1.(a) $\begin{cases} \dot{x}_1 = -x_1 + x_2^2 \\ \dot{x}_2 = -x_2 \end{cases}$

Lyapunov Function : $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\Rightarrow \dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1 + x_2^2) + x_2(-x_2)$$
$$= -x_1^2 + x_1x_2^2 - x_2^2$$

$\because$ $x_1x_2^2$ can dominate $x_1^2 + x_2^2$ in certain regions.

$\therefore$ negative semi-definite

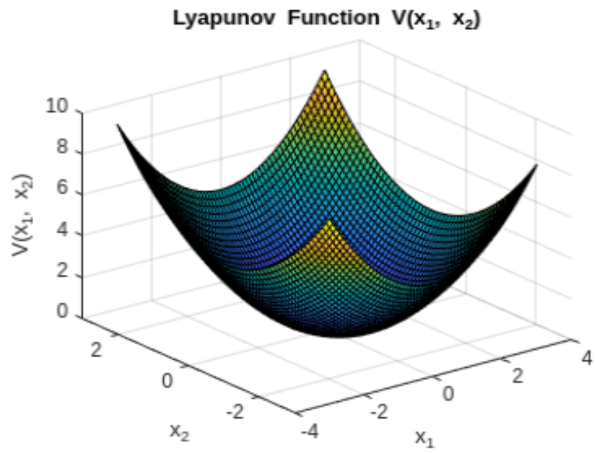
Near the origin, the negative terms dominate, so the origin appears to be locally asymptotically stable.

1.(b) Plot a Lyapunov function using MATLAB

```
% Define the grid
[x1, x2] = meshgrid(-3:0.1:3, -3:0.1:3);

% Lyapunov function
V = 0.5 * (x1.^2 + x2.^2);

% Plot
figure;
surf(x1, x2, V);
xlabel('x_1');
ylabel('x_2');
zlabel('V(x_1, x_2)');
title('Lyapunov Function V(x_1, x_2)');
grid on;
```



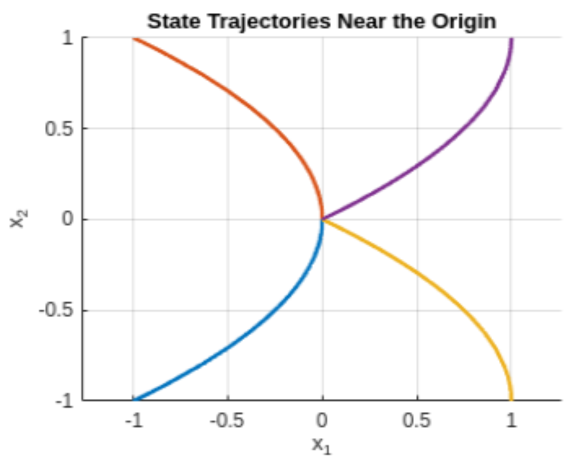
1.(c) Plot state trajectories of a dynamical system near the origin using MATLAB

```
% Define the system dynamics
f = @(t, x) [-x(1) + x(2)^2; -x(2)];

% Time span
tspan = [0 10];

% Initial conditions (near origin)
initial_conditions = [-1 -1; -1 1; 1 -1; 1 1];

% Plot phase portrait
figure; hold on;
for i = 1:size(initial_conditions,1)
    [t, x] = ode45(f, tspan, initial_conditions(i, :)');
    plot(x(:,1), x(:,2), 'LineWidth', 2);
end
xlabel('x_1'); ylabel('x_2');
title('State Trajectories Near the Origin');
grid on;
axis equal;
```



(d) $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$

Positive definite : $V(x) > 0$ for all $x \neq 0$, $V(0) = 0$.

Radially unbounded : $V(x) \to \infty$ as $\|x\| \to \infty$

$\Rightarrow \dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1(-x_1 + x_2^2) + x_2(-x_2) = -x_1^2 + x_1 x_2^2 - x_2^2$

$\dot{V}(x)$ is negative semi-definite.

$\because$ both $-x_1^2$, $-x_2^2$ dominate $x_1 x_2^2$ near the origin,
we can conclude stability of the origin,
but not asymptotic stability

$\therefore$ The origin is Lyapunov stable (but not necessarily asymptotically stable.

(e) If $x_1 < 0$, $x_1 x_2^2$ is negative, $\dot{V}$ become more negative.

If $x_1 > 0$, $x_1 x_2^2$ is positive, reducing how negative $\dot{V}$ is.

As $x_2 \to 0$, the nonlinear term $x_1 x_2^2 \to 0$, $\dot{V} \to -x_1^2 - x_2^2 < 0$.
so near the origin, asymptotic stability is preserved locally.

The origion is locally asymptotically stable using Lyapunov's direct method.

(f) To claim global asymptotic stability :

1. A positive definite $V(x)$, 2. A negative definite $\dot{V}(x)$ globally.
3. $\dot{V}(x) \to 0$ only at the origin.

$\dot{V}(x) = -x_1^2 + x_1 x_2^2 - x_2^2$ is not globally negative definite.

$\therefore$ The system is NOT globally asymptotically stable

2.

(d) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 - (2x_2 - x_1)(1 - x_2^2) \end{cases}$

set $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, it's positive definite, radially unbounded.

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2 + x_2\left[-x_1 - x_2 - (2x_2 + x_1)(1 - x_2^2)\right]$$

$$= x_1x_2 - x_1x_2 - x_2^2 - x_2(2x_2 + x_1)(1 - x_2^2)$$

$$= -x_2^2 - x_2(2x_2 + x_1)(1 - x_2^2)$$

$$= -x_2^2 - x_2(2x_2 + x_1)(1 - x_2^2)$$

$\Rightarrow$ Near the origin where $x_2 \approx 0$, $(1 - x_2^2) \approx 1$,

$\therefore \dot{V} \approx -x_2^2 - x_2(2x_2 + x_1) = -x_2^2 - 2x_2^2 - x_1x_2 = -3x_2^2 - x_1x_2$

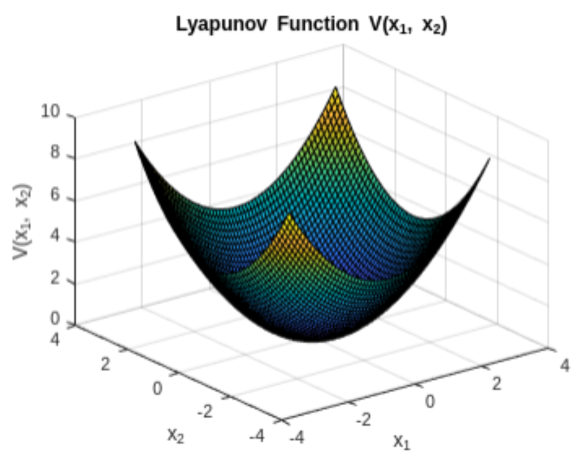
$\Rightarrow$ Lyapunov's direct method shows local asymptotically stability of the origin.

2.(e) Plot a Lyapunov function using MATLAB

```
% Define the grid
[x1, x2] = meshgrid(-3:0.1:3, -3:0.1:3);

% Lyapunov function
V = 0.5 * (x1.^2 + x2.^2);

% Plot
figure;
surf(x1, x2, V);
xlabel('x_1'); ylabel('x_2'); zlabel('V(x_1, x_2)');
title('Lyapunov Function V(x_1, x_2)');
grid on;
```



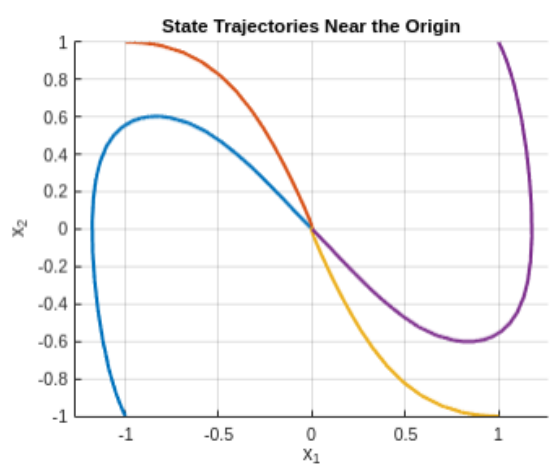
2.(f) Plot state trajectories of a dynamical system near the origin using MATLAB

```
% Define the system dynamics
f = @(t, x) [x(2);
             -x(1) - x(2) - (2*x(2) + x(1))*(1 - x(2)^2)];

% Time span
tspan = [0 20];

% Initial conditions
initial_conditions = [-1 -1; -1 1; 1 -1; 1 1];

% Plot state trajectories
figure; hold on;
for i = 1:size(initial_conditions,1)
    [t, x] = ode45(f, tspan, initial_conditions(i, :)');
    plot(x(:,1), x(:,2), 'LineWidth', 2);
end
xlabel('x_1'); ylabel('x_2');
title('State Trajectories Near the Origin');
grid on;
axis equal;
```



(b) To show Lyapunov stability:

$V(x)$ is positive definite.

$\dot{V}(x)$ is negative semi-definite or at least not positive in any neighborhood around the origin.

$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is positive definite.

$\dot{V}(x) = -x_2^2 - x_2(2x_2 + x_1)(1 - x_2^2)$

$\Rightarrow$ near the origion, where x_2 is small, $1 - x_2^2 \approx 1$

$\therefore \dot{V} \approx -x_2^2 - x_2(2x_2 + x_1) = -3x_2^2 - x_1 x_2$

$\therefore$ The origin is Lyapunov stable.

(c) To show asymptotic stability:

$V(x)$ is positive.

$\dot{V}(x)$ is negative definite or negative semi-definite with trajectories approaching the origin over time.

$\dot{V} \approx -3x_2^2 - x_1 x_2$

if $x_1 x_2$ is positive, still couldn't overcome $-3x_2^2$ in the local region, since the origin is the only equilibrium and energy tends to decrease in almost all directions, the trajectories are drawn toward the origin.

$\therefore$ The origin is locally asymptotically stable by Lyapunov's direct method.