1. $\begin{cases} \dot{x}_1 = -x_1 + 3x_2^3 \\ \dot{x}_2 = -x_2 + x_1 x_2 \end{cases}$. To estimate the region of attraction.

Use Lyapunov's direct method.

① Identify the equilibrium point, set $\dot{x}_1 = 0$, $\dot{x}_2 = 0$:

$\begin{cases} 0 = -x_1 + 3x_2^3 \\ 0 = -x_2 + x_1 x_2 \end{cases}$ $\Rightarrow$ If $x_2 = 0$, $x_1 = 0$

$\therefore$ $(0, 0)$ is an equilibrium point.

② Choose a Lyapunov Function Candidate.

$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, positive define.

③ Compute the derivative of $V(x)$.

$$\begin{aligned} \dot{V} &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_1(-x_1 + 3x_2^3) + x_2(-x_2 + x_1 x_2) \\ &= -x_1^2 + 3x_1 x_2^3 - x_2^2 + x_1 x_2^2 \\ &= -x_1^2 - x_2^2 + x_1 x_2^2 + 3x_1 x_2^3 \end{aligned}$$

④ Bound $\dot{V} = -x_1^2 - x_2^2 + x_1 x_2^2 (1 + 3x_2)$,

we want $\dot{V} < 0$, the Lyapunov function decreasing.

To estimate the region where this is true, bounding the nonlinear term:

$$|x_1 x_2^2 (1 + 3x_2)| \le |x_1| |x_2|^2 |1 + 3x_2|$$

$\therefore$ $\dot{V} < 0$ when $x_1^2 + x_2^2 > x_1 x_2^2 (1 + 3x_2)$

$\rightarrow$ hard to evaluate analytically.

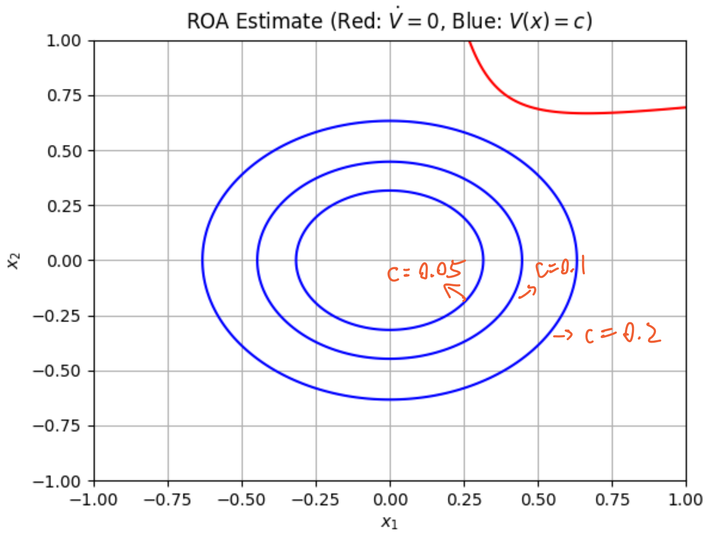
⇒ We can estimate an invariant set such as a level set:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) = c,$$

and try to find the largest $c > 0$ such that $\dot{V} < 0$ inside the level set.

⑤ Numerical estimation of ROA:

Try values of $c = 0.05, 0.1, 0.2, \cdots$, plot $\dot{V}$ in the region:



Red contour: where $\dot{V}(x_1, x_2) = 0$

Blue circles: level sets of $V(x) = \frac{1}{2}(x_1^2 + x_2^2) = c$

The innermost blue level set is fully inside the red curve,

⇒ Inside the region, $\dot{V} < 0$.

⇒ The Lyapunov function is decreasing.

⇒ Trajectories starting here will converge to the origin.

The innermost blue contour corresponds to:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) = 0.05,$$

then the ROA is approximately:

$$D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 0.1\}$$

This is a circular region of radius $\sqrt{0.1} \approx 0.316$ around the origin.

2. $$\begin{cases} \dot{x}_1 = (x_1x_2-1)x_1^3 + (x_1x_2-1+x_2^2)x_1 \\ \dot{x}_2 = x_2 \end{cases}$$

① set $\dot{x}_1 = 0$, $\dot{x}_2 = 0$ $\Rightarrow \dot{x}_2 = -x_2 = 0 \Rightarrow x_2 = 0$

$$x_2 = 0 \Rightarrow \dot{x}_1 = (-1)x_1^3 + (-1)x_1 = -x_1(x_1^2+1) = 0 \Rightarrow x_1 = 0$$

$\therefore$ the only equilibrium point is $(x_1, x_2) = (0.0)$

② Set Lyapunov function: $V(x) = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$

$\Rightarrow$ This function penalizes larger values of x_1 more strongly (due to the x_1^4 term), helping handle the nonlinear term in $\dot{x}_1$.

③ Compute $\dot{V}$:

$$V(x) = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \Rightarrow \dot{V}(x) = x_2\dot{x}_2 + x_1\dot{x}_1 + x_1^3\dot{x}_1$$

$$\Rightarrow \dot{x}_2 = -x_2, \quad \dot{x}_1 = (x_1x_2-1)x_1^3 + (x_1x_2-1+x_2^2)x_1$$

$$\Rightarrow \dot{V} = x_2(-x_2) + x_1\left[(x_1x_2-1)x_1^3 + (x_1x_2-1+x_2^2)x_1\right] + x_1^3\left[(x_1x_2-1)x_1^3 + (x_1x_2-1+x_2^2)x_1\right]$$

$$= -x_2^2 + x_1^4(x_1x_2-1) + x_1^2(x_1x_2-1+x_2^2) + x_1^6(x_1x_2-1) + x_1^4(x_1x_2-1+x_2^2)$$

We can now numerically evaluate and visualize $\dot{V}$ to identify a region where it is negative definite.

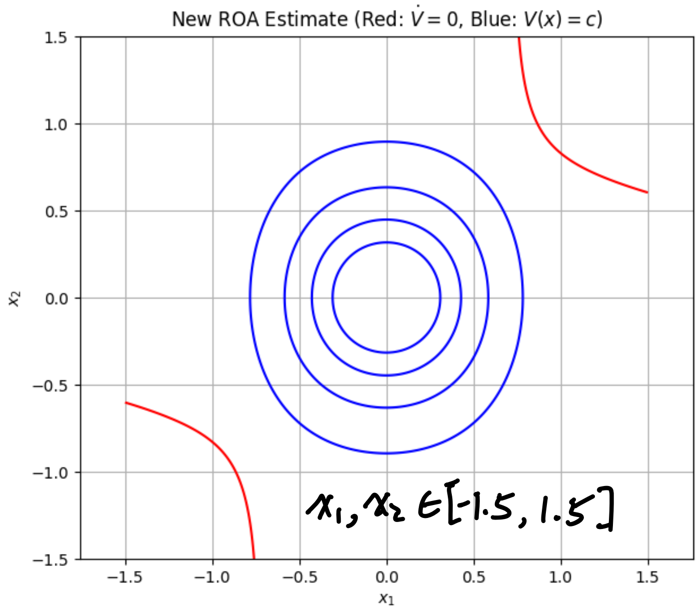
④ Bound the derivative of new Lyapunov function

$$V(x) = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \quad \text{(positive definite)}$$

$$\dot{V} = x_1\dot{x}_1 + x_1^3\dot{x}_1 + x_2\dot{x}_2 = (x_1 + x_1^3)\dot{x}_1 + x_2\dot{x}_2$$

$$= (x_1 + x_1^3)\left[(x_1x_2 - 1)x_1^3 + (x_1x_2 - 1 + x_2^2)x_1\right] - x_2^2$$

The expression is difficult to analyze symbolically, but we can numerically bound it by plotting it over a region.



Red curve: $\dot{V} = 0$, the boundary where the Lyapunov function derivative transitions from negative to possibly non-negative.

Blue contours: Level sets of the Lyapunov function:

$$V(x) = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$$

The largest blue contour that lies fully inside the red region gives an inner estimate of the ROA

This suggests that the ROA can be approximated by:

$$D = \{(x_1, x_2) \in \mathbb{R}^2 \mid V(x) < c\} \text{ for } c \approx 0.2$$