1. Is $g(t) = \left(\frac{t}{2}\right)^2$ Uniformly Continuous?

To determine whether the function $g(t) = \left(\frac{t}{2}\right)^2 = \frac{t^2}{4}$ is uniformly continuous on $\mathbb{R}$ using a counterexample method.

A function f is uniformly continuous on a domain D if:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in D$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Uniformly continuity means: the same δ must work everywhere in the domain, not just locally like in ordinary continuity.

Counterexample Strategy:

We will assume $g(t) = \frac{t^2}{4}$ is uniformly continuous on $\mathbb{R}$, and show this leads to a contradiction. To choose two points:

$$x_n = n, \quad y_n = n + \frac{1}{n}$$

Then: $|x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$

But compute the difference in function values:

$$|g(x_n) - g(y_n)| = \left|\frac{n^2}{4} - \frac{(n+\frac{1}{n})^2}{4}\right| = \left|\frac{n^2 - n^2 - 2 - \frac{1}{n^2}}{4}\right| = \frac{2 + \frac{1}{n^2}}{4} \to \frac{1}{2} \text{ as } n \to \infty$$

Conclusion:

Even though $|x_n - y_n| \to 0$, the value $|g(x_n) - g(y_n)| \to \frac{1}{2}$.

This violates the definition of uniform continuity.

Therefore:

$g(t) = \frac{t^2}{4}$ is not uniformly continuous on $\mathbb{R}$.

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2. $\begin{cases} \dot{e} = -e^3 + \theta w(t) & \text{(Tracking error dynamics)} \\ \dot{\theta} = -e w(t) & \text{(Parameter error dynamics)} \end{cases}$

Use Lyapunov-like function and Barbalat's Lemma to analyze the stability or boundedness of $e(t)$ and $\theta(t)$.

① Choose a Lyapunov-like function

$V(e, \theta) = \frac{1}{2} e^2 + \frac{1}{2} \theta^2$, this is positive definite and radially unbounded.

② Take the time derivative of V

$$\dot{V} = e\dot{e} + \theta\dot{\theta} = e(-e^3 + \theta w(t)) + \theta(-e w(t)) = -e^4 \le 0$$

This implies $V(t)$ is non-increasing $\Rightarrow V(t) \le V(0)$ for all $t \ge 0$.

Therefore, both $e(t)$ and $\theta(t)$ are bounded.

③ Apply Barbalat's Lemma

We know: $\dot{V}(t) = -e^4(t) \in \mathbb{R}$ is continuous.

$V(t)$ is bounded and non-increasing

$\Rightarrow \dot{V}(t) \in L_1$, i.e. integrable.

Since $\dot{V}(t) \in L_1$, and $\dot{V}(t)$ is uniformly continuous (due to smoothness of the system), we can apply Barbalat's Lemma:

$$\dot{V}(t) = -e^4(t) \to 0 \Rightarrow e^4(t) \to 0 \Rightarrow e(t) \to 0 \text{ as } t \to \infty$$

④ Behavior of $\theta(t)$:

From the equation : $\dot{\theta} = -e w(t)$

We know : $e(t) \to 0$, $w(t)$ is bounded and continuous.

Therefore: $\dot{\theta}(t) \to 0$ as $t \to \infty$

But we cannot conclude that $\theta(t) \to$ constant or converges to zero unless additional conditions (e.g. persistency of excitation of $w(t)$) are imposed.

From bounded $V(t)$, we have that:

$$\theta(t) \text{ is bounded and } \dot{\theta}(t) \to 0$$

Conclusion:

$e(t) \to 0$ as $t \to \infty$ (tracking error goes to zero),

$\theta(t)$ is bounded and its derivative vanishes,

Stability is guaranteed via Lyapunov analysis,

Barbalat's Lemma confirms asymptotic convergence of $e(t)$. *