➢System Analysis Based on Lyapunov's Direct Method

➢How to find a Lyapunov function for a specific problem

➢There is no general way of finding Lyapunov function for nonlinear system. Faced with specific systems, we have to use experience, intuition, and physical insights to search for an appropriate Lyapunov function.

➢Lyapunov analysis of linear time-invariant systems

Definition *A square matrix* $\mathbf{M}$ *is symmetric if* $\mathbf{M}=\mathbf{M}^T$ *(in other words, if* $\forall i,j \; M_{ij} = M_{ji}$*). A square matrix* $\mathbf{M}$ *is skew-symmetric if* $\mathbf{M} = -\mathbf{M}^T$ *(i.e.,* $\forall i,j \; M_{ij} = -M_{ji}$ *).*

➢System Analysis Based on Lyapunov's Direct Method

➢Lyapunov analysis of linear time-invariant systems

➢Remarks

⇨ *Any square* $n \times n$ *matrix can be represented as the sum of a symmetric and a skew-symmetric matrix. This can be shown in the following decomposition*

$$\mathbf{M} = \underbrace{\frac{\mathbf{M}+\mathbf{M}^T}{2}}_{symmetric} + \underbrace{\frac{\mathbf{M}-\mathbf{M}^T}{2}}_{skew-symmetric}$$

⇨ *The quadratic function associated with a skew-symmetric matrix is always zero. Let M be a* $n \times n$ *skew-symmetric matrix and x is an arbitrary* $n \times 1$ *vector. The definition of skew-symmetric matrix implies that* $\mathbf{x}^T\mathbf{M}\,\mathbf{x} = -\mathbf{x}^T\mathbf{M}^T\mathbf{x}$.

$\forall \mathbf{x},\; \mathbf{x}^T\mathbf{M}\,\mathbf{x} = \mathbf{0}$

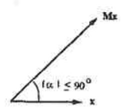
➢System Analysis Based on Lyapunov's Direct Method

➢Lyapunov analysis of linear time-invariant systems

➢Remarks

⇨ $\forall \mathbf{x},\; \mathbf{x}^T\mathbf{M}\,\mathbf{x} = \mathbf{0}$ is a necessary and sufficient condition for a matrix M to be skew-symmetric. In the designing some tracking control systems for robot, this fact is very useful because it can simplify the control law.

Definition *A square matrix* $\mathbf{M}$ *is positive definite (p.d.) if*
$\mathbf{x} \neq \mathbf{0} \;\Rightarrow\; \mathbf{x}^T\mathbf{M}\,\mathbf{x} > 0$.



➢Remarks

⇨ A necessary condition for a square matrix **M** to be p.d. is that its diagonal elements be strictly positive.

$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

A matrix can have positive diagonal elements but still fail to be positive definite

$[1 \;\; -1]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1(1) + (-1)(2) + 1(2) + (-1)(1) = -2 < 0.$

➢System Analysis Based on Lyapunov's Direct Method

➢Lyapunov analysis of linear time-invariant systems

➢Remarks

⇨ A necessary and sufficient condition for a symmetric matrix M to be p.d. is that all its eigenvalues be strictly positive.

⇨ A p.d. matrix is invertible.

⇨ A p.d. matrix M can always be decomposed as $\mathbf{M} = \mathbf{U}^T\Lambda\mathbf{U}$ *where* $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, Λ *is a diagonal matrix containing the eigenvalues of* $\mathbf{M}$

⇨ *There are some following facts*

- $\lambda_{\min}(\mathbf{M})\|\mathbf{x}\|^2 \le \mathbf{x}^T\mathbf{M}\mathbf{x} \le \lambda_{\max}(\mathbf{M})\|\mathbf{x}\|^2$
- $\mathbf{x}^T\mathbf{M}\mathbf{x} = \mathbf{x}^T\mathbf{U}^T\Lambda\mathbf{U}\mathbf{x} = \mathbf{z}^T\Lambda\mathbf{z}$ *where* $\mathbf{U}\mathbf{x} = \mathbf{z}$
- $\lambda_{\min}(\mathbf{M})\mathbf{I} \le \Lambda \le \lambda_{\max}(\mathbf{M})\mathbf{I}$
- $\mathbf{z}^T\mathbf{z} = \|\mathbf{x}\|^2$

➢System Analysis Based on Lyapunov's Direct Method

➢Lyapunov analysis of linear time-invariant systems

➢Remarks

⇨ Similar notations apply to the concepts of positive semi-definiteness, negative definiteness, and negative semi-definiteness.

➢Lyapunov equation of linear time-invariant systems

Given a linear system of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, let us consider a quadratic Lyapunov function candidate $V = \mathbf{x}^T\mathbf{P}\mathbf{x}$, where $\mathbf{P}$ is a given symmetric positive definite matrix. Its derivative yields

$$\dot{V} = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} \qquad Q>0$$

where

$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$ is so-called Lyapunov equation.

➢System Analysis Based on Lyapunov's Direct Method

➢Lyapunov analysis of linear time-invariant systems

➢A more useful way of studying a given linear system using quadratic functions is to derive a p.d. matrix P from a given p.d. matrix Q, i.e.,

➢1. choose a positive definite matrix Q

➢2. solve for P from the Lyapunov equation

➢3. check whether P is p.d.

➢4. If P is p.d., then 1/2x Px^T is a Lyapunov function for the linear system. And the global asymptotical stability is guaranteed.

➢Theorem: A necessary and sufficient condition for a LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ to be strictly stable is that, for any symmetric p.d. matrix Q, the unique matrix P solution of the Lyapunov equation $\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}=-\mathbf{Q}$ be symmetric positive definite.

➢System Analysis Based on Lyapunov's Direct Method

➢Lyapunov analysis of linear time-invariant systems

➢Ex: Consider a second-order linear system whose A matrix is $\mathbf{A} = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$

Let us take $\mathbf{Q} = \mathbf{I}$ and denote $\mathbf{P}$ by $\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$,

where due to the symmetry of $\mathbf{P}$, $p_{21} = p_{12}$. Then the Lyapunov equation is

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

whose solution is $p_{11} = 5$, $p_{12} = p_{22} = 1$. The corresponding matrix $\mathbf{P} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$ is p.d., and therefore the linear system is globally asymptotically stable.

➢System Analysis Based on Lyapunov's Direct Method

➢Krasovskii's method

➢Let us now come back to the problem of finding Lyapunov functions for general nonlinear systems. Krasovskii's method suggests a simple form of Lyapunov function candidate for autonomous nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

namely, $V = f^Tf$. The basic idea of the method is simply to check whether this particular choice indeed leads to a Lyapunov function.

➢System Analysis Based on Lyapunov's Direct Method

➢Krasovskii's method

➢Theorem (Krasovskii): Consider the autonomous system defined by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with the equilibrium point of interest being the origin. Let A(x) denote the Jacobian matrix of the system, i.e.,

$$\mathbf{A}(x) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

If the matrix $\mathbf{F} = \mathbf{A} + \mathbf{A}^T$ is negative definite in a neighborhood $\mathbf{\Omega}$, then the equilibrium point at the origin is asymptotically stable. A Lyapunov function for this system is

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\mathbf{f}(\mathbf{x})$$

If $\mathbf{\Omega}$ is the entire state space and, in addition, $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, then the equilibrium point is globally asymptotically stable.

➢System Analysis Based on Lyapunov's Direct Method

➢Krasovskii's method

➢Ex: Consider the nonlinear system

$$\dot{x}_1 = -6x_1 + 2x_2$$
$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3$$

We have

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} -6 & 2 \\ 2 & -6-6x_2^2 \end{bmatrix} \qquad \mathbf{F} = \mathbf{A} + \mathbf{A}^T = \begin{bmatrix} -12 & 4 \\ 4 & -12-12x_2^2 \end{bmatrix}$$

The matrix F is easily shown to be negative definite over the whole state space. Therefore, the origin is asymptotically stable, and a Lyapunov function candidate is

$$V(\mathbf{x}) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$$

Since $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, the equilibrium state at the origin is globally asymptotically stable.

➢System Analysis Based on Lyapunov's Direct Method

➢Krasovskii's method

➢The applicability of the above theorem is limited in practice, because the Jcobians of many systems do not satisfy the negative definiteness requirement. In addition, for systems of higher order, it is difficult to check the negative definiteness of the matrix F for all x.

Theorem (Generalized Krasovkii Theorem) *Consider the autonomous system defined by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with the equilibrium point of interest being the origin, and let $\mathbf{A}(\mathbf{x})$ denote the Jacobian matrix of the system. Then a sufficient condition for the origin to be asymptotically stable is that there exist two symmetric positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$, such that $\forall \mathbf{x} \neq 0$, the matrix*

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q}$$

is negative semi-definite in some neighborhood $\mathbf{\Omega}$ of the origin. The function $V(\mathbf{x}) = \mathbf{f}^T\mathbf{P}\mathbf{f}$ is then a Lyapunov function for this system. If the region $\mathbf{\Omega}$ is the whole state space, and if in addition, $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, then the system is globally asymptotically stable.

➢System Analysis Based on Lyapunov's Direct Method

➢The Variable Gradient Method

➢The variable gradient method is a formal approach to constructing Lyapunov functions.

To start with, let us note that a scalar function $V(\mathbf{x})$ is related to its gradient ∇V by the integral relation

$$V(\mathbf{x}) = \int_0^{\mathbf{x}} \nabla V \, d\mathbf{x}$$

where $\nabla V = \{\partial V / \partial x_1, \ldots, \partial V / \partial x_n\}^T$. In order to recover a unique scalar function V from the gradient ∇V , the gradient function has to satisfy the so-called curl conditions

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i} \qquad (i, j = 1,2,\ldots,n)$$

➢System Analysis Based on Lyapunov's Direct Method

➢The Variable Gradient Method

Note that the i^{th} component ∇V_i is simply the directional derivative $\partial V / \partial x_i$. For instance, in the case $n = 2$, the above simply means that

Gradient fields are curl-free: By definition, $\nabla \times (\nabla V) = 0$ (the curl of a gradient is always zero).

For any scalar function V, the identity $\nabla \times (\nabla V) = 0$ always holds because mixed partial derivatives commute:

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}, \quad \text{etc.}$$

This cancellation ensures the curl vanishes.

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

The principle of the variable gradient method is to assume a specific form for the gradient ∇V, instead of assuming a specific form for a Lyapunov function V itself. A simple way is to assume that the gradient function is of the form

$$\nabla V_i = \sum_{j=1}^{n} a_{ij} x_j$$

where the a_{ij} 's are coefficients to be determined. This leads to the following procedure for seeking a Lyapunov function V

➢System Analysis Based on Lyapunov's Direct Method

➢The Variable Gradient Method

⇨ assume that ∇V is given by

$$\nabla V_i = \sum_{j=1}^{n} a_{ij} x_j$$

⇨ solve for the coefficients a_{ij} so as to sastify the curl equations

⇨ restrict the coefficients so that $\dot{V}$ is negative semi-definite (at least locally)

⇨ compute V from ∇V by integration

⇨ check whether V is positive definite

➢System Analysis Based on Lyapunov's Direct Method

➢The Variable Gradient Method

➢ EX: Let us use the variable gradient method to find a Lyapunov function for the nonlinear system

$$\dot{x}_1 = -2x_1$$

$$\dot{x}_2 = -2x_2 + 2x_1 x_2^2$$

We assume that the gradient of the undetermined Lyapunov function has the following form

$$\nabla V_1 = a_{11}x_1 + a_{12}x_2$$
$$\nabla V_2 = a_{21}x_1 + a_{22}x_2$$

The curl equation is

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1} \quad \Rightarrow \quad a_{12} + x_2 \frac{\partial a_{12}}{\partial x_2} = a_{21} + x_1 \frac{\partial a_{21}}{\partial x_1}$$

➢System Analysis Based on Lyapunov's Direct Method

➢The Variable Gradient Method

➢ EX:

If the coefficients are chosen to be $a_{11} = a_{22} = 1, a_{12} = a_{21} = 0$ which leads to $\nabla V_1 = x_1, \nabla V_2 = x_2$ then V can be computed as

$$V(\mathbf{x}) = \int_0^{x_1} x_1\, dx_1 + \int_0^{x_2} x_2\, dx_2 = \frac{x_1^2 + x_2^2}{2}$$

$\boxed{\dot{V} = \nabla V \dot{\mathbf{x}} = -2x_1^2 - 2x_2^2(1 - x_1 x_2)}$ $\boxed{(1 - x_1 x_2) > 0}$

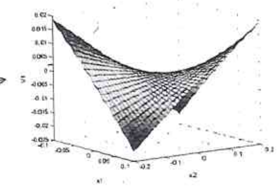
```
%matlab code
[x1,x2] = meshgrid(-0.1:0.01:0.1,-0.2:0.01:0.2);
V1=(x1.*x2-3*x1.^2.*x2.^2);
surf(x1,x2,V1);
```

This is indeed p.d., and therefore, the asymptotic stability is guaranteed.

If the coefficients are chosen to be $a_{11} = 1, a_{12} = x_2^2, \ a_{21} = 3x_2^2, a_{22} = 3$, we obtain the p.d. function

$$V(\mathbf{x}) = \frac{x_1^2}{2} + \frac{3}{2}x_2^2 + x_1 x_2^3$$

whose derivative is $\dot{V} = -2x_1^2 - 6x_2^2 - 2x_2^2\boxed{(x_1 x_2 - 3x_1^2 x_2^2)}$.



We can verify that $\dot{V}$ is a locally negative definite function

➢System Analysis Based on Lyapunov's Direct Method

➢Performance Analysis

➢ Lyapunov analysis can be used to determine the convergence rates of linear and nonlinear systems.

A simple convergence lemma

Lemma: If a real function $W(t)$ satisfies the inequality

$$\dot{W}(t)+\alpha W(t)\le 0$$

$$\boxed{Z(t)=\dot{W}+\alpha W}$$

$$\boxed{W(t)=W(0)e^{-\alpha t}+\int_0^t e^{-\alpha(t-r)}Z(t)dr}$$

where α is a real number. Then $W(t)\le W(0)e^{-\alpha t}$

The above Lemma implies that, if W is a non-negative function, the satisfaction guarantees the exponential convergence of W to zero.

➢System Analysis Based on Lyapunov's Direct Method

➢Performance Analysis

➢ Estimating convergence rates for linear system

➢ Let us denote the largest eigenvalue of the matrix P by $\lambda_{max}(P)$, the smallest eigenvalue of Q by $\lambda_{min}(Q)$, and their ratio $\lambda_{min}(Q)/\lambda_{max}(P)$ by Υ. The positive definiteness of P and Q implies that these scalars are all strictly positive.

Since matrix theory shows that $\mathbf{P}\le\lambda_{max}(\mathbf{P})\mathbf{I}$ and $\lambda_{min}(\mathbf{Q})\mathbf{I}\le\mathbf{Q}$, we have

$$\mathbf{x}^T\mathbf{Q}\mathbf{x}\ge\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}\mathbf{x}^T[\lambda_{max}(\mathbf{P})\mathbf{I}]\mathbf{x}\ge\gamma\mathbf{V}$$

This and $\boxed{V=\mathbf{x}^T\mathbf{P}\mathbf{x}}$

$$\dot{V}=\dot{\mathbf{x}}^T\mathbf{P}\mathbf{x}+\mathbf{x}^T\mathbf{P}\dot{\mathbf{x}}=-\mathbf{x}^T\mathbf{Q}\mathbf{x}$$

implies that $\dot{V}\le-\gamma V$. This means that $\boxed{\mathbf{x}^T\mathbf{P}\mathbf{x}}\le V(0)e^{-\gamma t}$. This together with the fact $\mathbf{x}^T\mathbf{P}\mathbf{x}\ge\lambda_{min}(\mathbf{P})\|\mathbf{x}(t)\|^2$, implies that the state $\mathbf{x}$ converges to the origin with a rate of at least $\gamma/2$.

➢System Analysis Based on Lyapunov's Direct Method

➢Performance Analysis

➢ Estimating convergence rates for nonlinear system

➢ The estimation of convergence rate for nonlinear systems also involves manipulating the expression of $\dot{V}$ so as to obtain an explicit estimate of V. The difference lies in that, for nonlinear systems, V and $\dot{V}$ are not necessarily quadratic functions of the states.

➢ Ex: Consider again the system

$$\dot{x}_1=x_1(x_1^2+x_2^2-2)-4x_1x_2^2$$
$$\dot{x}_2=4x_1^2x_2+x_2(x_1^2+x_2^2-2)$$

$$\dot{V}=2(x_1^2+x_2^2)(x_1^2+x_2^2-2)$$

Choose the Lyapunov function candidate $V=\|\mathbf{x}\|^2$, its derivative is $\dot{V}=2V(V-1)$.

That is $\frac{dV}{V(1-V)}=-2dt$.

➢System Analysis Based on Lyapunov's Direct Method

➢Performance Analysis

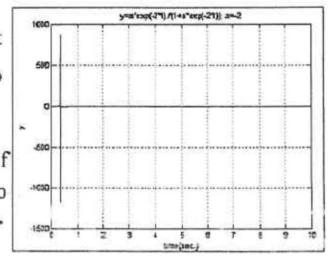
➢ Ex:

The solution of this equation is easily found to be

$$V(\mathbf{x})=\frac{\alpha e^{-2t}}{1+\alpha e^{-2t}},\ \text{where}\ \alpha=\frac{V(0)}{1-V(0)}.$$

⇨ If $\|\mathbf{x}(0)\|^2=V(0)<1$, *i.e.*, if the trajectory starts inside the unit circle, then $\alpha>0$, and $V(t)<\alpha e^{-2t}$. This implies that the norm $\|\mathbf{x}(t)\|$ of the state vector converges to zero exponentially, with a rate of 1.

⇨ However, if the trajectory starts outside the unit circle, *i.e.*, if $V(0)>1$, then $\alpha<0$, so that $V(t)$ and therefore $\|\mathbf{x}\|$ tend to infinity in a finite time.



➢System Analysis Based on Lyapunov's Direct Method

➢Control Design Based on Lyapunov's Direct Method

- ➢There are basically two ways of using Lyapunov's direct method for control design, and both have a trial and error flavor.
- ➢The first technique involves hypothesizing one form of control law and then finding a Lyapunov function to justify the choice.
- ➢The second technique, conversely, requires hypothesizing a Lyapunov function candidate and then finding a control law to make this candidate a real Lyapunov function.

➢Supplementary content

➢Estimating Region of Attraction

- ➢For asymptotically stable system, how far from the origin can the trajectory be and still converges to the origin as $t \to \infty$?
- ➢Let $\phi(t, x)$ be the solution of $\dot{x} = f(x)$ starting at x_0. Then, the Region of Attraction (RoA) is defined as the set of all points x such that $\lim_{t \to \infty} \phi(t, x) = 0$
- ➢Lyapunov function can be used to estimate the RoA
- ➢ If there is a Lyapunov function satisfying asymptotic stability over domain D, and set $\Omega_c = \{x \in R^n \mid V(x) \le c\}$ is bounded and contained in D, all trajectories starting in Ω_c remains there and converges to **0** at $t \to \infty$

➢Supplementary content

➢Estimating Region of Attraction (RoA)

- ➢Sometimes just knowing a system is a.s. is not enough. At least an estimation of RoA is required.

  Ex: Occurring fault and finding "critical clearance time"
- ➢Trajectory starting in D move from one Lyapunov surface to $V(x) = c_1$ to an inner surface $V(x) = c_2$ with $c_2 < c_1$. However, there is no guarantee that the trajectory will remain in D forever. Once, the trajectory leaves D, no guarantee that $\dot{V}$ remains negative. This problem does not occur in R_A since R_A is an invariant set.
- ➢The simplest estimate is given by the set $\Omega_c = \{x \in R_n \mid V(x) \le c\}$

➢Supplementary content

➢Estimating Region of Attraction (RoA)

- ➢To find RoA, first we need to find a domain D in which $\dot{V}$ is n.d. Then, a bounded set $\Omega_c \subset D$ shall be sought. We are interested in largest set Ω_c, i.e. the largest value of c since Ω_c is an estimate of R_A. V is p.d. everywhere in R^2 .
- ➢If $V(x) = x^T Px$, let $D = \{x \in R^2 \mid \|x\| \le r\}$ .Once, D is obtained, then select $\Omega_c \subset D$ by

$$c < \min_{\|x\|=r} V(x)$$

- ➢In words, the smallest $V(x) = c$ which fits into D.
- ➢Since $x^T Px \ge \lambda_{min}(P)\|x\|^2$, we can choose $c < \lambda_{min}(P) r^2$ To enlarge the estimate of $R_A \Rightarrow$ find largest ball on which $\dot{V}$ is n.d.

➢Supplementary content

➢Estimating Region of Attraction (RoA)

➢ Ex: Estimating Region of Attraction for the system

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2\end{aligned}$$

➢ Sol:

From the linearization $\frac{\partial f}{\partial x}|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ is stable

Taking $Q = I$ and solve the Lyap. equation:

$PA + A^T P = -I \implies P = \begin{bmatrix} 1.5 & -.5 \\ -.5 & 1 \end{bmatrix}$

$\lambda_{min}(P) = 0.69$

➢Supplementary content

➢Estimating Region of Attraction (RoA)

➢ Ex: Estimating Region of Attraction for the system

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2\end{aligned}$$

➢ Sol:

$\dot{V} = -(x_1^2 + x_2^2) - (x_1^3 x_2 - 2x_1^2 x_2^2) \leq -\|x\|_2^2 + |x_1||x_1 x_2||x_1 - 2x_2| \leq -\|x\|_2^2 + \frac{\sqrt{5}}{2}\|x\|_2^4$

- where $|x_1| \leq \|x\|_2$, $|x_1 x_2| \leq \|x\|_2^2/2$, $|x1 - 2x_2| \leq \sqrt{5}\|x\|_2$

$\dot{V}$ is n.d. on a ball D of radius

$r^2 = 2/\sqrt{5} = 0.894$

$c < 0.894 \times 0.69 = \boxed{0.617}$

$\Omega_c = \{x \in R^2 | \ V(x) \leq 0.6 \}$ is an estimate of R_A.

➢Supplementary content

➢Estimating Region of Attraction (RoA)

➢ Ex: Estimating Region of Attraction for the system

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➢ Sol:

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Taking $Q = I$ and solve the Lyap. equation:

$PA + A^T P = -I \implies P = \begin{bmatrix} 1.5 & -.5 \\ -.5 & 1 \end{bmatrix}$

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➢Supplementary content

➢Estimating Region of Attraction (RoA)

➢ Ex: Estimating Region of Attraction for the system

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2\end{aligned}$$

➢ Sol:

Let $x_1 = \rho\ cos\theta, \quad x_2 = \rho\ sin\theta$

$$\begin{aligned}\dot{V} &= -\rho^2 + \rho^4 cos^2\theta sin\theta\,(2sin\theta - cos\theta) \\ &\leq -\rho^2 + \rho^4 \left|cos^2\theta sin\theta\right| |2sin\theta - cos\theta| \\ &\leq -\rho^2 + \rho^4(.3849)(2.2361) \\ &\leq -\rho^2 + .861\rho^4 < 0 \quad \text{for } \rho^2 < \frac{1}{.861}\end{aligned}$$

$\boxed{c = .8 < \frac{.69}{.861} = .801}$

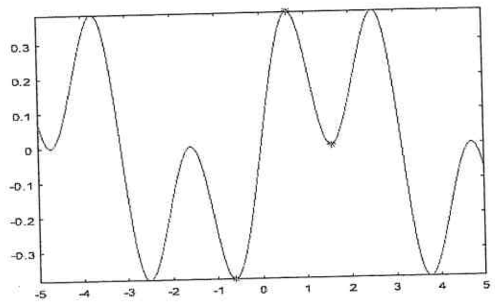
Thus the set:

$\Omega_c = \{x \in R^2 | \ V(x) \leq .8\}$ is an estimate of R_A.

➢Supplementary content

➢Estimating Region of Attraction (RoA)

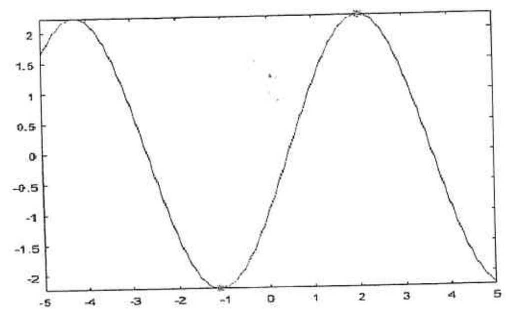
```
syms x
f = ((cos(x))^2*sin(x))
g = diff(f, x)
s=solve(g == 0, x, 'MaxDegree',4)
value=vpa(s,4)
vpa(subs(f,value),4)
figure(1)
fplot(f)
hold on
plot(value, subs(f,value), '*')
hold off
```



➢Supplementary content

➢Estimating Region of Attraction (RoA)

```
syms x
f = (2*sin(x)-cos(x))
g = diff(f, x)
s=solve(g == 0, x, 'MaxDegree',4)
vpa(subs(f,vpa(s,4)),4)
value=vpa(s,4)
vpa(subs(f,value),4)
figure(2)
fplot(f)
hold on
plot(value, subs(f,value), '*')
hold off
```



➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

$$\dot{x}_1 = -2x_1 + x_1x_2$$
$$\dot{x}_2 = -x_2 + x_1x_2$$

➢Sol:

There are two Equ. pts., $(0,0)$, $(1,2)$.

At $(1,2)$ $A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

$\Longrightarrow$ unstable $(\lambda_{1,2} = \pm\sqrt{2})$ (saddle pt.)

At $(0,0)$ $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow$ **a.s.**

➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

$$\dot{x}_1 = -2x_1 + x_1x_2$$
$$\dot{x}_2 = -x_2 + x_1x_2$$

➢Sol:

There are two Equ. pts., $(0,0)$, $(1,2)$.

At $(1,2)$ $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$\Longrightarrow$ unstable $(\lambda_{1,2} = \pm\sqrt{2})$ (saddle pt.)

At $(0,0)$ $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow$ **a.s.**

➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

```
clear
t1=0:0.1:10;
x0=[1.5 1.447];
normx=sqrt(x0*x0');
n=length(t1);
for i=1:n-1
[t x]=ode23('plant', [t1(i) t1(i+1)],x0);
y(i,:)=x0;
x0=x(length(x(:,1)),:)';
end
figure(1);
plot(t1(1:n-1),y(:,1),'r',t1(1:n-1),y(:,2),'g');
title(['Trajectory of States r=',num2str(normx)]);
xlabel('Time(secs)');
ylabel('Amplitude');
grid;
```

```
function dx=plant(t,x)
%global u
dx=zeros(2,1);
dx(1)=-2*x(1)+x(1)*x(2);
dx(2)=-1*x(2)+x(1)*x(2);
```

$$a^2+b^2 \geq \frac{1}{2}(a+b)^2$$

$$a^2+b^2 \geq 2ab$$

$$ab \leq \frac{a^2+b^2}{2}$$

$$ab \leq \left(\frac{a+b}{2}\right)^2$$

$$a+b \geq 2\sqrt{ab}$$

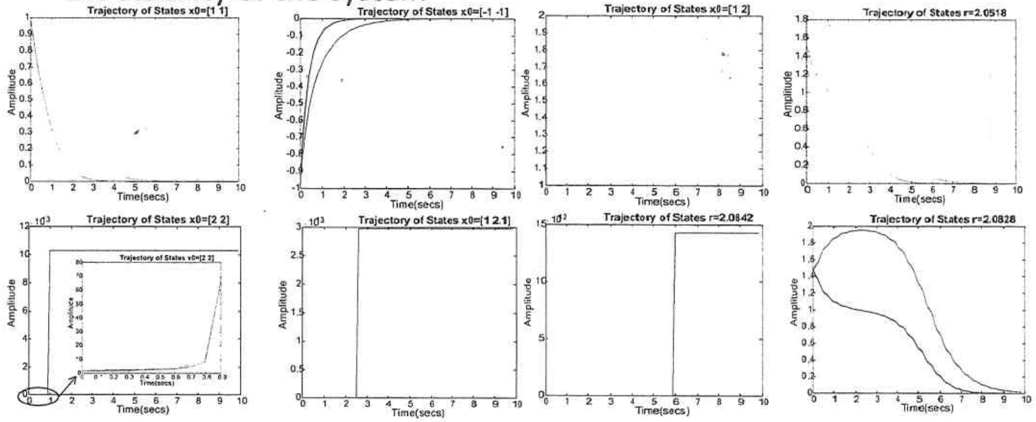
$$a+b \leq \sqrt{2\left(a^2+b^2\right)}$$

$$|\langle \mathbf{u},\mathbf{v}\rangle|^2 \leq \langle \mathbf{u},\mathbf{u}\rangle \cdot \langle \mathbf{v},\mathbf{v}\rangle$$

u, v : vectors

➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system



➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

$$\dot{x}_1 = -2x_1 + x_1x_2$$
$$\dot{x}_2 = -x_2 + x_1x_2$$

➢Sol:

Taking $Q = I$ and solving Lyap Eq. $A^TP + PA = -I \implies P = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

The Lyap. fcn is $V(x) = x^TPx$

We have $\dot{V} = -(x_1^2 + x_2^2) + \frac{1}{2}(x_1^2x_2 + 2x_1x_2^2)$

method 1:

➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

$$\dot{x}_1 = -2x_1 + x_1x_2$$
$$\dot{x}_2 = -x_2 + x_1x_2$$

➢Sol: $\dot{V} = -(x_1^2 + x_2^2) + \frac{1}{2}(x_1^2x_2 + 2x_1x_2^2)$

➢Because of $|x_1x_2| \leq \|x\|_2^2/2$, $\quad |\frac{1}{2}x_1 + x_2| \leq \frac{\sqrt{5}}{2}\|x\|_2$

$$\begin{aligned}\dot{V} &\leq -\|x\|_2^2 + \frac{\sqrt{5}}{4}\|x\|_2^3 \\ &\leq -\|x\|_2^2\,(1 - \frac{\sqrt{5}}{4}\|x\|_2) \end{aligned} \implies (1 - \frac{\sqrt{5}}{4}\|x\|_2) > 0$$

Since $\lambda_{min}(P) = \frac{1}{4} \implies$, we choose

$$c = .79 < \frac{1}{4} \times \left(\frac{4}{\sqrt{5}}\right)^2 = .8$$

➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

$$\dot{x}_1 = -2x_1 + x_1x_2$$
$$\dot{x}_2 = -x_2 + x_1x_2$$

➢Sol: method 2:

Let $x_1 = \rho\ cos\theta, \quad x_2 = \rho\ sin\theta$

$$\begin{aligned}\dot{V} &= -\rho^2 + \rho^3 cos\theta sin\theta \left(sin\theta + \frac{1}{2}cos\theta \right) \\ &\leq -\rho^2 + \frac{1}{2}\rho^3 \left|sin2\theta\right| \left| sin\theta + \frac{1}{2}cos\theta \right| \\ &\leq -\rho^2 + \frac{\sqrt{5}}{4}\rho^3 < 0 \quad \text{for } \rho < \frac{4}{\sqrt{5}}\end{aligned}$$

➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

$$\dot{x}_1 = -2x_1 + x_1x_2$$
$$\dot{x}_2 = -x_2 + x_1x_2$$

➢Sol:

Since $\lambda_{min}(P) = \frac{1}{4} \Longrightarrow$, we choose

$$c = .79 < \frac{1}{4} \times \left(\frac{4}{\sqrt{5}}\right)^2 = .8$$

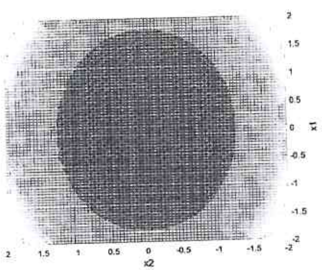
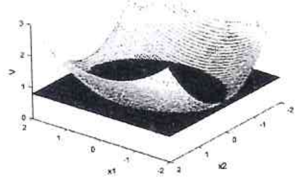
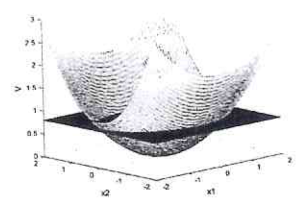
Thus the set:

$$\Omega_c = \{x \in R^2 |\ V(x)\ \leq\ .79\}\ \subset\ R_A.$$

➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

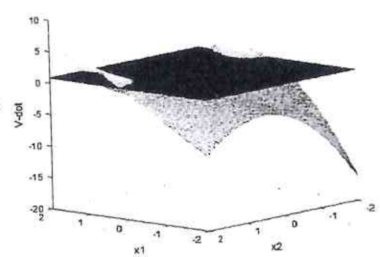
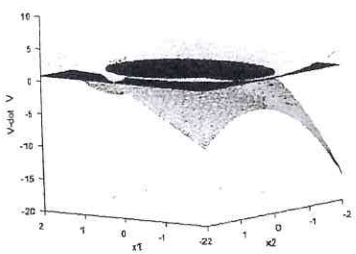
```
x1=-2:0.05:2;
x2=-2:0.05:2;
[xx, yy] = meshgrid(x1, x2);
v=(xx.^2)./4+(yy.^2)./2
vc=ones(length(v(:,1)),length(v(1,:)))*0.8
mesh(xx, yy, v);
hold on
surf(xx, yy,vc);
xlabel('x1')
ylabel('x2')
zlabel('V')
```

➢Supplementary content

➢Example: Consider the following second-order system, and discuss the stability of the system

```
x1=-2:0.05:2;
x2=-2:0.05:2;
[xx, yy] = meshgrid(x1, x2);
v=(xx.^2)./4+(yy.^2)./2
dV=-
(xx.^2+yy^2)+(yy.*xx.^2)./2+xx.*yy.^2
vc=ones(length(v(:,1)),length(v(1,:)))*0.8
mesh(xx, yy, v);
%hold on
mesh(xx, yy, dV);
hold on
surf(xx, yy,vc);
xlabel('x1')
ylabel('x2')
zlabel('V-dot V')
```

➢Advanced Stability Theory

➢It is to present stability analysis for non-autonomous systems. In many practical problems, however, we encounter non-autonomous systems. For instance, a rocket taking off is a non-autonomous system, because the parameters involved in its dynamic equations, such as air temperature and pressure, vary with time.

➢The concepts of stability for non-autonomous systems are quite similar to those of autonomous systems. However, due to the dependence of non-autonomous system behavior on initial time t_0, the definitions of these stability concepts include t_0 explicitly.

➢Concepts of Stability for Non-Autonomous Systems

For non-autonomous systems, of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$

equilibrium points $\mathbf{x}^*$ are defined by $\mathbf{f}(\mathbf{x}^*, t) \equiv 0 \quad \forall t \geq t_0$

Note that this equation must be satisfied $\forall t \geq t_0$, implying that the system should be able to stay at the point $\mathbf{x}^*$ all the time.

➢Advanced Stability Theory

➢Concepts of Stability for Non-Autonomous Systems

➢ Ex:

For instance, we can easily see that the linear time-varying system $\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x}$ has a unique equilibrium point at the origin $\mathbf{0}$ unless $\mathbf{A(t)}$ is always singular.

➢ Ex:

The system $\dot{x} = -\dfrac{a(t)x}{1+x^2}$ has an equilibrium point at $x = 0$.

However, the system $\dot{x} = -\dfrac{a(t)x}{1+x^2} + b(t)$ with $b(t) \neq 0$, does not have an equilibrium point.

It can be regarded as a system under external input or disturbance $b(t)$.

➢Advanced Stability Theory

➢Concepts of stability for non-autonomous systems

➢The definition of invariant set is the same for non-autonomous systems as for autonomous systems. Note that, unlike in autonomous systems, a system trajectory is generally not an invariant set for a non-autonomous system.

➢Extensions of the previous stability concepts

➢Extend the previously defined concepts of stability, instability, asymptotic stability, and exponential stability to non-autonomous systems. The key in doing so is to properly include the initial time t_0 in the definitions.

➢Advanced Stability Theory

➢Extensions of the previous stability concepts

➢The concept of stability can also be defined for non-autonomous systems

Definition *The equilibrium point* $\mathbf{0}$ *is* *stable* *at* t_0 *if for any* $R > 0$, *there exists a positive scalar* $r(R, t_0)$ such that

$$\|\mathbf{x}(t_0)\| < r \quad \Rightarrow \quad \|\mathbf{x}(t)\| < R \quad \forall t \geq t_0$$

Otherwise, the equilibrium point $\mathbf{0}$ *is unstable.*

➢The definition means that we can keep the state in a ball of arbitrarily small radius R by starting the state trajectory in a ball of sufficiently small radius r

➢Advanced Stability Theory

➢Extensions of the previous stability concepts

➢The concept of asymptotic stability can also be defined for non-autonomous systems

Definition *The equilibrium point* $\mathbf{0}$ *is* ***asymptotically stable*** *at* t_0 *if*

- *it is stable*
- $\exists r(t_0) > 0$ *such that* $\|\mathbf{x}(t_0)\| < r(t_0) \Rightarrow \|\mathbf{x}(t)\| \to \mathbf{0}$ as $t \to \infty$

The asymptotic stability requires that there exists an attractive region for every initial time t_0.

➢Advanced Stability Theory

➢Extensions of the previous stability concepts

➢The concept of exponentially stability can also be defined for non-autonomous systems

Definition *The equilibrium point* $\mathbf{0}$ *is* ***exponentially stable*** *if there exist two positive numbers,* α *and* λ, *such that for sufficiently small* $\mathbf{x}(t_0)$,

$$\|\mathbf{x}(t)\| \le \alpha \|\mathbf{x}_0\| e^{-\lambda(t-t_0)} \quad \forall t \ge t_0$$

➢The concept of globally asymptotical stability can also be defined for non-autonomous systems

Definition *The equilibrium point* $\mathbf{0}$ ***is global stable*** $\forall \mathbf{x}(t_0)$, $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$

➢Advanced Stability Theory

➢Extensions of the previous stability concepts

➢Ex:

Consider the first-order system $\dot{x}(t) = -a(t)x(t)$. Its solution is

$x(t) = x(t_0) e^{-\int_{t_0}^{t} a(r)dr}$. Thus system is stable if $a(t) \ge 0, \forall t \ge t_0$. It is asymptotically stable if $\int_0^{\infty} a(r)dr = +\infty$.

It is exponentially stable if there exists a strictly positive number T such that $\forall t \ge 0$, $\int_t^{t+T} a(r)dr \ge \gamma$, with γ being a positive constant.

➢Advanced Stability Theory

➢Extensions of the previous stability concepts

➢Ex:

- The system $\dot{x} = -x/(1+t)^2$ is stable (but not asymptotically stable)
- The system $\dot{x} = -x/(1+t)$ is asymptotically stable
- The system $\dot{x} = -t\,x$ is exponentially stable
- Another interesting example is the system $\dot{x}(t) = -\dfrac{x}{1+\sin x^2}$

The solution can be expressed as $x(t) = x(t_0) e^{-\int_{t_0}^{t} \frac{x}{1+\sin x^2(r)} dr}$

Since $\int_{t_0}^{t} \dfrac{x}{1+\sin x^2(r)} dr \ge \dfrac{t-t_0}{2}$, the system is exponentially convergent with rate $1/2$.

➢Advanced Stability Theory

➢Uniformity in stability concepts

- ➢The previous concepts of Lyapunov stability and asymptotic stability for non-autonomous systems both indicate the important effect of initial time.
- ➢In practice, it is usually desirable for the systems to have a certain uniformity in its behavior regardless of when the operation starts.
- ➢It is also useful to point out that, because the behavior of autonomous systems is independent of the initial time, all the stability properties of an autonomous system are uniform.

➢Advanced Stability Theory

➢Uniformity in stability concepts

Definition *The equilibrium point* $\mathbf{0}$ *is locally uniformly stable if the scalar* r *can be chosen independent of* t_0, *i.e., if* $r = r(R)$.

$$\boxed{r(R,t_0)}$$

Definition *The equilibrium point at the origin is locally uniformly asymptotically stable if*

- *it is uniformly stable*
- *there exits a ball of attraction* $\mathbf{B}_{R_0}$, *whose radius is independent of* t_0, *such that any trajectory with initial states in* $\mathbf{B}_{R_0}$ *converges to* $\mathbf{0}$ *uniformly in* t_0.

➢Advanced Stability Theory

➢Uniformity in stability concepts

By uniform convergence in terms of t_0, *we mean that for all* R_1 *and* R_2 *satisfying* $0 < R_2 < R_1 \le R_0$, $\exists T(R_1, R_2) > 0$ *such that,* $\forall t \ge t_0$

$$\|\mathbf{x}(t_0)\| < R_1 \quad \Rightarrow \quad \|\mathbf{x}(t)\| < R_2 \quad \forall t \ge t_0 + T(R_1, R_2)$$

i.e., the trajectory, starting from within a ball $\mathbf{B}_{R_1}$, will converges into a smaller ball $\mathbf{B}_{R_2}$ after a time period T which is independent of t_0.

➢Ex:

Consider the first-order system $\dot{x} = -\frac{x}{1+t}$. This system has general solution $x(t) = \frac{1+t_0}{1+t}x(t_0)$.

The solution asymptotically converges to zero. But the convergence is not uniform.

Intuitively, this is because a larger t_0 requires a longer time to get close to the origin.

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov's Direct Method for Non-Autonomous Systems

- ➢The basic idea of the direct method, i.e., concluding the stability of nonlinear systems using scalar Lyapunov functions, can be similarly applied to non-autonomous systems. Besides more mathematical complexity, a major difference in non-autonomous systems is that the powerful LaSalle's theorems do not apply. This drawback will partially be compensated by a simple result called Barbalat's lemma.

➢Time-varying positive definite functions

Definition *A scalar time-varying function* $V(\mathbf{x},t)$ *is locally positive definite if* $V(\mathbf{0},t) = 0$ *and there exits a time-invariant positive definite function* $V_0(\mathbf{x})$ *such that*

$$\forall t \ge t_0, \quad V(\mathbf{x},t) \ge V_0(\mathbf{x})$$

Thus, a time-variant function is locally positive definite if it *dominates a time-variant* locally positive definite function. Globally positive definite functions can be defined similarly.

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov's Direct Method for Non-Autonomous Systems

- ➢ Time-varying decrescent functions

Definition *A scalar time-varying function $V(\mathbf{x},t)$ is said to be __decrescent__ if $V(\mathbf{0},t)=0$ and if there exits a $\boxed{\text{time-invariant}}$ positive definite function $V_1(\mathbf{x})$ such that*

$$\forall t \geq 0, \qquad V_1(\mathbf{x}) \geq V(\mathbf{x},t)$$

In other word, a scalar function $V(\mathbf{x},t)$ *is __decrescent__ if it __is dominated by a time-invariant p.d. function__.*

- ➢ Ex: A simple example of a time-varying positive definite function is

$$V(\mathbf{x},t) = (1+\sin^2 t)(x_1^2+x_2^2) \qquad V_0(\mathbf{x}) = x_1^2+x_2^2 \qquad V_1(\mathbf{x}) = 2(x_1^2+x_2^2)$$

This function is also decrescent

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov's Direct Method for Non-Autonomous Systems

- ➢ Given a time-varying scalar function $V(x,t)$, its derivative along a system trajectory is

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}\dot{\mathbf{x}} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} f(\mathbf{x},t)$$

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov's Direct Method for Non-Autonomous Systems

- ➢ Lyapunov theorem for non-autonomous system stability

Theorem (Lyapunov theorem for non-autonomous systems)

Stability: *If, in a ball $\mathbf{B}_{R_0}$ around the equilibrium point $\mathbf{0}$, there exits a scalar function $V(\mathbf{x},t)$ with continuous partial derivatives such that*

1. V is positive definite

2. $\dot{V}$ is negative semi-definite

then the equilibrium point $\mathbf{0}$ is stable in the sense of Lyapunov.

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov's Direct Method for Non-Autonomous Systems

- ➢ Lyapunov theorem for non-autonomous system stability

Uniform stability and uniform asymptotic stability: *If, furthermore*

3. V is decrescent

then the origin is uniformly stable. If the condition 2 is strengthened by requiring that $\dot{V}$ be negative definite, then the equilibrium point is asymptotically stable.

➢ Advanced Stability Theory

➢ Lyapunov Analysis of Non-Autonomous Systems

➢ Lyapunov's Direct Method for Non-Autonomous Systems

➢ Lyapunov theorem for non-autonomous system stability

Global uniform asymptotic stability: *If, the ball $\mathbf{B}_{R_0}$ is replaced by the whole state space, and condition 1, the strengthened condition 2, condition 3, and the condition*

4. $V(\mathbf{x},t)$ is radially unbounded

are all satisfied, then the equilibrium point at $\mathbf{0}$ is globally uniformly asymptotically stable.

➢ Similarly to the case of autonomous systems, if, in a certain neighborhood of the equilibrium point, V is positive definite and $\dot{V}$, its derivative along the system trajectories, is negative semi-definite, then V is called a Lyapunov function for the non-autonomous system.

➢ Advanced Stability Theory

➢ Lyapunov Analysis of Non-Autonomous Systems

➢ Lyapunov's Direct Method for Non-Autonomous Systems

➢ Ex: Consider the system defined by

$$\dot{x}_1(t) = -x_1(t) - e^{-2t}x_2(t)$$
$$\dot{x}_2(t) = x_1(t) - x_2(t)$$

,Chose the Lyapunov function candidate

$$V(\mathbf{x},t) = x_1^2 + (1+e^{-2t})x_2^2$$

This function is p.d., because it dominates the time-invariant p.d. function $x_1^2 + x_2^2$. It is also decrescent, because it is dominated by the time-invariant p.d. function $x_1^2 + 2x_2^2$.

➢ Advanced Stability Theory

➢ Lyapunov Analysis of Non-Autonomous Systems

➢ Lyapunov's Direct Method for Non-Autonomous Systems

➢ Ex:

Furthermore,

$$\dot{V}(\mathbf{x},t) = -2[x_1^2 - x_1x_2 + x_2^2(1+e^{-2t})]$$

This shows that

$$\dot{V}(\mathbf{x},t) \le -2(x_1^2 - x_1x_2 + x_2^2) = -(x_1 - x_2)^2 - x_1^2 - x_2^2$$

Thus, $\dot{V}(\mathbf{x},t)$ is negative definite, and therefore, the point $\mathbf{0}$ is globally asymptotically stable.

➢ Advanced Stability Theory

➢ Lyapunov Analysis of Non-Autonomous Systems

➢ Lyapunov analysis of linear time-varying systems

➢ Consider linear time-varying systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$$

➢ Since LTI systems are asymptotically stable if their eigenvalues all have negative real parts ⇒ Will the system be stable if any time t ≥ 0 , the eigenvalues of A(t) all have negative parts?(This conjecture is not true)

➢ Ex Consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & e^{2t}\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Both eigenvalues of $\mathbf{A}(t)$ equal to -1 at all times. The solution can be rewritten as $x_2 = x_2(0)e^{-t}$, $\dot{x}_1 + x_1 = \boxed{x_2(0)e^{t}}$ Hence, the system is unstable.

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov analysis of linear time-varying systems

➢ A simple result, however, is that the time-varying system is asymptotically stable if the eigenvalues of the symmetric matrix $\mathbf{A}(t)+\mathbf{A}^T(t)$ (all of which are real) remain strictly in the left-half complex plane

$$\exists\lambda>0, \forall i, \forall t\geq 0, \lambda_i(\mathbf{A}(t)+\mathbf{A}^T(t))\leq-\lambda$$

This can be readily shown using the Lyapunov function $V=\mathbf{x}^T\mathbf{x}$, since

$$\dot{V}=\mathbf{x}^T\dot{\mathbf{x}}+\dot{\mathbf{x}}^T\mathbf{x}=\mathbf{x}^T(\mathbf{A}(t)+\mathbf{A}^T(t))\leq-\lambda\mathbf{x}^T\dot{\mathbf{x}}=-\lambda V$$

so that $\forall t\geq 0, 0\leq\mathbf{x}^T\mathbf{x}=V(t)\leq V(0)e^{-\lambda t}$ and therefore $\mathbf{x}$ tends to zero exponentially.

It is important to notice that the result provides a sufficient condition for any asymptotic stability.

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov-Like Analysis Using Barbalat's Lemma

➢ For autonomous systems, the invariant set theorems are powerful tools to study stability, because they allow asymptotic stability conclusions to be drawn even when $\dot{V}$ is only negative semi-definite. However, the invariant set theorems are not applicable to non-autonomous systems. Therefore, asymptotic stability analysis of non-autonomous systems is generally much harder than that of autonomous systems, since it is usually very difficult .to find Lyapunov functions with a negative definite derivative. An important and simple result which partially remedies this situation is Barbalat's lemma. Barbalat's lemma is a purely mathematical result concerning the asymptotic properties of functions and their derivatives. When properly used for dynamic systems, particularly non-autonomous systems, it may lead to the satisfactory solution of many asymptotic stability problems.

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov-Like Analysis Using Barbalat's Lemma

➢ Before discussing Barbalat's lemma itself, let us clarify a few points concerning the asymptotic properties of functions and their derivatives. Given a differentiable function f of time t , the following three facts are important to keep in mind

- $\dot{f}\to 0 \not\Rightarrow f$ converges

  The fact that $\dot{f}\to 0$ does not imply that $f(t)$ has a limit as $t\to\infty$.

- f converges $\not\Rightarrow \dot{f}\to 0$

  The fact that $f(t)$ has a limit as $t\to\infty$ does not imply that $\dot{f}\to 0$.

- If f is lower bounded and decreasing ($\dot{f}\leq 0$), then it converges to a limit.

$$\dot{f}(t)=\frac{\cos(\log t)}{t}\to 0 \text{ as } t\to\infty$$

$$\not\Rightarrow f(t)=\sin(\log t)$$

$$f(t)=e^{-t}\sin(e^{2t})$$

$$\not\Rightarrow \dot{f}(t)=-e^{-t}\sin(e^{2t})+2e^{t}\cos(e^{2t})$$

➢Advanced Stability Theory

➢Lyapunov Analysis of Non-Autonomous Systems

➢Lyapunov-Like Analysis Using Barbalat's Lemma

➢ **Lemma 4.2 (Barbalat)** *If the differentiable function $f(t)$ has a finite limit as $t\to\infty$, and if $\dot{f}$ is uniformly continuous, then $\dot{f}(t)\to 0$ as $t\to\infty$.*

➢Before proving this result, let us define what we mean by uniform continuity. Recall that a function g(t) is continuous on $[0,\infty)$ if

$$\forall t_1\geq 0, \forall R>0, \exists\eta(R,t_1)>0, \forall t\geq 0, \quad |t-t_1|<\eta \Rightarrow |g(t)-g(t_1)|<R$$

➢A function g is said to be uniformly continuous on $[0, \infty)$ if

$$\forall R>0, \exists\eta(R)>0, \forall t_1\geq 0, \forall t\geq 0, \quad |t-t_1|<\eta \Rightarrow |g(t)-g(t_1)|<R$$

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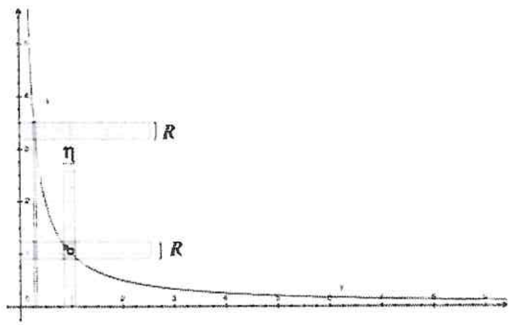
- ➢ Ex: Not uniformly continuous

$$\forall R>0, \exists \eta(R)>0, \forall t_1 \geq 0, \forall t \geq 0, \quad |t-t_1|<\eta \quad \Rightarrow \quad |g(t)-g(t_1)|<R$$



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- ➢ A very simple **sufficient condition** for a differentiable function to be uniformly continuous is that its derivative be bounded. This can be easily seen from the finite difference theorem

$$\forall t, \forall t_1, \quad \exists t_2 \text{ (between } t \text{ and } t_1\text{)} \quad \text{such that} \quad g(t)-g(t_1)=\dot{g}(t_2)(t-t_1)$$

and therefore, if $R_1>0$ is an upper bound on the function $|\dot{g}|$, one can always use $\eta = R/R_1$ independently of t_1 to verify the definition of uniform continuity.

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- ➢ Consider a strictly stable linear system whose input is bounded. Then the system output is uniformly continuous. Indeed, write the system in the standard form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

- ➢Since u is bounded and the linear system is strictly stable, thus the state x is bounded. This in turn implies from the first equation that $\dot{\mathbf{x}}$ is bounded, and therefore from the second equation that $\dot{\mathbf{y}} = \mathbf{C}\dot{\mathbf{x}}$ is bounded. Thus the system output y is uniformly continuous.

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- ➢ To apply Barbalat's lemma to the analysis of dynamic systems, one typically uses the following immediate corollary, which looks very much like an invariant set theorem in Lyapunov analysis:

Lemma (Lyapunov-Like Lemma) *If a scalar function $V(\mathbf{x},t)$ satisfies the following conditions*

- *$V(\mathbf{x},t)$ is lower bounded*
- *$\dot{V}(\mathbf{x},t)$ is negative semi-definite*
- *$\dot{V}(\mathbf{x},t)$ is uniformly continuous in time*

then $\dot{V}(\mathbf{x},t) \rightarrow 0$ as $t \rightarrow \infty$.

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➢Ex: Consider the closed-loop error dynamics of an adaptive control system for a first-order plant with unknown parameter

$$\dot{e} = -e + \theta w(t)$$
$$\dot{\theta} = -e w(t)$$

where e and θ are the two states of the closed-loop dynamics, representing tracking error and parameter error, and $w(t)$ is a bounded continuous function.
Consider the lower bounded function $V = e^2 + \theta^2$

Its derivative is

$$\dot{V} = 2e[-e + \theta w(t)] + 2\theta[-e w(t)] = -2e^2 \le 0$$

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➢Ex: This implies that $V(t) \le V(0)$, and therefore, that e and θ are bounded. But the invariant set cannot be used to conclude the convergence of e, because the dynamics is non-autonomous.
To use Barbalat's lemma, let us check the uniform continuity of $\dot{V}$. The derivative of $\dot{V}$ is $\ddot{V} = -4e(-e + \theta w)$. This shows that $\ddot{V}$ is bounded, since w is bounded by hypothesis, and e and θ were shown above to be bounded. Hence, $\dot{V}$ is uniformly continuous. Application of Babarlat's lemma then indicates that $e \to 0$ as $t \to \infty$.
Note that, although e converges to zero, the system is not asymptotically stable, because θ is only guaranteed to be bounded.

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➢Such above analysis based on Barbalat's lemma shall be called a Lyapunov-like analysis. There are two important differences with Lyapunov analysis:

- *The function V can simply be a lower bounded function of $\mathbf{x}$ and t instead of a positive definite function.*
- *The derivative $\dot{V}$ must be shown to be uniformly continuous, in addition to being negative or zero. This is typically done by proving that $\ddot{V}$ is bounded.*

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➢Lyapunov-Like Analysis Using Barbalat's Lemma

➢Lyapunov analysis can be used to show boundedness of the solution even when there is no equilibrium point:

➢Ex:

$$\dot{x} = -x + \delta \, sint, \quad x(t_0) = a, \quad a > \delta > 0$$

➢Sol:

The same properties can be obtained via Lyap. analysis. Let $V = x^2/2$

$$\dot{V} = x\dot{x} = -x^2 + x\delta \, sint \le -x^2 + \delta|x|$$

$\dot{V}$ is not **n.d.**, near the origin, positive linear term $\delta|x|$ is dominant.
However, $\dot{V}$ is negative outside the set $\{|x| \le \delta\}$.
Choose, $c > \delta^2/2$, solutions starting in the set $\{V(x) < c\}$ will remain there in for all future time since $\dot{V}$ is negative on the boundary $V = c$.
Hence, the solution is **ultimately bounded**.