

# Algebra and Discrete Mathematics

## ADM

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# Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces and matrix transformations
- Fundamental spaces and decompositions
- Eulerian tours
- Hamiltonian cycles
- Midterm
- Paths and spanning trees
- Trees and networks
- Matching

## Recommended reading

- Anton, Howard, and Chris Rorres. Elementary linear algebra: applications version. John Wiley & Sons, 2013.
  - Sections 1.8, 4.1 – 4.6, 4.9, 4.10
  - [Accessible online \(free copy\)](#)
  - [Alternative download link](#)

# Lecture outline

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
- Dimension
- Change of basis
- Matrix operators

# Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
- Dimension
- Change of basis
- Matrix operators

# Definition

## Definition

Let  $V$  be a nonempty set on which two operations are defined:

- Addition:  $V \times V \rightarrow V$ ,  $\mathbf{v} + \mathbf{u}$  is called the *sum* of  $\mathbf{v}$  and  $\mathbf{u}$
- Scalar multiplication:  $\mathbb{R} \times V \rightarrow V$ ,  $\alpha \mathbf{v}$  is called the *scalar multiple* of  $\mathbf{v}$  by  $\alpha$

$V$ , (together with the associated addition and scalar multiplication), is called a *vector space (over  $\mathbb{R}$ )* if  $\forall \mathbf{v}, \mathbf{u}, \mathbf{w} \in V$  (*vectors*) and  $\forall \alpha, \beta \in \mathbb{R}$  (*scalars*):

1.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2.  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3.  $\exists \mathbf{0} \in V$ , called *zero vector* or *additive identity*, s.t.  $\mathbf{0} + \mathbf{u} = \mathbf{u}$
4.  $\exists -\mathbf{u} \in V$ , called *negative* or *additive inverse* of  $\mathbf{u}$  s.t.  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5.  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
6.  $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
7.  $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
8.  $1\mathbf{u} = \mathbf{u}$

## Show that a set with two operations is a vector space

- Identify the set  $V$  and elements of  $V$  that will become vectors
- Identify the addition and scalar multiplication operations on  $V$
- Verify that  $v + u \in V$  (*closure under addition*) and  $\alpha u \in V$  (*closure under scalar multiplication*) for all  $v, u \in V, \alpha \in \mathbb{R}$
- Verify that all axioms are satisfied

### Remark

Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on  $\mathbb{R}^n$ .

## Vector space – example

### Example (The zero vector space)

Let  $V = \{\mathbf{0}\}$ , define

$$\mathbf{0} + \mathbf{0} = \mathbf{0}, \quad \alpha \mathbf{0} = \mathbf{0}, \quad \alpha \in \mathbb{R}$$

It is easy to check all axioms are satisfied and  $V$  together with the addition and scalar multiplication defined above,  $(V, +, \cdot)$ , is a vector space.



## Vector space – example

### Example

Consider  $\mathbb{R}^n$  with the usual operations of addition and scalar multiplication

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$\alpha \mathbf{u} = (\alpha u_1, \alpha u_2, \dots, \alpha u_n)$$

zero vector

$$\mathbf{0} = (0, 0, \dots, 0)$$

$(\mathbb{R}^n, +, \cdot)$  is a vector space.

## Vector space – example

### Example

Let  $V$  consist of objects of the form

$$\mathbf{u} = (u_1, u_2, \dots),$$

where  $u_1, u_2, \dots$  is an infinite sequence of real numbers.

We define two infinite sequences to be equal if their corresponding components are equal and we define addition and scalar multiplication component-wise by

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, \dots) + (v_1, v_2, \dots) = (u_1 + v_1, u_2 + v_2, \dots)$$

$$\alpha \mathbf{u} = (\alpha u_1, \alpha u_2, \dots)$$

$(V, +, \cdot)$  is a vector space, denoted by  $\mathbb{R}^\infty$

## Vector space – example

### Example (The vector space of $2 \times 2$ matrices)

Consider  $\mathcal{M}_{2 \times 2}$  together with matrix addition and multiplication by a scalar

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{pmatrix}$$

$$\alpha \mathbf{u} = \alpha \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha u_{11} & \alpha u_{12} \\ \alpha u_{21} & \alpha u_{22} \end{pmatrix}$$

$\mathcal{M}_{2 \times 2}$  is closed under addition and scalar multiplication. The additive identity is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

Additive inverse of  $\mathbf{u}$

$$\begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$$

## Vector space – example

### Example (The vector space of $2 \times 2$ matrices)

Consider  $\mathcal{M}_{2 \times 2}$  together with matrix addition and multiplication by a scalar

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{pmatrix}$$

$$\alpha \mathbf{u} = \alpha \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha u_{11} & \alpha u_{12} \\ \alpha u_{21} & \alpha u_{22} \end{pmatrix}$$

$$\begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$1\mathbf{u} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$(\mathcal{M}_{2 \times 2}, +, \cdot)$  is a vector space

## Vector space – example

### Example (The vector space of $m \times n$ matrices)

- Consider  $\mathcal{M}_{m \times n}$  with matrix addition and scalar multiplication.
- Similar to  $\mathcal{M}_{2 \times 2}$ , we can show that  $(\mathcal{M}_{m \times n}, +, \cdot)$  is also a vector space.
- The additive inverse is the zero matrix.

## Vector space – example

### Example (The vector space of real-valued functions)

- Let  $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{ f \mid f : \mathbb{R} \rightarrow \mathbb{R} \}$  be the set of real-valued functions that are defined at each  $x \in \mathbb{R}$
- Define addition and scalar multiplication as follows: for any  $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ , and any  $\alpha \in \mathbb{R}$ ,

$$(f + g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha(f(x))$$

- It is easy to see that  $f + g$  and  $\alpha f$  are real-valued functions defined at each  $x \in \mathbb{R}$  - closure under addition and scalar multiplication
- Additive identity: the function  $\mathbf{0}$  that outputs 0 for every  $x \in \mathbb{R}$
- Additive inverse: the additive inverse of  $f$  is the function defined by

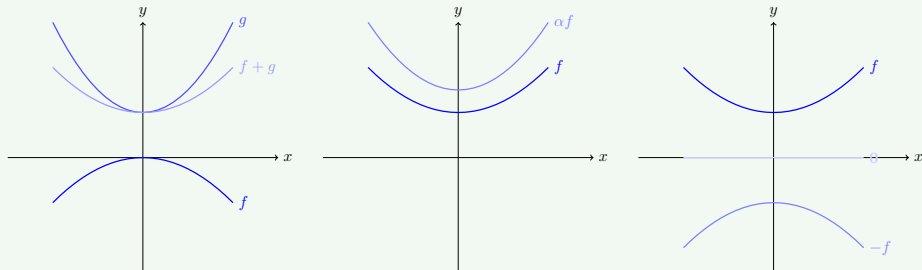
$$\begin{aligned} -f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto -f(x) \end{aligned}$$

## Vector space – example

### Example (The vector space of real-valued functions)

- $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{ f \mid f : \mathbb{R} \rightarrow \mathbb{R} \}$
- Validity of the axioms follow from the properties of real numbers
- For example, Axiom 1

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$$



## Vector space – example

### Example (Not a vector space)

- Consider  $\mathbb{R}^2$  with addition and scalar multiplication defined as follows:
- For any  $\mathbf{u} = (u_1, u_2)$   $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ ,  $\alpha \in \mathbb{R}$

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad \alpha \otimes \mathbf{u} = (\alpha u_1, 0).$$

- For example

$$(2, 4) + (-3, 5) = (-1, 9), \quad 7 \otimes (2, 4) = (14, 0)$$

- Not a vector space because for  $\mathbf{u} = (u_1, u_2)$  with  $u_2 \neq 0$

$$1 \otimes \mathbf{u} = (u_1, 0) \neq \mathbf{u}.$$



## Vector space – example

### Example (A special vector space)

- Consider  $\mathbb{R}_{>0}$ , the set of positive real numbers
- Define addition and scalar multiplication as follows: for any  $u, v \in \mathbb{R}_{>0}$  and any  $\alpha \in \mathbb{R}$

$$u \oplus v = uv, \quad \alpha \otimes u = u^\alpha$$

- The additive identity is the number 1

$$u \oplus 1 = u1 = u$$

- The additive inverse of an element  $u$  is its reciprocal

$$u \oplus \frac{1}{u} = u \frac{1}{u} = 1$$

because  $u > 0$ ,  $\frac{1}{u} > 0$  is an element of  $\mathbb{R}_{>0}$

## Vector space – example

### Example (A special vector space)

- Consider  $\mathbb{R}_{>0}$ , the set of positive real numbers
- Define addition and scalar multiplication as follows: for any  $u, v \in \mathbb{R}_{>0}$  and any  $\alpha \in \mathbb{R}$

$$u \oplus v = uv, \quad \alpha \otimes u = u^\alpha$$

- Other axioms are also satisfied
- For example, Axiom 6

$$(\alpha + \beta) \otimes u = u^{\alpha+\beta} = u^\alpha u^\beta = \alpha \otimes u + \beta \otimes u$$

- $(\mathbb{R}_{>0}, \oplus, \otimes)$  is a vector space

## Some properties of vectors

### Theorem 1

Let  $V$  be a vector space, for any  $\mathbf{u} \in V$  and  $\alpha \in \mathbb{R}$

(a)  $0\mathbf{u} = \mathbf{0}$

(b)  $\alpha\mathbf{0} = \mathbf{0}$

(c)  $(-1)\mathbf{u} = -\mathbf{u}$

(d) If  $\alpha\mathbf{u} = \mathbf{0}$ , then  $\alpha = 0$  or  $\mathbf{u} = \mathbf{0}$

### Proof.

We will show the proof of part (c)

$$\begin{aligned}\mathbf{u} + (-1)\mathbf{u} &= 1\mathbf{u} + (-1)\mathbf{u} \quad (\text{Axiom 8}) \\ &= (1 + (-1))\mathbf{u} \quad (\text{Axiom 6}) \\ &= 0\mathbf{u} \quad (\text{property of real numbers}) \\ &= \mathbf{0} \quad (\text{part (a)})\end{aligned}$$

## Remark

- Whenever we discover a new theorem about general vector spaces, we will at the same time be discovering a theorem about  $\mathbb{R}^n$ , matrices, etc.
- For example, consider the vector space  $\mathbb{R}_{>0}$  with the two operations defined in the previous example

$$0u = \mathbf{0}$$

translates to for  $u \in \mathbb{R}_{>0}$

$$u^0 = 1$$

# Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
- Dimension
- Change of basis
- Matrix operators

# Subspace – definition

## Definition

A subset  $W$  of a vector space  $V$  is called a *subspace* of  $V$  if  $W$  is a vector space under the addition and scalar multiplication defined on  $V$ .

- To show  $W$  is a vector space, certain properties are “inherited” from  $V$
- e.g.  $u + v = v + u$
- It remains to show
  - Closure of  $W$  under addition
  - Closure of  $W$  under scalar multiplication
  - Additive identity  $\in W$
  - Existence of additive inverse

# Subspace

## Theorem

Let  $V$  be a vector space, a nonempty set  $W \subseteq V$  is a subspace of  $V$  iff for any  $u, v \in W, \alpha \in \mathbb{R}$

1.  $u + v \in W$
2.  $\alpha u \in W$

## Proof.

$\implies$  by definition.

$\impliedby$  By the previous discussion, we just need to prove the existence of additive identity and additive inverse. By Axiom 2 and Theorem 1 (a), (c)

$$0u = \mathbf{0} \in W, \quad -u \in W$$



## Subspace – example

### Example

- Let  $V$  be any vector space and let  $W = \{\mathbf{0}\}$
- $W$  is closed under addition and scalar multiplication

$$\mathbf{0} + \mathbf{0} = \mathbf{0}, \quad \alpha \mathbf{0} = \mathbf{0}$$

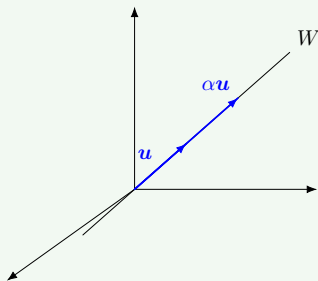
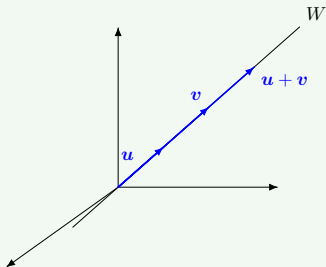
- $W$  is called the *zero subspace* of  $V$



## Subspace – example

Example (Lines through the origin are subspaces of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ )

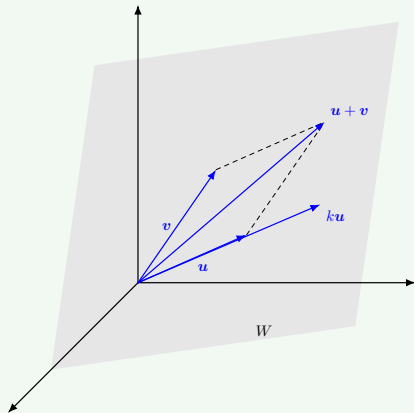
- $W$  = a line through the origin of either  $\mathbb{R}^2$  or  $\mathbb{R}^3$
- Adding two vectors on the line or multiplying a vector on the line by scalar gives another vector on the line - closed under addition and scalar multiplication



## Subspace – example

Example (Planes through the origin are subspaces of  $\mathbb{R}^3$ )

- $W$  = a plane through the origin of  $\mathbb{R}^3$
- Adding two vectors on the line or multiplying a vector on the line by scalar gives another vector in the same plane - closed under addition and scalar multiplication



# Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

Summary of what we have discussed

Subspaces of $\mathbb{R}^2$	Subspaces of $\mathbb{R}^3$
$\{\mathbf{0}\}$	$\{\mathbf{0}\}$
Lines through the origin	Lines through the origin
$\mathbb{R}^2$	Planes through the origin
	$\mathbb{R}^3$

## Subspace – example

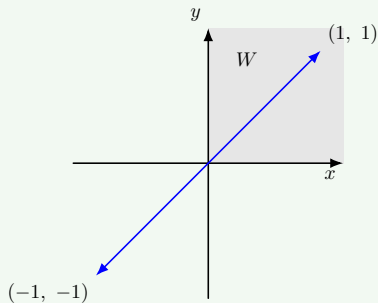
Example (A subset of  $\mathbb{R}^2$  that is not a subspace)

- Let

$$W = \{ (x, y) \mid x \geq 0, y \geq 0 \} \subseteq \mathbb{R}^2$$

- $W$  is not a subspace of  $\mathbb{R}^2$
- $W$  is not closed under scalar multiplication

$$(-1)(1, 1) = (-1, -1) \notin W$$



# Linear combination

## Definition

$\mathbf{u} \in V$  is a *linear combination* of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r \in V$  if

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_r \mathbf{v}_r,$$

where  $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{R}$  are called the *coefficients* of the linear combination.

## Linear combinations – example

### Example

- Consider  $\mathbf{u} = (1, 2, -1)$ ,  $\mathbf{v} = (6, 4, 2)$  from  $\mathbb{R}^3$
- $\mathbf{w} = (9, 2, 7)$  is a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ :  
Suppose

$$\mathbf{w} = \alpha_1 \mathbf{u} + \alpha_2 \mathbf{v}$$

$$(9, \ 2, \ 7) = (\alpha_1 + 6\alpha_2, \ 2\alpha_1 + 4\alpha_2, \ -\alpha_1 + 2\alpha_2)$$

gives

$$\alpha_1 + 6\alpha_2 = 9$$

$$2\alpha_1 + 4\alpha_2 = 2$$

$$-\alpha_1 + 2\alpha_2 = 7$$

Solving the linear system gives  $\alpha_1 = -3$ ,  $\alpha_2 = 2$

$$\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}.$$

## Linear combinations – example

### Example

- Consider  $\mathbf{u} = (1, 2, -1)$ ,  $\mathbf{v} = (6, 4, 2)$  from  $\mathbb{R}^3$
- $\mathbf{w}' = (4, -1, 8)$  is not a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ :  
Suppose

$$\mathbf{w}' = \alpha_1 \mathbf{u} + \alpha_2 \mathbf{v}$$

$$(4, -1, 8) = (\alpha_1 + 6\alpha_2, 2\alpha_1 + 4\alpha_2, -\alpha_1 + 2\alpha_2)$$

gives

$$\alpha_1 + 6\alpha_2 = 4$$

$$2\alpha_1 + 4\alpha_2 = -1$$

$$-\alpha_1 + 2\alpha_2 = 8$$

The linear system is inconsistent

## Subspace from a set of vectors

### Theorem

Let  $S = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r \} \subseteq V$ , let  $W$  be the set of all possible linear combinations of vectors in  $S$ . Then

- $W$  is a subspace of  $V$
- $W$  is the “smallest” subspace of  $V$  that contain  $S$  – any other subspace of  $V$  containing  $S$  contains  $W$

### Proof.

For any  $\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_r \mathbf{v}_r$ ,  $\mathbf{w} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \dots + \beta_r \mathbf{v}_r \in W$

$$\mathbf{u} + \mathbf{w} = (\alpha_1 + \beta_1) \mathbf{v}_1 + (\alpha_2 + \beta_2) \mathbf{v}_2 + \dots + (\alpha_r + \beta_r) \mathbf{v}_r \in W$$

$$\gamma \mathbf{u} = (\gamma \alpha_1) \mathbf{v}_1 + (\gamma \alpha_2) \mathbf{v}_2 + \dots + (\gamma \alpha_r) \mathbf{v}_r \in W$$

$W$  is closed under vector addition and scalar multiplication



## Subspace from a set of vectors

### Theorem

Let  $S = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r \} \subseteq V$ , let  $W$  be the set of all possible linear combinations of vectors in  $S$ . Then

- $W$  is a subspace of  $V$
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### Proof.

Let  $W'$  be any subspace of  $V$  containing  $S$ . Since  $W'$  is closed under vector addition and scalar multiplication, it contains all linear combinations of vectors in  $S$  and hence contains  $W$ . □

## Subspace generated by $S$

### Definition

Let  $S = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r \} \subseteq V$ , the subspace  $W$  of  $V$  that consists of all linear combinations of the vectors in  $S$  is called the subspace of  $V$  *generated* by  $S$ , and we say the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$  *span*  $W$ . We write

$$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}, \quad W = \text{span}(S), \quad W = \langle S \rangle.$$

## Subspace generated by $S$ – example

### Example (The standard unit vectors span $\mathbb{R}^n$ )

- The unit vectors

$$\mathbf{e}_1 := (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 := (0, 1, 0, \dots, 0), \quad \dots, \quad \mathbf{e}_n := (0, 0, 0, \dots, 1)$$

are called the *standard unit vectors* of  $\mathbb{R}^n$ .

- Any  $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$  can be expressed as

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \dots + v_n \mathbf{e}_n$$

- $\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$
- For example,

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1)$$

$\text{span } \mathbb{R}^3$

- e.g.  $\mathbf{v} = (2, 3, -2)$

$$\mathbf{v} = 2\mathbf{e}_1 + 3\mathbf{e}_2 + (-2)\mathbf{e}_3$$

# Test for spanning

## Example

- Determine if the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  span  $\mathbb{R}^3$

$$\mathbf{v}_1 = (1, 1, 2), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (2, 1, 3)$$

- We need to show every vector  $\mathbf{u} \in \mathbb{R}^3$  can be expressed as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

$$(u_1, u_2, u_3) = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3$$

$$(u_1, u_2, u_3) = (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_3, 2\alpha_1 + \alpha_2 + 3\alpha_3)$$

or

$$\alpha_1 + \alpha_2 + 2\alpha_3 = u_1$$

$$\alpha_1 + \alpha_3 = u_2$$

$$2\alpha_1 + \alpha_2 + 3\alpha_3 = u_3$$

## Test for spanning

### Example

$$\begin{aligned}\alpha_1 + \alpha_2 + 2\alpha_3 &= u_1 \\ \alpha_1 + \alpha_3 &= u_2 \\ 2\alpha_1 + \alpha_2 + 3\alpha_3 &= u_3\end{aligned}$$

- We need to show the linear system is consistent for all values of  $u_1, u_2, u_3$ , which is true iff the coefficient matrix is invertible

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \quad \det(A) = 0$$

- The vectors  $v_1, v_2, v_3$  do not span  $\mathbb{R}^3$

## Test for spanning – $\mathbb{R}^n$

From the previous examples, we have the following result for a special case

### Theorem

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \text{ spans } \mathbb{R}^n \text{ iff the determinant } \begin{vmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{vmatrix} \neq 0.$$

### Proof.

$S$  spans  $\mathbb{R}^n$  iff the following equation has solutions for  $\alpha_1, \alpha_2, \dots, \alpha_n$

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \mathbf{u}.$$

for any vector  $\mathbf{u} \in \mathbb{R}^n$ . This equation corresponds to a linear system in the unknowns  $\alpha_i$ 's with coefficient matrix  $A$ , whose columns are  $\mathbf{v}_j$ 's. Consequently, the equation has solutions for all  $\mathbf{u}$  iff  $\det(A) \neq 0$ . The result follows from that  $\det(A) = \det(A^\top)$ .

# Solution spaces of homogeneous systems

## Theorem

*The solution set of a homogeneous linear system  $Ax = \mathbf{0}$  of  $m$  equations in  $n$  unknowns is a subspace of  $\mathbb{R}^n$ .*

## Proof.

- Let  $W$  be the solution set
- $\mathbf{0} \in W$ ,  $W$  is not empty
- For any  $x_1, x_2 \in W$

$$A(x_1 + x_2) = Ax_1 + Ax_2 = \mathbf{0} \implies x_1 + x_2 \in W$$

$$A(\alpha x_1) = \alpha(Ax_1) = \mathbf{0} \implies \alpha x_1 \in W$$



## Definition

$W$  is called the *solution space* of the system.

## Solution space – example

### Example

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

General solution

$$x = 2s - 3t, \quad y = s, \quad z = t$$

it follows that

$$x - 2y + 2z = 0$$

Corresponds to a plane through the origin.



## Solution space – example

### Example

$$\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

General solution

$$x = -35t, \quad y = -t, \quad z = t$$

Corresponds to a line through the origin. The parametric equation for the line is

$$x = -35t, \quad y = -t, \quad z = t, \quad t \in \mathbb{R}.$$

## Solution space – example

### Example

$$\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Has a unique solution  $\mathbf{0}$ , corresponding to the zero subspace of  $\mathbb{R}^3$

## Solution space – example

### Example

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solution space =  $\mathbb{R}^3$

# Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- **Linear independence**
- Coordinates and basis
- Dimension
- Change of basis
- Matrix operators

# Linearly independent vectors

## Definition

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\} \subseteq V$  is said to be *linearly independent* if

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_r \mathbf{v}_r = \mathbf{0} \implies \alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_r = 0$$

Otherwise,  $S$  is said to be *linearly dependent*.

## Linearly independent vectors – $\mathbb{R}^n$

### Theorem

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq \mathbb{R}^n$  is linearly independent iff the determinant  $\begin{vmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{vmatrix} \neq 0$ .

### Proof.

$S$  is linearly independent iff the linear system

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_r \mathbf{v}_r = \mathbf{0}$$

has only the trivial solution. The coefficient matrix  $A$  has columns given by  $\mathbf{v}_i^\top$ .  
 $\det(A) = \det(A^\top)$



## Linearly independent vectors – example

### Example

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1)$$

$$\alpha_1(1, -2, 3) + \alpha_2(5, 6, -1) + \alpha_3(3, 2, 1) = (0, 0, 0)$$

$$\alpha_1 + 5\alpha_2 + 3\alpha_3 = 0$$

$$-2\alpha_1 + 6\alpha_2 + 2\alpha_3 = 0$$

$$3\alpha_1 - \alpha_2 + \alpha_3 = 0$$

The determinant of the coefficient matrix

$$\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 0 \implies \text{linearly dependent}$$

$$\text{Solution set is } \left\{ \left( -\frac{1}{2}t, -\frac{1}{2}t, t \right) \mid t \in \mathbb{R} \right\}$$

## Linearly independent vectors – example

### Example

- Determine whether  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$  are linearly independent

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1)$$

$$\begin{vmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

- In fact,  $\mathbf{v}_3 - \frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 = \mathbf{0}$



## Linearly independent vectors – example

### Example

- Determine whether  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^4$  are linearly independent

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

$$\alpha_1 (1, 2, 2, -1) + \alpha_2 (4, 9, 9, -4) + \alpha_3 (5, 8, 9, -5) = (0, 0, 0, 0)$$

$$\alpha_1 + 4\alpha_2 + 5\alpha_3 = 0$$

$$2\alpha_1 + 9\alpha_2 + 8\alpha_3 = 0$$

$$2\alpha_1 + 9\alpha_2 + 9\alpha_3 = 0$$

$$-\alpha_1 - 4\alpha_2 - 5\alpha_3 = 0$$

- The system has only the trivial solution  $\implies$  the vectors are linearly independent

# Linearly independent vectors

## Theorem

$S = \{v_1, v_2, \dots, v_r\} \subseteq V$ ,  $S$  is linearly independent iff no vector in  $S$  can be expressed as a linear combination of the others.

## Proof.

We will prove  $S$  is linearly dependent iff at least one vector in  $S$  can be expressed as a linear combination of the others.

$\implies$  Consider the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r = \mathbf{0}$$

If  $S$  is linearly dependent, then WLOG, suppose  $\alpha_1 \neq 0$ , and

$$v_1 = \left(-\frac{\alpha_2}{\alpha_1}\right) v_2 + \dots + \left(-\frac{\alpha_r}{\alpha_1}\right) v_r$$

# Linearly independent vectors

## Theorem

$S = \{v_1, v_2, \dots, v_r\} \subseteq V$ ,  $r \geq 2$ ,  $S$  is linearly independent iff no vector in  $S$  can be expressed as a linear combination of the others.

## Proof.

We will prove  $S$  is linearly dependent iff at least one vector in  $S$  can be expressed as a linear combination of the others.

$\Leftarrow$  Suppose  $v_1 = \beta_2 v_2 + \dots + \beta_r v_r$ , then

$$-v_1 + \beta_2 v_2 + \dots + \beta_r v_r = \mathbf{0}.$$



## Example

- $S = \{(2, 3), (1, 0), (0, 1)\}$ , linearly dependent
- $S = \{(1, 0), (0, 1)\}$ , linearly independent

## Special cases

### Theorem

- *A finite set that contains  $\mathbf{0}$  is linearly dependent*
- *A set with exactly one vector is linearly independent iff that vector is not  $\mathbf{0}$*
- *A set with exactly two vectors is linearly independent iff neither vector is a scalar multiple of the other*

### Proof.

We prove the first part. Let  $S = \{v_1, v_2, \dots, v_r, \mathbf{0}\}$

$$0v_1 + 0v_2 + \dots + 0v_r + 1\mathbf{0} = \mathbf{0}$$



## Linearly dependent vectors in $\mathbb{R}^n$

### Theorem

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\} \subseteq \mathbb{R}^n$ . If  $r > n$ , then  $S$  is linearly dependent.

### Proof.

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_r \mathbf{v}_r = \mathbf{0}$$

corresponds to the homogeneous system

$$\begin{array}{ccccccccc} v_{11}\alpha_1 & + & v_{21}\alpha_2 & + & \cdots & + & v_{r1}\alpha_r & = & 0 \\ v_{12}\alpha_1 & + & v_{22}\alpha_2 & + & \cdots & + & v_{r2}\alpha_r & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ v_{1n}\alpha_1 & + & v_{2n}\alpha_2 & + & \cdots & + & v_{rn}\alpha_r & = & 0 \end{array}$$

which has more equations than unknowns

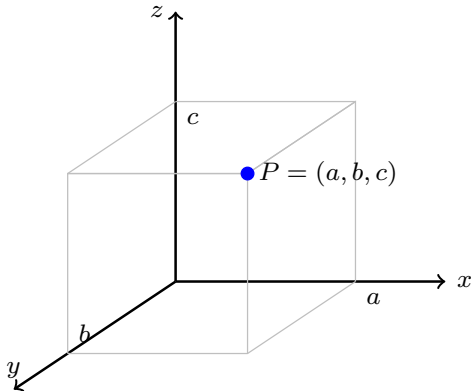
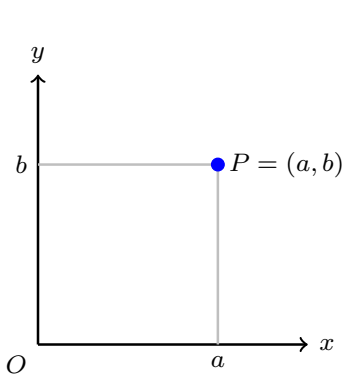


# Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- Linear independence
- **Coordinates and basis**
- Dimension
- Change of basis
- Matrix operators

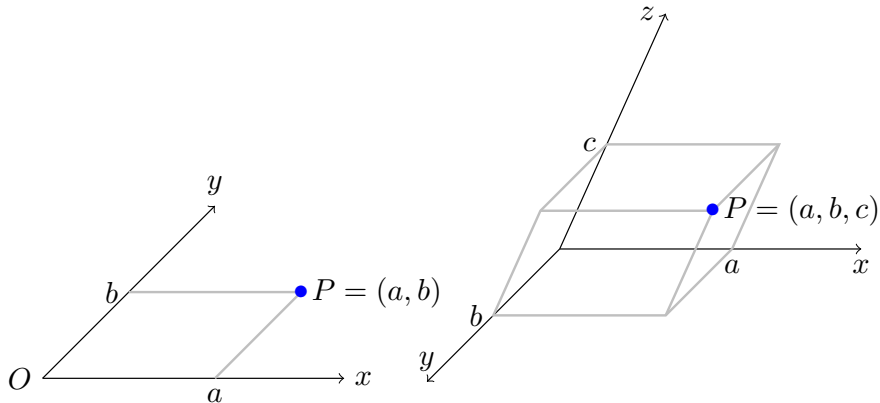
## Rectangular coordinate systems

- It is common to use rectangular coordinate systems to create a one-to-one correspondence between points in 2-D space and ordered pairs of real numbers and between points in 3-D space and ordered triple of real numbers



## Non-rectangular coordinate system

- Although rectangular coordinate systems are common, they are not essential.

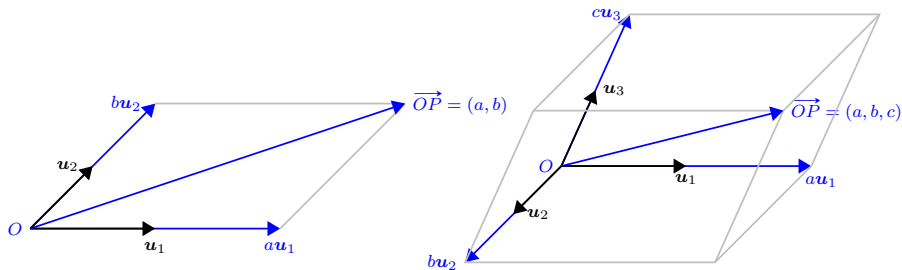




## Coordinate systems

- We can specify a coordinate system using vectors rather than coordinate axes
- Here, we have re-created the coordinate system from the previous slide by using unit vectors to identify the positive directions and then attaching coordinates to a point  $P$  using the scalar coefficients in the equations

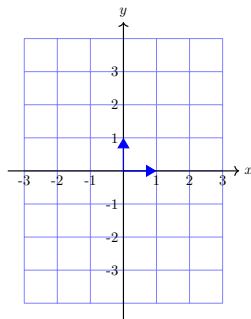
$$\overrightarrow{OP} = au_1 + bu_2, \quad \overrightarrow{OP} = au_1 + bu_2 + cu_3$$



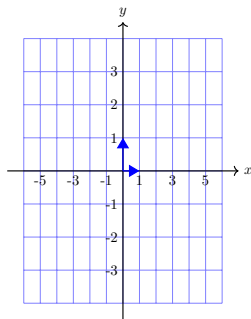
# Units of measurement

- Units of measurement are essential ingredients of any coordinate system
- In geometry problems one tries to use the same unit of measurement on all axes to avoid distorting the shapes of figures.
- This is less important in applications where coordinates represent physical quantities with diverse units (for example, time in seconds on one axis and temperature in degrees Celsius on another axis).
- To allow for this level of generality, we will relax the requirement that unit vectors be used to identify the positive directions and require only that those vectors be linearly independent.
- We will refer to these as the “basis vectors” for the coordinate system.

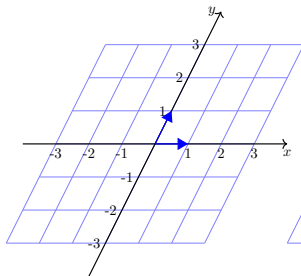
# Units of measurement



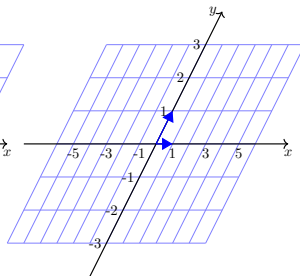
Equal spacing  
Perpendicular axes



Unequal spacing  
Perpendicular axes



Equal spacing  
Skew axes



Unequal spacing  
Skew axes

The directions of the basis vectors establish the positive directions, the lengths of the basis vectors establish the spacing between the integer points on the axes

## Dimension of vector space

- We will now extend the concept of “basis vectors” and “coordinate systems” to general vector spaces
- A vector space  $V$  is said to be *finite-dimensional* if there is a finite set of vectors in  $V$  that spans  $V$  and is said to be *infinite-dimensional* if no such set exists

# Basis

## Definition

If  $S = \{v_1, v_2, \dots, v_n\}$  is a set of vectors in a finite-dimensional vector space  $V$ , then  $S$  is called a *basis* for  $V$  if

- (a)  $S$  spans  $V$
- (b)  $S$  is linearly independent

## Basis – example

### Example

- We have discussed that the standard unit vectors of  $\mathbb{R}^n$

$$\mathbf{e}_1 := (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 := (0, 1, 0, \dots, 0), \quad \dots, \quad \mathbf{e}_n := (0, 0, 0, \dots, 1)$$

span  $\mathbb{R}^n$

- They are also linearly independent: the matrix  $(\mathbf{e}_1^\top \quad \mathbf{e}_2^\top \quad \dots \quad \mathbf{e}_n^\top) = I_n$
- Thus, they form a basis for  $\mathbb{R}^n$  – *standard basis for  $\mathbb{R}^n$* .

## Basis for $\mathbb{R}^n$

### Theorem

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for  $\mathbb{R}^n$  iff the determinant  $\begin{vmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{vmatrix} \neq 0$ .

### Proof.

We have proved that

### Theorem

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq \mathbb{R}^n$  is linearly independent iff the determinant  $\neq 0$

### Theorem

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  spans  $\mathbb{R}^n$  iff the determinant  $\neq 0$

## Basis – example

### Example (Another basis for $\mathbb{R}^3$ )

$v_1 = (1, 2, 1)$ ,  $v_2 = (2, 9, 0)$ ,  $v_3 = (3, 3, 4)$  The determinant

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 9 & 0 \\ 3 & 3 & 4 \end{vmatrix} = -1 \neq 0$$

Thus  $\{v_1, v_2, v_3\}$  is a basis for  $\mathbb{R}^3$ .



# Uniqueness of basis representation

## Theorem

*If  $S = \{v_1, v_2, \dots, v_n\}$  is a basis for a vector space  $V$ , then every vector  $v \in V$  has a unique representation as a linear combination of vectors in  $S$ :*

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n.$$

## Proof.

Suppose

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n,$$

then

$$(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \dots + (\alpha_n - \beta_n)v_n = \mathbf{0}$$

$S$  is linearly independent  $\implies \alpha_1 - \beta_1 = 0, \alpha_2 - \beta_2 = 0, \dots, \alpha_n - \beta_n = 0$



# Coordinates

## Definition

Let  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a basis for  $V$ . Suppose  $\mathbf{v} \in V$

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n,$$

then  $\alpha_1, \alpha_2, \dots, \alpha_n$  are called the *coordinates* of  $\mathbf{v}$  relative to the basis  $S$ . The vector  $(\alpha_1, \alpha_2, \dots, \alpha_n)^\top$  is called the *coordinate vector of  $\mathbf{v}$  relative to  $S$* , denoted by

$$[\mathbf{v}]_S = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

## Remark

The order of vectors matters for coordinate vectors – we assume the underlying basis is ordered without saying so explicitly

## Coordinates – example

### Example

- Let  $S$  be the standard basis for  $\mathbb{R}^n$
- The coordinate vector  $[v]_S$  and the vector  $v$  are the same

## Coordinates – example

### Example

- We have discussed that  $\mathbf{v}_1 = (1, 2, 1)$ ,  $\mathbf{v}_2 = (2, 9, 0)$ ,  $\mathbf{v}_3 = (3, 3, 4)$  form a basis for  $\mathbb{R}^3$
- Let  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
- $\mathbf{v} = (5, -1, 9)$
- Solve for  $\alpha_1, \alpha_2, \alpha_3$

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3$$

we get

$$\alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = 2$$

- Then

$$[\mathbf{v}]_S = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

# Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
- **Dimension**
- Change of basis
- Matrix operators

## Bases

### Lemma

Let  $S_1, S_2$  be subsets of  $V$ . If  $V = \text{span}(S_1)$  and vectors in  $S_2$  are linearly independent, then  $|S_1| \geq |S_2|$ .

### Proof.

Suppose  $S_1 = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{r_1} \}$  and  $S_2 = \{ \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{r_2} \}$ . Since  $V = \text{span}(S_1)$ ,

$$\mathbf{w}_1 = \sum_{j=1}^{r_1} \alpha_j \mathbf{v}_j$$

At least one of  $\alpha_j \neq 0$  as vectors in  $S_2$  are linearly independent –  $\mathbf{0} \notin S_2$ . WLOG, assume  $\alpha_1 \neq 0$ , then

$$\mathbf{v}_1 = - \sum_{j=2}^{r_1} \frac{\alpha_j}{\alpha_1} \mathbf{v}_j + \frac{1}{\alpha_1} \mathbf{w}_1 \implies V = \text{span}(\{ \mathbf{w}_1, \mathbf{v}_2, \dots, \mathbf{v}_{r_1} \})$$

## Bases

### Lemma

Let  $S_1, S_2$  be subsets of  $V$ . If  $V = \text{span}(S_1)$  and vectors in  $S_2$  are linearly independent, then  $|S_1| \geq |S_2|$ .

### Proof.

$V = \text{span}(\{ \mathbf{w}_1, \mathbf{v}_2, \dots, \mathbf{v}_{r_1} \})$ , then, we can write

$$\mathbf{w}_2 = \beta_1 \mathbf{w}_1 + \sum_{j=2}^{r_1} \beta_j \mathbf{v}_j,$$

at least one of  $\beta_j \neq 0$  for  $2 \leq j \leq r_1$ , otherwise  $\mathbf{w}_2$  is a linear combination of  $\mathbf{w}_1$ . Suppose  $\beta_2 \neq 0$ , We have

$$\mathbf{v}_2 = -\frac{\beta_1}{\beta_2} \mathbf{w}_1 - \sum_{j=3}^{r_1} \frac{\beta_j}{\beta_2} \mathbf{v}_j + \frac{1}{\beta_2} \mathbf{w}_2 \implies V = \text{span}(\{ \mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_3, \dots, \mathbf{v}_{r_1} \})$$

# Bases

## Lemma

*Let  $S_1, S_2$  be subsets of  $V$ . If  $V = \text{span}(S_1)$  and vectors in  $S_2$  are linearly independent, then  $|S_1| \geq |S_2|$ .*

## Proof.

We can continue in this manner, if  $r_1 < r_2$ , we will deduce that  $\{w_1, w_2, \dots, w_{r_1}\}$  spans  $V$  and  $w_{r_1+1}$  can be written as a linear combination of  $\{w_1, w_2, \dots, w_{r_1}\}$ , a contradiction. □



# Bases

## Theorem

*If  $B_1, B_2$  are bases of  $V$ , then  $|B_1| = |B_2|$*

## Proof.

By the previous lemma,

$$|B_1| \geq |B_2|, |B_2| \geq |B_1| \implies |B_1| = |B_2|.$$



# Dimension

## Definition

The *dimension* of a vector space  $V$ , denoted  $\dim(V)$ , is given by the cardinality of  $B$ ,  $|B|$ , where  $B$  is a basis of  $V$ . The zero vector space is defined to have dimension zero.

## Example

- $\dim(\mathbb{R}^n) = n$  - the standard basis has  $n$  vectors
- $S = \{v_1, v_2, \dots, v_r\}$ , linearly independent, then  $\text{span}(S)$  has dimension  $r$

## Dimension – example

### Example (Dimension of a solution space)

$$\begin{aligned}x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\5x_3 + 10x_4 + 15x_6 &= 0 \\2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0\end{aligned}$$

Solution is of the form

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_6) &= (-3r - 4s - 2t, r - 2s, s, t, 0) \\&= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)\end{aligned}$$

$\implies$  the vectors

$(-3, 1, 0, 0, 0, 0), (-4, 0, -2, 1, 0, 0), (-2, 0, 0, 0, 1, 0)$   
span the solution space. They are also linearly independent (verify), thus the solution space has dimension 3.

## Remark

- It can be shown that for any homogeneous linear system, the method of the last example always produces a basis for the solution space of the system.
- We omit the formal proof.

# Vector spaces and matrix transformations

- Real Vector Space
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# The change-of-basis problem

## The change-of-basis problem

$\mathbf{v} \in V$ ,  $\dim(V) < \infty$ . Let  $B_1$  and  $B_2$  be two bases for  $V$ . What is the relation between  $[\mathbf{v}]_{B_1}$  and  $[\mathbf{v}]_{B_2}$ ?

- $B_1$  old basis
- $B_2$  new basis
- Our objective is to find a relationship between the old and new coordinates of a fixed vector  $\mathbf{v} \in V$

## Two dimensional vector spaces

- Suppose  $\dim(V) = 2$   $B_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ ,  $B_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$  and

$$[\mathbf{w}_1]_{B_1} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad [\mathbf{w}_2]_{B_1} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\text{i.e. } \mathbf{w}_1 = a\mathbf{u}_1 + b\mathbf{u}_2, \quad \mathbf{w}_2 = c\mathbf{u}_1 + d\mathbf{u}_2$$

- Suppose  $[\mathbf{v}]_{B_2} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ , so

$$\mathbf{v} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 = k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) = (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2$$

$$[\mathbf{v}]_{B_1} = \begin{pmatrix} k_1a + k_2c \\ k_1b + k_2d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} [\mathbf{v}]_{B_2}$$

$$P := \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ([\mathbf{w}_1]_{B_1} \quad [\mathbf{w}_2]_{B_1})$$

## Change of basis and transition matrices

- $\mathbf{v} \in V$ ,  $B_1 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ ,  $B_2 = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  basis for  $V$
- Then

$$[\mathbf{v}]_{B_1} = P[\mathbf{v}]_{B_2}$$

where

$$P = ([\mathbf{w}_1]_{B_1} \quad [\mathbf{w}_2]_{B_1} \quad \cdots \quad [\mathbf{w}_n]_{B_1})$$

- $P$ : *transition matrix* from  $B_2$  to  $B_1$ , denoted  $P_{B_2 \rightarrow B_1}$

$$P_{B_2 \rightarrow B_1} = ([\mathbf{w}_1]_{B_1} \quad [\mathbf{w}_2]_{B_1} \quad \cdots \quad [\mathbf{w}_n]_{B_1})$$

$$P_{B_1 \rightarrow B_2} = ([\mathbf{u}_1]_{B_2} \quad [\mathbf{u}_2]_{B_2} \quad \cdots \quad [\mathbf{u}_n]_{B_2})$$

- The coordinate vector

$$[\mathbf{v}]_{B_1} = P_{B_2 \rightarrow B_1}[\mathbf{v}]_{B_2}, \quad [\mathbf{v}]_{B_2} = P_{B_1 \rightarrow B_2}[\mathbf{v}]_{B_1}$$



## Change of basis and transition matrices – example

### Example

- $B_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ ,  $B_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ , basis for  $\mathbb{R}^2$

$$\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1), \mathbf{w}_1 = (1, 1), \mathbf{w}_2 = (2, 1)$$

- Transition matrix  $P_{B_2 \rightarrow B_1}$

$$[\mathbf{w}_1]_{B_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, [\mathbf{w}_2]_{B_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \implies P_{B_2 \rightarrow B_1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

- Transition matrix  $P_{B_1 \rightarrow B_2}$

$$[\mathbf{u}_1]_{B_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, [\mathbf{u}_2]_{B_2} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \implies P_{B_1 \rightarrow B_2} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

- Suppose  $[\mathbf{v}]_{B_2} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ ,  $[\mathbf{v}]_{B_1} = P_{B_2 \rightarrow B_1}[\mathbf{v}]_{B_2} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

## Invertibility of transition matrices

$$P_{B_2 \rightarrow B_1} P_{B_1 \rightarrow B_2} = P_{B_1 \rightarrow B_1} = I$$

Using the previous example

$$P_{B_2 \rightarrow B_1} P_{B_1 \rightarrow B_2} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

### Theorem

*The transition matrix  $P_{B_1 \rightarrow B_2}$  is invertible and  $P_{B_1 \rightarrow B_2}^{-1} = P_{B_2 \rightarrow B_1}$ .*

# Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
- Dimension
- Change of basis
- **Matrix operators**

# Matrix transformations

- Consider the linear system  $Ax = w$
- We can view it as a transformation that maps a vector  $x \in \mathbb{R}^n$  into the vector  $w \in \mathbb{R}^m$
- We call this a *matrix transformation* (or *matrix operator* when  $m = n$ ), denoted

$$\begin{aligned} T_A : \mathbb{R}^n &\rightarrow \mathbb{R}^m \\ x &\mapsto w \end{aligned}$$

- We call the transformation  $T_A$  *multiplication by A*

## Example

- *Zero transformation:*

$$T_O(x) = Ox = \mathbf{0}$$

- *Identity operator:*

$$T_I(x) = Ix = x$$

# Properties of matrix transformations

## Theorem

For any  $A \in \mathcal{M}_{m \times n}$ ,  $\alpha \in \mathbb{R}$ ,  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  satisfies

- $T_A(\mathbf{0}) = \mathbf{0}$
- $T_A(\alpha \mathbf{u}) = \alpha T_A(\mathbf{u})$  (*Homogeneity property*)
- $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$  (*Additivity property*)
- $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

## Remark

When we deal with matrix transformations, vectors are assumed to be column vectors unless explicitly stated otherwise.

# Standard matrix

## Theorem

$T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $T_B : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . If  $T_A(\mathbf{x}) = T_B(\mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^n$ , then  $A = B$ .

## Proof.

Consider standard basis  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$

$$A\mathbf{e}_j = B\mathbf{e}_j, \quad j = 1, 2, \dots, n$$

$A\mathbf{e}_j$  (resp.  $B\mathbf{e}_j$ ) is the  $j$ th column of  $A$  (resp.  $B$ )



## Note

- Every  $A \in \mathcal{M}_{m \times n}$  produces exactly one matrix transformation (multiplication by  $A$ )
- Every matrix transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  arises from exactly one  $A \in \mathcal{M}_{m \times n}$  – *standard matrix* for the transformation

## Find standard matrix

- The standard matrix for  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is given by

$$A = (T(e_1) \quad T(e_2) \quad \cdots \quad T(e_n))$$

## Operators on $\mathbb{R}^2$ and $\mathbb{R}^3$

- There are many ways to transform the vector spaces  $\mathbb{R}^2$  and  $\mathbb{R}^3$
- Some can be accomplished by using a *matrix operator*  $T_A$ ,  $A \in \mathcal{M}_{2 \times 2}$  or  $\mathcal{M}_{3 \times 3}$
- e.g. rotations about the origin, reflections about lines and planes through the origin, and projections onto lines and planes through the origin
- *Reflection operators*: map each point into its symmetric image about a fixed line or a fixed plane that contains the origin



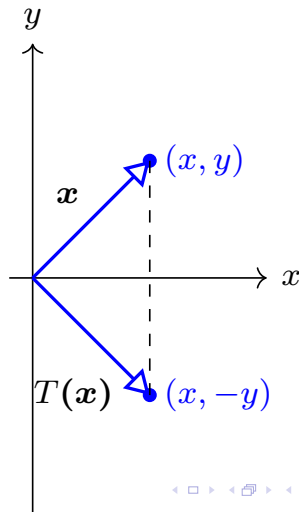
## Reflection operators on $\mathbb{R}^2$ – reflection about the $x$ -axis

- $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$
- Images of  $e_1$  and  $e_2$

$$T(e_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T(e_2) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

- Standard matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



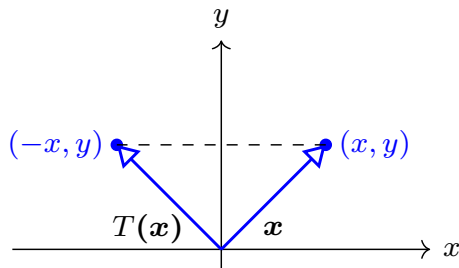
## Reflection operators on $\mathbb{R}^2$ – reflection about the $y$ -axis

- $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$
- Images of  $e_1$  and  $e_2$

$$T(e_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$T(e_2) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Standard matrix  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



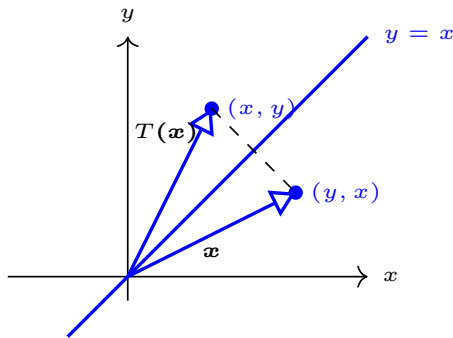
## Reflection operators on $\mathbb{R}^2$ – reflection about the line $y = x$

- $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$
- Images of  $e_1$  and  $e_2$

$$T(e_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T(e_2) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Standard matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



## Reflection operators on $\mathbb{R}^3$

Reflection about the  $xy$ -plane,  $xz$ -plane and  $yz$ -plane

$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$	$T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$	$T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$	$T(e_1) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

# Projection operators

- Projection operators/orthogonal projection operators: Map each point into its orthogonal projection onto a fixed line or plane through the origin

## Projection operators on $\mathbb{R}^2$

- Orthogonal projection onto the  $x$ -axis

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

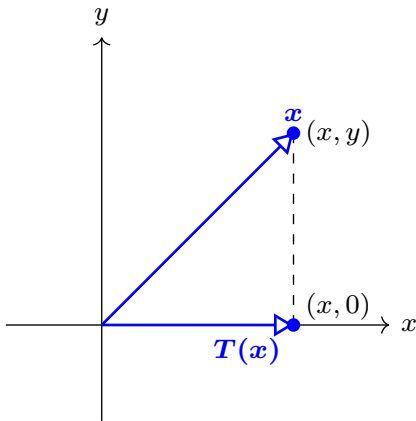
- Images of  $e_1$  and  $e_2$

$$T(e_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T(e_2) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Standard matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



## Projection operators on $\mathbb{R}^2$

- Orthogonal projection onto the  $y$ -axis

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

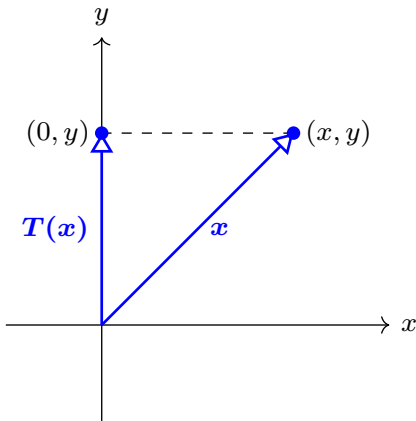
- Images of  $e_1$  and  $e_2$

$$T(e_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T(e_2) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Standard matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



## Projection operators on $\mathbb{R}^3$

Orthogonal projection onto the  $xy$ -plane,  $xz$ -plane and  $yz$ -plane

$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$	$T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$	$T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$	$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



# Rotation operators

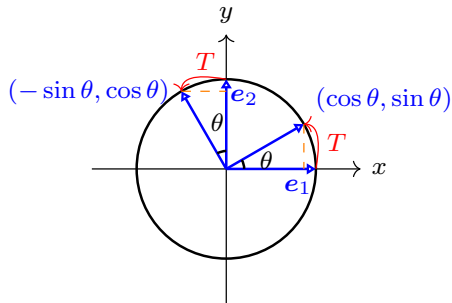
- Rotation operators: move points along arcs of circles centered at the origin
- Consider the rotation operator  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that moves points *counterclockwise* about the origin through a positive angle  $\theta$

$$T(e_1) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

- Standard matrix - *rotation matrix*

$$R_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- $\mathbf{x} = (x, y), \mathbf{w} = R_\theta \mathbf{x}^\top, \longrightarrow$



rotation equations:

$$w_1 = x \cos \theta - y \sin \theta$$

$$w_2 = x \sin \theta + y \cos \theta$$

## Rotation operators

- The rotation matrix for a clockwise rotation of  $\theta$ , or a rotation of  $-\theta$ , radians has rotation matrix

$$R_{-\theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

## Rotation operator on $\mathbb{R}^2$ – example

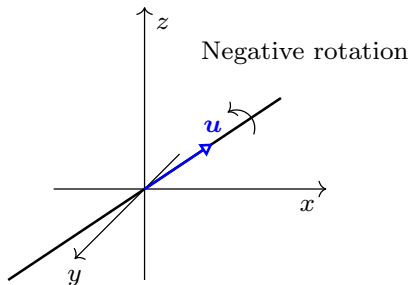
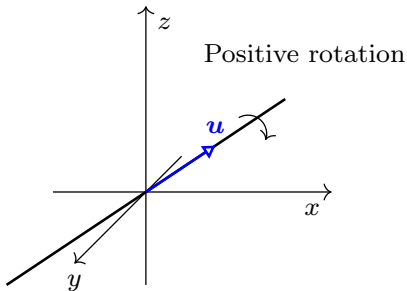
### Example

Find the image of  $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  under a rotation of  $\frac{\pi}{6}$  radians ( $= 30^\circ$ ) about the origin

$$R_{\frac{\pi}{6}} \mathbf{x} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{pmatrix} \approx \begin{pmatrix} 0.37 \\ 1.37 \end{pmatrix}$$

## Rotation operators on $\mathbb{R}^3$

- A rotation of vectors in  $\mathbb{R}^3$  is commonly described in relation to a line through the origin called the *axis of rotation* and unit vector  $u$  along that line
- Positive angle: counterclockwise looking toward the origin along the positive coordinate axis
- *right-hand-rule*: cup the fingers of right hand so they curl in the direction of the rotation, thumb points in the direction of  $u$  corresponds to positive angle



## Rotation operators on $\mathbb{R}^3$

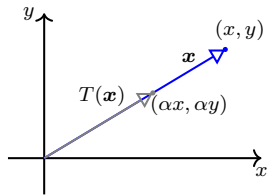
Operator	Rotation equations	Standard matrix
Counterclockwise rotation about the positive $x$ -axis through an angle $\theta$	$\begin{aligned} w_1 &= x \\ w_2 &= y \cos \theta - z \sin \theta \\ w_3 &= y \sin \theta + z \cos \theta \end{aligned}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$
Counterclockwise rotation about the positive $y$ -axis through an angle $\theta$	$\begin{aligned} w_1 &= x \cos \theta + z \sin \theta \\ w_2 &= y \\ w_3 &= -x \sin \theta + z \cos \theta \end{aligned}$	$\begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$
Counterclockwise rotation about the positive $z$ -axis through an angle $\theta$	$\begin{aligned} w_1 &= x \cos \theta - y \sin \theta \\ w_2 &= x \sin \theta + y \cos \theta \\ w_3 &= z \end{aligned}$	$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

# Dilations and contractions

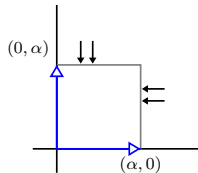
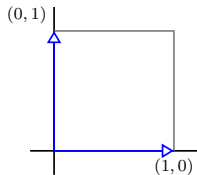
- $\alpha \in \mathbb{R}, \alpha \geq 0$
- $T(\mathbf{x}) = \alpha \mathbf{x}$  on  $\mathbb{R}^2$  or  $\mathbb{R}^3$  has the effect of increasing or decreasing the length of each vector by a factor of  $\alpha$
- *Contraction* with factor  $\alpha$ :  $0 \leq \alpha < 1$
- *Dilation* with factor  $\alpha$ :  $\alpha > 1$
- Identity operator:  $\alpha = 1$

# Dilations and contractions on $\mathbb{R}^2 - T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$

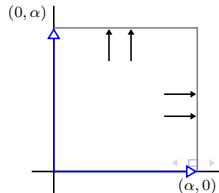
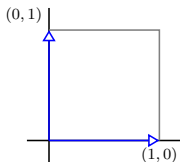
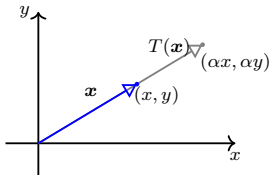
Illustration



Effect on the unit square



$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$



$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

## Dilations and contractions on $\mathbb{R}^3$

- Similarly, for dilation/contraction with factor  $\alpha$  on  $\mathbb{R}^3$

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$$

- The standard matrix is

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

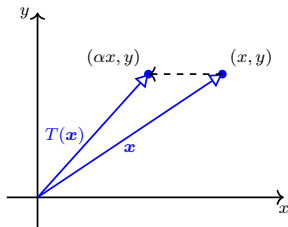


## Expansions and compressions

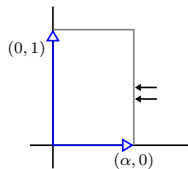
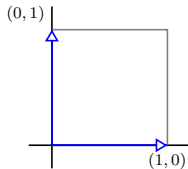
- Dilation or contraction: all coordinates are multiplied by a nonnegative factor
- *Compression/expansion*: only one coordinate is multiplied by  $\alpha$

# Expansions and compressions in the $x$ -direction – $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ y \end{pmatrix}$

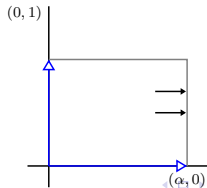
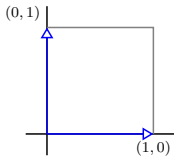
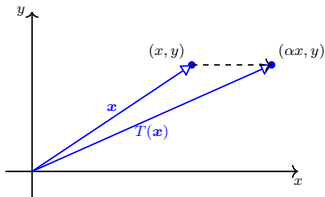
Illustration



Effect on the unit square



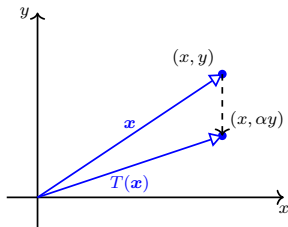
$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$



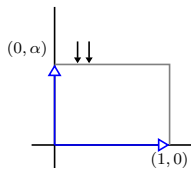
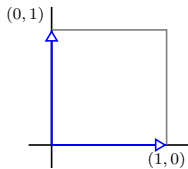
$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

# Expansions and compressions in the $y$ -direction – $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \alpha y \end{pmatrix}$

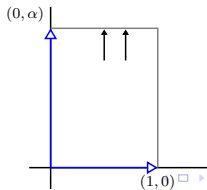
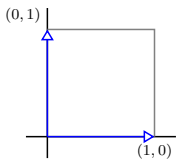
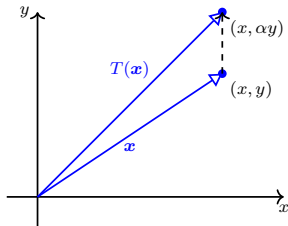
Illustration



Effect on the unit square



$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

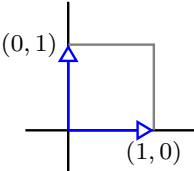
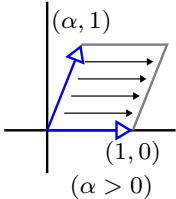
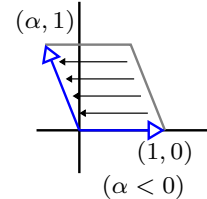
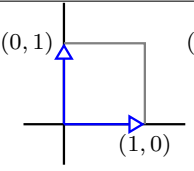
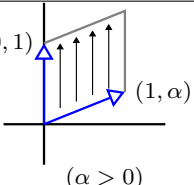
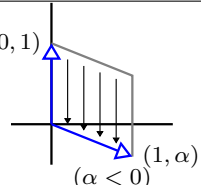


$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

# Shears

- $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y \end{pmatrix}$
- Translates a point  $\begin{pmatrix} x \\ y \end{pmatrix}$  in the  $xy$ -plane parallel to the  $x$ -axis by an amount  $\alpha y$  that is proportional to the  $y$ -coordinate of the point.
- Points on the  $x$ -axis are fixed ( $y = 0$ ), the translation distance increases as we progress away from the  $x$ -axis
- *Shear in the  $x$ -direction by a factor  $\alpha$*
- Similarly,  $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \alpha x \end{pmatrix}$  - *Shear in the  $y$ -direction by a factor  $\alpha$*

# Shears

Operator	Effect on the unit square			Standard matrix
$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y \end{pmatrix}$		 ( $\alpha > 0$ )	 ( $\alpha < 0$ )	$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$
$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \alpha x \end{pmatrix}$		 ( $\alpha > 0$ )	 ( $\alpha < 0$ )	$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$

## Shears – example

### Example

- $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ : shear in the  $x$ -direction by a factor 2
- $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ : shear in the  $x$ -direction by a factor  $-2$
- $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ : dilation with factor 2
- $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ : expansion in the  $x$ -direction with factor 2

## Orthogonal projections onto lines through the origin

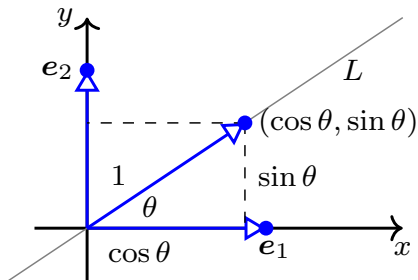
- Orthogonal projections onto the coordinate axes in  $\mathbb{R}^2$  are special cases of the operator that maps each point into its orthogonal projection onto a line  $L$  through the origin that makes an angle  $\theta$  with the positive  $x$ -axis
- Recall from tutorial 1

## Orthogonal projection on a line

- Find the orthogonal projections of the vectors  $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  on the line  $L$  that makes an angle  $\theta$  with the positive  $x$ -axis in  $\mathbb{R}^2$
- First we find the orthogonal projection of  $e_1$  onto  $a := \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$

$$\begin{aligned}\text{proj}_a e_1 &= \frac{e_1 \cdot a}{\|a\|^2} a = \frac{\cos \theta + 0}{1} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \theta \\ \sin \theta \cos \theta \end{pmatrix}\end{aligned}$$

- We note that for any other vector,  $u$  on the line  $L$ ,  $u = \alpha a$  for some  $\alpha \in \mathbb{R}$



- Similarly

$$\begin{aligned}\text{proj}_a e_2 &= \frac{e_2 \cdot a}{\|a\|^2} a \\ &= \begin{pmatrix} \sin \theta \cos \theta \\ \sin^2 \theta \end{pmatrix}\end{aligned}$$



## Orthogonal projection onto a line through the origin

- projection onto a line  $L$  through the origin that makes an angle  $\theta$  with the positive  $x$ -axis
- The images of  $e_1$  and  $e_2$  are

$$T(e_1) = \begin{pmatrix} \cos^2 \theta \\ \sin \theta \cos \theta \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} \sin \theta \cos \theta \\ \sin^2 \theta \end{pmatrix}$$

- Standard matrix is

$$P_\theta := \begin{pmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{pmatrix} = \begin{pmatrix} \cos^2 \theta & \frac{1}{2} \sin 2\theta \\ \frac{1}{2} \sin 2\theta & \sin^2 \theta \end{pmatrix}$$

## Orthogonal projection onto a line through the origin

### Example

- Find the orthogonal projection of the vector  $\mathbf{x} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$  onto the line through the origin that makes an angle of  $\frac{\pi}{6} (= 30^\circ)$  with the positive  $x$ -axis
- The standard matrix is

$$P_{\pi/6} = \begin{pmatrix} \cos^2(\pi/6) & \sin(\pi/6) \cos(\pi/6) \\ \sin(\pi/6) \cos(\pi/6) & \sin^2(\pi/6) \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

$$P_{\pi/6} \mathbf{x} = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3 + 5\sqrt{3}}{4} \\ \frac{\sqrt{3} + 5}{4} \end{pmatrix} \approx \begin{pmatrix} 2.91 \\ 1.68 \end{pmatrix}$$

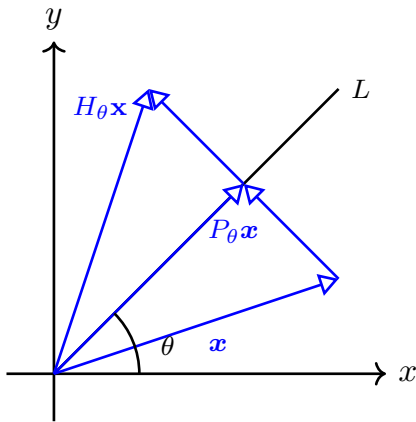
## Reflection about lines through the origin

- $H_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$
- Maps each point into its reflection about a line  $L$  through the origin that makes an angle  $\theta$  with the positive  $x$ -axis
- From the figure, we can see

$$P_\theta x - x = \frac{1}{2}(H_\theta x - x) \implies H_\theta x = (2P_\theta - I)x$$

- It follows from the theorem that

$$H_\theta = 2P_\theta - I = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$



### Theorem

$T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $T_B : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . If  $T_A(x) = T_B(x)$  for all  $x \in \mathbb{R}^n$ , then  $A = B$ .

## Reflection about lines through the origin

### Example

- Find the reflection of the vector  $\mathbf{x} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$  about the line through the origin that makes an angle  $\pi/6$  ( $= 30^\circ$ ) with the positive  $x$ -axis
- The standard matrix

$$H_{\pi/6} = \begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$H_{\pi/6} \mathbf{x}^\top = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{1+5\sqrt{3}}{2} \\ \frac{\sqrt{3}-5}{2} \end{pmatrix} \approx \begin{pmatrix} 4.83 \\ -1.63 \end{pmatrix}$$

## Compositions of matrix transformations

- $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^k$ ,  $T_B : \mathbb{R}^k \rightarrow \mathbb{R}^m$
- The composition of  $T_B$  and  $T_A$  is a matrix transformation

$$\begin{aligned} T_B \circ T_A : \mathbb{R}^n &\rightarrow \mathbb{R}^m \\ \mathbf{x} &\mapsto (BA)\mathbf{x} \end{aligned}$$

- Because

$$T_B \circ T_A(\mathbf{x}) = T_B(T_A(\mathbf{x})) = B(A\mathbf{x}) = (BA)\mathbf{x}$$

- Thus

$$T_B \circ T_A = T_{BA}$$

# Compositions of matrix transformations

- Consider

$$T_A : \mathbb{R}^n \rightarrow \mathbb{R}^k, \quad T_B : \mathbb{R}^k \rightarrow \mathbb{R}^\ell, \quad T_C : \mathbb{R}^\ell \rightarrow \mathbb{R}^m$$

- Then

$$T_C \circ T_B \circ T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

- and

$$(T_C \circ T_B \circ T_A)(\mathbf{x}) = T_C(T_B(T_A(\mathbf{x}))) = (CBA)\mathbf{x}$$

- i.e.

$$T_C \circ T_B \circ T_A = T_{CBA}$$

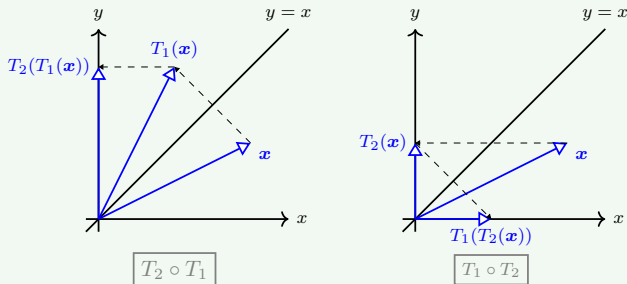
# Compositions of matrix transformations

- Let us denote the standard matrix for  $T$  by  $[T]$
- Then

$$[T_2 \circ T_1] = [T_2][T_1], \quad [T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$$

# Composition is not commutative

## Example (Composition is not commutative)



- $T_1$ : reflection about the line  $y = x$ ;  $T_2$ : orthogonal projection onto the  $y$ -axis

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



# Composition of rotations is commutative

## Example

- Rotation in  $\mathbb{R}^2$
- $T_1$ : rotate vectors about the origin through the angle  $\theta_1$
- $T_2$ : rotate vectors about the origin through the angle  $\theta_2$
- Standard matrix

$$[T_1] = \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \quad [T_2] = \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{pmatrix}$$

- With trigonometric identities, it can be verified that

$$[T_2 \circ T_1] = [T_1 \circ T_2] = \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}$$