1. 角动量守恒：$l = mr^2\dot{\theta}$ 或 $h = r^2\dot{\theta}$　$\dot{\theta} = \frac{l}{mr^2}$

2. 能量守恒：$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r)$ 或.

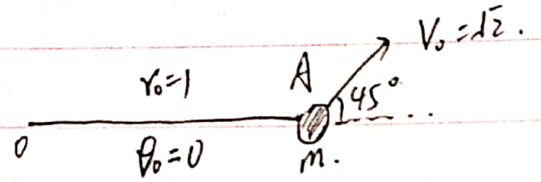
推导

3. 径向运动微分方程：$m(\ddot{r} - r\dot{\theta}^2) = f(r) \Rightarrow m\ddot{r} - \frac{l^2}{mr^3} = f(r)$.

4. 比耐公式　(1.65)

5. 降比耐公式

5.2. 解



$\underbrace{r(t), \theta(t)}_{消去t} \to r(\theta)$.

方法一：
$$\begin{cases} 能量守恒方程：E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) \\ 角动量守恒：l = mr^2\dot{\theta} \quad h = r^2\dot{\theta} \end{cases}$$

先求 V, E, l?

① $V = -\int_{\infty}^{r} F(r)\,dr = -\int_{\infty}^{r} -\frac{2m}{r^3}\,dr = -\frac{m}{r^2}$

② $E = \frac{1}{2}mV_0^2 + V_0(r) = \frac{1}{2}m(\sqrt{2})^2 + (-m) = 0$

③ $\vec{l} = \vec{r} \times m\vec{v} = |\vec{r}_0 \times mV_0| = r_0 mV_0 \sin 45° = \frac{\sqrt{2}}{2}r_0 mV_0 = m\vec{k} \Rightarrow h=1$.（垂直于纸面）.

④ $E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{m^2}{2mr^2} - \frac{m}{r^2} = 0$.

$\Rightarrow \dot{r}^2 - \frac{1}{r^2} = 0 \Rightarrow \dot{r}^2 = \frac{1}{r^2} \Rightarrow \dot{r} = \pm\frac{1}{r} \xrightarrow[定取正号]{由初始条件知} \dot{r} = \frac{1}{r}$

$r_0 = 1 > 0$, $V_0 = \sqrt{2}$.
$\dot{r}_0 > 0$

$\frac{dr}{dt} = \dot{r} = \frac{1}{r}$, $\int r\,dr = \int dt$

$\frac{1}{2}r^2 = t + C \xrightarrow[r_0=1]{t=0} C = \frac{1}{2}$

$r^2 = 2t+1$　$(r(t))$

⑤ $l = m$　$(h=1)$　$l = mr^2\dot{\theta} = m \Rightarrow r^2\dot{\theta} = 1 \Rightarrow \dot{\theta} = \frac{1}{2t+1} = \frac{d\theta}{dt}$

$\int\frac{dt}{2t+1} = \int d\theta$　$\frac{1}{2}\ln(2t+1) = \theta \Rightarrow e^{\theta} = (2t+1)^{\frac{1}{2}} = r \Rightarrow r = e^{\theta}$ 对数螺旋线

方法二：$\begin{cases} 径向微分方程：m(\ddot{r}-r\dot{\theta}^2)=F(r) \rightarrow m(\ddot{r}-\frac{h^2}{r^3})=F(r). \\ 角动量守恒：l=mr^2\dot{\theta} \end{cases}$

由前知 $h=1$ 代入 $m(\ddot{r}-\frac{1}{r^3})=-\frac{2m}{r^3} \Rightarrow \ddot{r}=-\frac{1}{r^3}$

$\ddot{r}=\frac{d\dot{r}}{dt}=\frac{d\dot{r}}{dr}\cdot\frac{dr}{dt}=\dot{r}\frac{d\dot{r}}{dt}=\frac{1}{2}\frac{d(\dot{r})^2}{dt}=\frac{-1}{r^3}$

$\int d(\dot{r})^2=\int\frac{-2}{r^3}dr \Rightarrow \dot{r}^2=\frac{1}{r^2} \Rightarrow \dot{r}=\frac{1}{r} \Rightarrow r=\sqrt{2t+1}$ $(r(t))$.

再由角守 $h=r^2\dot{\theta}$ 求 $\theta(t) \xrightarrow{r(t)} r(\theta)$.

方法三：比耐公式：$+mh^2u^2(\frac{d^2u}{d\theta^2}+u)=F(u)=+2mu$.

由 $h=1$ $\frac{d^2u}{d\theta^2}-u=0$ 通解：$u=C_1e^{\theta}+C_2e^{-\theta}$

$\theta_0=0, r_0=1, u_0=1$ 代入 $1=C_1+C_2$.

(P_{148} (5.2.4) $\dot{r}=-h\frac{du}{d\theta}$) $\dot{r}=\frac{dr}{dt}=\frac{d(\frac{1}{u})}{dt}=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$

$=-\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}=-r^2\dot{\theta}\frac{du}{d\theta}=-h\frac{du}{d\theta}$.

$\dot{r}=-h\frac{du}{d\theta}=-(C_1e^{\theta}-C_2e^{-\theta}) \xrightarrow[\theta_0=0,\ \dot{r}=1]{\theta_0=0,\ r_0=1} 1=-C_1+C_2 \Rightarrow C_1=0\ \ C_2=1$.

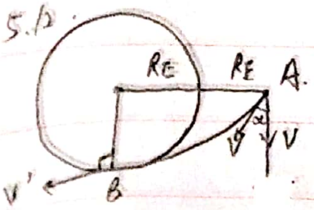
$u=e^{-\theta} \rightarrow r=e^{\theta}$

方法四：$\frac{1}{2}mh^2[(\frac{du}{d\theta})^2+u^2]+V=E$

求 V, E, h? $(\frac{du}{d\theta})^2=u^2 \rightarrow \frac{du}{d\theta}=\pm u$.

$\frac{du}{u}=\pm d\theta$ $\theta=\pm\ln u+C \longrightarrow C=0 \xrightarrow[\theta_0=0,\ \dot{r}_0>0,\ \dot{\theta}_0>0]{取正号} r=e^{\theta}$

5.12



E不变，T变

方法二：A点 $\frac{GMm}{(2R_E)^2}=m\frac{V^2}{2R_E} \Rightarrow V^2=\frac{GM}{2R_E}$.

改变后：横向速率 $V\cos\alpha=r\dot{\theta}=\frac{h}{r}2R_E$ $(h=r^2\dot{\theta})$

径向速率 $V\sin\alpha=\dot{r}$

B点：横向速率 $v'=\frac{h}{R_E}$

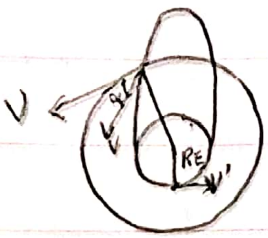
A→B：机械能守恒. $\frac{1}{2}mv^2 - \frac{GMm}{2R_E} = \frac{1}{2}mv'^2 - \frac{GMm}{R_E}$.

$\Longrightarrow v'^2 = \frac{3}{2}\frac{GM}{R_E}$

$v'^2 : v^2 = 3:1$　由 $v'^2 = \frac{h^2}{R_E^2}$，$v^2 = \frac{h^2}{4R_E^2}\frac{1}{\cos^2\alpha}$ $\Rightarrow \cos\alpha = \frac{\sqrt{3}}{2}$

$\alpha = 30^\circ$.

方法二：远近点方程.

3. 

质量为m的质点在中心力 $f(r) = -2ka^2r^{-5} + kr^{-3}$ 作用下半径为a的圆轨道运动，求其速率，并判断轨道稳定性.

圆轨道稳定性：$\frac{f'(r_0)}{f(r_0)} + \frac{3}{r_0} > 0$.

① 方法一：$-m\frac{v^2}{R} = f(r)$　　　$f(r) + m\frac{v^2}{R} = 0$.

$-m\frac{v^2}{a} = -2ka^{-3} + ka^{-3} = -ka^{-3} \Rightarrow v^2 = \frac{k}{m}a^{-2}$

$v = \frac{1}{a}\sqrt{\frac{k}{m}}$.

② 方法二：$-mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = F(u)$.

$u = \frac{1}{a}$，$\frac{du}{d\theta} = 0$，$h = |\vec{r}\times\vec{u}| = av$.

$-ma^2v^2a^{-3} = -ka^{-3} \longrightarrow v^2 = \frac{k}{ma^2}$

③ $f'(r) = 10ka^2r^{-6} - 3kr^{-4}$　　$f'(a) = 7ka^{-4}$

$f(a) = -2ka^{-3} + ka^{-3} = -ka^{-3}$

$\frac{f'(a)}{f(a)} + \frac{3}{a} = \frac{7ka^{-4}}{-ka^{-3}} + \frac{3}{a} = -\frac{7}{a} + \frac{3}{a} = -\frac{4}{a} < 0$ 不稳定.

4. P_{162}.