 5 onwards: as is immediately obvious, the PRDE RP is much more sparse than the PRSH RP, indicating that differential evolution is driving the trajectory through strategy phase-space onwards into previously unvisited territory, rather than meandering for long periods in already frequently-visited zones of phase-space, as was seen in the PRSH RP of Figure 4.

V. Discussion and Conclusion

The co-evolutionary population of traders in the BSE market simulator used here are, technically, a metapopulation: that is, a population (the set of traders) of independent subpopulations (the private set of NP candidate strategies used in each trader's internal DE strategy-optimization process). Previously authors have reported evolutionary optimization approaches that borrow ideas from metapopulations (e.g., [43]) and within the DE research literature there are papers that explore multi-population ensemble methods, such as [44] – although that is focused primarily on improving effective use of the mutation operator within DE. However, these systems are directed at creating multiple populations that collectively contribute to the solution of some overall problem. In contrast, the co-evolutionary system described here involves each DE sub-population being in zero-sum competition with every other sub-population in the market, and the overall system is not directed at solving any specific problem; rather it is intended as a minimal model of a contemporary real-world financial market populated by adaptive automated trading systems, each continuously co-evolving against the other. One of the contributions of this paper is the demonstration that when each trading entity within the market is using DE as its internal adaptation mechanism, the resultant system-level outcome is a doubling in the amount of available profit that is actually extracted via the traders' interactions within the market, relative to the baseline established by the market when the traders' adaptation mechanism is a stochastic hill-climber.

An additional contribution is the identification of the stark difference in the co-evolutionary dynamics between PRSH markets and PRDE markets, as evidenced by the two recurrence plots of figures 4 and 9. It seems likely that one major factor is the use here of a simple convergence-detection and countermeasure (whole-range re-randomization of one strategy chosen randomly from the trader's private set of NP candidate strategies) which in effect introduces a sudden mega-mutation to the trader's set of strategies: intuitively, this must surely greatly reduce the probability of recurrences, which would then explain the stark differences in the recurrence plots. The competitive co-evolution in the BSE system described here means that convergence is to be avoided *forever*, whereas other researchers working with DE in non-co-evolutionary applications have often either focused on convergence to a static population of genomes as a positive outcome, to be encouraged (e.g., [45]), or have explored principled ways of slowing its progress (e.g., [46], [47]), but not of permanently avoiding it.

To the best of my knowledge, this is the first paper to report on a system in which multiple DE-using entities are in a never-ending competitive co-evolutionary dynamic; and this is the first report on the behavior of the BSE system when the traders are competitively adapting and each using DE. In the spirit of minimalism associated with zero-intelligence and minimal-intelligence modelling of real-world financial-market dynamics, in this initial study there was a deliberate choice to use what is arguably the simplest from of DE, i.e. DE/rand/1, and that was then implemented in the most minimal fashion by keeping NP=4, the smallest viable value. The intriguing results presented here invite various lines of future research. One obvious line is to explore the effects on the market's dynamics of altering key parameters, such as increasing the value of NP homogeneously (i.e., with all traders in the market having an identical value of NP>4) and also of studying the dynamics of markets with heterogeneous distributions of NP (i.e., different traders having different values of NP); similar explorations of homo/heterogeneous parameter-sweeps of the key DE parameter F_i would also be worth exploring. Another appealing line of future research is to explore the co-evolutionary dynamics in the BSE system when DE/rand/1 is replaced by other, more sophisticated instances of DE such as DE/best/1; and/or JADE [48]; and/or SHADE [49]: various authors have offered comparative studies between a range of the more notable DE variants (see, e.g., [50], [51]) but as far as I am aware all such studies seem to be focused on stationary fitness landscapes, with static optima – it would be interesting to see the results of similar controlled comparison studies on the permanently shifting fitness landscapes of the co-evolutionary DE metapopulation described here.

Finally, again in the spirit of minimalism, the work reported here has studied adaptive traders whose trading strategy is completely specified by only a single scalar value: there are other trading strategies in the same broad class of minimal-intelligence models that have more parameters, e.g. the *Zero Intelligence Plus* family of strategies described in [52] has instances with parameter-counts that range from 8 to 60 real values, and studying the co-evolutionary dynamics of markets populated with these higher-dimensional minimal strategies would be interesting – again, either homogeneously populated with all strategies of the same type and dimensionality; or instead a heterogeneous mix of various strategy types and varying dimensionalities which would also be a step closer to the reality of present-day financial markets populated by heterogeneous mixes of automated trading strategies.

There is clearly plenty to explore: the Python source-code for the experiments described here has been made freely available on GitHub for other researchers to replicate and extend the results described in this paper: see [16].

Appendix

For the benefit of any readers unfamiliar with the recurrence plots (RPs) used in Figures 4 and 9, the diagrams in Figures 10 and 11 illustrate key aspects of this visualization technique for characterising high-dimensional dynamical systems: in their simplest incarnation, RPs are square arrays of cells or pixels, that are binary-shaded (e.g.: the pixels are either black or white), with a cell at column c and row r (denoted here by $C_{c,r}$) being shaded if the state of the system at the time associated with row r is a recurrence of a previously-observed system state that occurred at the time associated with column c; otherwise unshaded.

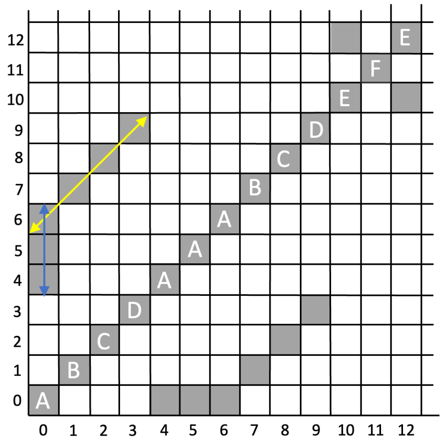

Fig. 10. Illustrative synthetic recurrence plot (RP) for a dynamical system that starts at time $t = 0$ in state A and then over the next 12 timesteps transitions through the following sequence of states: B, C, D, A, A, A, B, C, D, E, F, E. Let $C_{c,r}$ denote the cell/pixel at column c and row r: the cell is shaded if the state of the system at time $t = r$ is a *recurrence* of the state of the system at time $t = c$, and is otherwise unshaded. For D-dimensional dynamical systems where the state of the system at time t is $\vec{s}(t) \in \mathbb{R}^D$, recurrence is usually defined to occur when the distance $|\vec{s}(r) - \vec{s}(c)| < \epsilon$ for some suitably small ϵ. By convention, the RP origin point is at lower left, and the diagonal line of cells $C_{c,r:c=r}$ is referred to as the *Line of Identity* (LOI); cells on the LOI are shaded because the distance from any state to itself is zero. The LOI divides the RP into two right-triangles with mirror-symmetric patterns of blank and shaded cells. The figure shows two key features in RPs: the *diagonal line* of four shaded cells (i.e., $C_{0,6}, C_{1,7}, C_{2,8}$ and $C_{3,9}$) starting at time $t = 6$ when the state sequence A-B-C-D recurs, having first occurred at times $t = 0$ to $t = 3$; and the *vertical line* of three shaded cells (i.e., $C_{0,4}, C_{0,5}$, and $C_{0,6}$) starting at time $t = 4$ where the state A recurs three times, having first occurred at time $t = 0$. The quantitative analysis of RPs, an approach known as *Recurrence Quantification Analysis* (RQA), typically involves the calculation of statistics involving the distributions of vertical and/or diagonal lines in the RP.

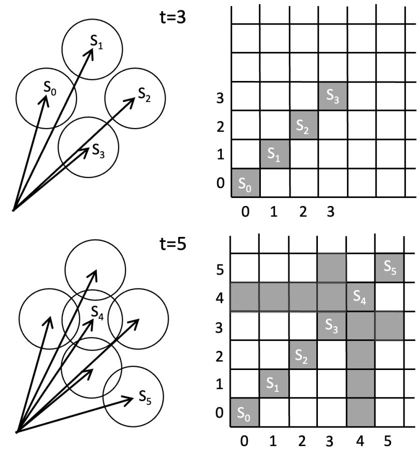

Fig. 11. Illustrative synthetic recurrence plot (RP) for a $D-$dimensional dynamical system with state vector $\vec{S}(t) \in \mathbb{R}^D$ that starts at time $t = 0$ in state $\vec{S}(0) = \mathsf{S}_0$ and then over the next three timesteps transitions through states S_1 to S_3 with no recurrences. The upper pair of figures, labelled $t = 3$, illustrates the set of non-recurring state-vectors on the left, and the corresponding RP on the right. Here the end-point of each state-vector is the centre of a $D-$ball (i.e., a solid $D-$dimensional hypersphere) of diameter ϵ, such that if any two balls intersect then the distance between the two vector end-points must be less than ϵ, which is thus counted as a recurrence. As there have been no recurrences by $t = 3$, the RP plot only shows shaded cells on the LOI. The lower pair of figures, labelled $t = 5$, illustrates the situation after the system has transitioned through state S_4 to state S_5: the ball for S_4 intersected with the balls for each of states S_0 to S_3, so the single state S_4 is recorded as a recurrence of each of the states S_0 to S_3, giving rise to a horizontal line of recurrences on the RP at cells $C_{0,4}$–$C_{3,4}$; then S_5 intersects only with S_3, shown on the RP as a single shaded cell at $C_{3,5}$.

In systems where the state at any one time is one of a small number of discrete values, recurrence would usually be defined as strict equality of states. But in many dynamical systems of practical interest, the system state at time t is a $D-$dimensional real-valued vector $\vec{S}(t)$, and for creating an RP any subsequent state $\vec{S}(t + \Delta_t)$ that is within a $D-$dimensional solid hypersphere (i.e., a $D-$ball) centered on $\vec{S}(t)$ with radius ϵ is considered to be a recurrence of $\vec{S}(t)$. Naturally, the choice of ϵ is significant: if too large, each new state is registered as a recurrence of all previous states; if too small, it is possible that no recurrences are ever recorded. The RP origin point is normally displayed at lower left, and the diagonal line of cells $C_{c,r:c=r}$, referred to as the *Line of Identity* (LOI), is always shaded because the distance from any state to itself is zero.

Once an $N \times N$ RP is created, summary statistics can be calculated by doing simple image-processing such as computing the frequency distribution of lengths of vertical and diagonal lines of shaded cells in the RP, and then calculating summary statistics from those distributions: this approach is known as *Recurrence Quantification Analysis* (RQA). For example, the *trapping time* statistic (conventionally denoted by TT), given $P(v)$ the frequency distribution of vertical lines of length v in the RP, measures the RP's average length of vertical lines at least as long as $v_{\min}$ (usually $v_{\min} = 2$):

$$TT = \left(\sum_{v=v_{\min}}^{N} vP(v) \right) / \left(\sum_{v=v_{\min}}^{N} P(v) \right)$$

So for example if an RP has a TT of 6, and the time delta between successive rows/columns on the RP is one hour, then the trapping time is six hours, indicating that on average the system remains within ϵ of any particular state for six hours.

For further details of RPs and RQA, see e.g. [53]–[60].

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