Algebra and Discrete Mathematics
ADM

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Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces and matrix transformations
- Fundamental spaces and decompositions
- Eulerian tours
- Hamiltonian cycles
- Midterm
- Paths and spanning trees
- Trees and networks
- Matching

Recommended reading

- Anton, Howard, and Chris Rorres. Elementary linear algebra: applications version. John Wiley & Sons, 2013.
    - Sections 1.5, 1.6, 2.1, 2.2, 2.3
    - Accessible online (free copy)
    - Alternative download link

Lecture outline

- Elementary row operations and elementary matrices
- Compute matrix inverse
- Linear systems and invertible matrices
- Determinants
- Evaluating determinants by row reduction
- Properties of determinants
- Cramer's rule
- Proofs and principles

Matrix inverse and determinants

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Elementary row operations

Recall elementary row operations

- Multiply a row by a nonzero constant β.
- Interchange two rows.
- Add a constant β times one row to another.

If we obtain B from A by one of the operations, then A can be recovered from B by one of the following

- Multiply the same row by $1/\beta$.
- Interchange the same two rows.
- If B resulted by adding β times row i of A to row j, then add $-\beta$ times row i to row j.

If B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A

Row equivalence and elementary matrices

Definition

Matrices A and B are said to be *row equivalent* if either can be obtained from the other by a sequence of elementary row operations.

Definition

A matrix E is called an *elementary matrix* if it can be obtained from an identity matrix by performing a single elementary row operation.

Elementary matrices – example

Example

$$\begin{pmatrix}1 & 0\\ 0 & -3\end{pmatrix}, \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

The elementary operations:

- $I_2 \xrightarrow{R_2 \to -3R_2}$
- $I_4 \xrightarrow{R_2 \leftrightarrow R_4}$
- $I_3 \xrightarrow{R_1 \to R_1 + 3R_3}$
- $I_3 \xrightarrow{R_1 \to 1R_1}$

Row operations by matrix multiplication

Theorem 1

Suppose the elementary matrix E results from performing a certain row operation on I_m. Take any $A \in \mathcal{M}_{m \times n}$, then EA is the matrix that results when this same row operation is performed on A.

Example

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

Note that $I_3 \xrightarrow{R_3 \to R_3 + 3R_1} E$, and

$$EA = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{pmatrix}$$

Inverse operation

- E: an elementary matrix that results from performing an elementary row operation on an identity matrix I
- There is another elementary row operation when applied to E produces I
- We call them *inverse operation* of each other

Row operations and inverse row operations – example

We first apply an elementary row operation on I_2 to get an elementary matrix and then apply the inverse row operation to get I_2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to 7R_2} \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{7}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 5R_2} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 5R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Inverse of an elementary matrix

Theorem 2

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

- Suppose E is obtained from I by performing some row operation
- Its inverse is the elementary matrix that results when the inverse of this operation is performed on I

Example

$$E = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equivalent statements

Theorem 3

For any $A \in \mathcal{M}_{n\times n}$, the following statements are equivalent.

(a) A *is invertible*

(b) $A\boldsymbol{x} = \mathbf{0}$ *has only the trivial solution*

(c) *The reduced row echelon form of* A *is* I_n

(d) A *is expressible as a product of elementary matrices*

Example

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Matrix inverse

Lemma 1

Given $A, B \in \mathcal{M}_{n\times n}$

- *If* $BA = I$, *then* $B = A^{-1}$
- *If* $AB = I$, *then* $B = A^{-1}$

Proof.

Assume $BA = I$. If A is invertible,

$$BAA^{-1} = IA^{-1} \Longrightarrow BI = IA^{-1} \Longrightarrow B = A^{-1}.$$

By Theorem 3, to show A is invertible, it suffices to show that the system $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. Let $\boldsymbol{x}_0$ be any solution.

$$BA\boldsymbol{x}_0 = B\mathbf{0} \Longrightarrow I\boldsymbol{x}_0 = \mathbf{0} \Longrightarrow \boldsymbol{x}_0 = \mathbf{0}.$$

□

Matrix inverse and determinants

- Elementary row operations and elementary matrices
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Inverse of a matrix and reduced row echelon form

- We will develop a procedure to determine if a matrix is invertible, if yes, find the inverse
- Suppose $A \in \mathcal{M}_{n\times n}$ is invertible, then its reduced row echelon form is I_n

$$E_k \cdots E_2 E_1 A = I_n.$$

Multiply both sides on the right by A^{-1}

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

i.e. the same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}

Inversion algorithm

To find the inverse of an invertible matrix $A \in \mathcal{M}_{n\times n}$

- Find a sequence of elementary row operations that reduces A to I_n
- Perform the same sequence of operations on I_n to find A^{-1}
- We adjoin I_n to the right side of A

$$\left(A \mid I_n\right)$$

and apply elementary row operations to get

$$\left(I_n \mid A^{-1}\right)$$

Inversion algorithm – example

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to 2R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \xrightarrow{R_3 \to -R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Note

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \dots$$

Inversion algorithm – example

Example

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow[R_2 \to 3R_3 + R_2]{R_1 \to -3R_3 + R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 \to -2R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Exercise

Verity that $AA^{-1} = I$, find $E_3, E_4, \ldots$ and verity that $I = E_1 E_2 E_3 \ldots A$

Show that a matrix is not invertible – example

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \to -2R_1 + R_2]{R_3 \to R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

Since the left side has a row of zeros, A is not invertible.

Analyzing homogeneous systems – example

Example

(a)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 + 8x_3 &= 0 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

- From Theorem 3, a homogeneous linear system has only the trivial solution iff its coefficient matrix is invertible
- From the previous example, we know that (a) has only the trivial solution and (b) has nontrivial solutions

Matrix inverse and determinants

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Solution of linear systems

Theorem

A system of linear equations has either zero, one, or infinitely many solutions.

Proof.

$A\boldsymbol{x} = \boldsymbol{b}$ has either (a) no solutions, (b) has exactly one solution, or (c) more than one solution. It can be shown that if the system has more than one solution, it has infinitely many solutions. $\square$

Solving linear systems

Theorem 4

Given $A \in \mathcal{M}_{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^n$, *if* A *is invertible, then the system of equations* $A\boldsymbol{x} = \boldsymbol{b}$ *has a unique solution* $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Solution of a linear system using matrix inverse

Example

$$\begin{aligned} x_1 + 2x_3 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17 \end{aligned}$$

In matrix form, the system can be written as $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}$$

We already calculated:

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Solution of a linear system using matrix inverse

Example

$$\begin{aligned} x_1 + 2x_3 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

The solution is given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Linear systems with a common coefficient matrix

- Frequently, one is concerned with solving a sequence of systems

$$A\boldsymbol{x} = \boldsymbol{b}_1, \quad A\boldsymbol{x} = \boldsymbol{b}_2, \quad \ldots, \quad A\boldsymbol{x} = \boldsymbol{b}_k$$

  each of which has the same square coefficient matrix A.
- If A is invertible, then the solutions

$$\boldsymbol{x}_1 = A^{-1}\boldsymbol{b}_1, \quad \boldsymbol{x}_2 = A^{-1}\boldsymbol{b}_2, \quad \ldots, \quad \boldsymbol{x}_k = A^{-1}\boldsymbol{b}_k$$

  can be obtained with one matrix inversion and k matrix multiplications.

Linear systems with a common coefficient matrix – example

Example

(a)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 + 8x_3 &= 9 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 + 8x_3 &= -6 \end{aligned}$$

The two solutions are given by

$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Equivalent statements

Theorem 5

For any $A \in \mathcal{M}_{n\times n}$, the following statements are equivalent.

(a) *A is invertible*

(b) *$A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution*

(c) *The reduced row echelon form of A is I_n*

(d) *A is expressible as a product of elementary matrices*

(e) *$A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\forall \boldsymbol{b} \in \mathbb{R}^n$*

(f) *$A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution $\forall \boldsymbol{b} \in \mathbb{R}^n$*

Invertible matrices and their product

Theorem 6

For any $A, B \in \mathcal{M}_{n\times n}$, if AB is invertible, then A and B are also invertible.

Example

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \quad AB = I_2$$

Matrix inverse and determinants

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Determinant of 2×2 matrices

- We have discussed in the previous lecture that the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible iff $ad - bc \neq 0$ and the expression $ad - bc$ is the *determinant* of A
- For consistency, we will denote

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- We write

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\text{or} \qquad \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- The inverse of A

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Determinant of 1×1 matrices

For

$$A = (a_{11}),$$

we define

$$\det(A) = a_{11}$$

Then we can write

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |a_{11}||a_{22}| - |a_{12}||a_{21}|$$

Minors and cofactors

Definition

Given $A = (a_{ij}) \in \mathcal{M}_{n\times n}$, the *minor* of entry a_{ij}, denoted by M_{ij}, is the determinant of the submatrix obtained from A by deleting the ith row and the jth column.

$$C_{ij} := (-1)^{i+j} M_{ij}$$

is called the *cofactor* of entry a_{ij}.

Example

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}, \quad M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16, \quad C_{11} = (-1)^2 M_{11} = 16$$

Minors and cofactors – example

Example

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}, \quad M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26, \quad C_{32} = (-1)^{3+2} M_{32} = -26$$

Minors and cofactors

- M_{ij} and C_{ij} are related by $(-1)^{i+j}$
- $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the "checkerboard" array

$$\begin{pmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- For example

$$C_{11} = M_{11}, \quad C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

Minors and cofactors - 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Minors and cofactors

$$C_{11} = M_{11} = a_{22}, \quad C_{12} = -M_{12} = -a_{21}$$

$$C_{21} = -M_{21} = -a_{12}, \quad C_{22} = M_{22} = a_{11}$$

We can verify that

$$\begin{aligned} \det(A) = a_{11}a_{22} - a_{12}a_{21} &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22} \end{aligned}$$

Cofactor expansion

Theorem

Given $A = (a_{ij}) \in \mathcal{M}_{n\times n}$*, regardless of which row or column of* A *is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same. For all* $1 \le i_1, i_2, j_1, j_2 \le n$,

$$\sum_{k=1}^{n} a_{i_1 k} C_{i_1 k} = \sum_{k=1}^{n} a_{i_2 k} C_{i_2 k} = \sum_{k=1}^{n} a_{k j_1} C_{k j_1} = \sum_{k=1}^{n} a_{k j_2} C_{k j_2}$$

Determinant

Definition

Given $A = (a_{ij}) \in \mathcal{M}_{n\times n}$, the *determinant* of A, denoted $\det(A)$ is given by

$$\det(A) = \sum_{k=1}^{n} a_{kj}C_{kj} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for some $1 \leq i \leq n$ (cofactor expansion along the jth column). Or equivalently,

$$\det(A) = \sum_{k=1}^{n} a_{ik}C_{ik} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

for some $1 \leq j \leq n$ (cofactor expansion along the ith row).

Determinant – example

Example

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Cofactor expansion along the first row

$$\det(A) = 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} = 3 \times (-4) - 1 \times (-11) + 0 = -1$$

Cofactor expansion along the first column

$$\det(A) = 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} = 3 \times (-4) - (-2) \times (-2) + 5 \times 3 = -1$$

Determinant – example

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

A smart choice of row/column - more zeros
Cofactor expansion along the second column

$$\det(A) = 1 \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 1 \times (-2) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -2(1+2) = -6$$

Determinant of a lower traingular matrix

Example

$$
\begin{aligned}
\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \\
&= a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix} \\
&= a_{11}a_{22}a_{33}a_{44}
\end{aligned}
$$

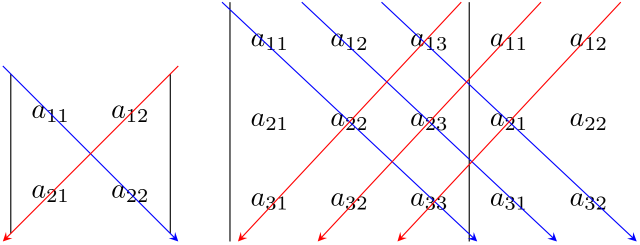
Determinant of a triangular matrix

Theorem

If $A \in \mathcal{M}_{n\times n}$ is a triangular matrix (upper triangular, lower triangular, diagonal), then $\det(A)$ is the product of the entries on the main diagonal of A

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Determinants of 2×2 and 3×3 matrices



$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Sarrus' rule

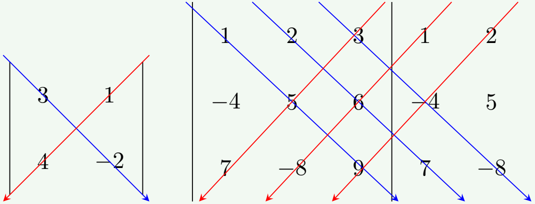
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Determinants of 2×2 and 3×3 matrices – example

Example



$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = 3 \times (-2) - 1 \times 4 = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = (45 + 84 + 96) - (105 - 48 - 72) = 240$$

Matrix inverse and determinants

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Special case

Theorem

For any $A \in \mathcal{M}_{n\times n}$, if A has a row (or column) of zeros, then $\det(A) = 0$.

Example

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\det(A) = 0C_{11} + 0C_{12} + 0C_{13} = 0$$

Determinant of transpose

Theorem

For any $A \in \mathcal{M}_{n\times n}$, $\det(A) = \det(A^\top)$

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Cofactor expansion along the second column of A

$$\det(A) = 1 \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 1 \times (-2) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -2(1+2) = -6$$

Cofactor expansion along the second row of $A^\top$, $\det(A^\top) = -6$

Row operations and determinants

$A \xrightarrow{R_1 \to \beta R_1} B$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to \begin{pmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The two matrices only differ in the first row, consider cofactor expansion along the first row, the cofactors C_{11}, C_{12}, C_{13} are the same

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det(B) = \beta a_{11}C_{11} + \beta a_{12}C_{12} + \beta a_{13}C_{13} = \beta(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) = \beta \det(A)$$

Row operations and determinants

$$A \xrightarrow{R_1 \leftrightarrow R_2} B$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}, \quad \det(B) = a_{21}a_{12} - a_{11}a_{22}$$

$$-\det(A) = \det(B)$$

For 2×2 matrices, swapping rows changes the sign of the determinant

Row operations and determinants

$A \xrightarrow{R_1 \leftrightarrow R_2} B$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

We have just observed that for 2×2 matrices, swapping rows changes the sign of the determinant. Hence

$$\det(B) = a_{31}(-C_{31}) + a_{32}(-C_{32}) + a_{33}(-C_{33}) = -\det(A)$$

Similar arguments hold for $R_1 \leftrightarrow R_3$, $R_2 \leftrightarrow R_3$

Row operations and determinants

Suppose for $(n-1)\times(n-1)$ matrix A, swapping two rows changes the sign of the determinant. Consider $n\times n$ matrix $A \xrightarrow{R_{i_1}\leftrightarrow R_{i_2}} B$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1 1} & a_{i_1 2} & \cdots & a_{i_1 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_2 1} & a_{i_2 2} & \cdots & a_{i_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{31} & a_{32} & \cdots & a_{3n} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_2 1} & a_{i_2 2} & \cdots & a_{i_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1 1} & a_{i_1 2} & \cdots & a_{i_1 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{31} & a_{32} & \cdots & a_{3n} \end{pmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$\det(B) = a_{11}(-C_{11}) + a_{12}(-C_{12}) + \cdots + a_{1n}(-C_{1n}) = -\det(A)$$

Remark

Note

If A has two identical rows R_{i_1}, R_{i_2}

$$A \xrightarrow{R_{i_1} \leftrightarrow R_{i_2}} B$$

Then

$$\det(B) = \det(A) = -\det(B) \Longrightarrow \det(A) = 0$$

Row operations and determinants

$$A \xrightarrow{R_1 \to R_1 + \beta R_2} B$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to \begin{pmatrix} a_{11} + \beta a_{21} & a_{12} + \beta a_{22} & a_{13} + \beta a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{aligned} \det(B) &= (a_{11} + \beta a_{21})C_{11} + (a_{12} + \beta a_{22})C_{12} + (a_{13} + \beta a_{23})C_{13} \\ &= (a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) + \beta(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}) \\ &= \det(A) + \beta \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \det(A) + 0 = \det(A) \end{aligned}$$

Row operations and determinants

Operation	Relationship	
$A \xrightarrow{R_1 \to \beta R_1} B$	$\begin{vmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \beta \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	$\det(B) = \beta \det(A)$
$A \xrightarrow{R_1 \leftrightarrow R_2} B$	$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	$\det(B) = -\det(A)$
$A \xrightarrow{R_1 \to R_1 + \beta R_2} B$	$\begin{vmatrix} a_{11} + \beta a_{21} & a_{12} + \beta a_{22} & a_{13} + \beta a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	$\det(B) = \det(A)$

Row/Column operations and determinants

Theorem

Let $A \in \mathcal{M}_{n\times n}$.

- *If* B *is the matrix that results when a single row or single column of* A *is multiplied by a scalar* β, *then* $\det(B) = \beta \det(A)$
- *If* B *is the matrix that results when two rows or two columns of* A *are interchanged, then* $\det(B) = -\det(A)$
- *If* B *is the matrix that results when a multiple of one row of* A *is added to another or when a multiple of one column is added to another, then* $\det(B) = \det(A)$

Determinants of elementary matrices

Theorem 7

Let $E \in \mathcal{M}_{n\times n}$ be an elementary matrix

- *If E results from multiplying a row of I_n by a nonzero scalar β, then $\det(E) = \beta$*
- *If E results from interchanging two rows of I_n, then $\det(E) = -1$*
- *If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$*

Example

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3, \quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

$$I_4 \xrightarrow{R_2 \to 3R_2}, \qquad I_4 \xrightarrow{R_1 \leftrightarrow R_4}, \qquad I_4 \xrightarrow{R_1 \to 7R_4 + R_1}$$

Determinants of elementary matrices

Theorem

Let $E \in \mathcal{M}_{n\times n}$ be an elementary matrix

- *If E results from multiplying a row of I_n by a nonzero scalar β, then $\det(E) = \beta$*
- *If E results from interchanging two rows of I_n, then $\det(E) = -1$*
- *If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$*

Remark

$\det(E) \neq 0$

Row operations and determinant

- Reduce the given matrix to upper triangular form by elementary row operations
- Compute the determinant of the upper triangular matrix (an easy computation)
- Relate that determinant to that of the original matrix

Row operations and determinant – example

Example

$$A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$$

$$\det(A) \overset{R_1 \leftrightarrow R_2}{=} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \overset{R_1 \to \frac{1}{3} R_1}{=} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \overset{R_3 \to -2R_1 + R_3}{=} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$$\overset{R_3 \to -10R_2 + R_3}{=} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = (-3) \times (-55 =)165$$

Column operations and determinant – example

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{pmatrix}$$

Add $-3\times$ the first column to the fourth

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{vmatrix} = -546$$

Row operations and cofactor expansion – example

Example

$$A = \begin{pmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{pmatrix}$$

By adding suitable multiples of the second row to the remaining rows

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \overset{\text{cofactor expasion first column}}{=} - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$\overset{R_3 \to R_1 + R_3}{=} - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \overset{\text{cofactor expasion first column}}{=} -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18$$

Matrix inverse and determinants

- Elementary row operations and elementary matrices
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Determinant of βA

Let $A \in \mathcal{M}_{n\times n}$. Since a common factor of any row of a matrix can be moved through the determinant sign, and since each of the n rows in βA has a common factor of β, we have

$$\det(\beta A) = \beta^n \det(A)$$

Example

$$\begin{aligned}
\begin{vmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{vmatrix} &= \beta \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{vmatrix} = \beta^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{vmatrix} \\
&= \beta^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Determinant of $A + B$

Example $(\det(A + B) \neq \det(A) + \det(B))$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad A + B = \begin{pmatrix} 4 & 3 \\ 3 & 8 \end{pmatrix}$$

We have

$$\det(A) = 1, \quad \det(B) = 8, \quad \det(A + B) = 23$$

Adding just one row

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{aligned} \det(A) + \det(B) &= (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21}) \\ &= a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix} \end{aligned}$$

Adding just one row

Theorem

Suppose $A, B, C \in \mathcal{M}_{n\times n}$ differ only in a single row, say the rth row, and assume that the rth row of C can be obtained by adding the corresponding rows of A and B. Then

$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns

Example

$$\begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{vmatrix}$$

Remark

A useful trick for computing determinants

Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n\times n}$, the following statements are equivalent.

(a) A *is invertible*

(b) $A\boldsymbol{x} = \mathbf{0}$ *has only the trivial solution*

(c) *The reduced row echelon form of* A *is* I_n

(d) A *is expressible as a product of elementary matrices*

(e) $A\boldsymbol{x} = \boldsymbol{b}$ *is consistent* $\forall \boldsymbol{b} \in \mathbb{R}^n$

(f) $A\boldsymbol{x} = \boldsymbol{b}$ *has exactly one solution* $\forall \boldsymbol{b} \in \mathbb{R}^n$

(g) $\det(A) \neq 0$

Determinant test for invertibility

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix}$$

$R_3 = 2R_1$, row operation $A \xrightarrow{R_3 \to R_3 - 2R_1}$ contains one row of zeros, hence $\det(A) = 0$ and A is not invertible

Determinant of matrix product

Theorem

For any $A, B \in \mathcal{M}_{n\times n}$, $\det(AB) = \det(A)\det(B)$.

Example

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 3 \\ 5 & 8 \end{pmatrix}, \quad AB = \begin{pmatrix} 2 & 17 \\ 3 & 14 \end{pmatrix}$$

We have

$$\det(A) = 1, \quad \det(B) = -23, \quad \det(AB) = -23$$

Determinant of inverse

Theorem

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof.

$$AA^{-1} = I \Longrightarrow \det(A)\det(A^{-1}) = \det(I) = 1$$

Since A is invertible, $\det(A) \neq 0$. □

Matrix inverse and determinants

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Cofactors

- In a cofactor expansion we compute determinant by multiplying the entries in a row or column by their cofactors and adding the resulting products.
- Multiply the entries in any row by the corresponding cofactors from a different row, the sum of these products is always zero - note that this corresponds to computing the determinant of a matrix with two identical rows
- This result also holds for columns

Cofactors

Example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

We have

$$\begin{array}{llllll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 & C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

Cofactor expansion of $\det(A)$ along the first row is

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$$

and along the first column is

$$\det(A) = 3C_{11} + C_{21} + 2C_{31} = 36 + 4 + 24 = 64$$

Cofactors

Example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{array}{llllll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 & C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 & & & \end{array}$$

Multiply the entries in the first row by the corresponding cofactors from the *second row* and add the resulting products

$$3C_{21} + 2C_{22} + (-1)C_{23} = 12 + 4 - 16 = 0$$

Note that

$$3C_{21} + 2C_{22} + (-1)C_{23} = \begin{vmatrix} 3 & 2 & -1 \\ 3 & 2 & -1 \\ 2 & -4 & 0 \end{vmatrix} = 0$$

Cofactors

Example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{array}{llllll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 & C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 & & & \end{array}$$

Multiply the entries in the first column by the corresponding cofactors from the *second column* and add the resulting products

$$3C_{12} + 1C_{22} + 2C_{32} = 18 + 2 - 20 = 0$$

Note that

$$3C_{12} + 1C_{22} + 2C_{32} = \begin{vmatrix} 3 & 3 & -1 \\ 1 & 1 & 3 \\ 2 & 2 & 0 \end{vmatrix} = 0$$

Adjugate

Definition

Let $A = (a_{ij}) \in \mathcal{M}_{n\times n}$. Let C_{ij} be the cofactor of a_{ij}, then the matrix

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

is called the *matrix of cofactors from* A. The transpose of this matrix is called the *adjugate of* A and is denoted by $\operatorname{adj}(A)$.

Adjugate – example

Example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{array}{llllll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 & C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

The matrix of cofactors is

$$\begin{pmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}$$

and the adjugate of A is

$$\operatorname{adj}(A) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

Inverse of a matrix using adjugate

Theorem

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Inverse of a matrix using adjugate – example

Example

With the same example

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{array}{llllll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 & C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 & & & \end{array}$$

Cofactor expansion along the first row

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$$

The inverse is given by

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{64} \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

Cramer's rule

Theorem (Cramer's Rule)

Given $A\boldsymbol{x} = \boldsymbol{b}$, a system of n linear equations in n unknowns. If $\det(A) \neq 0$, then the system has a unique solution given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by $\boldsymbol{b}$.

Cramer's rule – exmaple

Example

$$
\begin{aligned}
x_1 + 2x_3 &= 6 \\
-3x_1 + 4x_2 + 6x_3 &= 30 \\
-x_1 - 2x_2 + 3x_3 &= 8
\end{aligned}
$$

$$
A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix}, \; A_1 = \begin{pmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{pmatrix}
$$

The solution is given by

$$
x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},
$$

$$
x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}
$$

Matrix inverse and determinants

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Inverse of an elementary matrix

Theorem

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Proof.

Let E be any elementary matrix. By definition, E is obtained from I by performing some row operation. Let E_0 be the matrix that results when the inverse of this operation is performed on I. By Theorem 1 and the fact that inverse row operations cancel the effect of each other

$$E_0 E = I \quad \text{and} \quad E E_0 = I.$$

E_0 is the inverse of E. $\square$

Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n\times n}$, the following statements are equivalent.

(a) *A is invertible*

(b) *$A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution*

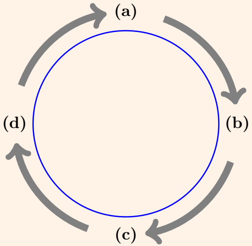
(c) *The reduced row echelon form of A is I_n*

(d) *A is expressible as a product of elementary matrices*

Proof.

We will prove

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$$



Equivalent statements – proof

(a) A is invertible

(b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution

Proof.

$(a) \Rightarrow (b)$ Suppose A is invertible and let $\boldsymbol{x}_0$ be a solution of $A\boldsymbol{x} = \mathbf{0}$. Multiplying both sides by A^{-1}

$$A^{-1}(A\boldsymbol{x}_0) = \mathbf{0} \Longrightarrow (A^{-1}A)\boldsymbol{x}_0 = \mathbf{0} \Longrightarrow I\boldsymbol{x}_0 = \mathbf{0} \Longrightarrow \boldsymbol{x}_0 = \mathbf{0}$$

Equivalent statements – proof

(b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution

(c) The reduced row echelon form of A is I_n

Proof.

$(b) \Rightarrow (c)$ Let the linear system corresponding to $A\boldsymbol{x} = \mathbf{0}$ be

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & 0 \end{array}$$

Since the linear system has only the trivial solution, the reduced row echelon form of its augmented matrix would represent

Equivalent statements – proof

(b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution

(c) The reduced row echelon form of A is I_n

Proof.

$(b) \Rightarrow (c)$ Since the linear system has only the trivial solution, the reduced row echelon form of its augmented matrix would represent

$$\begin{array}{ccccccc} x_1 & & & & = & 0 \\ & x_2 & & & = & 0 \\ & & \ddots & & = & 0 \\ & & & x_n & = & 0 \end{array} \xrightarrow{\text{augmented matrix}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Equivalent statements – proof

(c) The reduced row echelon form of A is I_n

(d) A is expressible as a product of elementary matrices

Proof.

$(c) \Rightarrow (d)$ Suppose the reduced row echelon form of A is I_n, then A can be reduced to I_n by a finite sequence of elementary row operations. By Theorem 1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus, $\exists$ $E_1, E_2, \ldots, E_k$ s.t.

$$E_k \cdots E_2 E_1 A = I_n.$$

By Theorem 2, $E_1, E_2, \ldots, E_k$ are invertible and their inverses are also elementary matrices.

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

Equivalent statements – proof

(a) A is invertible

(d) A is expressible as a product of elementary matrices

Proof.

$(d) \Rightarrow (a)$ Suppose

$$A = E_1 E_2 \cdots E_k,$$

then

$$A^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}.$$

$\square$

Solution of linear systems

Theorem

A system of linear equations has either zero, one, or infinitely many solutions.

Proof.

$A\boldsymbol{x} = \boldsymbol{b}$ has either (a) no solutions, (b) has exactly one solution, or (c) more than one solution. It suffices to show if the system has more than one solution, it has infinitely many solutions.

Assume two distinct solutions $\boldsymbol{x}_1, \boldsymbol{x}_2$, let $\boldsymbol{x}_0 = \boldsymbol{x}_1 - \boldsymbol{x}_2$. Then $\boldsymbol{x}_0 \neq \mathbf{0}$, and

$$A\boldsymbol{x}_0 = A(\boldsymbol{x}_1 - \boldsymbol{x}_2) = A\boldsymbol{x}_1 - A\boldsymbol{x}_2 = \boldsymbol{b} - \boldsymbol{b} = \mathbf{0}.$$

For any $\beta \in \mathbb{R}$,

$$A(\boldsymbol{x}_1 + \beta\boldsymbol{x}_0) = A\boldsymbol{x}_1 + A(\beta\boldsymbol{x}_0) = \boldsymbol{b} + \beta(A\boldsymbol{x}_0) = \boldsymbol{b} + \mathbf{0} = \boldsymbol{b},$$

showing that $\boldsymbol{x}_1 + \beta\boldsymbol{x}_0$ is a solution. Since $\boldsymbol{x}_0 \neq \mathbf{0}$ and there are infinitely many β, the system has infinitely many solutions. □

Solving linear systems

Theorem

Given $A \in \mathcal{M}_{n \times n}$, $\boldsymbol{b} \in \mathbb{R}^n$, if A is invertible, then the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Proof.

Since $A(A^{-1}\boldsymbol{b}) = \boldsymbol{b}$, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is a solution. If $\boldsymbol{x}_0$ is any solution of $A\boldsymbol{x} = \boldsymbol{b}$, then

$$A\boldsymbol{x}_0 = \boldsymbol{b} \xrightarrow{\text{mutiply by } A^{-1}} A^{-1}A\boldsymbol{x}_0 = A^{-1}\boldsymbol{b} \Longrightarrow \boldsymbol{x}_0 = A^{-1}\boldsymbol{b}$$

$\square$

Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n\times n}$, the following statements are equivalent.

(a) A *is invertible*

(b) $A\boldsymbol{x} = \mathbf{0}$ *has only the trivial solution*

(c) *The reduced row echelon form of* A *is* I_n

(d) A *is expressible as a product of elementary matrices*

(e) $A\boldsymbol{x} = \boldsymbol{b}$ *is consistent* $\forall \boldsymbol{b} \in \mathbb{R}^n$

(f) $A\boldsymbol{x} = \boldsymbol{b}$ *has exactly one solution* $\forall \boldsymbol{b} \in \mathbb{R}^n$

Proof.

- We have shown the equivalence of (a)-(d) in Theorem 3.
- It is sufficient to prove (a)$\Longrightarrow$(f)$\Longrightarrow$(e)$\Longrightarrow$(a).
- Theorem 4 proves (a)$\Longrightarrow$(f)
- (f)$\Longrightarrow$(e) is trivial.

Equivalent statements – proof

(a) A is invertible

(e) $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\forall \boldsymbol{b} \in \mathbb{R}^n$

Proof.

We will now show (e)$\Longrightarrow$(a). Assume (e) is true. Let $\boldsymbol{x}_1, \ \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be the solutions for each of the following linear systems:

$$A\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad A\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Let

$$C = \left(\boldsymbol{x}_1 \mid \boldsymbol{x}_2 \mid \cdots \mid \boldsymbol{x}_n \right)$$

Equivalent statements – proof

(a) A is invertible

(e) $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\forall \boldsymbol{b} \in \mathbb{R}^n$

Proof.

Then (discussion of matrix multiplication from the last lecture)

$$AC = \left(A\boldsymbol{x}_1 \mid A\boldsymbol{x}_2 \mid \cdots \mid A\boldsymbol{x}_n \right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I$$

By Lemma 1, C is the inverse of A. □

Invertible matrices and their product

Theorem 6

For any $A, B \in \mathcal{M}_{n\times n}$, if AB is invertible, then A and B are also invertible.

Proof.

Let $\boldsymbol{x}_0$ be any solution of $B\boldsymbol{x} = \mathbf{0}$, then

$$(AB)\boldsymbol{x}_0 = A(B\boldsymbol{x}_0) = A\mathbf{0} = \mathbf{0}.$$

By Theorem 5, since AB is invertible, $\boldsymbol{x}_0 = \mathbf{0}$. Applying Theorem 5 again, B is invertible. We have

$$(AB)B^{-1} = A(BB^{-1}) = AI = A,$$

implying A is invertible. □

Special case

Theorem

For any $A \in \mathcal{M}_{n\times n}$, if A has a row (or column) of zeros, then $\det(A) = 0$.

Proof.

Let $C_1,\ C_2,\ldots,\ C_n$ denote the cofactors of A along the row (or column) of zeros, we have

$$\det(A) = 0C_1 + 0C_2 + \cdots + 0C_n = 0.$$

$\square$

Determinant of transpose

Theorem

For any $A \in \mathcal{M}_{n\times n}$, $\det(A) = \det(A^\top)$

Proof.

Since transposing a matrix changes its columns to rows and its rows to columns, the cofactor expansion of A along any row is the same as the cofactor expansion of $A^\top$ along the corresponding column. $\square$

Determinant and multiplication by an elementary matrix

Lemma 2

Given $B, E \in \mathcal{M}_{n\times n}$, *where* E *is an elementary matrix, then*

$$\det(EB) = \det(E)\det(B).$$

Proof.

- If E results from multiplying a row of I_n by β, then by Theorem 1, EB results from B by multiplying the corresponding row by β and we have

  $$\det(EB) = \beta\det(B)$$

  From Theorem 7, $\det(E) = \beta$
- Similar arguments hold if E results from interchanging two rows of I_n or from adding a multiple of one row to another of I_n

□

Determinant and multiplication by an elementary matrix

- By repeated application of Lemma 2, if $A \in \mathcal{M}_{n\times n}$, and $E_1, E_2, \ldots, E_r \in \mathcal{M}_{n\times n}$ are elementary matrices, then

$$\det(E_1 E_2 \cdots E_r A) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(A)$$

Invertible matrix and determinant

Theorem

A square matrix A is invertible iff $\det(A) \neq 0$.

Proof.

Let R be the reduced row echelon form of A. Let $E_1, E_2, \dots, E_r$ be the elementary matrices that correspond to the elementary row operations that produce R from A. Then

$$R = E_r \cdots E_2 E_1 A.$$

And from what we just discussed

$$\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A) \Longrightarrow \det(R) = 0 \text{ iff } \det(A) = 0$$

Invertible matrix and determinant

Theorem

A square matrix A is invertible iff $\det(A) \neq 0$.

Proof.

Let R be the reduced row echelon form of A.

$$\det(R) = 0 \iff \det(A) = 0$$

If A is invertible, by Theorem 3, $R = I$, which implies that $\det(A) \neq 0$.
If $\det(A) \neq 0$, then $\det(R) \neq 0$, if R has one row of zeros, $\det(R) = 0$, thus $R = I$.
Apply Theorem 3 again we can conclude A is invertible. $\square$

Recall – theorem from last lecture

If R is the reduced row echelon form of a matrix $A \in \mathcal{M}_{n\times n}$, then either R has at least one row of zeros or $R = I_n$.

Determinant of matrix product

Theorem

For any $A, B \in \mathcal{M}_{n\times n}$, $\det(AB) = \det(A)\det(B)$.

Proof.

- If A is not invertible, by Theorem 8, AB is not invertible, $\det(AB) = \det(A)\det(B) = 0$.
- If A is invertible, by Theorem 5,

$$A = E_1E_2\cdots E_r \implies AB = E_1E_2\cdots E_rB$$

Then

$$\det(AB) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B) = \det(A)\det(B)$$

$\square$

Inverse of a matrix using adjugate

Theorem

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Proof.

We show first that

$$A \operatorname{adj}(A) = \det(A) I.$$

$$A \operatorname{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{pmatrix}$$

Inverse of a matrix using adjugate

Proof.

i, j-entry of $A \operatorname{adj}(A)$ is $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$

- If $i = j$, then the above is the cofactor expansion of $\det(A)$ along the ith row of A
- If $i \neq j$, it is 0

$$A \operatorname{adj}(A) = \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{pmatrix} = \det(A)I$$

Since A is invertible, $\det(A) \neq 0$, multiply by $\frac{1}{\det(A)}A^{-1}$ on the left

$$\frac{1}{\det(A)} \operatorname{adj}(A) = A^{-1}$$

□

Theorem (Cramer's Rule)

Given $A\boldsymbol{x} = \boldsymbol{b}$, a system of n linear equations in n unknowns. If $\det(A) \neq 0$, then the system has a unique solution given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by $\boldsymbol{b}$.

Proof.

Since A_j differs from A only in the jth column, the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are the same as the cofactors of the corresponding entries in the jth column of A

$$\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$$

Cramer's rule

Proof.

$$\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$$

We know that the unique solution is given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{\det(A)} \operatorname{adj}(A)\, \boldsymbol{b} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Then

$$x_j = \frac{b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}}{\det(A)} = \frac{\det(A_j)}{\det(A)}$$

$\square$