Constraints Handling for MOEAs
(Ch 7 by Deb)

Typical constrained MOOP

$$\left.\begin{array}{lll} \text{Minimize/Maximize} & f_m(\mathbf{x}), & m = 1, 2, \ldots, M; \\ \text{subject to} & g_j(\mathbf{x}) \geq 0, & j = 1, 2, \ldots, J; \\ & h_k(\mathbf{x}) = 0, & k = 1, 2, \ldots, K; \\ & x_i^{(L)} \leq x_i \leq x_i^{(U)}, & i = 1, 2, \ldots, n. \end{array}\right\}$$

- Constraints divide the search space into feasible and infeasible regions
- Constraint types: equality and inequality
- Consider only inequality constraint, because equality constraint can be converted into inequality constraint
- Smaller-than form can also be converted into greater-than form
- Constraint violation for $g_j\left(\mathrm{x}^{(i)}\right) < 0$ implies the amount of constraint violation is

$$\left|g_j\left(\mathrm{x}^{(i)}\right)\right|$$

EXAMPLE 9.3 THE DUAL OF AN LP PROBLEM WITH EQUALITY AND "≥ TYPE" CONSTRAINTS

Write the dual for the problem

Maximize

$$z_p = x_1 + 4x_2 \tag{a}$$

subject to

$$x_1 + 2x_2 \le 5 \tag{b}$$

$$\boxed{2x_1 + x_2 = 4} \tag{c}$$

$$x_1 - x_2 \ge 1 \tag{d}$$

$$x_1, x_2 \ge 0 \tag{e}$$

Solution

The equality constraint $2x_1 + x_2 = 4$ is equivalent to the two inequalities $2x_1 + x_2 \ge 4$ and $2x_1 + x_2 \le 4$. The "$\ge$ type" constraints are multiplied by -1 to convert them into the "$\le$" form. Thus, the standard primal for the given problem is

Maximize

$$z_p = x_1 + 4x_2 \tag{f}$$

subject to

$$x_1 + 2x_2 \le 5 \tag{g}$$

$$\boxed{2x_1 + x_2 \le 4} \tag{h}$$

$$\boxed{-2x_1 - x_2 \le -4} \tag{i}$$

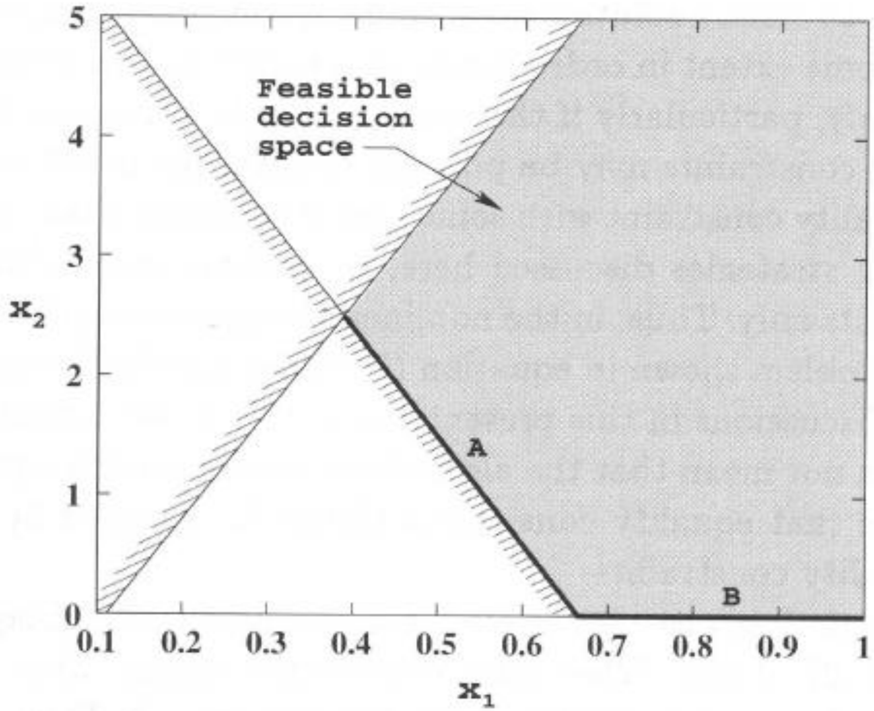
$$-x_1 + x_2 \le -1 \tag{j}$$

$$x_1, \; x_2 \ge 0 \tag{k}$$

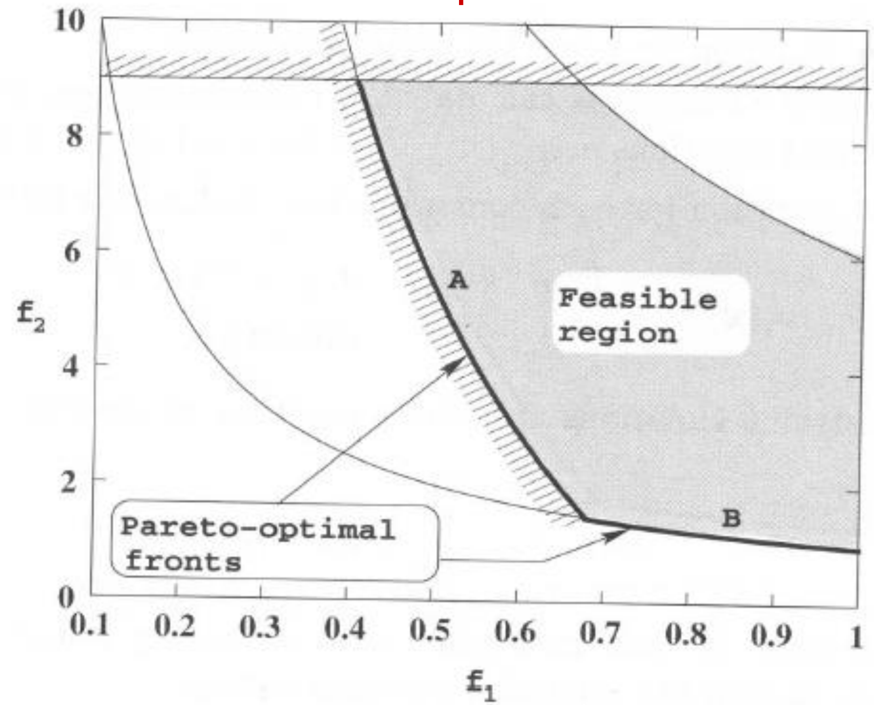
An Example

$$\text{Constr-Ex:} \begin{cases} \text{Minimize} & f_1(\mathbf{x}) = x_1, \\ \text{Minimize} & f_2(\mathbf{x}) = \dfrac{1 + x_2}{x_1}, \\ \text{subject to} & g_1(\mathbf{x}) \equiv x_2 + 9x_1 \geq 6, \\ & g_2(\mathbf{x}) \equiv -x_2 + 9x_1 \geq 1, \\ & 0.1 \leq x_1 \leq 1, \\ & 0 \leq x_2 \leq 5. \end{cases}$$

Variable space



Objective space



Ignoring infeasible solutions

- A common and simple way to handle constraints
- Ignore any solution that violates any of the assigned constraints (discard infeasible solutions)
- Simple to implement!
- A naïve approach!
- In reality, finding even one feasible solution is difficult, let alone finding a set of Pareto optimal solutions! That is, ignoring infeasible solutions might be time-consuming, even for finding a feasible one.

- As a result, we have to handle infeasible solutions!
- Measurement of constraint violation:

  Overall constraint violation of an infeasible solution.
- By assigning more selection pressure to solutions with less-violated constraints, an EA may provide a direction for reaching the feasible region.
- Once solutions reach the feasible region, a regular MOEA can be used to guide the search towards the Pareto optimal region.

Penalty Function Approach

- Popular constraint handling strategy
- All constraints are of the form $\underline{g}_j(\mathbf{x}^{(i)}) \geq 0$
- Steps:
  - Calculate the constraint violation for each solution **x**

$$\omega_j(\mathbf{x}^{(i)}) = \begin{cases} |\underline{g}_j(\mathbf{x}^{(i)})|, & \text{if } \underline{g}_j(\mathbf{x}^{(i)}) < 0; \\ 0, & \text{otherwise.} \end{cases}$$

  - All constraint violations are added together to get the overall violation

$$\Omega(\mathbf{x}^{(i)}) = \sum_{j=1}^{J} \omega_j(\mathbf{x}^{(i)}).$$

Penalty Function Approach

– Constraint violation is then multiplied with a penalty parameter R_m and the product is added to each of the objective function values

$$F_m(\mathbf{x}^{(i)}) = f_m(\mathbf{x}^{(i)}) + R_m \Omega(\mathbf{x}^{(i)}).$$

$$F_m(\mathbf{x}^{(i)}) = f_m(\mathbf{x}^{(i)}) + R_m\Omega(\mathbf{x}^{(i)}).$$

- $f_m \rightarrow F_m$, taking into account the constraint violation
- For a feasible solution, $f_m = F_m$
- For an infeasible solution, $F_m > f_m$
- Penalty parameter R_m: make both terms to have the same order of magnitude
- Any constrained optimization methods can be used with the newly established F_m

- Different objective functions have different magnitudes.

➔ Penalty parameter R_m must vary from one objective to another!

Example

$$\text{Constr-Ex:}\begin{cases} \text{Minimize} & f_1(\mathbf{x}) = x_1, \\ \text{Minimize} & f_2(\mathbf{x}) = \dfrac{1+x_2}{x_1}, \\ \text{subject to} & \boxed{\begin{array}{l} g_1(\mathbf{x}) \equiv x_2 + 9x_1 \geq 6, \\ g_2(\mathbf{x}) \equiv -x_2 + 9x_1 \geq 1, \end{array}} \\ & 0.1 \leq x_1 \leq 1, \\ & 0 \leq x_2 \leq 5. \end{cases}$$

normalized

$$\begin{aligned} \underline{g}_1(\mathbf{x}) &= \frac{9x_1 + x_2}{6} - 1 \geq 0, \\ \underline{g}_2(\mathbf{x}) &= \frac{9x_1 - x_2}{1} - 1 \geq 0. \end{aligned}$$

Table 25 Fitness assignment using the penalty function approach.

$$g_1(\mathbf{x}) = \frac{9x_1 + x_2}{6} - 1 \geq 0,$$
$$g_2(\mathbf{x}) = \frac{9x_1 - x_2}{1} - 1 \geq 0.$$

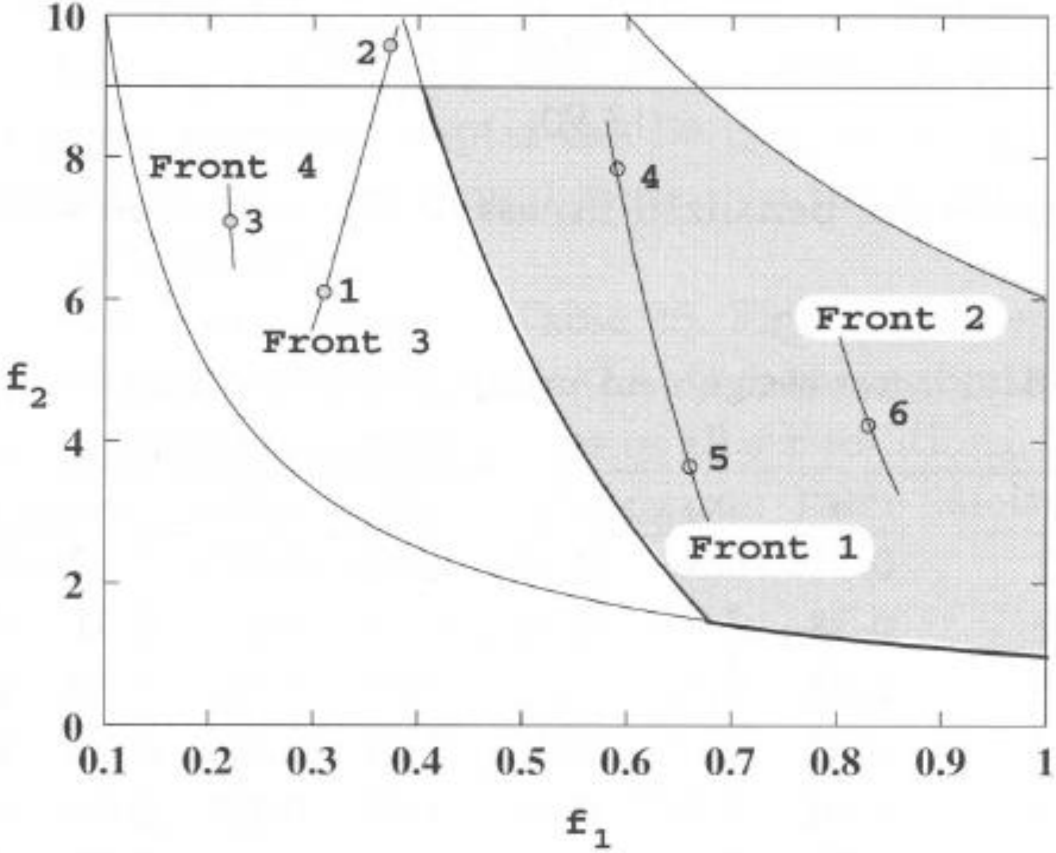
Solution	x_1	x_2	f_1	f_2	ω_1	ω_2	Ω
1	0.31	0.89	0.31	6.10	0.39	0.00	0.39
2	0.38	2.73	0.38	9.82	0.03	0.31	0.34
3	0.22	0.56	0.22	7.09	0.58	0.00	0.58
4	0.59	3.63	0.59	7.85	0.00	0.00	0.00
5	0.66	1.41	0.66	3.65	0.00	0.00	0.00
6	0.83	2.51	0.83	4.23	0.00	0.00	0.00

For Solution 1:

$$\begin{aligned} F_1 &= f_1 + R_1\Omega, \\ &= 0.31 + 2 \times 0.39, \\ &= 1.09. \\ F_2 &= f_2 + R_2\Omega, \\ &= 6.10 + 20 \times 0.39, \\ &= 13.90. \end{aligned}$$

Table 26 Penalized function values of all six solutions.

Solution	f_1	f_2	Ω	F_1	F_2	Front
1	0.31	6.10	0.39	1.09	13.90	3
2	0.38	9.82	0.34	1.06	16.62	3
3	0.22	7.09	0.58	1.38	18.69	4
4	0.59	7.85	0.00	0.59	7.85	1
5	0.66	3.65	0.00	0.66	3.65	1
6	0.83	4.23	0.00	0.83	4.23	2



Non-dominated sorting for unconstrained objective functions: [(1,3,5), (2,4,6)]

Non-dominated sorting for constrained objective functions: [(4,5), (6), (1,2),(3)]

- Infeasible solutions get de-emphasized by penalty terms!
- Feasible solutions close to the Pareto-optimal front are allocated in the best non-dominated front
- Infeasible solutions close to the constraint boundary have better fronts
- It is possible that infeasible solutions can be on the same front with a feasible solution!
- However, classification largely depends on the chosen penalty parameters R_m

➔ One of the desired ways to assign fitness in a constraint-handling MOEA!

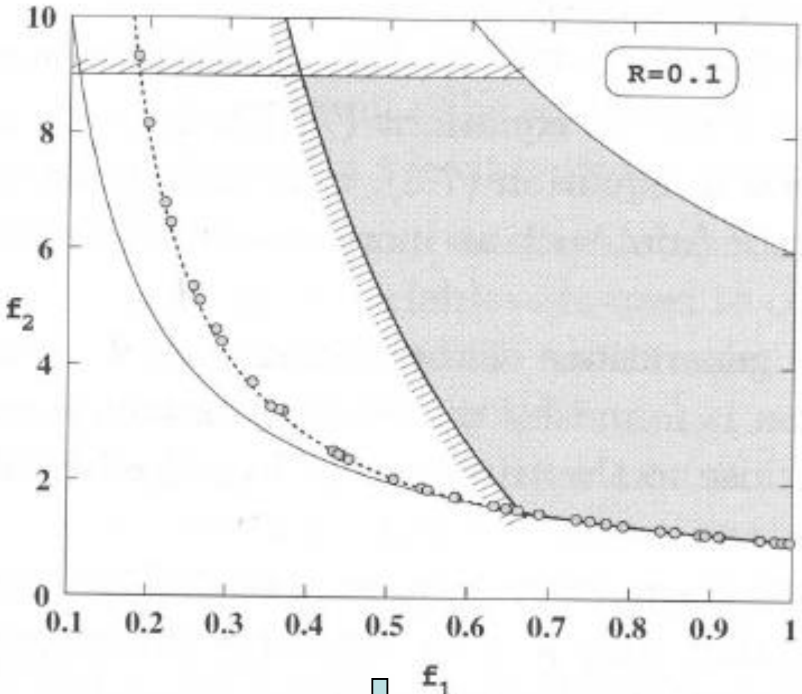
Simulations

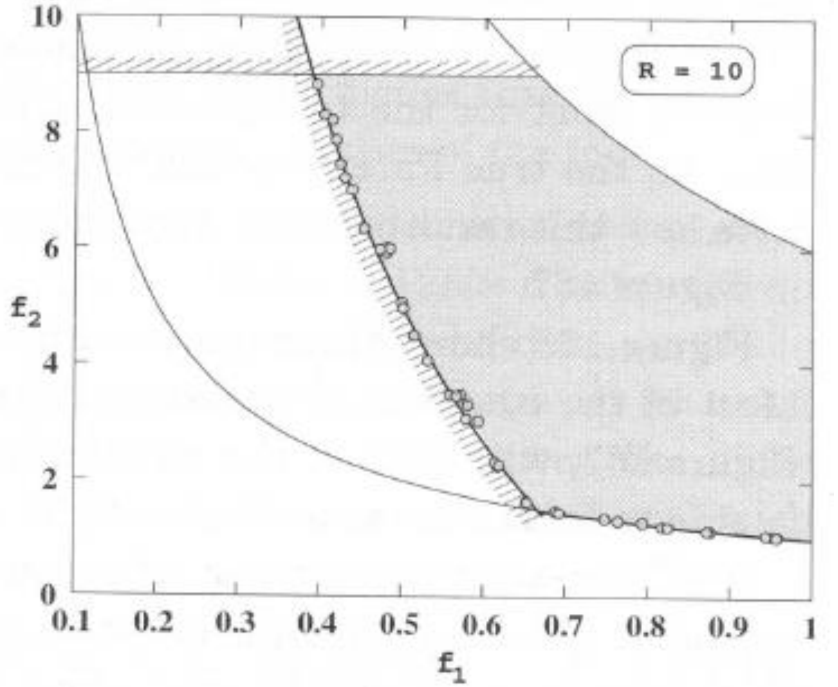
- Apply NSGA II
- GA parameters:
  - Population size: 40
  - Crossover prob.: 0.9
  - Mutation prob.: 0
  - Max gen. no.: 500

➔ Different penalty terms might form different Pareto-optimal front

Simulations

- R_1=R, and R_2=10R

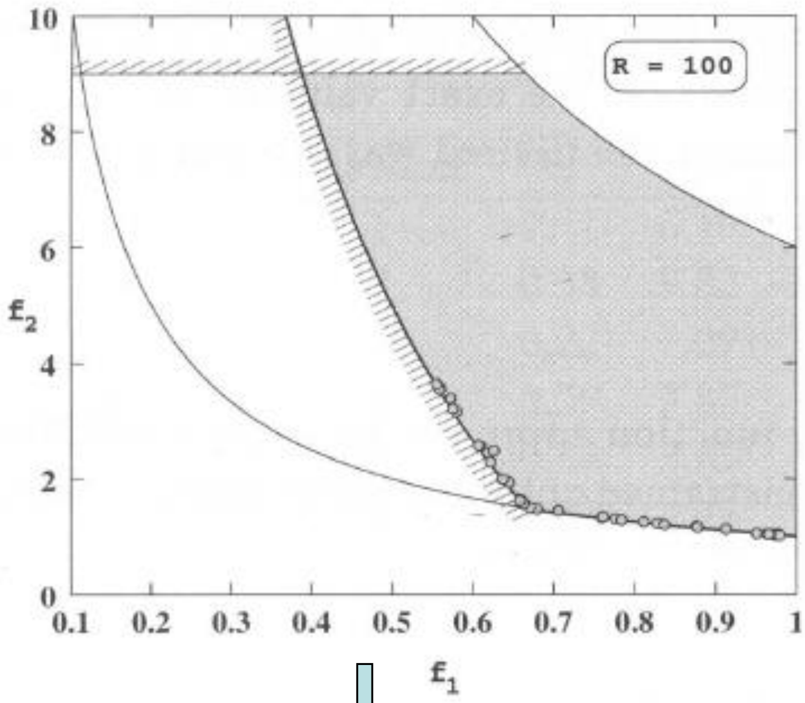


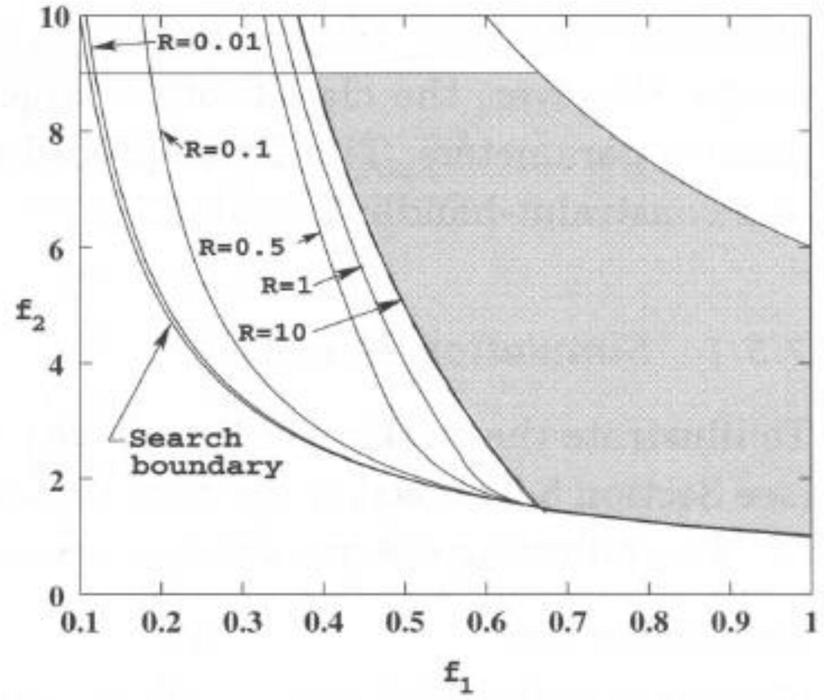


The penalty parameter is too small.

➔The front resides in the infeasible regions after 500 generations.

- If a smaller than adequate penalty parameter is chosen, the penalty effect is less and the resulting optimal solution may be infeasible ~ by Deb.





The constraints are over-emphasized!

➔Spread of the obtained solutions is not good.

➔Converge near a portion of the Pareto-optimal front!

- Optimization results depend on the choice of penalty parameters.
- If the choice of R_m is not adequate, either a set of infeasible solutions or poor distribution is likely to occur.