Integral Maths

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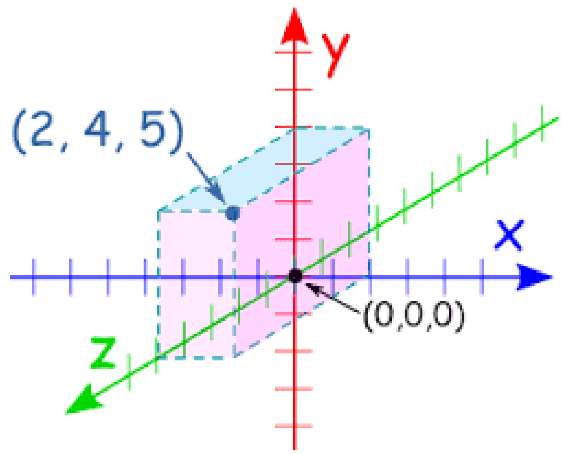
1. Coordinate Systems

1.1. A Need for Coordinate Systems

1.1.1. While it may be obvious to some, coordinate systems are required in any real application of math and science. When evaluating functions and properties, the results mean nothing unless there is a link between the math and the environment. Additionally, being able to convert between coordinate systems is extremely useful, as evaluating properties in some coordinate systems is much easier than others, where the dimensions can be changed to reflect the geometry of interest.

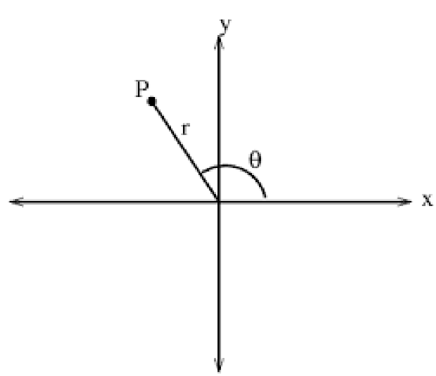
1.2. Rectangular/Cartesian Coordinates

1.2.1. These are the classic coordinates most are familiar with, where two axes are set perpendicular, and lay flat in a 2-D plane. Alternatively, a third axis can be laid perpendicular to these two, expanding the system to a 3-D domain. Axes are typically called x, y and z, or sometimes i, j and k.



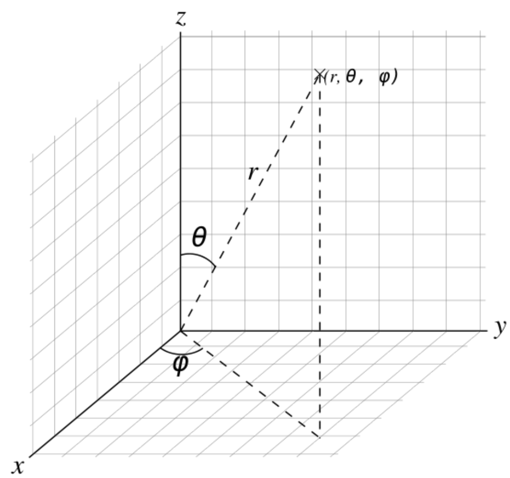
1.3. Polar Coordinates

1.3.1. Polar coordinates are extremely useful when dealing with circular shapes, as quantities can be evaluated much more conveniently. Polar coordinates consist of a radial length component r and a displacement angle from the horizontal θ.



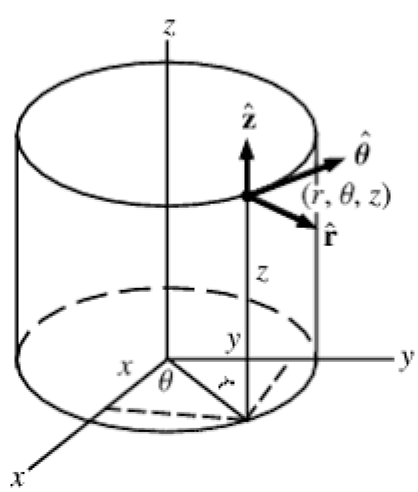
1.4.Spherical Coordinates

1.4.1. Similar to polar coordinates, spherical coordinates have a radial distance component from the center, as well as angular displacement values in the xy plane and from the z axis. This is somewhat difficult to picture, so a picture is very valuable.



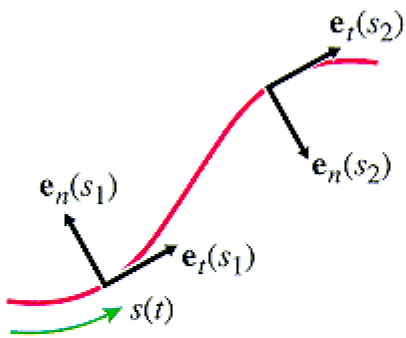
1.5.Cylindrical Coordinates

1.5.1. Not as well used, with applications in electronic wiring and magnetic fields, cylindrical coordinates are worth mentioning, as their applications can save a lot of work. Similar to polar coordinates, cylindrical coordinates are extended along the z axis.



1.6. Path Coordinates

1.6.1. Of use in thermodynamic, and cyclical applications, path coordinates are useful for irregular curves and shapes. The coordinates consist of a tangent component, and a normal component at some point along the path.

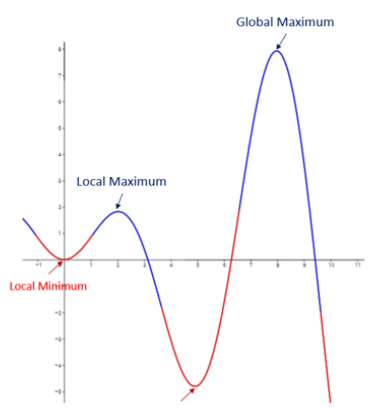


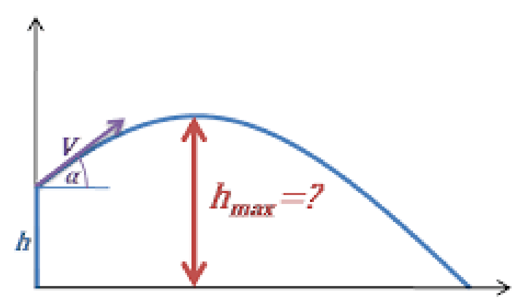
2. Derivatives

2.1. A Need for Derivatives

2.1.1. Assuming familiarity with a function that defines a given quantity with respect to some variable, such as time, position, etc, it is possible for a new function to be defined, which itself gives the rate of change of the initial function at any value of the independent variable. This new function is called the derivative of the initial function.

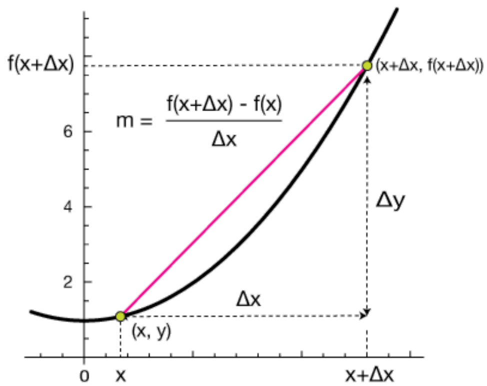
2.1.2. Uses arise when a maximum/minimum is desired, such as in business, or in physical kinematics, as well as in many other cases. Sometimes, simply the rate of change is desired at some known value of the independent variable, so the derivative can be calculated and gives a scalar number value representing the instantaneous rate of change of the function at that value.





2.1.3. From the concept of finding a slope *m* of a line by taking the change in $f(x)$ over the change in x, given by $\frac{\Delta f(x)}{\Delta x}$, *The Limit Definition of the Derivative* is a revolutionary idea that shows that as the change in x approaches 0, there is an instantaneous slope of the function at a value of x, equal to the slope of the tangent line at that point:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \frac{df(x)}{dx} = \frac{d}{dx} f(x)$$



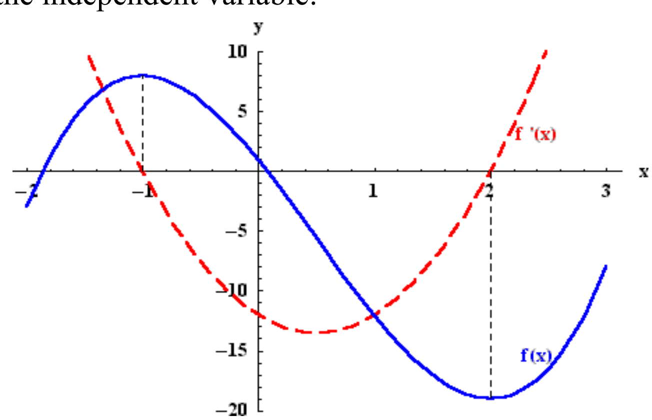
2.1.4. Here, the $\frac{d}{dx}$ term is called a "differential," and denotes a change in the following quantity with respect to some change in x so small it can be considered negligible. The derivative is denoted in the following ways for some function of the independent variable x, $f(x)$:

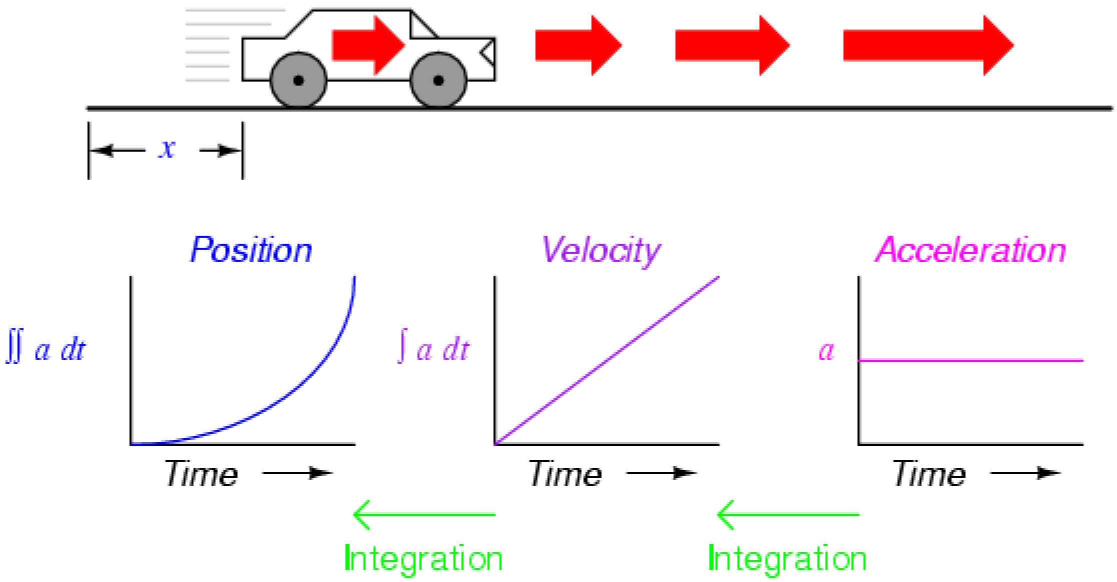
$$\frac{d}{dx} f(x) = f'(x)$$

2.1.5. Should the independent variable be time, which is very common, a shorthand "dot notation" is used:

$$\frac{d}{dt} f(t) = \dot{f}$$

2.1.6. This new graph can be plotted, showing all values of the derivative across the independent variable:




x
Position
Velocity
Acceleration
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a
Time
Time
Time
Integration
Integration

2.2. Standard Derivatives

2.2.1. These derivatives are the most basic and fundamental, and should be first found using the limit definition of the derivative, but can be referred to here:

$\frac{d}{dx}C = 0$

$\frac{d}{dx}x = C$

$\frac{d}{dx}x^n = n * x^{n-1}$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}ln(x) = \frac{1}{x}$

$\frac{d}{dx}sin(x) = cos(x)$

$\frac{d}{dx}cos(x) = -sin(x)$

$\frac{d}{dx}tan(x) = sec^2(x)$

$\frac{d}{dx}csc(x) = -cot(x) * csc(x)$

$\frac{d}{dx}cot(x) = -csc^2(x)$

2.3. Product Rule

2.3.1. When taking the derivative of a product of functions of an independent variable, a new approach must be found to find the derivative of the function. An unorganized mish-mash of split up derivatives will not do, and a more systematic approach is required. Taking the derivative of each, multiplied by the other, gives the following:

$$\frac{d}{dx}[f(x) * g(x)] = f'(x) * g(x) + g'(x) * f(x)$$

2.3.2. Some Examples:

$\frac{d}{dx}\left[x^2cos(x)\right] = (2x)cos(x) - (x^2)sin(x)$

$\frac{d}{dx}[e^xsin(x)] = e^xsin(x) + e^xcos(x) = e^x(sin(x) + cos(x))$

$\frac{d}{dx}\left[e^{2x}sin(x)\right] = 2e^xsin(x) + e^{2x}cos(x) = e^x(2sin(x) + e^xcos(x))$

2.4.Quotient Rule

2.4.1. Similar to the way the derivative of a product of functions must be taken systematically, so must the quotient. The following gives the general approach:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)*g(x)-g'(x)*f(x)}{(g(x))^2}$$

2.4.2. Some Examples:

$$\frac{d}{dx}\left[\frac{e^x}{x^2}\right] = \frac{e^x x^2 - e^x 2x}{x^4} = \frac{e^x(x-2)}{x^3}$$

$$\frac{d}{dx}\left[\frac{x^2}{x-2}\right] = \frac{2x(x-2)-x^2}{(x-2)^2} = \frac{x^2-4x}{(x-2)^2}$$

$$\frac{d}{dx}\left[\frac{x^2}{sin(x)}\right] = \frac{2xsin(x)-x^2cos(x)}{sin^2(x)} = x\left(\frac{2}{sin(x)} - \frac{xcos(x)}{sin^2(x)}\right)$$

2.5.Chain Rule

2.5.1. When a function contains within it another function of an independent variable, the chain rule allows the derivative of the entire function to be determined:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) * g'(x)$$

2.5.2. Some Examples:

$$\frac{d}{dx}\left[sin(x^2)\right] = cos(x^2)2x$$

$$\frac{d}{dx}\left[ln(x^3-5)\right] = \frac{1}{x^3-5}3x^2 = \frac{3x^2}{x^2-5}$$

$$\frac{d}{dx}\left[(2x^2+8)^2\right] = 2(2x^2+8)*(4x) = 8x(2x^2+8) = 16x^3+64x$$

2.6. Higher Derivatives

2.6.1. Previously, only the first derivatives were found, or $f'(x)$. As stated before, this is useful in finding maxima and minima, kinematic peaks, and other things. The next logical question is whether the derivative of a derivative makes sense to be found. Indeed, if the first derivative is thought of as the rate of change of the initial function as a function of the independent variable, then the second derivative can relay the rate of change of the first derivative, or how rapidly the slope of the original function is changing as a function of the independent variable.

2.6.2. This is a complex thought, and sometimes makes sense in terms of a quantity we are familiar with. The common example is position, given as a function of time $f(t)$.

Now, the derivative of position is given by $f'(t)$. The physical meaning of $f'(t)$ is the rate of change of position as a function of time. This is just velocity, which is understandable because how quickly something's position is changing is just the speed at which it is moving.

Furthermore, the second derivative of position, $f''(t)$ can be understood as the rate at which velocity is changing at any time *t*. The quantity we associate with this concept is acceleration, like when a car's gas pedal is pressed, it does not jump to 70mph, but accelerates so that the velocity of the car increases over time.

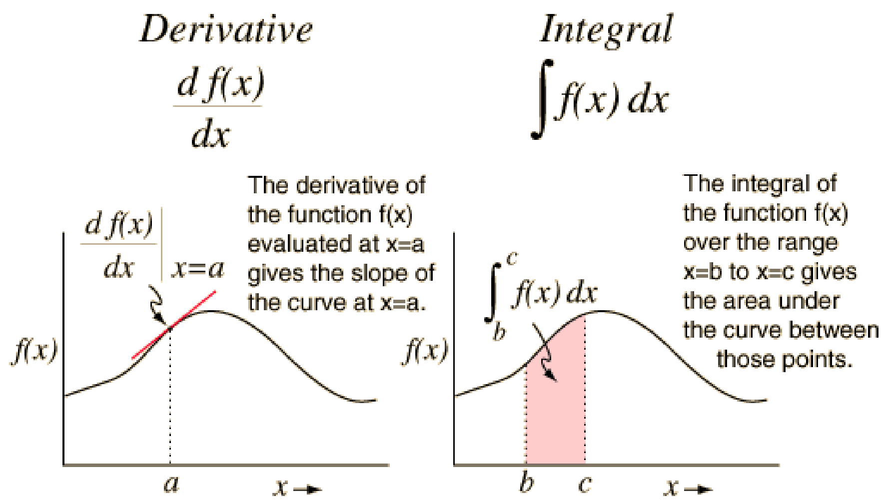
2.6.3. Furthermore, even higher derivatives can be calculated with similar interpretations, but are frankly harder to conceptualize. Higher derivatives are denoted as follows:

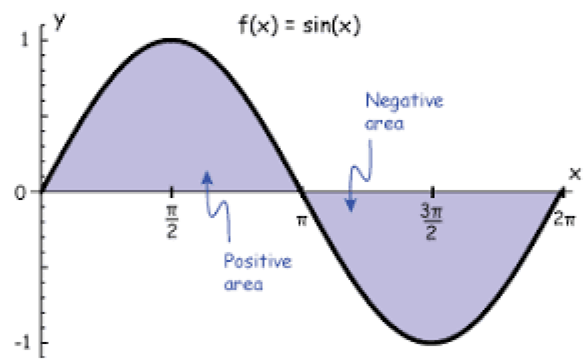
$$f'(t),\ f''(t),\ f'''(t),\ f^4(t),\ f^5(t),\ f^n(t)\ ...$$

3. Integrals

3.1. A Need For Integrals

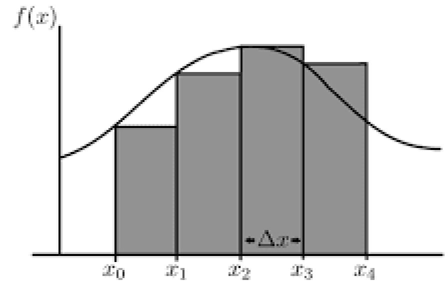
3.1.1. While derivatives show the rate of change in a quantity with respect to some independent variable, the integral shows the "magnitude" or area of a function with respect to some independent variable. This can be confusing, but a graph makes more sense:

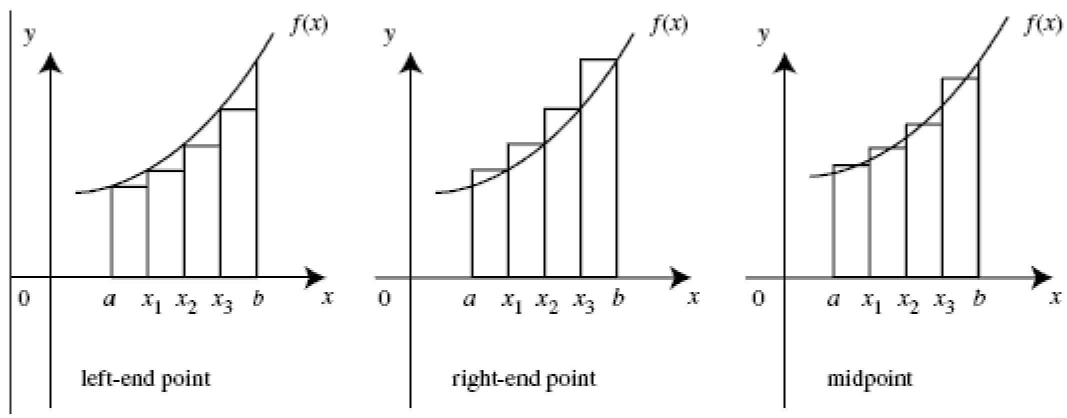




3.1.2. Integrals are important for determining areas that involve a product where one or more quantities change with respect to some independent variable. This is commonly used in mechanics when stresses and strains are analyzed along the length of a beam where quantities like force or material can change. Integration also has extensive uses in physics among other applications.

3.1.3. An integral can initially be thought of as the area under a curve where areas above the horizontal axis are positive and areas below the horizontal axis are negative. The mathematician Riemann came up with taking a summation of a series of rectangles created underneath the curve, called a *Riemann Sum*:





These sums are critical in understanding the origin of the integral. As in the limit definition of the derivative, should Δx approach 0, then the result is an infinite number of neighboring rectangles that sum to approximate the area under the curve.

3.1.4. The (left handed, which is most common) Riemann Sum can be defined as follows and gives a reflection of the actual value of the integral:

$$\sum_{n=a}^{b-1} f_n(\Delta x)_n \approx \int_a^b f(x)dx$$

3.1.5. An integral can be taken across certain bounds of the independent variable, infinite bounds, or found generally. To find an integral, similarly to a derivative, a set of laws can be used to define an integral, then the expression can be evaluated across the bounds of the integral.

3.1.6. Integrals that are across known bounds are "definite integrals", whereas integrals that involve any sort of infinite quantity, which are common, are called "improper integrals".

3.1.7. An important aspect of integration is that functions inside the integral can be separated and evaluated separately. This concept of "splitting up" an integral is called *Linearity*:

$$\int (\lambda f(x) + \mu g(x)) dx = \lambda \int f(x) dx + \mu \int g(x) dx$$

3.2. Standard Integrals

3.2.1. These Integrals are the most basic and fundamental components of any larger integral. Some are easily derivable, but shown here for reference.

$$\int 1 dx = x + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x} dx = ln\,|x| + C$$

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + C$$

$$\int sin(x) dx = -cos(x) + C$$

$$\int cos(x) dx = sin(x) + C$$

$$\int tan(x) dx = -ln\,|cos(x)| + C = ln\,|sec(x)| + C$$

$$\int cot(x) dx = ln\,|sin(x)| + C$$

$$\int sec^2(x) dx = tan(x) + C$$

$$\int csc^2(x) dx = -cot(x) + C$$

$$\int sec(x) dx = ln\,|sec(x) + tan(x)| + C$$

$$\int csc(x) dx = ln\,|csc(x) - cot(x)| + C$$

3.3.U-Substitution

3.3.1. When a function contains another function within it, and is multiplied by some form of the inner function's derivative, the integral can be calculated by substituting in a variable u for the inner function, then differentiating the outer function with respect to u, taking care to note the factor of $\frac{du}{dx}$ present.

3.3.2. General Case:

$$\int f(g(x)) * g'(x)dx \quad \{u = g(x),\ \tfrac{du}{dx} = g'(x)dx\}$$

$$\int f(g(x)) * g'(x)dx = \int f(u)du$$

3.3.3. Some Examples:

$$\int (x+3)^4 dx \quad \{u = (x+3), \tfrac{du}{dx} = 1dx\}$$

$$\int (u)^4 dx = \tfrac{1}{5}u^5 + C = \tfrac{1}{5}(x+3)^5 + C$$

$$\int cos(x^2)\ 6x\ dx \quad \{u = x^2, \tfrac{du}{dx} = 2xdx\}$$

$$3\int cos(u)du = 3sin(u) + C = 3sin(x^2) + C$$

3.4.Integration by Parts

3.4.1. Similar to u-substitution, when more complicated functions are involved, the integral may not be obvious. Many harder integrals can be solved using the strategy of integration by parts. With u-substitution, an inner function with a multiple of its derivative on the outside was sought. All integration by parts seeks is two functions of an independent variable, where one has a simpler derivative, and the other's integral should not be dramatically more complex. Integrating by parts typically takes longer than other integrals, but with some experience, it becomes easy to identify the parts and solve. Given two functions of x, the following derives the method of integration by parts:

$$\frac{d[u(x)v(x)]}{dx} = v(x)\frac{du(x)}{dx} + u(x)\frac{dv(x)}{dx} \quad (chain\ rule)$$

$$\int \frac{d[u(x)v(x)]}{dx}dx = \int v(x)\frac{du(x)}{dx}dx + \int u(x)\frac{dv(x)}{dx}dx \quad (integrating\ on\ x)$$

$$u(x)v(x) = \int v(x)du + \int u(x)dv \quad (simplified)$$

$$\int u(x)dv = uv - \int v(x)du \quad (final\ general\ formula)$$

3.4.2. From the general formula, it is standard to choose $u(x)$ to be a function with a simple derivative, and $v(x)$ to have a simple integral. Typically before evaluating, from the integral $\int u(x)dv$, it is useful to identify $u(x)$ and dv on the side and evaluate $\frac{du(x)}{dx}$ and $v(x)$ before writing the integral in terms of the general form.

3.4.3. Some Examples:

$$\int xe^{6x}dx \quad \{u = x,\ du = 1dx,\ dv = e^{6x},\ v = \int e^{6x}dx = \tfrac{1}{6}e^{6x}\}$$

$$\int xe^{6x}dx = uv - \int vdu = x * \tfrac{1}{6}e^{6x} - \int \tfrac{1}{6}e^{6x}1dx$$

$$= \tfrac{x}{6}e^{6x} - \tfrac{1}{36}e^{6x} + C = e^{6x}(\tfrac{x}{6} - \tfrac{1}{36}) + C$$

$$\int(3x+5)cos(\tfrac{x}{4})dx \quad \{u = 3x+5,\ du = 3dx,\ dv = cos(\tfrac{x}{4})dx,\ v = \int cos(\tfrac{x}{4})dx = 4sin(\tfrac{x}{4})$$

$$\int(3x+5)cos(\tfrac{x}{4})dx = uv - \int vdu = (3x+5)4sin(\tfrac{x}{4}) - \int 4sin(\tfrac{x}{4}) * 3dx$$

$$= (3x+5) * 4sin(\tfrac{x}{4}) + 12 * 4cos(\tfrac{x}{4}) + C$$

3.5.Trigonometric Integrals

3.5.1. Although the linearity principle, u-substitution, and integration by parts, can deal with most straightforward integrals, integrals involving many trigonometric functions can require a new approach.

3.5.2. Essentially, when evaluating an integral that has multiples of trig functions, the general approach is to simplify the expression, typically reducing to sines and cosines, then apply trigonometric identities so that u-substitution easily solves the integral.

3.5.3. Useful trigonometric identities:

$$sin^2(x) + cos^2(x) = 1$$

$$cos^2(x) = \tfrac{1}{2}(1 + cos(2x))$$

$$sin^2(x) = \tfrac{1}{2}(1 - cos(2x))$$

$$sin(x)cos(x) = \tfrac{1}{2}sin(2x)$$

3.5.4. Notice that it is important to recognize whether the exponents on the trig functions are odd or even, as that will determine which identity fits best.

3.5.5. Some Examples:

$$\int sin^6(x)cos^3(x)dx = \int sin^6(x)cos^2(x)cos(x)dx = \int sin^6(x)(1 - sin^2(x))cos(x)dx$$

$$\int sin^6(x)(1 - sin^2(x))cos(x)dx \quad \{u = sin(x),\ du = cos(x)\}$$

$$\int sin^6(x)(1 - sin^2(x))cos(x)dx = \int u^6(1 - u^2)du = \int u^6 - u^8 du$$

$$= \tfrac{1}{7}u^7 - \tfrac{1}{9}u^9 + C = \tfrac{1}{7}sin(x)^7 - \tfrac{1}{9}sin^9(x) + C$$

$$\int sec^9(x)tan^5(x)dx = \int sec^8(x)tan^4(x)sec(x)tan(x)dx$$

$$= \int sec^8(x)(sec^2(x) - 1)^2 sec(x)tan(x)dx \quad \{u = sec(x),\ du = sec(x)tan(x)\}$$

$$= \int u^8(u^4 - 2u^2 + 1)du = \int u^{12} - 2u^{10} + u^8 du = \tfrac{1}{13}u^{13} - 2\tfrac{1}{11}u^{11} + \tfrac{1}{9}u^9 + C$$

$$= \tfrac{1}{13}sec^{13}(x) - \tfrac{2}{11}sec^{11}(x) + \tfrac{1}{9}sec^9(x) + C$$

3.6.Trigonometric Substitution

3.6.1. Occasionally, a simple substitution is not quite enough, and functions that follow a certain form can be evaluated more easily by choosing a specific substitution. There are three forms of trigonometric substitution, with corresponding giveaways in the integral.

3.6.2. For integrals involving $(\alpha^2 - x^2)$, use $x = a * sin(\theta)$

3.6.3. For integrals involving $(x^2 - \alpha^2)$, use $x = a * sec(\theta)$

3.6.4. For integrals involving $(x^2 + \alpha^2)$, use $x = a * tan(\theta)$

3.6.5. Note this quantity is typically under a square root, but can be raised to other powers. By choosing a clever value of α, the integral can be greatly simplified so that a trigonometric identity can be utilized.

3.6.6. If the integral is definite, it is typically easiest to find the limits of integration in terms of θ and evaluate. Otherwise, and an option regardless, it is standard to return the integral to terms of x by using trigonometric relations like $sin(\theta) = \frac{x}{a}$.

3.6.7. There is no simple general form for trigonometric substitution integrals. Instead, practice, and recalling the three forms listed above is key.

3.6.8. Some Examples:

$$\int \frac{\sqrt{9-x^2}}{x^2}dx \quad \{noticing\ the\ (9-x^2)^{\frac{1}{2}}\ term,\ the\ substitution\ should\ be\ x = \alpha * sin(\theta)\}$$

$$\int \frac{\sqrt{9-x^2}}{x^2}dx \quad \{noticing\ that\ 9\ is\ a\ perfect\ square,\ \alpha\ should\ be\ chosen\ to\ be\ 3\}$$

$$\int \frac{\sqrt{9-x^2}}{x^2}dx \quad \{x = 3 * sin(\theta),\ dx = 3 * cos(\theta)d\theta\}$$

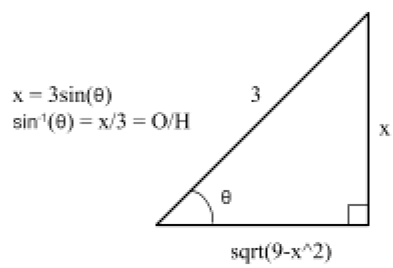
$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{\sqrt{9-(3sin(\theta))^2}}{(3sin(\theta))^2}3cos(\theta)d\theta = \int \frac{\sqrt{9-9sin^2(\theta)}}{9sin^2(\theta)}3cos(\theta)d\theta$$

$$= \int \frac{\sqrt{9cos^2(\theta)}}{9sin^2(\theta)}3cos(\theta)d\theta = \int \frac{\sqrt{9(1-sin^2(\theta))}}{9sin^2(\theta)}3cos(\theta)d\theta = \int \frac{\sqrt{9cos^2(\theta)}}{9sin^2(\theta)}3cos(\theta)d\theta$$

$$= \int \frac{3cos(\theta)}{9sin^2(\theta)}3cos(\theta)d\theta = \int \frac{cos^2(\theta)}{sin^2(\theta)}d\theta = \int cot^2(\theta)d\theta = \int(csc^2(\theta)-1)d\theta$$

$$= \ -cot(\theta) - \theta + C$$

$\{now\ use\ a\ triangle\ to\ convert\ back\ to\ terms\ of\ x\}$



From the triangle that was constructed from the original substitution $x = 3 * sin(\theta)$*, the solution* $-cot(\theta) - \theta + C$ *can be converted back to terms of* x :

$$-cot(\theta) - \theta + C = -\frac{cos(\theta)}{sin(\theta)} - \theta + C = -\frac{\sqrt{9-x^2}}{3} \div \frac{x}{3} - sin^{-1}(\tfrac{x}{3}) + C$$

$$= \ -\frac{\sqrt{9-x^2}}{x} - sin^{-1}(\tfrac{x}{3}) + C$$

$\int \frac{dx}{x^2\sqrt{x^2+4}}$ $\{noticing\ the\ (x^2+4)^{\frac{1}{2}}\ term,\ the\ substitution\ should\ be\ x = \alpha * tan(\theta)\}$

$\int \frac{dx}{x^2\sqrt{x^2+4}}$ $\{noticing\ that\ 4\ is\ a\ perfect\ square,\ \alpha\ should\ be\ chosen\ to\ be\ 2\}$

$\int \frac{dx}{x^2\sqrt{x^2+4}}$ $\{x = 2 * tan(\theta),\ dx = 2 * sec^2(\theta)d\theta\}$

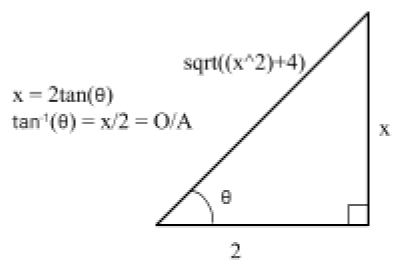
$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2sec^2(\theta)}{(2tan(\theta))^2\sqrt{(2tan(\theta))^2+4}}d\theta = \frac{1}{2}\int \frac{sec^2(\theta)}{tan^2(\theta)\sqrt{4(tan^2(\theta)+1)}}d\theta = \frac{1}{4}\int \frac{sec^2(\theta)}{tan^2(\theta)\sqrt{sec^2(\theta)}}d\theta$$

$$= \frac{1}{4}\int \frac{sec(\theta)}{tan^2(\theta)}d\theta = \frac{1}{4}\int \frac{cos(\theta)}{sin^2(\theta)}d\theta \quad \{u = sin(\theta),\ du = cos(\theta)d\theta\}$$

$$= \frac{1}{4}\int \frac{1}{u^2}du = \frac{1}{4}(-\frac{1}{u}) + C = -\frac{1}{4sin(\theta)} + C$$

$-\frac{1}{4sin(\theta)} + C$

$\{now\ use\ a\ triangle\ to\ convert\ back\ to\ terms\ of\ x\}$



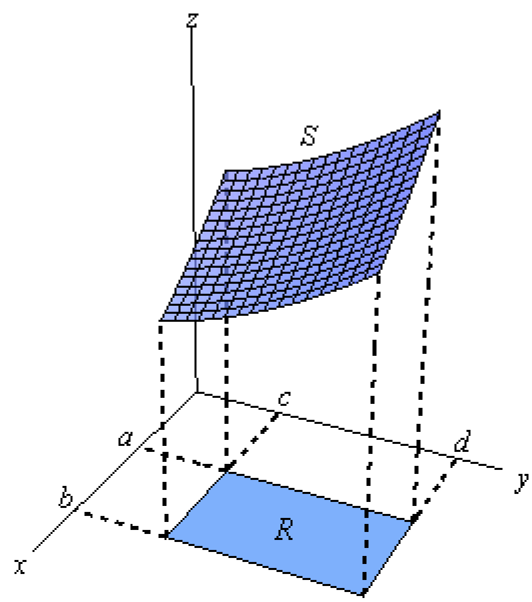
From the triangle that was constructed from the original substitution $x = 3 * sin(\theta)$, *the solution* $-\frac{1}{sin(\theta)} + C$ *can be converted back to terms of* x:

$$-\frac{1}{4sin(\theta)} + C = \frac{-1}{4(x)/\sqrt{x^2+4}} + C = \frac{-\sqrt{x^2+4}}{4x} + C$$

3.7.Double Integrals

3.7.1. If regular integration gives the area under a curve, then in order to calculate the volume under a surface, it's simple to see that an integral in two dimensions is needed. If it is imagined that a single integral evaluates the area on the plane of the $x-y$ *axis* or $f(x)$ along x, then if the axis perpendicular to both the x and y axes is z, the volume under $f(x,y)$ along $x-y$.

3.7.2. Simply this is taking the volume under of a plane in z which is defined by $f(x,y)$ which requires integration along the x and y axes.



3.7.3. The double integral under a surface S and on a region R can be defined in terms of the summations:

$$\iint_R f(x,y)dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\Delta A$$

3.7.4. Although it may seem straightforward, a double integral can be expressed by taking an integral with bounds on x or y then integrating again across the other bound. When doing this, similar to partial derivatives, terms with respect to the other variable are considered constants. Note that the order does not matter but when dealing with definite integrals, it is important to keep track. Note the Linearity principle of integration applies to double integrals. Additionally, and often a useful property, a double integral can be split across separate regions R_1 and R_2. This could be done for a regular integral, but this property can be quite useful when dealing with double integrals.

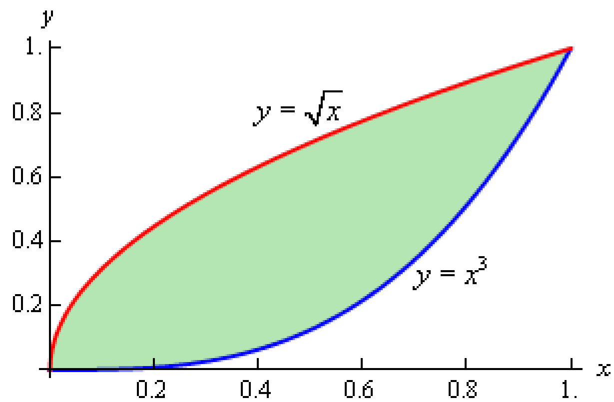
3.7.5. General Case:

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

3.7.6. Some Examples:

$$\iint_R 4xy - y^3\,dA \quad \{\textit{bounded by } y = \sqrt{x} \textit{ and } y = x^3,\ \textit{note intersections at } x = 0, 1\}$$

$\{$*this can be visualized by drawing the curves to help find their bounds*$\}$



$\{$*regardless of the independent variable, there is only one region*$\}$

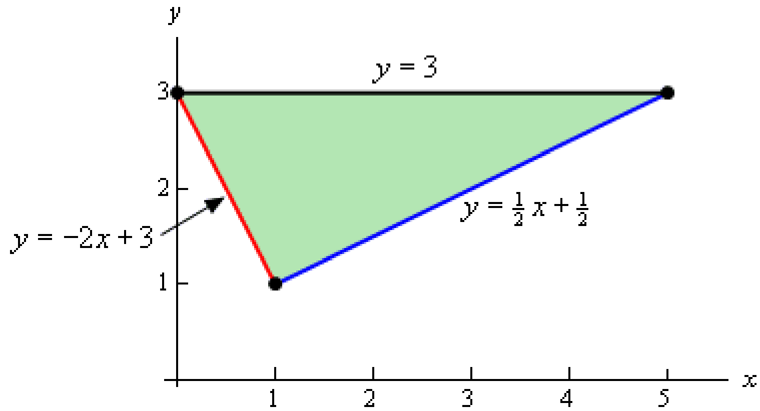
$\{$*sticking with y in terms of x from the sketch...*$\}$

$$\int_0^1\int_{x^3}^{\sqrt{x}} 4xy - y^3\,dy\,dx = \int_0^1 \left[2xy^2 - \tfrac{1}{4}y^4\right]_{x^3}^{\sqrt{x}} dy\,dx = \int_0^1 \tfrac{7}{4}x^2 - 2x^7 + \tfrac{1}{4}x^{12}\,dx$$

$$= \left[\tfrac{7}{12}x^3 - \tfrac{1}{4}x^8 + \tfrac{1}{52}x^{13}\right]_0^1 = \tfrac{55}{156}$$

$$\iint_R 6x^2 - 40y\,dA \quad \{\textit{bounded by vertices at } (0,3),(1,1),(5,3)\}$$

$\{$*this can be visualized by drawing the curves to help find their bounds*$\}$



$\{$*the region can be viewed as functions of x, or functions of y*$\}$

$\{$*if functions of x were used, two regions would have to be evaluated*$\}$

$\{$*since one integral is easier here, putting x in terms of y from the sketch...*$\}$

$\{x = -\frac{1}{2}y + \frac{3}{2},\ x = 2y - 1\}$

$$\iint_R 6x^2 - 40y\,dA = \int_1^3 \int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} (6x^2 - 40y)\,dx\,dy = \int_1^3 \left[2x^3 - 40xy\right]_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} dy$$

$$= \int_1^3 100y - 100y^2 + 2(2y-1)^3 - 2(-\tfrac{1}{2}y + \tfrac{3}{2})^3 dy$$

$$= \left[50y^2 - \tfrac{100}{3}y^3 + \tfrac{1}{4}(2y-1)^4 + (-\tfrac{1}{2}y + \tfrac{3}{2})^4\right]_1^3 = -\tfrac{935}{3}$$

3.8.Triple Integrals

3.8.1. Whereas double integrals were evaluated over two dimensions, triple integrals are evaluated across three dimensions. It is typically effective to think of a triple integral as a volume, just like double integrals, just with more adaptability.

3.8.2. Triple integrals are written basically the same as double integrals:

$$\iiint_D f(x,y,z)\,dV$$

3.8.3. Evaluating triple integrals gets complex, and varies based on what shape is of interest. Generally, a suitable coordinate system should be chosen, variables should be converted to this system, then the triple integral should be reduced to a double integral involving dA instead of dV , and a $Jacobian$ of the transform.

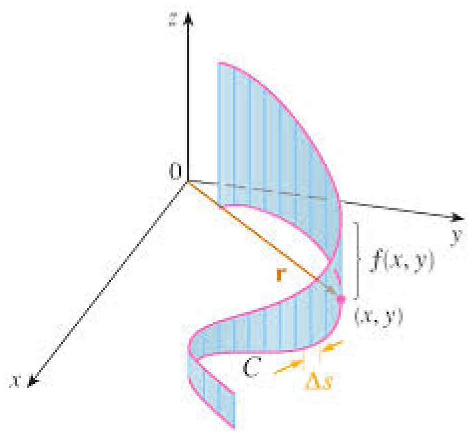
3.9.Surface Integrals

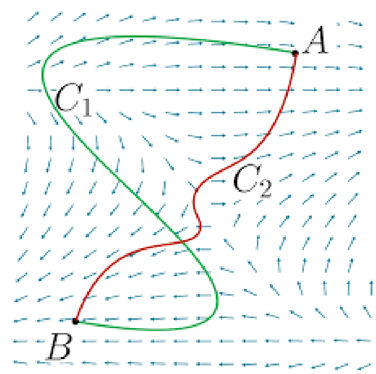
3.9.1. Surface integrals are of importance in electrical engineering - dealing with charge densities and distributions, cost analysis of materials, and other surface area applications.

3.9.2.

3.10. Path Integrals

3.10.1. Also called line integrals, path integrals are useful when calculating work and other quantities as a particle moves through space or a vector field. Important to note is the parameterization of the variables when completing a path integral. Additionally, the path in question must be smooth where it’s derivative is continuous and is never 0.





3.10.2. General form:

$$\int_C f(x,y)ds$$

3.10.3. Here ds denotes that the integral is across a curve C where from the concept of differential arc length, ds has the value:

$$ds = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt$$

3.10.4. So, parameterizing x and y as functions of t, results in the following more useful forms for 2D and 3D respectively:

$$\int_C f(x,y)ds = \int_a^b f(h(t),g(t))\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt$$

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt$$

3.10.5. Some Examples:

$$\int_C xy^4 ds \text{ where } C \text{ is right half of } x^2+y^2=16 \text{ counter}-\text{clockwise}$$

$$\text{parameterizing } x \text{ and } y: \; x=4cos(t), \; y=4sin(t)$$

$$\text{the bounds will be on } -\tfrac{\pi}{2} \le t \le \tfrac{\pi}{2}$$

$$\text{from the formula for } ds, \text{ derivatives of } x \text{ and } y:$$

$$\tfrac{dx}{dt} = -4sin(t), \; \tfrac{dy}{dt} = 4cos(t)$$

$$\text{now the integral is given by}: \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4cos(t)(4sin(t))^4 \sqrt{(-4sin(t))^2+(4cos(t))^2}\, dt$$

$$= 4096 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(t)sin^4(t)dt = 4096 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u^4 du = \left[\tfrac{4096}{5} sin^5(t)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \tfrac{8192}{5}$$

$$\int_C (xyz)ds \text{ where } C \text{ is the helix given by } \hat{r}(t) = \prec cos(t), sin(t), 3t \succ \text{ on } 0 \le t \le 4\pi$$

$$\text{parameterizing } x,y,z: \; x=cos(t), \; y=sin(t), \; z=3t$$

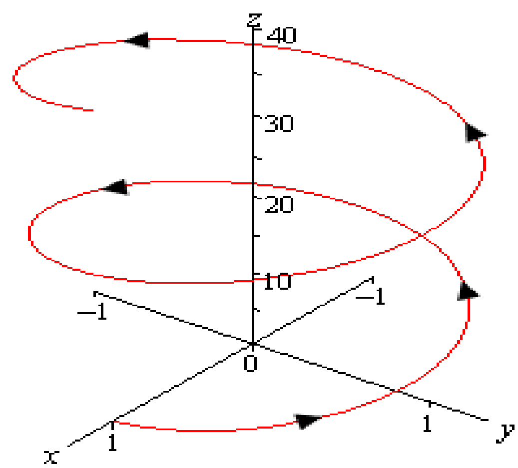
$$\text{from the formula for } ds, \text{ derivatives of } x \text{ and } y:$$

$$\tfrac{dx}{dt} =- sin(t), \; \tfrac{dy}{dt} = sin(t), \; \tfrac{dz}{dt} = 3$$

$$\text{now the integral is given by}: \int_0^{4\pi} cos(t)sin(t)3t\sqrt{(-sin(t))^2+(cos(t))^2+3^2}\, dt$$

$$= \int_0^{4\pi} \left(\tfrac{1}{2}sin(2t)\right) 3t\sqrt{1+9}\, dt = \tfrac{3\sqrt{10}}{2} \int_0^{4\pi} sin(2t)tdt = \tfrac{3\sqrt{10}}{2}\left[\tfrac{1}{4}sin(2t) - \tfrac{t}{2}cos(2t)\right]_0^{4\pi}$$

$$= -3\pi\sqrt{10}$$

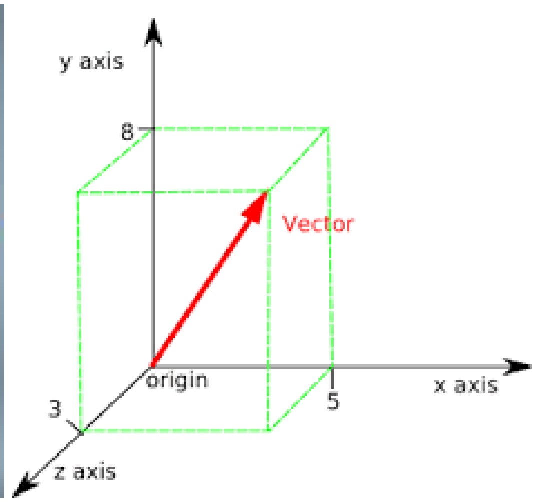


4. Vectors and Matrix Operations

4.1. A Need for Vectors and Matrix Operations

4.1.1. Very often it is useful to represent forces and speeds, etc, in terms of their direction. For example when a potted plant is hung from wires, or a concrete load pulled up by a crane, it is essential to know the forces on each individual wire. Now, this is possible by setting up the three axes and evaluating many trig functions and intermediate angles and lengths and such, but with the use of vectors, directional (force) problems and the like become much easier.

4.1.2. A vector is simply a quantity with a specific magnitude and direction. A vector is represented by an arrow, with the length of the arrow representing it's magnitude, and the placement of the arrow, it's direction.



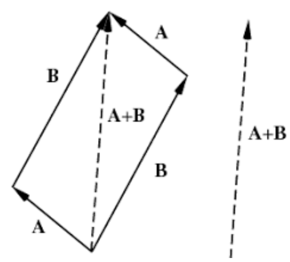
4.2. Vector Representation

4.2.1. A vector's magnitude and direction must be relayed completely in it's representation. This is done by using vector notation. Vector notation standards vary, but the concepts are the important takeaways here. In the 3-D space, which is not very different from the 2-D space, a vector "$\widehat{a}$" with a tail at the origin (0,0,0) and head at (5,8,3) would have the representation: $\widehat{a} = \ < 5, 8, 3 >$. Despite the simplicity of this representation, it's applications are powerful. Now, while addressing the representation, a good question is how the magnitude of the vector is relayed through this cartesian-like form. Recalling the immortal pythagorean theorem, a 2-D vector's magnitude is it's length, or the hypotenuse. In 3-D the same holds true, resulting in the following for the above example: $|\widehat{a}| = \sqrt{5^2 + 8^2 + 3^2}$. Finally, it is important to note that all a vector has is a magnitude and direction, it does not have an exact location.

4.2.2. Now, there is another common way of representing vectors, which is to split up the components into their cartesian forms, and multiply each by the unit vector, a vector of length 1, along the standard axes. This results in the vector from the first example, $\widehat{a} = <5,8,3>$, being represented as $\widehat{a} = 5i + 8j + 3k$. Note that the unit vectors of the $x,\ y,\ and\ z$ axes are $i,\ j,\ and\ k$, respectively.

4.3. Vector Addition

4.3.1. Vector addition can be conceptualized by taking 2, or any number of vectors and placing them head-to-tail to form a sort of chain. From here the resultant vector can be drawn by connecting the first tail to the last head. Mathematically, this is done by adding up all the $x,\ y,\ and\ z$ components of the vectors.



4.3.2. Some Examples:

$<1,2> \ + \ <5,3> \ = \ <6,5>$

$<4,6,-2> \ + \ <-5,4,3> \ = \ <-1,10,1>$

4.4. Vector Scalar Multiplication

4.4.1. The concept here is like magnifying the magnitude of a vector by some scale factor that only increases the magnitude, but has no impact on the direction. Here the scalar multiplier is simply distributed to each term.

4.4.2. Some Examples:

$5<1,2> \ = \ <5,10>$

$-2.5<3,4> \ = \ <-7.5,-10>$

4.5. Vector Dot Product

4.5.1. Given two vectors of length n, the dot product is used to find the angle between them, and is given by the equivalent identities:

$$\widehat{u} \bullet \widehat{v} = |\widehat{u}|\,|\widehat{v}|\,cos(\theta)$$

$$\widehat{u} \bullet \widehat{v} = u_1 v_1 + u_2 v_2 + \ldots\ + u_n v_n$$

4.5.2. From this equation it is trivial to see that the angle between the vectors is given by:

$$\theta = cos^{-1}\left(\frac{\widehat{u} \bullet \widehat{v}}{|\widehat{u}||\widehat{v}|}\right)$$

4.5.3. Note that the angle gives a lot of information about the two vectors, such as if they are orthogonal or parallel.

4.5.4. Some Examples:

$\widehat{a} = \ <5,8,3>,\ \widehat{b} = \ <-2,1,-3>$

$$\widehat{u} \bullet \widehat{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$
$$\widehat{a} \bullet \widehat{b} = (5 * -2) + (8 * 1) + (3 * -3) = -10 + 8 + -9 = -11$$
$$\theta = cos^{-1}\left(\frac{\widehat{a} \bullet \widehat{b}}{|\widehat{a}|\left|\widehat{b}\right|}\right) = cos^{-1}\left(\frac{-11}{\sqrt{5^2+8^2+3^2} * \sqrt{-2^2+1^2+-3^2}}\right) = cos^{-1}(-.296972) = 107.27^{\circ}$$

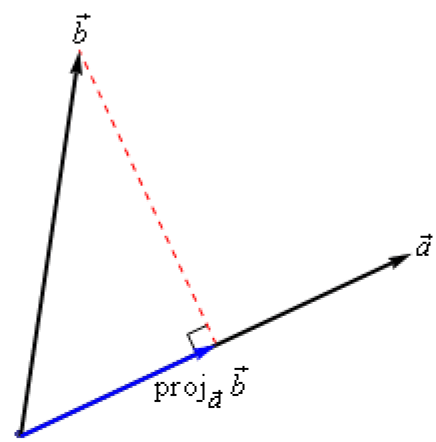
$\widehat{a} = \; <3,\ 3,\ 0>,\ \widehat{b} = \; <0,\ 0,\ 5>$

$$\widehat{u} \bullet \widehat{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$
$$\widehat{a} \bullet \widehat{b} = (3 * 0) + (3 * 0) + (0 * 5) = 0$$
$$\theta = cos^{-1}\left(\frac{\widehat{a} \bullet \widehat{b}}{|\widehat{a}|\left|\widehat{b}\right|}\right) = cos^{-1}\left(\frac{0}{\sqrt{3^2+3^2+0^2} * \sqrt{0^2+0^2+5^2}}\right) = cos^{-1}(0) = 90^{\circ}$$

4.6. Scalar Projection

4.6.1. The projection of one vector onto another gives the component of that vector that is parallel/collinear to the other. Convention is to read the following as "the projection of $\widehat{b}$ *onto* $\widehat{a}$":

$$proj_{\widehat{b}}\,\widehat{a} = \frac{\widehat{a} \bullet \widehat{b}}{|\widehat{a}|^2}\,\widehat{a}$$



4.6.2. Note if the angle between $\widehat{a}$ and $\widehat{b}$ is obtuse then the projection will be in the opposite direction.

4.6.3. Some Examples:

$\widehat{a} = \; <5, 8, 3>,\ \widehat{b} = \; <-2, 1, -3>$

$$proj_{\widehat{b}}\,\widehat{a} = \frac{\widehat{a} \bullet \widehat{b}}{|\widehat{a}|^2}\,\widehat{a} = \frac{(5*-2)+(8*1)+(3*-3)}{(5^2+8^2+3^2)} <5,\ 8,\ 3> \; = \frac{-11}{98} <5,\ 8,\ 3> \; = \; <\frac{-55}{98}, \frac{-88}{98}, \frac{-33}{98}>$$

$\widehat{a} = \; <3,\ 3,\ 0>,\ \widehat{b} = \; <0,\ 0,\ 5>$

$$proj_{\widehat{b}}\,\widehat{a} = \frac{\widehat{a} \bullet \widehat{b}}{|\widehat{a}|^2}\,\widehat{a} = \frac{(3*0)+(3*0)+(0*5)}{(3^2+3^2+0^2)} <3,\ 3,\ 0> \; = \frac{0}{18} <3,\ 3,\ 0> \; = \; <0,\ 0,\ 0>$$

4.7. The Cross Product

4.7.1. asf

4.8. Unit Vectors, and Matrix Representation

4.9. The Determinant