Algebra and Discrete Mathematics
ADM

Bc. Xiaolu Hou, PhD.

FIIT, STU
xiaolu.hou @ stuba.sk

Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces and matrix transformations
- Fundamental spaces and decompositions
- Eulerian tours
- Hamiltonian cycles
- Midterm
- Paths and spanning trees
- Trees and networks
- Matching
- Tutorial 12

Recommended reading

- Saoub, K. R. (2017). A tour through graph theory. Chapman and Hall/CRC.
  - Sections 5.4, 5.5
  - Free copy online

Lecture outline

- Matching and vertex cover
- Distinct representatives
- Matchings in Non-Bipartite
- Gale-Shapley Algorithm

Trees and networks

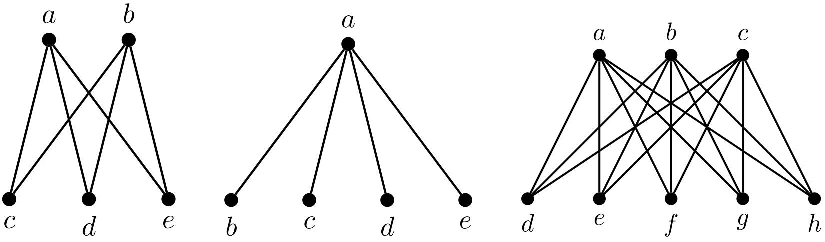
- Matching and vertex cover
- Distinct representatives
- Matchings in Non-Bipartite
- Gale-Shapley Algorithm

Question 1

Question

Draw the complete bipartite graphs $K_{2,3}$, $K_{1,4}$, and $K_{3,5}$.

Solution.



Question 2

Question

- Three student organizations (Student Government, Math Club, and the Equestrian Club) are holding meetings on Thursday afternoon.
- The only available rooms are 105, 201, 271, and 372.
- Based on membership and room size, the Student Government can only use 201 or 372, Equestrian Club can use 105 or 372, and Math Club can use any of the four rooms.
- Find a maximum matching for this scenario.

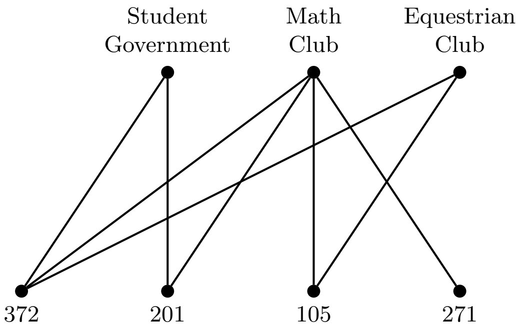
Solution. Each organization and room is represented by a vertex, and an edge denotes when an organization is able to use a room

Question 2

Question

- Student Government can only use 201 or 372, Equestrian Club can use 105 or 372, and Math Club can use any of the four rooms.

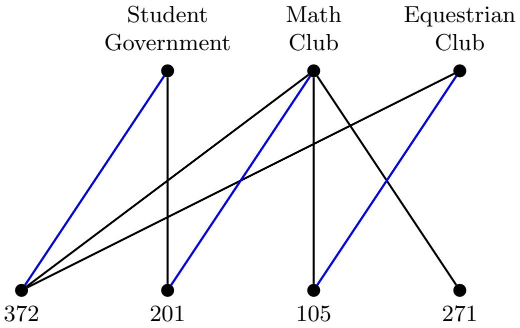
Solution. Each organization and room is represented by a vertex, and an edge denotes when an organization is able to use a room



Question 2

Solution.

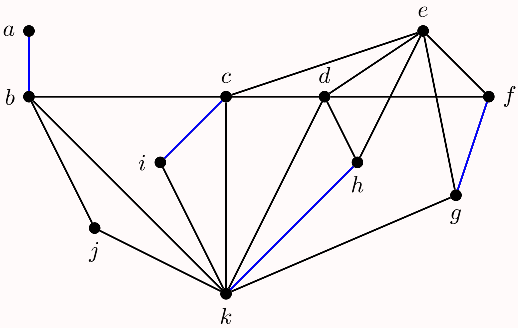
- A matching has at most 3 edges
- Maximum matching



Question 3

Question

Below is a graph with a matching M shown in blue.



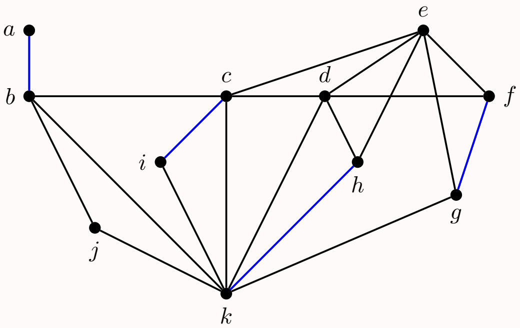
1. Find an alternating path starting at a. Is this path augmenting?

Solution.

1. $abcikhe$, augmenting: both endpoints of the path are unsaturated by M, it is not augmenting

Question 3

Question



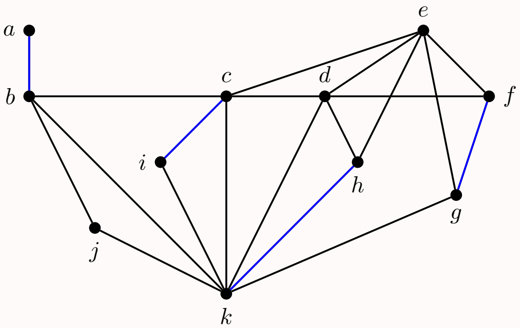
2. Find an augmenting path in the graph or explain why none exists.

Solution.

2. $jkhe$, $jkhd$

Question 3

Question



3. Is M a maximum matching? maximal matching? perfect matching?

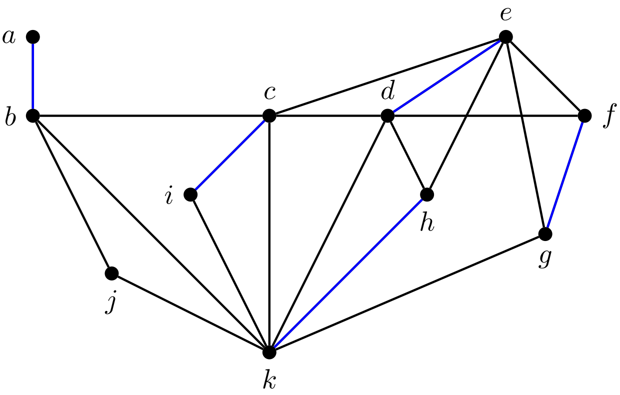
Solution.

- M is not maximum, since an augmenting path exists (Berge's Theorem)
- It is not maximal, because we can add an edge ed
- Not perfect because it does not saturate all vertices

Question 3

Solution.

- There are 11 vertices, maximum possible size of a matching is 5

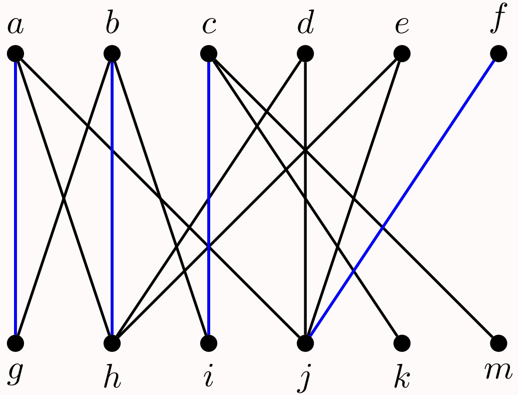


Question 4 - 1

Question

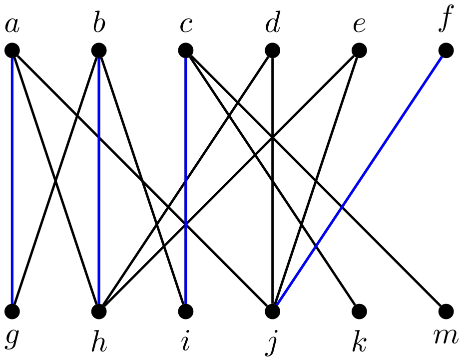
(iii) Use the Augmenting Path Algorithm to find a maximum matching.

(iv) Use the Vertex Cover Method to find a minimum vertex cover.



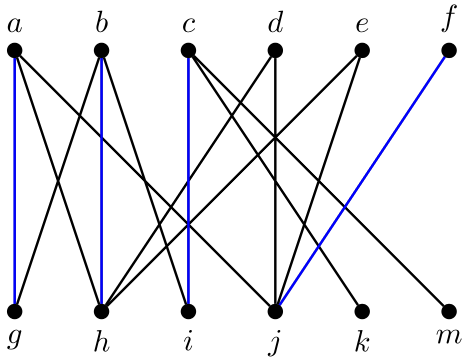
Question 4 - 1

- Step 2. $U = \{d, e\}$
- Step 3. select $d \in U$
- Step 4. neighbors of d: h, j
- Step 6. $dhbg$, $dhbicm$ - augmenting



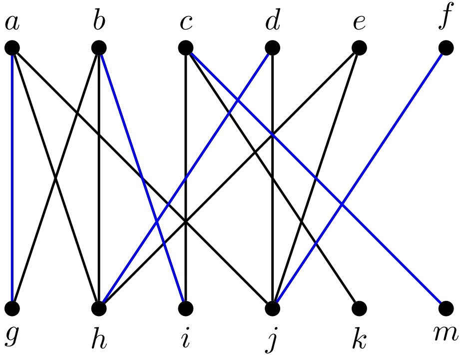
Question 4 - 1

- Step 2. $U = \{d, e\}$
- Step 3. select $d \in U$
- Step 4. neighbors of d: h, j
- Step 6. $dhbg$, $dhbicm$ - augmenting, switch edges, remove bh, ci, add dh, bi, cm



Question 4 - 1

- Step 2. $U = \{e\}$
- Step 3. select $e \in U$
- Step 4. neighbors of e: h, j
- Step 6. $ehdjf$, ejf
- Mark vertices: e, h, j, d, f
- Unmarked vertices in X: a, b, c
- Marked vertices in Y: h, j
- Minimum vertex cover: a, b, c, h, j

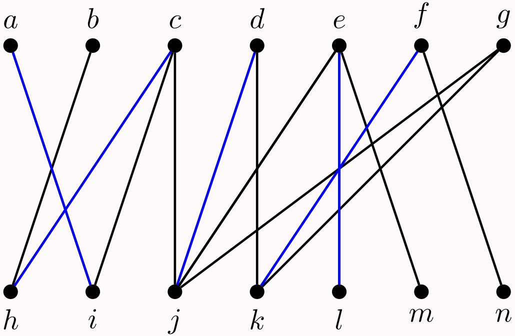


Question 4 - 2

Question

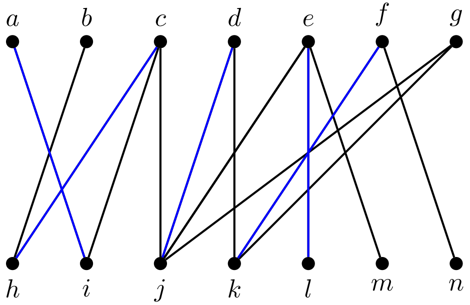
(iii) Use the Augmenting Path Algorithm to find a maximum matching.

(iv) Use the Vertex Cover Method to find a minimum vertex cover.



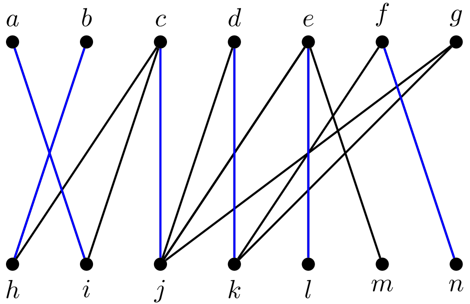
Question 4 - 2

- Step 2. $U = \{b, g\}$
- Step 3. select $b \in U$
- Step 4. neighbor of b: h
- Step 6. $bhcjdkfn$ - augmenting, remove edges ch, jd, kf, add edges bh, cj, dk, fn



Question 4 - 2

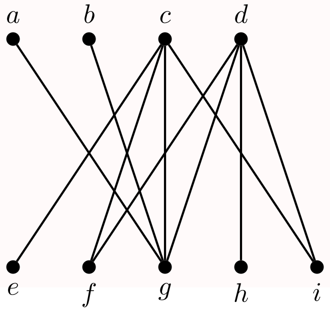
- Step 2. $U = \{g\}$
- Step 3. select $g \in U$
- Step 4. neighbors of g: j, k
- Step 6. $gjchb$, $gjcia$, $gkdjcia$, $gkdjchb$ – not augmenting
- Mark vertices: $g, j, k, c, h, b, i, a, d$
- Unmarked vertices in X: e, f
- Marked vertices in Y: h, i, j, k
- Minimum vertex cover: e, f, h, i, j, k



Questions 5 and 6 – 1

Question

- Find a maximum matching
- Is the graph bipartite?
- If yes, find a minimum vertex cover



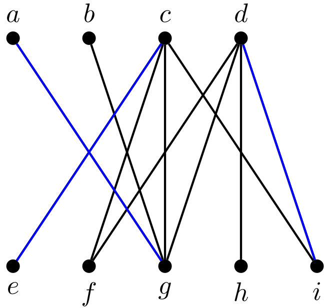
Solution.

- Since a, b are both only adjacent to g, the size of a matching is at most 3.
- A possible maximum matching: ag, ce, di
- The graph is bipartite
- Now, given the maximum matching, we can apply Augmenting Path Algorithm starting from step 2 to find a minimum vertex cover

Questions 5 and 6 – 1

Solution.

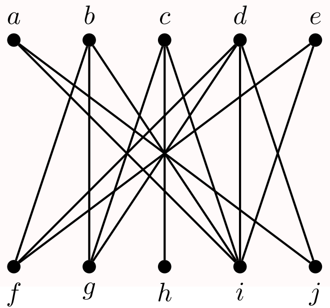
- A possible maximum matching: ag, ce, di
- $U = \{b\}$, neighbor of b is g
- Alternating path: bga
- Marked vertices: b, g, a
- Vertex cover: c, d, g



Questions 5 and 6 – 2

Question

- Find a maximum matching
- Is the graph bipartite?
- If yes, find a minimum vertex cover



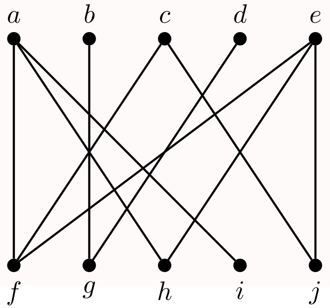
Solution.

- There are five vertices in each partition. Maximum possible size of a match is five.
- A possible maximum matching: bf, dg, aj, ch, ei
- The graph is bipartite, and we have a perfect matching
- Vertex cover: a, b, c, d, e

Questions 5 and 6 – 3

Question

- Find a maximum matching
- Is the graph bipartite?
- If yes, find a minimum vertex cover



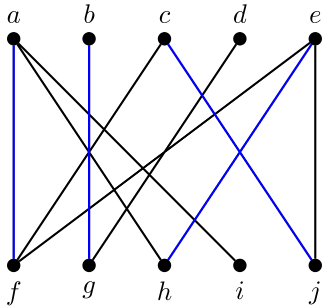
Solution.

- Both b and d are only adjacent to g, maximum possible size of a match is four.
- A possible maximum matching: bg, af, cj, eh
- The graph is bipartite
- Now, given the maximum matching, we can apply Augmenting Path Algorithm starting from step 2 to find a minimum vertex cover

Questions 5 and 6 – 3

Solution.

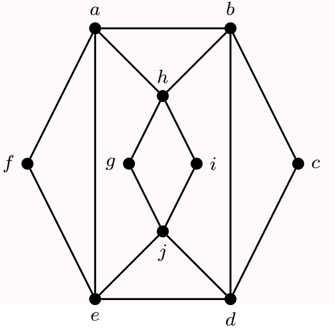
- A possible maximum matching: bg, af, cj, eh
- $U = \{d\}$
- Only augmenting path to consider is dgb
- Vertex cover is a, c, e, g



Questions 5 and 6 – 4

Question

- Find a maximum matching
- Is the graph bipartite?
- If yes, find a minimum vertex cover



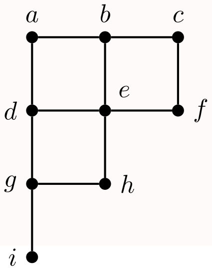
Solution.

- There are in total 10 vertices, maximum possible size of a match is five
- ab, fe, cd, hi, gj
- Not bipartite, odd cycle $afea$

Questions 5 and 6 – 5

Question

- Find a maximum matching
- Is the graph bipartite?
- If yes, find a minimum vertex cover

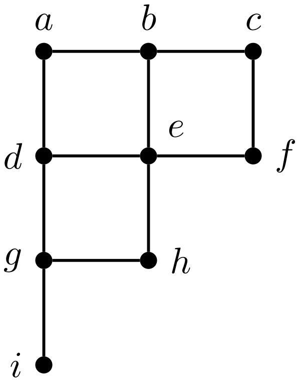


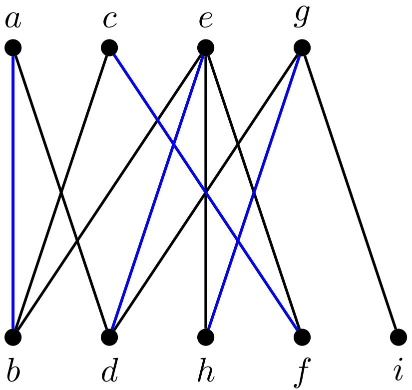
Solution.

- There are in total 9 vertices, maximum possible size of a match is four
- ab, cf, ed, gh
- Bipartite

Questions 5 and 6 – 5

Solution.



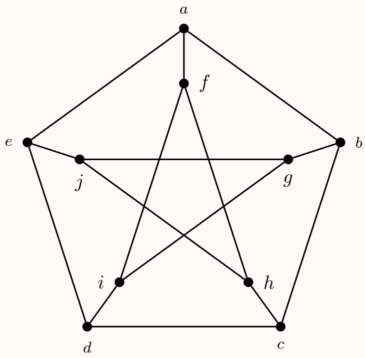


- ab, cf, ed, gh
- Take the lower vertices as X
- $U = \{i\}$, alternating path $ighedabcf \Longrightarrow$ vertex cover: a, c, e, g

Questions 5 and 6 – 6

Question

- Find a maximum matching
- Is the graph bipartite?
- If yes, find a minimum vertex cover



Solution.

- There are in total 10 vertices, maximum possible size of a match is five
- af, bg, hc, di, ej
- Not bipartite, odd cycle $aejgba$

Question 10

Question

Assign two courses to each professor

Professor	Preferred Courses		
Dave	Abstract Algebra	Real Analysis	Number Theory
	Calculus II	Calculus I	Statistics
Roland	Vector Calculus	Discrete Math	Statistics
	Calculus II	Geometry	Calculus I
Chris	Vector Calculus	Real Analysis	Discrete Math
	Statistics	Geometry	Calculus I
Adam	Statistics	Calculus I	Number Theory
	Geometry	Differential Equations	
Hannah	Abstract Algebra	Real Analysis	Number Theory
	Linear Algebra	Topology	
Maggie	Abstract Algebra	Real Analysis	Linear Algebra
	Geometry	Topology	Calculus II

Question 10

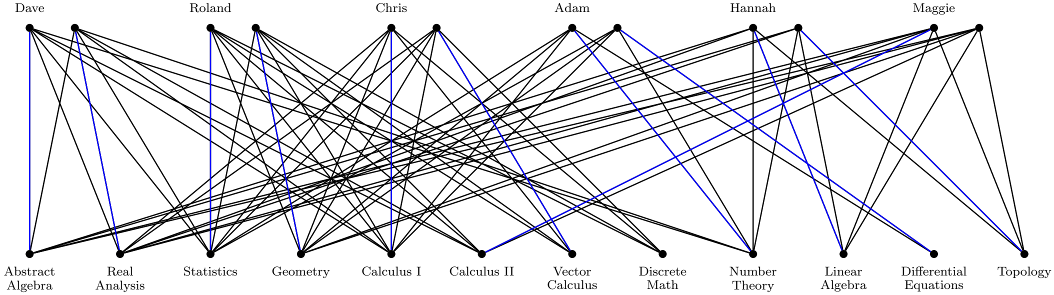
Solution. To model this as a perfect matching problem, we use the following approach:

- Vertices in X represent professor-copies for each professor, create two vertices
- Vertices in Y represent courses
- Draw and edge between a $x \in X$ and $y \in Y$ if y is on the preference list of x

There are six professors, hence 12 vertices in X, and 12 courses, hence 12 vertices in Y. The solution to the problem is a perfect matching for the bipartite graph.

Question 10

- One possible solution: we can start by matching the first vertex available for each course



- Discrete Math and one copy of Maggie are unsaturated
- Find a professor who can teach Discrete Math and currently matched to a course Maggie can teach
- Roland - Geometry

Trees and networks

- Matching and vertex cover
- Distinct representatives
- Matchings in Non-Bipartite
- Gale-Shapley Algorithm

Distinct representatives

Definition

Given a collection of finite nonempty sets $S_1, S_2, \ldots, S_n$ (where $n \geq 1$), a system of *distinct representatives* is a collection $r_1, r_2, \ldots, r_n$ such that

$$r_i \in S_i, \quad r_i \neq r_j \text{ if } i \neq j$$

for all $i, j = 1, 2, \ldots, n$.

- In less technical terms, the idea of distinct representatives is that a collection of groups each need their own representative and no two groups can have the same representative

Distinct representatives – example

Example

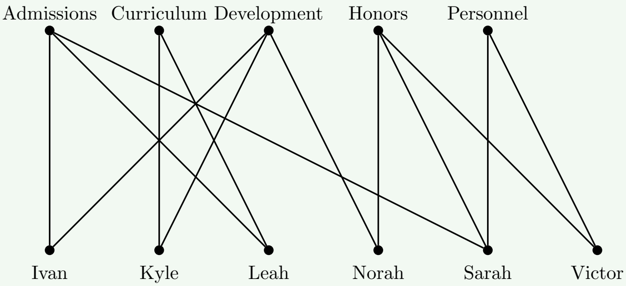
Committee	Members		
Admissions Council	Ivan	Leah	Sarah
Curriculum Committee	Kyle	Leah	
Development and Grants	Ivan	Kyle	Norah
Honors Program Council	Norah	Sarah	Victor
Personnel Committee	Sarah	Victor	

- Bipartite graph: X consists of the committees and Y the members
- Edges: a person is a member of that committee
- A collection of distinct representatives is a matching

Distinct representatives – example

Example

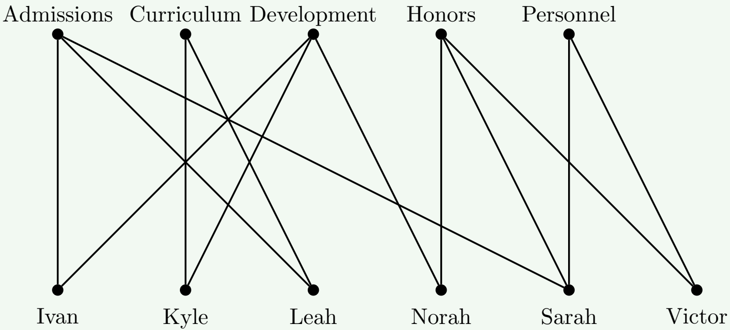
Committee	Members		
Admissions Council	Ivan	Leah	Sarah
Curriculum Committee	Kyle	Leah	
Development and Grants	Ivan	Kyle	Norah
Honors Program Council	Norah	Sarah	Victor
Personnel Committee	Sarah	Victor	



Distinct representatives – example

Example

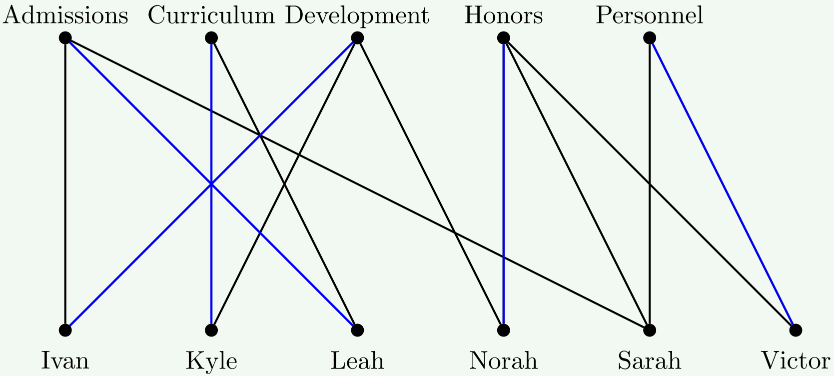
- A collection of distinct representatives is modeled as a matching
- Se are interested in each committee having a representative, not every person being a representative
- Thus we want an $X-$matching



Distinct representatives – example

Example

- A possible matching



Trees and networks

- Matching and vertex cover
- Distinct representatives
- Matchings in Non-Bipartite
- Gale-Shapley Algorithm

Canoe example

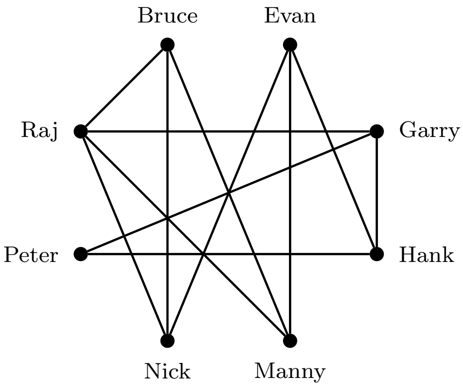
- Eight men, Y indicates a possible pair to share a canoe

	Bruce	Evan	Garry	Hank	Manny	Nick	Peter	Raj
Bruce	.	.	.	.	Y	Y	.	Y
Evan	.	.	.	Y	Y	Y	.	.
Garry	.	.	.	Y	.	.	Y	Y
Hank	.	Y	Y	.	.	.	Y	.
Manny	Y	Y	.	.	.	.	.	Y
Nick	Y	Y	.	.	.	.	.	Y
Peter	.	.	Y	Y	.	.	.	.
Raj	Y	.	.	Y	Y	Y	.	.

- We can model this information as a graph
- Perfect matching: solution to share canoes

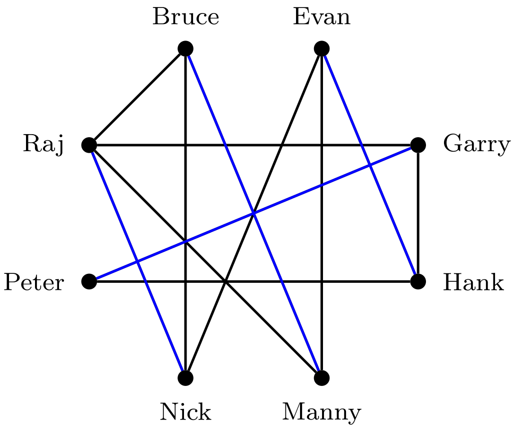
Canoe example

- Peter can only be paired with either Garry or Hank
- Choose to pair Peter and Garry, then Hank must be paired with Evan
- Nick and Manny: each be paired with one of Raj and Bruce



Canoe example

- One possible matching



Matching in general graphs

- Finding matchings in general graphs is often more complex than in bipartite graphs
- Berge' s Theorem holds (M is maximum if and only if G has no M-augmenting paths)
- Augmenting Path Algorithm only applies to bipartite graphs

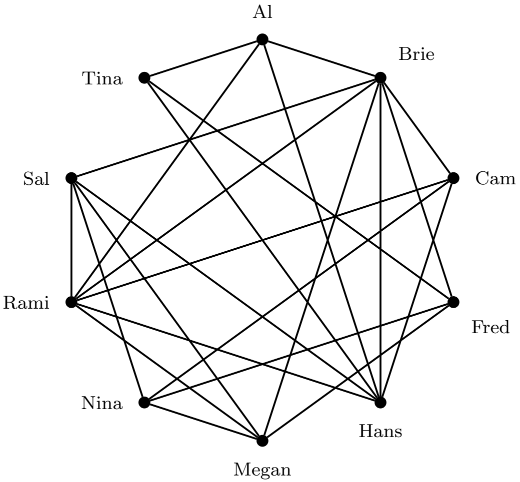
Question 11

Question

	Al	Brie	Cam	Fred	Hans	Megan	Nina	Rami	Sal	Tina
Al	.	Y	.	.	Y	.	.	Y	.	Y
Brie	Y	.	Y	Y	Y	Y	.	Y	Y	.
Cam	.	Y	.	.	Y	.	Y	Y	.	.
Fred	.	Y	.	.	.	Y	Y	.	.	Y
Hans	Y	Y	Y	.	.	.	.	Y	Y	Y
Megan	.	Y	.	Y	.	.	Y	Y	Y	.
Nina	.	.	Y	Y	.	Y	.	.	Y	.
Rami	Y	Y	Y	.	Y	Y	.	.	Y	.
Sal	.	Y	.	.	Y	Y	Y	Y	.	.
Tina	Y	.	.	Y	Y	.	.	.	.	.

Question 11

- We start by matching each vertex to the first vertex on the right
- Tina-Al, Brie-Cam, Fred-Mega, Hans-Rami, Nina-Sal



Trees and networks

- Matching and vertex cover
- Distinct representatives
- Matchings in Non-Bipartite
- Gale-Shapley Algorithm

Gale-Shapley Algorithm

- Named after David Gale and Lloyd Shapley, the two American mathematicians and economists who published this algorithm
- In addition, their work led to further studies on economic markets, one of which awarded Shapley (along with his collaborator Alvin Roth) the 2012 Nobel Prize in Economics
- Input: preference rankings of n women and n men
- Output: stable matching

Gale-Shapley Algorithm – steps

1. Each man proposes to the highest ranking woman on his list
2. If every woman receives only one proposal, this matching is stable. Otherwise move to step 3
3. Each woman
   i. accepts a proposal if it is from the man she prefers above all other currently available men and rejects the rest (if any); or
   ii. delays with a maybe to the highest ranked proposal and rejects the rest (if any)
4. Each man now proposes to the highest ranking unmatched woman on his list who has not rejected him
5. Repeat steps 2-4 until all people have been paired

Question 12 – men proposing

Question

Alice:	r	>	s	>	t	>	v		Rich:	a	>	d	>	b	>	c
Beth:	s	>	r	>	v	>	t		Stefan:	a	>	c	>	d	>	b
Cindy:	v	>	t	>	r	>	s		Tom:	c	>	b	>	d	>	a
Dahlia:	t	>	v	>	s	>	r		Victor:	c	>	d	>	b	>	a

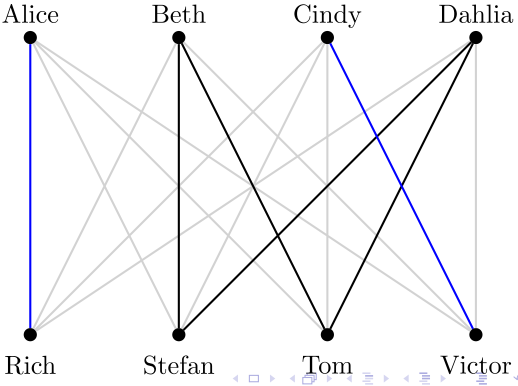
- Step 1. Rich-Alice, Stefan-Alice, Tom-Cindy, Victor-Cindy
- Step 3. Alice accepts Rich and reject Stefan; Cindy accepts Victor and rejects Tom

Question 12 – men proposing

Alice:	r	>	s	>	t	>	v
Beth:	s	>	r	>	v	>	t
Cindy:	v	>	t	>	r	>	s
Dahlia:	t	>	v	>	s	>	r

Rich:	a	>	d	>	b	>	c
Stefan:	a	>	c	>	d	>	b
Tom:	c	>	b	>	d	>	a
Victor:	c	>	d	>	b	>	a

- Step 4. Stefan - Dahlia, Tom - Beth
- Step 2. both proposals are different, both women accept the proposals
- Stable matching: Rich-Alice, Victor-Cindy, Stefan - Dahlia, Tom - Beth



Question 12 – women proposing

Question

Alice:	$r > s > t > v$	Rich:	$a > d > b > c$
Beth:	$s > r > v > t$	Stefan:	$a > c > d > b$
Cindy:	$v > t > r > s$	Tom:	$c > b > d > a$
Dahlia:	$t > v > s > r$	Victor:	$c > d > b > a$

- Step 1. Alice-Rich, Beth-Stefan, Cindy-Victor, Dahlia-Tom
- Step 2. All proposals are different, all men accept the proposals

Egalitarian cost

- The *pairwise egalitarian cost* of a man and a woman is the sum of the rankings they give each other
- The *egalitarian cost* of a stable matching is the sum of all the pairwise egalitarian costs of the married couples in the matching
- For comparison of matchings

Question 12 – egalitarian cost

Alice:	r > s > t > v	Rich:	a > d > b > c
Beth:	s > r > v > t	Stefan:	a > c > d > b
Cindy:	v > t > r > s	Tom:	c > b > d > a
Dahlia:	t > v > s > r	Victor:	c > d > b > a

- Men proposing: Rich-Alice, Victor-Cindy, Stefan-Dahlia, Tom-Beth

$$(1+1)+(1+1)+(3+3)+(2+4) = 2+2+6+6 = 16$$

- Women proposing: Alice-Rich, Beth-Stefan, Cindy-Victor, Dahlia-Tom

$$(1+1)+(1+4)+(1+1)+(1+3) = 2+5+2+4 = 13$$

Question 13 – men proposing

Question

Edith:	$l > n > o > m > p$	Liam:	$f > e > h > g > i$
Faye:	$n > l > m > o > p$	Malik:	$e > i > g > f > h$
Grace:	$p > m > o > n > l$	Nate:	$f > g > i > h > e$
Hanna:	$p > n > o > l > m$	Olaf:	$i > e > f > g > h$
Iris:	$p > o > m > n > l$	Pablo:	$f > h > g > e > i$

- Step 1. Liam-Faye, Malik-Edith, Nate-Faye, Olaf-Iris, Pablo-Faye
- Step 3. Faye accepts Nate, rejects the rest; Edith says maybe to Malik; Iris says maybe to Olaf

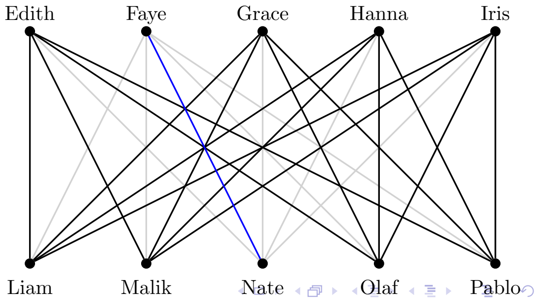
Question 13 – men proposing

Question

Edith:	$l > n > o > m > p$	
Faye:	$n > l > m > o > p$	
Grace:	$p > m > o > n > l$	
Hanna:	$p > n > o > l > m$	
Iris:	$p > o > m > n > l$	

Liam:	$f > e > h > g > i$
Malik:	$e > i > g > f > h$
Nate:	$f > g > i > h > e$
Olaf:	$i > e > f > g > h$
Pablo:	$f > h > g > e > i$

- Step 4. Liam-Edith, Malik-Edith, Olaf-Iris, Pablo-Hanna
- Step 3. Edith accepts Liam; Iris says maybe to Olaf; Hanna accepts Pablo

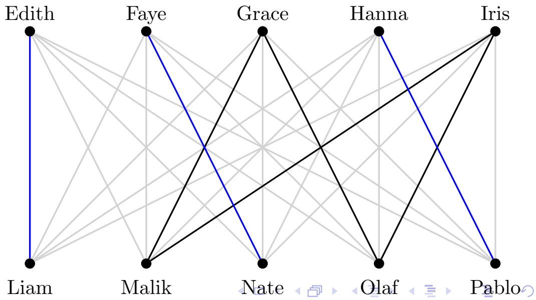


Question 13 – men proposing

Question

Edith:	l > n > o > m > p					Liam:	f > e > h > g > i
Faye:	n > l > m > o > p					Malik:	e > i > g > f > h
Grace:	p > m > o > n > l					Nate:	f > g > i > h > e
Hanna:	p > n > o > l > m					Olaf:	i > e > f > g > h
Iris:	p > o > m > n > l					Pablo:	f > h > g > e > i

- Step 4. Malik-Iris, Olaf-Iris
- Step 3. Iris accepts Olaf and rejects Malik

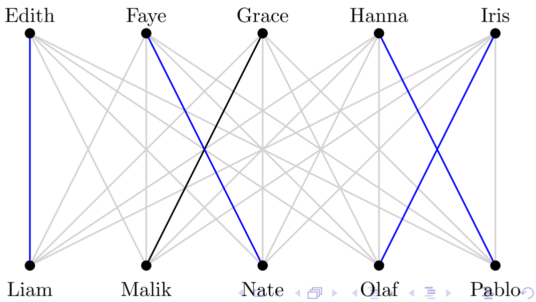


Question 13 – men proposing

Question

Edith:	l	>	n	>	o	>	m	>	p	Liam:	f	>	e	>	h	>	g	>	i
Faye:	n	>	l	>	m	>	o	>	p	Malik:	e	>	i	>	g	>	f	>	h
Grace:	p	>	m	>	o	>	n	>	l	Nate:	f	>	g	>	i	>	h	>	e
Hanna:	p	>	n	>	o	>	l	>	m	Olaf:	i	>	e	>	f	>	g	>	h
Iris:	p	>	o	>	m	>	n	>	l	Pablo:	f	>	h	>	g	>	e	>	i

- Step 4. Malik-Grace
- Step 2. Grace accepts
- Stable matching: Nate-Faye, Liam-Edith, Pablo-Hanna, Olaf-Iris, Malik-Grace



Question 13 – women proposing

Question

Edith:	l > n > o > m > p	Liam:	f > e > h > g > i
Faye:	n > l > m > o > p	Malik:	e > i > g > f > h
Grace:	p > m > o > n > l	Nate:	f > g > i > h > e
Hanna:	p > n > o > l > m	Olaf:	i > e > f > g > h
Iris:	p > o > m > n > l	Pablo:	f > h > g > e > i

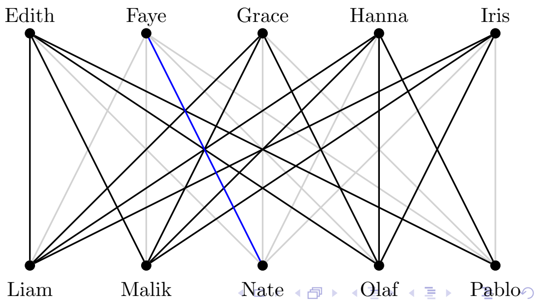
- Step 1. Edith-Liam, Faye-Nate, Grace-Pablo, Hanna-Pablo, Iris-Pablo
- Step 3. Nate accepts Faye; Liam says maybe to Edith; Pablo says maybe to Hanna, rejects Iris and Grace

Question 13 – women proposing

Question

Edith:	l > n > o > m > p	Liam:	f > e > h > g > i
Faye:	n > l > m > o > p	Malik:	e > i > g > f > h
Grace:	p > m > o > n > l	Nate:	f > g > i > h > e
Hanna:	p > n > o > l > m	Olaf:	i > e > f > g > h
Iris:	p > o > m > n > l	Pablo:	f > h > g > e > i

- Step 4. Edith-Liam, Grace-Malik, Hanna-Pablo, Iris-Olaf
- Step 2. All men accept the proposals
- Stable matching: Faye-Nate, Edith-Liam, Grace-Malik, Hanna-Pablo, Iris-Olaf



Question 13 – egalitarian cost

Edith:	l > n > o > m > p	Liam:	f > e > h > g > i
Faye:	n > l > m > o > p	Malik:	e > i > g > f > h
Grace:	p > m > o > n > l	Nate:	f > g > i > h > e
Hanna:	p > n > o > l > m	Olaf:	i > e > f > g > h
Iris:	p > o > m > n > l	Pablo:	f > h > g > e > i

- Men proposing: Nate-Faye, Liam-Edith, Olaf-Iris, Pablo-Hanna, Malik-Grace

$$(1+1)+(2+1)+(1+2)+(2+1)+(3+2) = 2+3+3+3+5 = 16$$

- Women proposing: Edith-Liam, Faye-Nate, Grace-Malik, Hanna-Pablo, Iris-Olaf

$$(1+2)+(1+1)+(2+3)+(1+2)+(2+1) = 3+2+5+3+3 = 16$$

Gale-Shapley Algorithm (with Unacceptable Partners)

- We are still looking for a matching in a bipartite graph, only now the graph is not complete
- We must adjust our notion of a stable matching, since it is possible that not all people could be matched
- Under these new conditions, a matching (with unacceptable partners) is *stable* if no unmatched pair x and y such that x and y are both acceptable to each other, and each is either single or prefers the other to their current partner
- Input: preference ranking of n women and n men
- Output: stable matching

Gale-Shapley Algorithm (with Unacceptable Partners) – steps

1. Each man proposes to the highest ranking woman on his list
2. If every woman receives only one proposal from someone they deem acceptable, they all accept and this matching is stable. Otherwise move to step 3
3. Each woman
   i. rejects a proposal if it is from an unacceptable man;
   ii. accepts if the proposal is from the man she prefers above all other currently available men and rejects the rest (if any); or
   iii. delays with a maybe to the highest ranked proposal and rejects the rest (if any)
4. Each man now proposes to the highest ranking unmatched woman on their list who has not rejected him
5. Repeat steps 2-4 until all people have been paired or until no unmatched man has any acceptable partners remaining

Remark

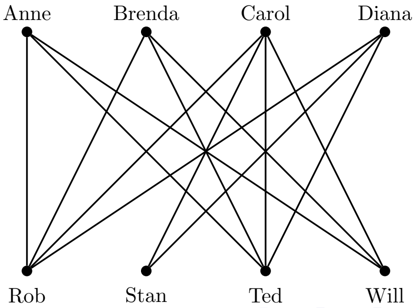
- Always produces a stable matching
- Can be modified so that the women are proposing

Question 14 – women proposing

Question

Anne:	t > r > w	Rob:	a > b > c > d
Brenda:	w > r > t	Stan:	a > b
Carol:	w > r > s > t	Ted:	c > d > a > b
Diana:	s > r > t	Will:	c > b > a

- Step 1. Anne-Ted, Brenda-Will, Carol-Will, Diana-Stan
- Step 3. Ted says maybe to Anne; Will accepts Carol and rejects Brenda; Stan rejects Diana

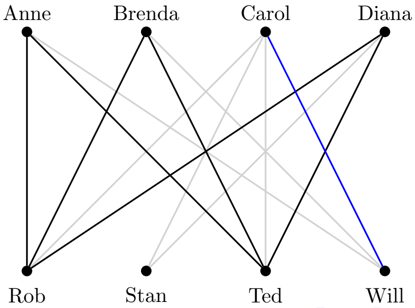


Question 14 – women proposing

Question

Anne:	$t > r > w$	Rob:	$a > b > c > d$
Brenda:	$w > r > t$	Stan:	$a > b$
Carol:	$w > r > s > t$	Ted:	$c > d > a > b$
Diana:	$s > r > t$	Will:	$c > b > a$

- Step 4. Anne-Ted, Brenda-Rob, Diana-Rob
- Step 3. Ted says maybe to Anne; Rob says maybe to Brenda and rejects Diana

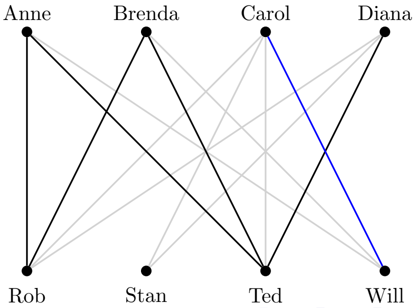


Question 14 – women proposing

Question

Anne:	t > r > w	Rob:	a > b > c > d
Brenda:	w > r > t	Stan:	a > b
Carol:	w > r > s > t	Ted:	c > d > a > b
Diana:	s > r > t	Will:	c > b > a

- Step 4. Anne-Ted, Brenda-Rob, Diana-Ted
- Step 3. Ted accepts Diana and rejects Anne; Rob says maybe to Brenda

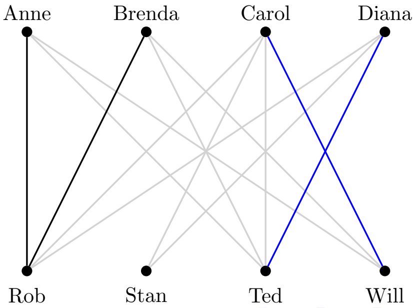


Question 14 – women proposing

Question

Anne:	t > r > w	Rob:	a > b > c > d
Brenda:	w > r > t	Stan:	a > b
Carol:	w > r > s > t	Ted:	c > d > a > b
Diana:	s > r > t	Will:	c > b > a

- Step 4. Anne-Rob, Brenda-Rob
- Step 3. Rob accepts Anne and rejects Brenda
- Stable matching: Carol-Will, Anne-Rob, Diana-Ted

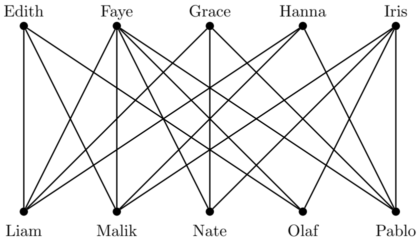


Question 15 – men proposing

Question

Edith:	$l > n > m$	Liam:	$f > e > h > g$
Faye:	$n > l > m > o > p$	Malik:	$e > h > i > f$
Grace:	$m > o > n > l$	Nate:	$g > f > i$
Hanna:	$p > o > l > m$	Olaf:	$i > e > f$
Iris:	$p > m > n > l$	Pablo:	$f > h > g > i$

- Step 1. Liam-Faye, Malik-Edith, Nate-Grace, Olaf-Iris, Pablo-Faye
- Step 3. Faye says maybe to Liam and rejects Pablo; Edith says maybe to Malik; Grace says maybe to Nate; Iris rejects Olaf

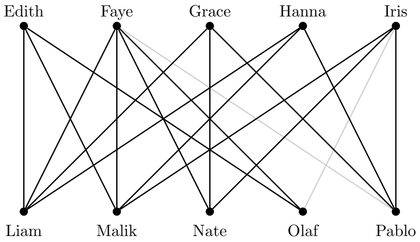


Question 15 – men proposing

Question

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Step 4. Liam-Faye, Malik-Edith, Nate-Grace, Olaf-Edith, Pablo-Hanna
- Step 3. Faye says maybe to Liam; Edith says maybe to Malik and rejects Olaf; Grace says maybe to Nate; Hanna accepts Pablo

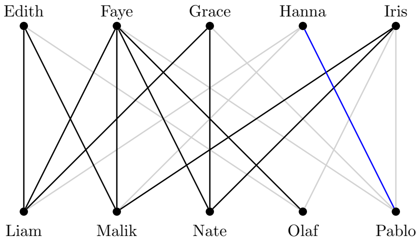


Question 15 – men proposing

Question

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Step 4. Liam-Faye, Malik-Edith, Nate-Grace, Olaf-Faye
- Step 3. Faye says maybe to Liam and rejects Olaf; Edith says maybe to Malik; Grace says maybe to Nate

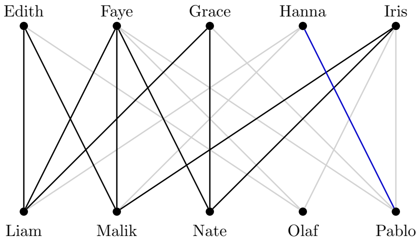


Question 15 – men proposing

Question

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Step 4. Liam-Faye, Malik-Edith, Nate-Grace
- Step 2. Every woman accetps the proposal
- Stable matching: Pablo-Hanna, Liam-Faye, Malik-Edith, Nate-Grace

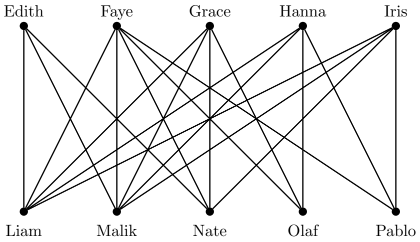


Question 15 – women proposing

Question

Edith: l > n > m	Liam: f > e > h > g
Faye: n > l > m > o > p	Malik: e > h > i > f
Grace: m > o > n > l	Nate: g > f > i
Hanna: p > o > l > m	Olaf: i > e > f
Iris: p > m > n > l	Pablo: f > h > g > i

- Step 1. Edith-Liam, Faye-Nate, Grace-Malik, Hanna-Pablo, Iris-Pablo
- Step 3. Liam says maybe to Edith; Nate says maybe to Faye; Malik rejects Grace; Pablo says maybe to Hanna and rejects Iris

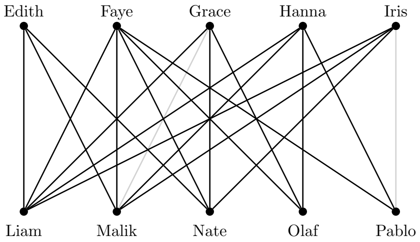


Question 15 – women proposing

Question

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Step 4. Edith-Liam, Faye-Nate, Grace-Olaf, Hanna-Pablo, Iris-Malik
- Step 3. Liam says maybe to Edith; Nate says maybe to Faye; Olaf rejects Grace; Pablo says maybe to Hanna; Malik says maybe to Iris

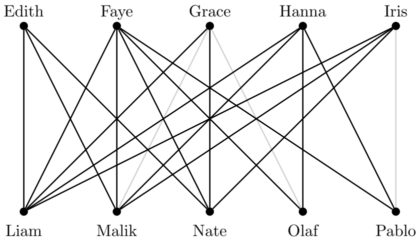


Question 15 – women proposing

Question

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Step 4. Edith-Liam, Faye-Nate, Grace-Olaf, Hanna-Pablo, Iris-Malik
- Step 3. Liam says maybe to Edith; Nate says maybe to Faye; Olaf rejects Grace; Pablo says maybe to Hanna; Malik says maybe to Iris

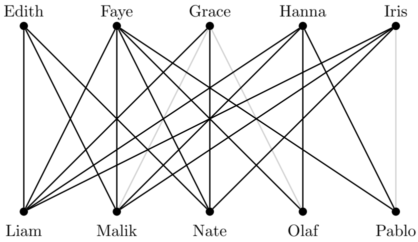


Question 15 – women proposing

Question

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Step 4. Edith-Liam, Faye-Nate, Grace-Nate, Hanna-Pablo, Iris-Malik
- Step 3. Liam says maybe to Edith; Nate accepts Grace and rejects Faye; Pablo says maybe to Hanna; Malik says maybe to Iris

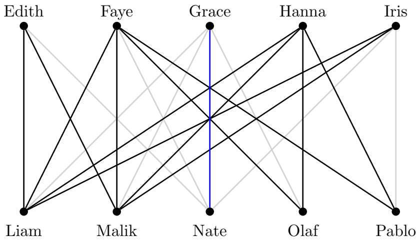


Question 15 – women proposing

Question

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Step 4. Edith-Liam, Faye-Liam, Hanna-Pablo, Iris-Malik
- Step 3. Liam accepts Faye and rejects Edith; Pablo says maybe to Hanna; Malik says maybe to Iris

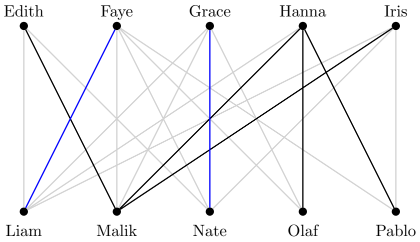


Question 15 – women proposing

Question

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Step 4. Edith-Malik, Hanna-Pablo, Iris-Malik
- Step 3. Malik accepts Edith and rejects Iris; Pablo accepts Hanna
- Stable matching: Faye-Liam, Grace-Nate, Edith-Malik, Hanna-Pablo



Question 15 – egalitarian cost

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

- Men proposing: Pablo-Hanna, Liam-Faye, Malik-Edith, Nate-Grace

$$(2+1)+(1+2)+(1+3)+(1+3)=3+3+4+4=14$$

- Women proposing: Faye-Liam, Grace-Nate, Edith-Malik, Hanna-Pablo

$$(2+1)+(3+1)+(3+1)+(1+2)=3+4+4+3=14$$

Question 16 – men proposing

Question

Alice:	$p > r > s > t$	Peter:	$b > a > c > d > e$
Beth:	$r > p > s > t$	Rich:	$c > b > e > d > a$
Carol:	$t > p > s > r$	Saul:	$a > b > c > d > e$
Diana:	$t > s > r > p$	Teddy:	$e > c > d > a > b$
Edith:	$r > s > t > p$		

- Step 1. Peter-Beth, Rich-Carol, Saul-Alice, Teddy-Edith
- Step 2. All proposals are different and acceptable, we have a stable matching

Question 16 – women proposing

Question

Alice:	$p > r > s > t$	Peter:	$b > a > c > d > e$
Beth:	$r > p > s > t$	Rich:	$c > b > e > d > a$
Carol:	$t > p > s > r$	Saul:	$a > b > c > d > e$
Diana:	$t > s > r > p$	Teddy:	$e > c > d > a > b$
Edith:	$r > s > t > p$		

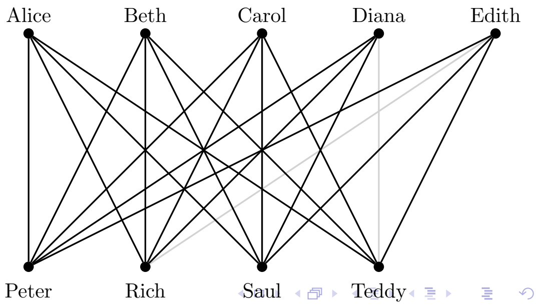
- Step 1. Alice-Peter, Beth-Rich, Carol-Teddy, Diana-Teddy, Edith-Rich
- Step 3. Peter says maybe to Alice; Rich says maybe to Beth and rejects Edith; Teddy says maybe to Carol and rejects Diana

Question 16 – women proposing

Question

Alice:	p > r > s > t		Peter:	b > a > c > d > e
Beth:	r > p > s > t		Rich:	c > b > e > d > a
Carol:	t > p > s > r		Saul:	a > b > c > d > e
Diana:	t > s > r > p		Teddy:	e > c > d > a > b
Edith:	r > s > t > p			

- Step 4. Alice-Peter, Beth-Rich, Carol-Teddy, Diana-Saul, Edith-Saul
- Step 3. Peter says maybe to Alice; Rich says maybe to Beth; Teddy says maybe to Carol; Saul says maybe to Diana and rejects Edith

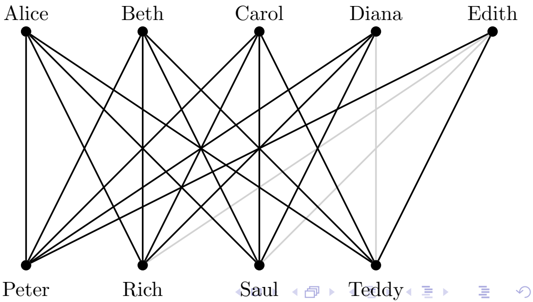


Question 16 – women proposing

Question

Alice:	p	>	r	>	s	>	t				Peter:	b	>	a	>	c	>	d	>	e
Beth:	r	>	p	>	s	>	t				Rich:	c	>	b	>	e	>	d	>	a
Carol:	t	>	p	>	s	>	r				Saul:	a	>	b	>	c	>	d	>	e
Diana:	t	>	s	>	r	>	p				Teddy:	e	>	c	>	d	>	a	>	b
Edith:	r	>	s	>	t	>	p													

- Step 4. Alice-Peter, Beth-Rich, Carol-Teddy, Diana-Saul, Edith-Teddy
- Step 3. Peter says maybe to Alice; Rich says maybe to Beth; Teddy accepts Edith and rejects Carol; Saul says maybe to Diana

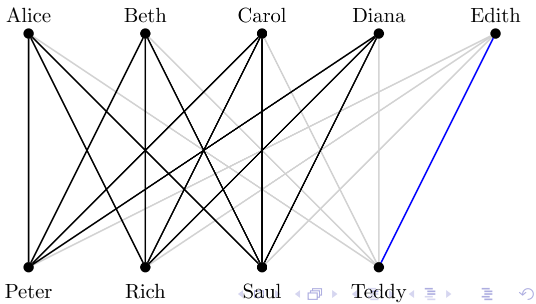


Question 16 – women proposing

Question

Alice:	p	>	r	>	s	>	t			Peter:	b	>	a	>	c	>	d	>	e
Beth:	r	>	p	>	s	>	t			Rich:	c	>	b	>	e	>	d	>	a
Carol:	t	>	p	>	s	>	r			Saul:	a	>	b	>	c	>	d	>	e
Diana:	t	>	s	>	r	>	p			Teddy:	e	>	c	>	d	>	a	>	b
Edith:	r	>	s	>	t	>	p												

- Step 4. Alice-Peter, Beth-Rich, Carol-Peter, Diana-Saul
- Step 3. Peter says maybe to Alice and rejects Carol; Rich says maybe to Beth; Saul says maybe to Diana

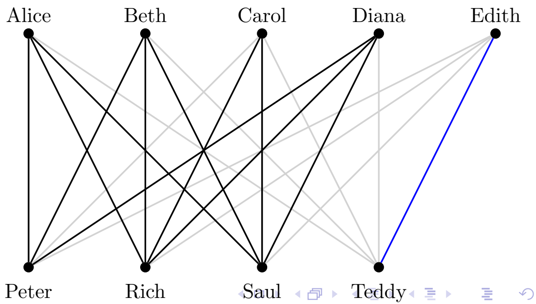


Question 16 – women proposing

Question

Alice:	p > r > s > t		Peter:	b > a > c > d > e
Beth:	r > p > s > t		Rich:	c > b > e > d > a
Carol:	t > p > s > r		Saul:	a > b > c > d > e
Diana:	t > s > r > p		Teddy:	e > c > d > a > b
Edith:	r > s > t > p			

- Step 4. Alice-Peter, Beth-Rich, Carol-Saul, Diana-Saul
- Step 3. Peter says maybe to Alice; Rich says maybe to Beth; Saul says maybe to Carol and rejects Diana



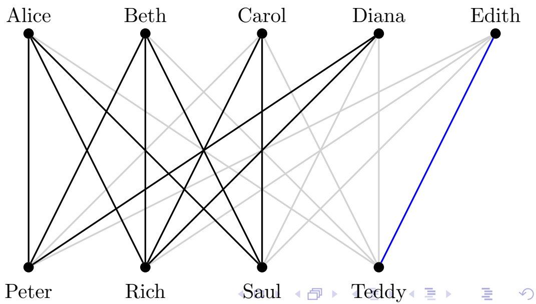
Question 16 – women proposing

Question

Alice:	p	>	r	>	s	>	t	
Beth:	r	>	p	>	s	>	t	
Carol:	t	>	p	>	s	>	r	
Diana:	t	>	s	>	r	>	p	
Edith:	r	>	s	>	t	>	p	

Peter:	b	>	a	>	c	>	d	>	e
Rich:	c	>	b	>	e	>	d	>	a
Saul:	a	>	b	>	c	>	d	>	e
Teddy:	e	>	c	>	d	>	a	>	b

- Step 4. Alice-Peter, Beth-Rich, Carol-Saul, Diana-Rich
- Step 3. Peter says maybe to Alice; Rich says maybe to Beth and rejects Diana; Saul says maybe to Carol

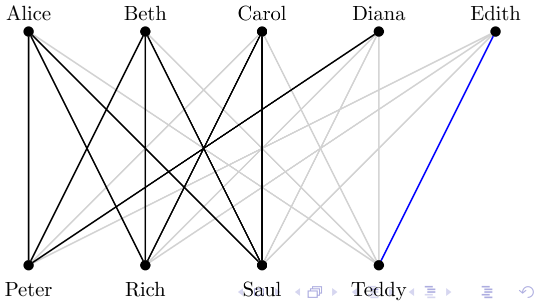


Question 16 – women proposing

Question

Alice:	p	>	r	>	s	>	t			Peter:	b	>	a	>	c	>	d	>	e
Beth:	r	>	p	>	s	>	t			Rich:	c	>	b	>	e	>	d	>	a
Carol:	t	>	p	>	s	>	r			Saul:	a	>	b	>	c	>	d	>	e
Diana:	t	>	s	>	r	>	p			Teddy:	e	>	c	>	d	>	a	>	b
Edith:	r	>	s	>	t	>	p												

- Step 4. Alice-Peter, Beth-Rich, Carol-Saul, Diana-Peter
- Step 3. Peter says maybe to Alice and rejects Diana; Rich says maybe to Beth; Saul says maybe to Carol

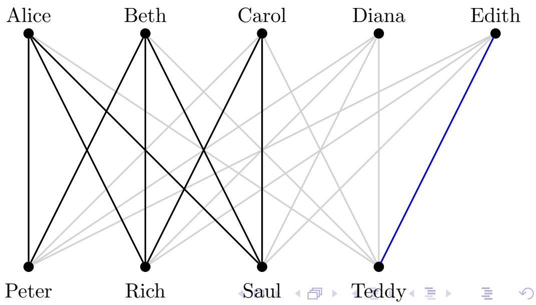


Question 16 – women proposing

Question

Alice:	p > r > s > t		Peter:	b > a > c > d > e
Beth:	r > p > s > t		Rich:	c > b > e > d > a
Carol:	t > p > s > r		Saul:	a > b > c > d > e
Diana:	t > s > r > p		Teddy:	e > c > d > a > b
Edith:	r > s > t > p			

- Step 4. Alice-Peter, Beth-Rich, Carol-Saul
- Step 2. All proposals are different
- Stable matching: Edith-Teddy, Alice-Peter, Beth-Rich, Carol-Saul



Question 16 – egalitarian cost

Alice:	p > r > s > t		Peter:	b > a > c > d > e
Beth:	r > p > s > t		Rich:	c > b > e > d > a
Carol:	t > p > s > r		Saul:	a > b > c > d > e
Diana:	t > s > r > p		Teddy:	e > c > d > a > b
Edith:	r > s > t > p			

- Men proposing: Peter-Beth, Rich-Carol, Saul-Alice, Teddy-Edith

$$(1 + 2) + (1 + 4) + (1 + 3) + (1 + 3) = 3 + 5 + 4 + 4 = 16$$

- Women proposing: Edith-Teddy, Alice-Peter, Beth-Rich, Carol-Saul

$$(3 + 1) + (1 + 2) + (1 + 2) + (3 + 3) = 4 + 3 + 3 + 6 = 16$$