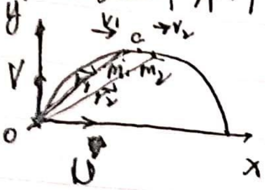
理力习题课 1.

法1,2 非惯性系下. 由 p. 角动量守恒 或王.

一、一炮弹质量为 $m=m_1+m_2$，射出时水平和竖直速度分量为 u V. 当炮弹到最高点，内部炸开，产生能量 E，分成 m_1, m_2 两块，$t=0$ 时，两者仍沿原方向飞行，试问落地时，m_1 m_2 相距多远？不计阻力，炸药能量仅供炮弹炸裂为 $m_1 \cdot m_2$ 且保持水平速度 u



法一：质心运动定理

$$\begin{cases} m_1\ddot{\vec r}_1 = m_1\vec g + \vec F_{12} & ① \\ m_2\ddot{\vec r}_2 = m_2\vec g + \vec F_{21} & ② \end{cases}$$

$①+②$ $\quad m_1\ddot{\vec r}_1 + m_2\ddot{\vec r}_2 = (m_1+m_2)\vec g$

$\Rightarrow (m_1+m_2)\ddot{\vec r}_c = (m_1+m_2)\vec g$

$\frac{①}{m_1} - \frac{②}{m_2}$ $\quad \ddot{\vec r}_1 - \ddot{\vec r}_2 = \frac{\vec F_{12}}{m_1} + \frac{\vec F_{12}}{m_2} = \frac{1}{\mu}\vec F_{12}$ $\qquad \mu = \frac{m_1m_2}{m_1+m_2}$ 折合质量

相对位置 $\vec r = \vec r_1 - \vec r_2 =$ $\qquad \mu\ddot{\vec r} = \vec F_{12}$ 相对运动微分方程

两边 $\cdot\, d\vec r$ $\qquad \int \ddot{\vec r}\cdot d\vec r = \int \frac{1}{\mu}\vec F_{12}\cdot d\vec r$

$\cancel{\ddot{\vec r}\,d\vec r = \frac{d\dot{\vec r}}{dt}\cdot d\vec r = d\dot{\vec r}\cdot d\vec r = \ddot{\vec r}\,d\vec r = \frac{1}{2}}$

$\ddot{\vec r}\,d\vec r = \frac{d\dot{\vec r}}{dt}\cdot d\vec r = d\dot{\vec r}\cdot\dot{\vec r} = \frac{1}{2}d(\dot r)^2$

$T_r = \frac{1}{2}\dot r^2 \qquad T_r = \frac{1}{\mu}E$

$dW_内 = \sum_i \vec F_i^{(i)} d\vec r_i = \vec F_{12}\,d\vec r_1 - \vec F_{12}\cdot\vec r_2 = \vec F_{12}\cdot d\vec r = \vec F_{12}(d\vec r_1 - d\vec r_2)$

$\Delta\ \frac{1}{2}\dot r^2 = \frac{E}{\mu}$

$\frac{1}{2}\dot x^2 = \frac{E}{\mu}$

$\dot x = \sqrt{\frac{2E}{\mu}}$

$S = \dot x t = \sqrt{\frac{2E}{\mu}}\,\frac{V}{g}$

法2. 水平 $\quad p = p' \qquad m_1\dot x_1 + m_2\dot x_2 = (m_1+m_2)u$ ① $\qquad \vec F_{炸}^{(i)}$ 为 $\vec F^{(i)}$

内

$\vec r = x\vec i + y\vec j$

动能定理 (考虑内力) $\quad dT = \sum_i \vec F_i^{(e)}\cdot d\vec r_i + \underbrace{\sum_i \vec F_i^{(i)}\,d\vec r_i}_{0}$ $\qquad$ 内力(炸炸力) 为外力.

$\frac{1}{2}m_1\dot x_1^2 + \frac{1}{2}m_2\dot x_2^2 - \frac{1}{2}(m_1+m_2)u^2 = \sum_i \vec F_i^{(i)}\,d\vec r_i$

$= E$ ②

①. ② 联立

$\dot x_1 = u + \sqrt{\frac{2Em_2}{m_1(m_1+m_2)}}$ $\qquad \dot x_2 = u - 2E$

$\dot x = \dot x_1 - \dot x_2 = \sqrt{\frac{2E}{\mu}}$

方法三 质心系 (快/非快)

v_1', v_2' $m_1v_1' + m_2v_2' = (m_1+m_2)v_c' = 0$ ①

水平动量守恒

能量方程: $\frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 - \underbrace{\frac{1}{2}(m_1+m_2)v_c'^2}_{0} = E$ ②

由①②联立

$v_1' = \pm\sqrt{\frac{2Em_2}{m_1(m_1+m_2)}}$ $v_2' = \mp\sqrt{\frac{2Em_1}{m_2(m_1+m_2)}}$ $v_1' - v_2'$

方法四: 化二体问题

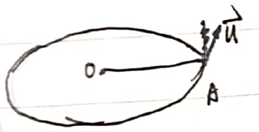
解 $\begin{cases} m_1\ddot{\vec r_1} = \vec F_{12} & ① \\ m_2\ddot{\vec r_2} = \vec F_{21} & ② \end{cases}$

$\frac{①}{m_1} - \frac{②}{m_2}$ $\ddot{\vec r} = \ddot{\vec r_1} - \ddot{\vec r_2} = \frac{\vec F_{12}}{m_1} - \frac{\vec F_{21}}{m_2} = \frac{\vec F_{12}}{\mu}$

$\mu\ddot{\vec r} = \vec F_{12}$

$\int \mu\ddot x\,dx = \int F_{12}\,dx$

$\frac{1}{2}\mu\dot x^2 = E$ $\dot x = \sqrt{\frac{2E}{\mu}}$



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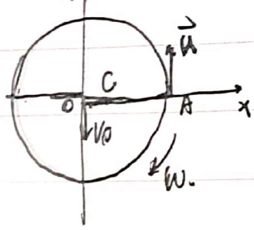


解: 分析

$\begin{cases} \text{水平方向不受外力 } p = p' \\ \text{铅垂方向 } M=0,\ z\text{方向角动量守恒,} \end{cases}$

动能不守恒 $\vec v_0$, ω.

求解: ①质心C, $x_C = \frac{mR}{m+M} =$ $\vec r_C = \frac{\sum m\vec r_i}{\sum_1 m_i}$

②由质心运动定理 $(m+M)\ddot x_C = 0$.

$\dot x_C = 0$ ∴C不动

绕质心C运动 选择坐标系和参考点C 选择静系下的动量和角动量守恒

水平方向动量守恒 人静系下 $v = v_{牵} + v_{相对}$

$v_{相对} = u$. $v_{牵} = v_{圆} = \vec\omega \times \vec r + \vec v_0$ $\vec v = \vec v_0 + \vec\omega\times\vec r + \vec u$

$M\vec v_0 + m(\vec v_0 + \vec\omega \times \vec{OA} + \vec u) = 0$

No.
Date

8. 子方向角动量守恒，1标系时C点，(背景中固定)

$\boxed{\vec{L} = \vec{L_c} + \vec{L'}}$ 系质点组角动量定理.

$\vec{L_c}$ = O点相对C点 = $\vec{CO} \times M\vec{v_0}$

$\vec{L'}$ = 绕质心O的角动量. = $I\vec{\omega}$ + $\vec{CA} \times m(\vec{u} + \vec{v_0} + \vec{\omega} \times \vec{r})$

$\vec{CO} \times M\vec{v_0} + I\omega + \vec{CA} \times m(\vec{v_0} + \vec{\omega} \times \vec{OA} + \vec{u}) = 0$

二. $(\alpha, \alpha, 0)$ 人

绕七 取v_0方向为分量方程.

$$Mv_0 + m(v_0 + \omega R - u) = 0$$

$$-Mv_0\frac{mR}{m+M} + I\omega + m(v_0 + \omega R - u)\frac{MR}{M+m} = 0$$

$I = \frac{1}{2}MR^2$ 圆盘时通过盘心的竖直轴转动惯量

人绕圆盘的相对角速度 $\dot{\varphi} = \frac{u}{R}$. 盘绕O自转角 $\omega = \dot{\theta}$

$\omega = \frac{d\theta}{dt} = \frac{2m}{3m+M}\frac{u}{R}$. $\quad t = \frac{2\pi R}{u}$ $\quad \theta = \omega t = \frac{2m}{2m+M}\frac{2\pi R}{R}$

若人转一圈 2π. $\theta = \frac{2m}{3m+M} \cdot 2\pi = \frac{4\pi m}{3m+M} = \frac{4\pi}{3+\frac{M}{m}}$

若 $\frac{M}{m} \to \infty$. 盘很重. 人转. $\theta \to 0$. 盘基本不动 $\theta = \frac{\pi}{3}$

人绕一圈. 盘转$\frac{4}{3}\pi$

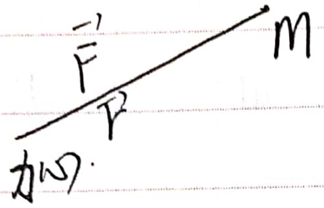
$\frac{M}{m} = 1$ $\quad \theta = \pi$.

$\frac{M}{m} \to 0$. $\quad \theta \to \frac{4}{3}\pi$. $\varphi - \theta = 2\pi - \frac{4}{3}\pi = \frac{2}{3}\pi$

总角动量 = 0

$\varphi = 0$, $\theta = 0$ $\quad$ 由动量守恒. 求v_0 $\quad \frac{v_0}{OC} \pm \dot{\theta} = \dot{\varphi} - \dot{\theta}$

第四章. 中心力—两体问题



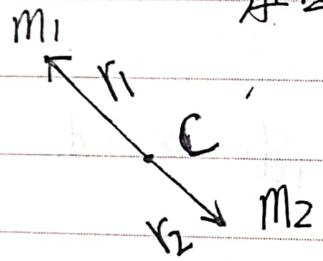
$$\vec F = \vec F(r) = f(r)\hat r \quad \begin{cases} f(r) > 0 \\ f(r) < 0 \end{cases}$$

万有引力: $f(r) = -\frac{GMm}{r^2}$

库仑力: $f(r) = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$

$\vec D = \varepsilon \vec E$

弹性力 $f(r) = -kr$



$$L = T - V$$

$$T = \frac{1}{2}(m_1+m_2)\underbrace{v_c^2}_{\text{质心系}=0} + \frac{1}{2}m_1\dot r_1^2 + \frac{1}{2}m_2\dot r_2^2$$

$$\frac{1}{2}M\dot r^2 - V(r)$$

$$0 = m_1\vec r_1 + m_2\vec r_2 \qquad M = \frac{m_1 m_2}{m_1+m_2}$$

$$\vec r_1 + \vec r_2 = r$$

4.1 物体在中心力场中运动规律

对力心O的力矩 $\vec r \times \vec F = \vec r \times f(r)\hat r = 0$

$$\frac{d\vec L}{dt} = \vec N = 0 \quad \vec L = \vec L_0 \quad \vec r \times m\vec v = \vec L_0$$

$$\vec r \times m\dot{\vec r} = \vec L_0$$

$\vec L_0 \perp \vec r$ $\vec L_0 \perp \vec v$ 有心力运动为平面运动

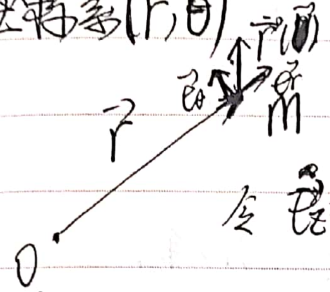
物体在中心力场中运动一般规律

1. 动量矩守恒，角动量守恒 $\vec{r}\times m\dot{\vec{r}}=\vec{L}_0$

$\vec{r}\perp\vec{L}_0$ $\dot{\vec{r}}\perp\vec{L}_0$ 平面运动

平面极坐标系 (r,θ)

极坐标？



$\vec{L}=\vec{r}\times m\dot{\vec{r}}=r\vec{e}_r\times m(\dot{r}\vec{e}_r+r\dot{\theta}\vec{e}_\theta)$

令 $\vec{e}_z=\vec{e}_r\times\vec{e}_\theta$

$=mr^2\dot{\theta}\vec{e}_z=\vec{L}_0$

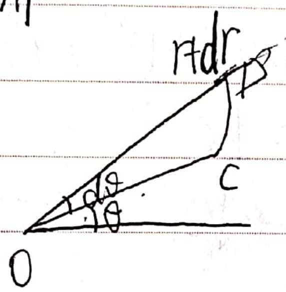
$=l\vec{e}_z=mh\vec{e}_z$

$\vec{e}_r\ \vec{e}_\theta,\ \vec{e}_z$ } 两组等价

$\hat{e}_r\ \hat{e}_\theta\ \hat{e}_z$ } 不区分

h：速度矩

速度矩 $\vec{r}\times\vec{v}$



$h=|\vec{r}\times\vec{v}|$

$=rv\sin\alpha$

$=2\dot{A}$

$\Delta t\to dt\to dA$

$dA=\frac{1}{2}rds$

$=\frac{1}{2}r^2d\theta$

$\dot{A}=\frac{dA}{dt}=\frac{1}{2}r^2\dot{\theta}$

面积速度 $=\frac{1}{2}h$

2. $\oint\vec{F}d\vec{l}=0=\iint\nabla\times\vec{F}d\vec{S}\cdot\ \nabla\times\vec{F}=0$

F为保守力

$\nabla\times\vec{F}(\vec{r})=\nabla\times[f(r)\hat{e}_r]$

$=0$

$\cancel{\vec{e}_r}\frac{\partial}{\partial r}$

$\nabla\times A=\frac{1}{r}\begin{vmatrix}\hat{e}_r & \hat{e}_\theta & \hat{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & A_\theta & A_z\end{vmatrix}$ $\mathrm{cof}[A_r]$

$V=-\int_{\vec{r}_0}^{\vec{r}}\vec{F}d\vec{r}=\int_{\vec{r}_0}^{\vec{r}}f(r)\vec{e}_rd\vec{r}=\int_{r_0}^{r}f(r)dr$

2. 机械能守恒 $T+V=E$ $\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V(r)=E$

3. 拉氏方程 $L=T-V=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-V(r)$

选广义坐标

$\frac{\partial L}{\partial\theta}=0$ $P_\theta=P_{\theta0}$ 角动量守恒

$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}=0$ $mr^2\dot{\theta}=$常数

还是角动量守恒

TODAY IS A HAPPY DAY

$(\frac{dr}{d\theta})^2 - \frac{2}{m} \quad \frac{1}{2}m(\dot r^2 \; r\dot\theta^2)$

$\frac{\partial L}{\partial t} = 0 \Rightarrow H = h = T_2 - T_0 + V = T_2 + V = T + V = E$

力学体系空间均匀性→动量守恒

各向同性　角动量守恒

时间均匀　能量守恒

二、运动微分方程、轨道方程　1.运动微分方程

牛、拉、哈（正、原——）

$\vec F = m\vec a = m\ddot{\vec r} = m(\ddot r - r\dot\theta^2,\ r\ddot\theta + 2\dot r\dot\theta)$

$m(\ddot r - r\dot\theta^2) = f(r)$

$m(r\ddot\theta + 2\dot r\dot\theta) = f_\theta = 0$

拉氏方程

$r:\ \frac{d}{dt}(\frac{\partial L}{\partial \dot r}) - \frac{\partial L}{\partial r} = 0 \quad m\ddot r - (mr\dot\theta^2 - \frac{\partial V}{\partial r}) = 0$

$\theta:\ \frac{d}{dt}(\frac{\partial L}{\partial \dot\theta}) - \frac{\partial L}{\partial \theta} = 0 \quad r^2\dot\theta = h$

$r^2\dot\theta = h \quad \dot\theta = \frac{h}{r^2} \qquad m(\ddot r - \frac{h^2}{r^3}) = f(r) \to r$

2.轨道方程　$r = r(t)\ \ \theta = \theta(t)$ 将时间消掉

$\frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + V(r) = E$

$r^2\dot\theta = h$

$\frac{1}{2}m(\dot r^2 + \frac{h^2}{r^2}) + V(r) = E \Rightarrow \frac{dr}{dt} = \dot r = \pm\sqrt{\frac{2(E-V)}{m} - \frac{h^2}{r^2}}$

由 扫描全能王 扫描创建

$\Rightarrow dt = \pm \frac{1}{\sqrt{\frac{2(E-V)}{m} - \frac{h^2}{r^2}}} dr$ ③ $t = t(r) \to r = r(t)$

$r = r(t, E, h, r_0)$ ①

$t=0$ 时 $\theta = \theta_0$

$d\theta = \frac{h}{r^2} dt \quad \theta = \theta_0 + \int_0^t \frac{h}{r^2} dt$ ④ $\theta = \theta(t)$ ②

① ② 为运动方程. ③ 代入 ②

$$\theta = \theta_0 \pm \int_{r_0}^{r} \frac{h}{r^2} \cdot \frac{1}{\sqrt{\frac{2(E-V)}{m} - \frac{h^2}{r^2}}} dr$$ 轨道方程

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{h}{r^2} = \pm\sqrt{\frac{2(E-V)}{m} - \frac{h^2}{r^2}}$$

积分 $\int_{\theta_0}^{\theta} d\theta = \pm \int_{r_0}^{r} \frac{\frac{h}{r^2} dr}{\sqrt{\frac{2(E-V)}{m} - \frac{h^2}{r^2}}}$ 轨道方程

θ_0, r_0 运动学初始条件

$E, h(e)$ 动力学初始条件

作业: 5.2, 5.7, 5.12

二、轨道微分方程 — 比耐方程

$m(\ddot{r} - r\dot{\theta}^2) = f(r)$ 由力可求轨道, 也可由轨道求力

$m(\ddot{r} - \frac{h^2}{r^3}) = f(r)$

$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right)$

$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{h}{r^2} = \frac{h}{r^2}\left[-h\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]'$

$= -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$

$\left\{ \begin{array}{l} r^2\dot{\theta} = h \\ \dot{\theta} = \frac{h}{r^2} \\ r\dot{\theta}^2 = \frac{h^2}{r^3} \\ \dot{\theta}^2 = \frac{h^2}{r^4} \end{array} \right.$

$m\left[-\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{h^2}{r^3}\right] = f(r)$

$-m\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{mh^2}{r^3} = f(r)$

令 $u=\frac{1}{r}$

$$-mh^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=f(u)$$

$$\frac{1}{2}m\left(\dot{r}^2+\frac{h^2}{r^2}\right)+V=E \qquad \dot{r}=-h\frac{du}{d\theta}$$

$$\frac{1}{2}m\left(h^2\left(\frac{du}{d\theta}\right)^2+h^2u^2\right)=E-V$$

$$\frac{1}{2}mh^2\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)=E-V$$ 降阶的比耐方程 Binet方程

$f(r)\to V\to r=r(\theta)$

运动微分方程

$$m\ddot{r}=f(r)+m\frac{h^2}{r^3}$$ 形式上是一维问题 → 将其当做一维来处理

$r^2\dot{\theta}=h$

有效力

惯性离心力

$$mr^2\dot{\theta}=m[-\vec{\omega}\times(\vec{\omega}\times\vec{r})]$$

$$=m[\omega^2\vec{r}-\vec{\omega}\cdot\vec{r}]$$

$$\frac{1}{2}m\dot{r}^2+\frac{1}{2}m\frac{h^2}{r^2}+V(r)=E$$

$\frac{L^2}{2mr^2}$ 看成势能，离心势能

V_{eff} 有效势能

等效势能

$$\frac{1}{2}m\dot{x}^2+V(x)=E \qquad T=E-V$$

$$\frac{1}{2}m\dot{r}^2+V_{eff}=E$$

$\frac{1}{2}m\dot{r}^2 = E - V_{eff}(>0)$

四、等效的一维问题

$m\ddot{r} = \frac{l^2}{mr^3} + f(r) = f_{eff}$

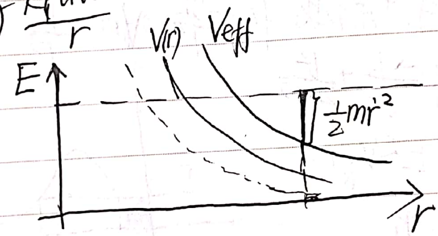
$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$

$V_{eff} = \frac{l^2}{2mr^2} + V(r)$

$\frac{1}{2}m\dot{r}^2 = E - V_{eff} > 0$ 势能曲线

(I) 与 r^2 成反比的排斥力 $f(r) = \frac{kq_1q_2}{r^2}$

$V(r) = \frac{kq_1q_2}{r}$



$V_{eff} = \frac{kq_1q_2}{r} + \frac{l^2}{2mr^2}$

拱点

$\frac{1}{2}m\dot{r}^2 = E - V_{eff} = 0$

$E = \frac{kq_1q_2}{r} + \frac{l^2}{2mr^2}$

$r_{min} \le r < \infty$

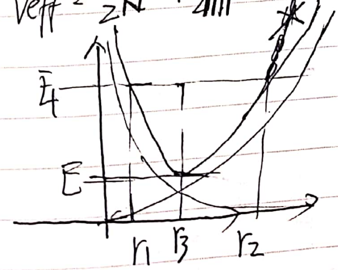
(II) 与 r^2 成正比的排斥势能 $f(r) = -kr$ $V(r) = \frac{1}{2}kr^2$

$V_{eff} = \frac{1}{2}kr^2 + \frac{l^2}{2mr^2}$

当 $E = E_1 > V_{eff}$ $r_0 \in (r_1, r_2)$

椭圆

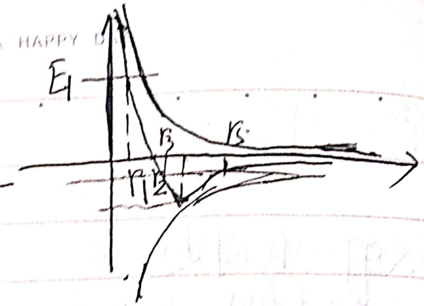
当 $E = E_2 = V_{eff}$ r_3 $r_0 = r_3$



(III) 与r^2成反比的引力

$f(r) = -\frac{GMm}{r^2}$　$V(r) = -\frac{GMm}{r}$

$V_{eff} = -\frac{GMm}{r} + \frac{l^2}{2mr^2}$



① $E = E_1 > 0$　$\infty > r \geq r_1$　双曲线

② $E = E_2 = 0$　$E - V_{eff} \geq 0$　$\infty > r \geq r_2$　抛物线

③ $E = E_3 < 0$　$E - V_{eff} = 0$ 有两"拐点" (r_3, r_5)
轨道为椭圆

④ $E = E_4 = V_{eff}$ —　$r = r_4$　圆

(IV) 与r^4成反比　$f(r) = \frac{3k}{r^4}$　$V(r) = \frac{k}{r^3}$

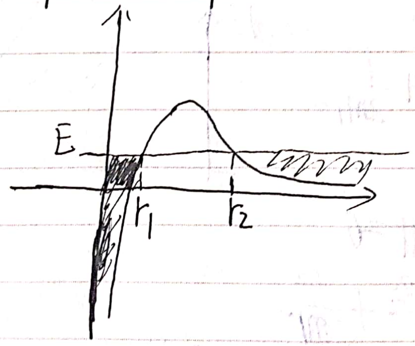
$V_{eff} = \frac{k}{r^3} + \frac{l^2}{2mr^2}$

$0 < E < V_{eff\,max}$

$0 \leq r \leq r_1$

$r_2 \leq r < \infty$

$r_1 \sim r_2$



4.2 Kopler's 问题