大数据科学与应用技术

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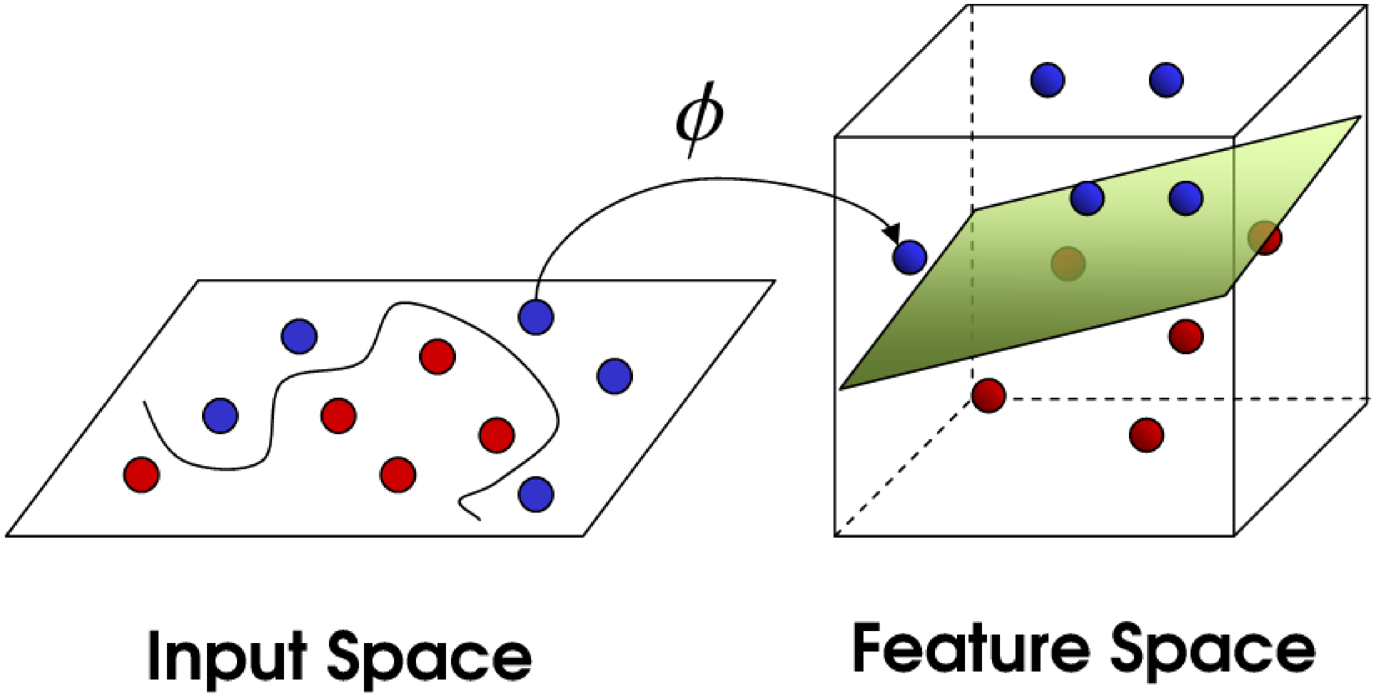
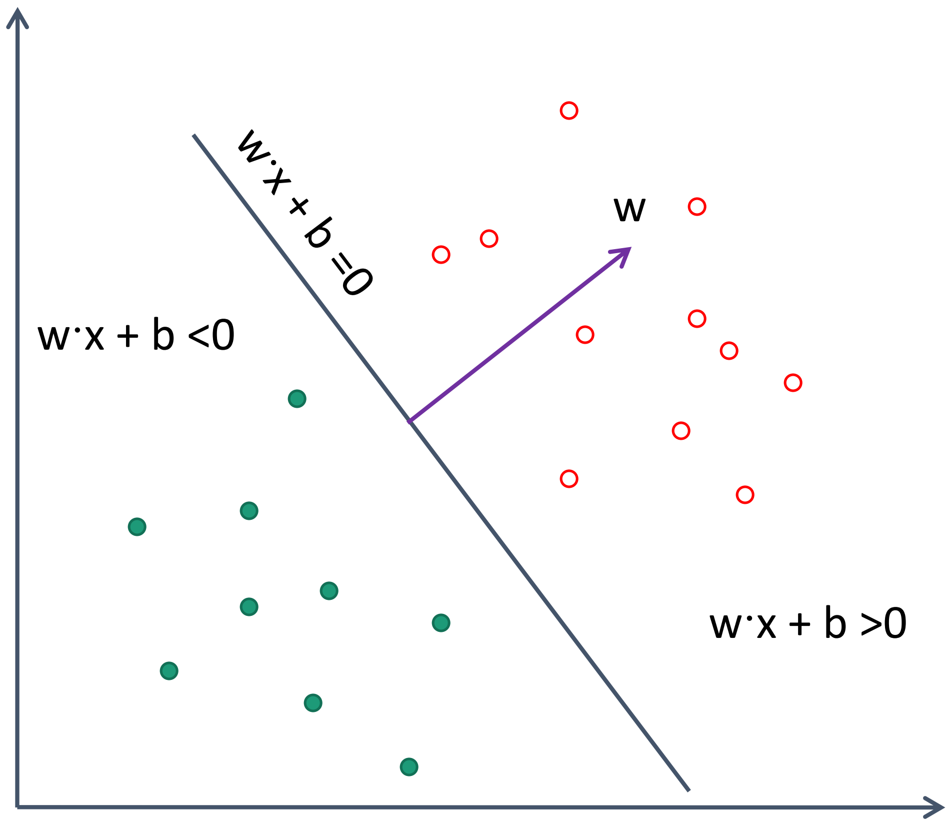
Overview



Linear Classifier



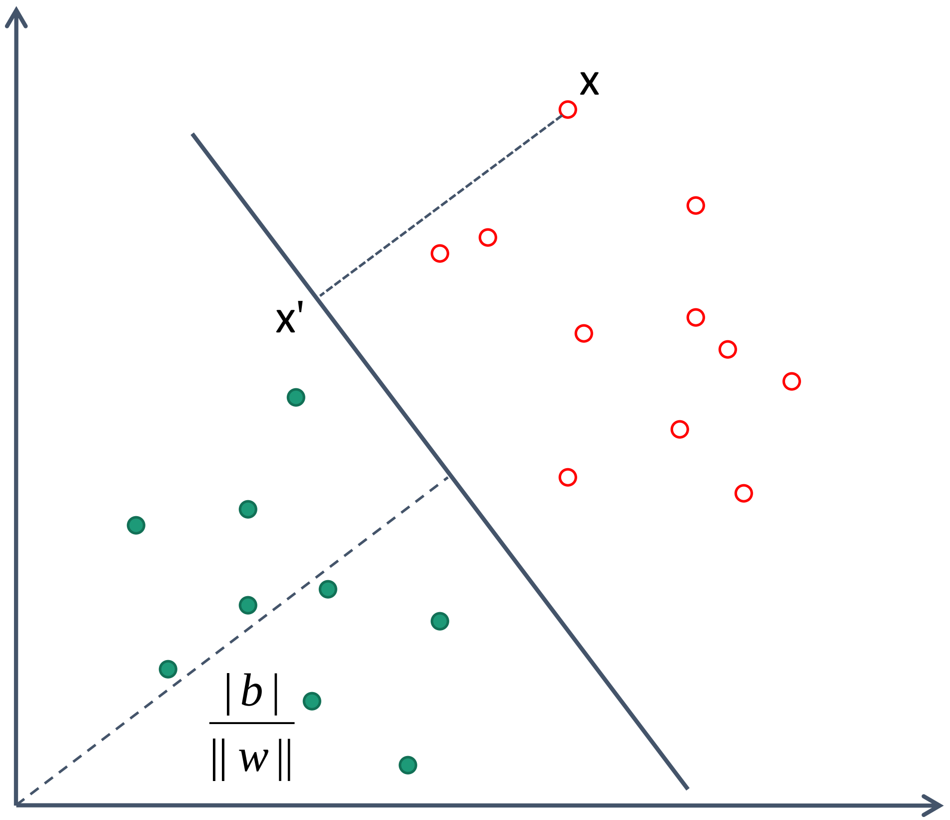
$$f(x, w, b) = sign(g(x))$$
$$= sign(w \cdot x + b)$$

Just in case ...

$$w \cdot x = \sum_{i=1}^{n} w_i x_i$$

$$w \cdot x_1 + b = w \cdot x_2 + b$$
$$w(x_1 - x_2) = 0$$

Distance to Hyperplane



$$g(x) = w \cdot x + b$$

$$x = x' + \lambda w$$

$$g(x) = w(x' + \lambda \cdot w) + b$$

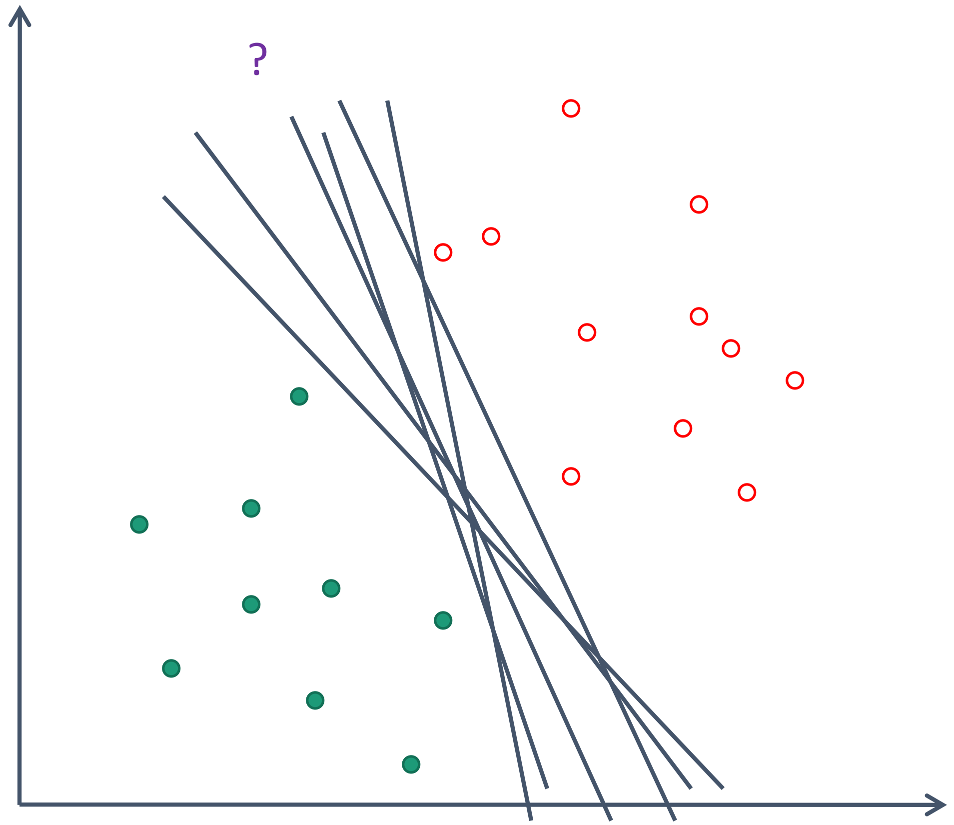
$$= w \cdot x' + b + \lambda w \cdot w$$

$$= \lambda w \cdot w$$

$$M = \| x - x' \| = \| \lambda w \|$$

$$= \frac{|g(x)| \times \| w \|}{w \cdot w} = \frac{|g(x)|}{\| w \|}$$

Selection of Classifiers

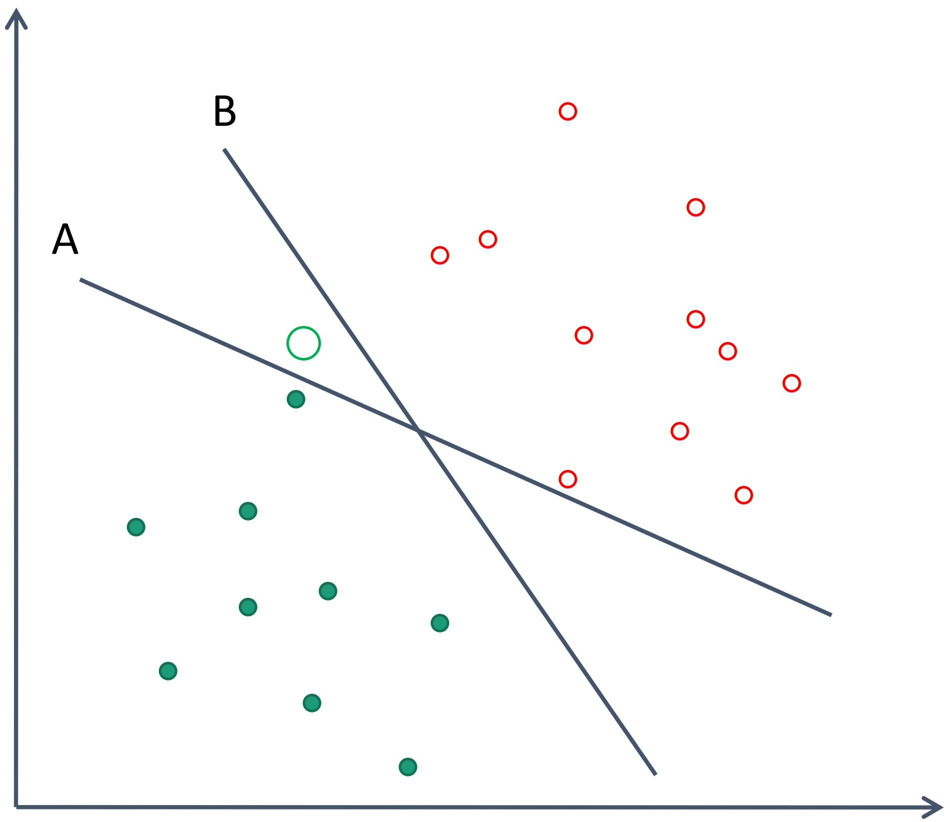


Which classifier is the best?

All have the same training error.

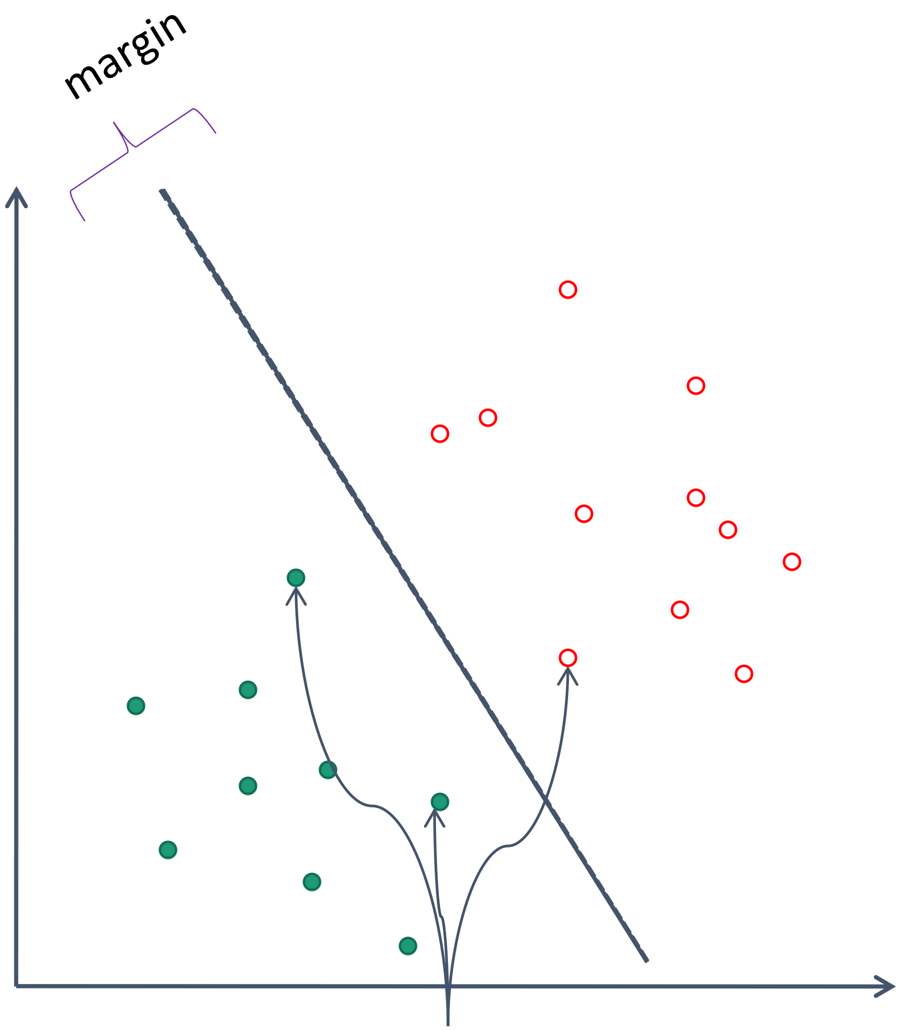
How about generalization?

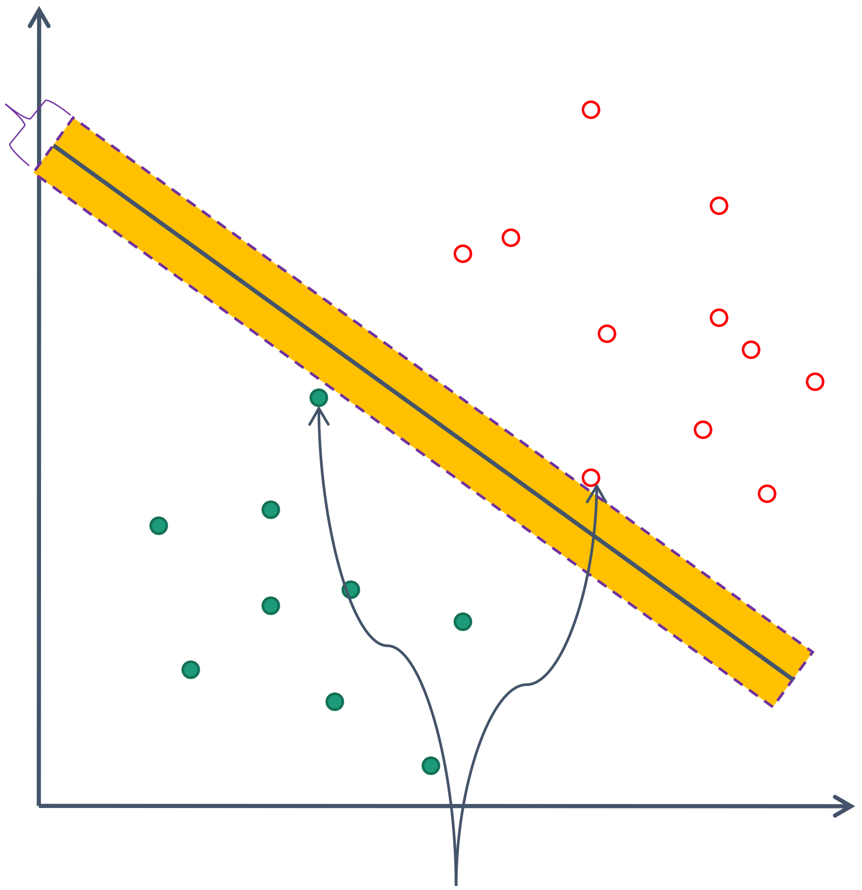
Unknown Samples



Classifier B divides the space more consistently (unbiased).

Margins

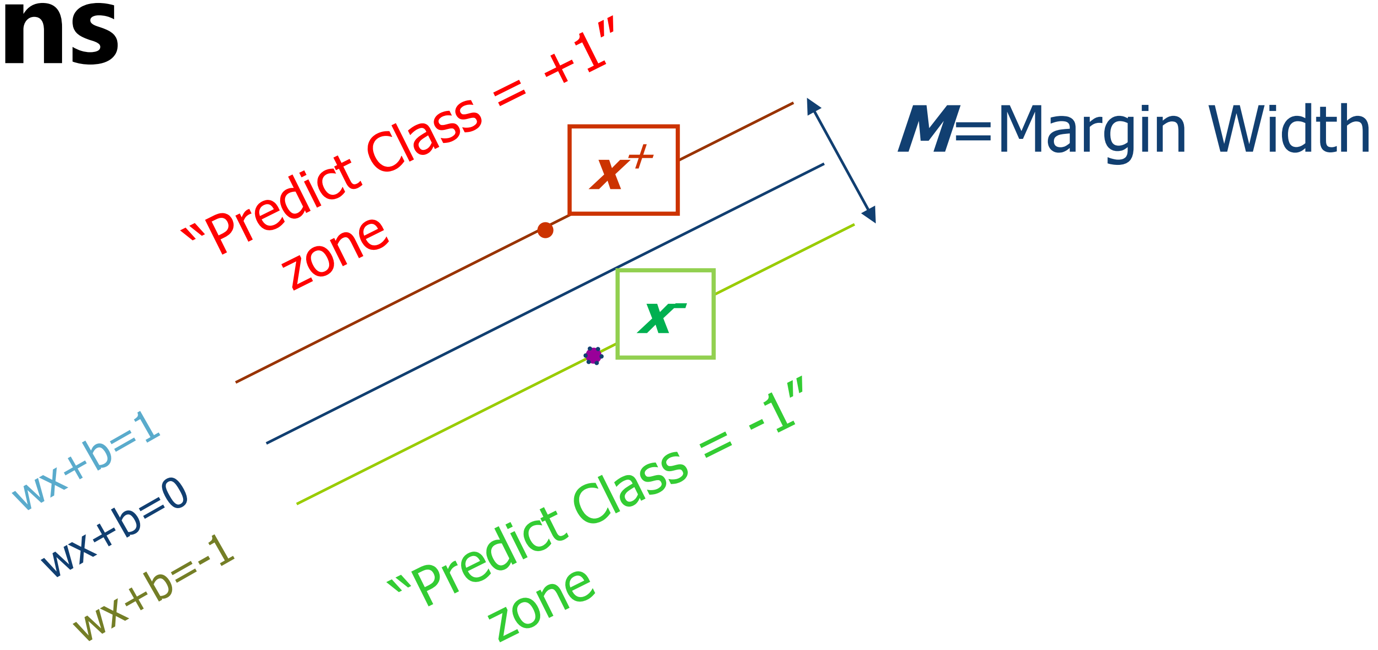




Margins

- The margin of a linear classifier is defined as the width that the boundary could be increased by before hitting a data point.
- Intuitively, it is safer to choose a classifier with a larger margin.
- Wider buffer zone for mistakes
- The hyperplane is decided by only a few data points.
  - Support Vectors
  - Others can be discarded!
- Select the classifier with the maximum margin.
  - Linear Support Vector Machines (LSVM)
- How to specify the margin formally?

Margins



$$M = \boxed{\frac{2}{\|w\|}}$$

Objective Function

- Correctly classify all data points:

$$w \cdot x_i + b \geq 1 \qquad if \;\; y_i = +1$$

$$w \cdot x_i + b \leq -1 \qquad if \;\; y_i = -1$$

$$y_i(w \cdot x_i + b) - 1 \geq 0$$

- Maximize the margin: $\max M = \dfrac{2}{\|w\|} \Rightarrow \min \dfrac{1}{2} w^T w$

- Quadratic Optimization Problem

  - Minimize $\Phi(w) = \dfrac{1}{2} w^t w$

  - Subject to $y_i(w \cdot x_i + b) \geq 1$

Lagrange Multipliers

$$y_i(w\cdot x_i+b)-1\geq 0$$
$$\alpha_i \geq 0$$
$$\alpha_i\left[y_i(w\cdot x_i+b)-1\right]=0$$

$$L_P \equiv \frac{1}{2}\|w\|^2 - \sum_{i=1}^{l}\alpha_i y_i(w\cdot x_i+b)+\sum_{i=1}^{l}\alpha_i$$

$$\frac{\partial L_p}{\partial w}=0 \Rightarrow w=\sum_{i=1}^{l}\alpha_i y_i x_i$$

$$\frac{\partial L_p}{\partial b}=0 \Rightarrow \sum_{i=1}^{l}\alpha_i y_i=0$$

Dual Problem

$$L_D \equiv \sum_i \alpha_i - \frac{1}{2}\sum_{i,j}\alpha_i\alpha_j y_i y_j x_i\cdot x_j$$

$$\equiv \sum_i \alpha_i - \frac{1}{2}\alpha^T H\alpha \; where \; H_{ij}=y_i y_j x_i\cdot x_j$$

Quadratic problem again!

$$subject\ to: \sum_i \alpha_i y_i = 0 \;\; \& \;\; \alpha_i \geq 0$$

Solutions of w & b

$Support\ Vectors: Samples\ with\ \ positive\ \alpha$

$$g(x) = \sum_{i=1}^{l} \alpha_i y_i x_i \cdot x + b$$

inner product

$$y_s(x_s \cdot w + b) = 1$$

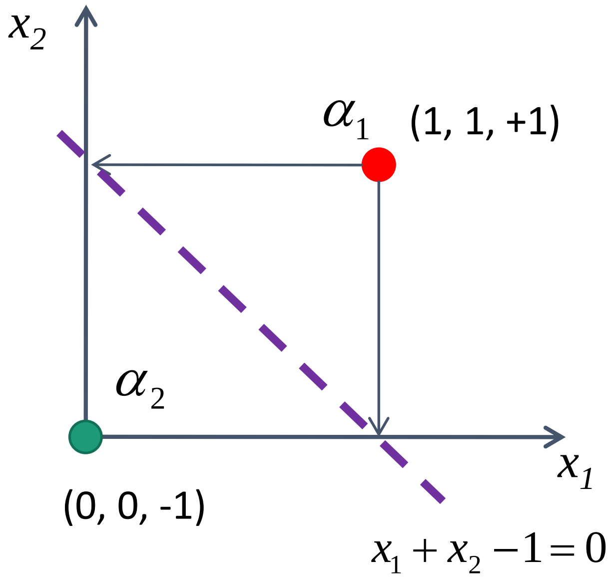
$$y_s\left(\sum_{m \in S} \alpha_m y_m x_m \cdot x_s + b\right) = 1$$

$$y_s^2\left(\sum_{m \in S} \alpha_m y_m x_m \cdot x_s + b\right) = y_s$$

$$b = y_s - \sum_{m \in S} \alpha_m y_m x_m \cdot x_s$$

$$b = \frac{1}{N_s} \sum_{s \in S} \left(y_s - \sum_{m \in S} \alpha_m y_m x_m \cdot x_s\right)$$

An Example



$$\sum_{i=1}^{2} \alpha_i y_i = 0 \Rightarrow \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2$$

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} y_1 y_1 x_1 \cdot x_1 & y_1 y_2 x_1 \cdot x_2 \\ y_2 y_1 x_2 \cdot x_1 & y_2 y_2 x_2 \cdot x_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$L_D \equiv \sum_{i=1}^{2} \alpha_i - \frac{1}{2} [\alpha_1, \alpha_2] H \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 2\alpha_1 - \alpha_1^2$$

$$w = \sum_{i=1}^{2} \alpha_i y_i x_i = 1 \times 1 \times [1,1] + 1 \times (-1) \times [0,0] = [1,1]$$

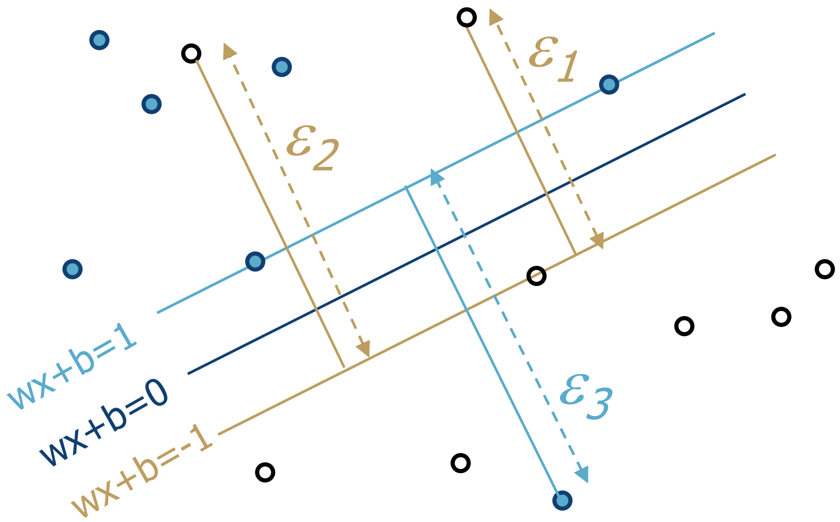
$$\boxed{\alpha_1 = 1; \ \alpha_2 = 1}$$

$$b = -w x_1 + 1 = -2 + 1 = -1$$

$$g(x) = wx + b = x_1 + x_2 - 1$$

$$M = \frac{2}{\|w\|} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Soft Margin



$$y_i(wx_i + b) - 1 + \xi_i \geq 0$$

$$\Phi(w) = \frac{1}{2} w^t w + C \sum_i \xi_i$$

$$\xi_i \geq 0$$

$$L_P \equiv \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{l} \xi_i - \sum_{i=1}^{l} \alpha_i [y_i(w \cdot x_i + b) - 1 + \xi_i] - \sum_{i=1}^{l} \mu_i \xi_i$$

Soft Margin

$$\frac{\partial L_p}{\partial w} = 0 \Rightarrow w = \sum_{i=1}^{l} \alpha_i y_i x_i$$

$$\frac{\partial L_p}{\partial b} = 0 \Rightarrow \sum_{i=1}^{l} \alpha_i y_i = 0$$

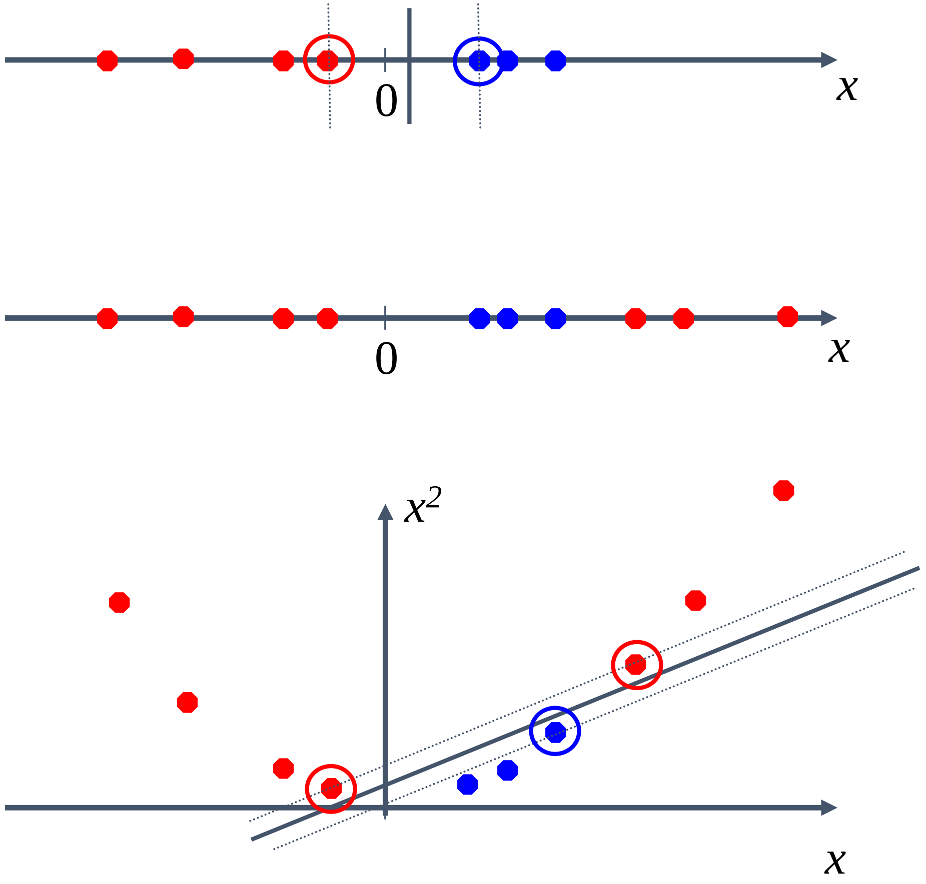
Same as before

$$\frac{\partial L_p}{\partial \xi_i} = 0 \Rightarrow C = \alpha_i + \mu_i$$

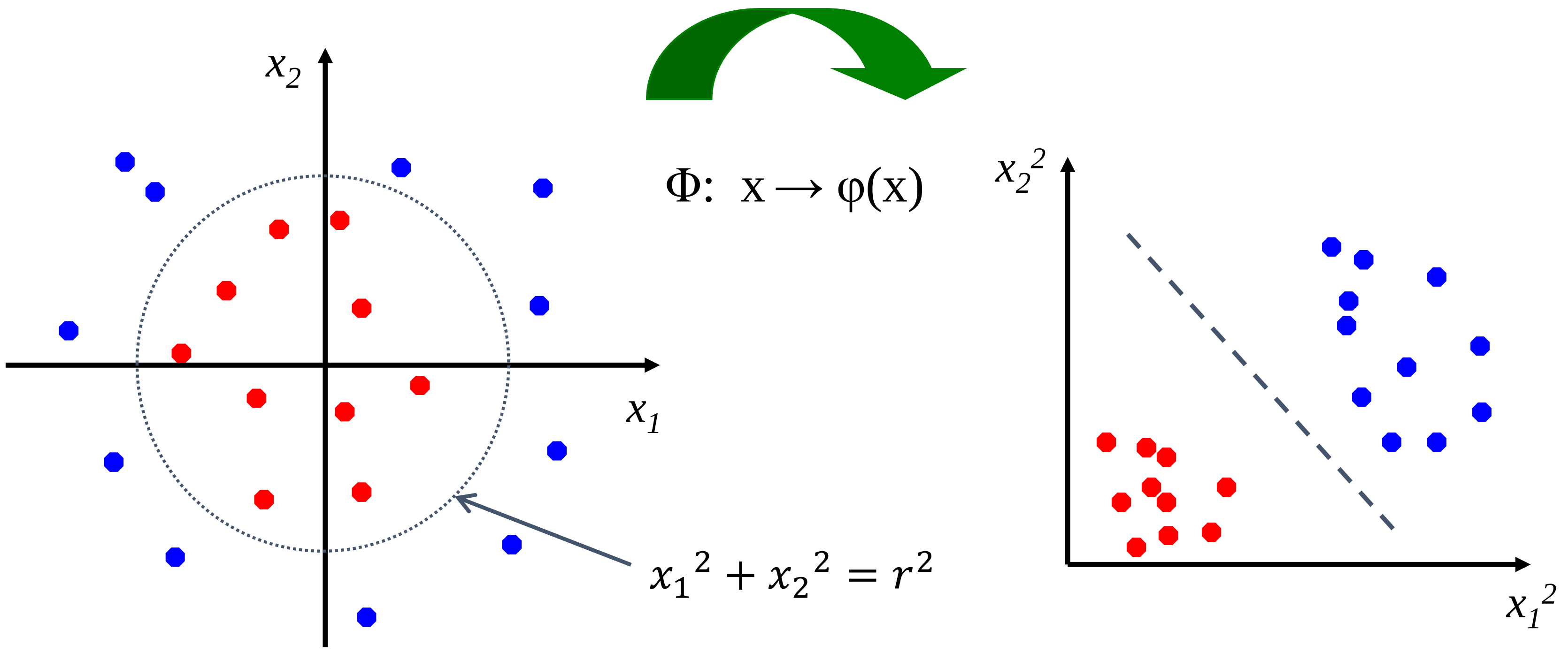
$$L_P \equiv \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{l} \xi_i - \sum_{i=1}^{l} \alpha_i [y_i (w \cdot x_i + b) - 1 + \xi_i] - \sum_{i=1}^{l} \mu_i \xi_i$$

$$\boxed{L_D \equiv \sum_i \alpha_i - \frac{1}{2}\alpha^T H \alpha \quad s.t.\, 0 \le \alpha_i \le C \quad and \ \sum_i \alpha_i y_i = 0}$$

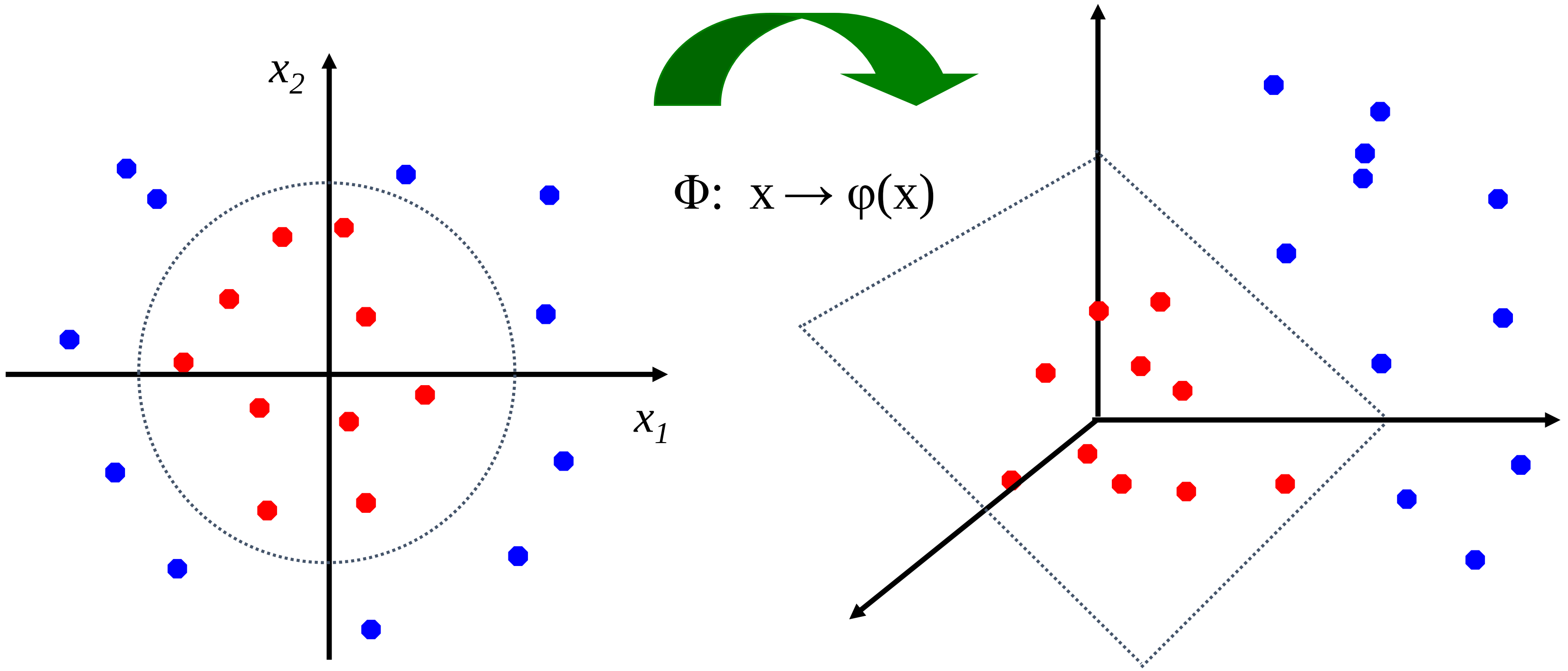
Non-linear SVMs



Feature Space



Feature Space



Quadratic Basis Functions

$$\Phi(x) = \begin{bmatrix} 1 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \\ \vdots \\ \sqrt{2}x_m \\ x_1^2 \\ x_2^2 \\ \vdots \\ x_m^2 \\ \sqrt{2}x_1x_2 \\ \sqrt{2}x_1x_3 \\ \vdots \\ \sqrt{2}x_1x_m \\ \sqrt{2}x_2x_3 \\ \vdots \\ \sqrt{2}x_2x_m \\ \vdots \\ \sqrt{2}x_{m-1}x_m \end{bmatrix}$$

Constant Terms

Linear Terms

Pure Quadratic Terms

Quadratic Cross-Terms

Number of terms

$$C_{m+2}^2 = \frac{(m+2)(m+1)}{2} \approx \frac{m^2}{2}$$

Calculation of Φ(xi)·Φ(xj)

$$\Phi(a)\cdot\Phi(b)=\begin{bmatrix}1\\ \sqrt{2}a_1\\ \sqrt{2}a_2\\ \vdots\\ \sqrt{2}a_m\\ a_1^2\\ a_2^2\\ \vdots\\ a_m^2\\ \sqrt{2}a_1a_2\\ \sqrt{2}a_1a_3\\ \vdots\\ \sqrt{2}a_1a_m\\ \sqrt{2}a_2a_3\\ \vdots\\ \sqrt{2}a_2a_m\\ \vdots\\ \sqrt{2}a_{m-1}a_m\end{bmatrix}\cdot\begin{bmatrix}1\\ \sqrt{2}b_1\\ \sqrt{2}b_2\\ \vdots\\ \sqrt{2}b_m\\ b_1^2\\ b_2^2\\ \vdots\\ b_m^2\\ \sqrt{2}b_1b_2\\ \sqrt{2}b_1b_3\\ \vdots\\ \sqrt{2}b_1b_m\\ \sqrt{2}b_2b_3\\ \vdots\\ \sqrt{2}b_2b_m\\ \vdots\\ \sqrt{2}b_{m-1}b_m\end{bmatrix}$$

$$\mathbf{1}$$

$$\sum_{i=1}^{m}2a_ib_i$$

$$\sum_{i=1}^{m}a_i^2b_i^2$$

$$\boxed{x_i\cdot x_j\Rightarrow\Phi(x_i)\cdot\Phi(x_j)}$$

$$\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}2a_ia_jb_ib_j$$

It turns out …

$$\Phi(a)\cdot\Phi(b) = \underline{1 + 2\sum_{i=1}^{m} a_i b_i + \sum_{i=1}^{m} a_i^2 b_i^2 + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} 2a_i a_j b_i b_j}$$

$$(a\cdot b+1)^2 = (a\cdot b)^2 + 2a\cdot b + 1 = (\sum_{i=1}^{m} a_i b_i)^2 + 2\sum_{i=1}^{m} a_i b_i + 1$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} a_i b_i a_j b_j + 2\sum_{i=1}^{m} a_i b_i + 1$$

$$= \underline{\sum_{i=1}^{m} (a_i b_i)^2 + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} a_i b_i a_j b_j + 2\sum_{i=1}^{m} a_i b_i + 1}$$

$$K(a,b) = (a\cdot b+1)^2 = \Phi(a)\cdot\Phi(b)$$

$O(m)$ $O(m^2)$

Kernel Trick

- The linear classifier relies on dot products $x_i \cdot x_j$ between vectors.
- If every data point is mapped into a high-dimensional space via some transformation Φ: $x \rightarrow \varphi(x)$, the dot product becomes: $\varphi(x_i) \cdot \varphi(x_j)$
- A *kernel function* is some function that corresponds to an inner product in some expanded feature space: $K(x_i,x_j) = \varphi(x_i) \cdot \varphi(x_j)$
- Example: $x=[x_1,x_2]$; $K(\mathrm{x_i},\mathrm{x_j}) = (1+x_i \cdot x_j)^2$

$$K(x_i,x_j) = (1+x_i \cdot x_j)^2 = 1 + x_{i1}^2 x_{j1}^2 + 2x_{i1}x_{j1}x_{i2}x_{j2} + x_{i2}^2 x_{j2}^2 + 2x_{i1}x_{j1} + 2x_{i2}x_{j2}$$

$$= [1, x_{i1}^2, \sqrt{2}x_{i1}x_{i2}, x_{i2}^2, \sqrt{2}x_{i1}, \sqrt{2}x_{i2}] \cdot [1, x_{j1}^2, \sqrt{2}x_{j1}x_{j2}, x_{j2}^2, \sqrt{2}x_{j1}, \sqrt{2}x_{j2}]$$

$$= \Phi(\mathrm{x_i}) \cdot \Phi(\mathrm{x_j}), \quad \text{where } \Phi(x) = [1, x_1^2, \sqrt{2}x_1x_2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2]$$

Kernels

$$Polynomial: K(x_i, x_j) = (x_i \cdot x_j + 1)^d$$

$$Gaussian: K(x_i, x_j) = \exp\left(-\frac{\left\|x_i - x_j\right\|^2}{2\sigma^2}\right)$$

$$Hyperbolic\ Tangent: K(x_i, x_j) = \tanh(\kappa x_i \cdot x_j + c)$$

Solutions of w & b

$$w = \sum_{i=1}^{l} \alpha_i y_i \Phi(x_i)$$

$$w \cdot \Phi(x_j) = \sum_{i=1}^{l} \alpha_i y_i \Phi(x_i) \cdot \Phi(x_j) = \sum_{i=1}^{l} \alpha_i y_i K(x_i, x_j)$$

$$b = \frac{1}{N_s} \sum_{s \in S} (y_s - \sum_{m \in S} \alpha_m y_m \Phi(x_m) \cdot \Phi(x_s)) = \frac{1}{N_s} \sum_{s \in S} (y_s - \sum_{m \in S} \alpha_m y_m K(x_m, x_s))$$

$$\boxed{g(x) = \sum_{i=1}^{l} \alpha_i y_i K(x_i, x) + b} \Longleftrightarrow g(x) = w \cdot x + b = \sum_{i=1}^{l} \alpha_i y_i x_i \cdot x + b$$

SVM Roadmap

Linear Classifier

Maximum Margin

↓

Linear SVM

Noise

↓

Soft Margin

Nonlinear Problem

↓

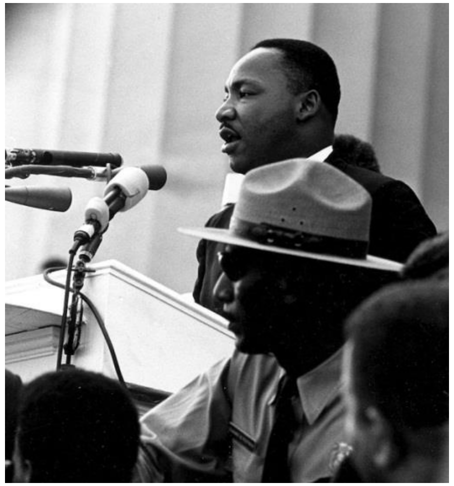
$a \cdot b \rightarrow \Phi(a) \cdot \Phi(b)$

High Computational Cost

↓

Kernel Trick

$K(a,b)=\Phi(a) \cdot \Phi(b)$



“I have a dream — one day there will be a classifier that can handle nonlinear problems …”

Reading Materials

- N. Cristianini and J. Shawe-Taylor, *An Introduction to Support Vector Machines and Other Kernel-Based Learning Methods*. Cambridge University Press, 2000.

- C. Burges, "A Tutorial on Support Vector Machines for Pattern Recognition". *Data Mining and Knowledge Discovery*, vol. 2, pp. 121-167, 1998.

- H. Lodhi et al., "Text Classification Using String Kernels". *The Journal of Machine Learning Research*, vol. 2, pp. 419-444, 2002.

- Online Resources
  - http://www.kernel-machines.org/
  - http://www.support-vector-machines.org/
  - http://www.tristanfletcher.co.uk/SVM%20Explained.pdf
  - http://www.csie.ntu.edu.tw/~cjlin/libsvm/


ALSO
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APPLIED
SENSOR
SHARED
EXAMPLES
EVERY
DEFINITION
CURRENT
MAY
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MANAGE
WITHIN
ARCHIVES
PRACTITIONERS
CAPTURE
BUSINESS
TECHNOLOGIES
USING
INTERNET
SETS
DISTRIBUTED
GENOMICS
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TIME
ONE
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