- By using backtracking we need to determine how to compute the set of possible vertices for x_k if x_1,x_2,x_3---x_{k-1} have already been chosen.
  If k=1 then x1 can be any of the n-vertices.

By using "NextValue" algorithm the recursive backtracking scheme to find all Hamiltoman cycles.

This algorithm is started by 1st initializing the adjacency matrix G[1:n, 1:n] then setting x[2:n] to zero & x[1] to 1, and then executing Hamiltonian (2)

Generating Next Vertex	Finding all Hamiltonian Cycles

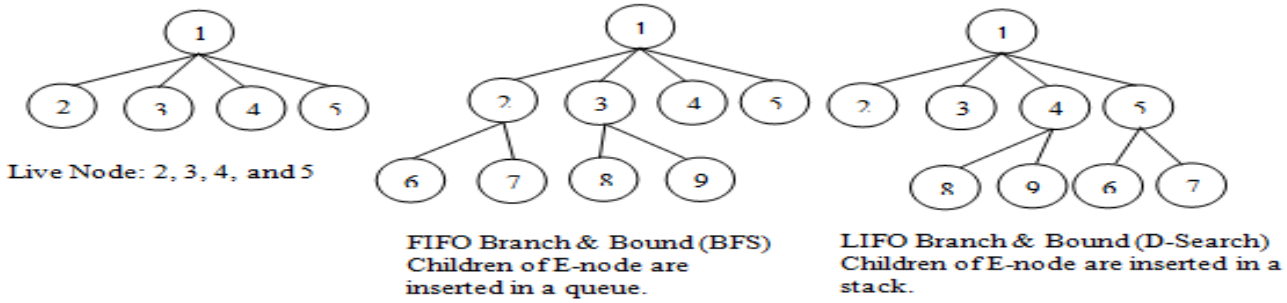
Algorithm NextValue(k)
{
// x[1: k-1]→ is path of k-1 distinct vertices.
// if x[k]=0, then no vertex has yet been assigned to x[k]
Repeat{
X[k]=(x[k]+1) mod (n+1); //Next vertex
If(x[k]=0) then return;
If(G[x[k-1], x[k]]≠0) then
{
For j:=1 to k-1 do if(x[j]=x[k]) then break;
//Check for distinctness
If(j=k) then //if true , then vertex is distinct
If((k<n) or (k=n) and G[x[n], x[1]]≠0))
Then return ;
}
}
Until (false);
}
	Algorithm Hamiltonian(k)
	{
	Repeat{
	NextValue(k); //assign a legal next value to x[k]
	If(x[k]=0) then return;
	If(k=n) then write(x[1:n]);
	Else Hamiltonian(k+1);
	} until(false)
	}

Branch & Bound

Branch & Bound (B & B) is general algorithm (or Systematic method) for finding optimal solution of various optimization problems, especially in discrete and combinatorial optimization.

- The B&B strategy is very similar to backtracking in that a state space tree is used to solve a problem.
- The differences are that the B&B method
  - ✓ Does not limit us to any particular way of traversing the tree.
  - ✓ It is used only for optimization problem
  - ✓ It is applicable to a wide variety of discrete combinatorial problem.
- B&B is rather general optimization technique that applies where the greedy method & dynamic programming fail.
- It is much slower, indeed (truly), it often (rapidly) leads to exponential time complexities in the worst case.
- The term B&B refers to all state space search methods in which all children of the "E-node" are generated before any other "live node" can become the "E-node"
  - ✓ **Live node→** is a node that has been generated but whose children have not yet been generated.
  - ✓ **E-node→**is a live node whose children are currently being explored.

✓ **Dead node→** is a generated node that is not to be expanded or explored any further. All children of a dead node have already been expanded.



➢ Two graph search strategies, BFS & D-search (DFS) in which the exploration of a new node cannot begin until the node currently being explored is fully explored.

➢ Both BFS & D-search (DFS) generalized to B&B strategies.

✓ **BFS→**like state space search will be called FIFO (First In First Out) search as the list of live nodes is "First-in-first-out" list (or queue).

✓ **D-search (DFS)→** Like state space search will be called LIFO (Last In First Out) search as the list of live nodes is a "last-in-first-out" list (or stack).

➢ In backtracking, bounding function are used to help avoid the generation of sub-trees that do not contain an answer node.

➢ We will use 3-types of search strategies in branch and bound

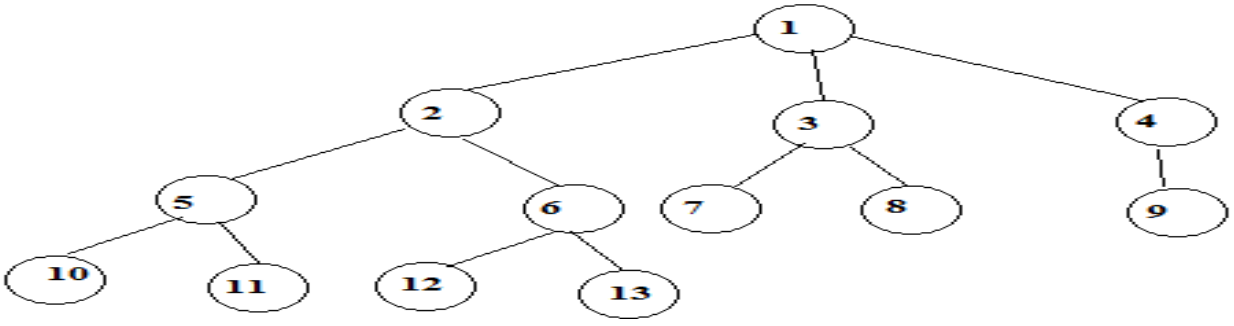
1) FIFO (First In First Out) search
2) LIFO (Last In First Out) search
3) LC (Least Count) search

FIFO B&B:

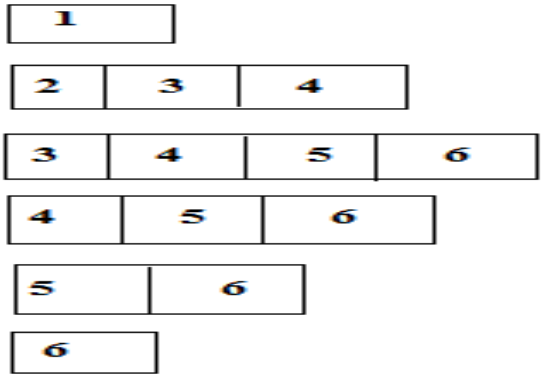
FIFO Branch & Bound is a BFS.
In this, children of E-Node (or Live nodes) are inserted in a queue.
Implementation of list of live nodes as a queue

✓ Least()→ Removes the head of the Queue

✓ Add()→ Adds the node to the end of the Queue

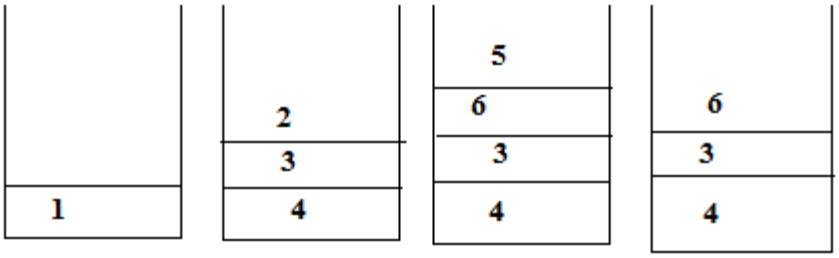


Assume that node '12' is an answer node in FIFO search, 1st we take E-node has '1'



LIFO B&B:
LIFO Brach & Bound is a D-search (or DFS).
In this children of E-node (live nodes) are inserted in a stack
Implementation of List of live nodes as a stack

- ✓ Least()→ Removes the top of the stack
- ✓ ADD()→Adds the node to the top of the stack.

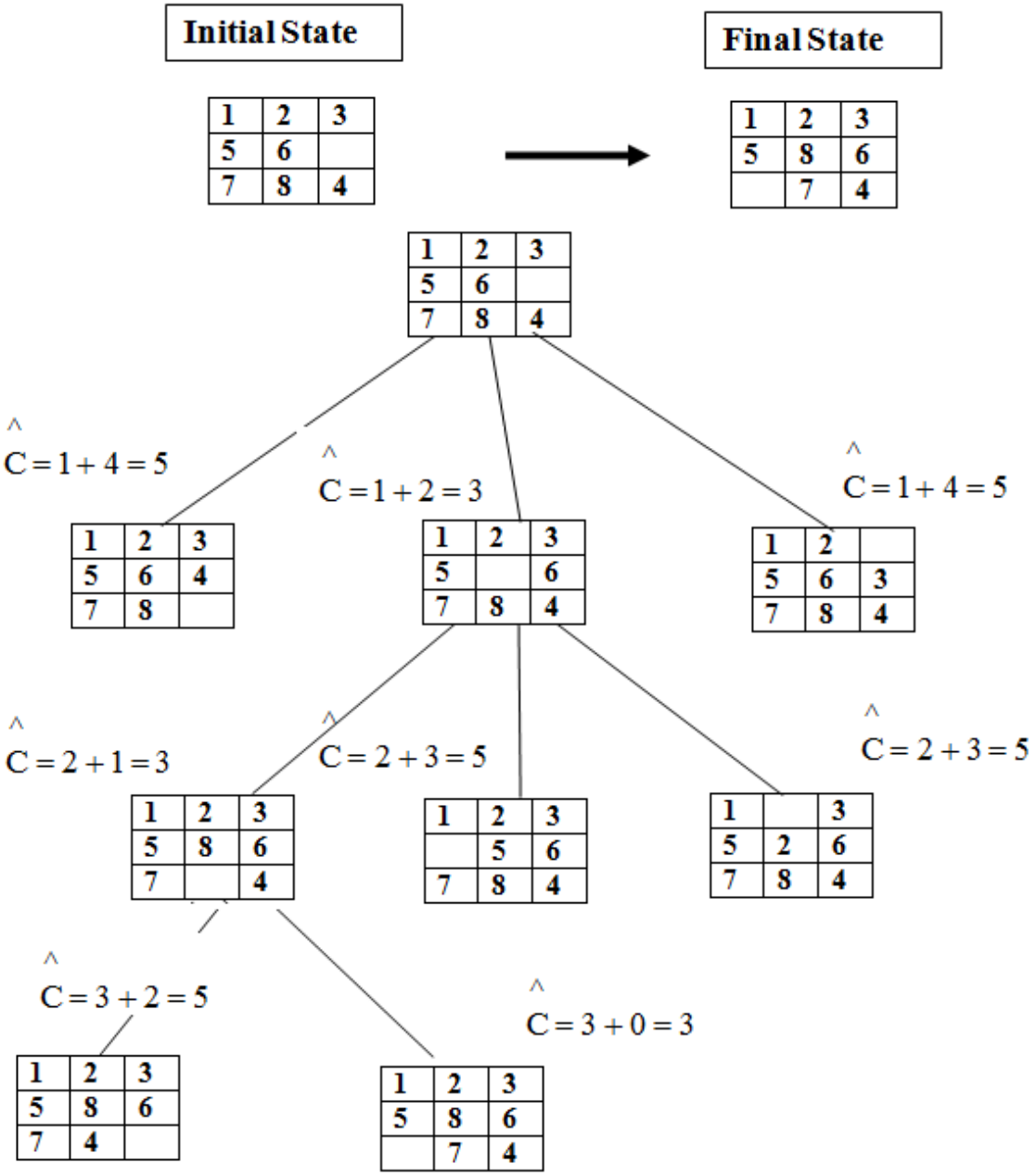


Least Cost (LC) Search:
The selection rule for the next E-node in FIFO or LIFO branch and bound is sometimes "blind". i.e., the selection rule does not give any preference to a node that has a very good chance of getting the search to an answer node quickly.

The search for an answer node can often be speeded by using an "intelligent" ranking function. It is also called an approximate cost function "**Ĉ**".
Expended node (E-node) is the live node with the best **Ĉ** value.
Branching: A set of solutions, which is represented by a node, can be partitioned into mutually (jointly or commonly) exclusive (special) sets. Each subset in the partition is represented by a child of the original node.
Lower bounding: An algorithm is available for calculating a lower bound on the cost of any solution in a given subset.

Each node X in the search tree is associated with a cost: **Ĉ(X)**
C=cost of reaching the current node, X(E-node) form the root + The cost of reaching an answer node form X.
Ĉ=g(X)+H(X).

Example:
8-puzzle
Cost function: **Ĉ** = g(x) +h(x)
where h(x) = the number of misplaced tiles
and g(x) = the number of moves so far
Assumption: move one tile in any direction cost 1.



Note: In case of tie, choose the leftmost node.

Travelling Salesman Problem:

Def:- Find a tour of minimum cost starting from a node S going through other nodes only once and returning to the starting point S.

Time Conmlexity of TSP for Dynamic Programming algorithm is $O(n^2 2^n)$

B&B algorithms for this problem, the worest case complexity will not be any better than $O(n^2 2^n)$ but good bunding functions will enables these B&B algorithms to solve some problem instances in much less time than required by the dynamic programming alogrithm.

Let $G=(V,E)$ be a directed graph defining an instances of TSP.

Let C_{ij} → cost of edge <i, j>

$C_{ij} = \infty$ if <i, j> $\notin$ E

$|V|=n$ → total number of vertices.

Assume that every tour starts & ends at vertex 1.

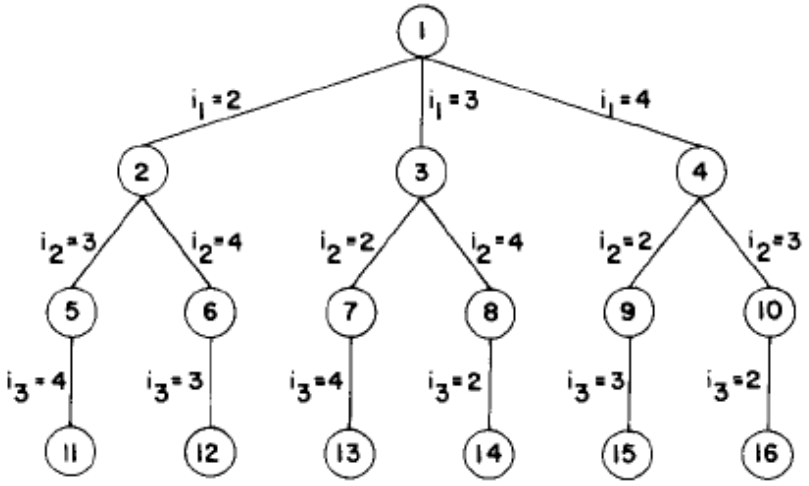
Solution Space S= {1, Π , 1 / Π is a permutation of (2, 3. 4. ----n) } then $|S|=(n-1)!$

The size of S reduced by restricting S

Sothat $(1, i_1, i2, \text{-----} i_{n-1}, 1\} \in S$ iff $<i_j, i_{j+1}> \in E$. $O \le j \le n-1$, $i_0 - i_n = 1$

S can be organized into "State space tree".

Consider the following Example



State space tree for the travelling salesperson problem with n=4 and $i_0=i_4=1$

The above diagram shows tree organization of a complete graph with |V|=4.

Each leaf node 'L' is a solution node and represents the tour defined by the path from the root to L.

Node 12 represents the tour.

$i_0=1, i_1=2, i_2=4, i_3=3, i_4=1$

Node 14 represents the tour.

$i_0=1, i_1=3, i_2=4, i_3=2, i_4=1.$

TSP is solved by using LC Branch & Bound:

To use LCBB to search the travelling salesperson "State space tree" first define a cost function C(.) and other 2 functions $\hat{C}$(.) & u(.)

Such that $\hat{C}(r) \le C(r) \le u(r)$ → for all nodes r.

Cost C(.)→ is the solution no 1 with least C(.) corresponds to a shortest tour in G.

C(A)={Length of tour defined by the path from root to A if A is leaf
Cost of a minimum-cost leaf in the sub-tree A, if A is not leaf }

From (1) $\hat{C}(r) \leq C(r)$ then $\hat{C}(r) \rightarrow$ is the length of the path defined at node A.

From previous example the path defined at node 6 is i_0, i_1, i_2=1, 2, 4 & it consists edge of <1,2> & <2,4>

Abetter $\hat{C}(r)$ can be obtained by using the reduced cost matrix corresponding to G.

- A row (column) is said to be reduced iff it contains at least one zero & remaining entries are non negative.
- A matrix is reduced iff every row & column is reduced.

$$\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix}$$

(a) Cost Matrix

$$\begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

(b) Reduced Cost Matrix
L = 25

Given the following cost matrix:

$$\begin{bmatrix} inf & 20 & 30 & 10 & 11 \\ 15 & inf & 16 & 4 & 2 \\ 3 & 5 & inf & 2 & 4 \\ 19 & 6 & 18 & inf & 3 \\ 16 & 4 & 7 & 16 & inf \end{bmatrix}$$

- The TSP starts from node 1: **Node 1**
- Reduced Matrix: To get the lower bound of the path starting at node 1

Row # 1: reduce by 10	Row #2: reduce 2	Row #3: reduce by 2
$\begin{bmatrix} inf & 10 & 20 & \\ 15 & inf & 16 & \\ 3 & 5 & inf & 2 \\ 19 & 6 & 18 & in \\ 16 & 4 & 7 & 16 \end{bmatrix}$	$\begin{bmatrix} inf & 10 & 20 & \\ 13 & inf & 14 & \\ 3 & 5 & inf & 2 \\ 19 & 6 & 18 & in \\ 16 & 4 & 7 & 16 \end{bmatrix}$	$\begin{bmatrix} inf & 10 & 20 & \\ 13 & inf & 14 & \\ 1 & 3 & inf & 0 \\ 19 & 6 & 18 & in \\ 16 & 4 & 7 & 16 \end{bmatrix}$
Row # 4: Reduce by 3:	Row # 5: Reduce by 4	Column 1: Reduce by 1

$\begin{bmatrix} inf & 10 & 20 \\ 13 & inf & 14 \\ 1 & 3 & inf \\ 16 & 3 & 15 \\ 16 & 4 & 7 \end{bmatrix}$	$\begin{bmatrix} inf & 10 & 20 \\ 13 & inf & 14 \\ 1 & 3 & inf \\ 16 & 3 & 15 \\ 12 & 0 & 3 \end{bmatrix}$	$\begin{bmatrix} inf & 10 & 20 \\ 12 & inf & 14 \\ 0 & 3 & inf \\ 15 & 3 & 15 \\ 11 & 0 & 3 \end{bmatrix}$
Column 2: It is reduced.	Column 3: Reduce by 3 $\begin{bmatrix} inf & 10 & 17 & 0 & 1 \\ 12 & inf & 11 & 2 & 0 \\ 0 & 3 & inf & 0 & 2 \\ 15 & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & 12 & inf \end{bmatrix}$	Column 4: It is reduced. Column 5: It is reduced.

The reduced cost is: RCL = 25
So the cost of node 1 is: Cost (1) = 25
The reduced matrix is:

Cost (1) = 25

$$\begin{bmatrix} inf & 10 & 17 & 0 & 1 \\ 12 & inf & 11 & 2 & 0 \\ 0 & 3 & inf & 0 & 2 \\ 15 & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & 12 & inf \end{bmatrix}$$

- **Choose to go to vertex 2: Node 2**

  - Cost of edge <1,2> is: A(1,2) = 10
  - Set row #1 = inf since we are choosing edge <1,2>
  - Set column # 2 = inf since we are choosing edge <1,2>
  - Set A(2,1) = inf
  - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & 11 & 2 & 0 \\ 0 & inf & inf & 0 & 2 \\ 15 & inf & 12 & inf & 0 \\ 11 & inf & 0 & 12 & inf \end{bmatrix}$$

  - The matrix is reduced:
  - RCL = 0
  - The cost of node 2 (Considering vertex 2 from vertex 1) is:

  Cost(2) = cost(1) + A(1,2) = 25 + 10 = 35

➢ **Choose to go to vertex 3: Node 3**

- Cost of edge <1,3> is: A(1,3) = 17 (In the reduced matrix
- Set row #1 = inf since we are starting from node 1
- Set column # 3 = inf since we are choosing edge <1,3>
- Set A(3,1) = inf
- The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & inf & 2 & 0 \\ inf & 3 & inf & 0 & 2 \\ 15 & 3 & inf & inf & 0 \\ 11 & 0 & inf & 12 & inf \end{bmatrix}$$

Reduce the matrix: Rows are reduced
The columns are reduced except for column # 1:
Reduce column 1 by 11:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 1 & inf & inf & 2 & 0 \\ inf & 3 & inf & 0 & 2 \\ 4 & 3 & inf & inf & 0 \\ 0 & 0 & inf & 12 & inf \end{bmatrix}$$

The lower bound is: RCL = 11
The cost of going through node 3 is:
cost(3) = cost(1) + RCL + A(1,3) = 25 + 11 + 17 = 53

➢ Choose to go to vertex 4: **Node 4**

Remember that the cost matrix is the one that was reduced at the starting vertex 1
Cost of edge <1,4> is: A(1,4) = 0
Set row #1 = inf since we are starting from node 1
Set column # 4 = inf since we are choosing edge <1,4>
Set A(4,1) = inf
The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & inf & 0 \\ 0 & 3 & inf & inf & 2 \\ inf & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & inf & inf \end{bmatrix}$$

Reduce the matrix: Rows are reduced

Columns are reduced
The lower bound is: RCL = 0
The cost of going through node 4 is:
cost(4) = cost(1) + RCL + A(1,4) = 25 + 0 + 0 = 25

- **Choose to go to vertex 5: Node 5**

- Remember that the cost matrix is the one that was reduced at starting vertex 1
- Cost of edge <1,5> is: A(1,5) = 1
- Set row #1 = inf since we are starting from node 1
- Set column # 5 = inf since we are choosing edge <1,5>
- Set A(5,1) = inf
- The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & 2 & inf \\ 0 & 3 & inf & 0 & inf \\ 15 & 3 & 12 & inf & inf \\ inf & 0 & 0 & 12 & inf \end{bmatrix}$$

Reduce the matrix:
Reduce rows:
Reduce row #2: Reduce by 2

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 10 & inf & 9 & 0 & inf \\ 0 & 3 & inf & 0 & inf \\ 15 & 3 & 12 & inf & inf \\ inf & 0 & 0 & 12 & inf \end{bmatrix}$$

Reduce row #4: Reduce by 3

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 10 & inf & 9 & 0 & inf \\ 0 & 3 & inf & 0 & inf \\ 12 & 0 & 9 & inf & inf \\ inf & 0 & 0 & 12 & inf \end{bmatrix}$$

Columns are reduced
The lower bound is: RCL = 2 + 3 = 5
The cost of going through node 5 is:
cost(5) = cost(1) + RCL + A(1,5) = 25 + 5 + 1 = 31

In summary:

So the live nodes we have so far are:

- ✓ 2: cost(2) = 35, path: 1->2
- ✓ 3: cost(3) = 53, path: 1->3
- ✓ 4: cost(4) = 25, path: 1->4
- ✓ 5: cost(5) = 31, path: 1->5

Explore the node with the lowest cost: Node 4 has a cost of 25
Vertices to be explored from node 4: 2, 3, and 5
Now we are starting from the cost matrix at node 4 is:

Cost (4) = 25

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & inf & 0 \\ 0 & 3 & inf & inf & 2 \\ inf & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & inf & inf \end{bmatrix}$$

- ➢ **Choose to go to vertex 2: Node 6 (path is 1->4->2)**

Cost of edge <4,2> is: A(4,2) = 3
Set row #4 = inf since we are considering edge <4,2>
Set column # 2 = inf since we are considering edge <4,2>
Set A(2,1) = inf
The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & 11 & inf & 0 \\ 0 & inf & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & inf & 0 & inf & inf \end{bmatrix}$$

Reduce the matrix: Rows are reduced
Columns are reduced
The lower bound is: RCL = 0
The cost of going through node 2 is:
cost(6) = cost(4) + RCL + A(4,2) = 25 + 0 + 3 = 28

- **Choose to go to vertex 3: Node 7 (path is 1->4->3)**

Cost of edge <4,3> is: A(4,3) = 12
Set row #4 = inf since we are considering edge <4,3>
Set column # 3 = inf since we are considering edge <4,3>
Set A(3,1) = inf
The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & inf & inf & 0 \\ inf & 3 & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & 0 & inf & inf & inf \end{bmatrix}$$

Reduce the matrix:

Reduce row #3: by 2:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & inf & inf & 0 \\ inf & 1 & inf & inf & 0 \\ inf & inf & inf & inf & inf \\ 11 & 0 & inf & inf & inf \end{bmatrix}$$

Reduce column # 1: by 11

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 1 & inf & inf & inf & 0 \\ inf & 1 & inf & inf & 0 \\ inf & inf & inf & inf & inf \\ 0 & 0 & inf & inf & inf \end{bmatrix}$$

The lower bound is: RCL = 13
So the RCL of node 7 (Considering vertex 3 from vertex 4) is:
Cost(7) = cost(4) + RCL + A(4,3) = 25 + 13 + 12 = 50

- Choose to go to vertex 5: **Node 8** (path is 1->4->5)

Cost of edge <4,5> is: A(4,5) = 0
Set row #4 = inf since we are considering edge <4,5>
Set column # 5 = inf since we are considering edge <4,5>
Set A(5,1) = inf
The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & inf & inf \\ 0 & 3 & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & 0 & 0 & inf & inf \end{bmatrix}$$

Reduce the matrix:
Reduced row 2: by 11

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 1 & inf & 0 & inf & inf \\ 0 & 3 & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & 0 & 0 & inf & inf \end{bmatrix}$$

Columns are reduced
The lower bound is: RCL = 11
So the cost of node 8 (Considering vertex 5 from vertex 4) is:
Cost(8) = cost(4) + RCL + A(4,5) = 25 + 11 + 0 = 36

In summary: So the live nodes we have so far are:

- ✓ 2: cost(2) = 35, path: 1->2
- ✓ 3: cost(3) = 53, path: 1->3
- ✓ 5: cost(5) = 31, path: 1->5
- ✓ 6: cost(6) = 28, path: 1->4->2
- ✓ 7: cost(7) = 50, path: 1->4->3
- ✓ 8: cost(8) = 36, path: 1->4->5
- Explore the node with the lowest cost: Node 6 has a cost of 28
- Vertices to be explored from node 6: 3 and 5
- Now we are starting from the cost matrix at node 6 is:

Cost (6) = 28

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & 11 & inf & 0 \\ 0 & inf & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & inf & 0 & inf & inf \end{bmatrix}$$

- **Choose to go to vertex 3: Node 9 (path is 1->4->2->3)**

Cost of edge <2,3> is: A(2,3) = 11
Set row #2 = inf since we are considering edge <2,3>
Set column # 3 = inf since we are considering edge <2,3>
Set A(3,1) = inf
The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & inf & inf & inf & inf \end{bmatrix}$$

Reduce the matrix: Reduce row #3: by 2

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & 0 \\ inf & inf & inf & inf & inf \\ 11 & inf & inf & inf & inf \end{bmatrix}$$

Reduce column # 1: by 11

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & 0 \\ inf & inf & inf & inf & inf \\ 0 & inf & inf & inf & inf \end{bmatrix}$$

The lower bound is: RCL = 2 +11 = 13
So the cost of node 9 (Considering vertex 3 from vertex 2) is:
Cost(9) = cost(6) + RCL + A(2,3) = 28 + 13 + 11 = 52

- **Choose to go to vertex 5: Node 10 (path is 1->4->2->5)**

Cost of edge <2,5> is: A(2,5) = 0
Set row #2 = inf since we are considering edge <2,3>
Set column # 3 = inf since we are considering edge <2,3>
Set A(5,1) = inf
The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ 0 & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & 0 & inf & inf \end{bmatrix}$$

Reduce the matrix: Rows reduced
Columns reduced
The lower bound is: RCL = 0
So the cost of node 10 (Considering vertex 5 from vertex 2) is:
Cost(10) = cost(6) + RCL + A(2,3) = 28 + 0 + 0 = 28

In summary: **So the live nodes we have so far are:**

- ✓ 2: cost(2) = 35, path: 1->2
- ✓ 3: cost(3) = 53, path: 1->3
- ✓ 5: cost(5) = 31, path: 1->5
- ✓ 7: cost(7) = 50, path: 1->4->3
- ✓ 8: cost(8) = 36, path: 1->4->5
- ✓ 9: cost(9) = 52, path: 1->4->2->3
- ✓ 10: cost(2) = 28, path: 1->4->2->5
- Explore the node with the lowest cost: Node 10 has a cost of 28
- Vertices to be explored from node 10: 3
- Now we are starting from the cost matrix at node 10 is:

Cost (10)=28

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ 0 & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & 0 & inf & inf \end{bmatrix}$$

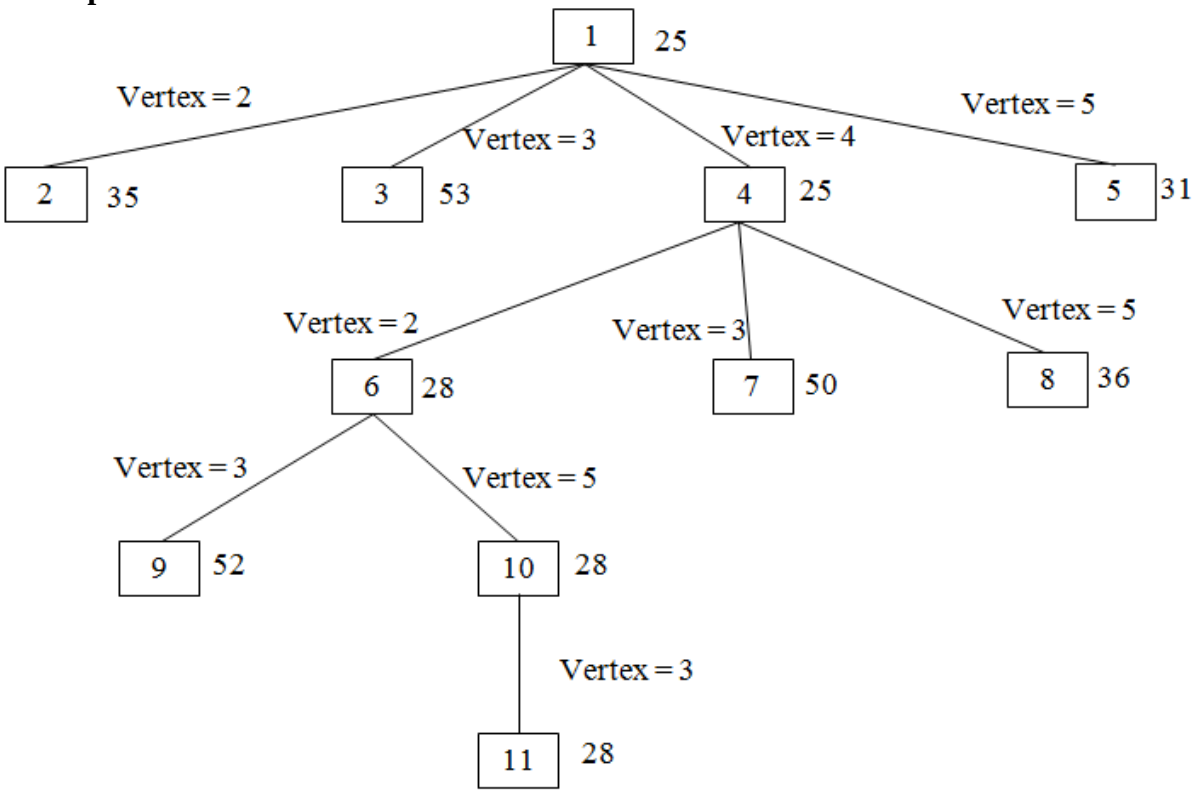
- **Choose to go to vertex 3: Node 11 (path is 1->4->2->5->3)**

Cost of edge <5,3> is: A(5,3) = 0
Set row #5 = inf since we are considering edge <5,3>
Set column # 3 = inf since we are considering edge <5,3>
Set A(3,1) = inf
The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \end{bmatrix}$$

Reduce the matrix: Rows reduced
Columns reduced
The lower bound is: RCL = 0
So the cost of node 11 (Considering vertex 5 from vertex 3) is:
Cost(11) = cost(10) + RCL + A(5,3) = 28 + 0 + 0 = 28

State Space Tree:



O/1 Knapsack Problem

What is Knapsack Problem: Knapsack problem is a problem in combinatorial optimization, Given a set of items, each with a mass & a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit & the total value is as large as possible.

O-1 Knapsack Problem can formulate as. Let there be n items, $\mathbf{Z_1}$ **to** $\mathbf{Z_n}$ where $\mathbf{Z_i}$ has value $\mathbf{P_i}$ & weight $\mathbf{w_i}$. The maximum weight that can carry in the bag is m.

All values and weights are non negative.

Maximize the sum of the values of the items in the knapsack, so that sum of the weights must be less than the knapsack's capacity m.

The formula can be stated as

$$\text{maximize} \sum_{1 \le i \le n} p_i x_i$$

$$\text{subject to} \sum_{1 \le i \le n} w_i x_i \le M$$

$\mathbf{X_i=0 \text{ or } 1\ 1 \le i \le n}$

To solve o/1 knapsack problem using B&B:

- Knapsack is a maximization problem
- Replace the objective function $\sum p_i x_i$ by the function $-\sum p_i x_i$ to make it into a minimization problem
- The modified knapsack problem is stated as

$$\text{Minimize } -\sum_{i=1}^{n} p_i x_i$$

$$\text{subject to } \sum_{i=1}^{n} w_i x_i < m,$$

$$x_i \in \{0, 1\}, 1 \le i \le n$$

- Fixed tuple size solution space:
  - Every leaf node in state space tree represents an answer for which $\sum_{1 \le i \le n} w_i x_i \leqslant m$ is an answer node; other leaf nodes are infeasible
  - For optimal solution, define

$$c(x) = -\sum_{1 \le i \le n} p_i x_i$$

for every answer node x

- For infeasible leaf nodes, $c(x) = \infty$
- For non leaf nodes

$$c(x) = \min\{c(\text{lchild}(x)), c(\text{rchild}(x))\}$$

- Define two functions $\hat{c}(x)$ and $u(x)$ such that for every node x,

$$\hat{c}(x) \le c(x) \le u(x)$$

- Computing ĉ(·) and u(·)

Let x be a node at level j, $1 \leq j \leq n+1$
Cost of assignment: $-\sum_{1 \leq i < j} p_i x_i$
$c(x) \leq -\sum_{1 \leq i < j} p_i x_i$
We can use $u(x) = -\sum_{1 \leq i < j} p_i x_i$

Using $q = -\sum_{1 \leq i < j} p_i x_i$, an improved upper bound function $u(x)$ is

$$u(x) = \text{ubound}(q, \sum_{1 \leq i < j} w_i x_i, j-1, m)$$

```
Algorithm  ubound ( cp, cw, k, m )
{
// Input:   cp: Current profit total
// Input:   cw: Current weight total
// Input:   k:  Index of last removed item
// Input: m:  Knapsack capacity
b=cp; c=cw;
for i:=k+1 to n do{
      if(c+w[i] ≤ m) then {
            c:=c+w[i]; b=b-p[i];
      }
}
return b;
}
```

UNIT V:
NP-Hard and NP-Complete problems: Basic concepts, non deterministic algorithms, NP - Hard and NPComplete classes, Cook's theorem.

Basic concepts:

NP, Nondeterministic Polynomial time

The problems has best algorithms for their solutions have "Computing times", that cluster into two groups

Group 1	Group 2
> Problems with solution time bound by a polynomial of a small degree. > It also called "Tractable Algorithms" > Most Searching & Sorting algorithms are polynomial time algorithms > **Ex:** Ordered Search **(O (log n)),** Polynomial evaluation **O(n)** Sorting **O(n.log n)**	> Problems with solution times not bound by polynomial (simply non polynomial) > These are hard or intractable problems > None of the problems in this group has been solved by any polynomial time algorithm > **Ex:** Traveling Sales Person **$O(n^2 2^n)$** Knapsack **$O(2^{n/2})$**

No one has been able to develop a polynomial time algorithm for any problem in the 2nd group (i.e., group 2)

So, it is compulsory and finding algorithms whose computing times are greater than polynomial very quickly because such vast amounts of time to execute that even moderate size problems cannot be solved.

Theory of NP-Completeness:

Show that may of the problems with no polynomial time algorithms are computational time algorithms are computationally related.

There are two classes of non-polynomial time problems

1. NP-Hard
2. NP-Complete

NP Complete Problem: A problem that is NP-Complete can solved in polynomial time if and only if (iff) all other NP-Complete problems can also be solved in polynomial time.

NP-Hard: Problem can be solved in polynomial time then all NP-Complete problems can be solved in polynomial time.

All NP-Complete problems are NP-Hard but some NP-Hard problems are not know to be NP-Complete.

Nondeterministic Algorithms:

Algorithms with the property that the result of every operation is uniquely defined are termed as deterministic algorithms. Such algorithms agree with the way programs are executed on a computer.

Algorithms which contain operations whose outcomes are not uniquely defined but are limited to specified set of possibilities. Such algorithms are called nondeterministic algorithms.

The machine executing such operations is allowed to choose any one of these outcomes subject to a termination condition to be defined later.

To specify nondeterministic algorithms, there are 3 new functions.

Choice(S), arbitrarily chooses one of the elements of sets S

Failure (), Signals an Unsuccessful completion

Success (), Signals a successful completion.

Example for Non Deterministic algorithms:

Algorithm Search(x){
//Problem is to search an element x
//output J, such that A[J]=x; or J=0 if x is not in A
J:=Choice(1,n);
if(A[J]:=x) then {
Write(J);
Success();
}
else{
write(0);
failure();
	Whenever there is a set of choices that leads to a successful completion then one such set of choices is always made and the algorithm terminates.
	A Nondeterministic algorithm terminates unsuccessfully if and only if (iff) there exists no set of choices leading to a successful signal.

}	

Nondeterministic Knapsack algorithm	

Algorithm DKP(p, w, n, m, r, x){
W:=0;
P:=0;
for i:=1 to n do{
x[i]:=choice(0, 1);
W:=W+x[i]*w[i];
P:=P+x[i]*p[i];
}
if((W>m) or (P<r)) then Failure();
else Success();
}
	p - given Profits
	w - given Weights
	n - Number of elements (number of p or w)
	m - Weight of bag limit
	P - Final Profit
	W - Final weight

The Classes NP-Hard & NP-Complete:

For measuring the complexity of an algorithm, we use the input length as the parameter. For example, An algorithm A is of polynomial complexity p() such that the computing time of A is O(p(n)) for every input of size n.

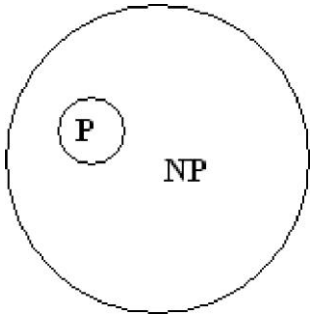
Decision problem/ Decision algorithm: Any problem for which the answer is either zero or one is decision problem. Any algorithm for a decision problem is termed a decision algorithm.

Optimization problem/ Optimization algorithm: Any problem that involves the identification of an optimal (either minimum or maximum) value of a given cost function is known as an optimization problem. An optimization algorithm is used to solve an optimization problem.

P - is the set of all decision problems solvable by deterministic algorithms in polynomial time.

NP - is the set of all decision problems solvable by nondeterministic algorithms in polynomial time.

Since deterministic algorithms are just a special case of nondeterministic, by this we concluded that **P ⊆ NP**



Commonly believed relationship between P & NP

The most famous unsolvable problems in Computer Science is Whether P=NP or P≠NP
In considering this problem, s.cook formulated the following question.

If there any single problem in NP, such that if we showed it to be in 'P' then that would imply that P=NP.

Cook answered this question with

Theorem: Satisfiability is in P if and only if (iff) P=NP

-)Notation of Reducibility

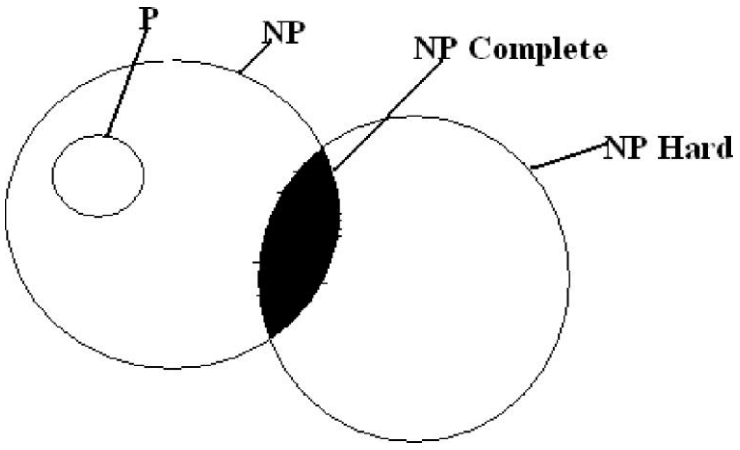
Let L_1 and L_2 be problems, Problem L_1 reduces to L_2 (written $\mathbf{L_1 \alpha L_2}$) iff there is a way to solve L_1 by a deterministic polynomial time algorithm using a deterministic algorithm that solves L_2 in polynomial time

This implies that, if we have a polynomial time algorithm for L_2, Then we can solve L_1 in polynomial time.

Here α-) is a transitive relation i.e., $\mathbf{L_1 \alpha L_2}$ and $\mathbf{L_2 \alpha L_3}$ then $\mathbf{L_1 \alpha L_3}$

A problem L is NP-Hard if and only if (iff) satisfiability reduces to L ie., **Statisfiability α L**

A problem L is NP-Complete if and only if (iff) L is NP-Hard and **L Є NP**



Commonly believed relationship among P, NP, NP-Complete and NP-Hard

Most natural problems in NP are either in P or NP-complete.

Examples of NP-complete problems:

- Packing problems: SET-PACKING, INDEPENDENT-SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, CLIQUE.
- Constraint satisfaction problems: SAT, 3-SAT.
- Numerical problems: SUBSET-SUM, PARTITION, KNAPSACK.

Cook's Theorem: States that satisfiability is in P if and only if P=NP If P=NP then satisfiability is in P
If satisfiability is in P, then P=NP
To do this

> A-) Any polynomial time nondeterministic decision algorithm.

I-)Input of that algorithm

Then formula Q(A, I), Such that Q is satisfiable iff '**A**' has a successful termination with Input **I**.

> If the length of '**I**' is '**n**' and the time complexity of **A** is **p(n)** for some polynomial **p()** then length of Q is $\mathbf{O(p^3(n) \log n)=O(p^4(n))}$

The time needed to construct Q is also $\mathbf{O(p^3(n) \log n).}$

> A deterministic algorithm '**Z**' to determine the outcome of '**A**' on any input '**I**'

Algorithm **Z** computes '**Q**' and then uses a deterministic algorithm for the satisfiability problem to determine whether '**Q**' is satisfiable.

> If **O(q(m))** is the time needed to determine whether a formula of length '**m**' is satisfiable then the complexity of '**Z**' is $\mathbf{O(p^3(n) \log n + q(p^3(n) \log n)).}$

> If satisfiability is '**p**', then '**q(m)**' is a polynomial function of '**m**' and the complexity of '**Z**' becomes '**O(r(n))**' for some polynomial '**r()**'.

> Hence, if satisfiability is in **p**, then for every nondeterministic algorithm **A** in **NP**, we can obtain a deterministic **Z** in **p.**

By this we shows that satisfiability is in **p** then **P=NP**