§5.4 贝塞尔函数习题

1、 利用 Bessel 函数的积分公式证明如下三个积分

（1）、$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2\phi e^{ix\cos(\phi-\theta)}d\phi=\left[\cos^2\theta J_0(x)-\frac{\cos 2\theta}{x}J_1(x)\right]$

（2）、$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\phi\cos\phi e^{ix\cos(\phi-\theta)}d\phi=\frac{1}{2}\sin 2\theta\left[J_0(x)-\frac{2}{x}J_1(x)\right]$

（3）、$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2\phi e^{ix\cos(\phi-\theta)}d\phi=\sin^2\theta J_0(x)+\frac{2\cos 2\theta}{x}J_1(x)$

三个证明思路一样，仅证明第一个等式，其余两个等式请自行证明。

（1）证明：$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2\varphi e^{ix\cos(\varphi-\theta)}d\varphi=\left[\cos^2\theta J_0(x)-\frac{\cos 2\theta}{x}J_1(x)\right]$

令$\varphi-\theta=\alpha$,即$\varphi=\alpha+\theta$，则

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi-\theta}^{\pi-\theta}\cos^2(\alpha+\theta)e^{ix\cos\alpha}d\alpha\\
&=\frac{1}{4\pi}\int_{-\pi-\theta}^{\pi-\theta}[1+\cos 2(\alpha+\theta)]e^{ix\cos\alpha}d\alpha\\
&=\frac{1}{4\pi}\int_{-\pi-\theta}^{\pi-\theta}(1+\cos 2\theta\cos 2\alpha-\sin 2\theta\sin 2\alpha)e^{ix\cos\alpha}d\alpha\\
&=\frac{1}{4\pi}\int_{-\pi-\theta}^{\pi-\theta}\left[e^{ix\cos\alpha}+\cos 2\theta\cos 2\alpha e^{ix\cos\alpha}-\sin 2\theta\sin 2\alpha e^{ix\cos\alpha}\right]d\alpha
\end{aligned}
$$

因为 $J_m(x)=\frac{(-i)^m}{2\pi}\int_{-\pi}^{\pi}\cos m\theta e^{ix\cos\theta}d\theta$

所以，上式$=\frac{1}{2}\left[J_0(x)+\cos 2\theta\frac{1}{(-i)^2}J_2(x)-0\right]$

$$=\frac{1}{2}J_0(x)-\frac{\cos 2\theta}{2}J_2(x)$$

因为，$J_{m+1}(x)-\frac{2m}{x}J_m(x)+J_{m-1}(x)=0$

$$J_2(x)-\frac{2}{x}J_1(x)+J_0(x)=0$$

$$J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$$

所以 上式$=\frac{1}{2}J_0(x)-\frac{\cos 2\theta}{2}\left[\frac{2}{x}J_1(x)-J_0(x)\right]$

$$=\frac{1}{2}J_0(x)-\frac{\cos 2\theta}{x}J_1(x)+\frac{1}{2}\cos 2\theta J_0(x)$$
$$=\cos^2\theta J_0(x)-\frac{\cos 2\theta}{x}J_1(x)$$

2、求如下 Bessel 函数的积分

（1）$\int_0^{x_0} x^3 J_0(x)\mathrm{d}x$ （2）$\int J_3(x)\mathrm{d}x$ （3）$\int_0^{x_0} xJ_0(x)\ln x\mathrm{d}x$

（4）$\int_0^{x_0} J_0(x)\cos x\mathrm{d}x$ （5）$\int_0^{x_0} J_0(x)\sin x\mathrm{d}x$ （6）$\int_0^{x_0} x^n J_n(x)\sin x\mathrm{d}x$

解：（1）

$$\int_0^{x_0} x^3 J_0(x)\mathrm{d}x=\int_0^{x_0} x^2 xJ_0(x)\mathrm{d}x=\int_0^{x_0} x^2\frac{\mathrm{d}}{\mathrm{d}x}[xJ_1(x)]\mathrm{d}x$$
$$=x^2 xJ_1(x)\Big|_0^{x_0}-2\int_0^{x_0} x^2 J_1(x)\mathrm{d}x=x_0{}^3J_1(x_0)-2x^2J_2(x)\Big|_0^{x_0}$$
$$=x_0{}^3J_1(x_0)-2x_0^2J_2(x_0)$$

由 $J_{m+1}(x)-\frac{2m}{x}J_m(x)+J_{m-1}(x)=0$

得到 $J_2(x)-\frac{2}{x}J_1(x)+J_0(x)=0$

即 $J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$

所以，

$$\int_0^{x_0} x^3J_0(x)dx=x_0{}^3J_1(x_0)-2x_0^2J_2(x_0)=x_0{}^3J_1(x_0)-2x_0{}^2\left[\frac{2}{x_0}J_1(x_0)-J_0(x_0)\right]$$
$$=x_0{}^3J_1(x_0)-4x_0J_1(x_0)+2x_0{}^2J_0(x_0)$$

（2）、

$$\int J_3(x)\mathrm{d}x=\int x^2\frac{1}{x^2}J_3(x)\mathrm{d}x=-\int x^2\mathrm{d}\left[\frac{1}{x^2}J_2(x)\right]\mathrm{d}x$$
$$=-x^2\frac{1}{x^2}J_2(x)+2\int\frac{1}{x}J_2(x)\mathrm{d}x+C=-J_2(x)-2\int\mathrm{d}\left[\frac{1}{x}J_1(x)\right]+C$$
$$=-J_2(x)-2\frac{J_1(x)}{x}+C'$$

由于 $J_{m+1}(x)-\frac{2m}{x}J_m(x)+J_{m-1}(x)=0$

得到 $J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$

$$\therefore\int J_3(x)\mathrm{d}x=-J_2(x)-2\frac{J_1(x)}{x}+C'=-\frac{2}{x}J_1(x)+J_0(x)-2\frac{J_1(x)}{x}+C'$$
$$=-\frac{4}{x}J_1(x)+J_0(x)+C'$$

(3)

$$\int_0^{x_0} x\mathrm{J}_0(x)\ln x\mathrm{d}x = \int_0^{x_0} \ln x\mathrm{d}[x\mathrm{J}_1(x)] = x\mathrm{J}_1(x)\ln x\Big|_0^{x_0} - \int_0^{x_0} \mathrm{J}_1(x)\mathrm{d}x$$

$$= x_0\mathrm{J}_1(x_0)\ln x_0 + \int_0^{x_0} \mathrm{d}[\mathrm{J}_0(x)] = x_0\mathrm{J}_1(x_0)\ln x_0 + \mathrm{J}_0(x)\Big|_0^{x_0}$$

$$= x_0\mathrm{J}_1(x_0)\ln x_0 + \mathrm{J}_0(x_0) - 1$$

$xJ_1(x)\ln x\big|_{x=0}$

$\dfrac{xJ_1(x)}{\frac{1}{\ln x}}$

$x\ln x\, J_1(x)$

$\dfrac{\ln x}{\frac{1}{x}} = \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = x J_1(x)$

(4) $\displaystyle\int_0^{x_0} \mathrm{J}_0(x)\cos x\mathrm{d}x = x\mathrm{J}_0(x)\cos x\Big|_0^{x_0} - \int_0^{x_0} x\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{J}_0(x)\cos x]\mathrm{d}x$

由递推公式 $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{\mathrm{J}_m(x)}{x^m}\right] = -\dfrac{\mathrm{J}_{m+1}(x)}{x^m}$，当 $m=0$ 时，$\dfrac{\mathrm{d}}{\mathrm{d}x}[\mathrm{J}_0(x)] = -\mathrm{J}_1(x)$

所以，积分

$$I = x_0\mathrm{J}_0(x_0)\cos x_0 + \int_0^{x_0} [x\mathrm{J}_1(x)\cos x + x\mathrm{J}_0(x)\sin x]\mathrm{d}x$$

由递推公式 $\dfrac{\mathrm{d}}{\mathrm{d}x}[x^m\mathrm{J}_m(x)] = x^m\mathrm{J}_{m-1}(x)$，当 $m=1$ 时，$\dfrac{\mathrm{d}}{\mathrm{d}x}[x\mathrm{J}_1(x)] = x\mathrm{J}_0(x)$

可得 $[x\mathrm{J}_1(x)\cos x + x\mathrm{J}_0(x)\sin x] = \dfrac{\mathrm{d}}{\mathrm{d}x}[x\mathrm{J}_1(x)\sin x]$

因此，积分

$$I = x_0\mathrm{J}_0(x_0)\cos x_0 + \int_0^{x_0} \frac{\mathrm{d}}{\mathrm{d}x}[x\mathrm{J}_1(x)\sin x]\mathrm{d}x$$

$$= x_0\mathrm{J}_0(x_0)\cos x_0 + x\mathrm{J}_1(x)\sin x\Big|_0^{x_0}$$

$$= x_0\mathrm{J}_0(x_0)\cos x_0 + x_0\mathrm{J}_1(x_0)\sin x_0$$

(5)、方法与（4）相似

$$\int_0^{x_0} \mathrm{J}_0(x)\sin x\mathrm{d}x = x\mathrm{J}_0(x)\sin x\Big|_0^{x_0} - \int_0^{x_0} x\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{J}_0(x)\sin x]\mathrm{d}x$$

$$= x_0\mathrm{J}_0(x_0)\sin x_0 + \int_0^{x_0} [x\mathrm{J}_1(x)\sin x - x\mathrm{J}_0(x)\cos x]\mathrm{d}x$$

$$= x_0\mathrm{J}_0(x_0)\sin x_0 - \int_0^{x_0} \frac{\mathrm{d}}{\mathrm{d}x}[x\mathrm{J}_1(x)\cos x]\mathrm{d}x$$

$$= x_0\mathrm{J}_0(x_0)\sin x_0 - x\mathrm{J}_1(x)\cos x\Big|_0^{x_0}$$

$$= x_0\mathrm{J}_0(x_0)\sin x_0 - x_0\mathrm{J}_1(x_0)\cos x_0$$

(6)、（4）和（5）的扩展积分

$$\int_0^{x_0} x^n \mathrm{J}_n(x)\sin x\mathrm{d}x = \int_0^{x_0} x^{2n}\left[x^{-n}\mathrm{J}_n(x)\sin x\right]\mathrm{d}x$$
$$=\frac{1}{2n+1}\int_0^{x_0}\left[x^{-n}\mathrm{J}_n(x)\sin x\right]\mathrm{d}x^{2n+1}$$
$$=\frac{1}{2n+1}x^{n+1}\mathrm{J}_n(x)\sin x\Big|_0^{x_0}-\frac{1}{2n+1}\int_0^{x_0}x^{2n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-n}\mathrm{J}_n(x)\sin x\right]\mathrm{d}x$$
$$=\frac{1}{2n+1}x_0^{\;n+1}\mathrm{J}_n(x_0)\sin x_0-\frac{1}{2n+1}\int_0^{x_0}x^{2n+1}\left[x^{-n}\mathrm{J}_n(x)\cos x-x^{-n}\mathrm{J}_{n+1}(x)\sin x\right]\mathrm{d}x$$
$$=\frac{1}{2n+1}x_0^{\;n+1}\mathrm{J}_n(x_0)\sin x_0-\frac{1}{2n+1}\int_0^{x_0}\left[x^{n+1}\mathrm{J}_n(x)\cos x-x^{n+1}\mathrm{J}_{n+1}(x)\sin x\right]\mathrm{d}x$$
$$=\frac{1}{2n+1}x_0^{\;n+1}\mathrm{J}_n(x_0)\sin x_0-\frac{1}{2n+1}\int_0^{x_0}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n+1}\mathrm{J}_{n+1}(x)\cos x\right]\mathrm{d}x$$
$$=\frac{1}{2n+1}x_0^{\;n+1}\mathrm{J}_n(x_0)\sin x_0-\frac{1}{2n+1}x^{n+1}\mathrm{J}_{n+1}(x)\cos x\Big|_0^{x_0}$$
$$=\frac{1}{2n+1}x_0^{\;n+1}\mathrm{J}_n(x_0)\sin x_0-\frac{1}{2n+1}x_0^{\;n+1}\mathrm{J}_{n+1}(x_0)\cos x_0$$

同理，可以求解积分

$$\int_0^{x_0}x^n\mathrm{J}_n(x)\cos x\mathrm{d}x=\frac{1}{2n+1}x_0^{\;n+1}\mathrm{J}_n(x_0)\cos x_0+\frac{1}{2n+1}x_0^{\;n+1}\mathrm{J}_{n+1}(x_0)\sin x_0$$

3、证明

$$\cos x=\mathrm{J}_0(x)+2\sum_{m=1}^{\infty}(-1)^m\mathrm{J}_{2m}(x)$$

$$\sin x=2\sum_{m=0}^{\infty}(-1)^n\mathrm{J}_{2m+1}(x)$$

由 Bessel 函数的生成函数 $\mathrm{e}^{\frac{x}{2}\left(z-z^{-1}\right)}=\sum_{m=-\infty}^{\infty}\mathrm{J}_m(x)z^m$，

令 $z=\mathrm{i}$，则有

$$\mathrm{e}^{\mathrm{i}x}=\sum_{m=-\infty}^{\infty}\mathrm{J}_m(x)\mathrm{i}^m$$

$$\Rightarrow\cos x+\mathrm{i}\sin x=\sum_{m=-\infty}^{-1}\mathrm{J}_m(x)\mathrm{i}^m+\mathrm{J}_0(x)+\sum_{m=1}^{\infty}\mathrm{J}_m(x)\mathrm{i}^m$$

$$\Rightarrow\cos x+\mathrm{i}\sin x=\sum_{m=1}^{\infty}\mathrm{J}_{-m}(x)\mathrm{i}^{-m}+\mathrm{J}_0(x)+\sum_{m=1}^{\infty}\mathrm{J}_m(x)\mathrm{i}^m$$

$$\Rightarrow\cos x+\mathrm{i}\sin x=\sum_{m=1}^{\infty}(-1)^m\mathrm{J}_m(x)\mathrm{i}^{-m}+\mathrm{J}_0(x)+\sum_{m=1}^{\infty}\mathrm{J}_m(x)\mathrm{i}^m$$

$$\Rightarrow\cos x+\mathrm{i}\sin x=\mathrm{J}_0(x)+2\sum_{m=1}^{\infty}\mathrm{J}_m(x)\mathrm{i}^m$$

$$\Rightarrow \cos x+\mathrm{i}\sin x=\mathrm{J}_0(x)+2\sum_{m=1}^{\infty}\mathrm{J}_{2m}(x)\mathrm{i}^{2m}+2\sum_{m=0}^{\infty}\mathrm{J}_{2m+1}(x)\mathrm{i}^{2m+1}$$

$$\Rightarrow \cos x+\mathrm{i}\sin x=\mathrm{J}_0(x)+2\sum_{m=1}^{\infty}(-1)^m\mathrm{J}_{2m}(x)+2\mathrm{i}\sum_{m=0}^{\infty}(-1)^m\mathrm{J}_{2m+1}(x)$$

比较上式的实部和虚部，得

$$\cos x=\mathrm{J}_0(x)+2\sum_{m=1}^{\infty}(-1)^m\mathrm{J}_{2m}(x)$$

$$\sin x=2\sum_{m=0}^{\infty}(-1)^n\mathrm{J}_{2m+1}(x)$$

4、已知 $x_n^{(1)}>0$ 是一阶 Bessel 方程的第 n 个零点，即 $\mathrm{J}_1\left(x_n^{(1)}\right)=0$，计算积分：

$$\int_0^a\rho^3\mathrm{J}_0\left(\frac{x_n^{(1)}}{a}\rho\right)\mathrm{d}\rho\quad (a>0\text{为常数})$$

解：

$$\int_0^a\rho^3\mathrm{J}_0\left(\frac{x_n^{(1)}}{a}\rho\right)\mathrm{d}\rho$$

$$=\left(\frac{a}{x_n^{(1)}}\right)^4\int_0^a\left(\frac{x_n^{(1)}}{a}\rho\right)^3\mathrm{J}_0\left(\frac{x_n^{(1)}}{a}\rho\right)\mathrm{d}\left(\frac{x_n^{(1)}}{a}\rho\right)=\left(\frac{a}{x_n^{(1)}}\right)^4\int_0^{x_n^{(1)}}x^3\mathrm{J}_0(x)\mathrm{d}x=\left(\frac{a}{x_n^{(1)}}\right)^4\int_0^{x_n^{(1)}}x^2x\mathrm{J}_0(x)\mathrm{d}x$$

$$=\left(\frac{a}{x_n^{(1)}}\right)^4\int_0^{x_n^{(1)}}x^2\mathrm{d}\left[x\mathrm{J}_1(x)\right]=\left(\frac{a}{x_n^{(1)}}\right)^4\left[x^3\mathrm{J}_1(x)\right]_0^{x_n^{(1)}}-2\left(\frac{a}{x_n^{(1)}}\right)^4\int_0^{x_n^{(1)}}x^2\mathrm{J}_1(x)\mathrm{d}x$$

$$=-2\left(\frac{a}{x_n^{(1)}}\right)^4\int_0^{x_n^{(1)}}\mathrm{d}\left[x^2\mathrm{J}_2(x)\right]=-2\left(\frac{a}{x_n^{(1)}}\right)^4x^2\mathrm{J}_2(x)\Big|_0^{x_n^{(1)}}$$

$$=-\frac{2a^4}{\left[x_n^{(1)}\right]^2}\mathrm{J}_2\left(x_n^{(1)}\right)$$

$$\because \mathrm{J}_2\left(x_n^{(1)}\right)-\frac{2}{x_n^{(1)}}\mathrm{J}_1\left(x_n^{(1)}\right)+\mathrm{J}_0\left(x_n^{(1)}\right)=0$$

$$\therefore \mathrm{J}_2\left(x_n^{(1)}\right)=-\mathrm{J}_0\left(x_n^{(1)}\right)$$

$$\int_0^a\rho^3\mathrm{J}_0\left(\frac{x_n^{(1)}}{a}\rho\right)\mathrm{d}\rho=\frac{2a^4}{\left[x_n^{(1)}\right]^2}\mathrm{J}_0\left(x_n^{(1)}\right)$$

5、将下列函数分别进行傅立叶-贝塞尔级数展开

（1）、在区间 $[0,a]$ 上，以 $\mathrm{J}_0(\sqrt{\mu_n^{(0)}}\rho)$ 为基（其中，$\mathrm{J}_0(\sqrt{\mu_n^{(0)}}a)=0$），把函数 $f(\rho)=u_0$ 展开傅立叶-贝塞尔级数。

解：$u_0=\sum_{n=1}^{\infty} f_n \mathrm{J}_0(\sqrt{u_n^{(0)}}\rho)$

$$f_n=\frac{1}{[N_n^{(0)}]^2}\int_0^a u_0 \mathrm{J}_0(\sqrt{u_n^{(0)}}\rho)\rho\mathrm{d}\rho$$

$\because \mathrm{J}_0(\sqrt{u_n^{(0)}}\rho)\Big|_{\rho=a}$ 满足第一类齐次边界条件

$\therefore [N_n^{(0)}]^2=\frac{1}{2}a^2[\mathrm{J}_1(\sqrt{u_n^{(0)}}a)]^2$

$$\therefore f_n=\frac{2u_0}{a^2[\mathrm{J}_1(\sqrt{u_n^{(0)}}a)]^2}\int_0^a \mathrm{J}_0(\sqrt{u_n^{(0)}}\rho)\rho\mathrm{d}\rho$$

$$=\frac{2u_0}{a^2[\mathrm{J}_1(x_n^{(0)})]^2}\left(\frac{a}{x_n^{(0)}}\right)^2\int_0^a \mathrm{J}_0\left(\frac{x_n^{(0)}}{a}\rho\right)\left(\frac{x_n^0}{a}\rho\right)\mathrm{d}\left(\frac{x_n^0}{a}\rho\right)$$

（其中，$\sqrt{u_n^{(0)}}a=x_n^{(0)},x_n^{(0)}$ 为 $J_0(x)$ 第 n 个零点）

令 $x=\frac{x_n^{(0)}}{a}\rho$

$$f_n=\frac{2u_0}{[x_n^{(0)}]^2[\mathrm{J}_1(x_n^{(0)})]^2}\int_0^{x_n^{(0)}}\mathrm{J}_0(x)x\mathrm{d}x$$

$$=\frac{2u_0}{[x_n^{(0)}]^2[\mathrm{J}_1(x_n^{(0)})]^2}x\mathrm{J}_1(x)\Big|_0^{x_n^{(0)}}$$

$$=\frac{2u_0}{x_n^{(0)}\mathrm{J}_1(x_n^{(0)})}$$

$$\therefore u_0=\sum_{n=1}^{\infty}\frac{2u_0}{x_n^{(0)}\mathrm{J}_1(x_n^{(0)})}\mathrm{J}_0\left(\frac{x_n^{(0)}}{a}\rho\right)$$

（2）、在第一类齐次边界条件下，把定义在[0，1]上的函数 $f(x)=1-x^2$ 展开为以零阶贝塞尔函数 $\mathrm{J}_0(\sqrt{\mu_n^{(0)}}x)$ 为基的傅立叶-贝塞尔级数。

解：$1-x^2=\sum_{n=1}^{\infty} f_n \mathrm{J}_0(\sqrt{u_n^{(0)}}x)$

$$\therefore f_n=\frac{1}{[N_n^{(0)}]^2}\int_0^1(1-x^2)\mathrm{J}_0(\sqrt{u_n^{(0)}}x)x\mathrm{d}x$$

第一类边界条件下，$\mathrm{J}_0(\sqrt{u_n^{(0)}}1)=0$，

设 $x_n^{(0)}$ 为 $\mathrm{J}_0(x)$ 的第 n 个零点，则 $\sqrt{u_n^{(0)}}=x_n^{(0)}$

$$[N_n]^2=\frac{1}{2}[J_1(x_n^{(0)})]^2$$

$$f_n=\frac{1}{\frac{1}{2}[J_1(x_n^{(0)})]^2}\int_0^1(1-x^2)J_0(x_n^{(0)}x)x\mathrm{d}x$$

$\therefore$

$$=\frac{2}{[J_1(x_n^{(0)})]}\frac{1}{[(x_n^{(0)})]^2}\int_0^1\left[1-\frac{1}{(x_n^{(0)})^2}(x_n^{(0)}x)^2\right]J_0(x_n^{(0)}x)(x_n^{(0)}x)\mathrm{d}(x_n^{(0)}x)$$

设 $x_n^{(0)}x=\rho$

$$f_n=\frac{2}{(x_n^{(0)})^2[J_1(x_n^{(0)})]^2}\int_0^{x_n^{(0)}}[1-\frac{\rho^2}{(x_n^{(0)})^2}]J_0(\rho)\rho\mathrm{d}\rho$$

$$=\frac{2}{(x_n^{(0)})^2[J_1(x_n^{(0)})]^2}\int_0^{x_n^{(0)}}[1-\frac{\rho^2}{(x_n^{(0)})^2}]\mathrm{d}[\rho J_1(\rho)]$$

$$=\frac{2}{(x_n^{(0)})^2[J_1(x_n^{(0)})]^2}\left\{[1-\frac{\rho^2}{(x_n^{(0)})^2}]\rho J_1(\rho)\Big|_0^{x_n^{(0)}}+2\int_0^{x_n^{(0)}}\rho J_1(\rho)\frac{\rho}{(x_n^{(0)})^2}\mathrm{d}\rho\right\}$$

$$=\frac{4}{(x_n^{(0)})^4[J_1(x_n^{(0)})]^2}\int_0^{x_n^{(0)}}\mathrm{d}[\rho^2J_2(\rho)]$$

$$=\frac{4}{(x_n^{(0)})^4[J_1(x_n^{(0)})]^2}\rho^2J_2(\rho)\Big|_0^{x_n^{(0)}}=\frac{4J_2(x_n^{(0)})}{(x_n^{(0)})^2[J_1(x_n^{(0)})]^2}$$

$$\because J_2(x_n^{(0)})-\frac{2J_1(x_n^{(0)})}{x_n^{(0)}}+J_0(x_n^{(0)})=0\qquad J_0(x_n^{(0)})=0$$

$$\therefore J_2(x_n^{(0)})=\frac{2J_1(x_n^{(0)})}{x_n^{(0)}}$$

$$\therefore f_n=\frac{8}{(x_n^{(0)})^3J_1(x_n^{(0)})}$$

$$\therefore(1-x^2)=\sum_{n=1}^{\infty}\frac{8}{(x_n^{(0)})^3J_1(x_n^{(0)})}J_n(x_n^{(0)}x)$$

（3）、将 $f(x)=x$ 在 $[0,a]$ 上展开以 $J_1\left(\frac{x_n^{(1)}}{a}x\right)$ 为基的 Fourier-Bessel 函数，其中，$x_n^{(1)}$ 为 $J_1(x)$ 的第 n 个零点。

解：$x=\sum_{n=1}^{\infty}f_nJ_1\left(\frac{x_n}{a}x\right)$

$$f_n=\frac{1}{[N_n^{(1)}]^2}\int_0^a xJ_1\left(\frac{x_n}{a}x\right)x\mathrm{d}x$$

$\because J_1\left(\frac{x_n}{a}x\right)\Big|_{x=a}=\mathrm{J}_1(x_n)=0$ 为第一类齐次边界条件

$\therefore \left[N_n^{(1)}\right]^2=\frac{1}{2}a^2\left[\mathrm{J}_2(x_n)\right]^2$

$$\therefore f_n=\frac{1}{\frac{1}{2}a^2\left[\mathrm{J}_2(x_n)\right]^2}\int_0^a x\mathrm{J}_1\left(\frac{x_n}{a}x\right)x\mathrm{d}x$$

$$=\frac{2}{a^2\left[\mathrm{J}_2(x_n)\right]^2}\left(\frac{a}{x_n}\right)^3\int_0^a\left(\frac{x_n}{a}x\right)^2\mathrm{J}_1\left(\frac{x_n}{a}x\right)\mathrm{d}\left(\frac{x_n}{a}x\right)$$

$$=\frac{2a}{\left[\mathrm{J}_2(x_n)\right]^2(x_n)^3}\int_0^{x_n}\rho^2\mathrm{J}_1(\rho)\mathrm{d}\rho$$

$$=\frac{2a}{\left[\mathrm{J}_2(x_n)\right]^2(x_n)^3}\int_0^{x_n}\mathrm{d}\left[\rho^2\mathrm{J}_2(\rho)\right]$$

$$=\frac{2a}{\left[\mathrm{J}_2(x_n)\right]^2(x_n)^3}\left[\rho^2\mathrm{J}_2(\rho)\right]_0^{x_n}$$

$$=\frac{2a}{\mathrm{J}_2(x_n)x_n}$$

$$\because \mathrm{J}_2(x_n)=\frac{2\mathrm{J}_1(x_n)}{x_n}-\mathrm{J}_0(x_n)=\mathrm{J}_0(x_n)$$

$$\therefore f_n=-\frac{2a}{\mathrm{J}_0(x_n)x_n}$$

$$\therefore x=-\sum_{n=1}^{\infty}\frac{2a}{\mathrm{J}_0(x_n)x_n}\mathrm{J}_1\left(\frac{x_n}{a}x\right)$$

6、 半径为 R 的圆形膜，边缘固定，初始形状$u(\rho,t)=H\left(1-\frac{\rho^2}{R^2}\right)$，初始速度为零，$H$ 为常数，求膜振动情况。

解：膜振动的定解方程为

$$\begin{cases}u_{tt}(\rho,t)=a^2\nabla^2u(\rho,t)\\ u|_{\rho=R}=0,\ \ u|_{\rho=0}=\text{有限值},\\ u|_{t=0}=H\cdot\left(1-\frac{\rho^2}{R^2}\right),\ \ u_t|_{t=0}=0\end{cases}$$

设$u(\rho,t)=R(\rho)T(t)$，代入泛定方程，得

$RT''=a^2T\nabla^2R$

$\frac{T''}{a^2T}=\frac{\nabla^2R}{R}=-k^2$

$\begin{cases}T''+a^2k^2T=0 & (1)\\ \nabla^2R+k^2R=0 & (2)\end{cases}$

常微分方程（1）的通解为

$T(t)=A\cos akt+B\sin akt$

由题可知，此薄膜振动位移只是 ρ,t 的函数，与 φ 无关

方程（2）写为极坐标形式

$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right)+k^2R=0$

即 $\rho^2\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}+k^2\rho^2R=0$

上述方程为零阶 Bessel 方程，其通解在 $R\big|_{\rho=0}=$ 有限值 条件下可写为

$R(\rho)=\mathrm{J}_0(k\rho)$

又因为 $R(\rho)\big|_{\rho=R}=0$ 满足第一类齐次边界条件

设 $x_n^{(0)}$ 为 $\mathrm{J}_0(x)$ 的第 n 个零点，则本征值

$k_n^{(0)}=\frac{x_n^{(0)}}{R}$

本征函数：$R_n(\rho)=\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)$

$$T_n(t)=A_n\cos\frac{x_n^{(0)}}{R}at+B_n\sin\frac{x_n^{(0)}}{R}at$$

$\therefore u(\rho,t)=\sum_{n=1}^{\infty}\left[A_n\cos\frac{x_n^{(0)}}{R}at+B_n\sin\frac{x_n^{(0)}}{R}at\right]\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)$

由 $u_t\big|_{t=0}=0\Rightarrow B_n=0$

通解退化为 $u(\rho,t)=\sum_{n=1}^{\infty}\left[A_n\cos\frac{x_n^{(0)}}{R}at\right]\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)$

由 $u\big|_{t=0} = H\cdot\left(1-\dfrac{\rho^2}{R^2}\right) \Rightarrow \sum\limits_{n=1}^{\infty} A_n \mathrm{J}_0\left(\dfrac{x_n^{(0)}}{R}\rho\right) = H\cdot\left(1-\dfrac{\rho^2}{R^2}\right)$

$$\therefore A_n = \frac{1}{\frac{1}{2}R^2\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}\int_0^R H\left(1-\frac{\rho^2}{R^2}\right)\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)\rho\,\mathrm{d}\rho$$

$$= \frac{2H}{R^2\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}\left(\frac{R}{x_n^{(0)}}\right)^2\int_0^R\left[1-\frac{1}{\left(x_n^{(0)}\right)^2}\left(\frac{x_n^{(0)}}{R}\rho\right)^2\right]\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)\cdot\left(\frac{x_n^{(0)}}{R}\rho\right)\mathrm{d}\left(\frac{x_n^{(0)}}{R}\rho\right) \quad \text{设} x = \frac{x_n^{(0)}}{R}\rho$$

$$= \frac{2H}{\left(x_n^{(0)}\right)^2\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}\int_0^{x_n^{(0)}}\left[1-\frac{1}{\left(x_n^{(0)}\right)^2}x^2\right]\mathrm{J}_0(x)x\,\mathrm{d}x$$

$$= \frac{2H}{\left(x_n^{(0)}\right)^2\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}\int_0^{x_n^{(0)}}\left[1-\frac{1}{\left(x_n^{(0)}\right)^2}x^2\right]\mathrm{d}\left[x\mathrm{J}_1(x)\right]$$

$$= \frac{2H}{\left(x_n^{(0)}\right)^2\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}\left\{x\mathrm{J}_1(x)\left[1-\frac{1}{\left(x_n^{(0)}\right)^2}x^2\right]_0^{x_n^{(0)}} + \frac{2}{\left(x_n^{(0)}\right)^2}\int_0^{x_n^{(0)}}x^2\mathrm{J}_1(x)\,\mathrm{d}x\right\}$$

$$= \frac{4H}{\left(x_n^{(0)}\right)^4\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}\int_0^{x_n^{(0)}}\mathrm{d}\left[x^2\mathrm{J}_2(x)\right]$$

$$= \frac{4H}{\left(x_n^{(0)}\right)^4\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}x^2\mathrm{J}_2(x)\Big|_0^{x_n^{(0)}}$$

$$= \frac{4H}{\left(x_n^{(0)}\right)^2\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}\mathrm{J}_2\left(x_n^{(0)}\right)$$

$\because \mathrm{J}_2\left(x_n^{(0)}\right)-\dfrac{2}{x_n^{(0)}}\mathrm{J}_1\left(x_n^{(0)}\right)+\mathrm{J}_0\left(x_n^{(0)}\right)=0$，且 $\mathrm{J}_0\left(x_n^{(0)}\right)=0$

$$\therefore \mathrm{J}_2\left(x_n^{(0)}\right) = \frac{2}{x_n^{(0)}}\mathrm{J}_1\left(x_n^{(0)}\right)$$

$$\therefore A_n = \frac{4H}{\left(x_n^{(0)}\right)^2\left[\mathrm{J}_1\left(x_n^{(0)}\right)\right]^2}\frac{2}{x_n^{(0)}}\mathrm{J}_1\left(x_n^{(0)}\right) = \frac{8H}{\left(x_n^{(0)}\right)^3\mathrm{J}_1\left(x_n^{(0)}\right)}$$

因此，定解问题的本征解为

$$u(\rho,t) = \sum_{n=1}^{\infty}\left[\frac{8H}{\left(x_n^{(0)}\right)^3\mathrm{J}_1\left(x_n^{(0)}\right)}\cos\frac{x_n^{(0)}}{R}at\right]\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)$$

7、一半径为 R 高为 H 的均匀圆柱体，上底有均匀分布的恒定热流（面积热流量为 q_0），垂直进入，下底则有同样的热流垂直流出，圆柱的侧面保持零度，求柱体内稳定温度分布。

解：定解方程为：

$$\begin{cases}\nabla^2 u(\rho,\varphi,z)=0\\ u|_{\rho=0}=\text{有限值},\ u|_{\rho=R}=0\\ u_z|_{z=0}=\dfrac{q_0}{k},u_z|_{z=H}=\dfrac{q_0}{k}\end{cases}$$

柱坐标系下 Laplace 方程分离变量，令 $u(\rho,\varphi,z)=R(\rho)\Phi(\varphi)Z(z)$，得到如下三个常微分方程

$$\begin{cases}\dfrac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+m^2\Phi=0 & (1)\\ \dfrac{\mathrm{d}^2 Z}{\mathrm{d}z^2}-\mu Z=0 & (2)\\ \rho\dfrac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\dfrac{\mathrm{d}R}{\mathrm{d}\rho}\right)+\left(\mu\rho^2-m^2\right)R=0 & (3)\end{cases}$$

由题意可知，圆柱体内温度沿 z 轴对称，与φ无关，即$m=0$

因此，方程（3）的退化为零阶 Bessel 方程，其解为零阶 Bessel 函数，即 $R(\rho)=\mathrm{J}_0\left(\sqrt{\mu}\rho\right)$，由第一类齐次边界条件 $R(\rho)|_{\rho=R}=0$，可得

本征值：$\mu_n^{(0)}=\left(\dfrac{x_n^{(0)}}{R}\right)^2$，其中 $x_n^{(0)}$ 是 $\mathrm{J}_0(x)$ 的第 n 个零点。

本征函数：$R_n(\rho)=\mathrm{J}_0\left(\dfrac{x_n^{(0)}}{R}\rho\right)$

方程（2）通解为 $Z(z)=C\mathrm{e}^{\sqrt{\mu}z}+D\mathrm{e}^{-\sqrt{\mu}z}=C\mathrm{e}^{\frac{x_n^{(0)}}{R}z}+D\mathrm{e}^{-\frac{x_n^{(0)}}{R}z}$

所以，定解问题通解可以写为：

$$u(\rho,z)=\sum_{n=1}^{\infty}\left[C_n\mathrm{e}^{\frac{x_n^{(0)}}{R}z}+D_n\mathrm{e}^{-\frac{x_n^{(0)}}{R}z}\right]\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)$$

再由边界条件 $u_z|_{z=0}=\dfrac{q_0}{k},u_z|_{z=H}=\dfrac{q_0}{k}$，的

$$\begin{cases}\displaystyle\sum_{n=1}^{\infty}(C_n-D_n)\frac{x_n^{(0)}}{R}\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)=\frac{q_0}{k}\\ \displaystyle\sum_{n=1}^{\infty}\left(C_n\mathrm{e}^{\frac{x_n^{(0)}}{R}H}-D_n\mathrm{e}^{-\frac{x_n^{(0)}}{R}H}\right)\frac{x_n^{(0)}}{R}\mathrm{J}_0\left(\frac{x_n^{(0)}}{R}\rho\right)=\frac{q_0}{k}\end{cases}$$

根据贝塞尔-傅里叶级数展开公式

$$C_n - D_n = \frac{R}{x_n^{(0)}} \frac{1}{\left[N_m^0\right]^2} \int_0^R \frac{q_0}{k} J_0\left(\frac{x_n^{(0)}}{R}\rho\right)\rho d\rho = \frac{Rq_0}{kx_n^{(0)} \frac{1}{2} R^2 \left[J_1\left(x_n^{(0)}\right)\right]^2} \int_0^R J_0\left(\frac{x_n^{(0)}}{R}\rho\right)\rho d\rho$$

$$= \frac{2Rq_0}{kx_n^{(0)} R^2 \left[J_1\left(x_n^{(0)}\right)\right]^2} \frac{R^2}{\left[x_n^{(0)}\right]^2} x_n^{(0)} J_1\left(x_n^{(0)}\right)$$

$$= \frac{2Rq_0}{k\left[x_n^{(0)}\right]^2 J_1\left(x_n^{(0)}\right)}$$

$$C_n e^{\frac{x_n^{(0)}}{R}H} - D_n e^{-\frac{x_n^{(0)}}{R}H} = \frac{2Rq_0}{k\left[x_n^{(0)}\right]^2 J_1\left(x_n^{(0)}\right)}$$

由以上两式可以求得

$$\begin{cases} C_n = \dfrac{1-e^{-\frac{x_n^{(0)}}{R}H}}{e^{\frac{x_n^{(0)}}{R}H} - e^{-\frac{x_n^{(0)}}{R}H}} \dfrac{2Rq_0}{k\left[x_n^{(0)}\right]^2 J_1\left(x_n^{(0)}\right)} \\ D_n = \dfrac{1-e^{\frac{x_n^{(0)}}{R}H}}{e^{\frac{x_n^{(0)}}{R}H} - e^{-\frac{x_n^{(0)}}{R}H}} \dfrac{2Rq_0}{k\left[x_n^{(0)}\right]^2 J_1\left(x_n^{(0)}\right)} \end{cases}$$

最终定解问题的解为:

$$u(\rho,z) = \frac{2Rq_0}{k} \sum_{n=1}^{\infty} \left[\frac{1-e^{-\frac{x_n^{(0)}}{R}H}}{e^{\frac{x_n^{(0)}}{R}H} - e^{-\frac{x_n^{(0)}}{R}H}} \frac{1}{\left[x_n^{(0)}\right]^2 J_1\left(x_n^{(0)}\right)} e^{\frac{x_n^{(0)}}{R}z} + \frac{1-e^{\frac{x_n^{(0)}}{R}H}}{e^{\frac{x_n^{(0)}}{R}H} - e^{-\frac{x_n^{(0)}}{R}H}} \frac{1}{\left[x_n^{(0)}\right]^2 J_1\left(x_n^{(0)}\right)} e^{-\frac{x_n^{(0)}}{R}z} \right] J_0\left(\frac{x_n^{(0)}}{R}\rho\right)$$

§5.5 虚宗量贝塞尔函数习题

1、匀质圆柱，半径为R，高为L，柱侧面有均匀分布的热流进入，其强度为q_0，圆柱上下两底面保持恒定温度u_0，求柱内温度分布？

解：采用柱坐标系，极点选在圆柱下底中心，z 轴沿圆柱的轴，柱内温度所满足定解方程为：

$$\begin{cases} \nabla^2 u(\rho,\varphi,z) = 0 \\ u\big|_{\rho=0} = \text{有限值},\ ku_\rho\big|_{\rho=R} = q_0 \\ u\big|_{z=0} = u_0,\ u\big|_{z=L} = u_0 \end{cases}$$

由于本征值都是由齐次边界条件所确定，因此，设$u = u_0 + v$，v满足如下定解方程

$$\begin{cases}\nabla^2 v(\rho,\varphi,z)=0\\ v\big|_{\rho=0}=\text{有限值},\ kv_\rho\big|_{\rho=R}=q_0\\ v\big|_{z=0}=0,\ v\big|_{z=L}=0\end{cases}$$

柱坐标系下 Laplace 方程分离变量，令 $v(\rho,\varphi,z)=R(\rho)\Phi(\varphi)Z(z)$，得到如下三个常微分方程

$$\begin{cases}\dfrac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+m^2\Phi=0 & (1)\\ \dfrac{\mathrm{d}^2 Z}{\mathrm{d}z^2}+\mu Z=0 & (2)\\ \rho\dfrac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\dfrac{\mathrm{d}R}{\mathrm{d}\rho}\right)-\left(\mu\rho^2+m^2\right)R=0 & (3)\end{cases}$$

由方程$\dfrac{\mathrm{d}^2 Z}{\mathrm{d}z^2}+\mu Z=0$及其齐次边界条件$Z(z)\big|_{z=0}=0,\ Z(z)\big|_{z=L}=0$ 构成本征值问题.

根据题意可知，球内温度分布关于 z 轴对称，因此所得解与φ无关，即$m=0$

$$Z(z)=C\cos\sqrt{\mu}z+D\sin\sqrt{\mu}z$$

由边界条件$Z(z)\big|_{z=0}=0\Rightarrow C=0$

$$Z(z)\Big|_{z=L}=0\Rightarrow \mu=\frac{n^2\pi^2}{L^2},\quad (n=1,2,\cdots)$$

因此方程（2）的本征解为：$Z_n(z)=\sin\dfrac{n\pi}{L}z,\quad (n=1,2,\cdots)$

方程（3）退化为$\rho\dfrac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\dfrac{\mathrm{d}R}{\mathrm{d}\rho}\right)-\mu\rho^2R=0$，为零阶虚宗量 Bessel 方程，其通解为

$$\begin{aligned}R(\rho)&=A\mathrm{I}_0\left(\sqrt{\mu}\rho\right)+B\mathrm{K}_0\left(\sqrt{\mu}\rho\right)\\&=A\mathrm{I}_0\left(\frac{n\pi}{L}\rho\right)+B\mathrm{K}_0\left(\frac{n\pi}{L}\rho\right)\end{aligned}$$

由$R(\rho)\big|_{\rho=0}=\text{有限值}\Rightarrow B=0$

所以，方程（3）的本征解$R(\rho)=\mathrm{I}_0\left(\dfrac{n\pi}{L}\rho\right)$

因此，定解问题通解可写为：

$$v(\rho,z)=\sum_{n=1}^{\infty}C_n\mathrm{I}_0\left(\frac{n\pi}{L}\rho\right)\sin\frac{n\pi}{L}z$$

由 $v_\rho\Big|_{\rho=R}=\dfrac{q_0}{k}$，可得

$$\sum_{n=1}^{\infty}C_n\frac{n\pi}{L}\mathrm{I'}_0\left(\frac{n\pi}{L}R\right)\sin\frac{n\pi}{L}z=\frac{q_0}{k}$$ 可以看作将 $\dfrac{q_0}{k}$ 以 $\sin\dfrac{n\pi}{L}z$ 为基展开傅里叶级数

$$C_n=\frac{L}{n\pi}\frac{1}{\mathrm{I'}_0\left(\frac{n\pi}{L}R\right)}\frac{2}{L}\int_0^L\frac{q_0}{k}\sin\left(\frac{n\pi}{L}z\right)\mathrm{d}z$$

$$=\frac{2Lq_0}{n^2\pi^2k}\frac{1}{\mathrm{I'}_0\left(\frac{n\pi}{L}R\right)}\int_0^L\sin\left(\frac{n\pi}{L}z\right)\mathrm{d}\left(\frac{n\pi}{L}z\right)$$

$$=-\frac{2Lq_0}{n^2\pi^2k}\frac{1}{\mathrm{I'}_0\left(\frac{n\pi}{L}R\right)}\cos\left(\frac{n\pi}{L}z\right)\Big|_0^L$$

$$=\frac{2Lq_0}{n^2\pi^2k}\frac{1}{\mathrm{I'}_0\left(\frac{n\pi}{L}R\right)}(1-\cos n\pi)$$

当 $n=2l+1$ 时，$C_{2l+1}=\dfrac{4Lq_0}{(2l+1)^2\pi^2k}\dfrac{1}{\mathrm{I'}_0\left(\frac{(2l+1)\pi}{L}R\right)}$

当 $n=2l$ 时，$C_{2l}=0$

最终，定解问题的解为：

$$u(\rho,z)=u_0+\frac{4Lq_0}{\pi^2k}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\frac{1}{\mathrm{I'}_0\left(\frac{(2l+1)\pi}{L}R\right)}\mathrm{I}_0\left(\frac{(2l+1)\pi}{L}\rho\right)\sin\frac{(2l+1)\pi}{L}z$$

§5.6 球贝塞尔函数习题

2、一半径为 r_0 的匀质球，初始时刻，球体温度均匀为 u_0，把它放入温度为 U_0 的烘箱中，使球体表面温度保持为 U_0。球球内各处温度分布？

解：定解问题为