

Algebra and Discrete Mathematics (ADM)

Tutorial 6 Matrix operators

Lecturer: Bc. Xiaolu Hou, PhD.
xiaolu.hou@stuba.sk

Matrix transformations

- Consider $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with standard matrix

$$[T] = \begin{pmatrix} -1 & -1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$$

- Find the image of $\mathbf{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$T(\mathbf{x}) = \begin{pmatrix} -1 & -1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 14 \\ 7 \end{pmatrix}$$

Matrix transformations

- Consider $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with standard matrix

$$[T] = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

- Find the image of $\mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

$$T(\mathbf{x}) = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ 16 \end{pmatrix}$$

Reflection operators on \mathbb{R}^2

- Reflection about the x -axis, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection about the y -axis, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection about the line $y = x$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Reflection about the line $y = \sqrt{3}x$
 - $y = \sqrt{3}x$ makes an angle $\pi/3$ ($= 60^\circ$) with positive x -axis

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Reflection operators on \mathbb{R}^3

- Reflection about the xy -plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- Reflection about the xz -plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Reflection about the yz -plane

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Projection operators on \mathbb{R}^2

- Orthogonal projection onto the x -axis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- Orthogonal projection onto the y -axis

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Projection operators on \mathbb{R}^3

- Orthogonal projection onto the xy -plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Orthogonal projection onto the xz -plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Orthogonal projection onto the yz -plane

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation operators on \mathbb{R}^2

- Moves points *counterclockwise* about the origin through a positive angle θ
- *Rotation matrix*

$$R_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- Clockwise about the origin through an angle θ

$$R_{-\theta} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

Rotation operators on \mathbb{R}^3

Operator	Rotation equations	Standard matrix
Counterclockwise rotation about the positive x -axis through an angle θ	$\begin{aligned} w_1 &= x \\ w_2 &= y \cos \theta - z \sin \theta \\ w_3 &= y \sin \theta + z \cos \theta \end{aligned}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$
Counterclockwise rotation about the positive y -axis through an angle θ	$\begin{aligned} w_1 &= x \cos \theta + z \sin \theta \\ w_2 &= y \\ w_3 &= -x \sin \theta + z \cos \theta \end{aligned}$	$\begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$
Counterclockwise rotation about the positive z -axis through an angle θ	$\begin{aligned} w_1 &= x \cos \theta - y \sin \theta \\ w_2 &= x \sin \theta + y \cos \theta \\ w_3 &= z \end{aligned}$	$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Dilations and contractions

- Dilation/contraction with factor α on \mathbb{R}^2 , $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

- Dilation/contraction with factor α on \mathbb{R}^3 , $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

Expansions and compressions on \mathbb{R}^2

- In the x -direction – $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ y \end{pmatrix}$
$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

- In the y -direction – $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \alpha y \end{pmatrix}$
$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

Shears on \mathbb{R}^2

- Shear in the x -direction by a factor α , $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y \end{pmatrix}$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

- Shear in the y -direction by a factor α , $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \alpha x \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

Composition of matrix transformations

- Consider a square $ABCD$ with vertices

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- Perform the following transformations
 - T_1 : shear in the x -direction by a factor 2
 - T_2 : reflection about the x -axis
 - T_3 : reflection about the y -axis

$$[T_1] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad [T_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [T_3] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[T_3][T_2][T_1] \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} -3 & -5 & -9 & -7 \\ -1 & -1 & -3 & -3 \end{pmatrix}$$

Midterm

- Room 1.37 (same as for lecture and tutorial)
- Written
- Time: next Friday (14th Nov), 9 : 30 – 11 : 30am
- 50 marks
- 8 questions
- Covers: Lectures 1 – 4, Tutorials 1 – 5
- Write your answers on the provided answer sheets. Additional sheets will be supplied upon request. Please ensure that your full name is clearly written on each page of the answer sheets.
- Include detailed computation steps for all solutions. Answers without supporting calculations will receive a score of zero.