Final Exam

- Time: 10:30-13:00
- Toilet breaks are NOT allowed
- Write your answers on the provided answer sheets. Additional sheets will be supplied upon request. Please ensure that your full name is clearly written on each page of the answer sheets.
- Include detailed computation steps for all solutions. Answers without supporting calculations will receive a score of zero.

Question 1. (15 marks) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \\ 0 \end{pmatrix}$$

Let A denote the standard matrix for T.

(a) (1 mark) Determine the matrix A.

(b) (2 marks) Find a basis for the row space of A.

(c) (1 mark) Find a basis for the range of T.

(d) (1 mark) Find a basis for the kernel of T.

(e) (1 mark) Determine the rank and nullity of A.

(f) (6 marks) Find the characteristic equation, the eigenvalues, and the bases for the eigenspaces of the matrix A.

(g) (1 mark) Determine whether A is diagonalizable, and if so, find a diagonalization of A.

(h) (2 marks) Use Doolittle's method to compute an LU decomposition of A.

Solution.

(a) The standard matrix is obtained by computing the images of the standard basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathbb{R}^3$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the standard matrix is

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{(1 mark)}$$

(b) The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (1 mark)}$$

Thus a basis for the row space of A is

$$\{ (1, \ 0, \ 0), (0, \ 1, \ 0) \}. \quad \text{(1 mark)}$$

(c) Since the range of T corresponds to the column space of A. By observing R, we know that a basis for the conlumn space of A is given by

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\} \quad \text{(1 mark)}$$

(d) From R, we can deduce that a vector form for the general solution of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ is

$$\boldsymbol{x} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}. \quad \text{(0.5 marks)}$$

Thus a basis for the kernel of T is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \quad \text{(0.5 marks)}$$

(e) $\text{rank}\,(A) = 2$, (0.5 marks) $\text{nullity}\,(A) = 1$. (0.5 marks)

(f) The characteristic equation of A is given by

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ -1 & \lambda - 2 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda((\lambda - 2)^2 - 11) = \lambda(\lambda - 3)(\lambda - 1) = 0 \quad \text{(1 mark)}$$

Solving for λ, we obtain the eigenvalues:

$$\lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = 0 \quad \text{(0.5 marks)}$$

To determine a basis for the eigenspace corresponding to $\lambda_1 = 3$, we solve the homogeneous system

$$(3I - A)\boldsymbol{x} = \mathbf{0},$$

where

$$3I - A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \quad \text{(0.5 marks)}$$

We find the general solution:

$$\boldsymbol{x} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \quad \text{(0.5 marks)}$$

Thus, a basis for the eigenspace corresponding to $\lambda_1 = 3$ is

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}. \quad \text{(0.5 marks)}$$

To determine a basis for the eigenspace corresponding to $\lambda_2 = 1$, we solve the homogeneous system

$$(I - A)\boldsymbol{x} = \mathbf{0},$$

where

$$I - A = \begin{pmatrix}-1 & -1 & 0\\-1 & -1 & 0\\0 & 0 & 1\end{pmatrix}. \quad \text{(0.5 marks)}$$

We find the general solution:

$$\boldsymbol{x} = t\begin{pmatrix}1\\-1\\0\end{pmatrix}. \quad \text{(0.5 marks)}$$

Thus, a basis for the eigenspace corresponding to $\lambda_2 = 1$ is

$$\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}. \quad \text{(0.5 marks)}$$

Similarly, to determine a basis for the eigenspace corresponding to $\lambda_3 = 0$, we solve the homogeneous system

$$\begin{pmatrix}-2 & -1 & 0\\-1 & -2 & 0\\0 & 0 & 0\end{pmatrix}\boldsymbol{x} = \mathbf{0}. \quad \text{(0.5 marks)}$$

We obtain the general solution:

$$\boldsymbol{x} = t\begin{pmatrix}0\\0\\1\end{pmatrix}. \quad \text{(0.5 marks)}$$

Thus, a basis for the eigenspace corresponding to $\lambda_3 = 0$ is

$$\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}. \quad \text{(0.5 marks)}$$

(g) Since A has three linearly independent eigenvectors, it is diagonalizable.

We form the matrix P whose column vectors are the 3 basis vectors for the eigenspaces of A

$$P = \begin{pmatrix}1 & 1 & 0\\1 & -1 & 0\\0 & 0 & 1\end{pmatrix} \quad \text{(0.5 marks)}$$

and

$$P^{-1}AP = \begin{pmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix} \quad \text{(0.5 marks)}$$

(h)

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \quad \text{(0.5 marks)}$$

First row

$$u_{11} = 2, \quad u_{12} = 1, \quad u_{13} = 0 \quad \text{(0.5 marks)}$$

Second row

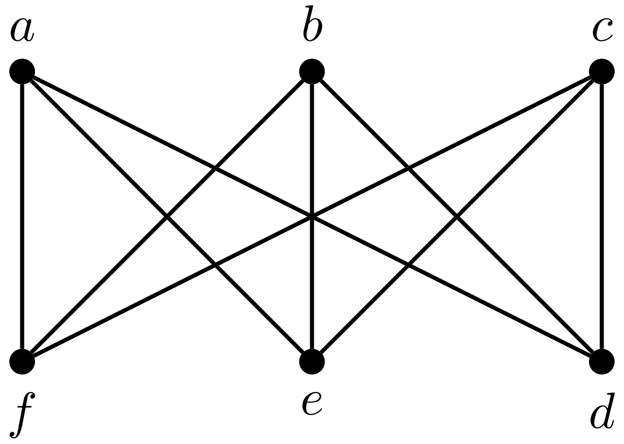
$$\begin{aligned} \ell_{21}u_{11} &= 1 \\ \ell_{21}u_{12} + u_{22} &= 2 \\ \ell_{21}u_{13} + u_{23} &= 0 \end{aligned} \implies \begin{aligned} \ell_{21} &= \frac{1}{2} \\ u_{22} &= 2 - \frac{1}{2} = \frac{3}{2} \\ u_{23} &= 0 \end{aligned} \quad \text{(0.5 marks)}$$

Third row

$$\begin{aligned} \ell_{31}u_{11} &= 0 \\ \ell_{31}u_{12} + \ell_{32}u_{22} &= 0 \\ \ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} &= 0 \end{aligned} \implies \begin{aligned} \ell_{31} &= 0 \\ \ell_{32} &= 0 \\ u_{33} &= 0 \end{aligned} \quad \text{(0.5 marks)}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Question 2. (9 marks) Consider the following graph:



(a) (4 marks) Determine an optimal Eulerization of the graph and justify its optimality (i.e., explain why the number of duplicated edges is minimal).

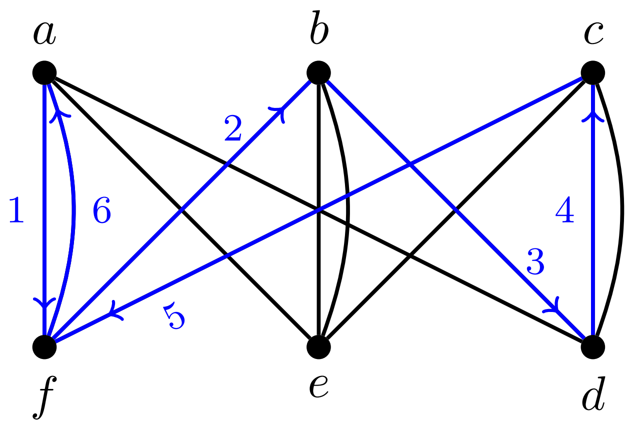
(b) (5 marks) Use Hierholzer's Algorithm to find an Eulerian circuit. Choose a to be the starting vertex in step 1 and provide visual illustrations of each step.

Solution.

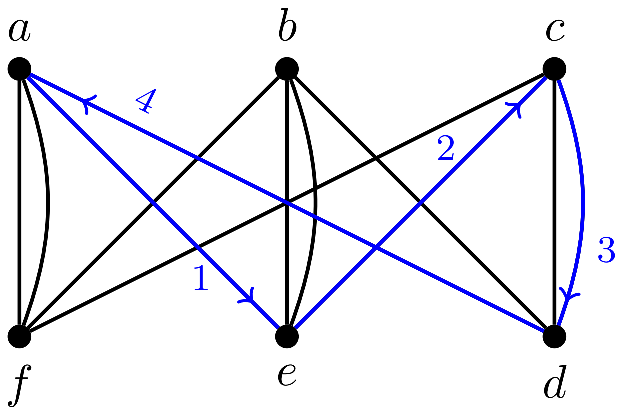
(a) In this graph, all six vertices have odd degree. To Eulerize the graph, we must add edges so that all vertices become even. The most efficient approach is to pair the odd-degree vertices into three adjacent pairs, requiring the duplication of exactly three edges. For example, we may add edges af, be, and cd. (3 marks) This is optimal because each odd vertex must have its degree increased by at least one, and adding three edges is the minimum possible to convert all six odd vertices into even-degree vertices. (1 mark)

(b) Following Hierholzer's Algorithm:

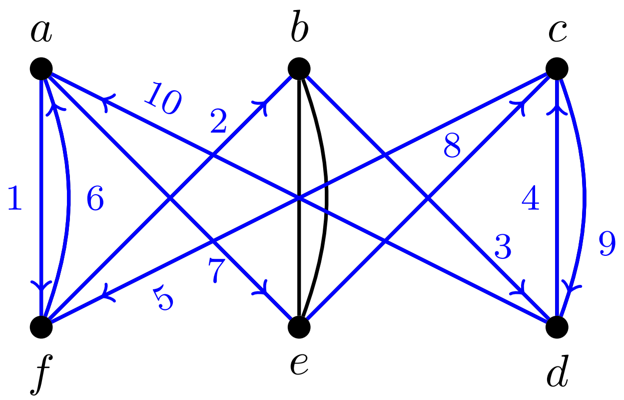
- Step 1: find an initial circuit starting at a: $afbdcfa$



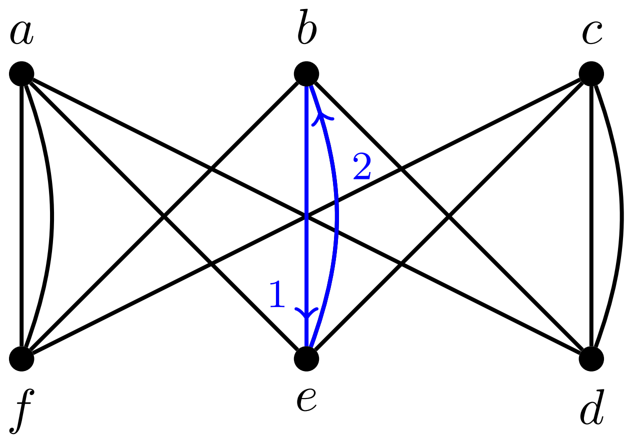
- Step 2: find a second circuit starting at a: $aecda$



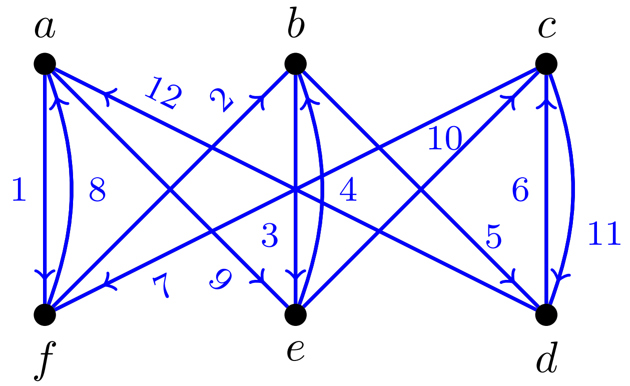
- Step 3: combine the two circuits: $afbdcfaecda$



- Step 2: find another circuit starting at b: beb



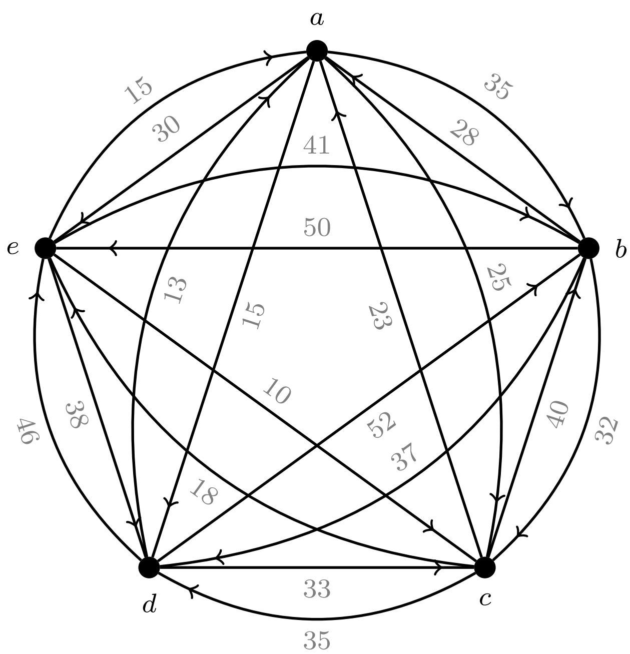
- Step 3: combine the circuits: $afbebdcfaecda$



Question 3. (5 marks) Using the digraph below,

(a) (2 marks) Apply the Undirecting Algorithm to construct the table of edge weights for the corresponding undirected clone graph.

(b) (3 marks) Using your result from 1, apply the Nearest Neighbor Algorithm with starting vertex a to find a Hamiltonian cycle. Convert your result to a directed Hamiltonian cycle in the original digraph and determine its total weight.



Solution.

(a) (2 marks) We first construct the table of edge weights for the digraph

	a	b	c	d	e
a	.	35	25	15	30
b	28	.	32	37	50
c	23	40	.	35	18
d	13	52	33	.	46
e	15	41	10	38	.

Then we can construct the table of edge weights for the undirected clone graph as follows

	a	b	c	d	e	a'	b'	c'	d'	e'
a	.	.	.	.	.	0	28	23	13	15
b	.	.	.	.	.	35	0	40	52	41
c	.	.	.	.	.	25	32	0	33	10
d	.	.	.	.	.	15	37	35	0	38
e	.	.	.	.	.	30	50	18	46	0
a'	0	35	25	15	30	.	.	.	.	.
b'	28	0	32	37	50	.	.	.	.	.
c'	23	40	0	35	18	.	.	.	.	.
d'	13	52	33	0	46	.	.	.	.	.
e'	15	41	10	38	0	.	.	.	.	.

(b)
- Step 1: starting vertex is a.
- Step 2: among all edges incident to a, aa' has the smallest weight
- Step 3: highlight a' (0.2 marks)
- Step 2: among all edges incident to a', $a'd$ has the smallest weight, highlight edge $a'd$
- Step 3: highlight vertex d (0.2 marks)
- Step 2: highlight edge dd'
- Step 3: highlight vertex d' (0.2 marks)
- Step 2: highlight edge $d'c$
- Step 3: highlight vertex c (0.2 marks)
- Step 2: highlight edge cc'
- Step 3: highlight vertex c' (0.2 marks)
- Step 2: highlight edge $c'e$
- Step 3: highlight vertex e (0.2 marks)
- Step 2: highlight edge ee'
- Step 3: highlight vertex e' (0.2 marks)
- Step 2: highlight edge $e'b$
- Step 3: highlight vertex b (0.2 marks)
- Step 2: highlight edge bb'
- Step 3: highlight vertex b (0.2 marks)
- Step 4: close the cycle by adding edge ba
- The cycle in the clone undirected graph is $aa'dd'cc'ee'bb'a$ (0.2 marks)

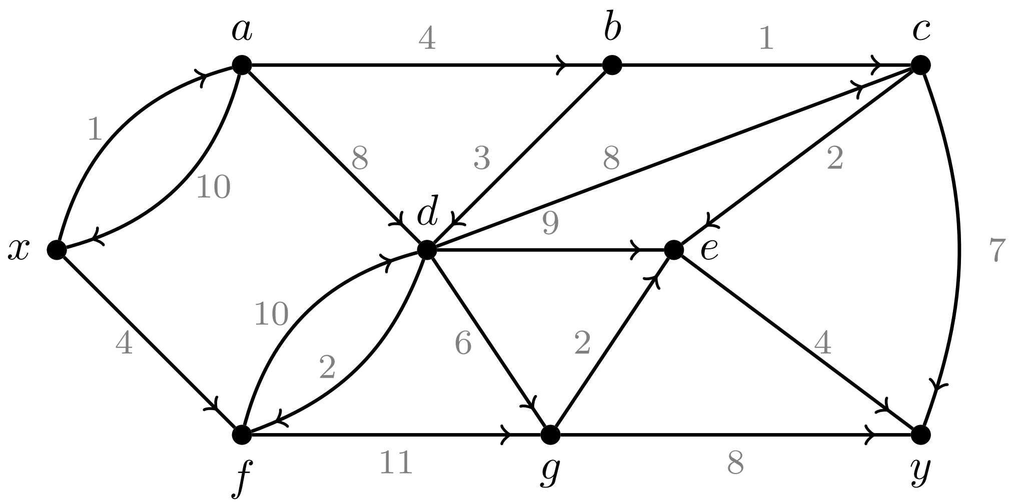
Converting to directed cycle:

$$adceba \quad \text{(0.5 marks)}$$

The total weight is

$$15 + 33 + 18 + 41 + 28 = 135 \quad \text{(0.5 marks)}$$

Question 4. (9 marks) Apply Dijkstra's Algorithm to the directed graph below, using x as the starting vertex and y as the ending vertex.

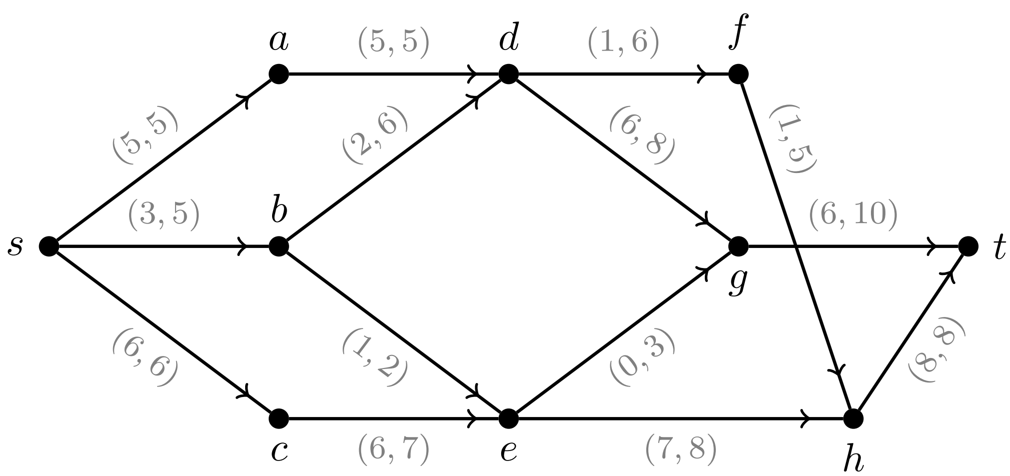


Solution.

	a	b	c	d	e	f	g	y
	∞	∞	∞	∞	∞	∞	∞	∞
$x(0)$	$(x,1)$	∞	∞	∞	∞	$(x,4)$	∞	∞
$a(1)$	$(x,1)$	$(a,5)$	∞	$(a,9)$	∞	$(x,4)$	∞	∞
$f(4)$	$(x,1)$	$(a,5)$	∞	$(a,9)$	∞	$(x,4)$	$(f,15)$	∞
$b(5)$	$(x,1)$	$(a,5)$	$(b,6)$	$(b,8)$	∞	$(x,4)$	$(f,15)$	∞
$c(6)$	$(x,1)$	$(a,5)$	$(b,6)$	$(b,8)$	$(c,8)$	$(x,4)$	$(f,15)$	$(c,13)$
$e(8)$	$(x,1)$	$(a,5)$	$(b,6)$	$(b,8)$	$(c,8)$	$(x,4)$	$(f,15)$	$(e,12)$
$d(8)$	$(x,1)$	$(a,5)$	$(b,6)$	$(b,8)$	$(c,8)$	$(x,4)$	$(d,14)$	$(e,12)$

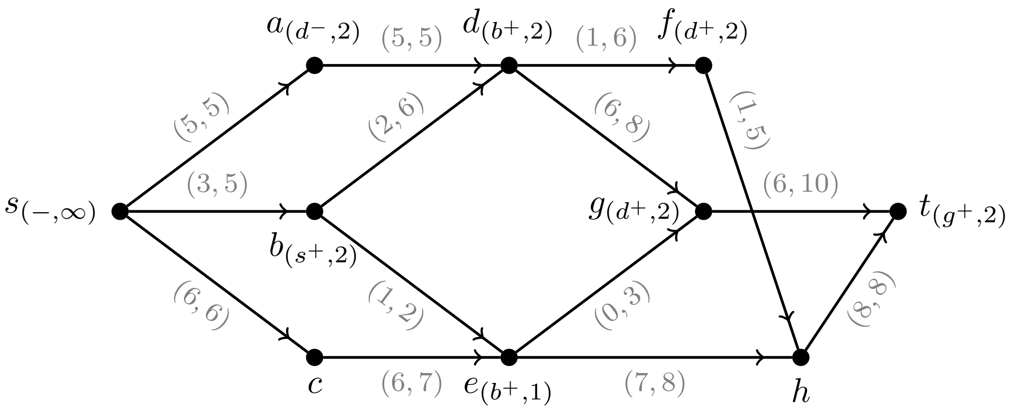
(6 marks)
path: $x \to a \to b \to c \to e \to y$ (2.5 marks), weight: 12 (0.5 marks)

Question 5. (12 marks) For the network below, use the Augmenting Flow Algorithm to maximize the flow and the Min-Cut Method to find a minimum cut.



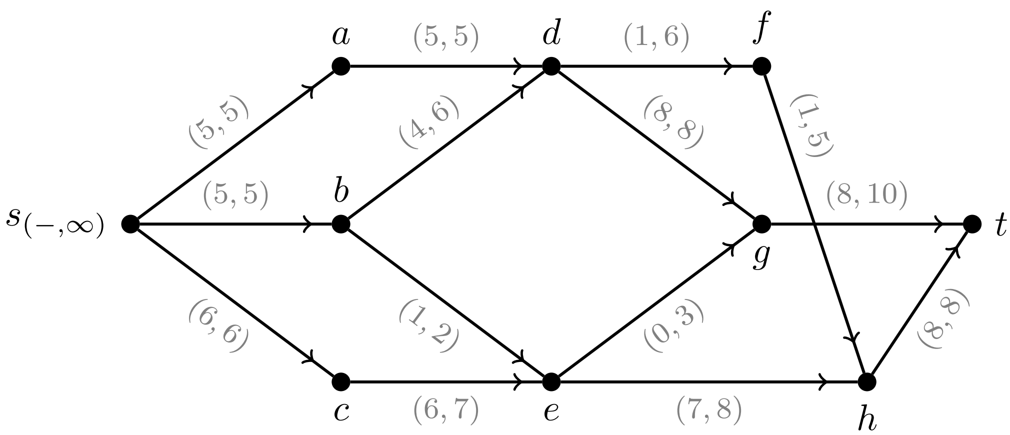
Solution.

- Step 1. Label s
- Step 2. Label b
- Step 3. Choose b
- Step 2. Label d, e
- Step 3. Choose d
- Step 2. Label a, f, g
- Step 3. Choose g
- Step 2. Label t



(4 marks)

- Step 3. t is labeled, go to step 4
- Step 4. find chain $sbdgt$ (1 mark)
  - Increase the flow by $\sigma(t) = 2$ units along each of these edges since all are in the forward direction. (1 mark)
  - Update the network flow and remove all labels except for s



(3 marks)

- No more vertex to label
- Value of flow $|f| = f^+(s) = 16$ (0.5 marks)

Min-Cut Method:

- $P = \{s\}$ (0.5 marks)
- $C(P, \overline{P}) = 5 + 5 + 6 = 16$ (2 marks)