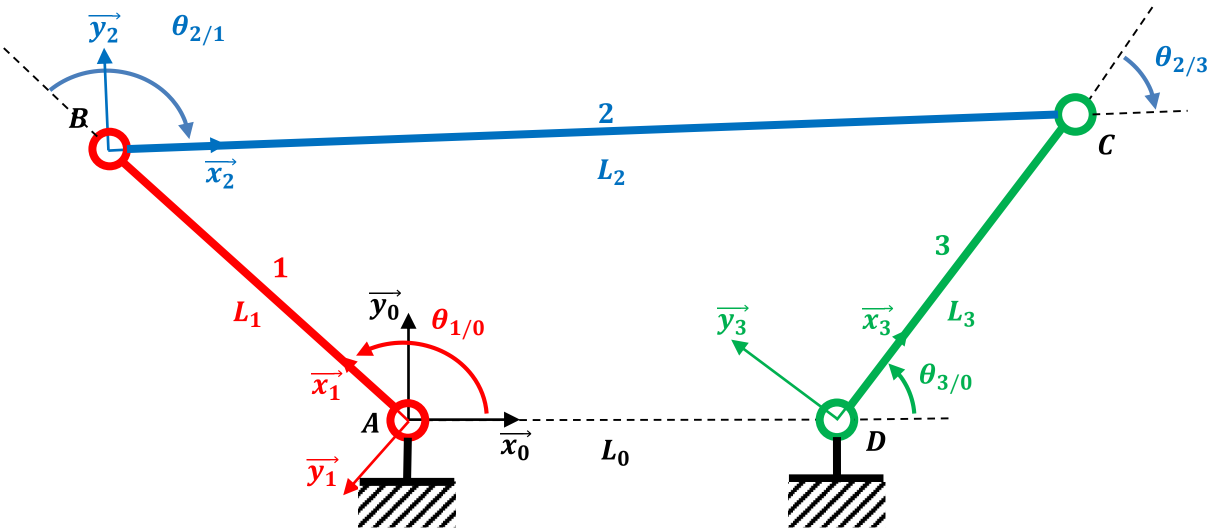
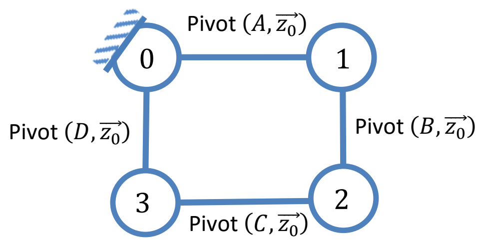
Fermeture cinématique

Exercice 1: Manège « Tapis Volant »



Cas général

Question 1: Faire le graphe des liaisons du mécanisme



Question 2: Identifier le nombre d'inconnues et d'équations du mécanisme et estimer sa mobilité.

$$I_c = 1 + 1 + 1 + 1 = 4$$
$$\gamma = L - P + 1 = 4 - 4 + 1 = 1$$
$$E_c = 3\gamma \ (Plan) = 3$$
$$m = 1$$

On aura donc 3 équations pour 3 inconnues.

Question 3: Ecrire la fermeture de chaîne cinématique du problème

$$\{\mathcal{V}_{32}\} + \{\mathcal{V}_{21}\} + \{\mathcal{V}_{10}\} + \{\mathcal{V}_{03}\} = 0$$

Question 4: Ecrire les torseurs cinématiques de chaque liaison en leurs points caractéristiques

$\{\mathcal{V}_{32}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ R_{32} & 0 \end{Bmatrix}_C^{\mathfrak{B}_0}$
$\{\mathcal{V}_{21}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ R_{21} & 0 \end{Bmatrix}_B^{\mathfrak{B}_0}$
$\{\mathcal{V}_{10}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ R_{10} & 0 \end{Bmatrix}_A^{\mathfrak{B}_0}$
$\{\mathcal{V}_{03}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ R_{03} & 0 \end{Bmatrix}_D^{\mathfrak{B}_0}$

Question 5: En déduire les deux équations vectorielles de la fermeture cinématique en B

Choix du point : pas de préférences

$\{\mathcal{V}_{32}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ R_{32} & 0 \end{Bmatrix}_C^{\mathfrak{B}_0} = \begin{Bmatrix} R_{32}\overrightarrow{z_0} \\ -L_2R_{32}\overrightarrow{y_2} \end{Bmatrix}_B$	$\vec{V}(B,3/2) = \vec{V}(C,3/2) + \overrightarrow{BC} \wedge \overrightarrow{\Omega_{32}}$ $= L_2\overrightarrow{x_2} \wedge R_{32}\overrightarrow{z_2} = -L_2R_{32}\overrightarrow{y_2}$
$\{\mathcal{V}_{21}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ R_{21} & 0 \end{Bmatrix}_B^{\mathfrak{B}_0} = \begin{Bmatrix} R_{21}\overrightarrow{z_0} \\ \vec{0} \end{Bmatrix}_B$	
$\{\mathcal{V}_{10}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ R_{10} & 0 \end{Bmatrix}_A^{\mathfrak{B}_0} = \begin{Bmatrix} R_{10}\overrightarrow{z_0} \\ L_1R_{10}\overrightarrow{y_1} \end{Bmatrix}_B$	$\vec{V}(B,1/0) = \vec{V}(A,1/0) + \overrightarrow{BA} \wedge \overrightarrow{\Omega_{10}}$ $= -L_1\overrightarrow{x_1} \wedge R_{10}\overrightarrow{z_1} = L_1R_{10}\overrightarrow{y_1}$
$\{\mathcal{V}_{03}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ R_{03} & 0 \end{Bmatrix}_D^{\mathfrak{B}_0}$ $= \begin{Bmatrix} R_{03}\overrightarrow{z_0} \\ -L_2R_{03}\overrightarrow{y_2} + L_3R_{03}\overrightarrow{y_3} \end{Bmatrix}_B$	$\vec{V}(B,0/3) = \vec{V}(D,0/3) + \overrightarrow{BD} \wedge \overrightarrow{\Omega_{03}}$ $= (L_2\overrightarrow{x_2} - L_3\overrightarrow{x_3}) \wedge R_{03}\overrightarrow{z_1}$ $= -L_2R_{03}\overrightarrow{y_2} + L_3R_{03}\overrightarrow{y_3}$

Question 6: Obtenir les deux équations vectorielles issues de la fermeture cinématique du système

Dans le TD « Fermeture cinématique » on met tout dans la base 2, donc là aussi.

$$\begin{Bmatrix} R_{32}\overrightarrow{z_0} \\ -L_2R_{32}\overrightarrow{y_2} \end{Bmatrix}_B + \begin{Bmatrix} R_{21}\overrightarrow{z_0} \\ \vec{0} \end{Bmatrix}_B + \begin{Bmatrix} R_{10}\overrightarrow{z_0} \\ L_1R_{10}\overrightarrow{y_1} \end{Bmatrix}_B + \begin{Bmatrix} R_{03}\overrightarrow{z_0} \\ -L_2R_{03}\overrightarrow{y_2} + L_3R_{03}\overrightarrow{y_3} \end{Bmatrix}_B = \{0\}$$

$$\begin{cases} (R_{32} + R_{21} + R_{10} + R_{03})\overrightarrow{z_0} = \vec{0} \\ L_1R_{10}\overrightarrow{y_1} - L_2(R_{03} + R_{32})\overrightarrow{y_2} + L_3R_{03}\overrightarrow{y_3} = \vec{0} \end{cases}$$

Question 7: Obtenir les 3 équations scalaires du problème par projection dans la base 2

Choix de la base : Les bases 1, 2 et 3 sont équivalentes – Choix base 2

$$\begin{cases} R_{32} + R_{21} + R_{10} + R_{03} = 0 \\ -L_1R_{10}\sin\theta_{12} - L_3R_{03}\sin\theta_{32} = 0 \\ L_1R_{10}\cos\theta_{12} - L_2(R_{03} + R_{32}) + L_3R_{03}\cos\theta_{32} = 0 \end{cases}$$

$$\begin{cases} R_{32} + R_{21} + R_{10} + R_{03} = 0 \\ L_1R_{10}\sin\theta_{12} + L_3R_{03}\sin\theta_{32} = 0 \\ L_1R_{10}\cos\theta_{12} - L_2(R_{03} + R_{32}) + L_3R_{03}\cos\theta_{32} = 0 \end{cases}$$

Question 8: Résoudre le système obtenu afin d'exprimer toutes les vitesses Ω_{21}, Ω_{32}, Ω_{30} en fonction de Ω_{10}

$$\begin{cases} R_{32} + R_{21} + \boldsymbol{R_{10}} + R_{03} = 0 \\ L_1 \boldsymbol{R_{10}} \sin\theta_{12} + L_3 R_{03} \sin\theta_{32} = 0 \\ L_1 \boldsymbol{R_{10}} \cos\theta_{12} - L_2(R_{03} + R_{32}) + L_3 R_{03} \cos\theta_{32} = 0 \end{cases}$$

$L_1 \boldsymbol{R_{10}} \sin\theta_{12} + L_3 R_{03} \sin\theta_{32} = 0$ $R_{03} = -\frac{L_1}{L_3}\frac{\sin\theta_{12}}{\sin\theta_{32}}\boldsymbol{R_{10}}$ $R_{30} = \frac{L_1}{L_3}\frac{\sin\theta_{12}}{\sin\theta_{32}}\boldsymbol{R_{10}}$
$L_1 \boldsymbol{R_{10}} \cos\theta_{12} - L_2(R_{03} + R_{32}) + L_3 R_{03} \cos\theta_{32} = 0$ $L_1 \boldsymbol{R_{10}} \cos\theta_{12} + L_2 R_{30} - L_2 R_{32} - L_3 R_{30} \cos\theta_{32} = 0$ $-L_2 R_{32} = (L_3 \cos\theta_{32} - L_2) R_{30} - L_1 \boldsymbol{R_{10}} \cos\theta_{12}$ $R_{32} = \left(\frac{L_2 - L_3\cos\theta_{32}}{L_2}\right) R_{30} + \frac{L_1}{L_2}\boldsymbol{R_{10}}\cos\theta_{12}$ $R_{32} = \left(\frac{L_2 - L_3\cos\theta_{32}}{L_2}\right)\frac{L_1}{L_3}\frac{\sin\theta_{12}}{\sin\theta_{32}}\boldsymbol{R_{10}} + \frac{L_1}{L_2}\boldsymbol{R_{10}}\cos\theta_{12}$ $R_{32} = \boldsymbol{R_{10}}\left[\left(\frac{L_2 - L_3\cos\theta_{32}}{L_2}\right)\frac{L_1}{L_3}\frac{\sin\theta_{12}}{\sin\theta_{32}} + \frac{L_1}{L_2}\cos\theta_{12}\right]$
$R_{32} + R_{21} + \boldsymbol{R_{10}} + R_{03} = 0$ $R_{21} = -\boldsymbol{R_{10}} - R_{03} - R_{32}$ $R_{21} = -\boldsymbol{R_{10}} + R_{30} - R_{32}$ $R_{21} = -\boldsymbol{R_{10}} + \frac{L_1}{L_3}\frac{\sin\theta_{12}}{\sin\theta_{32}}\boldsymbol{R_{10}} - \frac{1}{L_2}\boldsymbol{R_{10}}\left[(L_2 - L_3\cos\theta_{32})\frac{L_1}{L_3}\frac{\sin\theta_{12}}{\sin\theta_{32}} + L_1\cos\theta_{12}\right]$ $R_{21} = \boldsymbol{R_{10}}\left[-1 + \frac{L_1}{L_3}\frac{\sin\theta_{12}}{\sin\theta_{32}} - \frac{1}{L_2}\left[(L_2 - L_3\cos\theta_{32})\frac{L_1}{L_3}\frac{\sin\theta_{12}}{\sin\theta_{32}} + L_1\cos\theta_{12}\right]\right]$

Question 9: En projetant la fermeture de chaîne cinématique dans la base 0, montrer que :

$$R_{30} = \frac{L_1}{L_3}\frac{\tan(\theta_{21} + \theta_{10})\cos\theta_{10} - \sin\theta_{10}}{\tan(\theta_{21} + \theta_{10})\cos\theta_{30} - \sin\theta_{30}}\boldsymbol{R_{10}}$$

$$\begin{cases} (R_{32} + R_{21} + R_{10} + R_{03})\vec{z_0} = \vec{0} \\ L_1 R_{10}\vec{y_1} - L_2(R_{03} + R_{32})\vec{y_2} + L_3 R_{03}\vec{y_3} = \vec{0} \end{cases}$$

$$\begin{cases} R_{32} + R_{21} + R_{10} + R_{03} = 0 \\ -L_1 R_{10}\sin\theta_{10} + L_2(R_{03} + R_{32})\sin(\theta_{21} + \theta_{10}) - L_3 R_{03}\sin\theta_{30} = 0 \\ L_1 R_{10}\cos\theta_{10} - L_2(R_{03} + R_{32})\cos(\theta_{21} + \theta_{10}) + L_3 R_{03}\cos\theta_{30} = 0 \end{cases}$$

$$\begin{cases} R_{32} + R_{21} + R_{10} + R_{03} = 0 \\ -L_1 R_{10}\sin\theta_{10} + L_2 R_{03}\sin(\theta_{21} + \theta_{10}) + L_2 R_{32}\sin(\theta_{21} + \theta_{10}) - L_3 R_{03}\sin\theta_{30} = 0 \\ L_1 R_{10}\cos\theta_{10} - L_2 R_{03}\cos(\theta_{21} + \theta_{10}) - L_2 R_{32}\cos(\theta_{21} + \theta_{10}) + L_3 R_{03}\cos\theta_{30} = 0 \end{cases}$$

$$\begin{cases} R_{32}+R_{21}+R_{10}+R_{03}=0 \\ -L_1\sin\theta_{10}\,R_{10}+[L_2\sin(\theta_{21}+\theta_{10})-L_3\sin\theta_{30}]R_{03}+L_2\sin(\theta_{21}+\theta_{10})\,R_{32}=0 \\ L_1\cos\theta_{10}\,R_{10}+[L_3\cos\theta_{30}-L_2\cos(\theta_{21}+\theta_{10})]R_{03}-L_2\cos(\theta_{21}+\theta_{10})\,R_{32}=0 \end{cases}$$

$$\begin{cases} R_{32}+R_{21}+\boldsymbol{R_{10}}+R_{03}=0 \\ -L_1\sin\theta_{10}\,\boldsymbol{R_{10}}+[L_2\sin(\theta_{21}+\theta_{10})-L_3\sin\theta_{30}]R_{03}+L_2\sin(\theta_{21}+\theta_{10})\,R_{32}=0 \\ L_1\cos\theta_{10}\,\boldsymbol{R_{10}}+[L_3\cos\theta_{30}-L_2\cos(\theta_{21}+\theta_{10})]R_{03}-L_2\cos(\theta_{21}+\theta_{10})\,R_{32}=0 \end{cases}$$

$$L_2R_{32}=L_1\frac{\sin\theta_{10}}{\sin(\theta_{21}+\theta_{10})}\boldsymbol{R_{10}}-\left[L_2-L_3\frac{\sin\theta_{30}}{\sin(\theta_{21}+\theta_{10})}\right]R_{03}$$

$$L_1\cos\theta_{10}\,\boldsymbol{R_{10}}+[L_3\cos\theta_{30}-L_2\cos(\theta_{21}+\theta_{10})]R_{03}-\cos(\theta_{21}+\theta_{10})\left[L_1\frac{\sin\theta_{10}}{\sin(\theta_{21}+\theta_{10})}\boldsymbol{R_{10}}-\left[L_2-L_3\frac{\sin\theta_{30}}{\sin(\theta_{21}+\theta_{10})}\right]R_{03}\right]=0$$

$$L_1\cos\theta_{10}\,\boldsymbol{R_{10}}+[L_3\cos\theta_{30}-L_2\cos(\theta_{21}+\theta_{10})]R_{03}-L_1\frac{\cos(\theta_{21}+\theta_{10})\sin\theta_{10}}{\sin(\theta_{21}+\theta_{10})}\boldsymbol{R_{10}}+\cos(\theta_{21}+\theta_{10})\left[L_2-L_3\frac{\sin\theta_{30}}{\sin(\theta_{21}+\theta_{10})}\right]R_{03}=0$$

$$\left[L_1\cos\theta_{10}-L_1\frac{\cos(\theta_{21}+\theta_{10})\sin\theta_{10}}{\sin(\theta_{21}+\theta_{10})}\right]\boldsymbol{R_{10}}+\left[L_3\cos\theta_{30}-L_2\cos(\theta_{21}+\theta_{10})+L_2\cos(\theta_{21}+\theta_{10})-L_3\frac{\cos(\theta_{21}+\theta_{10})\sin\theta_{30}}{\sin(\theta_{21}+\theta_{10})}\right]R_{03}=0$$

$$\left[L_3\cos\theta_{30}-L_3\frac{\cos(\theta_{21}+\theta_{10})\sin\theta_{30}}{\sin(\theta_{21}+\theta_{10})}\right]R_{03}=\left[L_1\frac{\cos(\theta_{21}+\theta_{10})\sin\theta_{10}}{\sin(\theta_{21}+\theta_{10})}-L_1\cos\theta_{10}\right]\boldsymbol{R_{10}}$$

$$R_{03}=\frac{L_1}{L_3}\frac{\dfrac{\sin\theta_{10}}{\tan(\theta_{21}+\theta_{10})}-\cos\theta_{10}}{\cos\theta_{30}-\dfrac{\sin\theta_{30}}{\tan(\theta_{21}+\theta_{10})}}\boldsymbol{R_{10}}$$

$$R_{03}=\frac{L_1}{L_3}\frac{\sin\theta_{10}-\tan(\theta_{21}+\theta_{10})\cos\theta_{10}}{\tan(\theta_{21}+\theta_{10})\cos\theta_{30}-\sin\theta_{30}}\boldsymbol{R_{10}}$$

$$R_{30}=\frac{L_1}{L_3}\frac{\tan(\theta_{21}+\theta_{10})\cos\theta_{10}-\sin\theta_{10}}{\tan(\theta_{21}+\theta_{10})\cos\theta_{30}-\sin\theta_{30}}\boldsymbol{R_{10}}$$

Question 10: En réutilisant les équations de la fermeture géométrique, retrouver la relation suivante

$$\dot\theta_{30}=\frac{L_1}{L_3}\frac{L_3\sin(\theta_{30}+\theta_{01})-L_0\sin\theta_{10}}{L_1\sin(\theta_{30}+\theta_{01})-L_0\sin\theta_{30}}\dot\theta_{10}$$

En reprenant les équations dans 0 de la fermeture géométrique :

$$\begin{cases} \theta_{10} + \theta_{21} + \theta_{32} + \theta_{03} = 0 \\ L_1 \cos\theta_{10} + L_2 \cos(\theta_{21} + \theta_{10}) - L_3 \cos\theta_{30} - L_0 = 0 \\ L_1 \sin\theta_{10} + L_2 \sin(\theta_{21} + \theta_{10}) - L_3 \sin\theta_{30} = 0 \end{cases}$$

On peut exprimer :

$$\cos(\theta_{21} + \theta_{10}) = \frac{L_3 \cos\theta_{30} + L_0 - L_1 \cos\theta_{10}}{L_2}$$

$$\sin(\theta_{21} + \theta_{10}) = \frac{L_3 \sin\theta_{30} - L_1 \sin\theta_{10}}{L_2}$$

$$\tan(\theta_{21} + \theta_{10}) = \frac{\sin(\theta_{21} + \theta_{10})}{\cos(\theta_{21} + \theta_{10})} = \frac{L_3 \sin\theta_{30} - L_1 \sin\theta_{10}}{L_3 \cos\theta_{30} + L_0 - L_1 \cos\theta_{10}}$$

$$R_{30} = \frac{L_1}{L_3} \frac{\frac{L_3 \sin\theta_{30} - L_1 \sin\theta_{10}}{L_3 \cos\theta_{30} + L_0 - L_1 \cos\theta_{10}} \cos\theta_{10} - \sin\theta_{10}}{\frac{L_3 \sin\theta_{30} - L_1 \sin\theta_{10}}{L_3 \cos\theta_{30} + L_0 - L_1 \cos\theta_{10}} \cos\theta_{30} - \sin\theta_{30}} \boldsymbol{R_{10}}$$

$$R_{30} = \frac{L_1}{L_3} \frac{\frac{L_3 \sin\theta_{30} \cos\theta_{10} - L_1 \sin\theta_{10} \cos\theta_{10}}{L_3 \cos\theta_{30} + L_0 - L_1 \cos\theta_{10}} - \sin\theta_{10}}{\frac{L_3 \sin\theta_{30} \cos\theta_{30} - L_1 \sin\theta_{10} \cos\theta_{30}}{L_3 \cos\theta_{30} + L_0 - L_1 \cos\theta_{10}} - \sin\theta_{30}} \boldsymbol{R_{10}}$$

$$R_{30} = \frac{L_1}{L_3} \frac{\frac{L_3 \sin\theta_{30} \cos\theta_{10} - L_1 \sin\theta_{10} \cos\theta_{10} - \sin\theta_{10} L_3 \cos\theta_{30} - \sin\theta_{10} L_0 + \sin\theta_{10} L_1 \cos\theta_{10}}{L_3 \cos\theta_{30} + L_0 - L_1 \cos\theta_{10}}}{\frac{L_3 \sin\theta_{30} \cos\theta_{30} - L_1 \sin\theta_{10} \cos\theta_{30} - \sin\theta_{30} L_3 \cos\theta_{30} - \sin\theta_{30} L_0 + \sin\theta_{30} L_1 \cos\theta_{10}}{L_3 \cos\theta_{30} + L_0 - L_1 \cos\theta_{10}}} \boldsymbol{R_{10}}$$

$$R_{30} = \frac{L_1}{L_3} \frac{L_3 \sin\theta_{30} \cos\theta_{10} - \sin\theta_{10} L_3 \cos\theta_{30} - L_0 \sin\theta_{10}}{-L_1 \sin\theta_{10} \cos\theta_{30} + L_1 \sin\theta_{30} \cos\theta_{10} - L_0 \sin\theta_{30}} \boldsymbol{R_{10}}$$

$$R_{30} = \frac{L_1}{L_3} \frac{L_3 \sin\theta_{30} \cos\theta_{01} + \sin\theta_{01} L_3 \cos\theta_{30} - L_0 \sin\theta_{10}}{L_1 \sin\theta_{01} \cos\theta_{30} + L_1 \sin\theta_{30} \cos\theta_{01} - L_0 \sin\theta_{30}} \boldsymbol{R_{10}}$$

$$R_{30} = \frac{L_1}{L_3} \frac{L_3 \sin(\theta_{30} + \theta_{01}) - L_0 \sin\theta_{10}}{L_1 \sin(\theta_{30} + \theta_{01}) - L_0 \sin\theta_{30}} \boldsymbol{R_{10}}$$

$$cqfd$$

Remarque : c'est bien compliqué…

Cas du manège

Question 11: En utilisant la relation géométrique entre les angles et les formules précédemment établies en fermeture cinématique, montrer que $\boldsymbol{\Omega_{10}} = \boldsymbol{\Omega_{30}} = \boldsymbol{\Omega_{12}} = \boldsymbol{\Omega_{32}}$

On a :

$$\theta_{1/0} = \theta_{3/0} = \theta_{1/2} = \theta_{3/2}$$

$R_{30} = \frac{\sin\theta_{12}}{\sin\theta_{32}}\boldsymbol{R_{10}} = \boldsymbol{R_{10}}$
$R_{32} = \boldsymbol{R_{10}}\left[(1-\cos\theta_{32})\frac{\sin\theta_{12}}{\sin\theta_{32}} + \cos\theta_{12}\right] = \boldsymbol{R_{10}}[1-\cos\theta_{32}+\cos\theta_{12}] = \boldsymbol{R_{10}}$
$R_{21} = \boldsymbol{R_{10}}\left[-1+\frac{\sin\theta_{12}}{\sin\theta_{32}} - \left[(1-\cos\theta_{32})\frac{\sin\theta_{12}}{\sin\theta_{32}} + \cos\theta_{12}\right]\right]$ $= \boldsymbol{R_{10}}[-[1-\cos\theta_{32}+\cos\theta_{12}] = -\boldsymbol{R_{10}}]$

Soit :

$$\boldsymbol{R_{10}} = R_{30} = R_{12} = R_{32}$$