Tutorial 2

Solving linear systems and Gauss–Jordan elimination

Question 1. Determine whether the following equations are linear

1. $x_1 - 3x_2 - \sqrt{2}x_3 = 2$
2. $x_1 + 3x_2 - 2x_1x_2 = x_1$
3. $x_1 = 5x_2 - 3x_3$
4. $x_1^{1/2} - 2x_2 + x_1 = 1$
5. $x_1^2 + x_2 + 8x_3 = 5$
6. $\pi x_1 - \sqrt{2}x_2 = 3^{1/3}$
7. $2^{1/3}x + \sqrt{3}y = 1$
8. $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$
9. $xy = 3$
10. $2x^{1/3} + 3\sqrt{y} = 1$
11. $\frac{\pi}{7}\cos x - 4y = 0$
12. $y + 5 = 2x + 4^2$
13. $2x_1 - x_2 = \sqrt{x_1^2}$

Question 2. Find the system of linear equations whose augmented matrix has the following form

1. $\left[\begin{array}{cc|c} 2 & 0 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{array}\right]$
2. $\left[\begin{array}{cc|c} 3 & 0 & 2 \\ 1 & -5 & 0 \\ 0 & 1 & 2 \\ 2 & -4 & 3 \end{array}\right]$
3. $\left[\begin{array}{ccc|c} 0 & 3 & -1 & -1 \\ 2 & 3 & 0 & -5 \end{array}\right]$
4. $\left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & -4 \\ 3 & 0 & 2 & 1 & 7 \\ -1 & 3 & 0 & -2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right]$

Question 3. Find the augmented matrix for the following linear system

1.

$$\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$$

2.

$$\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$$

3.

$$\begin{aligned} 2x_2 - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 - x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$$

4.

$$\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$$

5.

$$\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 - x_2 - x_3 &= 0 \end{aligned}$$

6.

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$$

Question 4. Determine which of the following ordered triples is a solution to the given linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

1. $(3, 1, 1)$
2. $(3, -1, 1)$
3. $(13, 5, 2)$
4. $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$
5. $(17, 7, 5)$

Question 5. Solve each of the following systems of linear equations. In each case, indicate whether the system has one solution, infinitely many solutions, or no solutions. Give the complete solution set.

1.

$$\begin{aligned} 3x - 2y &= 4 \\ 6x - 4y &= 9 \end{aligned}$$

2.

$$\begin{aligned} 2x - 4y &= 1 \\ 4x - 8y &= 2 \end{aligned}$$

3.

$$\begin{aligned} x - 2y &= 0 \\ x - 4y &= 8 \end{aligned}$$

Question 6. Find the solution set of the following linear system.

1. $7x - 5y = 3$
2. $x + 10y = 3$
3. $x_1 - 5x_2 + 2x_3 = -1$
4. $3x_1 - 5x_2 + 4x_3 = 7$
5. $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$
6. $4x_1 + 2x_2 - 3x_3 - x_4 = 2$
7. $3v - 8w + 2x - y + 4z = 0$
8. $v + w + x - 5y + 7z = 0$

Question 7. Find the solution set for the given linear system.

1.
$$\begin{aligned} 2x - 3y &= 1 \\ 6x - 9y &= 3 \end{aligned}$$

2.
$$\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$$

3.
$$\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 3x_1 + 9x_2 - 3x_3 &= -12 \\ -x_1 - 3x_2 + x_3 &= 4 \end{aligned}$$

4.
$$\begin{aligned} 2x - y + 2z &= -4 \\ 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$$

Question 8. Find an elementary row operation that will create a leading 1 in the top left corner of the given matrix without introducing fractions in its first row.

1. $\begin{bmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 0 & -1 & -5 & 0 \\ 2 & -6 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$

Question 9. Determine whether the following matrices are in row echelon form and which are in reduced row echelon form.

1. $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
2. $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$
3. $\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$
4. $\begin{bmatrix}1&2&0\\0&1&0\\0&0&0\end{bmatrix}$
5. $\begin{bmatrix}1&0&0\\0&1&0\\0&2&0\end{bmatrix}$
6. $\begin{bmatrix}1&3&4\\0&1&1\\0&0&0\end{bmatrix}$
7. $\begin{bmatrix}1&0&3&1\\0&1&2&4\end{bmatrix}$
8. $\begin{bmatrix}1&-2&0&1\\0&0&1&-2\end{bmatrix}$
9. $\begin{bmatrix}1&5&-3\\0&1&1\\0&0&0\end{bmatrix}$
10. $\begin{bmatrix}0&0\\0&0\end{bmatrix}$
11. $\begin{bmatrix}1&-7&5&5\\0&1&3&2\end{bmatrix}$
12. $\begin{bmatrix}1&2&3\\0&0&0\end{bmatrix}$
13. $\begin{bmatrix}1&2&0&3&0\\0&0&1&1&0\\0&0&0&1&\end{bmatrix}$
14. $\begin{bmatrix}1&2&3&4&5\\1&0&7&1&3\\0&0&0&1&\end{bmatrix}$

For Questions 10 – 13 do the following
Suppose that each of the following is the augmented matrix for a linear system. Use the Gauss-Jordan elimination method to convert each matrix to the reduced row echelon form and give the complete solution set for the corresponding system of linear equations if the system is consistent.

Question 10.

1. $\left[\begin{array}{ccc|c}1&-3&4&7\\0&1&2&2\\0&0&1&5\end{array}\right]$
2. $\left[\begin{array}{ccc|c}1&-3&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]$
3. $\left[\begin{array}{ccc|c}1&0&0&-3\\0&1&0&0\\0&0&1&7\end{array}\right]$
4. $\left[\begin{array}{ccc|c}1&0&0&-3\\0&1&0&0\\0&0&1&8\end{array}\right]$
5. $\left[\begin{array}{ccc|c}1&-4&0&0\\0&0&1&0\\0&0&2&0\end{array}\right]$
6. $\left[\begin{array}{ccc|c}1&0&0&3\\0&2&0&-2\\0&0&3&0\end{array}\right]$

Question 11.

1. $\left[\begin{array}{cccc|c} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$

2. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{array}\right]$

3. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & -2 & 3 \\ 0 & 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 13 & 13 \end{array}\right]$

Question 12.

1. $\left[\begin{array}{ccccc|c} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{array}\right]$

3. $\left[\begin{array}{ccccc|c} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

4. $\left[\begin{array}{cccc|c} -5 & 10 & -19 & -17 & 20 \\ -3 & 6 & -11 & -11 & 14 \\ -7 & 14 & -26 & -25 & 31 \\ 9 & -18 & 34 & 31 & -37 \end{array}\right]$

5. $\left[\begin{array}{cc|c} 2 & -5 & -20 \\ 0 & 2 & 7 \\ 1 & -5 & -19 \\ -5 & 16 & 64 \\ 3 & 9 & -36 \end{array}\right]$

6. $\left[\begin{array}{cccc|c} -2 & 1 & -1 & -1 & 3 \\ 3 & 1 & -4 & -2 & -4 \\ 7 & 1 & -6 & -2 & -3 \\ -8 & -1 & 6 & 2 & 3 \\ -3 & 0 & 2 & 1 & 2 \end{array}\right]$

7. $\left[\begin{array}{ccccc|c} -3 & 6 & -1 & -5 & 0 & -5 \\ -1 & 2 & 3 & -5 & 10 & 5 \end{array}\right]$

For Questions 13 – 17 do the following

Solve the following linear systems using Gauss–Jordan elimination or Gaussian elimination. In each case, indicate whether the system is consistent or inconsistent. Give the complete solution set if the system is consistent.

Question 13.

1.

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= 8 \\
-x_1 - 2x_2 + 3x_3 &= 1 \\
3x_1 - 7x_2 + 4x_3 &= 10
\end{aligned}
$$

2.

$$
\begin{aligned}
2x_1 + 2x_2 + 2x_3 &= 0 \\
-2x_1 + 5x_2 + 2x_3 &= 1 \\
8x_1 + x_2 + 4x_3 &= -1
\end{aligned}
$$

3.

$$
\begin{aligned}
x - y + 2z - w &= -1 \\
2x + y - 2z - 2w &= -2 \\
-x + 2y - 4z - 2w &= 1 \\
3x - 3w &= -3
\end{aligned}
$$

4.

$$
\begin{aligned}
-2b + 3c &= 1 \\
3a + 6b - 3c &= -2 \\
6a + 6b + 3c &= 5
\end{aligned}
$$

Question 14.

1.

$$
\begin{aligned}
5x_1 - 5x_2 - 15x_3 &= 40 \\
4x_1 - 2x_2 - 6x_3 &= 19 \\
3x_1 - 6x_2 - 17x_3 &= 41
\end{aligned}
$$

2.

$$
\begin{aligned}
x_1 + 3x_2 - x_3 &= 0 \\
x_2 - 8x_3 &= 0 \\
4x_3 &= 0
\end{aligned}
$$

3.

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 0 \\
x_1 + 2x_2 &= 0 \\
x_2 + x_3 &= 0
\end{aligned}
$$

4.

$$
\begin{aligned}
2x - y - 3z &= 0 \\
-x + 2y - 3z &= 0 \\
x + y + 4z &= 0
\end{aligned}
$$

Question 15.

1.

$$
\begin{aligned}
-2x_1 + x_2 + 8x_3 &= 0 \\
7x_1 - 2x_2 - 22x_3 &= 0 \\
3x_1 - x_2 - 10x_3 &= 0
\end{aligned}
$$

2.

$$
\begin{aligned}
5x_1 - 2x_3 &= 0 \\
-15x_1 - 16x_2 - 9x_3 &= 0 \\
10x_1 + 12x_2 + 7x_3 &= 0
\end{aligned}
$$

Question 16.

1.

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$$

2.

$$\begin{aligned} -2x_1 - 3x_2 + 2x_3 - 13x_4 &= 0 \\ -4x_1 - 7x_2 + 4x_3 - 29x_4 &= 0 \\ +x_1 + 2x_2 - x_3 + 8x_4 &= 0 \end{aligned}$$

3.

$$\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

4.

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

5.

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

6.

$$\begin{aligned} 2x_1 + 6x_2 + 13x_3 + x_4 &= 0 \\ x_1 + 4x_2 + 10x_3 + x_4 &= 0 \\ 2x_1 + 8x_2 + 20x_3 + x_4 &= 0 \\ 3x_1 + 10x_2 + 21x_3 + 2x_4 &= 0 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 - 6x_2 + 3x_3 - 21x_4 &= 0 \\ 4x_1 - 5x_2 + 2x_3 - 24x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 10x_4 &= 0 \\ -2x_1 + 3x_2 - x_3 + 13x_4 &= 0 \end{aligned}$$

Question 17.

1.

$$\begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

2.

$$\begin{aligned} x_3 + x_4 + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \\ 2x_1 + 2x_2 - x_3 + x_5 &= 0 \end{aligned}$$

3.

$$\begin{aligned} 2x_1 + 4x_2 - x_3 + 5x_4 + 2x_5 &= 0 \\ 3x_1 + 3x_2 - x_3 + 3x_4 &= 0 \\ -5x_1 - 6x_2 + 2x_3 - 6x_4 - x_5 &= 0 \end{aligned}$$

4.

$$\begin{aligned} 7x_1 + 28x_2 + 4x_3 - 2x_4 + 10x_5 + 19x_6 &= 0 \\ -9x_1 - 36x_2 - 5x_3 + 3x_4 - 15x_5 - 29x_6 &= 0 \\ 3x_1 + 12x_2 + 2x_3 + 6x_5 + 11x_6 &= 0 \\ 6x_1 + 24x_2 + 3x_3 - 3x_4 + 10x_5 + 20x_6 &= 0 \end{aligned}$$

Question 18. The matrices below represent augmented matrices of linear systems, where $*$ denotes an arbitrary real number. For each case, determine whether the system is consistent. If it is consistent, establish whether the solution is unique. If the solution is not unique, analyze the structure of the solution set.

1. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{array}\right]$

5. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{array}\right]$

7. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right]$

Question 19. For each of the following linear systems, determine the value of α that results in the system having no solutions, a unique solution, or infinitely many solutions.

1.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 2x - y + 5z &= 2 \\ 4x + y + (\alpha^2 - 14)z &= \alpha + 2 \end{aligned}$$

2.

$$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - (\alpha^2 - 3)z &= \alpha \end{aligned}$$

Question 20. What conditions must the parameters a, b, c satisfy for the following linear system to be consistent?

1.

$$\begin{aligned} x + 3y - z &= a \\ x + y + 2z &= b \\ 2y - 3z &= c \end{aligned}$$

2.

$$\begin{aligned} x + 3y + z &= a \\ -x - 2y + z &= b \\ 3x + 7y - z &= c \end{aligned}$$

Question 21. Solve the following systems, where a, b, c are constants.

1.

$$\begin{aligned} 2x + y &= a \\ 3x + 6y &= b \end{aligned}$$

2.

$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ 2x_1 + 2x_3 &= b \\ 3x_2 + 3x_3 &= c \end{aligned}$$

Question 22. Show that the following nonlinear system for the unknowns u, v, w, where $0 \leq u, v, w \leq 2\pi$, has 18 solutions.

$$\begin{aligned} \sin u + 2\cos v + 3\tan w &= 0 \\ 2\sin u + 5\cos v + 3\tan w &= 0 \\ -\sin u - 5\cos v + 5\tan w &= 0 \end{aligned}$$

Question 23. Solve the following nonlinear system for the unknowns u, v, w, where $0 \leq u, v, w \leq 2\pi$.

$$\begin{aligned} 2\sin u - \cos v + 3\tan w &= 3 \\ 4\sin u + 2\cos v - 2\tan w &= 2 \\ 6\sin u - 3\cos v + \tan w &= 9 \end{aligned}$$

Question 24. Solve the following nonlinear system for the unknowns x, y, z.

1.
$$\begin{aligned} x^2+y^2+z^2 &= 6\\ x^2-y^2+z^2 &= 2\\ 2x^2+y^2-z^2 &= 3 \end{aligned}$$

2.
$$\begin{aligned} \frac{1}{x}+\frac{2}{y}-\frac{4}{z} &= 1\\ \frac{2}{x}+\frac{3}{y}+\frac{8}{z} &= 0\\ \frac{1}{x}+\frac{9}{y}+\frac{10}{z} &= 5 \end{aligned}$$

For Questions 25 – 27 do the following
Suppose that each of the following is the augmented matrix for a linear system. The matrices are all in reduced row echelon form. For each case, determine whether the system is consistent. If it is consistent, find the complete solution set.

Question 25.

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & \frac{1}{2}\\ 0 & 0 & 1 & 0 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3\\ 0 & 1 & -1 & 2\\ 0 & 0 & 0 & 0 \end{array}\right]$

5. $\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 4\\ 0 & 1 & 3 & \frac{4}{3}\\ 0 & 0 & 0 & 0 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0 \end{array}\right]$

7. $\left[\begin{array}{ccc|c} 1 & 5 & 5 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right]$

Question 26.

1. $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 5 & 3\\ 0 & 1 & -1 & 2 & 2 \end{array}\right]$, 2. $\left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & 1\\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$, 3. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 1\\ 0 & 1 & 0 & -1 & 7 \end{array}\right]$

Question 27.

1. $\left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & -1 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & \frac{4}{5} \end{array}\right]$, 2. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -1 & 7 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$, 3. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -1 & 7 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$

Question 28. Suppose that each of the following is the augmented matrix for a linear system. The matrices are all in row echelon form. Find the reduced row echelon form of each matrix, and determine whether the corresponding system is consistent. If it is consistent, find the complete solution set.

1. $\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 3 & 5 & -1 & 1 \\ 0 & 0 & 1 & 4 & 23 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

2. $\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 1 & 8 \\ 0 & 1 & 3 & -2 & 6 & -11 \\ 0 & 0 & 0 & 1 & -3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

3. $\left[\begin{array}{ccccc|c} 1 & -5 & 2 & 3 & -2 & -4 \\ 0 & 1 & -1 & -3 & -7 & -2 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

4. $\left[\begin{array}{ccccc|c} 1 & -7 & -3 & -2 & -1 & -5 \\ 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right]$

5. $\left[\begin{array}{cccccc|c} 1 & -3 & 6 & 0 & -2 & 4 & -3 \\ 0 & 0 & 1 & -2 & 8 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

6. $\left[\begin{array}{cccccc|c} 1 & 4 & -8 & -1 & 2 & -3 & -4 \\ 0 & 1 & -7 & 2 & -9 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

For Questions 29 – 32 do the following

Solve the following linear systems using Gauss–Jordan elimination or Gaussian elimination. In each case, indicate whether the system is consistent or inconsistent. Give the complete solution set if the system is consistent.

Question 29.

1.

$$\begin{aligned} 2x - 3y &= 5 \\ -x + y &= -3 \end{aligned}$$

2.

$$\begin{aligned} 2x - 2y &= 1 \\ 3x &= 1 \end{aligned}$$

3.

$$\begin{aligned} 2x - z &= 4 \\ x + 4y + z &= 2 \\ 4x + y - z &= 1 \end{aligned}$$

4.

$$\begin{aligned} -3x + y + z &= 2 \\ -4z &= 0 \\ -4x + 2y - 3z &= 1 \end{aligned}$$

5.

$$\begin{aligned} 2x_1 - x_3 &= 4 \\ x_1 + 4x_2 + x_3 &= 2 \end{aligned}$$

6.

$$\begin{aligned} 4x_1 + x_2 - 4x_3 &= 1 \\ 4x_1 - 4x_2 + 2x_3 &= -2 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 + 2x_4 &= -2 \\ 4x_1 - 2x_2 - 3x_3 - 2x_4 &= 2 \\ x_1 + 3x_2 + 3x_3 - 3x_4 &= -4 \end{aligned}$$

8.

$$\begin{aligned} 3x_1 - 3x_3 + 4x_4 &= -3 \\ -4x_1 + 2x_2 - 2x_3 - 4x_4 &= 4 \\ 4x_2 - 3x_3 + 2x_4 &= -3 \end{aligned}$$

Question 30.

1.

$$\begin{aligned} x + y &= 1 \\ 4x + 3y &= 2 \end{aligned}$$

2.

$$\begin{aligned} -3x + y &= 1 \\ 4x + 2y &= 0 \end{aligned}$$

3.

$$\begin{aligned} 3x - 3y &= 3 \\ 4x - y - 3z &= 3 \\ -2x - 2y &= -2 \end{aligned}$$

4.

$$\begin{aligned} 2x - 4z &= 1 \\ 4x + 3y - 2z &= 0 \\ 2x + 2z &= 0 \end{aligned}$$

5.

$$\begin{aligned} -3x_2 - x_3 &= 2 \\ x_1 + x_3 &= -2 \end{aligned}$$

6.

$$\begin{aligned} x + 2y + z &= 1 \\ 2x + 3y + 2z &= 0 \\ x + y + z &= 2 \end{aligned}$$

7.

$$\begin{aligned} 3x - 2z &= -3 \\ -2x + z &= -2 \\ -z &= 2 \end{aligned}$$

Question 31.

1.

$$\begin{aligned} 3x_1 + 2x_2 + 3x_3 &= -3 \\ x_1 + 2x_2 - x_3 &= -2 \end{aligned}$$

2.

$$\begin{aligned} -x_1 + 3x_3 + x_4 &= 2 \\ 2x_1 + 3x_2 - 3x_3 + x_4 &= 2 \\ 2x_1 - 2x_2 - 2x_3 - x_4 &= -2 \end{aligned}$$

3.

$$\begin{aligned} 3x_1 - x_2 + 3x_3 + 3x_4 &= -3 \\ x_1 - x_2 + x_3 + x_4 &= 3 \\ -3x_1 + 3x_2 - x_3 + 2x_4 &= 1 \end{aligned}$$

4.

$$\begin{aligned} 3x_1 - 3x_2 + x_3 + 3x_4 &= -3 \\ x_1 + x_2 - x_3 - 2x_4 &= 3 \\ 4x_1 - 2x_2 + x_4 &= 0 \end{aligned}$$

5.

$$\begin{aligned} -3x_1 + 2x_2 - x_3 - 2x_4 &= 2 \\ x_1 - x_2 - 3x_4 &= 3 \\ 4x_1 - 3x_2 + x_3 - x_4 &= 1 \end{aligned}$$

6.

$$\begin{aligned} 3x_1 + x_2 + 7x_3 + 2x_4 &= 13 \\ 2x_1 - 4x_2 + 14x_3 - x_4 &= -10 \\ 5x_1 + 11x_2 - 7x_3 + 8x_4 &= 59 \\ 2x_1 + 5x_2 - 4x_3 - 3x_4 &= 39 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 4x_4 &= 9 \\ x_1 - 2x_3 + 7x_4 &= 11 \\ 3x_1 - 3x_2 + x_3 + 5x_4 &= 8 \\ 2x_1 + x_2 + 4x_3 + 4x_4 &= 10 \end{aligned}$$

Question 32.

1.

$$\begin{aligned} -5x_1 - 2x_2 + 2x_3 &= 16 \\ 3x_1 + x_2 - x_3 &= -9 \\ 2x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

2.

$$\begin{aligned} 3x_1 - 3x_2 - 2x_3 &= 23 \\ -6x_1 + 4x_2 + 3x_3 &= -40 \\ -2x_1 + x_2 + x_3 &= -12 \end{aligned}$$

3.

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= -54 \\ -x_1 + x_2 - 2x_3 &= 20 \\ 5x_1 - 4x_2 + 8x_3 &= -83 \end{aligned}$$

4.

$$\begin{aligned} 4x_1 - 2x_2 - 7x_3 &= 5 \\ -6x_1 + 5x_2 + 10x_3 &= -11 \\ -2x_1 + 3x_2 + 4x_3 &= -3 \\ -3x_1 + 2x_2 + 5x_3 &= -5 \end{aligned}$$

5.

$$\begin{aligned} -2x_1 + 3x_2 - 4x_3 + x_4 &= -17 \\ 8x_1 - 5x_2 + 2x_3 - 4x_4 &= 47 \\ -5x_1 + 9x_2 - 13x_3 + 3x_4 &= -44 \\ -4x_1 + 3x_2 - 2x_3 + 2x_4 &= -25 \end{aligned}$$

6.

$$\begin{aligned} 5x_1 - x_2 - 9x_3 - 2x_4 &= 26 \\ 4x_1 - x_2 - 7x_3 - 2x_4 &= 21 \\ -2x_1 + 4x_3 + x_4 &= -12 \\ -3x_1 + 2x_2 + 4x_3 + 2x_4 &= -11 \end{aligned}$$

7.

$$\begin{aligned} 6x_1 - 12x_2 - 5x_3 + 16x_4 - 2x_5 &= -53 \\ -3x_1 + 6x_2 + 3x_3 - 9x_4 + x_5 &= 29 \\ -4x_1 + 8x_2 + 3x_3 - 10x_4 + x_5 &= 33 \end{aligned}$$

8.

$$\begin{aligned} 5x_1 - 5x_2 - 15x_3 - 3x_4 &= -34 \\ -2x_1 + 2x_2 + 6x_3 + x_4 &= 12 \end{aligned}$$

Question 33.

Find the reduced row echelon form of the given matrix

1. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 & 0 & 4 & \frac{2}{3} \\ 0 & 1 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$

9. $\begin{bmatrix} -2 & 2 & -1 & 2 \\ 0 & 3 & 3 & -3 \\ 1 & -4 & 2 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 4 & -3 & -4 & -2 \\ -4 & 2 & 1 & -4 \\ -1 & -3 & 1 & -4 \end{bmatrix}$

11. $\begin{bmatrix} -4 & 1 & 4 \\ 3 & 4 & -3 \end{bmatrix}$

12. $\begin{bmatrix} -4 & -2 & -1 \\ -2 & -3 & 0 \end{bmatrix}$

Question 34.
Consider each of the following matrices as the augmented matrix of a linear system:

$$1.\ \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 2 & 3 & -2 & b \\ -1 & -1 & 1 & c \end{array}\right], \quad 2.\ \left[\begin{array}{cc|c} a & 1 & 1 \\ 2 & a-1 & 1 \end{array}\right], \quad 3.\ \left[\begin{array}{ccc|c} -2 & 3 & 1 & a \\ 1 & 1 & -1 & b \\ 0 & 5 & -1 & c \end{array}\right]$$

For each case,

(a) Determine the values of a, b, c for which this linear system is *inconsistent.*

(b) Determine the values of a, b, c for which this linear system is *consistent.*

(c) If this system is consistent, determine whether it has a unique solution or infinitely many solutions. Express the solution(s) in terms of a, b, c.

(d) Choose a specific set of values for a, b, c that ensures the system is consistent and solve the system.

Question 35. Suppose $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are two distinct solutions of the linear system

$$A\boldsymbol{x} = \boldsymbol{b},$$

where $A \in \mathcal{M}_{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$.

1. Show that for any $\alpha \in \mathbb{R}$, $\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1)$ is also a solution of the linear system.

2. Show that if $\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{x}_1 + \beta(\boldsymbol{x}_2 - \boldsymbol{x}_1)$, where $\alpha,\ \beta \in \mathbb{R}$, then $\alpha = \beta$.

3. Prove that if a linear system has two distinct solutions, then it must have infinitely many solutions.