The system:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \frac{-fcosx_1 - \mathrm{mL}x_2{}^2\mathrm{sin}x_1\mathrm{cos}x_1 + (M+m)\mathrm{gsin}x_1}{L\left(M + m\mathrm{sin}^2 x_1\right)},$$

$$y = x_1$$

where $m = 0.1, g = 9.81, M = 1.0, L = 1$ and f is control law.

1. Based on input-output linearization, design a tracking controller

Since $y = x_1$, we compute its derivatives until the input f appears.

$$\dot{y} = \dot{x_1} = x_2$$

$$\ddot{y} = \dot{x_2} = \frac{-f\ \mathrm{cos}x_1 - m\,L\,x_2{}^2\mathrm{sin}x_1\mathrm{cos}x_1 + (M+m)g\ \mathrm{sin}x_1}{L\left(M + m\ \mathrm{sin}^2 x_1\right)}$$

Now we define the feedback law: $\ddot{y} = v = \text{desired linear output dynamics}$

Set: $f = -\dfrac{L\left(M + m\ \mathrm{sin}^2 x_1\right)v + m\,L\ \ x_2{}^2\ \mathrm{sin}x_1\ \mathrm{cos}x_1 - (M+m)g\ \mathrm{sin}x_1}{\mathrm{cos}x_1}$

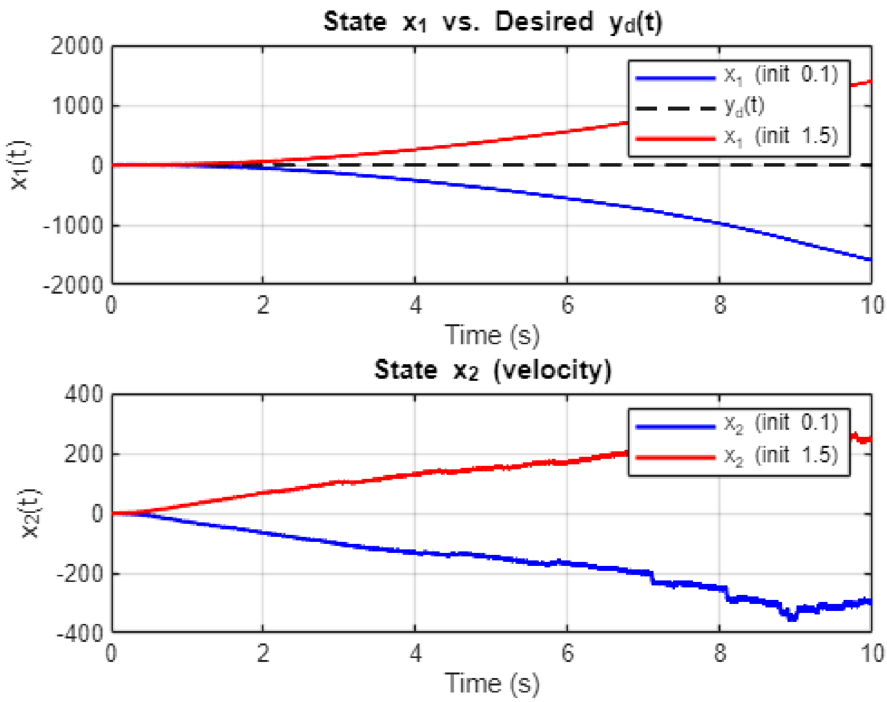
Then, define tracking law v: To make $y(t) \to y_d(t)$, use a second-order tracking law:

$v = \ddot{y_d} + k_1\dot{e} + k_0 e$, where $\begin{array}{l} e = y - y_d \\ \dot{e} = x_2 - \dot{y_d} \end{array}$. Choose gains for good convergence (e.g., $k_1 = 4, k_0 = 4$)

```
% Parameters
m = 0.1;
g = 9.81;
M = 1.0;
L = 1.0;

% Tracking controller gains (critically damped)
k0 = 4;
k1 = 4;

% Desired trajectory and derivatives
yd = @(t) cos(2*t);
dyd = @(t) -2*sin(2*t);
ddyd = @(t) -4*cos(2*t);

% System dynamics with feedback linearization
nonlinear_system = @(t, x) [
    x(2);
    compute_x2_dot(t, x, m, g, M, L, k0, k1, yd, dyd, ddyd)
];
```

```
% Compute x2_dot with control input
function dx2 = compute_x2_dot(t, x, m, g, M, L, k0, k1, yd, dyd, ddyd)
    x1 = x(1); x2 = x(2);
    e = x1 - yd(t);
    de = x2 - dyd(t);
    v = ddyd(t) + k1*de + k0*e;

    cosx1 = max(min(cos(x1), 0.5), -0.5);  % Clamp to avoid exploding division


    % Feedback linearized control law
    f = (-L*(M + m*sin(x1)^2)*v - m*L*x2^2*sin(x1)*cosx1 + (M + m)*g*sin(x1)) /
cosx1;
    f = max(min(f, 100), -100);  % Saturate to ±100 for stability

    % Second state derivative
    dx2 = (-f*cosx1 - m*L*x2^2*sin(x1)*cosx1 + (M + m)*g*sin(x1)) / (L*(M +
m*sin(x1)^2));
end

% Simulation time
tspan = [0 10];

% Initial states
x0_1 = [0.1; 0];    % Initial state 1
x0_2 = [1.5; 0];    % Initial state 2

% Simulate both
[t1, x1] = ode45(nonlinear_system, tspan, x0_1);
[t2, x2] = ode45(nonlinear_system, tspan, x0_2);

figure;
subplot(2,1,1);
plot(t1, x1(:,1), 'b', 'LineWidth', 1.5); hold on;
plot(t1, yd(t1), 'k--', 'LineWidth', 1.2);
plot(t2, x2(:,1), 'r', 'LineWidth', 1.5);
xlabel('Time (s)');
ylabel('x_1(t)');
title('State x_1 vs. Desired y_d(t)');
legend('x_1 (init 0.1)', 'y_d(t)', 'x_1 (init 1.5)');
grid on;

subplot(2,1,2);
plot(t1, x1(:,2), 'b', 'LineWidth', 1.5); hold on;
plot(t2, x2(:,2), 'r', 'LineWidth', 1.5);
xlabel('Time (s)');
ylabel('x_2(t)');
title('State x_2 (velocity)');
legend('x_2 (init 0.1)', 'x_2 (init 1.5)');
```

```
grid on;
```



- State $x_1(t)$ is diverging instead of tracking the reference $y_d(t) = \cos(2t)$.
- State $x_2(t)$ is growing liearly and not stabilizing.

2. Assume f=0 and sampling step=0.01 seconds. Based on Koopman theory, complete the following tasks.

System with $f = 0$, given dynamics: $\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \dfrac{-m\,L\,x_2^2 \sin x_1 \cos x_1 + (M+m)g \sin x_1}{L\left(M + m\sin^2 x_1\right)} \end{cases}$

with: $\begin{array}{l} m = 0.1, M = 1.0, L = 1.0, g = 9.81 \\ \text{Time step}: 0.01s \\ f = 0, \text{so no control input} \end{array}$

2.1 Find the observable function

To find the observable function, we need to lift the system into a higher-dimensional space:

$g(x) = \left[x_1, x_2, \sin(x_1), \cos(x_1), x_2^2\right]$

```
%% 2.1 - Define Observable Function
g = @(x) [x(1); x(2); sin(x(1)); cos(x(1)); x(2)^2];
```

2.2 Find Koopman Operator

Using state transition data:

1. Simulate trajectory over time (5 seconds, 0.01s step→500steps)
2. For each time t_k, compute: $z_k = g(x_k)$, $z_{k+1} = g(x_{k+1})$
3. Construct matrices: $Z = [z_0, z_1, \ldots, z_{N-1}]$, $\dot{Z} = [z_1, z_1, \ldots, z_N]$
4. Solve: $K = \dot{Z} \cdot Z_{\text{pseudoinverse}}$

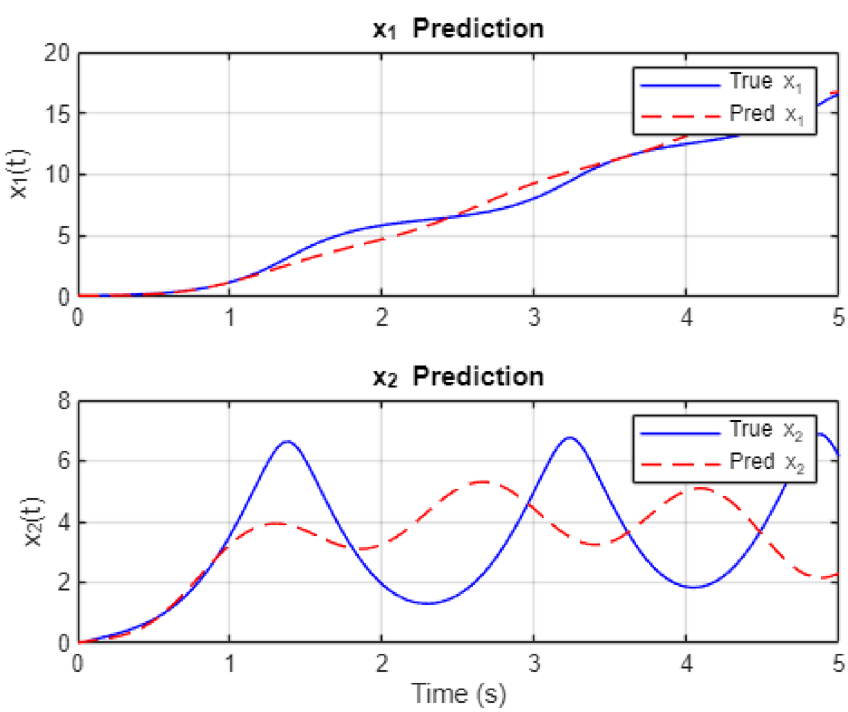
```
%% 2.2 - Simulate System and Compute Koopman Operator
% System Parameters
m = 0.1; g_const = 9.81; M = 1.0; L = 1.0;
dt = 0.01; T = 5; N = T / dt;

% Define dynamics with f = 0
f = @(t, x) [
    x(2);
    (- m * L * x(2)^2 * sin(x(1)) * cos(x(1)) + (M + m) * g_const * sin(x(1))) / (L
* (M + m * sin(x(1))^2))
];

% Simulate trajectory for initial state x0 = [0.1; 0]
x0 = [0.1; 0];
x = zeros(2, N+1); x(:,1) = x0;

for k = 1:N
    x(:,k+1) = x(:,k) + dt * f((k-1)*dt, x(:,k));
end

% Build snapshot matrices Z and Z_next
z_dim = 5;
Z = zeros(z_dim, N);
Z_next = zeros(z_dim, N);
for k = 1:N
    Z(:,k) = g(x(:,k));
    Z_next(:,k) = g(x(:,k+1));
end

% Compute Koopman operator
K = Z_next * pinv(Z);
```

2.3 Koopman Prediction and Plotting

Using: $z_{k+1}^{\text{pred}} = K z_k$. Then recover approximate x_k by inverse mapping (e.g., from first two entries of $g(x)$).

Plot: Actual $x_1(t), x_2(t)$, Predicted $x_1^{\text{koop}}, x_2^{\text{koop}}$

```
%% 2.3 - Koopman Prediction and Plotting
% Predict from initial observable
z_pred = zeros(z_dim, N+1);
x_pred = zeros(2, N+1);
z_pred(:,1) = g(x0);
x_pred(:,1) = x0;

for k = 1:N
    z_pred(:,k+1) = K * z_pred(:,k);
    x_pred(:,k+1) = z_pred(1:2,k+1); % Approximate inverse
end

t = 0:dt:T;
figure;
subplot(2,1,1);
plot(t, x(1,:), 'b', 'DisplayName','True x_1'); hold on;
plot(t, x_pred(1,:), 'r--', 'DisplayName','Pred x_1');
ylabel('x_1(t)'); legend; title('x_1 Prediction'); grid on;

subplot(2,1,2);
plot(t, x(2,:), 'b', 'DisplayName','True x_2'); hold on;
plot(t, x_pred(2,:), 'r--', 'DisplayName','Pred x_2');
ylabel('x_2(t)'); xlabel('Time (s)'); legend; title('x_2 Prediction'); grid on;
```



2.4 Compute RMSE

$\text{RMSE} = \sqrt{\frac{1}{T}\sum_{k=1}^{T} \|x_k - \hat{x}_k\|^2}$. Implement this for both initial conditions.

```
%% 2.4 - RMSE Calculation
rmse_x1 = sqrt(mean((x(1,:) - x_pred(1,:)).^2));
rmse_x2 = sqrt(mean((x(2,:) - x_pred(2,:)).^2));

fprintf('RMSE for x1: %.4f\n', rmse_x1);
```

```
RMSE for x1: 0.8531
```

```
fprintf('RMSE for x2: %.4f\n', rmse_x2);
```

```
RMSE for x2: 2.1717
```

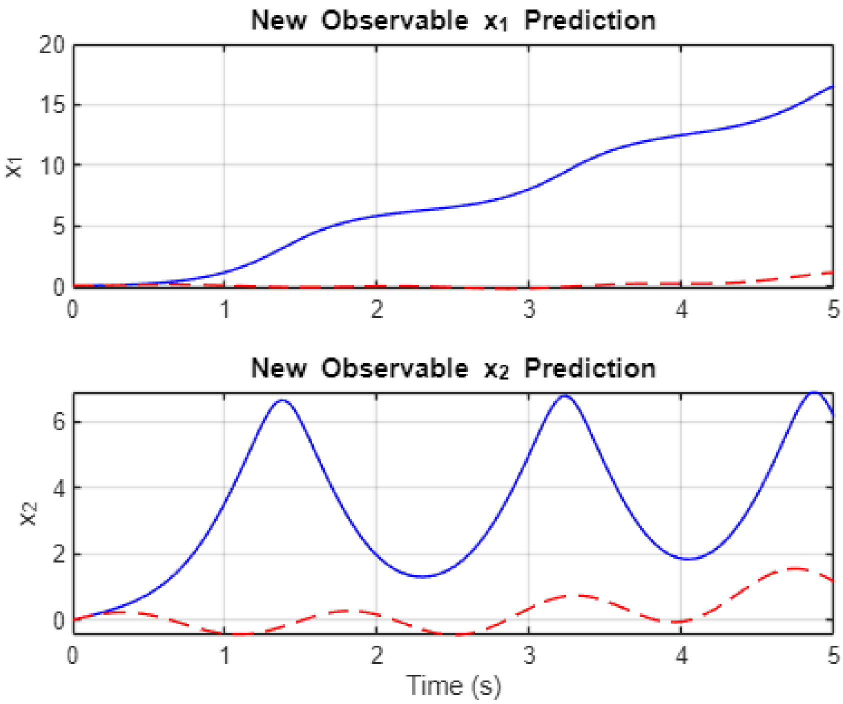
2.5 Try a New Observable Function and Re-Do

Try a different observable set with cross-terms and more nonlinear features:

$$g_{\text{new}}(x) = \left[x_1, x_2, {x_1}^2,\ {x_2}^2, x_1x_2, \sin(x_1)\right]$$

```
%% 2.5 - Try New Observable Function and Recompute
% New observable function
g_new = @(x) [x(1); x(2); x(1)^2; x(2)^2; x(1)*x(2); sin(x(1))];

% Regenerate Z and Z_next with new g_new
z_dim_new = 6;
Z_new = zeros(z_dim_new, N);
Z_next_new = zeros(z_dim_new, N);
for k = 1:N
    Z_new(:,k) = g_new(x(:,k));
    Z_next_new(:,k) = g_new(x(:,k+1));
end

K_new = Z_next_new * pinv(Z_new);

% Prediction
z_pred_new = zeros(z_dim_new, N+1);
x_pred_new = zeros(2, N+1);
z_pred_new(:,1) = g_new(x0);
x_pred_new(:,1) = x0;

for k = 1:N
    z_pred_new(:,k+1) = K_new * z_pred_new(:,k);
    x_pred_new(:,k+1) = z_pred_new(1:2,k+1);
end

% Plot
figure;
subplot(2,1,1);
plot(t, x(1,:), 'b', t, x_pred_new(1,:), 'r--');
title('New Observable x_1 Prediction'); ylabel('x_1'); grid on;

subplot(2,1,2);
```

```
plot(t, x(2,:), 'b', t, x_pred_new(2,:), 'r--');
title('New Observable x_2 Prediction'); ylabel('x_2'); xlabel('Time (s)'); grid on;
```




```
% RMSE
rmse_x1_new = sqrt(mean((x(1,:) - x_pred_new(1,:)).^2));
rmse_x2_new = sqrt(mean((x(2,:) - x_pred_new(2,:)).^2));

fprintf('New RMSE for x1: %.4f\n', rmse_x1_new);
```

```
New RMSE for x1: 8.4705
```

```
fprintf('New RMSE for x2: %.4f\n', rmse_x2_new);
```

```
New RMSE for x2: 3.5889
```

Using artificial intelligence models: 50%