# Tutorial 6

## Bases and matrix operators

**Homework**

1. Provide detailed solutions to Q2, part 2.
2. Provide a detailed solution to Q6.
3. Provide a detailed solution to Q7, part 2.
4. Provide a detailed solution to Q9, part 2.
5. Provide a detailed solution to *one* selected part of Q10.
6. Provide a detailed solution to Q19, part (b).
7. Provide a detailed solution to Q20, choosing either part (a) or part (b).
8. Provide a detailed solution to Q23.
9. Provide a detailed solution to Q28.

**Question 1.** Show that the following set of vectors forms a basis for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

1. $\{ \begin{pmatrix}2, & 1\end{pmatrix}, \begin{pmatrix}3, & 0\end{pmatrix} \}$
2. $\{ \begin{pmatrix}3, & 1, & -4\end{pmatrix}, \begin{pmatrix}2, & 5, & 6\end{pmatrix} \}, \begin{pmatrix}1, & 4, & 8\end{pmatrix}$

**Question 2.** Show that the following matrices form a basis for $\mathcal{M}_{2\times 2}$.

1. $\begin{pmatrix}3 & 6\\ 3 & -6\end{pmatrix}, \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & -8\\ -12 & -4\end{pmatrix}, \begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$
2. $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

**Question 3.** In each part, show that the set of vectors is not a basis for $\mathbb{R}^3$

1. $\{ \begin{pmatrix}2, & -3, & 1\end{pmatrix}, \begin{pmatrix}4, & 1, & 1\end{pmatrix}, \begin{pmatrix}0, & -7, & 1\end{pmatrix} \}$
2. $\{ \begin{pmatrix}1, & 6, & 4\end{pmatrix}, \begin{pmatrix}2, & 4, & -1\end{pmatrix}, \begin{pmatrix}-1, & 2, & 5\end{pmatrix} \}$

**Question 4.** Show that the following matrices do not form a basis for $\mathcal{M}_{2\times 2}$.

$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad \begin{pmatrix}2 & -2\\ 3 & 2\end{pmatrix}, \quad \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$$

**Question 5.** Find the coordinate vector of $\boldsymbol{w}$ relative to the basis $S = \{\, \boldsymbol{u}_1,\ \boldsymbol{u}_2 \,\}$ for $\mathbb{R}^2$.

1. $\boldsymbol{u}_1 = \begin{pmatrix} 2, & -4 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 3, & 8 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 1, & 1 \end{pmatrix}$
2. $\boldsymbol{u}_1 = \begin{pmatrix} 1, & 1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 0, & 2 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} a, & b \end{pmatrix}$
3. $\boldsymbol{u}_1 = \begin{pmatrix} 1, & -1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1, & 1 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 1, & 0 \end{pmatrix}$
4. $\boldsymbol{u}_1 = \begin{pmatrix} 1, & -1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1, & 1 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 0, & 1 \end{pmatrix}$

**Question 6.** Find the coordinate vector of $\boldsymbol{u}$ relative to the basis $S = \{\, \boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3 \,\}$ for $\mathbb{R}^3$

1. $\boldsymbol{v}_1 = \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 2, & 2, & 0 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 3, & 3, & 3 \end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix} 2, & -1, & 3 \end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix} 1, & 2, & 3 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} -4, & 5, & 6 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 7, & -8, & 9 \end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix} 5, & -12, & 3 \end{pmatrix}$

**Question 7.** For each case, first show that the set $S = \{\, A_1, A_2, A_3, A_4 \,\}$ is a basis for $\mathcal{M}_{2\times 2}$, then express $A$ as a linear combination of the matrices in $S$, and then find the coordinate vector of $A$ relative to $S$.

1. $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$
2. $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

**Question 8.** In each part, let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by $A$, and let $\{\, \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3 \,\}$ be the standard basis for $\mathbb{R}^3$. Determine whether the set $\{\, T_A(\boldsymbol{e}_1),\ T_A(\boldsymbol{e}_2),\ T_A(\boldsymbol{e}_3) \,\}$ is linearly independent in $\mathbb{R}^3$

1. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{pmatrix}$
2. $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}$

**Question 9.** In each part, let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by $A$, and let $\boldsymbol{u} = \begin{pmatrix} 1, & -2, & -1 \end{pmatrix}$. Find the coordinate vector of $T_A(\boldsymbol{u})$ relative to the basis $S = \left\{ \begin{pmatrix} 1, & 1, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 1, & 1 \end{pmatrix}, \begin{pmatrix} 1, & 1, & 1 \end{pmatrix} \right\}$ for $\mathbb{R}^3$.

1. $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

**Question 10.** Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

1.
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

2.
$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

3.
$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 5x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

4.
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

5.
$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

6.
$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

**Question 11.** In each part, find a basis for the given subspace of $\mathbb{R}^3$, and sate its dimension

1. The plane $3x - 2y + 5z = 0$.
2. The plane $x - y = 0$.
3. The line $x = 2t$, $y = -t$, $z = 4t$.
4. All vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a + c$.

**Question 12.** In each part, find a basis for the given subspace of $\mathbb{R}^4$, and state its dimension.

1. All vectors of the form $\begin{pmatrix} a, & b, & c, & 0 \end{pmatrix}$.
2. All vectors of the form $\begin{pmatrix} a, & b, & c, & d \end{pmatrix}$, where $d = a + b$ and $c = a - b$.

c) All vectors of the form $\begin{pmatrix} a, & b, & c, & d \end{pmatrix}$, where $a = b = c = d$.

**Question 13.** Show that the matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for $\mathcal{M}_{2\times 2}$.

**Question 14.** Find the dimension of each of the following vector spaces.

1. The vector space of all diagonal $n \times n$ matrices.
2. The vector space of all symmetric $n \times n$ matrices.
3. The vector space of all upper triangular $n \times n$ matrices.
4. The vector space of all lower triangular $n \times n$ matrices.

**Question 15.** Find a standard basis vector in $\mathbb{R}^3$ that can be added to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to produce a basis for $\mathbb{R}^3$.

1. $\boldsymbol{v}_1 = \begin{pmatrix} -1, & 2, & 3 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 1, & 2, & -2 \end{pmatrix}$.
2. $\boldsymbol{v}_1 = \begin{pmatrix} 1, & -1, & 0 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 3, & 1, & -2 \end{pmatrix}$.

**Question 16.** Find standard basis vectors for $\mathbb{R}^4$ that can be added to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to produce a basis for $\mathbb{R}^4$.

$$v_1 = \begin{pmatrix} 1, & -4, & 2, & -3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3, & 8, & -4, & 6 \end{pmatrix}$$

**Question 17.** The vectors $\boldsymbol{v}_1 = \begin{pmatrix} 1, & -2, & 3 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} 0, & 5, & -3 \end{pmatrix}$ are linearly independent. Enlarge the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to a basis for $\mathbb{R}^3$.

**Question 18.** The vectors $\boldsymbol{v}_1 = \begin{pmatrix} 1, & 0, & 0, & 0 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} 1, & 1, & 0, & 0 \end{pmatrix}$ are linearly independent. Enlarge the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to a basis for $\mathbb{R}^4$.

**Question 19.** Consider the bases $B_1 = \{\boldsymbol{u}_1, \ \boldsymbol{u}_2\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ for $\mathbb{R}^2$, where

(a)

$$\boldsymbol{u}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

(b)

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

1. Find the transition matrix from $B_2$ to $B_1$
2. Find the transition matrix from $B_1$ to $B_2$
3. Compute the coordinate vector $[\boldsymbol{w}]_{B_1}$, where

$$\boldsymbol{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

   and compute $[\boldsymbol{w}]_{B_2}$ using the transition matrix from $B_1$ to $B_2$
4. Compute $[\boldsymbol{w}]_{B_2}$ directly

**Question 20.** Consider the bases $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for $\mathbb{R}^3$, where

(a)

$$\boldsymbol{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

(b)

$$\boldsymbol{u}_1 = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

1. Find the transition matrix from $B_2$ to $B_1$
2. Find the transition matrix from $B_1$ to $B_2$
3. Compute the coordinate vector $[\boldsymbol{w}]_{B_1}$, where

$$\boldsymbol{w} = \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix}$$

   and compute $[\boldsymbol{w}]_{B_2}$ using the transition matrix from $B_1$ to $B_2$
4. Compute $[\boldsymbol{w}]_{B_2}$ directly

**Question 21.** Let $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the bases for $\mathbb{R}^2$, where

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

An efficient way to compute the transition matrix $P_{B_1 \to B_2}$ is as follows

**Step 1.** Form the matrix $\left(B_2 \mid B_1\right)$

**Step 2.** Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form

**Step 3.** The resulting matrix will be $\left(I \mid P_{B_1 \to B_2}\right)$

**Step 4.** Extract the matrix $P_{B_1 \to B_2}$ from the right side of the matrix in Step 3

In diagram

$$\left(\text{new basis} \mid \text{old basis}\right) \xrightarrow{\text{row operations}} \left(I \mid \text{transition from old to new}\right) \tag{1}$$

1. Apply the above procedure to find the transition matrix $P_{B_2 \to B_1}$
2. Apply the above procedure to find the transition matrix $P_{B_1 \to B_2}$
3. Confirm that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another
4. Let $\boldsymbol{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find $[\boldsymbol{w}]_{B_1}$ and then use the matrix $P_{B_1 \to B_2}$ to compute $[\boldsymbol{w}]_{B_2}$ from $[\boldsymbol{w}]_{B_1}$
5. Let $\boldsymbol{w} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Find $[\boldsymbol{w}]_{B_2}$ and then use the matrix $P_{B_2 \to B_1}$ to compute $[\boldsymbol{w}]_{B_1}$ from $[\boldsymbol{w}]_{B_2}$

**Question 22.** Let $S$ be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis in which

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

1. Find the transition matrix $P_{B \to S}$ by inspection
2. Use Formula (1) to find the transition matrix $P_{S \to B}$
3. Confirm that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another
4. Let $\boldsymbol{w} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Find $[\boldsymbol{w}]_B$ and then use the matrix $P_{B \to S}$ to compute $[\boldsymbol{w}]_S$ from $[\boldsymbol{w}]_B$
5. Let $\boldsymbol{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Find $[\boldsymbol{w}]_S$ and then use the matrix $P_{S \to B}$ to compute $[\boldsymbol{w}]_B$ from $[\boldsymbol{w}]_S$

**Question 23.** Let $S$ be the standard basis for $\mathbb{R}^3$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be the basis in which

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}$$

1. Find the transition matrix $P_{B \to S}$ by inspection
2. Use Formula (1) to find the transition matrix $P_{S \to B}$
3. Confirm that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another

4. Let $\boldsymbol{w} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$. Find $[\boldsymbol{w}]_B$ and then use the matrix $P_{B\to S}$ to compute $[\boldsymbol{w}]_S$ from $[\boldsymbol{w}]_B$

5. Let $\boldsymbol{w} = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}$. Find $[\boldsymbol{w}]_S$ and then use the matrix $P_{S\to B}$ to compute $[\boldsymbol{w}]_B$ from $[\boldsymbol{w}]_S$

**Question 24.** Let $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis that results when the vectors in $S$ are reflected about the line $y = x$.

1. Find the transition matrix $P_{B\to S}$
2. Show that $P_{B\to S}^\top = P_{S\to B}$

**Question 25.** Let $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis that results when the vectors in $S$ are reflected about the line that makes an angle $\theta$ with the positive $x-$axis.

1. Find the transition matrix $P_{B\to S}$
2. Show that $P_{B\to S}^\top = P_{S\to B}$

**Question 26.** Find the domain and codomain of the transformation $T_A(\boldsymbol{x}) = A\boldsymbol{x}$

1. $A \in \mathcal{M}_{3\times 2}$
2. $A \in \mathcal{M}_{2\times 3}$
3. $A \in \mathcal{M}_{3\times 3}$
4. $A \in \mathcal{M}_{1\times 6}$
5. $A \in \mathcal{M}_{4\times 5}$
6. $A \in \mathcal{M}_{5\times 4}$
7. $A \in \mathcal{M}_{4\times 4}$
8. $A \in \mathcal{M}_{3\times 1}$

**Question 27.** Find the domain and codomain of the transformation defined by the equations

1.
$$\begin{aligned} w_1 &= 4x_1 + 5x_2 \\ w_2 &= x_1 - 8x_2 \end{aligned}$$

2.
$$\begin{aligned} w_1 &= 5x_1 - 7x_2 \\ w_2 &= 6x_1 + x_2 \\ w_3 &= 2x_1 + 3x_2 \end{aligned}$$

3.
$$\begin{aligned} w_1 &= x_1 - 4x_2 + 8x_3 \\ w_2 &= -x_1 + 4x_2 + 2x_3 \\ w_3 &= -3x_1 + 2x_2 - 5x_3 \end{aligned}$$

4.
$$\begin{aligned} w_1 &= 2x_1 + 7x_2 - 4x_3 \\ w_2 &= 4x_1 - 3x_2 + 2x_3 \end{aligned}$$

**Question 28.** Find the standard matrix for the transformation defined below

1.
$$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 \\ w_2 &= 3x_1 + 5x_2 - x_3 \end{aligned}$$

2.
$$\begin{aligned} w_1 &= 7x_1 + 2x_2 - 8x_3 \\ w_2 &= -x_2 + 5x_3 \\ w_3 &= 4x_1 + 7x_2 - x_3 \end{aligned}$$

3. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix}$

4. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7x_1 + 2x_2 - x_3 + x_4 \\ x_2 + x_3 \\ -x_1 \end{pmatrix}$

5. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

6. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{pmatrix}$

**Question 29.** Find $T_A(\boldsymbol{x})$.

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

2. $A = \begin{pmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

3. $A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

4. $A = \begin{pmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

**Question 30.** The images of the standard basis vectors for $\mathbb{R}^3$ are given for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Find the standard matrix for the transformation, and find $T(\boldsymbol{x})$.

1. $T(\boldsymbol{e}_1) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $T(\boldsymbol{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $T(\boldsymbol{e}_3) = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

2. $T(\boldsymbol{e}_1) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $T(\boldsymbol{e}_2) = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$, $T(\boldsymbol{e}_3) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

**Question 31.** Use matrix multiplication to find the reflection of $\begin{pmatrix} -1, & 2 \end{pmatrix}$ about the

1. $x-$axis
2. $y-$axis
3. line $y = x$

**Question 32.** Use matrix multiplication to find the reflection of $\begin{pmatrix} a, & b \end{pmatrix}$ about the

1. $x-$axis
2. $y-$axis
3. line $y = x$

**Question 33.** Use matrix multiplication to find the reflection of $\begin{pmatrix} 2, & -5, & 3 \end{pmatrix}$ about the

1. $xy-$plane
2. $xz-$plane
3. $yz-$plane

**Question 34.** Use matrix multiplication to find the reflection of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ about the

1. $xy$-plane
2. $xz$-plane
3. $yz$-plane

**Question 35.** Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ onto the

1. $x$-axis
2. $y$-axis

**Question 36.** Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} a \\ b \end{pmatrix}$ onto the

1. $x$-axis
2. $y$-axis

**Question 37.** Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ onto the

1. $xy$-plane
2. $xz$-plane
3. $yz$-plane

**Question 38.** Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ onto the

1. $xy$-plane
2. $xz$-plane
3. $yz$-plane

**Question 39.** Use matrix multiplication to find the image of the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ when it is rotated about the origin through an angle of

1. $\theta = 30°$
2. $\theta = -60°$
3. $\theta = 45°$
4. $\theta = 90°$

**Question 40.** Use matrix multiplication to find the image of the nonzero vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ when it is rotated about the origin through

1. a positive angle $\theta$
2. a negative angle $-\theta$

**Question 41.** Use matrix multiplication to find the image of the vector $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ if it is rotated

1. $30°$ clockwise about the positive $x$-axis.
2. $30°$ counterclockwise about the positive $y$-axis.
3. $45°$ clockwise about the positive $y$-axis.
4. $90°$ counterclockwise about the positive $z$-axis.

**Question 42.** Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with factor $\alpha = \frac{1}{2}$.
2. The dilation of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with factor $\alpha = 3$.

**Question 43.** Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} a \\ b \end{pmatrix}$ with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
2. The dilation of $\begin{pmatrix} a \\ b \end{pmatrix}$ with factor $\alpha$, where $\alpha > 1$.

**Question 44.** Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with factor $\frac{1}{4}$.
2. The dilation of $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with factor 2.

**Question 45.** Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
2. The dilation of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with factor $\alpha$, where $\alpha > 1$.

**Question 46.** Use matrix multiplication to find:

1. The compression of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the $x$-direction with factor $\frac{1}{2}$.
2. The compression of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the $y$-direction with factor $\frac{1}{2}$.

**Question 47.** Use matrix multiplication to find:

1. The expansion of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the $x$-direction with factor 3.
2. The expansion of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the $y$-direction with factor 3.

**Question 48.** Use matrix multiplication to find:

1. The compression of $\begin{pmatrix} a \\ b \end{pmatrix}$ in the $x$-direction with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
2. The expansion of $\begin{pmatrix} a \\ b \end{pmatrix}$ in the $y$-direction with factor $\alpha$, where $\alpha > 1$.

**Question 49.** In each part, determine whether the operators $T_1$ and $T_2$ commute, i.e. whether $T_1 \circ T_2 = T_2 \circ T_1$.

1. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the line $y = x$, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the $x$-axis.
2. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the $x$-axis, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the line $y = x$.
3. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the $x$-axis, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the $y$-axis.
4. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation about the origin through an angle of $\frac{\pi}{4}$, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the $y$-axis.
5. $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is a dilation with factor $\alpha$, and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is a contraction with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
6. $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation about the $x$-axis through an angle $\theta_1$, and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation about the $z$-axis through an angle $\theta_2$.

**Question 50.** Find the standard matrix for the stated composition in $\mathbb{R}^2$.

1. A rotation of $90°$, followed by a reflection about the line $y = x$.
2. An orthogonal projection onto the $y$-axis, followed by a contraction with factor $\frac{1}{2}$.
3. A reflection about the $x$-axis, followed by a dilation with factor 3, followed by a rotation about the origin of $60°$.
4. A rotation about the origin of $60°$, followed by an orthogonal projection onto the $x$-axis, followed by a reflection about the line $y = x$.

5. A dilation with factor 2, followed by a rotation about the origin of $45°$, followed by a reflection about the $y$-axis.

6. A rotation about the origin of $15°$, followed by a rotation about the origin of $105°$, followed by a rotation about the origin of $60°$.

**Question 51.** Find the standard matrix for the stated composition in $\mathbb{R}^3$.

1. A reflection about the $yz$-plane, followed by an orthogonal projection onto the $xz$-plane.

2. A rotation of $45°$ about the $y$-axis, followed by a dilation with factor $\sqrt{2}$.

3. An orthogonal projection onto the $xy$-plane, followed by a reflection about the $yz$-plane.

4. A rotation of $30°$ about the $x$-axis, followed by a rotation of $30°$ about the $z$-axis, followed by a contraction with factor $\frac{1}{4}$.

5. A reflection about the $xy$-plane, followed by a reflection about the $xz$-plane, followed by an orthogonal projection onto the $yz$-plane.

6. A rotation of $270°$ about the $x$-axis, followed by a rotation of $90°$ about the $y$-axis, followed by a rotation of $180°$ about the $z$-axis.

**Question 52.** Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector in $\mathbb{R}^2$. Consider the linear transformations $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_1(\boldsymbol{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}, \quad T_2(\boldsymbol{x}) = \begin{pmatrix} 3x_1 \\ 2x_1 + 4x_2 \end{pmatrix}$$

1. Find the standard matrices for $T_1$ and $T_2$.

2. Find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$.

3. Find the standard matrices for $T_1 \circ T_2 \circ T_1$ and $T_1 \circ T_2 \circ T_2$.

4. Use the matrices obtained in part 2 to find formulas for $T_1\left(T_2\left(\boldsymbol{x}\right)\right)$ and $T_2\left(T_1\left(\boldsymbol{x}\right)\right)$

**Question 53.** Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be a vector in $\mathbb{R}^3$. Consider the linear transformations $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T_1(\boldsymbol{x}) = \begin{pmatrix} 4x_1 \\ -2x_1 + x_2 \\ -x_1 - 3x_3 \end{pmatrix}, \quad T_2(\boldsymbol{x}) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_3 \\ 4x_1 - x_3 \end{pmatrix}$$

1. Find the standard matrices for $T_1$ and $T_2$.

2. Find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$.
3. Find the standard matrices for $T_1 \circ T_2 \circ T_1$ and $T_1 \circ T_2 \circ T_2$.
4. Use the matrices obtained in part 2 to find formulas for $T_1(T_2(\boldsymbol{x}))$ and $T_2(T_1(\boldsymbol{x}))$