Algebra and Discrete Mathematics
ADM

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Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces and matrix transformations
- Fundamental spaces and decompositions
- Eulerian tours
- Hamiltonian cycles
- Midterm
- Paths and spanning trees
- Trees and networks
- Matching

Recommended reading

- Anton, Howard, and Chris Rorres. Elementary linear algebra: applications version. John Wiley & Sons, 2013.
  - Sections 1.1, 1.2, 1.3, 1.4
  - Accessible online (free copy)
  - Alternative download link

Lecture outline

- Introduction
- Augmented matrices and echelon forms
- Elimination methods
- Homogeneous linear system
- Matrices and their inverses

System of linear equations

- Introduction
- Augmented matrices and echelon forms
- Elimination methods
- Homogeneous linear system
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Linear equations

- In two dimensions, a line in a rectangular $xy-$coordinate system can be represented by an equation of the form

$$ax + by = c,$$

where $a, b, c \in \mathbb{R}$ and a, b are not both 0.

- In three dimensions, a plane in a rectangular $xyz-$coordinate system can be represented by an equation of the form

$$ax + by + cz = d,$$

where $a, b, c, d \in \mathbb{R}$ and a, b, c are not all 0.

- These are examples of "linear equations," - a linear equation in the variables x and y, and in the variables x, y and z

Linear equations

- A *linear equation* in the n variables $x_1, x_2, \ldots, x_n$ is an equation in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

  where $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$, not all a_i are 0
- When $b = 0$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

  is called a *homogeneous linear equation* in the variables $x_1, x_2, \ldots, x_n$.

Example

$$\begin{array}{rl} x + 3y = 7 & x_1 - 2x_2 - 3x_3 + x_4 = 0 \\ \frac{1}{2}x - y + 3z = -1 & x_1 + x_2 + \cdots + x_n = 1 \end{array}$$

Not linear equations:

$$\begin{array}{rl} x + 3y^2 = 4 & 3x + 2y - xy = 5 \\ \sin x + y = 0 & \sqrt{x_1} + 2x_2 + x_3 = 1 \end{array}$$

System of linear equations

- A *system of linear equations* or a *linear system*: finite set of linear equations
- The variables are called *unknowns*
- A linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- a_i's are called *coefficients*, b_j's are *constants*
- A *solution* of a linear system in n unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n real numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \quad \ldots, \quad x_n = s_n$$

makes each equation hold

System of linear equations – example

Example

The linear system

$$5x + y = 3, \quad 2x - y = 4$$

has the solution

$$x = 1, \quad y = -2$$

The linear system

$$4x_1 - x_2 + 3x_3 = -1, \quad 3x_1 + x_2 + 9x_3 = -4$$

has the solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$

These solutions can be written as

$$\begin{pmatrix} 1, & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1, & 2, & -1 \end{pmatrix}$$

Solution to a linear system

- A solution
$$x_1 = s_1, \quad x_2 = s_2, \quad \ldots, \quad x_n = s_n$$
of a linear system in n unknowns can be written as
$$\left(s_1, \quad s_2, \quad \ldots, \quad s_n\right)$$
which is called an *ordered* $n-$*tuple*
- With this notation it is understood that all variables appear in the same order in each equation

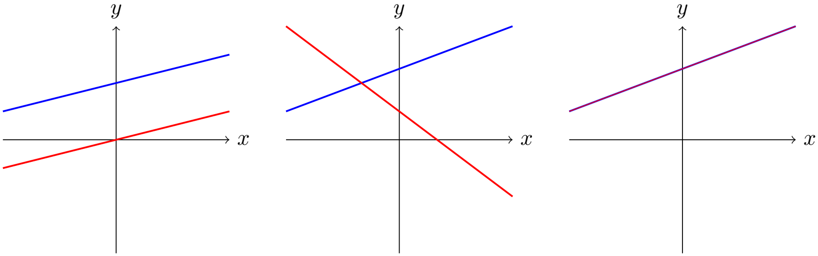
Linear system in two unknowns

- Consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

- The graphs of the equations are lines in the $xy-$plane
- Each solution of this system corresponds to a point of intersection of the lines
- There are three possibilities
  - Lines parallel and distinct - no solution
  - Lines intersect at only one point - one solution
  - Lines coincide - infinitely many solutions

Linear system of two equations in two unknowns



No solution One solution Infinitely many solutions

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

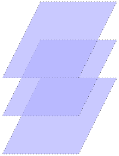
Solution of a linear system

- We say a linear system is *consistent* if it has at least one solution and *inconsistent* if it has no solutions
- In particular, a consistent linear system of two equations in two unknowns has either one solution or infinitely many solutions – there are no other possibilities.
- The same is true for a linear system of three equations in three unknowns

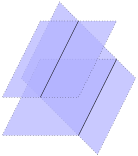
$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities – no solutions, one solution, or infinitely many solutions

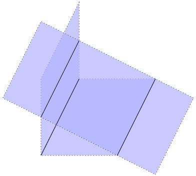
Linear system of three equations in three unknowns



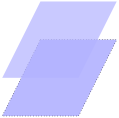
No solutions (three parallel planes; no common intersection)



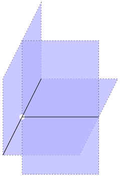
No solutions (two parallel planes; third intersects both)



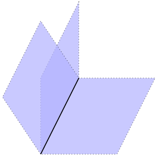
No solutions (no common intersection)



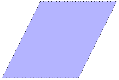
No solutions (two coincident planes parallel to the third; no common intersection)



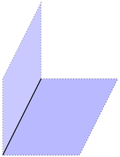
One solution (intersection is a point)



Infinitely many solutions (intersection is a line)



Infinitely many solutions (planes are all coincident; intersection is a plane)



Infinitely many solutions (two coincident planes; intersection is a line)

Three possibilities – no solutions, one solution, or infinitely many solutions

Solution of a linear system – example

Example (A linear system with one solution)

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Add those two equations, we get

$$3x = 7 \Longrightarrow x = \frac{7}{3},$$

then

$$y = x - 1 = \frac{4}{3}.$$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3}, \quad \frac{4}{3}\right)$

Solution of a linear system – example

Example (A linear system with no solution)

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Add $-3\times$ the first equation to the second one, we get

$$0 = -6,$$

a contradiction.
Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct.

Solution of a linear system – example

Example (A linear system with infinitely many solutions)

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Add $-4\times$ the first equation to the second one, we get

$$0 = 0,$$

which is always true. Thus the solutions are those values of x and y that satisfy the equation

$$4x - 2y = 1.$$

Geometrically, the lines corresponding to the two equations in the original system coincide.

Solution of a linear system – example

Example (A linear system with infinitely many solutions)

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

The solutions are those values of x and y that satisfy the equation

$$4x - 2y = 1.$$

We can describe the solution as follows: for any $t \in \mathbb{R}$ (*parameter*), the solution is given by (*parametric equations*)

$$x = \frac{1}{4} + \frac{1}{2}t, \quad y = t.$$

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter t, e.g. $t = 0$ gives solution $\left(\frac{1}{4}, \quad 0\right)$

Solution of a linear system – example

Example (A linear system with infinitely many solutions)

$$
\begin{aligned}
4x - 2y &= 1 \\
16x - 8y &= 4
\end{aligned}
$$

We can describe the solution as follows: for any $t \in \mathbb{R}$ (*parameter*), the *general solution* is given by

$$x = \frac{1}{4} + \frac{1}{2}t, \quad y = t$$

Or the *(complete) solution set* is equal to

$$\left\{ \left(\frac{1}{4} + \frac{1}{2}t,\ t \right) \;\middle|\; t \in \mathbb{R} \right\}$$

System of linear equations

- Introduction
- Augmented matrices and echelon forms
- Elimination methods
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Augmented matrices

- As the number of equations and unknowns in a linear system increases, so does the complexity of the algebra involved in finding solutions.
- By mentally keeping track of the location of the $+$'s, the x's, and the $=$'s in the linear system, we can abbreviate the system using the *augmented matrix*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \longrightarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

- Or, recall our discussions about matrices from last week

Matrix form of a linear system

Consider a system of m linear equations in n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Two matrices of the same size are equal iff their corresponding entries are equal, we can write

$$\begin{pmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & \ddots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Matrix form of a linear system

$$\begin{pmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & \ddots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The $m \times 1$ matrix on the left side can be written as a product

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Represent these matrices by A, $\boldsymbol{x}$, $\boldsymbol{b}$, then we can replace the original system of m equations in n unknowns by the single matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

Matrix form of a linear system

$$A\boldsymbol{x} = \boldsymbol{b}$$

- A is called the *coefficient matrix* of the system
- The *augmented matrix* is given by adjoining $\boldsymbol{b}$ to A as the last column

$$\begin{pmatrix} A & \mid & \boldsymbol{b} \end{pmatrix} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

Augmented matrices – example

Example

The augmented matrix for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

is

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Operations on the linear system

- To solve a linear system, we will perform algebraic operations on the system
- Those operations do not alter the solution set and produce simpler systems
- Until one point when we are certain whether the system is consistent (has at least one solution), if yes, what are the solutions
- Typical algebraic operations are
  - Multiply an equation by a nonzero constant
  - Interchange two equations
  - Add a constant times one equation to another

Elementary row operations

Typical algebraic operations on linear systems are

- Multiply an equation by a nonzero constant
- Interchange two equations
- Add a constant times one equation to another

These correspond to the following *elementary row operations* on the augmented matrix

- Multiply a row by a nonzero constant
- Interchange two rows
- Add a constant times one row to another

Elementary row operations – example

Example

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Add $-2\times$ the first equation to the second $\qquad R_2 \to R_2 - 2R_1$

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Add $-3\times$ the first equation to the third $\qquad R_3 \to R_3 - 3R_1$

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right)$$

Elementary row operations – example

Example

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right)$$

$\frac{1}{2}\times$ the second equation $\qquad R_2 \to \frac{1}{2}R_2$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ 3y - 11z &= -27 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right)$$

Add $-3\times$ the second equation to the third $\qquad R_3 \to R_3 - 3R_2$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ -\frac{1}{2}z &= -\frac{3}{2} \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right)$$

Elementary row operations – example

Example

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ -\frac{1}{2}z &= -\frac{3}{2} \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right)$$

$-2\times$ the third equation $\qquad R_3 \to -2R_3$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Elementary row operations – example

Example

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= -3 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Add $-1\times$ the second equation to the first $\qquad R_1 \to R_1 - R_2$

$$\begin{aligned} x + \frac{11}{2}z &= \frac{35}{2} \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Elementary row operations – example

Example

$$x + \frac{11}{2}z = \frac{35}{2}$$
$$y - \frac{7}{2}z = -\frac{17}{2}$$
$$z = 3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Add $-\frac{11}{2}\times$ the third equation to the first $\quad R_1 \to R_1 - \frac{11}{2}R_3$

Add $\frac{7}{2}\times$ the third equation to the second $\quad R_2 \to R_2 + \frac{7}{2}R_3$

$$x = 1$$
$$y = 2$$
$$z = 3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Echelon forms

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

The matrix we have arrived at is in *reduced row echelon form*.

- If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1 - *leading* 1/*pivot position*
- If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
- In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
- Each column that contains a leading 1 (*pivot column*) has zeros everywhere else in that column.

With only the first three properties - *row echlon form*

Echelon forms – example

Example

row echelon form

$$\begin{pmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Echelon forms – example

Example

row echelon form

$$\begin{pmatrix}1&*&*&*\\0&1&*&*\\0&0&1&*\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&*&*&*\\0&1&*&*\\0&0&1&*\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&*&*&*&*&*&*&*&*\\0&0&0&1&*&*&*&*&*&*\\0&0&0&0&1&*&*&*&*&*\\0&0&0&0&0&1&*&*&*&*\\0&0&0&0&0&0&0&0&1&*\end{pmatrix}$$

reduced row echelon form

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&*\\0&1&0&*\\0&0&1&*\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&*&0&0&0&*&*&0&*\\0&0&0&1&0&0&*&*&0&*\\0&0&0&0&1&0&*&*&0&*\\0&0&0&0&0&1&*&*&0&*\\0&0&0&0&0&0&0&0&1&*\end{pmatrix}$$

Reduced row echelon form and solution

- If, by a sequence of elementary row operations, the augmented matrix for a system of linear equations is put in reduced row echelon form, then the solution set can be obtained either by inspection or by converting certain linear equations to parametric form

Example

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

$$\begin{array}{ccccccr} x_1 & & & & = & 3 \\ & x_2 & & & = & -1 \\ & & x_3 & & = & 0 \\ & & & x_4 & = & 5 \end{array}$$

Reduced row echelon form and solution

Example

Suppose by elementary row operations, the augmented matrix for a linear system in the unknowns x, y, z has been reduced to the given reduced row echelon form

$$(a)\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad (b)\left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad (c)\left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(a) $0x + 0y + 0z = 1$, a contradiction. No solution

(b) Last row: $0x + 0y + 0z = 0$, no restrictions on x, y, z, can be omitted. We have

$$\begin{aligned} x + 3z &= -1 \\ y - 4z &= 2 \end{aligned}$$

Reduced row echelon form and solution

Example

$$(b)\ \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \begin{array}{ccccccc} x & + & & & 3z & = & -1 \\ & & y & - & 4z & = & 2 \end{array}$$

- x and y correspond to the leading 1's in the augmented matrix: *leading variables*.
- The remaining variables (in this case z) are called *free variables*.
- Solving for the leading variables in terms of the free variables gives

$$x = -1 - 3z, \quad y = 2 + 4z$$

- z can be treated as a parameter and when assigned an arbitrary value t, determines values for x and y

$$\text{solution set} = \{\, (-1 - 3t,\ 2 + 4t,\ t) \ \mid\ t \in \mathbb{R} \,\}.$$

Reduced row echelon form and solution

Example

$$(c)\ \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow x - 5y + z = 4$$

- The solution is a plane in the three-dimensional space
- Leading variable: x, free variables: $y,\ z$
- The solutions are given by

$$\{\,(4 + 5s - t,\ s,\ t)\ \mid\ s,\ t \in \mathbb{R}\,\}.$$

System of linear equations

- Introduction
- Augmented matrices and echelon forms
- Elimination methods
- Homogeneous linear system
- Matrices and their inverses

Elimination methods

We will illustrate the step-by-step elimination procedure to reduce any matrix to reduced row echelon form using an example

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{pmatrix} \color{red}{0} & 0 & -2 & 0 & 7 & 12 \\ \color{red}{2} & 4 & -10 & 6 & 12 & 28 \\ \color{red}{2} & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{pmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Elimination methods

$$\begin{pmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{pmatrix}$$

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row echelon form.

find leftmost nonzero column

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{pmatrix}$$

make leading 1 in R_1

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{pmatrix}$$

Elimination methods

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{pmatrix}$$

0 below the leading 1

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

return to step 1, find the leftmost nonzero column

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

Elimination methods

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

leading 1 in R_1

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

The entire matrix is now in row echelon form - this procedure is called *Gaussian elimination*.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

The whole procedure is called *Gauss–Jordan elimination*

Forward and backward phases

- Gauss–Jordan elimination consists of two parts
- *forward phase*: zeros are introduced below the leading 1's - row echelon form, Gaussian elimination
- *backward phase*: zeros are introduced above the leading 1's

Gauss–Jordan elimination – example

Example

$$\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}$$

$$\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right)$$

$R_2 \to R_2 - 2R_1$, $R_4 \to R_4 - 2R_1$

$$\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right)$$

Gauss–Jordan elimination – example

Example

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right) \xrightarrow{R_2 \to -1R_2} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

$$\xrightarrow[R_4 \to R_4 - 4R_2]{R_3 \to R_3 - 5R_2} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to \frac{1}{6} R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dfrac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now we have used Gaussian elimination to reach row echelon form

Gauss–Jordan elimination – example

Example

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The matrix is now in reduced row echelon form. The corresponding linear system is

$$\begin{array}{ccccccccccc} x_1 & + & 3x_2 & & & + & 4x_4 & + & 2x_5 & & = & 0 \\ & & & & x_3 & + & 2x_4 & & & & = & 0 \\ & & & & & & & & & x_6 & = & \frac{1}{3} \end{array}$$

Gauss–Jordan elimination – example

Example

$$\begin{array}{rcrcrcrcrcrcl} x_1 & + & 3x_2 & & & + & 4x_4 & + & 2x_5 & & & = & 0 \\ & & & & x_3 & + & 2x_4 & & & & & = & 0 \\ & & & & & & & & & & x_6 & = & \frac{1}{3} \end{array}$$

Solving for the leading variables

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \frac{1}{3} \end{aligned}$$

We have three free variables and hence three parameters. The solution set is given by

$$\left\{ \left(-3r - 4s - 2t,\ r,\ -2s,\ s,\ t,\ \frac{1}{3} \right) \;\middle|\; r,\ s,\ t \in \mathbb{R} \right\}$$

Gaussian elimination – example

Example

Note that, with just Gaussian elimination, we have

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This corresponds to the following linear system:

$$\begin{array}{ccccccccccccc} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ & & & & x_3 & + & 2x_4 & & & + & 3x_6 & = & 1 \\ & & & & & & & & & & x_6 & = & \frac{1}{3} \end{array}$$

Gaussian elimination – example

Example

$$\begin{array}{rcrcrcrcrcrcl} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ & & & & x_3 & + & 2x_4 & & & + & 3x_6 & = & 1 \\ & & & & & & & & & & x_6 & = & \frac{1}{3} \end{array}$$

Solving for the leading variables

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \frac{1}{3} \end{aligned}$$

Let $x_2 = r,\ x_4 = s,\ x_5 = t$, then

$$x_3 = 1 - 2s - 1 = -2s, \quad x_1 = -3r + 2(-2s) - 2t = -3r - 4r - 2t.$$

We get the same solution set.

Solutions for linear systems – example

Example

Linear systems with augmented matrix in row echelon form

$$\left(\begin{array}{cccc|c} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

The last row corresponds to

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1,$$

a contradiction, no solution

Solutions for linear systems – example

Example

Linear systems with augmented matrix in row echelon form

$$\left(\begin{array}{cccc|c} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- The last row has no restrictions on the variables
- Three leading variables: x_1, x_2, x_3. One free variable: x_4

$$x_4 = t, \quad x_3 = 9 - 6t, \quad x_2 + 2x_3 = 1 + 4t \Longrightarrow x_2 = 1 + 4t - 2(9 - 6t) = -17 + 16t$$

$$x_1 - 3x_2 + 7x_3 + 2x_4 = 5 \Longrightarrow x_1 = 5 - 2t - 7(9 - 6t) + 3(-17 + 16t)$$

Infinitely many solutions

Solutions for linear systems – example

Example

Linear systems with augmented matrix in row echelon form

$$\left(\begin{array}{cccc|c} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$x_4 = 0, \quad x_3 = 9, \quad x_2 + 2x_3 = 1 \Longrightarrow x_2 = 1 - 18 = -17$$

$$x_1 - 3x_2 + 7x_3 = 5 \Longrightarrow x_1 = 5 - 7 \times 9 + 3 \times (-17) = -109$$

A unique solution.

System of linear equations

- Introduction
- Augmented matrices and echelon forms
- Elimination methods
- Homogeneous linear system
- Matrices and their inverses

Homogeneous linear system

Definition

A system of linear equations is said to be *homogeneous* if the constant terms are all zero.

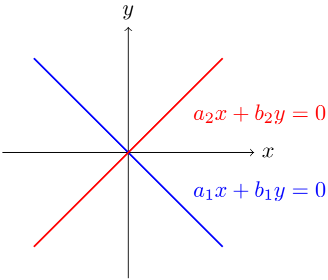
$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad &\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}$$

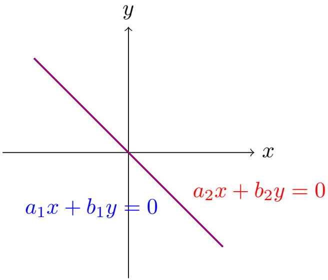
- Every homogeneous system of linear equations is consistent: $\mathbf{0}$ as a solution.
- $\mathbf{0}$: *trivial solution*
- Other solutions: *nontrivial solutions*
- There are only two possibilities for a homogeneous system:
  - Only the trivial solution
  - Infinitely many solutions, including the trivial one

Homogeneous linear system of two equations in two unknowns

$$\begin{aligned} a_1x + b_1y &= 0 \quad (a_1, b_1 \text{ not both zero}) \\ a_2x + b_2y &= 0 \quad (a_2, b_2 \text{ not both zero}) \end{aligned}$$

The graphs of the equations are lines through the origin, and the trivial solution corresponds to the point of intersection at the origin





Only the trivial solution

Infinitely many solutions

Homogeneous linear system – example

Example

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & 0 \\ & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 0 \\ 2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 0 \end{array}$$

- The coefficients of the unknowns in this system are the same as in the previous example. Only constant terms differ
- The elementary row operations do not affect a column of zeros
  - Multiply a row by a nonzero constant
  - Interchange two rows
  - Add a constant times one row to another
- Hence the reduced row echelon form is

$$\left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Homogeneous linear system – example

Example

$$\left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{ccccccccccc} x_1 & + & 3x_2 & & & + & 4x_4 & + & 2x_5 & & & = & 0 \\ & & & & x_3 & + & 2x_4 & & & & & = & 0 \\ & & & & & & & & & & x_6 & = & 0 \end{array}$$

Solving for the leading variables

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned}$$

Homogeneous linear system – example

Example

$$
\begin{aligned}
x_1 &= -3x_2 - 4x_4 - 2x_5 \\
x_3 &= -2x_4 \\
x_6 &= 0
\end{aligned}
$$

We have three free variables and hence three parameters. The solutions are given by

$$\{\, (-3r - 4s - 2t,\ r,\ -2s,\ s,\ t,\ 0) \ \mid\ r, s, t \in \mathbb{R} \,\}$$

Homogeneous linear system

The example shows

- Elementary row operations do not alter columns of zeros in a matrix, so the reduced row echelon form of the augmented matrix for a homogeneous linear system has a final column of zeros. This implies that the linear system corresponding to the reduced row echelon form is homogeneous, just like the original system.
- The last row is ignored because it corresponds to

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 = 0,$$

  which does not impose any restrictions on the unknowns. Depending on whether or not the reduced row echelon form of the augmented matrix for a homogeneous linear system has any rows of zero, the linear system corresponding to that reduced row echelon form will either have the same number of equations as the original system or it will have fewer.

Homogeneous linear system

- Consider a homogeneous linear system with n unknowns
- Suppose the reduced row echelon form of the augmented matrix has r nonzero rows

$$\begin{array}{ccccccc} x_{k_1} & & & & +\sum() & = & 0 \\ & x_{k_2} & & & +\sum() & = & 0 \\ & & \ddots & & \vdots & = & 0 \\ & & & x_{k_r} & +\sum() & = & 0 \end{array}$$

- Each nonzero row corresponds to a leading $1 \to$ a leading variable
- Thus the system has r leading variables and $n - r$ free variables

Free Variable Theorem for Homogeneous Systems

Theorem (Free Variable Theorem for Homogeneous Systems)

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

- If a homogeneous linear system has m equations in n unknowns, where $m < n$
- Number of nonzero rows cannot be more than $m \implies r < n$
- Thus, there will be at least one free variable $\implies$ infinitely many solutions

Corollary

A homogeneous linear system with more unknowns than equations has infinitely many solutions.

Facts about row echelon forms

- Every matrix has a unique reduced row echelon form
- Row echelon forms are not unique - different sequences of elementary row operations can result in different row echelon forms
- All row echelon forms of a matrix A have the same number of zero rows and the leading 1's always occur in the same positions (*pivot positions*) of A

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Can we divide by a matrix?

- We discussed that a linear system can be expressed as

$$A\boldsymbol{x} = \boldsymbol{b}$$

- It is tempting to compute

$$\boldsymbol{x} = \frac{\boldsymbol{b}}{A}$$

- We haven't discussed how to divide by matrix and if it is even possible.

Properties of zero matrices

Theorem

For any matrix A and any $\alpha \in \mathbb{R}$

- $A + O = O + A = A$
- $A - O = A$
- $A - A = A + (-A) = O$
- $0A = O$
- *If* $\alpha A = O$, *then* $\alpha = 0$ *or* $A = O$

Identity matrices

Recall

Theorem

Let $A \in \mathcal{M}_{m\times n}$ be any matrix, $I_n \in \mathcal{M}_{n\times n}$ and $I_m \in \mathcal{M}_{m\times m}$ be identity matrices. We have

$$AI_n = I_m A = A$$

We have

Theorem

If R is the reduced row echelon form of a matrix $A \in \mathcal{M}_{n\times n}$, then either R has at least one row of zeros or $R = I_n$.

Reduced row echelon form and identity matrix

Theorem

If R is the reduced row echelon form of a matrix $A \in \mathcal{M}_{n\times n}$, then either R has at least one row of zeros or $R = I_n$.

Proof.

Suppose

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix}.$$

If R does not contain any row of zeros, each of the n rows has a leading 1. Since these leading 1's occur progressively farther to the right as we move down the matrix, each of them must occur on the main diagonal. Furthermore, other entries in the same column as one of these 1's are zero, we have $R = I_n$. □

Multiplicative inverse

- Given any $\alpha \in \mathbb{R}$, $\alpha \neq 0$, α has a *multiplicative inverse* $\alpha^{-1} = 1/\alpha$ such that

$$\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1.$$

- How about matrices?

Invertible matrices

Definition

Given $A \in \mathcal{M}_{n\times n}$, if there exists $B \in \mathcal{M}_{n\times n}$ such that

$$AB = BA = I_n,$$

then A (also B) is said to be *invertible* (or *nonsingular*) and B is called an *inverse* of A. If no such matrix B can be found, then A is said to be *singular*.

Remark

If B is an inverse of A, then A is an inverse of B.

Example

An identity matrix I_n is invertible.

Invertible matrices – example

Example

$$A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

Then

$$AB = I, \quad BA = I$$

A, B are invertible they are inverses of one another.

Singular matrices – example

Example

Let

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{pmatrix}.$$

Take any matrix $B \in \mathcal{M}_{3\times 3}$, let $\boldsymbol{b}_1^\top,\ \boldsymbol{b}_2^\top,\ \boldsymbol{b}_3^\top$ denote the columns of B.
The 3rd column of BA can be expressed as a linear combination of the columns of B in which the coefficients in the linear combination are the entries from the 3rd column of A:

$$0\boldsymbol{b}_1^\top + 0\boldsymbol{b}_2^\top + 0\boldsymbol{b}_3^\top = \mathbf{0}.$$

Thus $BA \neq I_3$. And A is a singular matrix.

Remark

Similarly, it can be shown that any square matrix with a row or column of zeros is singular.

Uniqueness of inverse

Theorem

If B, C are both inverses of A, then $B = C$.

Proof.

Since B is an inverse of A, we have $BA = I$. Multiply both sides on the right by C

$$BAC = IC = C.$$

By associative law of matrix multiplicaiton

$$BAC = B(AC) = BI = B.$$

$\square$

Remark

- As a consequence, we now speak of "the" inverse of a matrix A.
- If A is invertible, we denote its inverse by A^{-1}

Solve linear system

If A is invertible and

$$A\boldsymbol{x} = \boldsymbol{b},$$

then

$$A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b} \Longrightarrow I\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

Note that

$$I_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad I_n \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Consequently, we get the solution for the system

Inverse of 2×2 matrices

Theorem

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible iff $ad - bc \neq 0$*, in which case the inverse is given by*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Note

The quantity $ad - bc$ is called the *determinant* of A. We write

$$\det(A) = ad - bc, \quad \text{or} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Inverse of 2×2 matrices – example

Example

$$A = \begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix}$$

The determinant of A is

$$\det(A) = 6 \times 2 - 1 \times 5 = 7,$$

and

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ -5 & 6 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{pmatrix}$$

We can verify that

$$AA^{-1} = A^{-1}A = I_2$$

Inverse of 2×2 matrices – example

Example

$$A = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}$$

A is not invertible since

$$\det(A) = (-1) \times (-6) - 2 \times 3 = 0$$

Solution of a linear system by matrix inversion

Consider the linear system

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned}$$

It can be represented as

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{where,} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Suppose A is invertible, we have

$$A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The solution is given by

$$\left(\frac{du - bv}{ad - bc}, \quad \frac{-cu + av}{ad - bc} \right)$$

Inverse of matrix product

Theorem

Suppose $A, B \in \mathcal{M}_{n \times n}$ *are both invertible, then* AB *is invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof.

We have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Similarly $(B^{-1}A^{-1})(AB) = I$ □

Remark

It can be proven that: A product of any number of invertible matrices is invertible,and the inverse of the product is the product of the inverses in the reverse order.

Inverse of matrix product – example

Consider

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 7 & 6 \\ 9 & 8 \end{pmatrix}, \quad (AB)^{-1} = \begin{pmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{pmatrix}$$

And

$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{pmatrix}, \quad B^{-1}A^{-1} = \begin{pmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{pmatrix}$$

Recall – Powers of a square matrix

- Square matrices are the only matrices that can be multiplied by themselves
- $A \in \mathcal{M}_{m\times n}$, AA can be computed iff $m = n$

Definition

For $A \in \mathcal{M}_{n\times n}$, the *(nonnegative) powers* of A are given by

$$A^0 = I_n, \quad A^1 = A, \quad A^k = A^{k-1}A \text{ for } k \geq 2.$$

Example

$$A = \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}, \quad A^2 = AA = \begin{pmatrix} 0 & 5 \\ -20 & 5 \end{pmatrix}, \quad A^3 = A^2A = \begin{pmatrix} -20 & 15 \\ -60 & -5 \end{pmatrix}.$$

Note

For any nonnegative integers r and s, $A^rA^s = A^{r+s}$, $(A^r)^s = A^{rs}$

Properties of exponents

If A is invertible, for a positive integer n, define

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1}$$

Theorem

If A is invertible and n is a nonnegative integer, then

- A^{-1} *is invertible and* $(A^{-1})^{-1} = A$
- A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$
- αA *is invertible for any nonzero* $\alpha \in \mathbb{R}$, *and* $(\alpha A)^{-1} = \alpha^{-1}A^{-1}$

Proof.

$$(\alpha A)(\alpha^{-1}A^{-1}) = \alpha\alpha^{-1}AA^{-1} = I$$

□

Properties of exponents – example

Example

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

Then

$$A^{-3} = (A^{-1})^3 = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 41 & -30 \\ -15 & 11 \end{pmatrix}$$

And

$$A^3 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 11 & 30 \\ 15 & 41 \end{pmatrix}$$

As shown from the previous theorem

$$(A^3)^{-1} = \frac{1}{11 \times 41 - 30 \times 15} \begin{pmatrix} 41 & -30 \\ -15 & 11 \end{pmatrix} = \begin{pmatrix} 41 & -30 \\ -15 & 11 \end{pmatrix}$$

Properties of exponents – example

Example

$$A = \begin{pmatrix} 4 & 8 \\ 12 & 16 \end{pmatrix}, \quad A = 4\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Then inverse of A simplifies as

$$A^{-1} = \frac{1}{4} \times \frac{1}{1 \times 4 - 2 \times 3} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{8} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} \end{pmatrix}$$

We can verify that

$$A^{-1} = \frac{1}{4 \times 16 - 8 \times 12} \begin{pmatrix} 16 & -8 \\ -12 & 4 \end{pmatrix} = -\frac{1}{32} \begin{pmatrix} 16 & -8 \\ -12 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} \end{pmatrix}$$

Properties of exponents – example

Example

With real number arithmetic, we have a commutative law for multiplication, we can write

$$(\alpha + \beta)^2 = \alpha^2 + \alpha\beta + \beta\alpha + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2, \quad \alpha, \beta \in \mathbb{R}$$

However, for matrices A, B

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Only when A and B commute, we have

$$(A + B)^2 = A^2 + 2AB + B^2$$

Matrix polynomial

$A \in \mathcal{M}_{n\times n}$

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_m x^m$$

is a polynomial ($\alpha_i \in \mathbb{R}$). We define the $n \times n$ matrix $p(A)$ to be

$$p(A) = \alpha_0 I_n + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_m A^m,$$

such an expression is called a *matrix polynomial in* A

Example

$$p(x) = x^2 - 2x - 3, \quad A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$p(A) = A^2 - 2A - 3I = \begin{pmatrix} 1 & 4 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} -2 & 4 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Inverse of transpose

Theorem

Given an invertible matrix A, $A^\top$ is also invertible, and

$$(A^\top)^{-1} = (A^{-1})^\top.$$

Proof.

$$(A^\top)(A^{-1})^\top = (A^{-1}A)^\top = I^\top = I$$

□

Recall:

$$(AB)^\top = B^\top A^\top$$

Inverse of transpose – example

Example

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^\top = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Suppose A is invertible $\det(A) = ad - bc \neq 0$, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad (A^\top)^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

We can see that

$$(A^\top)^{-1} = (A^{-1})^\top$$