

Tutorial 6

Bases and matrix operators

Homework

1. Provide detailed solutions to Q2, part 2.
2. Provide a detailed solution to Q6.
3. Provide a detailed solution to Q7, part 2.
4. Provide a detailed solution to Q9, part 2.
5. Provide a detailed solution to *one* selected part of Q10.
6. Provide a detailed solution to Q19, part (b).
7. Provide a detailed solution to Q20, choosing either part (a) or part (b).
8. Provide a detailed solution to Q23.
9. Provide a detailed solution to Q28.

Question 1. Show that the following set of vectors forms a basis for \mathbb{R}^2 and \mathbb{R}^3 respectively.

1. $\{ (2, 1), (3, 0) \}$
2. $\{ (3, 1, -4), (2, 5, 6) \}, (1, 4, 8)$

Solution.

- The determinant

$$\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = -3 \neq 0.$$

thus, the set of vectors forms a basis for \mathbb{R}^2 .

- The determinant

$$\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = 26 \neq 0$$

thus, the set of vectors forms a basis for \mathbb{R}^3 .

Question 2. Show that the following matrices form a basis for $\mathcal{M}_{2 \times 2}$.

1. $\begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Solution.

1. We must show that the matrices are linearly independent and span $\mathcal{M}_{2 \times 2}$. To prove linear independence we must show that the equation

$$\alpha_1 \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has only the trivial solution. Expanding both sides and equating corresponding entries, we obtain the following system of linear equations:

$$\begin{aligned} 3\alpha_1 + \alpha_4 &= 0 \\ 6\alpha_1 - \alpha_2 - 8\alpha_3 &= 0 \\ 3\alpha_1 - \alpha_2 - 12\alpha_3 - \alpha_4 &= 0 \\ -6\alpha_1 - 4\alpha_3 + 2\alpha_4 &= 0 \end{aligned}$$

The coefficient matrix of the linear system is

$$A = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{pmatrix}.$$

Computing the determinant, we get $\det(A) = 48 \neq 0$. Since the determinant is nonzero, the system has only the trivial solution, which confirms that the matrices are linearly independent.

To prove that the matrices span $\mathcal{M}_{2 \times 2}$ we must show that every 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be expressed as

$$\beta_1 \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \beta_3 \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix} + \beta_4 \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This corresponds to a system of linear equations in the unknowns $\beta_1, \beta_2, \beta_3, \beta_4$ with the same coefficient matrix A as before. Since we have computed that $\det(A) \neq 0$, the system always has a unique solution for any given values a, b, c, d . Consequently, the given matrices span $\mathcal{M}_{2 \times 2}$. Since the given set of matrices is both linearly independent and spans $\mathcal{M}_{2 \times 2}$, it forms a basis for $\mathcal{M}_{2 \times 2}$.

2. Using a similar approach as in Part 1, we consider the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Computing its determinant, we obtain $\det(A) = 1 \neq 0$.

Question 3. In each part, show that the set of vectors is not a basis for \mathbb{R}^3

1. $\{ (2, -3, 1), (4, 1, 1), (0, -7, 1) \}$
2. $\{ (1, 6, 4), (2, 4, -1), (-1, 2, 5) \}$

Solution.

1. The determinant

$$\begin{vmatrix} 2 & -3 & 1 \\ 4 & 1 & 1 \\ 0 & -7 & 1 \end{vmatrix} = 0.$$

Since the determinant is zero, the matrix is singular, implying that its column vectors are linearly dependent. Consequently, the given set of vectors does not form a basis for \mathbb{R}^3 .

2. The determinant

$$\begin{vmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{vmatrix} = 0,$$

thus, the set of vectors is not a basis for \mathbb{R}^3 .

Question 4. Show that the following matrices do not form a basis for $\mathcal{M}_{2 \times 2}$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Solution.

$$-\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

demonstrates that the zero matrix can be expressed as a nontrivial linear combination of the given matrices. This implies that the matrices are linearly dependent and, therefore, do not form a basis for $\mathcal{M}_{2 \times 2}$.

Question 5. Find the coordinate vector of \mathbf{w} relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for \mathbb{R}^2 .

1. $\mathbf{u}_1 = (2, -4)$, $\mathbf{u}_2 = (3, 8)$, $\mathbf{w} = (1, 1)$
2. $\mathbf{u}_1 = (1, 1)$, $\mathbf{u}_2 = (0, 2)$, $\mathbf{w} = (a, b)$
3. $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$, $\mathbf{w} = (1, 0)$
4. $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$, $\mathbf{w} = (0, 1)$

Solution.

1. To find $[\mathbf{w}]_S$, we must find values $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 = \mathbf{w} \implies \alpha_1 (2, -4) + \alpha_2 (3, 8) = (1, 1)$$

Equating corresponding components gives

$$\begin{aligned} 2\alpha_1 + 3\alpha_2 &= 1 \\ -4\alpha_1 + 8\alpha_2 &= 1 \end{aligned}$$

The augmented matrix of the system is

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array} \right)$$

which has reduced row echelon form

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{array} \right)$$

We have

$$\alpha_1 = \frac{5}{28}, \quad \alpha_2 = \frac{3}{14}.$$

Therefore

$$[\mathbf{w}]_S = \begin{pmatrix} \frac{5}{28} \\ \frac{3}{14} \end{pmatrix}$$

2. To find $[\mathbf{w}]_S$, we must find values $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 = \mathbf{w} \implies \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Equating corresponding components gives

$$\begin{aligned} \alpha_1 + \alpha_2 &= a \\ 2\alpha_2 &= b \end{aligned}$$

We have

$$\alpha_1 = \frac{2a - b}{2}, \quad \alpha_2 = \frac{b}{2}.$$

Therefore

$$[\mathbf{w}]_S = \begin{pmatrix} \frac{2a - b}{2} \\ \frac{b}{2} \end{pmatrix}$$

3.

$$[\mathbf{w}]_S = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

4.

$$[\mathbf{w}]_S = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Question 6. Find the coordinate vector of \mathbf{u} relative to the basis $S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$ for \mathbb{R}^3

$$1. \mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (2, 2, 0), \mathbf{v}_3 = (3, 3, 3), \mathbf{u} = (2, -1, 3)$$

$$2. \mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (-4, 5, 6), \mathbf{v}_3 = (7, -8, 9), \mathbf{u} = (5, -12, 3)$$

Solution.

1. $\mathbf{u} = 3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$, and

$$[\mathbf{u}]_S = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

2. $\mathbf{u} = -2\mathbf{v}_1 + \mathbf{v}_3$, and

$$[\mathbf{u}]_S = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Question 7. For each case, first show that the set $S = \{A_1, A_2, A_3, A_4\}$ is a basis for $\mathcal{M}_{2 \times 2}$, then express A as a linear combination of the matrices in S , and then find the coordinate vector of A relative to S .

1. $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

2. $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

Solution.

1. To show S is a basis for $\mathcal{M}_{2 \times 2}$, we must show that S linearly independent and span $\mathcal{M}_{2 \times 2}$. To prove linear independence we must show that the equation

$$\alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has only the trivial solution. Expanding both sides and equating corresponding entries, we obtain the following system of linear equations:

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \end{aligned}$$

The coefficient matrix of the linear system is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Computing the determinant, we get $\det(M) = 1 \neq 0$. Since the determinant is nonzero, the system has only the trivial solution, which confirms that the matrices are linearly independent.

To prove that the matrices span $\mathcal{M}_{2 \times 2}$ we must show that every 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be expressed as

$$\beta_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \beta_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \beta_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This corresponds to a system of linear equations with coefficient matrix M . Since $\det(M) \neq 0$, the system always has a unique solution for any given values a, b, c, d , proving that the given matrices span $\mathcal{M}_{2 \times 2}$. Since S is both linearly independent and spans $\mathcal{M}_{2 \times 2}$, it forms a basis for $\mathcal{M}_{2 \times 2}$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, we get the following system of linear equations in unknowns $\beta_1, \beta_2, \beta_3, \beta_4$

$$\begin{aligned} \beta_1 &= 1 \\ \beta_1 + \beta_2 &= 0 \\ \beta_1 + \beta_2 + \beta_3 &= 1 \\ \beta_1 + \beta_2 + \beta_3 + \beta_4 &= 0 \end{aligned}$$

Solving the system gives

$$\beta_1 = 1, \quad \beta_2 = -1, \quad \beta_3 = 1, \quad \beta_4 = -1$$

Thus, A can be expressed as

$$A = A_1 - A_2 + A_3 - A_4$$

Therefore, the coordinate vector of A relative to S is

$$[A]_S = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

2. The proof follows a similar argument as in Part 1. In this case, the coefficient matrix is

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which has determinant $\det(M) = -1$. Furthermore,

$$A = 1A_1 + 2A_2 + 3A_3 + 4A_4$$

and

$$[A]_S = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Question 8. In each part, let $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be multiplication by A , and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for \mathbb{R}^3 . Determine whether the set $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_3)\}$ is linearly independent in \mathbb{R}^3

$$1. \ A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

Solution. Since $T_A(\mathbf{e}_i)$ is equal to the i th column of A , the set $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_3)\}$ is linearly independent in \mathbb{R}^3 iff $\det(A) \neq 0$

1. $\det(A) = 10 \neq 0$, the set $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_3)\}$ is linearly independent in \mathbb{R}^3
2. $\det(A) = 0$, the set $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_3)\}$ is linearly dependent in \mathbb{R}^3

Question 9. In each part, let $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be multiplication by A , and let $\mathbf{u} = (1, -2, -1)$. Find the coordinate vector of $T_A(\mathbf{u})$ relative to the basis $S = \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ for \mathbb{R}^3 .

$$1. A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution.

1.

$$T_A(\mathbf{u}) = A\mathbf{u}^\top = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$$

By solving for $\alpha_1, \alpha_2, \alpha_3$ in

$$\alpha_1(1, 1, 0) + \alpha_2(0, 1, 1) + \alpha_3(1, 1, 1) = (4, -2, 0),$$

we get

$$T_A(\mathbf{u}) = -2(1, 1, 0) - 6(0, 1, 1) + 6(1, 1, 1)$$

Hence the coordinate vector is given by

$$[T_A(\mathbf{u})]_S = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix}$$

2.

$$T_A(\mathbf{u}) = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}, \quad [T_A(\mathbf{u})]_S = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

Question 10. Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

1.

$$\begin{aligned}x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0\end{aligned}$$

2.

$$\begin{aligned}3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0\end{aligned}$$

3.

$$\begin{aligned}2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 5x_3 &= 0 \\ x_2 + x_3 &= 0\end{aligned}$$

4.

$$\begin{aligned}x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0\end{aligned}$$

5.

$$\begin{aligned}x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0\end{aligned}$$

6.

$$\begin{aligned}x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0\end{aligned}$$

Solution.

1. The reduced row echelon form of the augmented matrix is $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. The solution is given by

$$(x_1, x_2, x_3) = (t, 0, t) = t(1, 0, 1)$$

Thus the solution space is equal to $\text{span}(\{(1, 0, 1)\})$ and has dimension 1

2. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{7} & 0 & 0 \\ 0 & 1 & \frac{3}{7} & 1 & 0 \end{array}\right).$$

The solution is given by

$$(x_1, x_2, x_3, x_4) = \left(-\frac{2t}{7}, -s - \frac{3t}{7}, t, s\right) = t\left(-\frac{2}{7}, -\frac{3}{7}, 1, 0\right) + s(0, -1, 0, 1)$$

A basis for the solution space is

$$\left\{\left(-\frac{2}{7}, -\frac{3}{7}, 1, 0\right), (0, -1, 0, 1)\right\}$$

and the solution space has dimension 2.

3. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

The system has only the trivial solution. The solution space is the zero vector space and has dimension zero.

4. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

The solution is given by

$$(x_1, x_2, x_3) = (3t - s, t, s) = t(3, 1, 0) + s(-1, 0, 1)$$

A basis for the solution space is

$$\{ (3, 1, 0), (-1, 0, 1) \}$$

and the solution space has dimension 2.

5. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

The solution is given by

$$(x_1, x_2, x_3, x_4) = (4t - 3s + r, t, s, r) = t(4, 1, 0, 0) + s(-3, 0, 1, 0) + r(1, 0, 0, 1)$$

A basis for the solution space is

$$\{ (4, 1, 0, 0), (-3, 0, 1, 0) \}, (1, 0, 0, 1)$$

and the solution space has dimension 3.

6. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

The solution is given by

$$(x_1, x_2, x_3) = (4t, -5t, t) = t(4, -5, 1)$$

A basis for the solution space is

$$\{ (4, -5, 1) \}$$

and the solution space has dimension 1.

Question 11. In each part, find a basis for the given subspace of \mathbb{R}^3 , and state its dimension

1. The plane $3x - 2y + 5z = 0$.
2. The plane $x - y = 0$.

3. The line $x = 2t, y = -t, z = 4t$.
4. All vectors of the form (a, b, c) , where $b = a + c$.

Solution.

1. The points in the plane are of the form

$$(x, y, z) = \left(\frac{2t}{3} - \frac{5s}{3}, t, s \right) = t \left(\frac{2}{3}, 1, 0 \right) + s \left(-\frac{5}{3}, 0, 1 \right)$$

A basis for the plane is:

$$\left\{ \left(\frac{2}{3}, 1, 0 \right), \left(-\frac{5}{3}, 0, 1 \right) \right\}$$

Since the basis consists of two vectors, the dimension is 2.

2. The points in the plane are of the form

$$(x, y, z) = (t, t, s) = t(1, 1, 0) + s(0, 0, 1)$$

A basis for the plane is:

$$\{ (1, 1, 0), (0, 0, 1) \}$$

Since the basis consists of two vectors, the dimension is 2.

3. The points on the line if of the form

$$(x, y, z) = (2t, -t, 4t) = t(2, -1, 4)$$

A basis for the line is:

$$\{ (2, -1, 4) \}$$

The dimension is 1.

4. A vector of the form (a, b, c) where $b = a + c$ can be rewritten as:

$$(a, b, c) = (a, a + c, c) = a(1, 1, 0) + c(0, 1, 1).$$

Thus, a basis is:

$$\{ (1, 1, 0), (0, 1, 1) \}$$

The dimension is 2.

Question 12. In each part, find a basis for the given subspace of \mathbb{R}^4 , and state its dimension.

1. All vectors of the form $(a, b, c, 0)$.
2. All vectors of the form (a, b, c, d) , where $d = a + b$ and $c = a - b$.
- c) All vectors of the form (a, b, c, d) , where $a = b = c = d$.

Solution.

1. We can express any vector in this subspace as

$$(a, b, c, 0) = a(1, 0, 0, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0)$$

Therefore, a basis for this subspace is:

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$$

The dimension of the subspace is 3.

2. We can express any vector in this subspace as

$$(a, b, a-b, a+b) = a(1, 0, 1, 1) + b(0, 1, -1, 1)$$

Therefore, a basis for this subspace is:

$$\{(1, 0, 1, 1), (0, 1, -1, 1)\}$$

The dimension of the subspace is 2.

3. We can express any vector in this subspace as

$$(a, a, a, a) = a(1, 1, 1, 1)$$

Therefore, a basis for this subspace is:

$$\{(1, 1, 1, 1)\}$$

The dimension of the subspace is 1.

Question 13. Show that the matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for $\mathcal{M}_{2 \times 2}$.

Solution. We must show that the matrices are linearly independent and span $\mathcal{M}_{2 \times 2}$. To prove linear independence we must show that the equation

$$\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = O$$

has only the trivial solution. To prove that the matrices span $\mathcal{M}_{2 \times 2}$ we must show that every 2×2 matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can be expressed as

$$\beta_1 M_1 + \beta_2 M_2 + \beta_3 M_3 + \beta_4 M_4 = B.$$

The matrix forms for the two equations are

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\beta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \beta_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \beta_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which can be rewritten as

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since the first equation has only the trivial solution

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

the matrices are linearly independent, and since the second equation has the solution

$$\beta_1 = a, \quad \beta_2 = b, \quad \beta_3 = c, \quad \beta_4 = d$$

the matrices space $\mathcal{M}_{2 \times 2}$. This proves that the matrices M_1, M_2, M_3 and M_4 form a basis for $\mathcal{M}_{2 \times 2}$.

More generally

Definition 1 The mn different matrices whose entries are zero except for a single entry of 1 form a basis for $\mathcal{M}_{m \times n}$ called the standard basis for $\mathcal{M}_{M \times n}$.

Question 14. Find the dimension of each of the following vector spaces.

1. The vector space of all diagonal $n \times n$ matrices.
2. The vector space of all symmetric $n \times n$ matrices.
3. The vector space of all upper triangular $n \times n$ matrices.
4. The vector space of all lower triangular $n \times n$ matrices.

Solution. Let E_{ij} denote the $n \times n$ matrix with a 1 in the (i, i) -entry and 0 elsewhere, for $1 \leq i, j \leq n$.

1. The vector space of all diagonal $n \times n$ matrices.

A diagonal matrix A is of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}E_{11} + a_{22}E_{22} + \cdots + a_{nn}E_{nn}.$$

Therefore, a basis for the vector space of diagonal matrices is given by the set:

$$\{ E_{11}, E_{22}, \dots, E_{nn} \}.$$

The dimension of this vector space is n .

2. A symmetric matrix A satisfies $A^\top = A$, meaning that $a_{ij} = a_{ji}$ for all i, j . It has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{pmatrix} = \sum_{i=1}^n a_{ii}E_{ii} + \sum_{i=1}^{n-1} \sum_{j=i+1}^n a_{ij}(E_{ij} + E_{ji})$$

Therefore, a basis consists of n diagonal matrices E_{ii} and $\frac{n(n-1)}{2}$ symmetric matrices of the form $E_{ij} + E_{ji}$ for $i < j$ which correspond to the $\frac{n(n-1)}{2}$ entries above (or below) the diagonal. The dimension of this vector space is

$$\frac{n(n+1)}{2}.$$

3. An upper triangular matrix A satisfies $a_{ij} = 0$ for all $i > j$, meaning it has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = \sum_{i=1}^n \sum_{j=i}^n a_{ij} E_{ij}$$

Therefore, a basis consists of $\frac{n(n+1)}{2}$ matrices E_{ij} for $i \leq j$, and the dimension of this vector space is

$$\frac{n(n+1)}{2}.$$

4. A lower triangular matrix A satisfies $a_{ij} = 0$ for all $i < j$, meaning it has the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \sum_{i=1}^n \sum_{j=1}^i a_{ij} E_{ij}$$

Therefore, a basis consists of $\frac{n(n+1)}{2}$ matrices E_{ij} for $i \geq j$, and the dimension of this vector space is

$$\frac{n(n+1)}{2}.$$

Question 15. Find a standard basis vector in \mathbb{R}^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for \mathbb{R}^3 .

1. $\mathbf{v}_1 = (-1, 2, 3), \mathbf{v}_2 = (1, 2, -2).$
2. $\mathbf{v}_1 = (1, -1, 0), \mathbf{v}_2 = (3, 1, -2).$

Solution.

1. We observe that both vectors share the same y -coordinate, indicating that they lie in a plane where the standard basis vector \mathbf{e}_2 is linearly independent from them. To confirm that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2\}$ forms a basis, we compute the determinant of the matrix whose columns are these vectors:

$$\begin{vmatrix} -1 & 2 & 3 \\ 1 & 2 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

2. To determine a standard basis vector that can be added to \mathbf{v}_1 and \mathbf{v}_2 to form a basis of \mathbb{R}^3 , we compute the following determinants:

$$\begin{vmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 2, \quad \begin{vmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2, \quad \begin{vmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = 4.$$

Since all of these determinants are nonzero, it follows that any of the standard basis vectors can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to form a basis for \mathbb{R}^3 .

Question 16. Find standard basis vectors for \mathbb{R}^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for \mathbb{R}^4 .

$$\mathbf{v}_1 = (1, -4, 2, -3), \quad \mathbf{v}_2 = (-3, 8, -4, 6)$$

Solution. To determine the necessary standard basis vectors, it suffices to find two standard basis vectors that are not contained in $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$. A vector belongs to $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$ if and only if it can be written as a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . To identify which standard basis vectors satisfy this condition, we construct the following augmented matrix, where the first two columns represent \mathbf{v}_1 and \mathbf{v}_2 , and the last four columns correspond to the four standard basis vectors

$$\left(\begin{array}{cc|cccc} 1 & -3 & 1 & 0 & 0 & 0 \\ -4 & 8 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 \end{array} \right).$$

Performing row reduction, we obtain the reduced row echelon form:

$$\left(\begin{array}{cc|cccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} \end{array} \right)$$

From the reduced matrix, we observe that \mathbf{e}_1 is a linear combination of \mathbf{v}_1 and \mathbf{v}_2 while $\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ do not belong to $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$. This means that any two of $\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for \mathbb{R}^4 .

Question 17. The vectors $\mathbf{v}_1 = (1, -2, 3)$ and $\mathbf{v}_2 = (0, 5, -3)$ are linearly independent. Enlarge the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for \mathbb{R}^3 .

Solution. Following a similar approach to Question 16, we seek a standard basis vector to add to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to form a basis for \mathbb{R}^3 . To determine which standard basis vectors are not contained in $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$, we construct the following augmented matrix, where the first two columns represent \mathbf{v}_1 and \mathbf{v}_2 , and the last three columns correspond to the three standard basis vectors

$$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ -2 & 5 & 0 & 1 & 0 \\ 3 & -3 & 0 & 0 & 1 \end{array} \right)$$

Applying row reduction, we obtain the reduced row echelon form:

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{9} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{9} \end{array} \right)$$

Since the last three columns remain pivot columns, this confirms that any standard basis vector can be added to $\{\mathbf{v}_1, \mathbf{v}_2\}$ to form a basis for \mathbb{R}^3 .

Question 18. The vectors $\mathbf{v}_1 = (1, 0, 0, 0)$ and $\mathbf{v}_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for \mathbb{R}^4 .

Solution. Observing that both \mathbf{v}_1 and \mathbf{v}_2 have zero entries in the third and fourth components, we select the standard basis vectors \mathbf{e}_3 and \mathbf{e}_4 to add to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$. To confirm that the resulting set forms a basis for \mathbb{R}^4 , we verify that the corresponding matrix is invertible by computing its determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

Since the determinant is nonzero, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3, \mathbf{e}_4\}$ forms a basis for \mathbb{R}^4 .

Question 19. Consider the bases $B_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for \mathbb{R}^2 , where

(a)

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

(b)

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

1. Find the transition matrix from B_2 to B_1
2. Find the transition matrix from B_1 to B_2
3. Compute the coordinate vector $[\mathbf{w}]_{B_1}$, where

$$\mathbf{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

and compute $[\mathbf{w}]_{B_2}$ using the transition matrix from B_1 to B_2

4. Compute $[\mathbf{w}]_{B_2}$ directly

Solution.

(a) Firstly, we express \mathbf{v}_1 and \mathbf{v}_2 as linear combinations of \mathbf{u}_1 and \mathbf{u}_2

$$\mathbf{v}_1 = \frac{13}{10}\mathbf{u}_1 - \frac{2}{5}\mathbf{u}_2, \quad \mathbf{v}_2 = -\frac{1}{2}\mathbf{u}_1$$

And \mathbf{u}_1 and \mathbf{u}_2 as linear combinations of \mathbf{v}_1 and \mathbf{v}_2

$$\mathbf{u}_1 = -2\mathbf{v}_2, \quad \mathbf{u}_2 = -\frac{5}{2}\mathbf{v}_1 - \frac{13}{2}\mathbf{v}_2.$$

We have

$$[\mathbf{v}_1]_{B_1} = \begin{pmatrix} \frac{13}{10} \\ 2 \\ -\frac{5}{2} \end{pmatrix}, \quad [\mathbf{v}_2]_{B_1} = \begin{pmatrix} -\frac{1}{2} \\ 2 \\ 0 \end{pmatrix}, \quad [\mathbf{u}_1]_{B_2} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad [\mathbf{u}_2]_{B_2} = \begin{pmatrix} -\frac{5}{2} \\ \frac{13}{2} \end{pmatrix},$$

1.

$$P_{B_2 \rightarrow B_1} = \begin{pmatrix} \frac{13}{10} & -\frac{1}{2} \\ 2 & 0 \\ -\frac{5}{2} & 0 \end{pmatrix}$$

2.

$$P_{B_1 \rightarrow B_2} = \begin{pmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{pmatrix}$$

3.

$$[\mathbf{w}]_{B_1} = \begin{pmatrix} -\frac{17}{10} \\ 8 \\ \frac{5}{5} \end{pmatrix}, \quad [\mathbf{w}]_{B_2} = P_{B_1 \rightarrow B_2}[\mathbf{w}]_{B_1} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$$

4. To find the coordinate vector for \mathbf{w} relative to B_2 , we solve for α_1, α_2 in the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \mathbf{w},$$

This corresponds to the system of equations:

$$\begin{aligned} 2\alpha_1 - 3\alpha_2 &= 3 \\ \alpha_1 + 4\alpha_2 &= -5 \end{aligned}$$

We obtain $\alpha_1 = -4, \alpha_2 = -7$. Thus

$$[\mathbf{w}]_{B_2} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}.$$

(b)

$$P_{B_1 \rightarrow B_2} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \\ -\frac{1}{11} & \frac{1}{11} \end{pmatrix}, \quad P_{B_2 \rightarrow B_1} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}, \quad [\mathbf{w}]_{B_1} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \quad [\mathbf{w}]_{B_2} = \begin{pmatrix} -\frac{3}{11} \\ \frac{13}{11} \\ -\frac{1}{11} \end{pmatrix}$$

$$P_{B_1 \rightarrow B_2}[\mathbf{w}]_{B_1} = [\mathbf{w}]_{B_2}$$

Question 20. Consider the bases $B_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for \mathbb{R}^3 , where

(a)

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

(b)

$$\mathbf{u}_1 = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

1. Find the transition matrix from B_2 to B_1
2. Find the transition matrix from B_1 to B_2
3. Compute the coordinate vector $[\mathbf{w}]_{B_1}$, where

$$\mathbf{w} = \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix}$$

and compute $[\mathbf{w}]_{B_2}$ using the transition matrix from B_1 to B_2

4. Compute $[\mathbf{w}]_{B_2}$ directly

Solution.

(a)

$$P_{B_2 \rightarrow B_1} = \begin{pmatrix} \frac{35}{2} & \frac{19}{2} & -\frac{13}{2} \\ -\frac{19}{2} & -\frac{11}{2} & \frac{7}{2} \\ -13 & -7 & 5 \end{pmatrix}, \quad P_{B_1 \rightarrow B_2} = \begin{pmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{pmatrix},$$

$$[\mathbf{w}]_{B_1} = \begin{pmatrix} 9 \\ -9 \\ -5 \end{pmatrix}, \quad [\mathbf{w}]_{B_2} = \begin{pmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{pmatrix}.$$

(b)

$$P_{B_2 \rightarrow B_1} = \begin{pmatrix} 0 & -\frac{4}{3} & -\frac{17}{6} \\ \frac{3}{2} & \frac{3}{2} & 3 \\ -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}, \quad P_{B_1 \rightarrow B_2} = \begin{pmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix},$$

$$[\mathbf{w}]_{B_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad [\mathbf{w}]_{B_2} = \begin{pmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{pmatrix}.$$

Question 21. Let $B_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the bases for \mathbb{R}^2 , where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

An efficient way to compute the transition matrix $P_{B_1 \rightarrow B_2}$ is as follows

Step 1. Form the matrix $(B_2 \mid B_1)$

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form

Step 3. The resulting matrix will be $(I \mid P_{B_1 \rightarrow B_2})$

Step 4. Extract the matrix $P_{B_1 \rightarrow B_2}$ from the right side of the matrix in Step 3

In diagram

$$(\text{new basis} \mid \text{old basis}) \xrightarrow{\text{row operations}} (I \mid \text{transition from old to new}) \quad (1)$$

1. Apply the above procedure to find the transition matrix $P_{B_2 \rightarrow B_1}$
2. Apply the above procedure to find the transition matrix $P_{B_1 \rightarrow B_2}$
3. Confirm that $P_{B_2 \rightarrow B_1}$ and $P_{B_1 \rightarrow B_2}$ are inverses of one another
4. Let $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find $[\mathbf{w}]_{B_1}$ and then use the matrix $P_{B_1 \rightarrow B_2}$ to compute $[\mathbf{w}]_{B_2}$ from $[\mathbf{w}]_{B_1}$
5. Let $\mathbf{w} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Find $[\mathbf{w}]_{B_2}$ and then use the matrix $P_{B_2 \rightarrow B_1}$ to compute $[\mathbf{w}]_{B_1}$ from $[\mathbf{w}]_{B_2}$

Solution.

1. Following the procedure, we first form the matrix

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array} \right)$$

The reduced row echelon form of the matrix is

$$\left(\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array} \right)$$

Thus the transition matrix

$$P_{B_2 \rightarrow B_1} = \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}$$

2. Following the procedure, we first form the matrix

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array} \right)$$

The reduced row echelon form of the matrix is

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array} \right)$$

Thus the transition matrix

$$P_{B_1 \rightarrow B_2} = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$$

- 3.

$$P_{B_1 \rightarrow B_2} P_{B_2 \rightarrow B_1} = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- 4.

$$[\mathbf{w}]_{B_1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad [\mathbf{w}]_{B_2} = P_{B_1 \rightarrow B_2} [\mathbf{w}]_{B_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

- 5.

$$[\mathbf{w}]_{B_2} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad [\mathbf{w}]_{B_1} = P_{B_2 \rightarrow B_1} [\mathbf{w}]_{B_2} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Question 22. Let S be the standard basis for \mathbb{R}^2 , and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis in which

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

1. Find the transition matrix $P_{B \rightarrow S}$ by inspection
2. Use Formula (1) to find the transition matrix $P_{S \rightarrow B}$
3. Confirm that $P_{B \rightarrow S}$ and $P_{S \rightarrow B}$ are inverses of one another
4. Let $\mathbf{w} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Find $[\mathbf{w}]_B$ and then use the matrix $P_{B \rightarrow S}$ to compute $[\mathbf{w}]_S$ from $[\mathbf{w}]_B$
5. Let $\mathbf{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Find $[\mathbf{w}]_S$ and then use the matrix $P_{S \rightarrow B}$ to compute $[\mathbf{w}]_B$ from $[\mathbf{w}]_S$

Solution.

1. Since the columns of $P_{B \rightarrow S}$ are given by the coordinate vectors of \mathbf{v}_1 and \mathbf{v}_2 relative to the standard basis, which are themselves, we have

$$P_{B \rightarrow S} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

2. We first form the matrix

$$\left(\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array} \right).$$

The reduced row echelon form of the matrix is

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array} \right)$$

Therefore

$$P_{S \rightarrow B} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix}$$

3.

$$P_{S \rightarrow B} P_{B \rightarrow S} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4.

$$[\mathbf{w}]_B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad [\mathbf{w}]_S = P_{B \rightarrow S} [\mathbf{w}]_B = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

5.

$$[\mathbf{w}]_S = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \quad [\mathbf{w}]_B = P_{S \rightarrow B} [\mathbf{w}]_S = \begin{pmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{pmatrix}$$

Question 23. Let S be the standard basis for \mathbb{R}^3 , and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis in which

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}$$

1. Find the transition matrix $P_{B \rightarrow S}$ by inspection

2. Use Formula (1) to find the transition matrix $P_{S \rightarrow B}$

3. Confirm that $P_{B \rightarrow S}$ and $P_{S \rightarrow B}$ are inverses of one another

4. Let $\mathbf{w} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$. Find $[\mathbf{w}]_B$ and then use the matrix $P_{B \rightarrow S}$ to compute $[\mathbf{w}]_S$ from $[\mathbf{w}]_B$

5. Let $\mathbf{w} = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}$. Find $[\mathbf{w}]_S$ and then use the matrix $P_{S \rightarrow B}$ to compute $[\mathbf{w}]_B$ from $[\mathbf{w}]_S$

Solution.

$$1. P_{B \rightarrow S} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

$$2. P_{S \rightarrow B} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

$$3. P_{S \rightarrow B} P_{B \rightarrow S} = I$$

$$4. [\mathbf{w}]_B = \begin{pmatrix} -239 \\ 77 \\ 30 \end{pmatrix}, [\mathbf{w}]_S = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$$

$$5. [\mathbf{w}]_S = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}, [\mathbf{w}]_B = \begin{pmatrix} -200 \\ 64 \\ 25 \end{pmatrix}$$

Question 24. Let $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for \mathbb{R}^2 , and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis that results when the vectors in S are reflected about the line $y = x$.

1. Find the transition matrix $P_{B \rightarrow S}$
2. Show that $P_{B \rightarrow S}^\top = P_{S \rightarrow B}$

Solution.

1. The standard matrix for the reflection about the line $y = x$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where the first column is the image of \mathbf{e}_1 under the reflection operator and the second column is the image of \mathbf{e}_2 under the reflection operator. Thus $\mathbf{v}_1 = (0, 1)$, $\mathbf{v}_2 = (1, 0)$. Then

$$P_{B \rightarrow S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. Since

$$\mathbf{e}_1 = \mathbf{v}_2, \quad \mathbf{e}_2 = \mathbf{v}_1,$$

$$\text{we have } P_{S \rightarrow B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P_{B \rightarrow S}^\top = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = P_{S \rightarrow B}$$

Question 25. Let $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for \mathbb{R}^2 , and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis that results when the vectors in S are reflected about the line that makes an angle θ with the positive x -axis.

1. Find the transition matrix $P_{B \rightarrow S}$
2. Show that $P_{B \rightarrow S}^\top = P_{S \rightarrow B}$

Solution.

1. The standard matrix for the reflection about the line that makes an angle θ with the positive x -axis is given by

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Thus $\mathbf{v}_1 = (\cos 2\theta, \sin 2\theta)$, $\mathbf{v}_2 = (\sin 2\theta, -\cos 2\theta)$, and

$$P_{B \rightarrow S} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

2. To find $P_{S \rightarrow B}$, we first express the standard basis vectors \mathbf{e}_1 and \mathbf{e}_2 in terms of the basis vectors \mathbf{v}_1 and \mathbf{v}_2 . That is, we solve for the scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$ in the equations:

$$\mathbf{e}_1 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2, \quad \mathbf{e}_2 = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2.$$

Using trigonometric identities, we obtain the relationships:

$$\mathbf{e}_1 = \cos 2\theta \mathbf{v}_1 + \sin 2\theta \mathbf{v}_2, \quad \mathbf{e}_2 = \sin 2\theta \mathbf{v}_1 - \cos 2\theta \mathbf{v}_2$$

Thus, the transition matrix from S to B is given by:

$$P_{S \rightarrow B} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Furthermore,

$$P_{B \rightarrow S}^\top = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = P_{S \rightarrow B}.$$

Question 26. Find the domain and codomain of the transformation $T_A(\mathbf{x}) = A\mathbf{x}$

- | | |
|-------------------------------------|-------------------------------------|
| 1. $A \in \mathcal{M}_{3 \times 2}$ | 2. $A \in \mathcal{M}_{2 \times 3}$ |
| 3. $A \in \mathcal{M}_{3 \times 3}$ | 4. $A \in \mathcal{M}_{1 \times 6}$ |
| 5. $A \in \mathcal{M}_{4 \times 5}$ | 6. $A \in \mathcal{M}_{5 \times 4}$ |
| 7. $A \in \mathcal{M}_{4 \times 4}$ | 8. $A \in \mathcal{M}_{3 \times 1}$ |

Solution.

- | | |
|--|--|
| 1. $A\mathbf{x} \in \mathbb{R}^3$, thus $T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ - domain is \mathbb{R}^2 , codomain is \mathbb{R}^3 | |
| 2. $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ | 3. $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ |
| 4. $T_A : \mathbb{R}^6 \rightarrow \mathbb{R}$ | 5. $T_A : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ |
| 6. $T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^5$ | 7. $T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ |
| 8. $T_A : \mathbb{R} \rightarrow \mathbb{R}^3$ | |

Question 27. Find the domain and codomain of the transformation defined by the equations

1.

$$\begin{aligned}w_1 &= 4x_1 + 5x_2 \\w_2 &= x_1 - 8x_2\end{aligned}$$

2.

$$\begin{aligned}w_1 &= 5x_1 - 7x_2 \\w_2 &= 6x_1 + x_2 \\w_3 &= 2x_1 + 3x_2\end{aligned}$$

3.

$$\begin{aligned}w_1 &= x_1 - 4x_2 + 8x_3 \\w_2 &= -x_1 + 4x_2 + 2x_3 \\w_3 &= -3x_1 + 2x_2 - 5x_3\end{aligned}$$

4.

$$\begin{aligned}w_1 &= 2x_1 + 7x_2 - 4x_3 \\w_2 &= 4x_1 - 3x_2 + 2x_3\end{aligned}$$

*Solution.*1. Domain: \mathbb{R}^2 . Codomain: \mathbb{R}^2 2. Domain: \mathbb{R}^2 . Codomain: \mathbb{R}^3 3. Domain: \mathbb{R}^3 . Codomain: \mathbb{R}^3 4. Domain: \mathbb{R}^3 . Codomain: \mathbb{R}^2 **Question 28.** Find the standard matrix for the transformation defined below

1.

$$\begin{aligned}w_1 &= 2x_1 - 3x_2 + x_3 \\w_2 &= 3x_1 + 5x_2 - x_3\end{aligned}$$

2.

$$\begin{aligned}w_1 &= 7x_1 + 2x_2 - 8x_3 \\w_2 &= -x_2 + 5x_3 \\w_3 &= 4x_1 + 7x_2 - x_3\end{aligned}$$

$$3. \quad T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix}$$

$$4. \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7x_1 + 2x_2 - x_3 + x_4 \\ x_2 + x_3 \\ -x_1 \end{pmatrix}$$

$$5. \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$6. \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{pmatrix}$$

Solution.

$$1. \begin{pmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{pmatrix}$$

$$3. \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix}$$

$$5. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$2. \begin{pmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{pmatrix}$$

$$4. \begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$6. \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

Question 29. Find $T_A(\mathbf{x})$.

$$1. A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$3. A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$4. A = \begin{pmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution.

$$1. \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$2. \begin{pmatrix} 3 \\ 13 \end{pmatrix}$$

$$3. \begin{pmatrix} -2x_1 + x_2 + 4x_3 \\ 3x_1 + 5x_2 + 7x_3 \\ 6x_1 - x_3 \end{pmatrix}$$

$$4. \begin{pmatrix} -x_1 + x_2 \\ 2x_1 + 4x_2 \\ 7x_1 + 8x_2 \end{pmatrix}$$

Question 30. The images of the standard basis vectors for \mathbb{R}^3 are given for a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

$$1. T(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, T(\mathbf{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, T(\mathbf{e}_3) = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$2. T(\mathbf{e}_1) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, T(\mathbf{e}_2) = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}, T(\mathbf{e}_3) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Solution.

$$1. [T] = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 0 & -3 \\ 0 & 1 & -1 \end{pmatrix}, T(\mathbf{x}) = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$$

$$2. [T] = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 3 & 0 & 2 \end{pmatrix}, T(\mathbf{x}) = \begin{pmatrix} 1 \\ 1 \\ 11 \end{pmatrix}$$

Question 31. Use matrix multiplication to find the reflection of $(-1, 2)$ about the

1. x -axis
2. y -axis
3. line $y = x$

Solution.

$$1. \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

$$2. \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$3. \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Question 32. Use matrix multiplication to find the reflection of (a, b) about the

1. x -axis
2. y -axis
3. line $y = x$

Solution.

$$1. \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}.$$

$$2. \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}$$

$$3. \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

Question 33. Use matrix multiplication to find the reflection of $(2, -5, 3)$ about the

1. xy -plane
2. xz -plane
3. yz -plane

Solution.

$$1. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix}$$

$$2. \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$$

$$3. \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 3 \end{pmatrix}$$

Question 34. Use matrix multiplication to find the reflection of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ about the

1. xy -plane

2. xz -plane

3. yz -plane

Solution.

$$1. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ -c \end{pmatrix}$$

$$2. \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ -b \\ c \end{pmatrix}$$

$$3. \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ b \\ c \end{pmatrix}$$

Question 35. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ onto the

1. x -axis

2. y -axis

Solution.

$$1. \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$2. \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

Question 36. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} a \\ b \end{pmatrix}$ onto the

1. x -axis

2. y -axis

Solution.

$$1. \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$2. \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Question 37. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ onto the

1. xy -plane

2. xz -plane

3. yz -plane

Solution.

$$1. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$2. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

$$3. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Question 38. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ onto the

1. xy -plane

2. xz -plane

3. yz -plane

Solution.

$$1. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$2. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ c \end{pmatrix}$$

$$3. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$$

Question 39. Use matrix multiplication to find the image of the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ when it is rotated about the origin through an angle of

1. $\theta = 30^\circ$

2. $\theta = -60^\circ$

3. $\theta = 45^\circ$

4. $\theta = 90^\circ$

Solution. Recall

$$R_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$1. \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{4+3\sqrt{3}}{2} \\ \frac{3-4\sqrt{3}}{2} \end{pmatrix} \approx \begin{pmatrix} 4.598 \\ -1.964 \end{pmatrix}$$

$$2. \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{3-4\sqrt{3}}{2} \\ -\frac{3\sqrt{3}+4}{2} \end{pmatrix} \approx \begin{pmatrix} -1.964 \\ -4.598 \end{pmatrix}$$

$$3. \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} \approx \begin{pmatrix} 4.950 \\ -0.7071 \end{pmatrix}$$

$$4. \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Question 40. Use matrix multiplication to find the image of the nonzero vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ when it is rotated about the origin through

1. a positive angle θ
2. a negative angle $-\theta$

Solution.

$$1. \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \cos \theta - v_2 \sin \theta \\ v_1 \sin \theta + v_2 \cos \theta \end{pmatrix}$$

$$2. \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \cos \theta + v_2 \sin \theta \\ -v_1 \sin \theta + v_2 \cos \theta \end{pmatrix}$$

Question 41. Use matrix multiplication to find the image of the vector $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ if it is rotated

1. 30° clockwise about the positive x -axis.
2. 30° counterclockwise about the positive y -axis.
3. 45° clockwise about the positive y -axis.
4. 90° counterclockwise about the positive z -axis.

Solution.

1. 30° clockwise corresponds to -30°

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right) \\ 0 & \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 - \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 2 \\ 0.1340 \\ 2.2321 \end{pmatrix}$$

- 2.

$$\begin{pmatrix} \cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{6} & 0 & \cos\frac{\pi}{6} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + \sqrt{3} \\ -1 \\ -1 + \sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 2.7321 \\ -1 \\ 0.7321 \end{pmatrix}$$

3. 45° clockwise corresponds to -45°

$$\begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & 0 & \sin\left(-\frac{\pi}{4}\right) \\ 0 & 1 & 0 \\ -\sin\left(-\frac{\pi}{4}\right) & 0 & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2\sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 0 \\ -1 \\ 2.8284 \end{pmatrix}$$

- 4.

$$\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Question 42. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with factor $\alpha = \frac{1}{2}$.
2. The dilation of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with factor $\alpha = 3$.

Solution.

1. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$
2. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

Question 43. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} a \\ b \end{pmatrix}$ with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
2. The dilation of $\begin{pmatrix} a \\ b \end{pmatrix}$ with factor α , where $\alpha > 1$.

Solution.

$$\begin{aligned}
 1. \quad & \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{\alpha} \\ \frac{b}{\alpha} \end{pmatrix} \\
 2. \quad & \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}
 \end{aligned}$$

Question 44. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with factor $\frac{1}{4}$.
2. The dilation of $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with factor 2.

Solution.

$$\begin{aligned}
 1. \quad & \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix} \\
 2. \quad & \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}
 \end{aligned}$$

Question 45. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
2. The dilation of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with factor α , where $\alpha > 1$.

Solution.

$$1. \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a}{\alpha} \\ -\frac{b}{\alpha} \\ \frac{c}{\alpha} \end{pmatrix}$$

$$2. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \\ \alpha c \end{pmatrix}$$

Question 46. Use matrix multiplication to find:

1. The compression of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the x -direction with factor $\frac{1}{2}$.
2. The compression of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the y -direction with factor $\frac{1}{2}$.

Solution.

$$1. \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$$

$$2. \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Question 47. Use matrix multiplication to find:

1. The expansion of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the x -direction with factor 3.
2. The expansion of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the y -direction with factor 3.

Solution.

$$1. \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$2. \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

Question 48. Use matrix multiplication to find:

1. The compression of $\begin{pmatrix} a \\ b \end{pmatrix}$ in the x -direction with factor $\frac{1}{\alpha}$, where $\alpha > 1$.

2. The expansion of $\begin{pmatrix} a \\ b \end{pmatrix}$ in the y -direction with factor α , where $\alpha > 1$.

Solution.

1. $\begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{\alpha} \\ b \end{pmatrix}$
2. $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ \alpha b \end{pmatrix}$

Question 49. In each part, determine whether the operators T_1 and T_2 commute, i.e. whether $T_1 \circ T_2 = T_2 \circ T_1$.

1. $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the reflection about the line $y = x$, and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the orthogonal projection onto the x -axis.
2. $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the reflection about the x -axis, and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the reflection about the line $y = x$.
3. $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the orthogonal projection onto the x -axis, and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the orthogonal projection onto the y -axis.
4. $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the rotation about the origin through an angle of $\frac{\pi}{4}$, and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the reflection about the y -axis.
5. $T_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a dilation with factor α , and $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a contraction with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
6. $T_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the rotation about the x -axis through an angle θ_1 , and $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the rotation about the z -axis through an angle θ_2 .

Solution.

1. $[T_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, [T_2] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

T_1 and T_2 do not commute.

2. $[T_1] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, [T_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

T_1 and T_2 do not commute.

$$3. [T_1] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, [T_2] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

T_1 and T_2 commute.

$$4. [T_1] = \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, [T_2] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

T_1 and T_2 do not commute.

$$5. [T_1] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, [T_2] = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix}$$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

T_1 and T_2 commute.

$$6. [T_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{pmatrix}, [T_2] = \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 \cos \theta_1 & \cos \theta_1 \cos \theta_2 & -\sin \theta_1 \\ \sin \theta_1 \sin \theta_2 & \sin \theta_1 \cos \theta_2 & \cos \theta_1 \end{pmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 \cos \theta_1 & \sin \theta_1 \sin \theta_2 \\ \sin \theta_2 & \cos \theta_1 \cos \theta_2 & -\sin \theta_1 \cos \theta_2 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{pmatrix}$$

T_1 and T_2 do not commute.

Question 50. Find the standard matrix for the stated composition in \mathbb{R}^2 .

1. A rotation of 90° , followed by a reflection about the line $y = x$.
2. An orthogonal projection onto the y -axis, followed by a contraction with factor $\frac{1}{2}$.
3. A reflection about the x -axis, followed by a dilation with factor 3, followed by a rotation about the origin of 60° .

4. A rotation about the origin of 60° , followed by an orthogonal projection onto the x -axis, followed by a reflection about the line $y = x$.
5. A dilation with factor 2, followed by a rotation about the origin of 45° , followed by a reflection about the y -axis.
6. A rotation about the origin of 15° , followed by a rotation about the origin of 105° , followed by a rotation about the origin of 60° .

Solution.

1.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2.

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

3.

$$\begin{aligned} \begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3\sqrt{3}}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix} \end{aligned}$$

4.

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \end{aligned}$$

5.

$$\begin{aligned} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix} \end{aligned}$$

6. The total rotated angle is

$$15^\circ + 105^\circ + 60^\circ = 180^\circ.$$

Hence the transition matrix is given by

$$\begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Question 51. Find the standard matrix for the stated composition in \mathbb{R}^3 .

1. A reflection about the yz -plane, followed by an orthogonal projection onto the xz -plane.
2. A rotation of 45° about the y -axis, followed by a dilation with factor $\sqrt{2}$.
3. An orthogonal projection onto the xy -plane, followed by a reflection about the yz -plane.
4. A rotation of 30° about the x -axis, followed by a rotation of 30° about the z -axis, followed by a contraction with factor $\frac{1}{4}$.
5. A reflection about the xy -plane, followed by a reflection about the xz -plane, followed by an orthogonal projection onto the yz -plane.
6. A rotation of 270° about the x -axis, followed by a rotation of 90° about the y -axis, followed by a rotation of 180° about the z -axis.

Solution.

1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.

$$\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{4} & 0 & \sin \frac{\pi}{4} \\ 0 & 1 & 0 \\ -\sin \frac{\pi}{4} & 0 & \cos \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

3.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4.

$$\begin{aligned}
\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{\pi}{6} & -\sin \frac{\pi}{6} \\ 0 & \sin \frac{\pi}{6} & \cos \frac{\pi}{6} \end{pmatrix} &= \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{8} & -\frac{1}{8} \\ 0 & \frac{1}{8} & \frac{\sqrt{3}}{8} \end{pmatrix}
\end{aligned}$$

5.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

6.

$$\begin{aligned}
&\begin{pmatrix} \cos \pi & -\sin \pi & 0 \\ \sin \pi & \cos \pi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{2} & 0 & \sin \frac{\pi}{2} \\ 0 & 1 & 0 \\ -\sin \frac{\pi}{2} & 0 & \cos \frac{\pi}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{3\pi}{2} & -\sin \frac{3\pi}{2} \\ 0 & \sin \frac{3\pi}{2} & \cos \frac{3\pi}{2} \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Question 52. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector in \mathbb{R}^2 . Consider the linear transformations $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T_1(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}, \quad T_2(\mathbf{x}) = \begin{pmatrix} 3x_1 \\ 2x_1 + 4x_2 \end{pmatrix}$$

1. Find the standard matrices for T_1 and T_2 .
2. Find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$.
3. Find the standard matrices for $T_1 \circ T_2 \circ T_1$ and $T_1 \circ T_2 \circ T_2$.
4. Use the matrices obtained in part 2 to find formulas for $T_1(T_2(\mathbf{x}))$ and $T_2(T_1(\mathbf{x}))$

Solution.

$$1. [T_1] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, [T_2] = \begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix}$$

2.

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 5 & 4 \\ 1 & -4 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 3 & 3 \\ 6 & -2 \end{pmatrix}$$

3.

$$[T_1 \circ T_2 \circ T_1] = [T_1][T_2][T_1] = \begin{pmatrix} 9 & 1 \\ -3 & 5 \end{pmatrix}, \quad [T_1 \circ T_2 \circ T_2] = [T_1][T_2][T_2] = \begin{pmatrix} 23 & 16 \\ -5 & -16 \end{pmatrix}$$

4.

$$T_1(T_2(\mathbf{x})) = \begin{pmatrix} 5x_1 + 4x_2 \\ x_1 - 4x_2 \end{pmatrix}, \quad T_2(T_1(\mathbf{x})) = \begin{pmatrix} 3x_1 + 3x_2 \\ 6x_1 - 2x_2 \end{pmatrix}$$

Question 53. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be a vector in \mathbb{R}^3 . Consider the linear transformations $T_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T_1(\mathbf{x}) = \begin{pmatrix} 4x_1 \\ -2x_1 + x_2 \\ -x_1 - 3x_3 \end{pmatrix}, \quad T_2(\mathbf{x}) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_3 \\ 4x_1 - x_3 \end{pmatrix}$$

1. Find the standard matrices for T_1 and T_2 .
2. Find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$.
3. Find the standard matrices for $T_1 \circ T_2 \circ T_1$ and $T_1 \circ T_2 \circ T_2$.
4. Use the matrices obtained in part 2 to find formulas for $T_1(T_2(\mathbf{x}))$ and $T_2(T_1(\mathbf{x}))$

Solution.

$$1. [T_1] = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & -3 \end{pmatrix}, \quad [T_2] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

2.

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 4 & 8 & 0 \\ -2 & -4 & 2 \\ -13 & -2 & 3 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & -6 \\ 17 & 0 & 3 \end{pmatrix}$$

3.

$$[T_1 \circ T_2 \circ T_1] = [T_1][T_2][T_1] = \begin{pmatrix} 0 & 8 & 0 \\ -2 & -4 & -6 \\ -51 & -2 & -9 \end{pmatrix}, \quad [T_1 \circ T_2 \circ T_2] = [T_1][T_2][T_2] = \begin{pmatrix} 4 & 8 & 16 \\ 6 & -4 & -10 \\ -1 & -26 & -7 \end{pmatrix}$$

4.

$$T_1(T_2(\mathbf{x})) = \begin{pmatrix} 4x_1 + 8x_2 \\ -2x_1 - 4x_2 + 2x_3 \\ -13x_1 - 2x_2 + 3x_3 \end{pmatrix}, \quad T_2(T_1(\mathbf{x})) = \begin{pmatrix} 2x_2 \\ -2x_1 - 6x_3 \\ 17x_1 + 3x_3 \end{pmatrix}$$