# Midterm

- Time: 9:00-12:00
- NO toilet breaks are allowed
- Phone calculators are NOT allowed
- Write your answers on the provided answer sheets. Additional sheets will be supplied upon request. Please ensure that your full name is clearly written on each page of the answer sheets.
- Include detailed computation steps for all solutions. Answers without supporting calculations will receive a score of zero.

**Question 1.** (4 marks) Find vectors $\boldsymbol{a}$ such that it is orthogonal to the given vector $\boldsymbol{b}$ and a norm equal to the specified value $d$.

1. (2 marks) $\boldsymbol{b} = \begin{pmatrix} 3, & 4 \end{pmatrix}$, $d = 15$
2. (2 marks) $\boldsymbol{b} = \begin{pmatrix} -3, & 2 \end{pmatrix}$, $d = 10$

*Solution.* Let $\boldsymbol{a} = \begin{pmatrix} a_1, & a_2 \end{pmatrix}$

1. By the given conditions, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 3a_1 + 4a_2 = 0, \quad \|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2} = 15.$$

Then we have the following two equations:

$$a_1 = -\frac{4}{3}a_2 \tag{1}$$
$$a_1^2 + a_2^2 = 15^2 \tag{2}$$

Substitute $a_1$ using Equation (1) in Equation (2) gives

$$a_2^2\left(\frac{16}{9} + 1\right) = 225 \Longrightarrow a_2^2 = 225 \times \frac{9}{25} = 81 \Longrightarrow a_2 = \pm 9 \quad \text{(1 mark)}$$

Together with Equation (1) we have

$$a_1 = -\frac{4}{3} \times 9 = -12, \text{ or } a_1 = \frac{4}{3} \times 9 = 12.$$

Thus

$$\boldsymbol{a} = \begin{pmatrix} -12, & 9 \end{pmatrix} \text{ or } \begin{pmatrix} 12, & -9 \end{pmatrix}. \quad \text{(1 mark)}$$

2. By the given conditions, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = -3a_1 + 2a_2 = 0, \quad \|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2} = 10.$$

Then we have the following two equations:

$$a_1 = \frac{2}{3}a_2 \tag{3}$$
$$a_1^2 + a_2^2 = 100 \tag{4}$$

Substitute $a_1$ using Equation (3) in Equation (4) gives

$$a_2^2\left(\frac{4}{9}+1\right)=100 \Longrightarrow a_2^2=100\times\frac{9}{13}=\frac{900}{13}\Longrightarrow a_2=\pm\frac{30\sqrt{13}}{13} \quad \text{(1 mark)}$$

Together with Equation (3) we have

$$a_1=\frac{2}{3}\times\frac{30\sqrt{13}}{13}=\frac{20\sqrt{13}}{13},\ \text{or}\ a_1=-\frac{2}{3}\times\frac{30\sqrt{13}}{13}=-\frac{20\sqrt{13}}{13}. \quad \text{(1 mark)}$$

Thus

$$\boldsymbol{a}=\left(\frac{20\sqrt{13}}{13},\ \frac{30\sqrt{13}}{13}\right)\ \text{or}\ \left(-\frac{20\sqrt{13}}{13},\ -\frac{30\sqrt{13}}{13}\right).$$

**Question 2.** (6 marks) Let

$$A=\begin{pmatrix}3&0\\-1&2\\1&1\end{pmatrix},\quad B=\begin{pmatrix}4&-1\\0&2\end{pmatrix},\quad C=\begin{pmatrix}1&4&2\\3&1&5\end{pmatrix}$$

$$D=\begin{pmatrix}1&5&2\\-1&0&1\\3&2&4\end{pmatrix},\quad E=\begin{pmatrix}6&1&3\\-1&1&2\\4&1&3\end{pmatrix}$$

Compute the following matrices.

1. (1 mark) $AB$
2. (1 mark) $A(BC)$
3. (2 marks) $2A^\top + C$
4. (2 marks) $2E^\top - 3D^\top$

*Solution.*

1. $AB=\begin{pmatrix}12&-3\\-4&5\\4&1\end{pmatrix}$ (1 mark)
2. $A(BC)=\begin{pmatrix}3&45&9\\11&-11&17\\7&17&13\end{pmatrix}$ (1 mark)
3. $2A^\top + C=\begin{pmatrix}7&2&4\\3&5&7\end{pmatrix}$ (2 marks)
4. $2E^\top - 3D^\top=\begin{pmatrix}9&1&-1\\-13&2&-4\\0&1&-6\end{pmatrix}$ (2 marks)

- computation of wrong transpose: ($-0.5$ marks);
- missing details for matrix multiplication ($-0.2$ marks)
- An incorrect detailed computation for the matrix multiplication, even if the final answer is correct, will not receive any points.
- Each incorrect entry in the final answer results in ($-0.2$ marks)

**Question 3.** (7 marks) Consider the following matrix as the augmented matrix of a linear system:

$$\left(\begin{array}{ccc|c}1&2&-1&a\\2&3&-2&b\\-1&-1&1&c\end{array}\right)$$

1. (2 marks) Apply the Gauss–Jordan elimination method to the given matrix to obtain its reduced row echelon form.
2. (1 mark) Determine the values of $a, b, c$ for which this linear system is *inconsistent*.
3. (1 mark) Determine the values of $a, b, c$ for which this linear system is *consistent*.
4. (2 marks) If this system is consistent, determine whether it has a unique solution or infinitely many solutions. Express the solution(s) in terms of $a, b, c$.
5. (1 mark) Choose a specific set of values for $a, b, c$ that ensures the system is consistent and solve the system.

*Solution.*

1. (2 marks) Applying Gauss-Jordan elimination, we have

$$\xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & 0 & b-2a \\ 0 & 1 & 0 & c+a \end{array}\right) \xrightarrow{R_2 \to -R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & 1 & 0 & 2a-b \\ 0 & 1 & 0 & c+a \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & 1 & 0 & 2a-b \\ 0 & 0 & 0 & c-a+b \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -3a+b \\ 0 & 1 & 0 & 2a-b \\ 0 & 0 & 0 & c-a+b \end{array}\right)$$

   - Not following the procedure ($-0.2$ marks)
   - Not labelled properly ($-0.2$ marks)

2. If

$$c - a + b \neq 0$$

   then the system is inconsistent. (1 mark)

3. If

$$c - a + b = 0$$

   the system is consistent. (1 mark)

4. In the case $c - a + b = 0$, the system has infinitely many solutions given by

$$\{\, (-3a + b + t,\ 2a - b,\ t) \ \mid\ t \in \mathbb{R} \,\}. \quad \text{(1 mark)}$$

   No matter what values $a,\ b,\ c$ take, the system will not have a unique solution. (1 mark)

5. Let

$$a = 1,\ b = 1,\ c = 0,$$

   which satisfies $c - a + b = 0$. Then the solution set for the system becomes

$$\{\, (-2 + t,\ 1,\ t) \ \mid\ t \in \mathbb{R} \,\}. \quad \text{(1 mark)}$$

**Question 4.** (6 marks) Solve the following linear systems using Gauss–Jordan elimination or Gaussian elimination. In each case, indicate whether the system is consistent or inconsistent. Give the complete solution set if the system is consistent.

1. (3 marks)

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 + 2x_4 &= -2 \\ 4x_1 - 2x_2 - 3x_3 - 2x_4 &= 2 \\ x_1 + 3x_2 + 3x_3 - 3x_4 &= -4 \end{aligned}$$

2. (3 marks)

$$\begin{aligned} 3x_1 - 3x_3 + 4x_4 &= -3 \\ -4x_1 + 2x_2 - 2x_3 - 4x_4 &= 4 \\ 4x_2 - 3x_3 + 2x_4 &= -3 \end{aligned}$$

*Solution.*

1. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 2 & 4 & 2 & 2 & -2 \\ 4 & -2 & -3 & -2 & 2 \\ 1 & 3 & 3 & -3 & -4 \end{array}\right) \quad \text{(0.5 marks)}$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{21}{13} & -\frac{7}{13} \\ 0 & 1 & 0 & \frac{40}{13} & \frac{9}{13} \\ 0 & 0 & 1 & -\frac{46}{13} & -\frac{24}{13} \end{array}\right) \quad \text{(2 marks)}$$

The solution set is

$$\left\{ \left( -\frac{7}{13} + \frac{21t}{13},\ \frac{9}{13} - \frac{40t}{13},\ -\frac{24}{13} + \frac{46t}{13},\ t \right) \ \middle|\ t \in \mathbb{R} \right\}. \quad \text{(0.5 marks)}$$

2. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 3 & 0 & -3 & 4 & -3 \\ -4 & 2 & -2 & -4 & 4 \\ 0 & 4 & -3 & 2 & -3 \end{array}\right) \quad \text{(0.5 marks)}$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{34}{27} & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{4}{9} & -1 \\ 0 & 0 & 1 & -\frac{2}{27} & -\frac{1}{3} \end{array}\right) \quad \text{(2 marks)}$$

The solution set is

$$\left\{ \left( -\frac{4}{3} - \frac{34t}{27},\ -1 - \frac{4t}{9},\ -\frac{1}{3} + \frac{2t}{27},\ t \right) \ \middle|\ t \in \mathbb{R} \right\}. \quad \text{(0.5 marks)}$$

**Question 5.** (6 marks) Solve the given linear system by inverting the coefficient matrix.

1. (3 marks)

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

2. (3 marks)

$$\begin{aligned} 5x_1 - 2x_3 &= 0 \\ -15x_1 - 16x_2 - 9x_3 &= 0 \\ 10x_1 + 12x_2 + 7x_3 &= 0 \end{aligned}$$

*Solution.*

1. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right) \quad \text{(0.5 marks)}$$

   Inverse of the coefficient matrix is:

$$\begin{pmatrix} \frac{1}{4} & -\frac{9}{26} & \frac{7}{52} \\ \frac{1}{4} & -\frac{1}{26} & -\frac{5}{52} \\ \frac{1}{4} & \frac{5}{26} & -\frac{1}{52} \end{pmatrix} \quad \text{(2 marks)}$$

   Solution of the system is given by:

$$\boldsymbol{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \quad \text{(0.5 marks)}$$

2. The augmented matrix is

$$\begin{pmatrix} 5 & 0 & -2 & 0 \\ -15 & -16 & -9 & 0 \\ 10 & 12 & 7 & 0 \end{pmatrix} \quad \text{(0.5 marks)}$$

   The inverse of the coefficient matrix is

$$\begin{pmatrix} -\frac{1}{5} & -\frac{6}{5} & -\frac{8}{5} \\ \frac{3}{4} & \frac{11}{4} & \frac{15}{4} \\ -1 & -3 & -4 \end{pmatrix} \quad \text{(2 marks)}$$

   The solution of the system is

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{(0.5 marks)}$$

**Question 6.** (4 marks) Find the determinant of the following matrices using some combination of row operations, column operations, and cofactor expansion.

1. (2 marks) $\begin{pmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{pmatrix}$

2. (2 marks) $\begin{pmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & -1 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{pmatrix}$

*Solution.*

1.

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \overset{R_2 \to R_2 - 5R_1,\ R_3 \to R_3 + R_1,\ R_4 \to R_4 - 2R_1}{=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix} \overset{R_4 \to R_4 - 12R_2}{=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix}$$

$$\overset{R_4 \to R_4 + 36R_3}{=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{vmatrix} = 39 \quad \text{(2 marks)}$$

2.

$$\begin{vmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & -1 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{vmatrix} \overset{R_1 \to R_1 + 4R_2,\ R_4 \to R_4 - R_2,\ R_3 \to R_3 + R_2}{=} \begin{vmatrix} 0 & 8 & -7 & -2 \\ 2 & 1 & -1 & -1 \\ -1 & -4 & 3 & -1 \\ 0 & -5 & 4 & 0 \end{vmatrix} \overset{R_2 \to R_2 + 2R_3}{=}$$

$$\begin{vmatrix} 0 & 8 & -7 & -2 \\ 0 & -7 & 5 & -3 \\ -1 & -4 & 3 & -1 \\ 0 & -5 & 4 & 0 \end{vmatrix} \overset{R_1 \to R_1 + R_2}{=} \begin{vmatrix} 0 & 1 & -2 & -5 \\ 0 & -7 & 5 & -3 \\ -1 & -4 & 3 & -1 \\ 0 & -5 & 4 & 0 \end{vmatrix} \overset{R_2 \to R_2 + 7R_1,\ R_4 \to R_4 + 5R_1}{=} \begin{vmatrix} 0 & 1 & -2 & -5 \\ 0 & 0 & -9 & -38 \\ -1 & -4 & 3 & -1 \\ 0 & 0 & -6 & -25 \end{vmatrix}$$

$$\overset{R_2 \to R_1,\ R_3 \to R_2,\ R_1 \to R_3}{=} \begin{vmatrix} -1 & -4 & 3 & -1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & -9 & -38 \\ 0 & 0 & -6 & -25 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & -5 \\ 0 & -9 & -38 \\ 0 & -6 & -25 \end{vmatrix} = -\begin{vmatrix} -9 & -38 \\ -6 & -25 \end{vmatrix} = 3 \quad \text{(2 marks)}$$

**Question 7.** (12 marks) Solve the following systems of equations using Cramer's rule

1. (6 marks)

$$\begin{aligned} -x_1 - 4x_2 + 2x_3 + x_4 &= -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 &= 14 \\ -x_1 + x_2 + 3x_3 + x_4 &= 11 \\ x_1 - 2x_2 + x_3 - 4x_4 &= -4 \end{aligned}$$

2. (6 marks)

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 4 \\ -x_1 + 7x_2 - 3x_3 &= 1 \\ 2x_1 + 6x_2 - x_3 &= 5 \end{aligned}$$

*Solution.*

1. The coefficient matrix is

$$A = \begin{pmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & -4 \end{pmatrix} \quad \text{(0.5 marks)}$$

$$\det(A) = -423, \quad \det(A_1) = -2115, \quad \det(A_2) = -3384,$$
$$\det(A_3) = -1269, \quad \det(A_4) = 423. \quad \text{(5 marks)}$$

The solution of the linear system is $\begin{pmatrix} 5, & 8, & 3, & -1 \end{pmatrix}$. (0.5 marks)

2. The coefficient matrix is

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 7 & -3 \\ 2 & 6 & -1 \end{pmatrix} \quad \text{(0.5 marks)}$$

$$\det(A) = 20, \quad \det(A_1) = 29, \quad \det(A_2) = 7, \quad \det(A_3) = 0. \quad \text{(5 marks)}$$

The solution of the linear system is $\begin{pmatrix} \frac{29}{20}, & \frac{7}{20}, & 0 \end{pmatrix}$. (0.5 marks)

**Question 8.** (5 marks) Recall the definition of vector spaces

**Definition 1.** Let $V$ be a nonempty set on which two operations are defined:

- Addition: $V \times V \to V$, $\boldsymbol{v} + \boldsymbol{u}$ is called the *sum* of $\boldsymbol{v}$ and $\boldsymbol{u}$
- Scalar multiplication: $\mathbb{R} \times V \to V$, $\alpha\boldsymbol{v}$ is called the *scalar multiple* of $\boldsymbol{v}$ by $\alpha$

$V$, (together with the associated addition and scalar multiplication), is called a *vector space (over $\mathbb{R}$)* if $\forall \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w} \in V$ (*vectors*) and $\forall \alpha, \beta \in \mathbb{R}$ (*scalars*):

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
2. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
3. $\exists \boldsymbol{0} \in V$, called *zero vector* or *additive identity*, s.t. $\boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$
4. $\exists -\boldsymbol{u} \in V$, called *negative* or *additive inverse* of $\boldsymbol{u}$ s.t. $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$
5. $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}$
6. $(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u}$
7. $\alpha(\beta\boldsymbol{u}) = (\alpha\beta)\boldsymbol{u}$
8. $1\boldsymbol{u} = \boldsymbol{u}$

For each of the following sets with the specified operations, identify one vector space axiom that is not satisfied. Clearly state the violated axiom and provide an example showing that it fails.

1. (2.5 marks) The set of all pairs of real numbers of the form $\begin{pmatrix} x, & y \end{pmatrix}$ such that $x \geq 0$, with the standard vector addition and scalar multiplication in $\mathbb{R}^2$.
2. (2.5 marks) The set of all invertible $2 \times 2$ matrices, together with the standard matrix addition and scalar multiplication.

*Solution.*

1. Let $V$ denote the subset of $\mathbb{R}^2$ of the form $\begin{pmatrix} x, & y \end{pmatrix}$ with $x \geq 0$. $V$ is not closed under scalar multiplication.

   For example, take $\begin{pmatrix} 1, 0 \end{pmatrix} \in V$ and scalar $-1 \in \mathbb{R}$

$$(-1)\begin{pmatrix} 1, & 0 \end{pmatrix} = \begin{pmatrix} -1, & 0 \end{pmatrix} \notin V.$$

2. Let $V$ be the set of all invertible $2 \times 2$ matrices.

   Axiom: $V$ is closed under addition.

   $V$ is not closed under addition: for example, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are both invertible matrices, but

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   is not an invertible matrix.

   Axiom: $\exists \mathbf{0} \in V$, called *zero vector* or *additive identity*, s.t. $\mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$ for any $\boldsymbol{u} \in V$.

   The axiom does not hold because the zero matrix is not an element of $V$.

A correct identification of the failed axiom is worth (1 mark), and a proper example is worth (1.5 marks).