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Architectural Caustics — Controlling Light with Geometry

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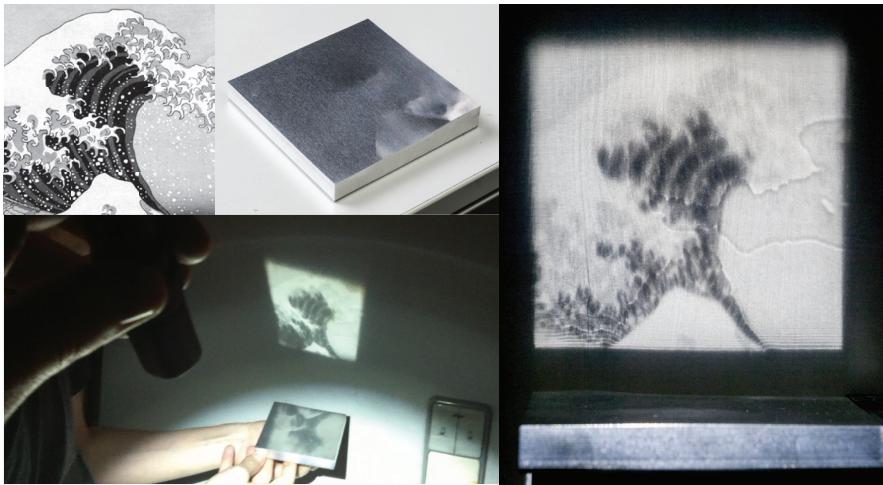


Figure 1: Designing caustics: the photograph shows an aluminum piece that was optimized such that, when lit by a simple flashlight from a certain angle, it produces a caustic image of a wave from the Hokusai print “The Great Wave off Kanagawa” shown on the left. The image on the wall is produced by the aluminum surface alone, focusing and diverting light onto the wall. See also the video on lgg.epfl.ch/caustics. This paper describes how to create such controlled caustic patterns and explores their application in architectural design.

Abstract. *Caustics are captivating light patterns created by materials bundling or diverting light by refraction or reflection. We know caustics as random side effects, appearing, for example, at the bottom of a swimming pool. In this paper we show that it is possible to control caustic patterns to form almost any desired shape by optimizing the geometry of the reflective or refractive surface generating the caustic. We demonstrate how this surprising result offers a new perspective on light control and the use of caustics as an inspiring architectural design element. Several produced prototypes illustrate that physical realizations of such optimized geometry are feasible.*

1 Introduction

Light is a principal design element in architecture. Light has a major impact on inhabitant comfort, aesthetics, and environmental efficiency of a building. Besides the choice of materials, a good understanding and careful design of geometry are the most effective means to control light and shadows within a site-specific environment.

One of the most fascinating light phenomena are caustics: light gets continuously focused and diverted by a reflective or refractive material, like metal or water, creating an intriguing pattern of varying light intensity (Figure 2). Caustics are ubiquitous but easily overlooked. Once aware of the wondrous effect, one suddenly spots caustics everywhere: cast from a window to the living room wall, from a wine glass to the dining table, from raindrops on the windshield to the car seat.

Caustics can be subtle and calm, or intense and dynamic, like the water caustics on the bottom of a swimming pool, but they almost without exception appear accidentally, as an unintentional, collateral presence of certain materials. How can that intriguing, raw, and wild nature of caustics be tamed to cast caustic patterns intentionally? How can the random caustic patterns be controlled to produce desired forms? In this paper we propose a new geometry optimization method to transform caustics from an “accidental”, “random” event to a powerful design element.

We show that by optimizing the surface geometry of a refractive or reflective object, the cast caustic can theoretically form any image, from that of a simple shape like a star to that of a human face. Prototypical tests with acrylic glass and aluminum prove that even with non-perfect machining quality and light setup, convincing results can be achieved. We see this paper as the start of a dialogue with researchers, designers, and manufacturers to instantiate caustics as an intriguing tool for architectural design.

To demonstrate the potential of caustic control in architecture, we provide a series of example applications and installations, highlighting the versatility of caustics as a design element. We wish to show to practitioners what is possible and hope to inspire designers to employ caustics in their own work. We believe that our modest set of examples already indicates the dynamics, the enchanting interplay of light and shadow, and the transparent elegance possible with architectural caustics.



Figure 2: Caustics created through reflection or refraction of light can lead to vivid, often seemingly random patterns. Center image courtesy of the art@James Gurney (<http://gurneyjourney.blogspot.ch/>)

1.1 Technical Report and Video

This paper is accompanied by a technical report that discusses the mathematical and algorithmic details of our caustic design method [Kiser and Pauly 2012]. Here we focus on the application of this tool in the context of architectural design and only provide a high-level overview of the technical aspects of our approach. Since we believe that still images are insufficient to capture the full beauty of caustics, we have also put together a video. Both the technical report and the video, as well as further details on the project can be found on the project website lgg.epfl.ch/caustics.

1.2 Related Work

The literature about light and architecture is abundant, as light is a key component in the understanding of space and therefore architecture [Fontoynont 1999; Millet 1996; Plummer 1995; Plummer 2003; Steane 2011]. Altogether these books present natural lighting of buildings either as a typology of principles, e.g. [Millet 1996], or as a strategy used in a specific project to serve the architectural intent, e.g. [Plummer 2003; Plummer 2009; Steane 2011]. Very few consider the attraction about light under the cultural aspect such as [Plummer 1995] and even more rarely, these books are written by architects to explain the philosophy of their practice such as [Kahn 1979]. In current practice, natural lighting of buildings has regained attention as it provides a mean for better use of natural resources. Light is considered mostly through its functional aspect, e.g. reduction of glare, using simulation or measurement technics such as in [Fontoynont 1999].

We are not aware of literature mentioning the phenomenon of caustics as an architectural material property, let alone as a design tool. Caustic patterns are found accidentally, as a collateral effect of the use of some specific materials. The first family, the reflective caustics, are created by the use of mirrors or mirror-polished stainless steel or aluminum. They appear either because the sheets are curved or show some usually uncontrolled surface imperfections. Some rare examples such as the Founders room of the Disney concert hall by Frank Ghery in Los Angeles can be found in architecture, a few others are shown in Figure 3, but the use of caustics as intentional effect is mostly witnessed in the work of artists like Julio Le Parc or Adolf Luther of the Group Zero in the 70s. Some artists also made use of the second family, refractive caustics, such as Alberto Biasi, or more recently Petrusson Finnbogi, using glass with texture or vibrating water lit from below. However, the use of caustics in art dates back much further. Already in the Han Dynasty more than two thousand years ago, Chinese magic mirrors created pictorial reflective light patterns that result from a specific fabrication process [Mak and Yip 2001].

From a technical point of view, the task of reproducing a prespecified light distribution by a specular surface also arises in the field of inverse reflector design, which concentrates on reflectors for lamps. A survey on inverse reflector design is given by Patow and Pueyo [Patow and Pueyo 2005]. Methods for inverse reflector design typically employ an analysis-by-synthesis approach: Some surface representation is chosen to parametrize the reflector, such as NURBS. The light distribution caused by a surface is evaluated and rated against the desired one, which is used

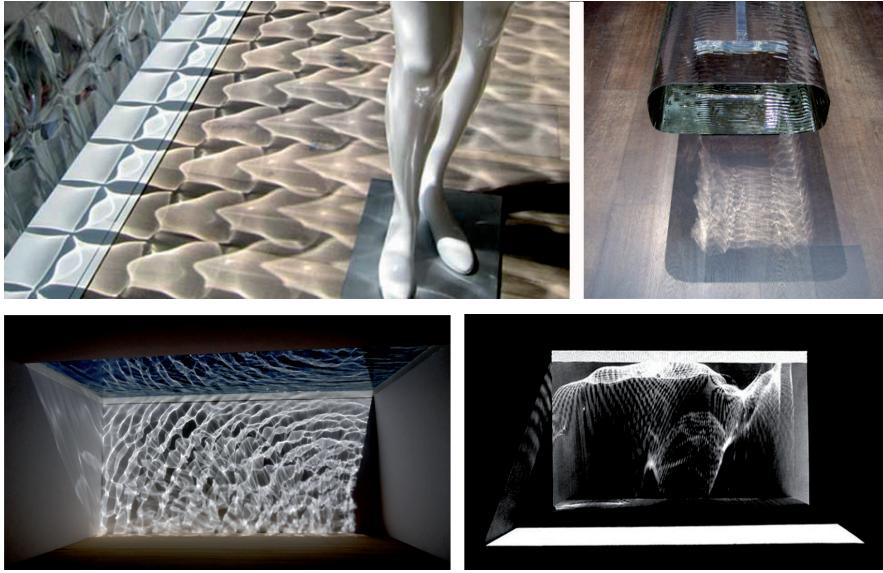


Figure 3: Reflective or refractive caustics are used in current architectural design practice mainly as organic patterns created by undulating surfaces. Precisely controlling the desired caustic images has not been feasible up to now. Top row: Renfrew Holt Ottawa (Nathan Allan Glass) - Tokujin Yoshioka: Glass Bench. Bottom row: Philippe Bompas: Light Room Experiments (philippebompas.com and www.l-00.net).

to iteratively optimize the surface parameters [Neubauer 1997; Finckh et al. 2010; Anson et al. 2008; Doyle et al. 1999]. The simplifications imposed on the scene vary; assumptions of perfect specularity and only one bounce of light without interreflections or occlusions (as used in our method) are common, though there are exceptions to both [Patow et al. 2007; Mas et al. 2009]. This previous work mostly focuses on reflective surfaces, though many readily extend to refraction. One noteworthy example investigating the refractive problem is the work by Finckh et al. [Finckh et al. 2010]. They use GPU computations to speed up the caustic evaluation, and a stochastic approximation algorithm to find a global optimum. Methods to obtain controlled light reflection and refraction have also been investigated in the context of optical systems design, e.g. [Ries and Muschaweck 2002; Oliker 2007].

Weyrich et al. [Weyrich et al. 2009] generate a set of sloped, planar microfacets to realize the desired distribution of ray directions and arrange the facets in a regular array using simulated annealing to minimize the resulting discontinuities. Strongly related to Weyrich et al.'s work is the system proposed by Papas et al. [Papas et al. 2011]. They extend the notion of microfacets to curved micropatches, which are used to produce specks of light with an anisotropic Gaussian distribution. To demonstrate their results, they also physically manufacture refractive surfaces from acrylic glass (PMMA, trademarked Plexiglas). As discussed in detail in the accompa-

nying technical report [Kiser and Pauly 2012], there are some technical similarities but also fundamental differences between the work of Papas et al. and our approach. Most importantly, our approach creates a smooth reflective or refractive surface and supports the generation of folds, which are important for the visual appearance of caustics. In addition, to our knowledge, none of these previous methods has been applied in the context of architectural geometry, where the design of caustics is as of now unheard of. With our work, we provide first case studies that highlight the immense design opportunities that become feasible when harnessing the untamed nature of caustics.

2 Caustics

The term *caustic* is derived from the latin word *causticus* and the greek *kaustikos* meaning “burned”. In optics, caustics refer to concentrations of light that can indeed lead to burns, as anyone who has experimented with a lens in bright sunlight might confirm. In our context, we consider as a caustic a pattern of light on a (mostly diffuse) surface that is created by focusing and diverting light through a reflective or refractive object (Figure 2). To control the shape of a caustic pattern generated by a specular surface, we need to know how to perform two central computations:

1. How to simulate, i.e. compute, the caustic generated by a given surface.
2. How to change the surface geometry such that it focuses and diverts incoming light to produce a desired caustic image.

These are the two central ingredients of our method and are explained in the following two subsections.

2.1 Computing Caustics

Creating a digital image of a virtual scene containing caustics requires an accurate simulation of light emission, scattering, and transport. In particular, the interaction of light with specular surfaces, such as refractive glass or reflective metal, needs to be simulated accurately. When computing a caustic on a digital scene representation, we make several simplifying assumptions to make the shape optimization tractable. We represent the reflective or refractive surface creating the caustic as a height field that is discretized as a regular grid. For some operations, we ignore the actual geometry of the surface and reduce it to the corresponding field of surface normals. The caustic receiver, that is, the diffuse surface on which the caustic image is visible, is assumed to be planar (see Figure 4).

To simplify the computations, we ignore shadowing, interreflection, and dispersion, and assume that the caustic forms a continuous pattern, which means that the normal field will be continuous and the corresponding height field continuously differentiable. With these simplifications, a simple and efficient way to compute a caustic from a given surface is as follows (see Figure 4): Trace a ray from the light source to each node of the height field. Compute the ray refractions or reflections,

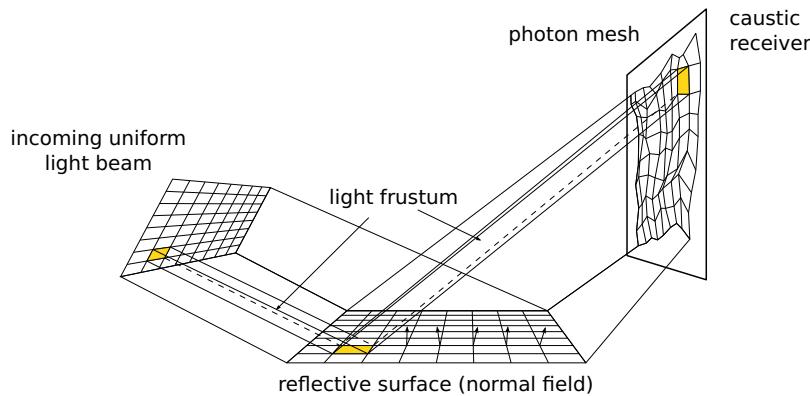


Figure 4: Computing a (reflective) caustic. The incoming light beam on the left is discretized using a mesh that matches the normal field associated with the reflective surface in the middle. Each mesh cell defines a light frustum that is deformed as the light is reflected and projected onto the receiver on the right.

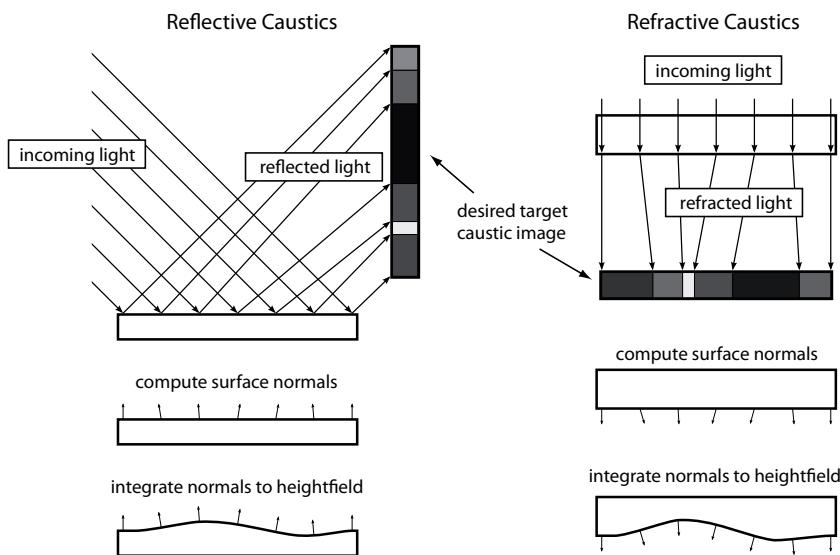


Figure 5: Sketch of the optimization method. We first compute the required surface normal vectors (middle row) such that the reflection or refraction of the uniform incoming light creates the desired caustic image. From the normals, we obtain the actual 3D surface through integration (bottom row).

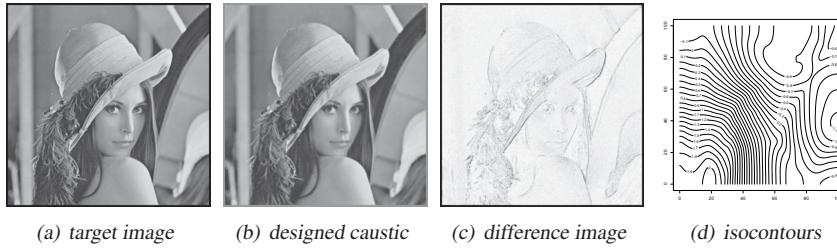


Figure 6: Result of the caustic optimization. The goal intensity image is shown in (a), the actual produced caustic computed with a global illumination light simulation tool is shown in (b). As the difference image (c) illustrates, the differences are minor. The caustic in (b) is produced by a height-field surface whose iso-contour lines are shown in (d). Surprisingly, a very smooth refractive surface can create complex detailed caustic images.

and intersect the resulting rays with the receiver. We shall call the mesh formed by the computed intersections the *photon mesh*. Then the amount of light transported through each frustum defined by four adjacent rays remains constant. This allows to assign a brightness value to each quadrilateral of the photon mesh. In the case of a caustic with folds, the photon mesh will exhibit overlaps; the values of the corresponding faces are then simply summed up.

2.2 Designing Caustics

To design a caustic image, we need to solve the inverse problem, i.e. compute a surface such that the reflection or refraction off that surface generates the desired caustic image. We achieve this by deforming the photon mesh to reproduce a given image. In order to reflect or refract rays on the specular surface such that they intersect the receiver at the designated points, we invert the laws of reflection and refraction to adjust the normal field accordingly. Once the normals are computed, we solve for the continuous surface that best fits the normal field (see Figure 5).

It is important to note that not all normal fields can be reproduced faithfully by a height field. For example, think of a closed loop on the surface where the normals indicate a permanent incline in height: this is physically impossible. Therefore, it is essential that the deformation of the photon mesh leads to a meaningful normal field. Our solution to this problem is an optimization algorithm that combines the requirement of an integrable normal field with the goal of reproducing an arbitrary intensity target image by the caustic created through reflection or refraction of the computed object. A typical result is shown in Figure 6. More examples and a detailed description of the technical details of the algorithm are given in the accompanying technical report [Kiser and Pauly 2012]. The optimization algorithm enables the designer to specify arbitrary target intensity images and obtain a reflective or refractive surface that, under the provided geometric configuration with respect to the caustic receiver and given incident lighting direction, produces a caustic pattern

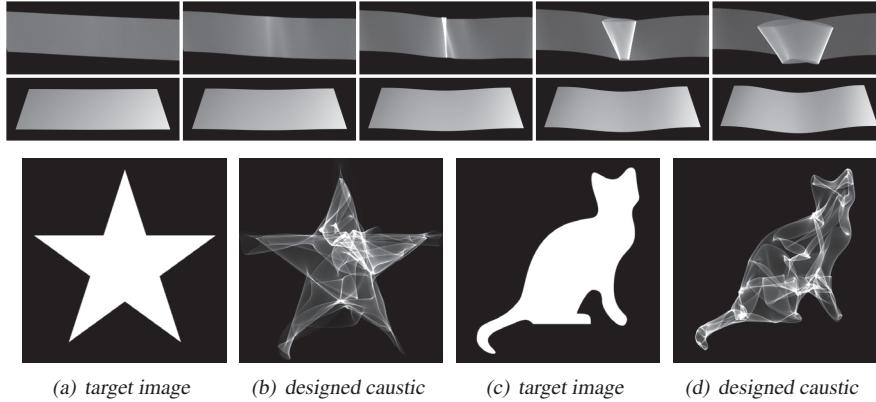


Figure 7: Caustics with folds. Top row: The caustics of a reflective strip start to overlap itself as the strip is bent, creating a fold. Bottom row: Our work in progress allows the computation of reflective or refractive surfaces that match the shape of the input target images (a, c), while creating a lively caustic pattern with folds as shown in the light simulations (b, d).

that is very close to the target image. This method, therefore, enables full control of *caustic design* as opposed to the random or accidental caustic patterns we are used to. The basic algorithm assumes a continuous, fold-free caustic image. The focusing effect of folds, however, can be a very interesting design element as well. Our method can be extended to explicitly control folds. Since this extension is work in progress, we only report a preliminary result in Figure 7 and leave the finalization of this approach to future work.

3 Results and Applications

3.1 Basic Examples

Figure 8 shows basic examples of designed caustics: Two transparent objects refract the incoming light in such a way that the caustic image of a forest and the folded caustic pattern of a cat appear on the wall. Interesting dynamic effects can be observed as the light source is moved away from the initial position as shown on the bottom row of the figure. The accompanying video (lgg.epfl.ch/caustics) gives a better impression of this effect.

3.2 The Alan Turing Tribute – Meaningful patterns within apparent noise. Or: Decoding caustics.

The project shown in Figures 9 and 10 is a tribute to Alan Turing and is a full scale architectural study employing caustics as central design element. Alan Turing is considered as one of the fathers of computing and 2012 is his 100th birth anniversary. The singularity of this building does not result from its shape, which is rather

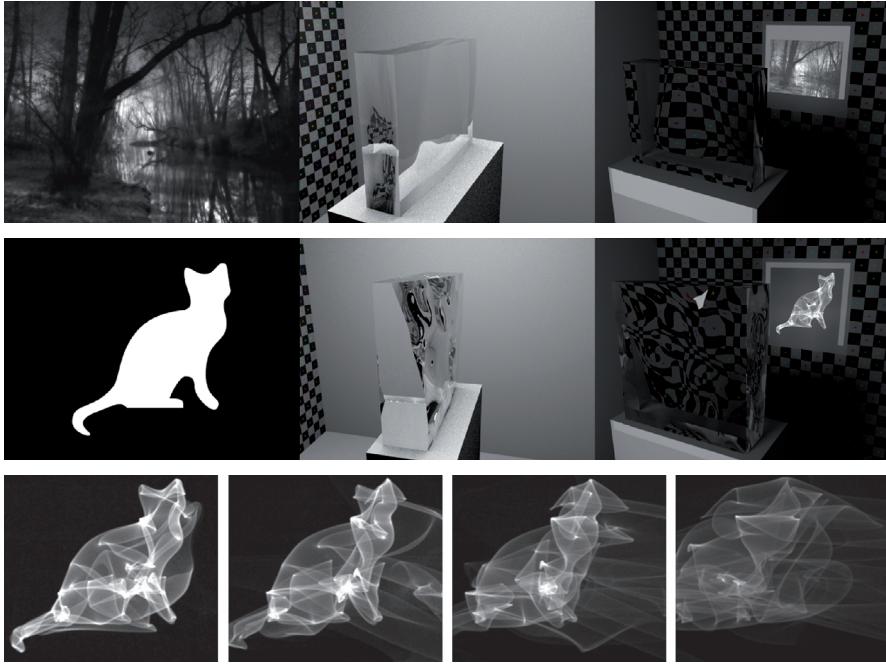


Figure 8: Two examples of caustics created by refraction of uniform light through a virtual glass block whose surface has been computed with our algorithm. The goal image is shown on the left, the glass block in the center, a light simulation showing the created caustic on the right. The bottom row shows the dynamic behavior of the caustic of the glass block in the middle when the light source is moved away from its initial position.

generic, but from the way it is lit, utilizing the research results described in this paper. The geometrical complexity lies in the reflectors and refractors which shape the sunlight, projecting it into the building and creating various kinds of caustic patterns.

The first family of caustics employed are random refractive and reflective patterns. Some are generated by the glass blocks above the stair entrance to the underground building. These are very static and change slowly as the sun travels through the sky. Others are due to a thin layer of water that covers the central skylight and the glazing just above the exhibition ring. These are very dynamic, because the water surface moves with the wind. Finally, the linear circular cuts on the ceiling and side walls are clad by mirror-polished stainless steel plates, which reflect not only the sky but also the sunlight into the main room, creating a network of overlapping circles of light.

The second family of caustic patterns form designed shapes utilizing the computational design techniques described in this paper: At the stair entrance, among the random caustics, the visitor can distinguish some numbers and formulas related to

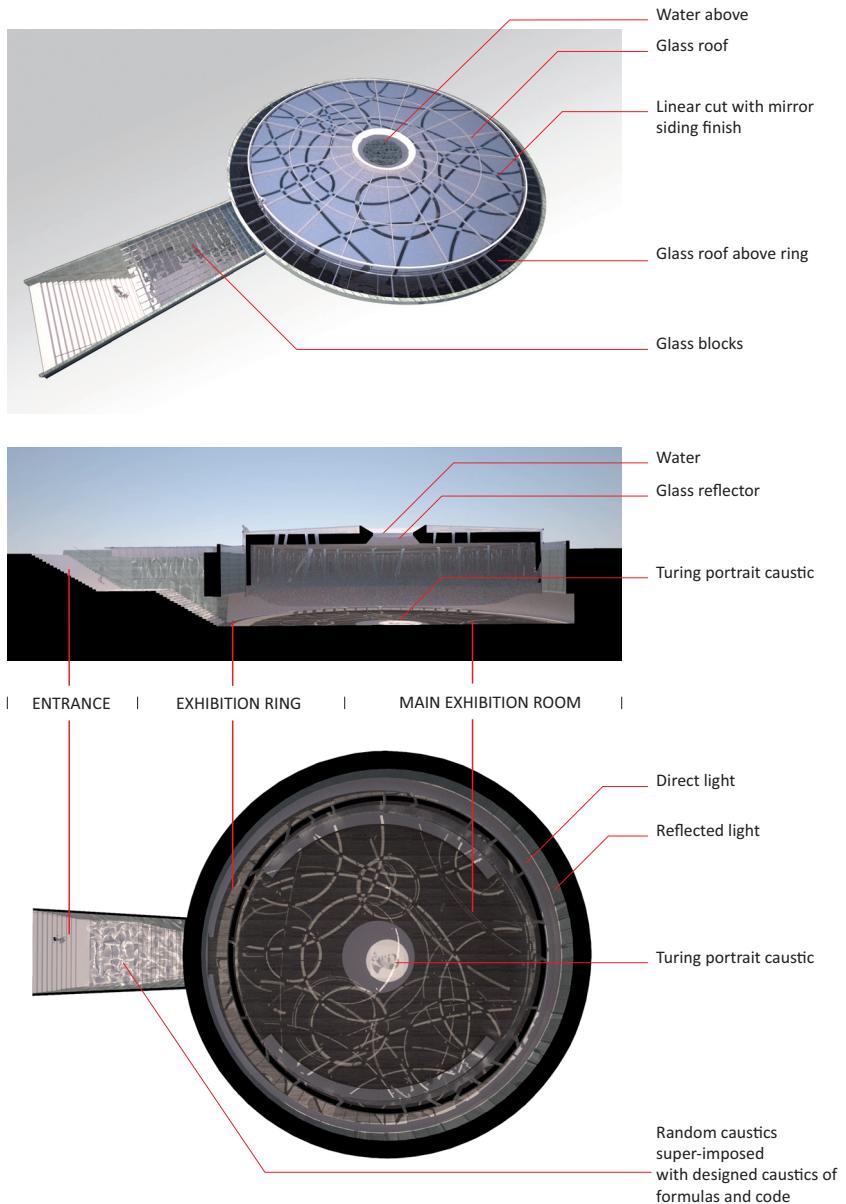
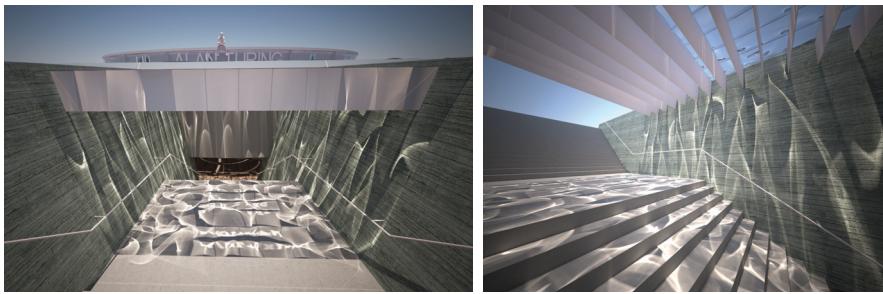
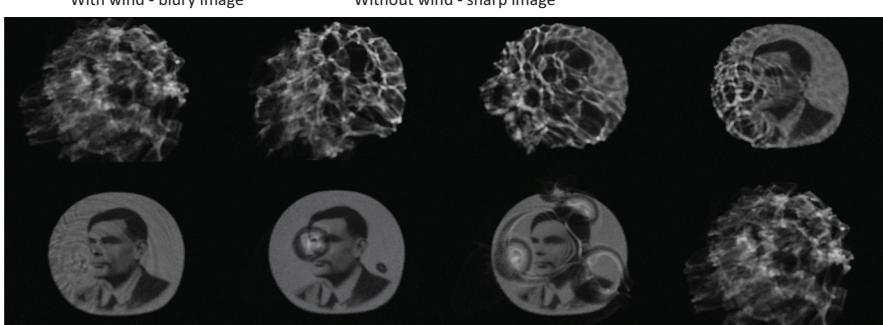
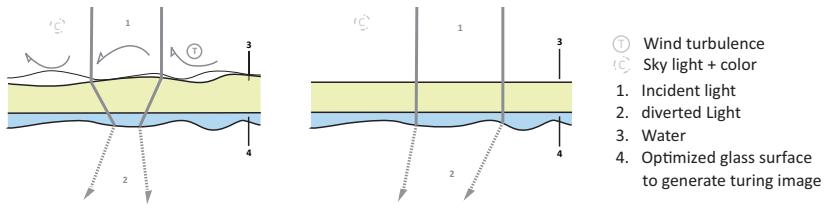


Figure 9: Layout of the Turing Memorial design study.

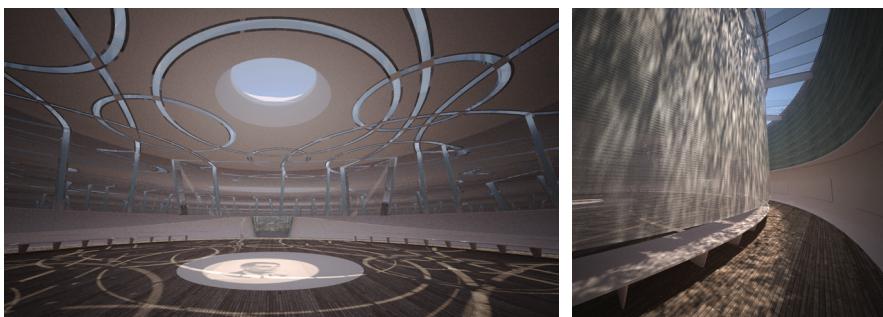
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Random caustics super-imposed with designed caustics of formulas and code



Turing portrait caustic with different water turbulences



Inside the exhibition room

Figure 10: A light simulation of the refractive caustics generated by the skylight in the Turing Memorial design study. As the water surface moves, the chaotic pattern caused by refraction at the water dominates the caustic image. As the water calms down and the water surface becomes flat, the caustic created by the curved glass surface becomes apparent: A portrait of Alan Turing. When rain drops hit the water, the image disappears again.

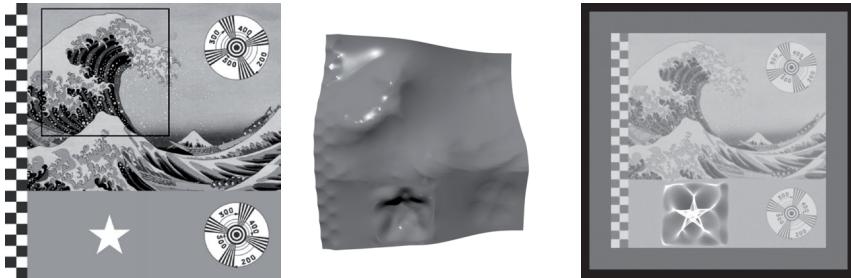


Figure 11: The woodblock print “The Great Wave off Kanagawa” by the Japanese artist Hokusai was used as the main test image for our physical prototypes. The left image shows the target image for the refractive prototype with some test patterns and the folded star added to obtain as much information on the accuracy and properties of the physical prototype as possible. For the reflective aluminum prototype, only the zoom on one of the waves indicated by the black square was used. The middle image shows the resulting surface geometry, optimized to reproduce the given target image by reflecting sunlight. The right image is a light simulation of the resulting caustic produced by the optimized surface. For better visualization of the surface geometry, the depth variation of the surface was scaled by a factor of 10.

Alan Turing’s famous Turing Machine. In the center of the main exhibition room, there is a large caustic image formed by the light entering through the central skylight. At first, it appears random due to the water that covers the circular glass window. As the water becomes still in moments of little wind, however, an unexpected order emerges from the chaos of the untamed water caustics: the optimized surface of the glass window produces a designed caustic, the portrait of Alan Turing (see Figure 10 and the accompanying video).

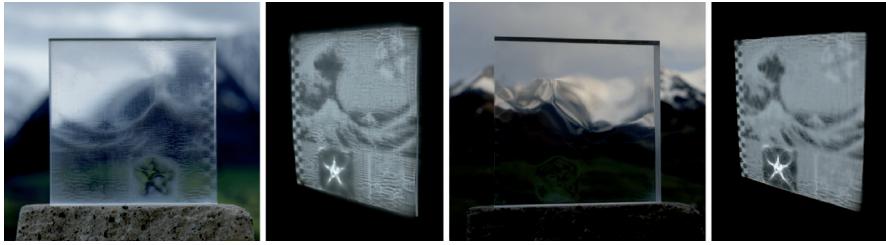
3.3 Physical Prototypes

To test the reproducibility of our simulated results with real materials, we manufactured two physical prototypes: A refractive acrylic glass piece (Figure 12) and a reflective aluminum plate (Figure 1). The test image to be reproduced by the two prototypes is based on the color woodblock print “The Great Wave off Kanagawa” by the Japanese artist Hokusai (Figure 11). We chose the Hokusai print because it is well known and easily recognized but contains a lot of details, which are a challenge for both computation and production and thus help to visually verify the achieved accuracy. For the acrylic glass prototype we also added some test patterns and the folded star from Figure 7 to gather as much information on the quality and behavior of the physical caustic image as possible. For the aluminum prototype we only used a zoom on one of the waves.

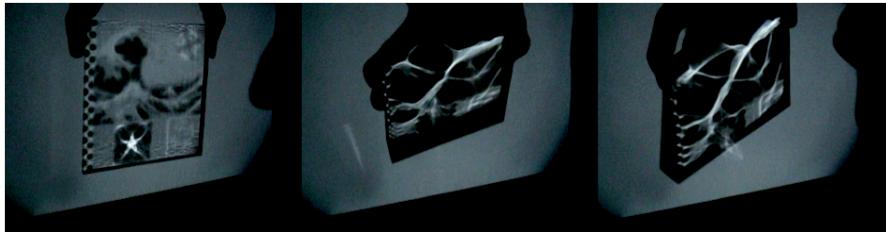
Figure 11 shows a light simulation of the designed caustic and a visualization of the surface geometry that was optimized by our method to generate this caustic image. The height field of the depicted surface, however, was multiplied by a factor of 10 to better show the geometry. The real surface has a maximal height variation



(a) The acrylic glass block as it came from milling and the simple lighting setup we used.



(b) The left two photographs show the milled acrylic block, after only a very rough polishing pass, photographed outside in front of a mountain scenery, and the caustic image that is cast when the block is lit. The right two images show the acrylic piece and the caustic image after fine polishing. Polishing reduced the milling artifacts in both the acrylic piece and its caustic.



(c) The three photographs show how the caustic image changes as the piece is rotated with respect to the fixed light source. See also accompanying video for a better live impression.

Figure 12: A physical prototype of a refractive caustic generator produced from acrylic glass.

of 4.38mm. Such shallow surfaces demand a relatively high production accuracy.

The refractive acrylic piece shown in Figure 12 was machined by a Roland MDX-40A milling machine with a ball nose end mill of 3mm diameter for the final pass. The milling time for the 10 by 10cm piece was around 1.5 hours. After milling we manually polished the surface to reduce machining artifacts using the headlight restorer kit by Turtle Wax.

The reflective aluminum plate was machined using a Spinner U5-620 milling machine with a ball nose end mill of 4mm diameter and a step size of 0.1mm. The total production time of the 10 by 10cm plate was around 3 hours. The milled surface was again polished to reduce milling artifacts.

Figures 1 and 12 show and discuss real photographs of the prototypes and the

generated caustic patterns. The accompanying video (lgg.epfl.ch/caustics) contains real video footage providing a better live impression of these prototypical results. These are the two first and only physical surfaces we have produced so far, with manufacturing processes that were easily accessible to us at short notice. We were surprised of the quality already achieved with these first tests with relatively low end, affordable machining.

We already have gained some insights on how to improve the quality even more in the future. We are, however, not manufacturing experts and one goal of this paper is to start a discussion with interested researchers and manufacturers on how to reach the best accuracy at a still affordable price. One way to produce accurate but cost-efficient reflective or refractive caustic generators could be to produce a mold with high-end machining and then cast several caustic generators using the same mold.

4 Conclusion

In this paper we demonstrated that it is both theoretically and practically possible to control the caustic pattern created by a reflective or refractive surface. We strived to provide an inspiring set of examples and installations to present caustics as an architectural design element.

In some way, caustics can be seen as a material property that can now be changed and controlled, almost like the color or finish of a surface. We believe that this surprising result is opening up a wide range of applications that we have only begun to discover. The design potential does not only lie in the possibility to make an image with light but more generally in new means for light control: being able to produce any desired light pattern and focusing light in desired directions and areas offers numerous applications from the graphic use of light to energy control. The scale of caustic design ranges from small consumer or art products, e.g., caustic text encoded on a drinking glass, to windows, facade elements, and even to an entire facade itself reflecting or refracting light in a controlled way.

Besides continuing to explore applications of caustic design in architecture and beyond, we plan to produce further prototypes and together with manufacturing experts and researchers investigate the ideal technologies to produce reflective and refractive caustic generators at various scales. Other interesting future work is to investigate the interplay of caustics and shadows, for example by combining our optimization for caustic images with the one presented for shadows by [Mitra and Pauly 2009].

The surface generating the caustic is computed through a complex geometry optimization that only a computer can perform. The surface is produced with high precision milling or molding to achieve an optimal resolution of the caustic image. The final product, the striking caustic pattern, however, appears when digital becomes analog, with no electricity involved, out of pure geometry and light. To us, this is the true magic behind architectural caustics.

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