Final Exam

- Time: 10:30-13:00
- Toilet breaks are NOT allowed
- Write your answers on the provided answer sheets. Additional sheets will be supplied upon request. Please ensure that your full name is clearly written on each page of the answer sheets.
- Include detailed computation steps for all solutions. Answers without supporting calculations will receive a score of zero.

Question 1. (15 marks) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}3x_1+2x_2\\x_1+x_2\\0\end{pmatrix}$$

Let A denote the standard matrix for T.

(a) (1 mark) Determine the matrix A.

(b) (2 marks) Find a basis for the row space of A.

(c) (1 mark) Find a basis for the range of T.

(d) (1 mark) Find a basis for the kernel of T.

(e) (1 mark) Determine the rank and nullity of A.

(f) (6 marks) Find the characteristic equation, the eigenvalues, and the bases for the eigenspaces of the matrix A.

(g) (1 mark) Determine whether A is diagonalizable, and if so, find a diagonalization of A.

(h) (2 marks) Use Doolittle's method to compute an LU decomposition of A.

Solution.

(a) The standard matrix is obtained by computing the images of the standard basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathbb{R}^3$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\1\\0\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Therefore, the standard matrix is

$$A = \begin{pmatrix}3&2&0\\1&1&0\\0&0&0\end{pmatrix} \quad \text{(1 mark)}$$

(b) The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (1 mark)}$$

Thus a basis for the row space of A is

$$\{ (1, \ 0, \ 0), (0, \ 1, \ 0) \}. \quad \text{(1 mark)}$$

(c) Since the range of T corresponds to the column space of A. By observing R, we know that a basis for the conlumn space of A is given by

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{(1 mark)}$$

(d) From R, we can deduce that a vector form for the general solution of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ is

$$\boldsymbol{x} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}. \quad \text{(0.5 marks)}$$

Thus a basis for the kernel of T is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \quad \text{(0.5 marks)}$$

(e) $\operatorname{rank}(A) = 2$, (0.5 marks) $\operatorname{nullity}(A) = 1$. (0.5 marks)

(f) The characteristic equation of A is given by

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -2 & 0 \\ -1 & \lambda - 1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda((\lambda-3)(\lambda-1)-2) = \lambda(\lambda-(2-\sqrt{3}))(\lambda-(2+\sqrt{3})) = 0 \quad \text{(1 mark)}$$

Solving for λ, we obtain the eigenvalues:

$$\lambda_1 = 2 + \sqrt{3}, \quad \lambda_2 = 2 - \sqrt{3}, \quad \lambda_3 = 0 \quad \text{(0.5 marks)}$$

To determine a basis for the eigenspace corresponding to $\lambda_1 = 2 + \sqrt{3}$, we solve the homogeneous system

$$((2 + \sqrt{3})I - A)\boldsymbol{x} = \mathbf{0},$$

where

$$(2 + \sqrt{3})I - A = \begin{pmatrix} \sqrt{3} - 1 & -2 & 0 \\ -1 & \sqrt{3} + 1 & 0 \\ 0 & 0 & 2 + \sqrt{3} \end{pmatrix}. \quad \text{(0.5 marks)}$$

We find the general solution:

$$\boldsymbol{x} = t \begin{pmatrix} 1 + \sqrt{3} \\ 1 \\ 0 \end{pmatrix}. \quad \text{(0.5 marks)}$$

Thus, a basis for the eigenspace corresponding to $\lambda_1 = 2 + \sqrt{3}$ is

$$\left\{ \begin{pmatrix} 1+\sqrt{3} \\ 1 \\ 0 \end{pmatrix} \right\}. \quad \text{(0.5 marks)}$$

To determine a basis for the eigenspace corresponding to $\lambda_2 = 2 - \sqrt{3}$, we solve the homogeneous system

$$((2-\sqrt{3})I - A)\boldsymbol{x} = \mathbf{0},$$

where

$$(2-\sqrt{3})I - A = \begin{pmatrix} -1-\sqrt{3} & -2 & 0 \\ -1 & 1-\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{pmatrix}. \quad \text{(0.5 marks)}$$

We find the general solution:

$$\boldsymbol{x} = t\begin{pmatrix} 1-\sqrt{3} \\ 1 \\ 0 \end{pmatrix}. \quad \text{(0.5 marks)}$$

Thus, a basis for the eigenspace corresponding to $\lambda_2 = 2 - \sqrt{3}$ is

$$\left\{ \begin{pmatrix} 1-\sqrt{3} \\ 1 \\ 0 \end{pmatrix} \right\}. \quad \text{(0.5 marks)}$$

Similarly, to determine a basis for the eigenspace corresponding to $\lambda_3 = 0$, we solve the homogeneous system

$$\begin{pmatrix} -3 & -2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x} = \mathbf{0}. \quad \text{(0.5 marks)}$$

We obtain the general solution:

$$\boldsymbol{x} = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad \text{(0.5 marks)}$$

Thus, a basis for the eigenspace corresponding to $\lambda_3 = 0$ is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \quad \text{(0.5 marks)}$$

(g) Since A has three linearly independent eigenvectors, it is diagonalizable.

We form the matrix P whose column vectors are the 3 basis vectors for the eigenspaces of A

$$P = \begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(0.5 marks)}$$

and

$$P^{-1}AP = \begin{pmatrix} 2+\sqrt{3} & 0 & 0 \\ 0 & 2-\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(0.5 marks)}$$

(h)
$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \quad \text{(0.5 marks)}$$

First row

$$u_{11} = 3, \quad u_{12} = 2, \quad u_{13} = 0 \quad \text{(0.5 marks)}$$

Second row

$$\begin{aligned} \ell_{21}u_{11} &= 1 \\ \ell_{21}u_{12} + u_{22} &= 1 \\ \ell_{21}u_{13} + u_{23} &= 0 \end{aligned} \implies \begin{aligned} \ell_{21} &= \frac{1}{3} \\ u_{22} &= 1 - \frac{2}{3} = \frac{1}{3} \\ u_{23} &= 0 \end{aligned} \quad \text{(0.5 marks)}$$

Third row

$$\begin{aligned} \ell_{31}u_{11} &= 0 \\ \ell_{31}u_{12} + \ell_{32}u_{22} &= 0 \\ \ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} &= 0 \end{aligned} \implies \begin{aligned} \ell_{31} &= 0 \\ \ell_{32} &= 0 \\ u_{33} &= 0 \end{aligned} \quad \text{(0.5 marks)}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Question 2. (10 marks) The table below lists 9 tasks that comprise a project, as well as their processing times and precedence relationships.

Task	**Processing Time**	**Precedence Relationships**
a	2	
b	5	
c	1	a
d	2	a, b
e	4	b
f	6	c, e
g	7	d
h	6	e
i	2	f, g

(a) (2 marks) Draw the project digraph.

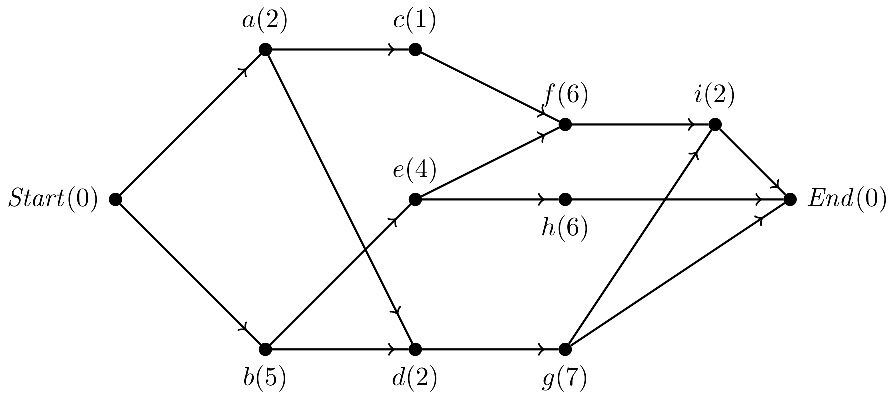
(b) (5 marks) Use the Critical Path Algorithm to find a schedule with 2 processors.

(c) (2 marks) Use the Critical Path Algorithm to find a schedule with 3 processors.

(d) (1 mark) Determine if either schedule is optimal.

Solution.

(a) Project digraph (2 marks)



(b) Critical time (2 marks)

$$\begin{aligned}
\text{ct}[i] &= \text{pt}(i) + \text{ct}[end] = 2 + 0 = 2,\\
\text{ct}[h] &= \text{pt}(h) + \text{ct}[end] = 6 + 0 = 6,\\
\text{ct}[f] &= \text{pt}(f) + \text{ct}[i] = 6 + 2 = 8,\\
\text{ct}[g] &= \text{pt}(g) + \text{ct}[i] = 7 + 2 = 9,\\
\text{ct}[e] &= \text{pt}(e) + \text{ct}[f] = 4 + 8 = 12,\\
\text{ct}[d] &= \text{pt}(d) + \text{ct}[g] = 2 + 9 = 11,\\
\text{ct}[c] &= \text{pt}(c) + \text{ct}[f] = 1 + 8 = 9,\\
\text{ct}[b] &= \text{pt}(b) + \text{ct}[e] = 5 + 12 = 17,\\
\text{ct}[a] &= \text{pt}(a) + \text{ct}[d] = 2 + 11 = 13,\\
\text{ct}[start] &= \text{pt}(start) + \text{ct}[b] = 0 + 17 = 17
\end{aligned}$$

Critical path: $Start \to b \to e \to f \to i \to End$. (0.5 marks)

Critical path priority list: $b - a - e - d - c - g - f - h - i$. (0.5 marks)

Schedule (2 marks)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
P_1	b	b	b	b	b	e	e	e	e	f	f	f	f	f	f	i	$*$	$*$	$*$	$*$
P_2	a	a	c	$*$	$*$	d	d	g	g	g	g	g	g	g	h	h	h	h	h	h

(c) Schedule (2 marks)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
P_1	b	b	b	b	b	e	e	e	e	f	f	f	f	f	f	i	i
P_2	a	a	c	$*$	$*$	d	d	g	g	g	g	g	g	g	$*$	$*$	$*$
P_3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	h	h	h	h	h	h	$*$	$*$

(d) For 2 processors, we have

$$OPT \geq \frac{\text{sum of processing times}}{\text{number of processors}} \implies OPT \geq \frac{35}{2} = 17.5 \implies OPT \geq 18$$

We also know that

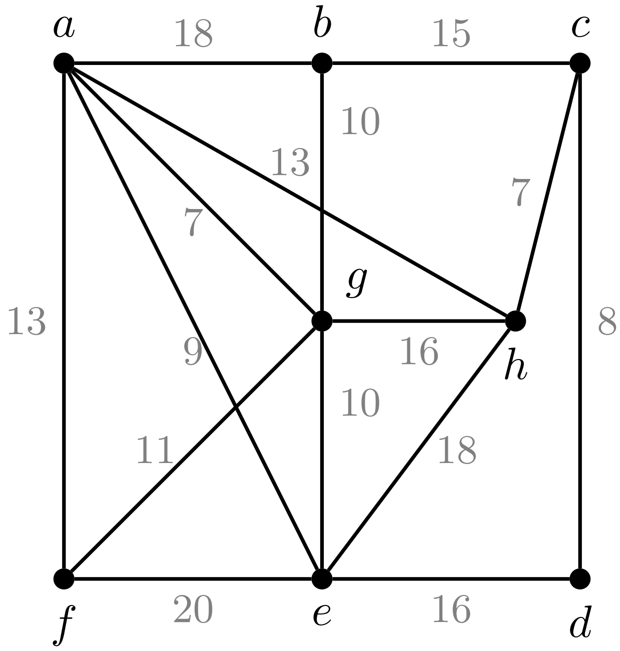
$$OPT \geq \text{ct}[Start] \implies OPT \geq 17.$$

Thus the schedule with 2 processors is not optimal (0.5 marks) but the schedule with 3 processors is optimal (0.5 marks).

Question 3. (7 marks) Find a minimum spanning tree for the graph below using

(a) (3.5 marks) Kruskal's Algorithm

(b) (3.5 marks) Prim's Algorithm



Solution. Weight of MST: 65

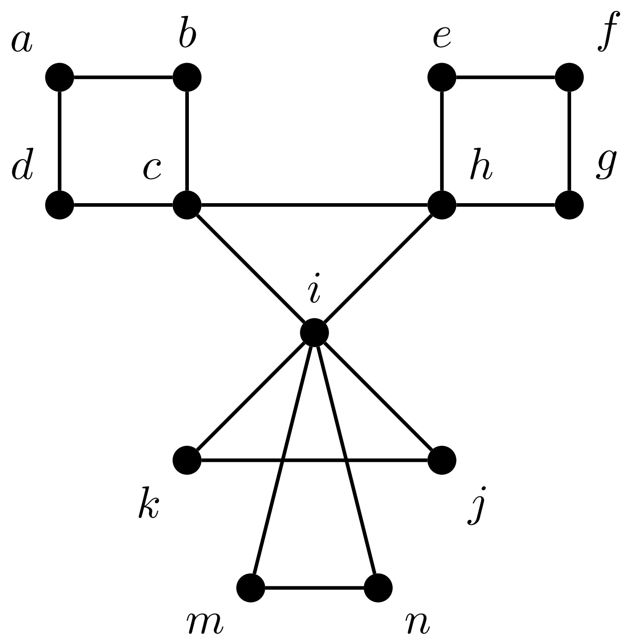
(a) Kruskal's Algorithm: $ag, ch, cd, ae, bg, gf, ah$

(b) Prim's Algorithm: $ag, ae, gb, gf, ah, hc, cd$

Question 4. (9 marks) Complete each of the following on the graph shown below.

(a) (5 marks) Find the breadth-first search tree with root a.

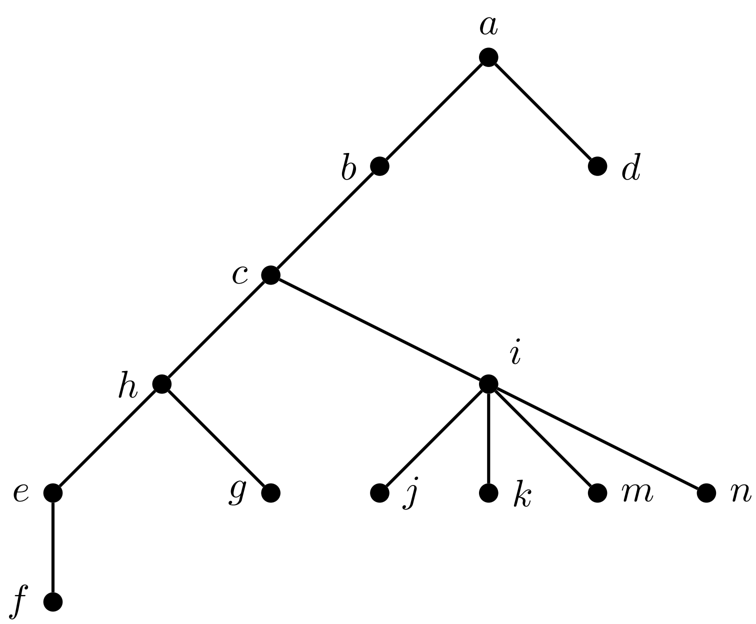
(b) (4 marks) Find the depth-first search tree with root a.



Solution.

(a) BFS tree with root a

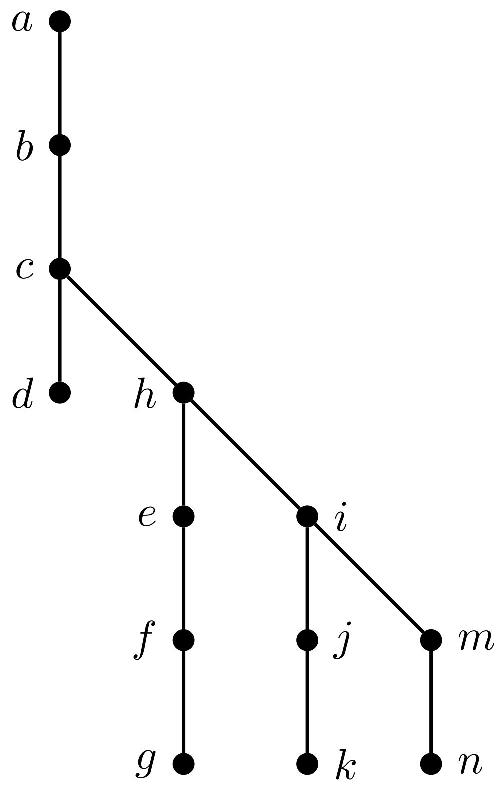
- Level 1: bd (1 mark)
- Level 2: c (1 mark)
- Level 3: hi (1 mark)
- Level 4: $egjkmn$ (1 mark)
- Level 5: f (1 mark)



(b) DFS tree with root a

- $abcd$ (1 mark)
- $chefg$ (1 mark)

- $hijk$ (1 mark)
- imn (1 mark)



Question 5. (9 marks) Apply the Gale–Shapley algorithm to find a stable matching based on the set of preferences provided below, considering both scenarios:

1. Men propose
2. Women propose

Alice:	r	>	s	>	t	>	v		Rich:	a	>	d	>	b	>	c
Beth:	s	>	r	>	v	>	t		Stefan:	a	>	c	>	d	>	b
Cindy:	v	>	t	>	r	>	s		Tom:	c	>	b	>	d	>	a
Dahlia:	t	>	v	>	s	>	r		Victor:	c	>	d	>	b	>	a

Solution.

1.
   - Step 1. Rich-Alice, Stefan-Alice, Tom-Cindy, Victor-Cindy (2 marks)
   - Step 3. Alice accepts Rich and reject Stefan; Cindy accepts Victor and rejects Tom (2 marks)
   - Step 4. Stefan - Dahlia, Tom - Beth (1 mark)
   - Step 2. both proposals are different, both women accept the proposals (0.5 marks)
   - Stable matching: Rich-Alice, Victor-Cindy, Stefan - Dahlia, Tom - Beth (0.5 marks)
2.
   - Step 1. Alice-Rich, Beth-Stefan, Cindy-Victor, Dahlia-Tom (2 marks)
   - Step 2. All proposals are different, all men accept the proposals (1 mark)