NONLINEAR SYSTEMS

THIRD EDITION

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APPLIED NONLINEAR CONTROL

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➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

⇨ The central idea of the approach is to algebraically transform a nonlinear system dynamics in to a fully or partly one, so that the linear control theory can be applied.

⇨ This differs entirely from conventional linearization (such as Jacobian linearization) in that the feedback, rather than by linear approximations of the dynamics.

⇨ Feedback linearization technique can be view as ways of **transforming** original system models into equivalent models of a simpler form.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Feedback linearization and the canonical form

➢Ex: A simple example $A(h)\,\dot{h} = u - a\sqrt{2gh}$

Sol: If $u(t)$ is chosen as $u(t) = a\sqrt{2gh} + A(h)v$

with v being an "*equivalent input*" to be specified, the resulting dynamics is linear $\dot{h} = v$

Choosing v as $v = -\alpha\,\tilde{h}$

with $\tilde{h} = h(t) - h_d$ is the level error, α is a strictly positive constant. Now, the close loop dynamics is $\dot{h} + \alpha\,\tilde{h} = 0$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Feedback linearization and the canonical form

➢Ex: A simple example $A(h)\,\dot{h} = u - a\sqrt{2gh}$

Sol: This implies that $\tilde{h}(t) \to 0$ as $t \to \infty$. The nonlinear control law

$$u(t) = a\sqrt{2gh} - A(h)\,\alpha(h)$$

If the desired level is a known time-varying function $h_d(t)$, the equivalent input v can be chosen as $v = \dot{h}_d(t) - \alpha\,\tilde{h}$ so as to still yield $\tilde{h}(t) \to 0$ when $t \to \infty$.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Feedback linearization and the canonical form

➢The idea of feedback linearization is to cancel the nonlinearities and imposing the desired linear dynamics.

➢Feedback linearization can be applied to a class of nonlinear system described by the so-called companion form, or controllability canonical form.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Feedback linearization and the canonical form

Consider the system in companion form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(x)+b(x)u \end{bmatrix}$$

where

x : the state vector

$f(x), b(x)$: nonlinear function of the state

u : scalar control input

For this system, using the control input of the form $u=(v-f)/b$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Feedback linearization and the canonical form

we can cancel the nonlinearities and obtain the simple input-output relation (multiple-integrator form) $x^{(n)}=v$. Thus, the control law $v=-k_0x-k_1\dot{x}-\ldots-k_{n-1}x^{(n-1)}$ with the k_i chosen so that the polynomial $p^n+k_{n-1}p^{n-1}+\ldots+k_0$ has its roots strictly in the left-half complex plane, lead to exponentially stable dynamics $x^{(n)}+k_{n-1}x^{(n-1)}+\ldots+k_0x=0$ which implies that $x(t)\to 0$. For tasks involving the tracking of the desired output $x_d(t)$, the control law

$$v=x_d^{(n)}-k_0e-k_1\dot{e}-\ldots-k_{n-1}e^{(n-1)} \qquad e(t)=x(t)-x_d(t)$$

leads to exponentially convergent tracking.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Feedback linearization and the canonical form

Consider the two-link robot as

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1-h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}+\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}=\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

where,

$q=[q_1 \;\; q_2]^T$: joint angles

$\tau=[\tau_1 \;\; \tau_2]^T$: joint inputs (torques)

$H_{11}=m_1l_{c1}^2+I_1+m_2(l_1^2+l_{c2}^2+2l_1l_{c2}\cos q_2)+I_2$

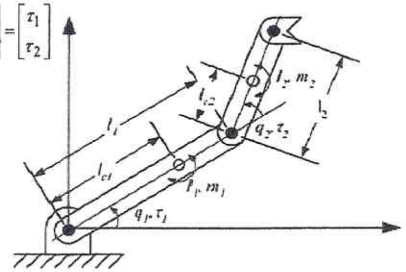
$H_{12}=H_{21}=m_2l_1c_2\cos q_2+m_2l_{c2}^2+I_2$

$H_{22}=m_2l_{c2}^2+I_2$

$h=m_2l_1l_{c2}\sin q_2$

$g_1=m_1l_{c1}g\cos q_1+m_2g[l_{c2}\cos(q_1+q_2)+l_1\cos q_1]$

$g_2=m_2l_{c2}g\cos(q_1+q_2)$



Feedback linearization

- Feedback linearization is an approach to nonlinear control design
  - Feedback linearization and the canonical form

Control objective: to make the joint position q_1 and q_2 follows desired trajectories $q_{d1}(t)$ and $q_{d2}(t)$

To achieve tracking control tasks, one can use the follow control law

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}=\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}+\begin{bmatrix}-h\dot{q}_2 & -h\dot{q}_1-h\dot{q}_2\\ h\dot{q}_1 & 0\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}+\begin{bmatrix}g_1\\ g_2\end{bmatrix}$$

where,

$\dot{v}=\ddot{q}_d-2\lambda\dot{\tilde{q}}-\lambda^2\tilde{q}$

$v=[v_1\ \ v_2]^T$: the equivalent input

$\tilde{q}=q-q_d$: position tracking error

λ : a positive number

The tracking error satisfies the equation $\ddot{\tilde{q}}+2\lambda\dot{\tilde{q}}+\lambda^2\tilde{q}=0$ and therefore converges to zeros exponentially.

Feedback linearization

- Feedback linearization is an approach to nonlinear control design
  - Feedback linearization and the canonical form
    - When the nonlinear dynamics is not in a controllability canonical form, one may have to use algebraic transforms to first put the dynamics into the controllability canonical form before using the above feedback linearization design.
    - We have assumed that the dynamics is linear in terms of the control input u (although nonlinear in the states). However, the approach can be easily extended to the case when u is replaced by an invertible function g(u). For example, in systems involving flow control by a valve, the dynamics may be dependent on u^4 rather than directly on u, with u being the valve opening diameter. Then, by defining $w = u^4$, one can first design w similarly to the previous procedure and then compute the input by $u = w^4$. This means that the nonlinearity is simply undone in the control computation.

Feedback linearization

- Feedback linearization is an approach to nonlinear control design
  - Input-State Linearization
    - Consider the problem of designing the control input u for a single-input nonlinear system of the form

$$\dot{x}=f(x,u)$$

    - The technique of input-state linearization solves this problem into two steps:
    - 1. Find a state transformation $z = z(x)$ and an input transformation $u = u(x,v)$, so that the nonlinear system dynamics is transformed into an equivalent linear time invariant dynamics, in the familiar form $\dot{z}=Az+bv$
    - 2. One uses standard linear techniques to design *v*.

Feedback linearization

- Feedback linearization is an approach to nonlinear control design
  - Input-State Linearization
    - Ex: Consider a simple second order system

$$\dot{x}_1=-2x_1+ax_2+\sin x_1$$
$$\dot{x}_2=-x_2\cos x_1+u\cos(2x_1)$$

    - Even though linear control design can stabilize the system in a small region around the equilibrium point (0,0), it is not obvious at all what controller can stabilize it in a large region. A specific difficulty is the nonlinearity in the first equation, which cannot be directly cancelled by the control input u.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Ex:

Consider the following state transformation

$$z_1 = x_1$$
$$z_2 = a x_2 + \sin x_1$$

which transforms the system into

$$\dot{z}_1 = -2 z_1 + z_2$$
$$\dot{z}_2 = -2 z_1 \cos z_1 + \cos z_1 \sin z_1 + a u \cos(2 z_1)$$

The new state equations also have an equilibrium point at (0,0).

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Ex:

Now the nolinearities can be canceled by the control law of the form

$$u = \frac{1}{a \cos(2 z_1)} (v - \cos z_1 \sin z_1 + 2 z_1 \cos z_1)$$

where v is equivalent input to be designed (equivalent in the sense that determining v amounts to determining u, and vice versa), leading to a linear input-state relation

$$\dot{z}_1 = -2 z_1 + z_2$$
$$\dot{z}_2 = v$$

The new state equations also have an equilibrium point at (0,0). Now the nolinearities can be canceled by the control law of the form

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Ex:

Thus, through the state transformation and input transformation, the problem of stabilizing the original nonlinear dynamics using the original control input u has been transformed into the problem of stabilizing the new dynamics using the new input *v*.

Since the new dynamics is linear and controllable, it is well known that the linear state feedback control law.

$$v = -k_1 z_1 - k_2 z_2$$

Let

$$v = -2 z_2$$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Ex:

resulting in the stable closed-loop

$$\dot{z}_1 = -2 z_1 + z_2$$
$$\dot{z}_2 = -2 z_2$$

whose poles are both placed at - 2. In terms of the original state x_1 and x_2, this control law corresponds to the original input

$$u = \frac{1}{a \cos(2 x_1)} (-2 a x_2 - 2 \sin x_1 - \cos x_1 \sin x_1 + 2 x_1 \cos x_1)$$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Ex:

The original state x is given from z by

$$x_1 = z_1$$
$$x_2 = (z_2 - \sin z_1)/a$$

Since both z_1 and z_2 converge to zero, the original state x converges to zero.

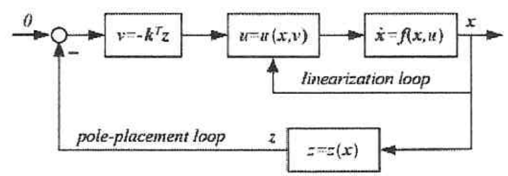
➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Ex:

The closed-loop system under the above control law is represented in the block diagram



➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Remarks:

➢1. The result, though valid in a large region of the state space, is not global. The control law is not well defined when $x_1 = (\pi/4 \pm k\pi/2),\ k = 1, 2, \ldots$ Obviously, when the initial state is at such singularity points, the controller cannot bring the system to the equilibrium point.

➢2. The input-state linearization is achieved by a combination of a state transformation and an input transformation, with state feedback used in both. Thus, it is a linearization by feedback, or feedback linearization. This is fundamentally different from a Jacobian linearization for small range operation, on which linear control is based.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Remarks:

➢3. In order to implement the control law, the new state components (z_1, z_2)must be available. If they are not physically meaningful or cannot be measured directly, the original state x must be measured and used to compute them .

➢4. In general, we rely on the system model both for the controller design and for the computation of z. If there is uncertainty in the model, e.g., uncertainty on the parameter a, this uncertainty will cause error in the computation of both the new state z and of the control input u.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢Remarks:

➢5. Tracking control can also be considered. However, the desired motion then needs to be expressed in terms of the full new state vector. Complex computations may be needed to translate the desired motion specification (in terms of physical output variables) into specifications in terms of the new states.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-State Linearization

➢To generalize input-state linearization method , there are two questions:

➢1. What classes of nonlinear systems can be transformed into linear systems?

➢2. How to find the proper transformations for those which can?

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Background: Lie derivative

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

$$y^{(1)} = L_f h(x) + \overset{=0,\ \rho>1}{\boxed{\cancel{L_g h(x)}}}u$$

$$y^{(2)} = L_f^2 h(x) + \overset{=0,\ \rho>2}{\boxed{\cancel{L_g L_f h(x)}}}u$$

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u$$

the relative degree ρ

The **Lie derivative** of a scalar function h(x) along a vector field f(x) is defined as:

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

$$L_f^2 h(x) = L_f\left(L_f h(x)\right) = \frac{\partial (L_f h)}{\partial x} f(x)$$

$$L_f^k h(x) = L_f\left(L_f^{k-1} h(x)\right)$$

$$L_f^0 h(x) = h(x)$$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Background: Lie derivative

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

$$y^{(1)} = L_f h(x) + \overset{=0,\ \rho>1}{\boxed{\cancel{L_g h(x)}}}u$$

$$y^{(2)} = L_f^2 h(x) + \overset{=0,\ \rho>2}{\boxed{\cancel{L_g L_f h(x)}}}u$$

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u$$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Background: Lie derivative

$$\begin{cases} \dot{x}_1 = -x_1 + \frac{2+x_3^2}{1+x_3^2}u \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_1 x_3 + u \\ y = x_2 \end{cases} \qquad f(x) = \begin{bmatrix} -x_1 \\ x_3 \\ x_1 x_3 \end{bmatrix}, \quad g(x) = \begin{bmatrix} \frac{2+x_3^2}{1+x_3^2} \\ 0 \\ 1 \end{bmatrix}$$

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

$$y^{(1)} = L_f h(x) + \boxed{\cancel{L_g h(x)}}\,u \qquad = 0,\ \rho>1$$

$$y^{(2)} = L_f^2 h(x) + \boxed{\cancel{L_g L_f h(x)}}\,u \qquad = 0,\ \rho>2$$

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u$$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Let us now consider a tracking control problem. Consider the

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

Control objective: to make the output y(t) track a desired trajectory $y_d(t)$ while keeping the whole state bounded. y_d (t) and its time derivatives are assumed to be known and bounded.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Ex:

Consider the third-order system

$$\dot{x}_1 = \sin x_2 + (x_2+1)x_3$$
$$\dot{x}_2 = x_1^5 + x_3$$
$$\dot{x}_3 = x_1^2 + u$$
$$y = x_1$$

To generate a direct relationship between the output and input, let us differentiate the output

$$\dot{y} = \dot{x}_1 = \sin x_2 + (x_2+1)x_3 \ .$$

Since $\dot{y}$ is still not directly relate to the input u, let us differentiate again. We now obtain

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Ex:

$$\ddot{y} = (x_2+1)u + f_1(x)$$
$$f_1(x) = (x_1^5 + x_3)(x_3 + \cos x_2) + (x_2+1)x_1^2$$

Clearly, it represents an explicit relationship between y and u. If we choose the control input to be in the form

$$u = \frac{1}{x_2+1}(v - f_1)$$

where v is a new input to be determined, the nonlinearity is canceled, and we obtain a simple linear double-integrator relationship between the output and the new input v $\qquad \ddot{y} = v$.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Ex:

The design of a tracking controller for this double-integrator relation is simple, because of the availability of linear control techniques. For instance, letting e = y(t)-y_d(t) be the racking error, and choosing the new input v as

$$v = \ddot{y}_d - k_1 e - k_2 \dot{e}$$

where k_1 , k_2 are positive constant. The tracking error of the closed-loop system is given by

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0$$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Ex:

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0 \Rightarrow$$

which represents an exponentially stable error dynamics. Therefore, if initially $e(0) = \dot{e}(0) = 0$, then $e(t) = 0, \forall t \geq 0$, i.e., perfect tracking is achieved; otherwise, $e(t)$ converge to zero exponentially.

Note that:

1. The control law is defined anywhere, except at the singularity point such that $x_2 = -1$
2. Full state measurement is necessary in implementing the control law.
3. The above controller does not guarantee the stability of internal dynamics.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

- ➢The above control design strategy of first generating a linear input-output relation and then formulating a controller based on linear control is referred to as the input-output linearization approach.
- ➢If we need to differentiate the output of a system *r* times to generate an explicit relationship between the output y and input u, the system is said to have relative degree r. Thus, the system in the above example has relative degree 2.
- ➢A part of the system dynamics (described by one state component) has been rendered "unobservable" in the input-output linearization. This part of the dynamics will be called the internal dynamics, because it cannot be seen from the external input output relationship. For the above example, the internal state can be chosen to be x_3 (because x_3 , y , and $\dot{y}$ constitute a new set of states), and the internal dynamics is represented by the equation $\dot{x}_3 = x_1^2 + \frac{1}{x_2 + 1}(\ddot{y}_d(t) - k_1 e - k_2 \dot{e} + f_1)$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢If this internal dynamics is stable (by which we actually mean that the states remain bounded during tracking, i.e., stability in the BIBO sense), our tracking control design problem has indeed been solved. Otherwise, the above tracking controller is practically meaningless, because the instability of the internal dynamics would imply undesirable phenomena such as the burning-up of fuses or the violent vibration of mechanical members. Therefore, the effectiveness of the above control design, based on the reduced-order model

$$\ddot{y} = (x_2 + 1)u + f_1(x)$$

hinges upon the stability of the internal dynamics.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Ex: Internal dynamics

Consider the nonlinear control system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 + u \\ u \end{bmatrix}$$

$$y = x_1$$

$e = y(t) - y_d(t)$

Control objective: to make y track to $y_d(t)$

$$\dot{y} = \dot{x}_1 = x_2^3 + u \quad \Rightarrow \quad u = -x_2^3 - e(t) + \dot{y}_d(t)$$

yields exponential convergence of e to zero.

$$\dot{e} + e = 0$$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Ex: Internal dynamics

Apply the same control law to the second dynamic equation, leading to the internal dynamics

$$\dot{x}_2 + x_2^3 = \dot{y}_d - e$$

which is non-autonomous and nonlinear. However, in view of the facts that e is guaranteed to be bound and $\dot{y}_d$ is assumed to be bounded, we have

$$|\dot{y}_d(t) - e| \leq D$$

where D is a positive constant.

$$|x_2| \leq D^{1/3}, \text{ since } \dot{x}_2 < 0 \text{ when } x_2 > D^{1/3}, \text{ and } \dot{x}_2 > 0 \text{ when } x_2 < -D^{1/3}.$$

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢Ex: Internal dynamics

Therefore, a satisfactory tracking control law is represented, given any trajectory $y_d(t)$ whose derivative is bounded.

Note: if the second state equation of the system is replaced by $\dot{x}_2 = -u$, the resulting internal dynamics is unstable.

➢The zero-dynamics

Definition: The zeros-dynamics is defined to be the internal dynamics of the systems when the system output is kept at zero by the input.

➢Feedback linearization

➢Feedback linearization is an approach to nonlinear control design

➢Input-Output Linearization

➢The zero-dynamics

The reason for defining and studying the zero-dynamics is that we want to find a simpler way of determining the stability of the internal dynamics.

- In linear systems, the stability of the zero-dynamics implies the global stability of the internal dynamics.
- In nonlinear systems, if the zero-dynamics is globally exponentially stable only local stability is guaranteed for the internal dynamics.

➢Feedback linearization

- ➢Feedback linearization is an approach to nonlinear control design
  - ➢Input-Output Linearization
    - ➢The zero-dynamics

For instance, for the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 + u \\ u \end{bmatrix} \qquad y = x_1$$

the output $y = x_1 \equiv 0 \rightarrow \dot{y} = \dot{x}_1 \equiv 0 \rightarrow u \equiv -x_2^3$, hence the zero-dynamics is

$\dot{x}_2 + x_2^3 = 0$

This zero-dynamics is easily seen to be asymptotically stable by using Lyapunov function $V = x_2^2$.

➢Feedback linearization

- ➢Feedback linearization is an approach to nonlinear control design
  - ➢Input-Output Linearization
    - ➢To summarize, control design based on input-output linearization can be made in three steps:
      - ➢1. differentiate the output y until the input **u** appears.
      - ➢2. choose u to cancel the nonlinearities and guarantee tracking convergence.
      - ➢3. study the stability of the internal dynamics.

➢Feedback linearization

- ➢Feedback linearization is an approach to nonlinear control design
  - ➢Summary
    - ➢Feedback linearization cancels the nonlinearities in a nonlinear system such that the closed-loop dynamics is in a linear form.
    - ➢Canceling the nonlinearities and imposing a desired linear dynamics, can be applied to a class of nonlinear systems, named companion form, or controllability canonical form.
    - ➢When the relative degree of a system is the same as its order:
      - ➢There is no internal dynamics
      - ➢The problem will be input-state linearization
    - ➢For any controllable system of order *n*, by taking at most *n* differentiations, the control input will appear to any output, i.e., r ≤ n. If the control input never appears after more than n differentiations, the system would not be controllable.

➢Feedback linearization

- ➢Feedback linearization is an approach to nonlinear control design
  - ➢Summary
    - ➢When the nonlinear dynamics is not in a controllability canonical form, input-state linearization technique is employed:
      - ➢1. Transform input and state into companion canonical form
      - ➢2. Use standard linear techniques to design controller
    - ➢ For tracking a desired trajectory, when y is not directly related to **u**, input-output linearizaton is applied:
      - ➢1. Generating a linear input-output relation (take derivative of y r ≤ n times)
      - ➢2. Formulating a controller based on linear control

➢Feedback linearization

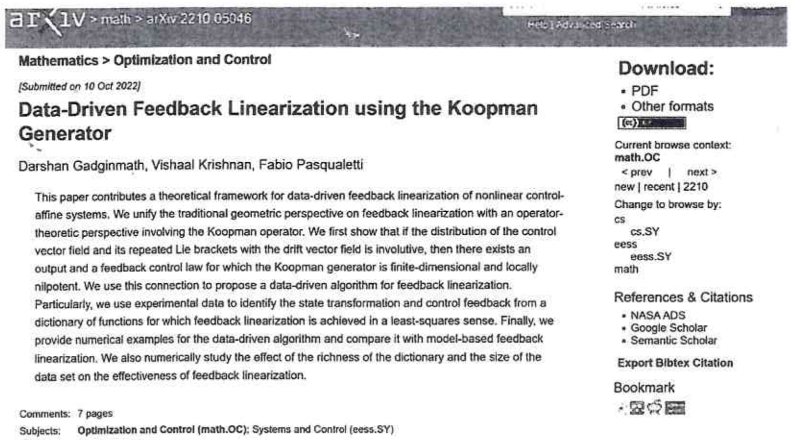
- ➢Feedback linearization is an approach to nonlinear control design
  - ➢Summary
    - ➢Relative degree: number of differentiating y to find explicate relation to u.
    - ➢If r is not equal to n, there are n – r internal dynamics that their stability be checked.
    - ➢In general, directly determining the stability of the internal dynamics is not easy since it is nonlinear, nonautonomous, and coupled to the "external" closed-loop dynamics.
    - ➢Zero-dynamics is an intrinsic feature of a nonlinear system, which does not depend on the choice of control law or the desired trajectories.
    - ➢Examining the stability of zero-dynamics is easier than examining the stability of internal dynamics, but the result is local.
      - ➢Zero-dynamics only involves the internal states
      - ➢Internal dynamics is coupled to the external dynamics and desired trajectories.

➢Feedback linearization

- ➢Feedback linearization is an approach to nonlinear control design
  - ➢Summary
    - ➢If the zero-dyiamics is unstable, different control strategies should be sought
    - ➢If the relative degree associated with the input-output linearization is the same as the order of the system -> **the** nonlinear system is fully linearized ->satisfactory controller
      - ➢Otherwise, the nonlinear system is only partly linearized whether or not the controller is applicable depends on the stability of the internal dynamics.

➢Feedback linearization

- ➢Koopman Theory (Koopman operator)



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Data-Driven Feedback Linearization using the Koopman Generator

Darshan Gadginmath, Vishaal Krishnan, Fabio Pasqualetti

This paper contributes a theoretical framework for data-driven feedback linearization of nonlinear control-affine systems. We unify the traditional geometric perspective on feedback linearization with an operator-theoretic perspective involving the Koopman operator. We first show that if the distribution of the control vector field and its repeated Lie brackets with the drift vector field is involutive, then there exists an output and a feedback control law for which the Koopman generator is finite-dimensional and locally nilpotent. We use this connection to propose a data-driven algorithm for feedback linearization. Particularly, we use experimental data to identify the state transformation and control feedback from a dictionary of functions for which feedback linearization is achieved in a least-squares sense. Finally, we provide numerical examples for the data-driven algorithm and compare it with model-based feedback linearization. We also numerically study the effect of the richness of the dictionary and the size of the data set on the effectiveness of feedback linearization.

Comments: 7 pages

Subjects: Optimization and Control (math.OC); Systems and Control (eess.SY)

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➢Feedback linearization

- ➢Koopman Theory (Koopman operator)
  - ➢Feedback linearization emerged as a popular model-based technique to linearize nonlinear control systems. However, it requires the knowledge of the system or prior system identification for implementation. Current robotic and cyber-physical systems are high dimensional and complex. Therefore, errors in system identification can yield poor control performance
  - ➢On the contrary, machine learning methods provide a powerful alternative as they can utilize experimental data from the system to implement feedback control on the system without much knowledge about the system. However, they often fail to provide an insight into the system and their performance. Further, their limitations are not fully understood. A systematic approach to nonlinear data-driven control is still an unsolved problem.
  - ➢Recently, the Koopman operator has gathered interest as it provides a global infinite-dimensional linear representation of autonomous nonlinear systems.
  - ➢The model-based feedback linearization approach and the modern data-driven Koopman operator approach are both linearization techniques for control and autonomous systems respectively.

➢Feedback linearization

➢Koopman Theory (Koopman operator)

$$\dot{x} = f(x, t) \Longrightarrow \dot{x} \approx AX$$

Koopman Theory

Let $y = g(x) \Rightarrow g(x_{k+1}) = K g(x_k)$

$$\Rightarrow \dot{y} = Ky$$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

Ex 1: $\dot{x}_1 = \mu x_1$

$\dot{x}_2 = \lambda(x_2 - x_1^2)$

sol: $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1^2$

$\dot{y}_1 = \mu y_1$, $\dot{y}_2 = \lambda(y_2 - y_3)$, $\dot{y}_3 = 2x_1\dot{x}_1 = 2\mu y_3$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

$$\dot{y} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

Ex2: $\dot{x}_1 = \mu x_1$

$\dot{x}_2 = \lambda(x_2^2 - x_1)$

sol: $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_2^2$

$\dot{y}_1 = \mu y_1$, $\dot{y}_2 = \lambda(y_3 - y_1)$, $\dot{y}_3 = 2\lambda x_2(x_2^2 - x_1)$

$= 2\lambda(x_2^3 - x_1 x_2)$

so, $y_4 = x_2^3$, $y_5 = x_1 x_2$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

$$\dot{y} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mu & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -\lambda & 0 & \lambda & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 2\lambda & -2\lambda & 0 \cdots & 0 \\ & & & & \vdots & & \ddots & \\ & & & & \vdots & & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \end{bmatrix}$$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

- ➢DMD: Dynamic Mode Decomposition
  - ➢How to analyze nonlinear complex systems through data has always been a very important issue.
  - ➢The development of Koopman theory provides a strategy for solving this problem
  - ➢The most common algorithm implemented is called Dynamic Mode Decomposition (DMD)
  - ➢The algorithm has a major limitation: a sufficient number of nonlinear measurement data
  - ➢Koopman theory is essentially a "coordinate transformation"

➢Feedback linearization

➢Koopman Theory (Koopman operator)

- ➢DMD: Dynamic Mode Decomposition
  - ➢J.Proctor,S.L. Brunton, and J.N. Kutz, "Dynamic Mode Decomposition with Control", SIAM J. APPLIED DYNAMICAL SYSTEM, Vol. 15, No. 1, pp. 142–161,2016.
- ➢Dynamical systems and data

$$\mathbf{x}_{k+1} \approx \mathbf{A}\mathbf{x}_k,$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{m-1} \\ | & | & & | \end{bmatrix}, \quad \mathbf{X}' = \begin{bmatrix} | & | & & | \\ \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_m \\ | & | & & | \end{bmatrix}$$

where m is the total number of snapshots and X is the time-shifted snapshot matrix of X, i.e., X' = AX.

➢Feedback linearization

➢Koopman Theory (Koopman operator)

- ➢DMD: Dynamic Mode Decomposition

$$\mathbf{X}' \approx \mathbf{A}\mathbf{X}.$$

- ➢The DMD of the measurement matrix pair X and X is the eigendecomposition of the matrix A. The operator A is defined by the following:

$$\mathbf{A} = \mathbf{X}'\mathbf{X}^{\dagger}$$

- ➢where † is the pseudoinverse. A computationally efficient and accurate method for finding the pseudoinverse is via the SVD. The SVD of X results in the well-known decomposition

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

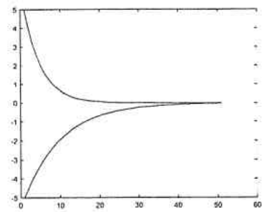
```
clear all
A=[0.9 0; 0 1.1];
B=[0;1];
K=[0 0.3];
CL=A-B*K;
x(:,1)=[-5,5];
for k=1:50
    u(:,k)=-K*x(:,k);
    x(:,k+1)=A*x(:,k)+B*u(:,k);
end
plot(x')
%DMD
X=x(:,1:end-1);
X2=x(:,2:end);
Ac=X2*pinv(X)

Ai=(X2-B*u)*pinv(X)
```

```
Ac =

    0.9000   -0.0000
   -0.0000    0.8000

Ai =

    0.9000   -0.0000
    0.0000    1.1000
```

$$\mathbf{X}' \approx \mathbf{A}\mathbf{X}.$$

⇩

$$\mathbf{A} = \mathbf{X}'\mathbf{X}^{\dagger}$$



➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^* = \begin{bmatrix}\tilde{\mathbf{U}} & \tilde{\mathbf{U}}_{\text{rem}}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{\Sigma}} & 0 \\ 0 & \boldsymbol{\Sigma}_{\text{rem}}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{V}}^* \\ \tilde{\mathbf{V}}^*_{\text{rem}}\end{bmatrix}$$
$$\approx \tilde{\mathbf{U}}\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{V}}^*.$$

where $\mathbf{U} \in \mathbb{R}^{n\times n}$, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times m-1}$, $\tilde{\mathbf{V}}^* \in \mathbb{R}^{m-1\times m-1}$, $\tilde{\mathbf{U}} \in \mathbb{R}^{n\times r}$, $\tilde{\boldsymbol{\Sigma}} \in \mathbb{R}^{r\times r}$, $\tilde{\mathbf{V}}^* \in \mathbb{R}^{r\times m-1}$, $_{\text{rem}}$ indicates the remaining $m-1-r$ singular values, and * denotes the complex conjugate transpose. Equation demonstrates how to reduce the dimension of the data matrix $\mathbf{X}$ by appropriately choosing a truncation value r of the singular values, thus eliminating the remainder (rem) terms and allowing for the psuedoinverse to be accomplished since $\tilde{\boldsymbol{\Sigma}}$ is square.

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

$$\mathbf{A} \approx \bar{\mathbf{A}} = \mathbf{X}'\tilde{\mathbf{V}}\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\mathbf{U}}^*$$

➢A dynamic model of the process can be constructed given by the following

$$\mathbf{x}_{k+1} = \bar{\mathbf{A}}\mathbf{x}_k$$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

If $r \ll n$, a more compact and computationally efficient model can be found by projecting $\mathbf{x}_k$ onto a linear subspace of dimension r. This basis transformation takes the form $\mathbf{P}\mathbf{x} = \tilde{\mathbf{x}}$. As previously shown by DMD, a convenient transformation has already been computed via the SVD of $\mathbf{X}$, given by $\mathbf{P} = \tilde{\mathbf{U}}^*$. The reduced-order model can be derived as follows:

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1} &= \tilde{\mathbf{U}}^*\bar{\mathbf{A}}\tilde{\mathbf{U}}\tilde{\mathbf{x}}_k \\ &= \tilde{\mathbf{U}}^*\mathbf{X}'\tilde{\mathbf{V}}\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\mathbf{x}}_k \\ &= \tilde{\mathbf{A}}\tilde{\mathbf{x}}_k.\end{aligned}$$

The reduced-order model is given by the following:

$$\tilde{\mathbf{A}} = \tilde{\mathbf{U}}^*\mathbf{X}'\tilde{\mathbf{V}}\tilde{\boldsymbol{\Sigma}}^{-1}.$$

The eigendecomposition of $\tilde{\mathbf{A}}$ defined by $\tilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\boldsymbol{\Lambda}$ yields eigenvalues and eigenvectors that can be investigated for fundamental properties of the underlying system such as growth modes and resonance frequencies. In addition, the computation is efficient since $\tilde{\mathbf{A}} \in \mathbb{R}^{r\times r}$ and $r \ll n$.

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

For DMD, the eigenvalues of them are equivalent, and the eigenvectors are related via a linear transformation.

Remark. Computing the eigendecomposition of $\bar{\mathbf{A}}$ versus $\bar{\mathbf{A}}$ can be a computationally crucial step for efficiency. For example, the domain discretization of a fluids or epidemiological problem can have an arbitrarily large set of dimensions n. The direct solution of the $n \times n$ eigenvalue problem might not be feasible; thus solving the $r \times r$ is substantially more attractive.

$$\mathbf{\Phi} = \mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{W}.$$

The diagonal elements of Λ are eigenvalues of A with corresponding eigenvectors given by columns of Φ.

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

➢Dynamic mode decomposition with control

$$\mathbf{X}' = \begin{bmatrix} | & | & & | \\ \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_m \\ | & | & & | \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{m-1} \\ | & | & & | \end{bmatrix}$$

$$\mathbf{\Upsilon} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{m-1} \\ | & | & & | \end{bmatrix}$$

$$\mathbf{x}_{k+1} \approx \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k,$$

$$\mathbf{X}' \approx \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{\Upsilon}.$$

➢Suppose that the map B is known

$$\mathbf{X}' - \mathbf{B}\mathbf{\Upsilon} \approx \mathbf{A}\mathbf{X}.$$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

$$\mathbf{A} \approx \bar{\mathbf{A}} = (\mathbf{X}' - \mathbf{B}\mathbf{\Upsilon})\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{U}}^*.$$

$$\mathbf{x}_{k+1} = \bar{\mathbf{A}}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k,$$

➢If $r \ll n$, though, a more compact and computationally efficient model can be found using the same basis transformation $\mathbf{P}\mathbf{x} = \tilde{\mathbf{x}}$, given by $\mathbf{P} = \tilde{\mathbf{U}}^*$.

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1} &= \tilde{\mathbf{U}}^*\bar{\mathbf{A}}\tilde{\mathbf{U}}\tilde{\mathbf{x}}_k + \tilde{\mathbf{U}}^*\mathbf{B}\mathbf{u}_k \\ &= \tilde{\mathbf{U}}^*(\mathbf{X}' - \mathbf{B}\mathbf{\Upsilon})\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{x}}_k + \tilde{\mathbf{U}}^*\mathbf{B}\mathbf{u}_k \\ &= \tilde{\mathbf{A}}\tilde{\mathbf{x}}_k + \tilde{\mathbf{B}}\mathbf{u}_k.\end{aligned}$$

$$\tilde{\mathbf{A}} = \tilde{\mathbf{U}}^*(\mathbf{X}' - \mathbf{B}\mathbf{\Upsilon})\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}.$$

$$\phi = (\mathbf{X}' - \mathbf{B}\mathbf{\Upsilon})\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\mathbf{w}.$$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

➢Ex: B is known

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{k+1} = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k,$$

where $u_k = K[x_1]_k$ and $K = -1$.

$$\mathbf{X} = \begin{bmatrix} 4 & 2 & 1 & 0.5 \\ 7 & 0.7 & 0.07 & 0.007 \end{bmatrix},$$

$$\mathbf{X}' = \begin{bmatrix} 2 & 1 & 0.5 & 0.25 \\ 0.7 & 0.07 & 0.007 & 0.0007 \end{bmatrix},$$

$$\mathbf{\Upsilon} = \begin{bmatrix} -4 & -2 & -1 & -0.5 \end{bmatrix}.$$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

We use the MATLAB economy-sized SVD algorithm to give the following matrix factorization of X:

```
%DMD_control; B is known.
X=[4 2 1 0.5; 7 0.7 0.07  0.007];
XP=[2 1 0.5 0.25 ;0.7 0.07  0.007  0.0007];
Ga=[-4 -2 -1 -0.5];
B=[1 0]';
[U,S,V] =svd(X,'econ');
At=conj(U')*(XP-B*Ga)*V*inv(S)
```


➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

$$\tilde{\mathbf{U}} = \begin{bmatrix} -0.5239 & -0.8462 \\ -0.8462 & 0.5329 \end{bmatrix},$$

$$\tilde{\mathbf{\Sigma}} = \begin{bmatrix} 8.2495 & 0 \\ 0 & 1.6402 \end{bmatrix},$$

$$\tilde{\mathbf{V}} = \begin{bmatrix} -0.9764 & 0.2105 \\ -0.2010 & -0.8044 \\ -0.0718 & -0.4932 \\ -0.0330 & -0.2557 \end{bmatrix}.$$

➢Feedback linearization

➢Koopman Theory (Koopman operator)

➢DMD: Dynamic Mode Decomposition

$$\mathbf{A} \approx \bar{\mathbf{A}} = (\mathbf{X}' - \mathbf{B}\mathbf{\Upsilon})\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{U}}^*.$$

$$\bar{\mathbf{A}} = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.1 \end{bmatrix}$$