

# Algebra and Discrete Mathematics

## ADM

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# Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces and matrix transformations
- Fundamental spaces and decompositions
- Eulerian tours
- Hamiltonian cycles
- Midterm
- Paths and spanning trees
- Trees and networks
- Matching

## Recommended reading

- Anton, Howard, and Chris Rorres. Elementary linear algebra: applications version. John Wiley & Sons, 2013.
  - Sections 1.5, 1.6, 2.1, 2.2, 2.3
  - [Accessible online \(free copy\)](#)
  - [Alternative download link](#)

# Lecture outline

- Elementary row operations and elementary matrices
- Compute matrix inverse
- Linear systems and invertible matrices
- Determinants
- Evaluating determinants by row reduction
- Properties of determinants
- Cramer's rule

# Matrix inverse and determinants

- Elementary row operations and elementary matrices
- Compute matrix inverse
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# Elementary row operations

Recall elementary row operations

- Multiply a row by a nonzero constant  $\beta$ .
- Interchange two rows.
- Add a constant  $\beta$  times one row to another.

If we obtain  $B$  from  $A$  by one of the operations, then  $A$  can be recovered from  $B$  by one of the following

- Multiply the same row by  $1/\beta$ .
- Interchange the same two rows.
- If  $B$  resulted by adding  $\beta$  times row  $i$  of  $A$  to row  $j$ , then add  $-\beta$  times row  $i$  to row  $j$ .

If  $B$  is obtained from  $A$  by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to  $B$  recovers  $A$

# Row equivalence and elementary matrices

## Definition

Matrices  $A$  and  $B$  are said to be *row equivalent* if either can be obtained from the other by a sequence of elementary row operations.

## Definition

A matrix  $E$  is called an *elementary matrix* if it can be obtained from an identity matrix by performing a single elementary row operation.

## Elementary matrices – example

### Example

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The elementary operations:

- $I_2 \xrightarrow{R_2 \rightarrow -3R_2}$
- $I_4 \xrightarrow{R_2 \leftrightarrow R_4}$
- $I_3 \xrightarrow{R_1 \rightarrow R_1 + 3R_3}$
- $I_3 \xrightarrow{R_1 \rightarrow 1R_1}$



# Row operations by matrix multiplication

## Theorem 1

*Suppose the elementary matrix  $E$  results from performing a certain row operation on  $I_m$ . Take any  $A \in \mathcal{M}_{m \times n}$ , then  $EA$  is the matrix that results when this same row operation is performed on  $A$ .*

## Example

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

Note that  $I_3 \xrightarrow{R_3 \rightarrow R_3 + 3R_1} E$ , and

$$EA = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{pmatrix}$$

## Inverse operation

- $E$ : an elementary matrix that results from performing an elementary row operation on an identity matrix  $I$
- There is another elementary row operation when applied to  $E$  produces  $I$
- We call them *inverse operation* of each other

## Row operations and inverse row operations – example

We first apply an elementary row operation on  $I_2$  to get an elementary matrix and then apply the inverse row operation to get  $I_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow 7R_2} \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{7}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + 5R_2} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 - 5R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

# Inverse of an elementary matrix

## Theorem 2

*Every elementary matrix is invertible, and the inverse is also an elementary matrix.*

### Proof.

Let  $E$  be any elementary matrix. By definition,  $E$  is obtained from  $I$  by performing some row operation. Let  $E_0$  be the matrix that results when the inverse of this operation is performed on  $I$ . By Theorem 1 and the fact that inverse row operations cancel the effect of each other

$$E_0E = I \quad \text{and} \quad EE_0 = I.$$

$E_0$  is the inverse of  $E$ . □

# Equivalent statements

## Theorem 3

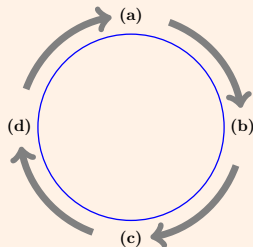
For any  $A \in \mathcal{M}_{n \times n}$ , the following statements are equivalent.

- (a)  $A$  is invertible
- (b)  $Ax = \mathbf{0}$  has only the trivial solution
- (c) The reduced row echelon form of  $A$  is  $I_n$
- (d)  $A$  is expressible as a product of elementary matrices

## Proof.

We will prove

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$$



## Equivalent statements – proof

- (a)  $A$  is invertible
- (b)  $Ax = \mathbf{0}$  has only the trivial solution

Proof.

(a)  $\Rightarrow$  (b) Suppose  $A$  is invertible and let  $x_0$  be a solution of  $Ax = \mathbf{0}$ . Multiplying both sides by  $A^{-1}$

$$A^{-1}(Ax_0) = \mathbf{0} \implies (A^{-1}A)x_0 = \mathbf{0} \implies Ix_0 = \mathbf{0} \implies x_0 = \mathbf{0}$$

## Equivalent statements – proof

- (b)  $Ax = \mathbf{0}$  has only the trivial solution
- (c) The reduced row echelon form of  $A$  is  $I_n$

Proof.

(b)  $\Rightarrow$  (c) Let the linear system corresponding to  $Ax = \mathbf{0}$  be

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & 0 \end{array}$$

Since the linear system has only the trivial solution, the reduced row echelon form of its augmented matrix would represent

## Equivalent statements – proof

(b)  $Ax = 0$  has only the trivial solution

(c) The reduced row echelon form of  $A$  is  $I_n$

Proof.

(b)  $\Rightarrow$  (c) Since the linear system has only the trivial solution, the reduced row echelon form of its augmented matrix would represent

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \\ & \ddots & \\ x_n & = & 0 \end{array} \xrightarrow{\text{augmented matrix}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$



## Equivalent statements – proof

- (c) The reduced row echelon form of  $A$  is  $I_n$
- (d)  $A$  is expressible as a product of elementary matrices

### Proof.

(c)  $\Rightarrow$  (d) Suppose the reduced row echelon form of  $A$  is  $I_n$ , then  $A$  can be reduced to  $I_n$  by a finite sequence of elementary row operations. By Theorem 1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus,  $\exists E_1, E_2, \dots, E_k$  s.t.

$$E_k \cdots E_2 E_1 A = I_n.$$

By Theorem 2,  $E_1, E_2, \dots, E_k$  are invertible and their inverses are also elementary matrices.

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

## Equivalent statements – proof

- (a)  $A$  is invertible
- (d)  $A$  is expressible as a product of elementary matrices

Proof.

(d)  $\Rightarrow$  (a) Suppose

$$A = E_1 E_2 \cdots E_k,$$

then

$$A^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}.$$



# Matrix inverse and determinants

- Elementary row operations and elementary matrices
- **Compute matrix inverse**
- Linear systems and invertible matrices
- Determinants
- Evaluating determinants by row reduction
- Properties of determinants
- Cramer's rule

## Inverse of a matrix and reduced row echelon form

- We will develop a procedure to determine if a matrix is invertible, if yes, find the inverse
- Suppose  $A \in \mathcal{M}_{n \times n}$  is invertible, then its reduced row echelon form is  $I_n$

$$E_k \cdots E_2 E_1 A = I_n.$$

Multiply both sides on the right by  $A^{-1}$

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

i.e. the same sequence of row operations that reduces  $A$  to  $I_n$  will transform  $I_n$  to  $A^{-1}$

# Inversion algorithm

To find the inverse of an invertible matrix  $A \in \mathcal{M}_{n \times n}$

- Find a sequence of elementary row operations that reduces  $A$  to  $I_n$
- Perform the same sequence of operations on  $I_n$  to find  $A^{-1}$
- We adjoin  $I_n$  to the right side of  $A$

$$(A \mid I_n)$$

and apply elementary row operations to get

$$(I_n \mid A^{-1})$$

## Inversion algorithm – example

### Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array} \right) \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{R_3 \rightarrow 2R_2 + R_3} \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right) \xrightarrow{R_3 \rightarrow -R_3} \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right)$$

Note

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \dots$$

## Inversion algorithm – example

### Example

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right) \xrightarrow[R_2 \rightarrow 3R_3 + R_2]{R_1 \rightarrow -3R_3 + R_1} \left( \begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right)$$

$$\xrightarrow{R_1 \rightarrow -2R_2 + R_1} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

### Exercise

Verity that  $AA^{-1} = I$ , find  $E_3, E_4, \dots$  and verity that  $I = E_1 E_2 E_3 \dots A$

## Show that a matrix is not invertible – example

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6 & 4 & | & 1 & 0 & 0 \\ 2 & 4 & -1 & | & 0 & 1 & 0 \\ -1 & 2 & 5 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 \rightarrow -2R_1 + R_2]{R_3 \rightarrow R_1 + R_3} \begin{pmatrix} 1 & 6 & 4 & | & 1 & 0 & 0 \\ 0 & -8 & -9 & | & -2 & 1 & 0 \\ 0 & 8 & 9 & | & 1 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_3 \rightarrow R_2 + R_3} \begin{pmatrix} 1 & 6 & 4 & | & 1 & 0 & 0 \\ 0 & -8 & -9 & | & -2 & 1 & 0 \\ 0 & 0 & 0 & | & -1 & 1 & 1 \end{pmatrix}$$

Since the left side has a row of zeros,  $A$  is not invertible.



## Analyzing homogeneous systems – example

### Example

(a)

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 0 \\2x_1 + 5x_2 + 3x_3 &= 0 \\x_1 + 8x_3 &= 0\end{aligned}$$

(b)

$$\begin{aligned}x_1 + 6x_2 + 4x_3 &= 0 \\2x_1 + 4x_2 - x_3 &= 0 \\-x_1 + 2x_2 + 5x_3 &= 0\end{aligned}$$

- From Theorem 3, a homogeneous linear system has only the trivial solution iff its coefficient matrix is invertible
- From the previous example, we know that (a) has only the trivial solution and (b) has nontrivial solutions

# Matrix inverse and determinants

- Elementary row operations and elementary matrices
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# Solution of linear systems

## Theorem

*A system of linear equations has either zero, one, or infinitely many solutions.*

## Proof.

$Ax = b$  has either (a) no solutions, (b) has exactly one solution, or (c) more than one solution. It suffices to show if the system has more than one solution, it has infinitely many solutions.

Assume two distinct solutions  $x_1, x_2$ , let  $x_0 = x_1 - x_2$ . Then  $x_0 \neq 0$ , and

$$Ax_0 = A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0.$$

For any  $\beta \in \mathbb{R}$ ,

$$A(x_1 + \beta x_0) = Ax_1 + A(\beta x_0) = b + \beta(Ax_0) = b + 0 = b,$$

showing that  $x_1 + \beta x_0$  is a solution. Since  $x_0 \neq 0$  and there are infinitely many  $\beta$ , the system has infinitely many solutions.



# Solving linear systems

## Theorem 4

*Given  $A \in \mathcal{M}_{n \times n}$ ,  $\mathbf{b} \in \mathbb{R}^n$ , if  $A$  is invertible, then the system of equations  $A\mathbf{x} = \mathbf{b}$  has a unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$ .*

## Proof.

Since  $A(A^{-1}\mathbf{b}) = \mathbf{b}$ ,  $\mathbf{x} = A^{-1}\mathbf{b}$  is a solution. If  $\mathbf{x}_0$  is any solution of  $A\mathbf{x} = \mathbf{b}$ , then

$$A\mathbf{x}_0 = \mathbf{b} \xrightarrow{\text{multiply by } A^{-1}} A^{-1}A\mathbf{x}_0 = A^{-1}\mathbf{b} \implies \mathbf{x}_0 = A^{-1}\mathbf{b}$$



## Solution of a linear system using matrix inverse

### Example

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 5 \\2x_1 + 5x_2 + 3x_3 &= 3 \\x_1 + 8x_3 &= 17\end{aligned}$$

In matrix form, the system can be written as  $A\mathbf{x} = \mathbf{b}$ , where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}$$

We already calculated:

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

## Solution of a linear system using matrix inverse

### Example

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 5 \\2x_1 + 5x_2 + 3x_3 &= 3 \\x_1 + 8x_3 &= 17\end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

The solution is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

## Linear systems with a common coefficient matrix

- Frequently, one is concerned with solving a sequence of systems

$$Ax = b_1, \quad Ax = b_2, \quad \dots, \quad Ax = b_k$$

each of which has the same square coefficient matrix  $A$ .

- If  $A$  is invertible, then the solutions

$$x_1 = A^{-1}b_1, \quad x_2 = A^{-1}b_2, \quad \dots, \quad x_k = A^{-1}b_k$$

can be obtained with one matrix inversion and  $k$  matrix multiplications.

## Linear systems with a common coefficient matrix – example

### Example

(a)

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 4 \\2x_1 + 5x_2 + 3x_3 &= 5 \\x_1 + 8x_3 &= 9\end{aligned}$$

(b)

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 1 \\2x_1 + 5x_2 + 3x_3 &= 6 \\x_1 + 8x_3 &= -6\end{aligned}$$

The two solutions are given by

$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$



# Matrix inverse

## Lemma 1

Given  $A, B \in \mathcal{M}_{n \times n}$

- If  $BA = I$ , then  $B = A^{-1}$
- If  $AB = I$ , then  $B = A^{-1}$

## Proof.

Assume  $BA = I$ . If  $A$  is invertible,

$$BAA^{-1} = IA^{-1} \implies BI = IA^{-1} \implies B = A^{-1}.$$

By Theorem 3, to show  $A$  is invertible, it suffices to show that the system  $Ax = \mathbf{0}$  has only the trivial solution. Let  $x_0$  be any solution.

$$BAx_0 = B\mathbf{0} \implies Ix_0 = \mathbf{0} \implies x_0 = \mathbf{0}.$$



# Equivalent statements

## Theorem 5

*For any  $A \in \mathcal{M}_{n \times n}$ , the following statements are equivalent.*

- (a)  *$A$  is invertible*
- (b)  *$Ax = \mathbf{0}$  has only the trivial solution*
- (c) *The reduced row echelon form of  $A$  is  $I_n$*
- (d)  *$A$  is expressible as a product of elementary matrices*
- (e)  *$Ax = \mathbf{b}$  is consistent  $\forall \mathbf{b} \in \mathbb{R}^n$*
- (f)  *$Ax = \mathbf{b}$  has exactly one solution  $\forall \mathbf{b} \in \mathbb{R}^n$*

## Proof.

- We have shown the equivalence of (a)-(d) in Theorem 3.
- It is sufficient to prove  $(a) \implies (f) \implies (e) \implies (a)$ .
- Theorem 4 proves  $(a) \implies (f)$
- $(f) \implies (e)$  is trivial.

## Equivalent statements – proof

(a)  $A$  is invertible

(e)  $A\mathbf{x} = \mathbf{b}$  is consistent  $\forall \mathbf{b} \in \mathbb{R}^n$

Proof.

We will now show (e) $\implies$ (a). Assume (e) is true. Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  be the solutions for each of the following linear systems:

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad A\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Let

$$C = (\mathbf{x}_1 \mid \mathbf{x}_2 \mid \dots \mid \mathbf{x}_n)$$

## Equivalent statements – proof

(a)  $A$  is invertible

(e)  $Ax = b$  is consistent  $\forall b \in \mathbb{R}^n$

Proof.

Then (discussion of matrix multiplication from the last lecture)

$$AC = (Ax_1 \mid Ax_2 \mid \cdots \mid Ax_n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I$$

By Lemma 1,  $C$  is the inverse of  $A$ . □

# Invertible matrices and their product

## Theorem 6

*For any  $A, B \in \mathcal{M}_{n \times n}$ , if  $AB$  is invertible, then  $A$  and  $B$  are also invertible.*

### Proof.

Let  $x_0$  be any solution of  $Bx = \mathbf{0}$ , then

$$(AB)x_0 = A(Bx_0) = A\mathbf{0} = \mathbf{0}.$$

By Theorem 5, since  $AB$  is invertible,  $x_0 = \mathbf{0}$ . Applying Theorem 5 again,  $B$  is invertible. We have

$$(AB)B^{-1} = A(BB^{-1}) = AI = A,$$

implying  $A$  is invertible. □

# Matrix inverse and determinants

- Elementary row operations and elementary matrices
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- **Determinants**
- Evaluating determinants by row reduction
- Properties of determinants
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## Determinant of $2 \times 2$ matrices

- We have discussed in the previous lecture that the  $2 \times 2$  matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is invertible iff  $ad - bc \neq 0$  and the expression  $ad - bc$  is the *determinant* of  $A$
- For consistency, we will denote

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- We write

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\text{or} \quad \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- The inverse of  $A$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

## Determinant of $1 \times 1$ matrices

For

$$A = (a_{11}),$$

we define

$$\det(A) = a_{11}$$

Then we can write

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |a_{11}||a_{22}| - |a_{12}||a_{21}|$$



# Minors and cofactors

## Definition

Given  $A = (a_{ij}) \in \mathcal{M}_{n \times n}$ , the *minor* of entry  $a_{ij}$ , denoted by  $M_{ij}$ , is the determinant of the submatrix obtained from  $A$  by deleting the  $i$ th row and the  $j$ th column.

$$C_{ij} := (-1)^{i+j} M_{ij}$$

is called the *cofactor* of entry  $a_{ij}$ .

## Example

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}, \quad M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16, \quad C_{11} = (-1)^2 M_{11} = 16$$

## Minors and cofactors – example

### Example

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}, \quad M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26, \quad C_{32} = (-1)^{3+2} M_{32} = -26$$

## Minors and cofactors

- $M_{ij}$  and  $C_{ij}$  are related by  $(-1)^{i+j}$
- $(-1)^{i+j}$  is either  $+1$  or  $-1$  in accordance with the pattern in the “checkerboard” array

$$\begin{pmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- For example

$$C_{11} = M_{11}, \quad C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

## Minors and cofactors - $2 \times 2$ matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Minors and cofactors

$$C_{11} = M_{11} = a_{22}, \quad C_{12} = -M_{12} = -a_{21}$$

$$C_{21} = -M_{21} = -a_{12}, \quad C_{22} = M_{22} = a_{11}$$

We can verify that

$$\begin{aligned} \det(A) = a_{11}a_{22} - a_{12}a_{21} &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22} \end{aligned}$$

# Cofactor expansion

## Theorem

*Given  $A = (a_{ij}) \in \mathcal{M}_{n \times n}$ , regardless of which row or column of  $A$  is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same. For all  $1 \leq i_1, i_2, j_1, j_2 \leq n$ ,*

$$\sum_{k=1}^n a_{i_1 k} C_{i_1 k} = \sum_{k=1}^n a_{i_2 k} C_{i_2 k} = \sum_{k=1}^n a_{k j_1} C_{k j_1} = \sum_{k=1}^n a_{k j_2} C_{k j_2}$$

# Determinant

## Definition

Given  $A = (a_{ij}) \in \mathcal{M}_{n \times n}$ , the *determinant* of  $A$ , denoted  $\det(A)$  is given by

$$\det(A) = \sum_{k=1}^n a_{kj} C_{kj} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

for some  $1 \leq j \leq n$  (cofactor expansion along the  $j$ th column). Or equivalently,

$$\det(A) = \sum_{k=1}^n a_{ik} C_{ik} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

for some  $1 \leq i \leq n$  (cofactor expansion along the  $i$ th row).

## Determinant – example

### Example

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Cofactor expansion along the first row

$$\det(A) = 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} = 3 \times (-4) - 1 \times (-11) + 0 = -1$$

Cofactor expansion along the first column

$$\det(A) = 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} = 3 \times (-4) - (-2) \times (-2) + 5 \times 3 = -1$$

## Determinant – example

### Example

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

A smart choice of row/column - more zeros

Cofactor expansion along the second column

$$\det(A) = 1 \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 1 \times (-2) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -2(1 + 2) = -6$$



# Determinant of a lower triangular matrix

## Example

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \\ = a_{11} a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix} \\ = a_{11} a_{22} a_{33} a_{44}$$

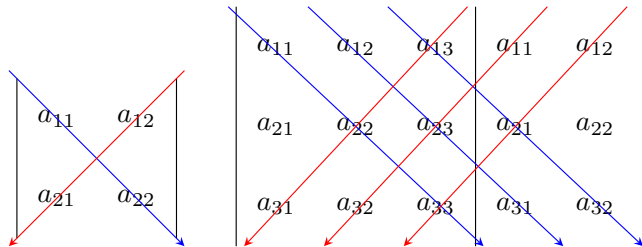
# Determinant of a triangular matrix

## Theorem

*If  $A \in \mathcal{M}_{n \times n}$  is a triangular matrix (upper triangular, lower triangular, diagonal), then  $\det(A)$  is the product of the entries on the main diagonal of  $A$*

$$\det(A) = a_{11}a_{22} \cdots a_{nn}.$$

## Determinants of $2 \times 2$ and $3 \times 3$ matrices



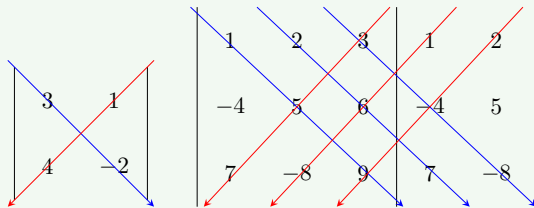
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

### Sarrus' rule

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{aligned}$$

## Determinants of $2 \times 2$ and $3 \times 3$ matrices – example

### Example



$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = 3 \times (-2) - 1 \times 4 = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = (45 + 84 + 96) - (105 - 48 - 72) = 240$$

# Matrix inverse and determinants

- Elementary row operations and elementary matrices
- Compute matrix inverse
- Linear systems and invertible matrices
- Determinants
- Evaluating determinants by row reduction
- Properties of determinants
- Cramer's rule

## Special case

### Theorem

*For any  $A \in \mathcal{M}_{n \times n}$ , if  $A$  has a row (or column) of zeros, then  $\det(A) = 0$ .*

### Proof.

Let  $C_1, C_2, \dots, C_n$  denote the cofactors of  $A$  along the row (or column) of zeros, we have

$$\det(A) = 0C_1 + 0C_2 + \cdots + 0C_n = 0.$$



# Determinant of transpose

## Theorem

For any  $A \in \mathcal{M}_{n \times n}$ ,  $\det(A) = \det(A^T)$

## Proof.

Since transposing a matrix changes its columns to rows and its rows to columns, the cofactor expansion of  $A$  along any row is the same as the cofactor expansion of  $A^T$  along the corresponding column. □

## Row operations and determinants

$$A \xrightarrow{R_1 \rightarrow \beta R_1} B$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The two matrices only differ in the first row, consider cofactor expansion along the first row, the cofactors  $C_{11}, C_{12}, C_{13}$  are the same

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det(B) = \beta a_{11}C_{11} + \beta a_{12}C_{12} + \beta a_{13}C_{13} = \beta(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) = \beta \det(A)$$



## Row operations and determinants

$$A \xrightarrow{R_1 \leftrightarrow R_2} B$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}, \quad \det(B) = a_{21}a_{12} - a_{11}a_{22}$$

$$-\det(A) = \det(B)$$

For  $2 \times 2$  matrices, swapping rows changes the sign of the determinant

## Row operations and determinants

$$A \xrightarrow{R_1 \leftrightarrow R_2} B$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

We have just observed that for  $2 \times 2$  matrices, swapping rows changes the sign of the determinant. Hence

$$\det(B) = a_{31}(-C_{31}) + a_{32}(-C_{32}) + a_{33}(-C_{33}) = -\det(A)$$

Similar arguments hold for  $R_1 \leftrightarrow R_3$ ,  $R_2 \leftrightarrow R_3$

## Row operations and determinants

Suppose for  $(n - 1) \times (n - 1)$  matrix  $A$ , swapping two rows changes the sign of the determinant. Consider  $n \times n$  matrix  $A \xrightarrow{R_{i_1} \leftrightarrow R_{i_2}} B$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1 1} & a_{i_1 2} & \cdots & a_{i_1 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_2 1} & a_{i_2 2} & \cdots & a_{i_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{31} & a_{32} & \cdots & a_{3n} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_2 1} & a_{i_2 2} & \cdots & a_{i_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1 1} & a_{i_1 2} & \cdots & a_{i_1 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{31} & a_{32} & \cdots & a_{3n} \end{pmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$\det(B) = a_{11}(-C_{11}) + a_{12}(-C_{12}) + \cdots + a_{1n}(-C_{1n}) = -\det(A)$$

## Remark

### Note

If  $A$  has two identical rows  $R_{i_1}, R_{i_2}$

$$A \xrightarrow{R_{i_1} \leftrightarrow R_{i_2}} B$$

Then

$$\det(B) = \det(A) = -\det(B) \implies \det(A) = 0$$

## Row operations and determinants

$$A \xrightarrow{R_1 \rightarrow R_1 + \beta R_2} B$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} + \beta a_{21} & a_{12} + \beta a_{22} & a_{13} + \beta a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{aligned} \det(B) &= (a_{11} + \beta a_{21})C_{11} + (a_{12} + \beta a_{22})C_{12} + (a_{13} + \beta a_{23})C_{13} \\ &= (a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) + \beta(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}) \\ &= \det(A) + \beta \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \det(A) + 0 = \det(A) \end{aligned}$$

## Row operations and determinants

Operation	Relationship
$A \xrightarrow{R_1 \rightarrow \beta R_1} B$	$\begin{vmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \beta \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \det(B) = \beta \det(A)$
$A \xrightarrow{R_1 \leftrightarrow R_2} B$	$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \det(B) = -\det(A)$
$A \xrightarrow{R_1 \rightarrow R_1 + \beta R_2} B$	$\begin{vmatrix} a_{11} + \beta a_{21} & a_{12} + \beta a_{22} & a_{13} + \beta a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \det(B) = \det(A)$

# Row/Column operations and determinants

## Theorem

Let  $A \in \mathcal{M}_{n \times n}$ .

- If  $B$  is the matrix that results when a single row or single column of  $A$  is multiplied by a scalar  $\beta$ , then  $\det(B) = \beta \det(A)$
- If  $B$  is the matrix that results when two rows or two columns of  $A$  are interchanged, then  $\det(B) = -\det(A)$
- If  $B$  is the matrix that results when a multiple of one row of  $A$  is added to another or when a multiple of one column is added to another, then  $\det(B) = \det(A)$

# Determinants of elementary matrices

## Theorem 7

Let  $E \in \mathcal{M}_{n \times n}$  be an elementary matrix

- If  $E$  results from multiplying a row of  $I_n$  by a nonzero scalar  $\beta$ , then  $\det(E) = \beta$
- If  $E$  results from interchanging two rows of  $I_n$ , then  $\det(E) = -1$
- If  $E$  results from adding a multiple of one row of  $I_n$  to another, then  $\det(E) = 1$

## Example

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3, \quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$
$$I_4 \xrightarrow{R_2 \rightarrow 3R_2}, \quad I_4 \xrightarrow{R_1 \leftrightarrow R_4}, \quad I_4 \xrightarrow{R_1 \rightarrow 7R_4 + R_1}$$



# Determinants of elementary matrices

## Theorem

Let  $E \in \mathcal{M}_{n \times n}$  be an elementary matrix

- If  $E$  results from multiplying a row of  $I_n$  by a nonzero scalar  $\beta$ , then  $\det(E) = \beta$
- If  $E$  results from interchanging two rows of  $I_n$ , then  $\det(E) = -1$
- If  $E$  results from adding a multiple of one row of  $I_n$  to another, then  $\det(E) = 1$

## Remark

$\det(E) \neq 0$

## Row operations and determinant

- Reduce the given matrix to upper triangular form by elementary row operations
- Compute the determinant of the upper triangular matrix (an easy computation)
- Relate that determinant to that of the original matrix

## Row operations and determinant – example

### Example

$$A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$$

$$\begin{aligned} \det(A) &\stackrel{R_1 \leftrightarrow R_2}{=} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \stackrel{R_1 \rightarrow \frac{1}{3}R_1}{=} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \stackrel{R_3 \rightarrow -2R_1 + R_3}{=} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \\ &\stackrel{R_3 \rightarrow -10R_2 + R_3}{=} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = (-3) \times (-55) = 165 \end{aligned}$$

## Column operations and determinant – example

### Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{pmatrix}$$

Add  $-3\times$  the first column to the fourth

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{vmatrix} = -546$$

## Row operations and cofactor expansion – example

### Example

$$A = \begin{pmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{pmatrix}$$

By adding suitable multiples of the second row to the remaining rows

$$\begin{aligned} \det(A) &= \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \text{ cofactor expansion } \underline{\text{first column}} - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \\ &\quad \underline{R_3 \rightarrow R_1 + R_3} - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \text{ cofactor expansion } \underline{\text{first column}} - (-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18 \end{aligned}$$

# Matrix inverse and determinants

- Elementary row operations and elementary matrices
- Compute matrix inverse
- Linear systems and invertible matrices
- Determinants
- Evaluating determinants by row reduction
- **Properties of determinants**
- Cramer's rule

## Determinant of $\beta A$

Let  $A \in \mathcal{M}_{n \times n}$ . Since a common factor of any row of a matrix can be moved through the determinant sign, and since each of the  $n$  rows in  $\beta A$  has a common factor of  $\beta$ , we have

$$\det(\beta A) = \beta^n \det(A)$$

### Example

$$\begin{aligned} \begin{vmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{vmatrix} &= \beta \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{vmatrix} = \beta^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{vmatrix} \\ &= \beta^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

## Determinant of $A + B$

Example ( $\det(A + B) \neq \det(A) + \det(B)$ )

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad A + B = \begin{pmatrix} 4 & 3 \\ 3 & 8 \end{pmatrix}$$

We have

$$\det(A) = 1, \quad \det(B) = 8, \quad \det(A + B) = 23$$



## Adding just one row

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{aligned} \det(A) + \det(B) &= (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21}) \\ &= a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix} \end{aligned}$$

## Adding just one row

### Theorem

*Suppose  $A, B, C \in \mathcal{M}_{n \times n}$  differ only in a single row, say the  $r$ th row, and assume that the  $r$ th row of  $C$  can be obtained by adding the corresponding rows of  $A$  and  $B$ . Then*

$$\det(C) = \det(A) + \det(B)$$

*The same result holds for columns*

### Example

$$\begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{vmatrix}$$

### Remark

A useful trick for computing determinants

# Determinant and multiplication by an elementary matrix

## Lemma 2

Given  $B, E \in \mathcal{M}_{n \times n}$ , where  $E$  is an elementary matrix, then

$$\det(EB) = \det(E) \det(B).$$

### Proof.

- If  $E$  results from multiplying a row of  $I_n$  by  $\beta$ , then by Theorem 1,  $EB$  results from  $B$  by multiplying the corresponding row by  $\beta$  and we have

$$\det(EB) = \beta \det(B)$$

From Theorem 7,  $\det(E) = \beta$

- Similar arguments hold if  $E$  results from interchanging two rows of  $I_n$  or from adding a multiple of one row to another of  $I_n$



## Determinant and multiplication by an elementary matrix

- By repeated application of Lemma 2, if  $A \in \mathcal{M}_{n \times n}$ , and  $E_1, E_2, \dots, E_r \in \mathcal{M}_{n \times n}$  are elementary matrices, then

$$\det(E_1 E_2 \cdots E_r A) = \det(E_1) \det(E_2) \cdots \det(E_r) \det(A)$$

# Invertible matrix and determinant

## Theorem

*A square matrix  $A$  is invertible iff  $\det(A) \neq 0$ .*

## Proof.

Let  $R$  be the reduced row echelon form of  $A$ . Let  $E_1, E_2, \dots, E_r$  be the elementary matrices that correspond to the elementary row operations that produce  $R$  from  $A$ . Then

$$R = E_r \cdots E_2 E_1 A.$$

And from what we just discussed

$$\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A) \implies \det(R) = 0 \text{ iff } \det(A) = 0$$

# Invertible matrix and determinant

## Theorem

A square matrix  $A$  is invertible iff  $\det(A) \neq 0$ .

## Proof.

Let  $R$  be the reduced row echelon form of  $A$ .

$$\det(R) = 0 \iff \det(A) = 0$$

If  $A$  is invertible, by Theorem 3,  $R = I$ , which implies that  $\det(A) \neq 0$ .

If  $\det(A) \neq 0$ , then  $\det(R) \neq 0$ , if  $R$  has one row of zeros,  $\det(R) = 0$ , thus  $R = I$ .

Apply Theorem 3 again we can conclude  $A$  is invertible. □

## Recall – theorem from last lecture

If  $R$  is the reduced row echelon form of a matrix  $A \in \mathcal{M}_{n \times n}$ , then either  $R$  has at least one row of zeros or  $R = I_n$ .

# Equivalent statements

## Theorem

*For any  $A \in \mathcal{M}_{n \times n}$ , the following statements are equivalent.*

- (a)  $A$  is invertible*
- (b)  $Ax = \mathbf{0}$  has only the trivial solution*
- (c) The reduced row echelon form of  $A$  is  $I_n$*
- (d)  $A$  is expressible as a product of elementary matrices*
- (e)  $Ax = \mathbf{b}$  is consistent  $\forall \mathbf{b} \in \mathbb{R}^n$*
- (f)  $Ax = \mathbf{b}$  has exactly one solution  $\forall \mathbf{b} \in \mathbb{R}^n$*
- (g)  $\det(A) \neq 0$*

## Determinant test for invertibility

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix}$$

$R_3 = 2R_1$ , row operation  $A \xrightarrow{R_3 \rightarrow R_3 - 2R_1}$  contains one row of zeros, hence  $\det(A) = 0$  and  $A$  is not invertible



# Determinant of matrix product

## Theorem

For any  $A, B \in \mathcal{M}_{n \times n}$ ,  $\det(AB) = \det(A) \det(B)$ .

## Proof.

- If  $A$  is not invertible, by Theorem 6,  $AB$  is not invertible,  $\det(AB) = \det(A) \det(B) = 0$ .
- If  $A$  is invertible, by Theorem 5,

$$A = E_1 E_2 \cdots E_r \implies AB = E_1 E_2 \cdots E_r B$$

Then

$$\det(AB) = \det(E_1) \det(E_2) \cdots \det(E_r) \det(B) = \det(A) \det(B)$$



## Determinant of matrix product – example

### Example

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 3 \\ 5 & 8 \end{pmatrix}, \quad AB = \begin{pmatrix} 2 & 17 \\ 3 & 14 \end{pmatrix}$$

We have

$$\det(A) = 1, \quad \det(B) = -23, \quad \det(AB) = -23$$

# Determinant of inverse

## Theorem

*If  $A$  is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

*Proof.*

$$AA^{-1} = I \implies \det(A) \det(A^{-1}) = \det(I) = 1$$

Since  $A$  is invertible,  $\det(A) \neq 0$ .



# Matrix inverse and determinants

- Elementary row operations and elementary matrices
- Compute matrix inverse
- Linear systems and invertible matrices
- Determinants
- Evaluating determinants by row reduction
- Properties of determinants
- Cramer's rule

# Cofactors

- In a cofactor expansion we compute determinant by multiplying the entries in a row or column by their cofactors and adding the resulting products.
- Multiply the entries in any row by the corresponding cofactors from a different row, the sum of these products is always zero - note that this corresponds to computing the determinant of a matrix with two identical rows
- This result also holds for columns

## Cofactors

### Example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

We have

$$\begin{aligned} C_{11} &= 12 & C_{12} &= 6 & C_{13} &= -16 & C_{21} &= 4 & C_{22} &= 2 & C_{23} &= 16 \\ C_{31} &= 12 & C_{32} &= -10 & C_{33} &= 16 \end{aligned}$$

Cofactor expansion of  $\det(A)$  along the first row is

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$$

and along the first column is

$$\det(A) = 3C_{11} + C_{21} + 2C_{31} = 36 + 4 + 24 = 64$$

## Cofactors

### Example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{aligned} C_{11} &= 12 & C_{12} &= 6 & C_{13} &= -16 & C_{21} &= 4 & C_{22} &= 2 & C_{23} &= 16 \\ C_{31} &= 12 & C_{32} &= -10 & C_{33} &= 16 \end{aligned}$$

Multiply the entries in the first row by the corresponding cofactors from the *second row* and add the resulting products

$$3C_{21} + 2C_{22} + (-1)C_{23} = 12 + 4 - 16 = 0$$

Note that

$$3C_{21} + 2C_{22} + (-1)C_{23} = \begin{vmatrix} 3 & 2 & -1 \\ 3 & 2 & -1 \\ 2 & -4 & 0 \end{vmatrix} = 0$$

## Cofactors

### Example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{aligned} C_{11} &= 12 & C_{12} &= 6 & C_{13} &= -16 & C_{21} &= 4 & C_{22} &= 2 & C_{23} &= 16 \\ C_{31} &= 12 & C_{32} &= -10 & C_{33} &= 16 \end{aligned}$$

Multiply the entries in the first column by the corresponding cofactors from the *second column* and add the resulting products

$$3C_{12} + 1C_{22} + 2C_{32} = 18 + 2 - 20 = 0$$

Note that

$$3C_{12} + 1C_{22} + 2C_{32} = \begin{vmatrix} 3 & 3 & -1 \\ 1 & 1 & 3 \\ 2 & 2 & 0 \end{vmatrix} = 0$$



# Adjugate

## Definition

Let  $A = (a_{ij}) \in \mathcal{M}_{n \times n}$ . Let  $C_{ij}$  be the cofactor of  $a_{ij}$ , then the matrix

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

is called the *matrix of cofactors from  $A$* . The transpose of this matrix is called the *adjugate of  $A$*  and is denoted by  $\text{adj}(A)$ .

## Adjugate – example

### Example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{array}{llll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 & C_{21} = 4 \quad C_{22} = 2 \quad C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 & \end{array}$$

The matrix of cofactors is

$$\begin{pmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}$$

and the adjugate of  $A$  is

$$\text{adj}(A) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

# Inverse of a matrix using adjugate

## Theorem

If  $A$  is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

## Proof.

We show first that

$$A \operatorname{adj}(A) = \det(A)I.$$

$$A \operatorname{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{pmatrix}$$

## Inverse of a matrix using adjugate

Proof.

$i, j$ -entry of  $A \operatorname{adj}(A)$  is  $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$

- If  $i = j$ , then the above is the cofactor expansion of  $\det(A)$  along the  $i$ th row of  $A$
- If  $i \neq j$ , it is 0

$$A \operatorname{adj}(A) = \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{pmatrix} = \det(A)I$$

Since  $A$  is invertible,  $\det(A) \neq 0$ , multiply by  $\frac{1}{\det(A)}A^{-1}$  on the left

$$\frac{1}{\det(A)} \operatorname{adj}(A) = A^{-1}$$



## Inverse of a matrix using adjugate – example

### Example

With the same example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{array}{llll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 & C_{21} = 4 \quad C_{22} = 2 \quad C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 & \end{array}$$

Cofactor expansion along the first row

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$$

The inverse is given by

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{64} \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

# Cramer's rule

## Theorem (Cramer's Rule)

Given  $Ax = b$ , a system of  $n$  linear equations in  $n$  unknowns. If  $\det(A) \neq 0$ , then the system has a unique solution given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where  $A_j$  is the matrix obtained by replacing the entries in the  $j$ th column of  $A$  by  $b$ .

## Proof.

Since  $A_j$  differs from  $A$  only in the  $j$ th column, the cofactors of entries  $b_1, b_2, \dots, b_n$  in  $A_j$  are the same as the cofactors of the corresponding entries in the  $j$ th column of  $A$

$$\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \dots + b_n C_{nj}$$

## Cramer's rule

Proof.

$$\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$$

We know that the unique solution is given by

$$\mathbf{x} = A^{-1} \mathbf{b} = \frac{1}{\det(A)} \operatorname{adj}(A) \mathbf{b} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Then

$$x_j = \frac{b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}}{\det(A)} = \frac{\det(A_j)}{\det(A)}$$



## Cramer's rule – example

### Example

$$\begin{aligned}x_1 + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8\end{aligned}$$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix}, A_1 = \begin{pmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{pmatrix}$$

The solution is given by

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$