

Algebra and Discrete Mathematics

ADM

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Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces and matrix transformations
- Fundamental spaces and decompositions
- Eulerian tours
- Hamiltonian cycles
- Midterm
- Paths and spanning trees
- Trees and networks
- Matching

Recommended reading

- Anton, Howard, and Chris Rorres. Elementary linear algebra: applications version. John Wiley & Sons, 2013.
 - Sections 4.7, 4.8, 4.10, 5.1, 5.2, 9.1
 - [Accessible online \(free copy\)](#)
 - [Alternative download link](#)

Lecture outline

- Row space, column space, and null space
- Rank and nullity
- Eigenvalues and Eigenvectors
- Diagonalization
- LU-Decompositions
- Find LU-Decompositions
- Proofs and principles

Fundamental spaces and decompositions

- Row space, column space, and null space
- Rank and nullity
- Eigenvalues and Eigenvectors
- Diagonalization
- LU-Decompositions
- Find LU-Decompositions
- Proofs and principles

Definitions

Definition

Let $A \in \mathcal{M}_{m \times n}$

- *Row vectors* of A : rows of A
- *Row space* of A : the subspace of \mathbb{R}^m spanned by the row vectors of A
- *Column vectors* of A : columns of A
- *Column space* of A : the subspace of \mathbb{R}^n spanned by the column vectors of A
- *Null space* of A : the solution space of $Ax = \mathbf{0}$

Elementary row operations

- $Ax = 0$ has augmented matrix $(A \mid 0)$
- By performing row operations, we do not change the solution set of $Ax = 0$
- Thus

Theorem

Elementary row operations do not change the null space of a matrix.

By analyzing every elementary row operation, it can be shown that

Theorem

Elementary row operations do not change the row space of a matrix.

Find basis for null space

Example

$$A = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{pmatrix}$$

$$3x_1 + x_2 + x_3 + x_4 = 0$$

$$5x_1 - x_2 + x_3 - x_4 = 0$$

The general solution in *vector form* is given by

$$\left(-\frac{t}{4}, -s - \frac{t}{4}, t, s\right) = t \left(-\frac{1}{4}, -\frac{1}{4}, 1, 0\right) + s (0, -1, 0, 1)$$

A basis for the solution space, i.e. null space of A is

$$\left\{ \left(-\frac{1}{4}, -\frac{1}{4}, 1, 0\right), (0, -1, 0, 1) \right\}$$

Row space and column space of matrix in reduced row echelon form

Theorem

If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R , and the pivot columns form a basis for the column space of R .

The theorem implies that

Remark

dimension of row space of R = dimension of column space of R = no. of leading 1's

Row space and column space of matrix in reduced row echelon form

Example

$$R = \begin{pmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Basis for row space:

$$(1, -2, 5, 0, 3), (0, 1, 3, 0, 0), (0, 0, 0, 1, 0)$$

- Basis for column space:

$$(1, 0, 0, 0), (-2, 1, 0, 0), (0, 0, 1, 0)$$

Find basis for row space

Note

Since elementary row operations do not change the row space, we can find a basis for row space by row reduction.

Example

$$A = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix} \xrightarrow{\text{row reduction}} R = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the row space of A :

$$(1, -3, 4, -2, 5, 4), (0, 0, 1, 3, -2, -6), (0, 0, 0, 0, 1, 5)$$

Elementary row operations and column spaces

By, e.g. analyzing each elementary operation:

- Elementary row operations do not alter dependence relationships or linear independence among the column vectors

Then we have

Theorem

If A, B are row equivalent matrices, then

- *A given set of column vectors of A is linearly independent iff the corresponding column vectors of B are linearly independent*
- *A given set of column vectors of A forms a basis for the column space of A iff the corresponding column vectors of B forms a basis for the column space of B*

Recall

Definition

Matrices A and B are said to be *row equivalent* if either can be obtained from the other by a sequence of elementary row operations.

Basis for a column space by row reduction

Example

$$A = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix} \xrightarrow{\text{row reduction}} R = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the column space of A :

$$(1, 2, 2, -1), (4, 9, 9, -4), (5, 8, 9, -5)$$

Fundamental spaces and decompositions

- Row space, column space, and null space
- Rank and nullity
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Rank

Theorem

The row space and the column space of A have the same dimension.

Definition

The dimension of the row space (or column space) of a matrix A is called the *rank* of A , denoted by $\text{rank}(A)$. The dimension of the null space of A is called the *nullity* of A , denoted by $\text{nullity}(A)$

Rank and nullity – example

Example

$$A = \begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\text{rank}(A) = 2$
- Nullity: dimension of solution space of $Ax = 0$

$$\begin{aligned} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 &= 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 &= 0 \end{aligned}$$

Rank and nullity – example

Example

- $\text{rank}(A) = 2$
- Nullity: dimension of solution space of $Ax = 0$

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$

$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

A vector form of the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = r \begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{nullity}(A) = 4$$

Dimension theorem for matrices

Theorem

If A has n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Proof.

number of leading variables = number of leading 1's = $\text{rank}(A)$

number of free variables in general solution of $A\mathbf{x} = \mathbf{0}$ = number of parameters in the solution = $\text{nullity}(A)$

number of leading variables + number of free variables = n



Rank and nullity

From the proof we have established

Theorem

$$A \in \mathcal{M}_{m \times n}$$

- $\text{rank}(A)$ = *the number of leading variables in the general solution of $Ax = 0$*
- $\text{nullity}(A)$ = *the number of parameters in the general solution of $Ax = 0$*

Example

$$A \in \mathcal{M}_{5 \times 7}$$

- $\text{rank}(A) = 3$, find the number of parameters in the general solution of $Ax = 0$

$$\text{number of parameters} = \text{nullity}(A) = 7 - 3 = 4$$

- $Ax = 0$ has a two-dimensional solution space, what is $\text{rank}(A)$?

$$\text{rank}(A) = 7 - \text{nullity}(A) = 7 - 2 = 5$$

Dimension theorem for matrices – example

Example

$$A = \begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank}(A) + \text{nullity}(A) = 2 + 4 = 6$$

Orthogonal complement

Definition

If W is a subspace of \mathbb{R}^n , the the set of all vectors in \mathbb{R}^n that are orthogonal to every vector in W is called the *orthogonal complement* of W , denoted by W^\perp , i.e.

$$W^\perp = \{ \mathbf{v} \mid \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \cdot \mathbf{w} = 0 \ \forall \mathbf{w} \in W \}$$

Theorem

If W is a subspace of \mathbb{R}^n , then

- W^\perp is a subspace of \mathbb{R}^n
- $W \cap W^\perp = \{\mathbf{0}\}$
- $(W^\perp)^\perp = W$

Orthogonal complement

Theorem

$A \in \mathcal{M}_{m \times n}$, the null space of A and the row space of A are orthogonal complements in \mathbb{R}^n .

Example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies \text{a basis for row space: } \{ (1, 0, 1), (0, 1, 1) \}.$$

Solving $Ax = 0$ gives solutions of the form $(-t, -t, t)$, so

$$\text{a basis for null space: } = \{ (-1, -1, 1) \}.$$

We can check that

$$(1, 0, 1) \cdot (-1, -1, 1) = (0, 1, 1) \cdot (-1, -1, 1) = 0$$

Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n \times n}$, the following statements are equivalent.

- (a) *A is invertible*
- (b) *$Ax = 0$ has only the trivial solution*
- (c) *The reduced row echelon form of A is I_n*
- (d) *A is expressible as a product of elementary matrices*
- (e) *$Ax = b$ is consistent $\forall b \in \mathbb{R}^n$*
- (f) *$Ax = b$ has exactly one solution $\forall b \in \mathbb{R}^n$*
- (g) *$\det(A) \neq 0$*
- (h) *The column vectors of A are linearly independent*
- (i) *The row vectors of A are linearly independent*
- (j) *The column vectors of A span \mathbb{R}^n*
- (k) *The row vectors of A span \mathbb{R}^n*

- (l) The column vectors of A form a basis for \mathbb{R}^n
- (m) The row vectors of A form a basis for \mathbb{R}^n
- (n) A has rank n
- (o) A has nullity 0
- (p) The orthogonal complement of the null space of A is \mathbb{R}^n
- (q) The orthogonal complement of the row space of A is $\{0\}$

Proof.

Let S be the set of row vectors of A , in the last lecture we have proved that

- S spans \mathbb{R}^n iff $\det(A) \neq 0$
- S is a basis for \mathbb{R}^n iff $\det(A) \neq 0$
- S is linearly independent iff $\det(A) \neq 0$

We have the equivalence of (g)(i), (g)(k), (g)(m)

Equivalence statements

- (b) $Ax = 0$ has only the trivial solution
- (n) A has rank n
- (o) A has nullity 0

Proof.

$\text{rank}(A) + \text{nullity}(A) = n$ proves the equivalence of (n)(o)

(b) \Rightarrow (o) If $Ax = 0$ has only the trivial solution, then the null space is the zero space

(o) \Rightarrow (b) We have proved that $\text{nullity}(A)$ = the number of parameters in the general solution of $Ax = 0$. $\text{nullity}(A) = 0$ implies there are no free variables and the trivial solution is the only solution.

Kernel and range of T_A

Definition

$T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation, the *kernel* of T_A , denoted $\ker(T_A)$, is the null space of A . The *range* of T_A , denoted $\mathcal{R}(T_A)$, is the column space of A

- $\ker(T_A)$: the set of all vectors in \mathbb{R}^n that T_A maps into $\mathbf{0}$
- $\mathcal{R}(T_A)$: the set of all vectors in \mathbb{R}^m that are images of at least one vector from \mathbb{R}^n

T_A and A

Theorem

$A \in \mathcal{M}_{n \times n}$, $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the following statements are equivalent

- (a) A is invertible
- (b) $\ker(T_A) = \{\mathbf{0}\}$
- (c) $\text{R}(T_A) = \mathbb{R}^n$
- (d) T_A is bijective

Proof.

(d) \Rightarrow (a) T_A is surjective \Rightarrow for any $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has a solution \Rightarrow (a) □

Remark

With a bit modification of the proof, we can prove that

$$T_A \text{ is surjective} \Leftrightarrow T_A \text{ is injective} \Leftrightarrow T_A \text{ is bijective}$$

The rotation operator on \mathbb{R}^2 is surjective

Example

- T : Rotation about the origin through an angle θ

$$[T] = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- $\det([T]) = 1 \neq 0$

Fundamental spaces and decompositions

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Definition

Definition

Let $A \in \mathcal{M}_{n \times n}$, a nonzero $x \in \mathbb{R}^n$ is called an *eigenvector* of A (or of the matrix operator T_A) if Ax is a scalar multiple of x , i.e.

$$Ax = \lambda x$$

for some $\lambda \in \mathbb{R}$. λ is called an *eigenvalue* of A (or T_A) and x is said to be an *eigenvector corresponding to λ* .

- In general Ax differs from x in both magnitude and direction
- For eigenvectors, either same direction or opposite direction

Eigenvector – example

Example

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 3$

$$Ax = 3x$$

Characteristic equation

- $Ax = \lambda x$ can be rewritten as $Ax = \lambda Ix$ or

$$(\lambda I - A)x = \mathbf{0}$$

- For λ to be an eigenvalue of A , the equation must have a **nonzero** solution for x
- Which is true iff $\det(\lambda I - A) = 0$

Theorem

Let $A \in \mathcal{M}_{n \times n}$, λ is an eigenvalue of A iff

$$\det(\lambda I - A) = 0$$

This is called the characteristic equation of A .

Finding eigenvalues – example

Example

Consider $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$, $\det(\lambda I - A) = 0$ gives

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0 \implies (\lambda - 3)(\lambda + 1) = 0$$

Thus, the eigenvalues of A are 3 and -1 .

Characteristic polynomial

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

- With cofactor expansion, the highest power of λ appears when multiplying all diagonal entries
- Characteristic equation of A takes the form

$$\lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = 0$$

- The polynomial (left side of the equation)

$$p(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n$$

is called the *characteristic polynomial* of A

- Degree n polynomial has at most n distinct roots \implies at most n distinct eigenvalues

Characteristic polynomial – example

Example

$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ has characteristic polynomial

$$p(\lambda) = (\lambda - 3)(\lambda + 1) = \lambda^2 - 2\lambda - 3$$

Characteristic polynomial – example

Example

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$$

Characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

- First we observe that an integer solution (if any) of a polynomial equation with integer coefficients must be a divisor of the constant term
- In our case, divisors of -4 : $\pm 1, \pm 2, \pm 4$
- $\lambda = 4$ is a solution

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

Characteristic polynomial – example

Example

Characteristic polynomial of A is $\lambda^3 - 8\lambda^2 + 17\lambda - 4$

- $\lambda = 4$ is a solution

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

- Solve the quadratic equation by the quadratic formula

$$\lambda = \frac{4 \pm \sqrt{4^2 - 4}}{2} = 2 \pm \sqrt{3}$$

- Eigenvalues of A are

$$\lambda_1 = 4, \quad \lambda_2 = 2 + \sqrt{3}, \quad \lambda_3 = 2 - \sqrt{3}$$

Eigenvalues of an upper triangular matrix

Example

$A = \begin{pmatrix} 1 & 8 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$, the characteristic equation of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 8 & 0 \\ 0 & \lambda - 2 & 6 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Eigenvalues of triangular matrices

Theorem

If $A \in \mathcal{M}_{n \times n}$ is a triangular matrix (upper triangular, lower triangular, diagonal), then the eigenvalues of A are the entries on the main diagonal of A .

Equivalent statements

Theorem

Let $A \in \mathcal{M}_{n \times n}$, the following statements are equivalent

- *λ is an eigenvalue of A*
- *λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$*
- *The system of equations $(\lambda I - A)x = \mathbf{0}$ has nontrivial solutions*
- *There is a nonzero vector x s.t. $Ax = \lambda x$*

Eigenspace

- Eigenvectors of A corresponding to an eigenvalue λ are the **nonzero** vectors that satisfy

$$(\lambda I - A)x = \mathbf{0}$$

- The solution space is called the *eigenspace* of A corresponding to λ consist of eigenvectors of A corresponding to λ and $\mathbf{0}$
- Eigenspace can also be reviewed as
 - the null space of the matrix $\lambda I - A$
 - the kernel of the matrix operator $T_{\lambda I - A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$
 - the set of vectors for which $Ax = \lambda x$

Find bases for eigenspaces – example

Example

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}, \quad \begin{vmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{vmatrix} = (\lambda - 2)(\lambda + 3) = 0$$

Eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -3$.

$$\begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus the eigenspace corresponding to $\lambda_1 = 2$ has basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenspace corresponding to $\lambda_2 = -3$ has basis $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n \times n}$, the following statements are equivalent.

- (a) *A is invertible*
- (b) *$Ax = \mathbf{0}$ has only the trivial solution*
- (c) *The reduced row echelon form of A is I_n*
- (d) *A is expressible as a product of elementary matrices*
- (e) *$Ax = \mathbf{b}$ is consistent $\forall \mathbf{b} \in \mathbb{R}^n$*
- (f) *$Ax = \mathbf{b}$ has exactly one solution $\forall \mathbf{b} \in \mathbb{R}^n$*
- (g) *$\det(A) \neq 0$*
- (h) *The column vectors of A are linearly independent*
- (i) *The row vectors of A are linearly independent*
- (j) *The column vectors of A span \mathbb{R}^n*
- (k) *The row vectors of A span \mathbb{R}^n*

- (l) The column vectors of A form a basis for \mathbb{R}^n
- (m) The row vectors of A form a basis for \mathbb{R}^n
- (n) A has rank n
- (o) A has nullity 0
- (p) The orthogonal complement of the null space of A is \mathbb{R}^n
- (q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$
- (r) $\ker(T_A) = \{\mathbf{0}\}$
- (s) $\operatorname{R}(T_A) = \{\mathbb{R}^n\}$
- (t) T_A is surjective
- (u) $\lambda = 0$ is not an eigenvalue of A

Fundamental spaces and decompositions

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Similarity transformation

- $A, P \in \mathcal{M}_{n \times n}$, P invertible
- *similarity transformation*: $A \mapsto P^{-1}AP$

$$\det(P^{-1}AP) = \det(A)$$

i.e. similarity transformation preserves determinant

- Any property that is preserved by a similarity transformation is called a *similarity invariant* and is said to be *invariant under similarity*

Similarity invariants

Determinant	$\det(A) = \det(P^{-1}AP)$
Invertibility	A is invertible iff $P^{-1}AP$ is invertible
Rank	$\text{rank}(A) = \text{rank}(P^{-1}AP)$
Nullity	$\text{nullity}(A) = \text{nullity}(P^{-1}AP)$
Trace	$\text{tr}(A) = \text{tr}(P^{-1}AP)$
Characteristic polynomial	
Eigenvalues	
Eigenspace dimension	Eigenspace of A corresponding to λ has same dimension as eigenspace of $P^{-1}AP$ corresponding to λ

Similar matrices

Definition

$A, B \in \mathcal{M}_{n \times n}$, if $\exists P$ invertible s.t. $B = P^{-1}AP$, then we say B is similar to A .

- If $B = P^{-1}AP$, let $Q = P^{-1}$
- $A = Q^{-1}BQ$
- We usually say that A and B are similar matrices

Diagonalizable

Definition

$A \in \mathcal{M}_{n \times n}$ is said to be *diagonalizable* if it is similar to some diagonal matrix, i.e. $A = P^{-1}DP$ for a diagonal matrix D . P is said to *diagonalize* A .

Diagonalization

- Find n linearly independent eigenvectors of A : $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$
- Construct

$$P = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$$

- Then

$$AP = (A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n) = (\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n) = PD,$$

where D is the diagonal matrix that has $\lambda_1, \lambda_2, \dots, \lambda_n$

- Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, P is invertible, we have $P^{-1}AP = D$

Diagonalization – example

Example

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Characteristic equation of A is $(\lambda - 1)(\lambda - 2)^2 = 0$

Bases for the eigenspace

$$\lambda_1 = 2 : \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \lambda_2 = 1 : \mathbf{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonalizable matrices and linearly independent eigenvectors

Theorem

$A \in \mathcal{M}_{n \times n}$, A is diagonalizable iff A has n linearly independent eigenvectors.

Theorem

- If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of A , and if v_1, v_2, \dots, v_k are corresponding eigenvectors, then $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set
- If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of A , and if S_1, S_2, \dots, S_k are corresponding sets of linearly independent eigenvectors, then the union of these sets is linearly independent.
- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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Solving linear systems

- Gaussian elimination (reduction to row echelon form) and Gauss–Jordan elimination (reduction to reduced row echelon form)
- Fine for small-scale problems, not suitable for large-scale problems in which computer roundoff error, memory usage, and speed are concerns
- We will discuss a method for solving linear systems based on factoring its coefficient matrix into a product of lower and upper triangular matrices
- LU-decomposition is the basis for many computer algorithms in common use

LU-decomposition

Definition

A factorization of a square matrix A as

$$A = LU,$$

where L is lower triangular and U is upper triangular, is called an *LU-decomposition* (or *LU-factorization*) of A .

Solving linear systems by LU-decomposition

Step 1. Rewrite the system $Ax = b$ as

$$LUx = b \quad (1)$$

Step 2. Define a new $n \times 1$ matrix y by

$$Ux = y \quad (2)$$

Step 3. Rewrite (1) as

$$Ly = b$$

and solve this system for y

Step 4. Substitute y in (2) and solve for x .

Solving linear systems by LU-decomposition

Example

$$A = LU$$

$$\begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Ax = b$$

$$\begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$LUx = b$$

$$\begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Solving linear systems by LU-decomposition

Example

$$U\mathbf{x} = \mathbf{y} \text{ and } L\mathbf{y} = \mathbf{b}$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Solve for \mathbf{y}

$$\begin{aligned} 2y_1 &= 2 \\ -3y_1 + y_2 &= 2 \implies y_1 = 1, \quad y_2 = 5, \quad y_3 = 2 \\ 4y_1 - 3y_2 + 7y_3 &= 3 \end{aligned}$$

Substitute \mathbf{y} to $U\mathbf{x} = \mathbf{y}$ and solve for \mathbf{x}

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \implies x_1 = 2, \quad x_2 = -1, \quad x_3 = 2$$

Solving linear systems by LU-decomposition

- Consider $A = LU \in \mathcal{M}_{3 \times 3}$
- $Ly = b$

$$\begin{pmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- Solving for y

$$\begin{aligned} \ell_{11}y_1 &= b_1 & y_1 &= \frac{b_1}{\ell_{11}} \\ \ell_{21}y_1 + \ell_{22}y_2 &= b_2 & \implies y_2 &= \frac{1}{\ell_{22}}(b_2 - \ell_{21}y_1) \\ \ell_{31}y_1 + \ell_{32}y_2 + \ell_{33}y_3 &= b_3 & y_3 &= \frac{1}{\ell_{33}}(b_3 - \ell_{31}y_1 - \ell_{32}y_2) \end{aligned}$$

- General formula

$$y_k = \frac{1}{\ell_{kk}} \left(b_k - \sum_{s=1}^{k-1} \ell_{ks}y_s \right)$$

Solving linear systems by LU-decomposition

- Consider $A = LU \in \mathcal{M}_{3 \times 3}$
- $Ux = y$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- Solving for x
- General formula

$$x_k = \frac{1}{u_{kk}} \left(b_k - \sum_{s=k+1}^n u_{ks} x_s \right)$$

A sufficient condition

- Not every square matrix has an LU-decomposition

Definition

$A = (a_{ij}) \in \mathcal{M}_{n \times n}$, the *leading principal minors* of A are $\det(A_1), \det(A_2), \dots, \det(A_n)$, where A_k is the top-left $k \times k$ submatrix of A

$$A_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

Theorem

Let $A = (a_{ij}) \in \mathcal{M}_{n \times n}$, if the leading principal minors of A are all nonzero, then there exists an LU-decomposition $A = LU$.

Remark

If A is invertible, then the sufficient condition is also necessary.

Another sufficient condition

Theorem

If A is a square matrix that can be reduced to a row echelon form U (upper triangular) by Gaussian elimination without row interchanges, then A can be factored as $A = LU$, where L is a lower triangular matrix.

Proof.

- U : row echelon form, upper triangular
- Row operations on A can be accomplished by multiplying A on the left by an appropriate sequence of elementary matrices

$$E_k \cdots E_2 E_1 A = U$$

- Elementary matrices are invertible

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

A sufficient condition

Proof.

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

- U : row echelon form, upper triangular
- $L := E_1^{-1} E_2^{-1} \cdots E_k^{-1}$
- Since row interchanges are excluded, each E_j results by
 - adding a scalar multiple of one row of an identity matrix to a row below
 - multiplying one row of an identity matrix by a nonzero scalar
- E_j is lower triangular $\xrightarrow{\text{Tutorial 4}}$ E_j^{-1} is lower triangular $\xrightarrow{\text{Tutorial 1}}$ L is lower triangular



Fundamental spaces and decompositions

- Row space, column space, and null space
- Rank and nullity
- Eigenvalues and Eigenvectors
- Diagonalization
- LU-Decompositions
- **Find LU-Decompositions**
- Proofs and principles

Find LU-decomposition – example

Example

$$A = \begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix}$$

To obtain LU-decomposition, we reduce A to a row echelon form U using Gaussian elimination

$$\text{Step 1} \quad \xrightarrow{R_1 \rightarrow \frac{1}{2}R_1} \begin{pmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \quad E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_1^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Step 2} \quad \xrightarrow{R_2 \rightarrow 3R_1 + R_2} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 4 & 9 & 2 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find LU-decomposition – example

Example

$$\text{Step 3} \quad \xrightarrow{R_3 \rightarrow -4R_1 + R_3} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

$$\text{Step 4} \quad \xrightarrow{R_3 \rightarrow 3R_2 + R_3} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$\text{Step 5} \quad \xrightarrow{R_3 \rightarrow \frac{1}{7}R_3} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix} \quad E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$L = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix}$$

Find LU-decomposition

Example

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix}$$

- $\ell_{11} = 2$: a multiplier of $\frac{1}{2}$ was needed in Step 1 to introduce a leading 1 in the first row
- $\ell_{33} = 7$: a multiplier of $\frac{1}{7}$ was needed in Step 5 to introduce a leading 1 in the third row
- $\ell_{22} = 1$: a multiplier of 1 to introduce a leading 1 in the second row
- To introduce 0 below the leading 1's:
- $\ell_{21} = -3$: Step 2 $R_2 \rightarrow 3R_1 + R_2$
- $\ell_{31} = 4$: Step 3 $R_3 \rightarrow -4R_1 + R_3$
- $\ell_{32} = -3$: Step 4 $R_3 \rightarrow 3R_2 + R_3$

Find LU-decomposition

- Each position along the main diagonal of L : reciprocal of the multiplier that introduced the leading 1 in that position in U
- Each position below the main diagonal of L : the negative of the multiplier used to introduce the zero in that position in U

Find LU-decomposition – example

Example

$$A = \begin{pmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{pmatrix}$$

$$\xrightarrow{R_1 \rightarrow \frac{1}{6}R_1} \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{pmatrix}$$

multiplier $\frac{1}{6}$, first row

$$\begin{pmatrix} 6 & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$$

$$\xrightarrow[\begin{matrix} R_2 \rightarrow -9R_1 + R_2 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix}]{\begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{pmatrix}}$$

multiplier -9 , position $(2, 1)$,
multiplier -3 , position $(3, 1)$

$$\begin{pmatrix} 6 & 0 & 0 \\ 9 & * & 0 \\ 3 & * & * \end{pmatrix}$$

Find LU-decomposition – example

Example

$$\xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 5 \end{pmatrix}$$

multiplier $\frac{1}{2}$, second row

$$\begin{pmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & * & * \end{pmatrix}$$

$$\xrightarrow{R_3 \rightarrow -8R_2 + R_3} \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

multiplier -8 , position $(3, 2)$

$$\begin{pmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & * \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

multiplier 1 , third row

$$L = \begin{pmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{pmatrix}$$

Find LU-decomposition – Doolittle decomposition

Example

$$A = \begin{pmatrix} 2 & 5 & 6 \\ 4 & 13 & 19 \\ 6 & 27 & 50 \end{pmatrix} = LU, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

First row

$$u_{11} = 2, \quad u_{12} = 5, \quad u_{13} = 6$$

Second row

$$\begin{aligned} \ell_{21}u_{11} &= 4 & \ell_{21} &= 2 \\ \ell_{21}u_{12} + u_{22} &= 13 & \implies u_{22} &= 13 - 2 \times 5 = 3 \\ \ell_{21}u_{13} + u_{23} &= 19 & u_{23} &= 19 - 2 \times 6 = 7 \end{aligned}$$

Third row

$$\begin{aligned} \ell_{31}u_{11} &= 6 & \ell_{31} &= 3 \\ \ell_{31}u_{12} + \ell_{32}u_{22} &= 27 & \implies \ell_{32} &= \frac{27 - 3 \times 5}{3} = 4 \\ \ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} &= 50 & u_{33} &= 50 - 3 \times 6 - 4 \times 7 = 4 \end{aligned}$$

Find LU-decomposition – Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{21} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}.$$

For each row i

- Computing U : for $j \geq i$:

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} \ell_{ik} u_{kj}. \quad (3)$$

- Computing L : for $j < i$:

$$\ell_{ij} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{i-1} \ell_{ik} u_{kj} \right). \quad (4)$$

Find LU-decomposition – Crout decomposition

$$L = \begin{pmatrix} \ell_{11} & 0 & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & \ell_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

For each column:

- Computing L : for $i \geq j$:

$$\ell_{ij} = a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} u_{kj}. \quad (5)$$

- Computing U : for $i > j$:

$$u_{ij} = \frac{1}{\ell_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} \ell_{jk} u_{ki} \right). \quad (6)$$

Same result as using Gaussian elimination

LU-dcomposition is not unique

Example

- Note that Crout decomposition and Doolittle decomposition produce different matrices

- Another example: $A = \begin{pmatrix} 2 & 10 \\ 7 & 44 \end{pmatrix}$

- Doolittle:

$$L = \begin{pmatrix} 1 & 0 \\ 7 & \frac{1}{2} \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 10 \\ 0 & 9 \end{pmatrix}$$

- Crout:

$$L = \begin{pmatrix} 2 & 0 \\ 7 & 9 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$

- Another possible decomposition:

$$L = \begin{pmatrix} 2 & 0 \\ 7 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}$$

What can we do with LU-decomposition

- Solving linear systems
- Find inverse

$$A = LU, \quad A^{-1} = U^{-1}L^{-1}$$

- Compute determinant

$$\det(A) = \det(L) \det(U)$$

where $\det(L)$ and $\det(U)$ are easy to compute - product of diagonal entries

Fundamental spaces and decompositions

- Row space, column space, and null space
- Rank and nullity
- Eigenvalues and Eigenvectors
- Diagonalization
- LU-Decompositions
- Find LU-Decompositions
- Proofs and principles

Rank

Theorem

The row space and the column space of A have the same dimension.

Proof.

- It follows from the previous discussions that elementary row operations do not change the dimension of the row space or of the column space of a matrix.
- Let R be a reduced row echelon form of A .
- We have shown that the row and column spaces of R have the same dimension.



Definition

The dimension of the row space (or column space) of a matrix A is called the *rank* of A , denoted by $\text{rank}(A)$. The dimension of the null space of A is called the *nullity* of A , denoted by $\text{nullity}(A)$

Orthogonal complement

Theorem

If W is a subspace of \mathbb{R}^n , then

- W^\perp is a subspace of \mathbb{R}^n
- $W \cap W^\perp = \{\mathbf{0}\}$
- $(W^\perp)^\perp = W$

Proof.

- Take any $\mathbf{u}, \mathbf{v} \in W^\perp$ and any $\alpha \in \mathbb{R}^n$. Then for any $\mathbf{w} \in W$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = 0 + 0 = 0, \quad (\alpha \mathbf{u}) \cdot \mathbf{w} = \alpha(\mathbf{u} \cdot \mathbf{w}) = \alpha \times 0 = 0$$

W^\perp is closed under addition and scalar multiplication

- $W \cap W^\perp = \{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{x} = 0\} = \{\mathbf{0}\}$
- We will only show $W \subseteq (W^\perp)^\perp$: take any $\mathbf{w} \in W$, we have $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in W^\perp$, thus $\mathbf{w} \in (W^\perp)^\perp$.

Orthogonal complement

Theorem

$A \in \mathcal{M}_{m \times n}$, the null space of A and the row space of A are orthogonal complements in \mathbb{R}^n .

Proof.

- Each solution of $Ax = 0$ satisfies $a_i \cdot x = 0$ for all i , where a_i is the i th row of A .
- The solution set of $Ax = 0$ consists of all vectors in \mathbb{R}^n that are orthogonal to every row vector of A



Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n \times n}$, the following statements are equivalent.

- (a) *A is invertible*
- (b) *$Ax = \mathbf{0}$ has only the trivial solution*
- (c) *The reduced row echelon form of A is I_n*
- (d) *A is expressible as a product of elementary matrices*
- (e) *$Ax = \mathbf{b}$ is consistent $\forall \mathbf{b} \in \mathbb{R}^n$*
- (f) *$Ax = \mathbf{b}$ has exactly one solution $\forall \mathbf{b} \in \mathbb{R}^n$*
- (g) *$\det(A) \neq 0$*
- (h) *The column vectors of A are linearly independent*
- (i) *The row vectors of A are linearly independent*
- (j) *The column vectors of A span \mathbb{R}^n*
- (k) *The row vectors of A span \mathbb{R}^n*

- (l) The column vectors of A form a basis for \mathbb{R}^n
- (m) The row vectors of A form a basis for \mathbb{R}^n
- (n) A has rank n
- (o) A has nullity 0
- (p) The orthogonal complement of the null space of A is \mathbb{R}^n
- (q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$

Proof.

Let S be the set of row vectors of A , in the last lecture we have proved that

- S spans \mathbb{R}^n iff $\det(A) \neq 0$
- S is a basis for \mathbb{R}^n iff $\det(A) \neq 0$
- S is linearly independent iff $\det(A) \neq 0$

Since $\det(A) = \det(A^\top)$ and the row vectors of A are the column vectors of A^\top , we have the equivalence of (g)(h), (g)(i), (g)(j), (g)(k), (g)(l), (g)(m), gives the equivalence from (a)-(m)

Equivalence statements

- (b) $Ax = \mathbf{0}$ has only the trivial solution
- (n) A has rank n
- (o) A has nullity 0

Proof.

$\text{rank}(A) + \text{nullity}(A) = n$ proves the equivalence of (n)(o)

(b) \Rightarrow (o) If $Ax = \mathbf{0}$ has only the trivial solution, then the null space is the zero space

(o) \Rightarrow (b) We have proved that $\text{nullity}(A) =$ the number of parameters in the general solution of $Ax = \mathbf{0}$. $\text{nullity}(A) = 0$ implies there are no free variables and the trivial solution is the only solution.

Now we have equivalence (a)-(o)

Equivalent statements

- (k) The row vectors of A span \mathbb{R}^n
- (p) The orthogonal complement of the null space of A is \mathbb{R}^n
- (q) The orthogonal complement of the row space of A is $\{0\}$

Proof.

We have just proved “the null space of A and the row space of A are orthogonal complements in \mathbb{R}^n ,” which shows the equivalence of (k)(p) and (k)(q) □

T_A and A

Theorem

$A \in \mathcal{M}_{n \times n}$, $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the following statements are equivalent

- (a) A is invertible
- (b) $\ker(T_A) = \{\mathbf{0}\}$
- (c) $\mathcal{R}(T_A) = \mathbb{R}^n$
- (d) T_A is bijective

Proof.

$$(a) \Leftrightarrow (b) \quad (a) \Leftrightarrow \text{nullity}(A) = 0$$

$$(a) \Leftrightarrow (c) \quad (a) \Leftrightarrow \text{column vectors of } A \text{ span } \mathbb{R}^n$$

$(a) \Rightarrow (d)$ $(a) \Rightarrow$ for any $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has a solution $\Rightarrow T_A$ is injective. If T_A is not injective, $\exists \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ s.t.

$$T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2) \Rightarrow T_A(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0} \Rightarrow \text{nullity}(A) \neq 0$$

a contradiction. Thus, T_A is injective.

Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n \times n}$, the following statements are equivalent.

- (a) *A is invertible*
- (b) *$Ax = \mathbf{0}$ has only the trivial solution*
- (c) *The reduced row echelon form of A is I_n*
- (d) *A is expressible as a product of elementary matrices*
- (e) *$Ax = \mathbf{b}$ is consistent $\forall \mathbf{b} \in \mathbb{R}^n$*
- (f) *$Ax = \mathbf{b}$ has exactly one solution $\forall \mathbf{b} \in \mathbb{R}^n$*
- (g) *$\det(A) \neq 0$*
- (h) *The column vectors of A are linearly independent*
- (i) *The row vectors of A are linearly independent*
- (j) *The column vectors of A span \mathbb{R}^n*
- (k) *The row vectors of A span \mathbb{R}^n*

- (l) The column vectors of A form a basis for \mathbb{R}^n
- (m) The row vectors of A form a basis for \mathbb{R}^n
- (n) A has rank n
- (o) A has nullity 0
- (p) The orthogonal complement of the null space of A is \mathbb{R}^n
- (q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$
- (r) $\ker(T_A) = \{\mathbf{0}\}$
- (s) $\text{R}(T_A) = \{\mathbb{R}^n\}$
- (t) T_A is surjective
- (u) $\lambda = 0$ is not an eigenvalue of A

Proof.

The equivalence of (a)(r)(s)(t) was proved just now

We will prove $(g) \Leftrightarrow (u)$

Equivalent statements

(g) $\det(A) \neq 0$

(u) $\lambda = 0$ is not an eigenvalue of A

Proof.

We will prove $(g) \Leftrightarrow (u)$

$\lambda = 0$ is a solution of the characteristic equation

$$\lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = 0$$

iff $\alpha_n = 0$. On the other hand, setting $\lambda = 0$

$$\det(\lambda I - A) = \det(-A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = \alpha_n$$

i.e. $(-1)^n \det(A) = \alpha_n$. □

Diagonalizable matrices

Theorem

$A \in \mathcal{M}_{n \times n}$, A is diagonalizable iff A has n linearly independent eigenvectors.

Proof.

We have just proved \Leftarrow .

\Rightarrow Assume $AP = PD$, P has columns $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, D has diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$

$$AP = (A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n), \quad PD = (\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n)$$

P invertible implies that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent. □