Tutorial 2

Linear systems and Gauss–Jordan elimination

Question 1. Determine whether the following equations are linear

1. $x_1 - 3x_2 - \sqrt{2}x_3 = 2$
2. $x_1 + 3x_2 - 2x_1x_2 = x_1$
3. $x_1 = 5x_2 - 3x_3$
4. $x_1^{1/2} - 2x_2 + x_1 = 1$
5. $x_1^2 + x_2 + 8x_3 = 5$
6. $\pi x_1 - \sqrt{2}x_2 = 3^{1/3}$
7. $2^{1/3}x + \sqrt{3}y = 1$
8. $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$
9. $xy = 3$
10. $2x^{1/3} + 3\sqrt{y} = 1$
11. $\frac{\pi}{7}\cos x - 4y = 0$
12. $y + 5 = 2x + 4^2$
13. $2x_1 - x_2 = \sqrt{x_1^2}$

Solution. The linear equations are 1, 3, 6, 7, 8, 12.

Question 2. Find the system of linear equations whose augmented matrix has the following form

1. $\left(\begin{array}{cc|c} 2 & 0 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{array}\right)$

2. $\left(\begin{array}{cc|c} 3 & 0 & 2 \\ 1 & -5 & 0 \\ 0 & 1 & 2 \\ 2 & -4 & 3 \end{array}\right)$

3. $\left(\begin{array}{ccc|c} 0 & 3 & -1 & -1 \\ 2 & 3 & 0 & -5 \end{array}\right)$

4. $\left(\begin{array}{cccc|c} 3 & 0 & 0 & 1 & -4 \\ 3 & 0 & 2 & 1 & 7 \\ -1 & 3 & 0 & -2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right)$

Solution.

1.
$$\begin{aligned} 2x_1 &= -1 \\ 3x_1 + 2x_2 &= 0 \\ x_2 &= 2 \end{aligned}$$

2.
$$\begin{aligned} 3x_1 &= 2 \\ x_1 - 5x_2 &= 0 \\ x_2 &= 2 \\ 2x_1 - 4x_2 &= 3 \end{aligned}$$

3.
$$\begin{aligned} 3x_2 - x_3 &= -1 \\ 2x_1 + 3x_2 &= -5 \end{aligned}$$

4.
$$\begin{aligned} 3x_1 + x_4 &= -4 \\ 3x_1 + 2x_3 + x_4 &= 7 \\ -x_1 + 3x_2 - 2x_4 &= 4 \\ -x_3 + 2x_4 &= 1 \end{aligned}$$

Question 3. Find the augmented matrix for the following linear system

1.

$$\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$$

2.

$$\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$$

3.

$$\begin{aligned} 2x_2 - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 - x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$$

4.

$$\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$$

5.

$$\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 - x_2 - x_3 &= 0 \end{aligned}$$

6.

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$$

Solution.

1. $\left(\begin{array}{c|c} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{array}\right)$

2. $\left(\begin{array}{ccc|c} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{array}\right)$

3. $\left(\begin{array}{ccccc|c} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & -1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{array}\right)$

4. $\left(\begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array}\right)$

5. $\left(\begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & -1 & -1 & 0 \end{array}\right)$

6. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$

Question 4. Determine which of the following ordered triples is a solution to the given linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

1. $(3, 1, 1)$
2. $(3, -1, 1)$
3. $(13, 5, 2)$
4. $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$
5. $(17, 7, 5)$

Solution.

1. Let $x_1 = 3,\ x_2 = 1,\ x_3 = 1$, we get

$$2x_1 - 4x_2 - x_3 = 1, \quad x_1 - 3x_2 + x_3 = 1, \quad 3x_1 - 5x_2 - 3x_3 = 1.$$

$(3, 1, 1)$ is a solution to the given linear system.

2. Let $x_1 = 3,\ x_2 = -1,\ x_3 = 1$, we get
$$2x_1 - 4x_2 - x_3 = 9.$$
$(3, -1, 1)$ is not a solution to the given linear system.

3. Let $x_1 = 13,\ x_2 = 5,\ x_3 = 2$, we get
$$2x_1 - 4x_2 - x_3 = 4.$$
$(13, 5, 2)$ is not a solution to the given linear system.

4. Let $x_1 = \frac{13}{2},\ x_2 = \frac{5}{2},\ x_3 = 2$, we get
$$2x_1 - 4x_2 - x_3 = 1, \quad x_1 - 3x_2 + x_3 = 1, \quad 3x_1 - 5x_2 - 3x_3 = 1.$$
$\left(\frac{13}{2}, \frac{5}{2}, 2\right)$ is a solution to the given linear system.

5. Let $x_1 = 17,\ x_2 = 7,\ x_3 = 5$, we get
$$2x_1 - 4x_2 - x_3 = 1, \quad x_1 - 3x_2 + x_3 = 1, \quad 3x_1 - 5x_2 - 3x_3 = -\frac{11}{2}.$$
$(17, 7, 5)$ is not a solution to the given linear system.

Question 5. Solve each of the following systems of linear equations. In each case, indicate whether the system has one solution, infinitely many solutions, or no solutions. Give the complete solution set.

1.
$$\begin{aligned} 3x - 2y &= 4 \\ 6x - 4y &= 9 \end{aligned}$$

2.
$$\begin{aligned} 2x - 4y &= 1 \\ 4x - 8y &= 2 \end{aligned}$$

3.
$$\begin{aligned} x - 2y &= 0 \\ x - 4y &= 8 \end{aligned}$$

Solution.

1. Add $-2\times$ the first equation to the second equation gives
$$\begin{aligned} 3x - 2y &= 4 \\ 0 &= 1, \end{aligned}$$
a contradiction. This linear system has no solutions.

2. Add $-2\times$ the first equation to the second equation gives
$$\begin{aligned} 2x - 4y &= 1 \\ 0 &= 0 \end{aligned}$$
This system has infinitely many solutions. The solution set is given by
$$\left\{ \left(\frac{1}{2} + 2t,\ t\right) \;\middle|\; t \in \mathbb{R} \right\}$$

3. Add $-1\times$ the first equation to the second equation gives

$$\begin{aligned} x - 2y &= 0 \\ -2y &= 8 \end{aligned}$$

We have

$$y = -4, \quad x = -8.$$

The system has the unique solution $(-8, -4)$.

Question 6. Find the solution set of the following linear system.

1. $7x - 5y = 3$
2. $x + 10y = 3$
3. $x_1 - 5x_2 + 2x_3 = -1$
4. $3x_1 - 5x_2 + 4x_3 = 7$
5. $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$
6. $4x_1 + 2x_2 - 3x_3 - x_4 = 2$
7. $3v - 8w + 2x - y + 4z = 0$
8. $v + w + x - 5y + 7z = 0$

Solution.

1. $\left\{ \left(\frac{7}{3} + \frac{5}{7}t,\ t\right) \;\middle|\; t \in \mathbb{R} \right\}$
2. $\{ (3 - 10t,\ t) \mid t \in \mathbb{R} \}$
3. $\{ (-1 + 5t - 2s,\ t,\ s) \mid t,\ s \in \mathbb{R} \}$
4. $\left\{ \left(\frac{7}{3} + \frac{5}{3}t - \frac{4}{3}s,\ t,\ s\right) \;\middle|\; t,\ s \in \mathbb{R} \right\}$
5. $\left\{ \left(-\frac{1}{8} + \frac{3}{4}r - \frac{5}{8}s + \frac{1}{4}t,\ t,\ s,\ r\right) \;\middle|\; t,\ s,\ r \in \mathbb{R} \right\}$
6. $\left\{ \left(\frac{1}{4} + \frac{1}{4}r + \frac{3}{4}s - \frac{1}{2}t,\ t,\ s,\ r\right) \;\middle|\; t,\ s,\ r \in \mathbb{R} \right\}$
7. $\left\{ \left(-\frac{4}{3}r + \frac{1}{3}s - \frac{2}{3}t + \frac{8}{3}u,\ u,\ t,\ s,\ r\right) \;\middle|\; u,\ t,\ s,\ r \in \mathbb{R} \right\}$
8. $\{ (-7r + 5s - t - u,\ u,\ t,\ s,\ r) \mid u,\ t,\ s,\ r \in \mathbb{R} \}$

Question 7. Find the solution set for the given linear system.

1.
$$\begin{aligned} 2x - 3y &= 1 \\ 6x - 9y &= 3 \end{aligned}$$

2.
$$\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$$

3.
$$\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 3x_1 + 9x_2 - 3x_3 &= -12 \\ -x_1 - 3x_2 + x_3 &= 4 \end{aligned}$$

4.
$$\begin{aligned} 2x - y + 2z &= -4 \\ 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$$

Solution.

1. Add $-3\times$ the first equation to the second gives

$$\begin{aligned} 2x - 3y &= 1 \\ 0 &= 0. \end{aligned}$$

The solution set is given by

$$\left\{ \left(\frac{1}{2} + \frac{3}{2}t,\ t \right) \ \middle|\ t \in \mathbb{R} \right\}.$$

2. Add $-2\times$ the second equation to the first gives

$$\begin{aligned} 0 &= 0 \\ 3x_1 + x_2 &= -4. \end{aligned}$$

The solution set is given by

$$\left\{ \left(-\frac{4}{3} - \frac{1}{3}t,\ t \right) \ \middle|\ t \in \mathbb{R} \right\}.$$

3. Add the first equation to the third equation and add $-3\times$ the first equation to the second equation, gives

$$\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 0 &= 0 \\ 0 &= 0. \end{aligned}$$

The solution set is given by

$$\{ (-4 + s - 3t,\ t,\ s) \mid t,\ s \in \mathbb{R} \}.$$

4. Add $2\times$ the first equation to the third equation, and add $-3\times$ the first equation to the second equation, gives

$$\begin{aligned} 2x - y + 2z &= -4 \\ 0 &= 0 \\ 0 &= 0. \end{aligned}$$

The solution set is given by

$$\left\{ \left(-2 - 1s + \frac{t}{2},\ t,\ s \right) \ \middle|\ t,\ s \in \mathbb{R} \right\}.$$

Question 8. Find an elementary row operation that will create a leading 1 in the top left corner of the given matrix without introducing fractions in its first row.

1. $\begin{pmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 0 & -1 & -5 & 0 \\ 2 & -6 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{pmatrix}$

4. $\begin{pmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{pmatrix}$

Solution.

1. Add $2\times$ the second row to the first row
2. Add the third row to the first row
3. Add the third row to $3\times$ the first row
4. Add the third row to the first row

Question 9. Determine whether the following matrices are in row echelon form and which are in reduced row echelon form.

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
5. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
7. $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix}$
8. $\begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$
9. $\begin{pmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
10. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
11. $\begin{pmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{pmatrix}$
12. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$
13. $\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & \end{pmatrix}$
14. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 7 & 1 & 3 \\ 0 & 0 & 0 & 1 & \end{pmatrix}$

Solution.

- Row echelon form: $1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13$
- Reduced row echelon form: $1, 2, 3, 7, 8, 10, 12$

For Questions 10 – 13 do the following

Suppose that each of the following is the augmented matrix for a linear system. Use the Gauss-Jordan elimination method to convert each matrix to the reduced row echelon form and give the complete solution set for the corresponding system of linear equations if the system is consistent.

Question 10.

1. $\left(\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right)$

2. $\left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

3. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 \end{array}\right)$

5. $\left(\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$

6. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 \end{array}\right)$

Solution.

1. By the Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right) \xrightarrow{R_1 \to R_1 + 3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 10 & 13 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right) \xrightarrow[R_1 \to R_1 - 10R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -37 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

   The unique solution of the corresponding linear system is $(-37,\ -8,\ 5)$.

2. The matrix is already in the reduced row echelon form and the linear system has no solutions.

3. The matrix is already in the reduced row echelon form and the solution is $(-3,\ 0,\ 7)$.

4. The matrix is already in the reduced row echelon form and the solution is $(-3,\ 0,\ 8)$.

5. By the Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   The solution set is

$$\{\, (4t,\ t,\ 0) \ \mid\ t \in \mathbb{R} \,\}.$$

6. By the Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow[R_2 \to \frac{1}{2}R_2]{R_3 \to \frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

   The system has no solutions.

Question 11.

1. $\left(\begin{array}{cccc|c} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$

2. $\left(\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{array}\right)$

3. $\left(\begin{array}{cccc|c} 1 & -2 & 0 & -2 & 3 \\ 0 & 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 13 & 13 \end{array}\right)$

Solution.

1. The solution set is given by
$$\{\, (13t-10,\ 13t-5,\ 2-t, t) \ \mid\ t \in \mathbb{R} \,\}$$
2. The solution set is given by
$$\{\, (7t+8,\ 2-3t,\ -t-5, t) \ \mid\ t \in \mathbb{R} \,\}$$
3. The linear system has no solutions.
4. The solution set is given by
$$\{\, (-4t-13,\ t,\ -3) \ \mid\ t \in \mathbb{R} \,\}$$

Question 12.

1. $\left(\begin{array}{ccccc|c} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

2. $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{array}\right)$

3. $\left(\begin{array}{ccccc|c} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

4. $\left(\begin{array}{cccc|c} -5 & 10 & -19 & -17 & 20 \\ -3 & 6 & -11 & -11 & 14 \\ -7 & 14 & -26 & -25 & 31 \\ 9 & -18 & 34 & 31 & -37 \end{array}\right)$

5. $\left(\begin{array}{cc|c} 2 & -5 & -20 \\ 0 & 2 & 7 \\ 1 & -5 & -19 \\ -5 & 16 & 64 \\ 3 & 9 & -36 \end{array}\right)$

6. $\left(\begin{array}{cccc|c} -2 & 1 & -1 & -1 & 3 \\ 3 & 1 & -4 & -2 & -4 \\ 7 & 1 & -6 & -2 & -3 \\ -8 & -1 & 6 & 2 & 3 \\ -3 & 0 & 2 & 1 & 2 \end{array}\right)$

7. $\left(\begin{array}{ccccc|c} -3 & 6 & -1 & -5 & 0 & -5 \\ -1 & 2 & 3 & -5 & 10 & 5 \end{array}\right)$

Solution.

1. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccccc|c} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \end{array}\right)$$

The solution set is

$$\{\, (6s-3t-2,\ s,\ 7-4t,\ 8-5t,\ t) \ \mid\ s,\ t \in \mathbb{R} \,\}$$

2. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The linear system is inconsistent.

3. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccccc|c} 1 & 7 & 0 & 0 & -2 & -11 \\ 0 & 0 & 1 & 0 & 3 & -4 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is

$$\{\, (-7s+2t-11,\ s,\ -3t-4,\ 9-3t,\ t) \ \mid\ s,\ t \in \mathbb{R} \,\}$$

4. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 11 & -23 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is

$$\{\, (2s-11t-23,\ s,\ 2t+5,\ 9-3t,\ t) \ \mid\ s,\ t \in \mathbb{R} \,\}$$

5. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The linear system is inconsistent.

6. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

The linear system is inconsistent.

7. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -1 & 3 & 2 \end{array}\right)$$

The solution set is

$$\{\ (2r-2s+t+1,\ r,\ s-3t+2,\ s,\ t)\ \mid\ r,\ s,\ t\in\mathbb{R}\ \}$$

For Questions 13 – 17 do the following
Solve the following linear systems using Gauss–Jordan elimination or Gaussian elimination. In each case, indicate whether the system is consistent or inconsistent. Give the complete solution set if the system is consistent.

Question 13.

1.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

2.

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

3.

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z - 2w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

4.

$$\begin{aligned} -2y + 3z &= 1 \\ 3x + 6y - 3z &= -2 \\ 6x + 6y + 3z &= 5 \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right)$$

By Gauss–Jordan elimination, the reduced row echelon form of the matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

The unique solution is given by $(3, 1, 2)$.

2. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right)$$

By Gauss–Jordan elimination, the reduced row echelon form of the matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution set is given by

$$\left\{ \left(-\frac{3t}{7} - \frac{1}{7},\ \frac{1}{7} - \frac{4t}{7},\ t \right) \ \middle|\ t \in \mathbb{R} \right\}$$

3. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{ (-1,\ 2t,\ t,\ 0) \ \mid\ t \in \mathbb{R} \}.$$

4. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The linear system has no solutions.

Question 14.

1.

$$\begin{aligned} 5x_1 - 5x_2 - 15x_3 &= 40 \\ 4x_1 - 2x_2 - 6x_3 &= 19 \\ 3x_1 - 6x_2 - 17x_3 &= 41 \end{aligned}$$

2.

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

3.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

4.

$$\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

Solution.

1. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right)$$

The unique solution is given by

$$\left(\frac{3}{2},\ 1,\ -\frac{5}{2}\right)$$

2. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system only has the trivial solution $(0,\ 0,\ 0)$.

3. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system only has the trivial solution $(0,\ 0,\ 0)$.

4. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system only has the trivial solution $(0,\ 0,\ 0)$.

Question 15.

1.

$$\begin{aligned} -2x_1 + x_2 + 8x_3 &= 0 \\ 7x_1 - 2x_2 - 22x_3 &= 0 \\ 3x_1 - x_2 - 10x_3 &= 0 \end{aligned}$$

2.

$$\begin{aligned} 5x_1 - 2x_3 &= 0 \\ -15x_1 - 16x_2 - 9x_3 &= 0 \\ 10x_1 + 12x_2 + 7x_3 &= 0 \end{aligned}$$

Solution.

1. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\,(2t,\ -4t,\ t)\ \mid\ t \in \mathbb{R}\,\}.$$

2. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system only has the trivial solution $(0,\ 0,\ 0)$.

Question 16.

1.

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$$

2.

$$\begin{aligned} -2x_1 - 3x_2 + 2x_3 - 13x_4 &= 0 \\ -4x_1 - 7x_2 + 4x_3 - 29x_4 &= 0 \\ x_1 + 2x_2 - x_3 + 8x_4 &= 0 \end{aligned}$$

3.

$$\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

4.

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

5.

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

6.

$$\begin{aligned} 2x_1 + 6x_2 + 13x_3 + x_4 &= 0 \\ x_1 + 4x_2 + 10x_3 + x_4 &= 0 \\ 2x_1 + 8x_2 + 20x_3 + x_4 &= 0 \\ 3x_1 + 10x_2 + 21x_3 + 2x_4 &= 0 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 - 6x_2 + 3x_3 - 21x_4 &= 0 \\ 4x_1 - 5x_2 + 2x_3 - 24x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 10x_4 &= 0 \\ -2x_1 + 3x_2 - x_3 + 13x_4 &= 0 \end{aligned}$$

Solution.

1. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{46}{83} & 0 \\ 0 & 1 & 0 & -\frac{15}{83} & 0 \\ 0 & 0 & 1 & -\frac{55}{83} & 0 \end{array}\right)$$

The solution set is given by

$$\left\{ \left(-\frac{46}{83},\ \frac{15}{83},\ \frac{55t}{83},\ t\right) \;\middle|\; t \in \mathbb{R} \right\}.$$

2. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\, (s - 2t,\ -3t,\ s,\ t) \mid s,\ t \in \mathbb{R} \,\}.$$

3. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\left\{ \left(\frac{7s}{2} - \frac{5t}{2},\ -3t + 2s,\ s,\ t\right) \;\middle|\; s,\ t \in \mathbb{R} \right\}.$$

4. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\, (t,\ -t,\ t,\ 0) \mid t \in \mathbb{R} \,\}.$$

5. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right)$$

The solution set is given by

$$\left\{ \left(-\frac{s}{4},\ -\frac{s}{4} - t,\ -t,\ s,\ t\right) \;\middle|\; s,\ t \in \mathbb{R} \right\}.$$

6. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

The system only has the trivial solution $(0,\ 0,\ 0, 0)$.

7. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\, (3t,\ -2t,\ t,\ t) \ \mid\ t \in \mathbb{R} \,\}.$$

Question 17.

1.

$$\begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

2.

$$\begin{aligned} x_3 + x_4 + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \\ 2x_1 + 2x_2 - x_3 + x_5 &= 0 \end{aligned}$$

3.

$$\begin{aligned} 2x_1 + 4x_2 - x_3 + 5x_4 + 2x_5 &= 0 \\ 3x_1 + 3x_2 - x_3 + 3x_4 &= 0 \\ -5x_1 - 6x_2 + 2x_3 - 6x_4 - x_5 &= 0 \end{aligned}$$

4.

$$\begin{aligned} 7x_1 + 28x_2 + 4x_3 - 2x_4 + 10x_5 + 19x_6 &= 0 \\ -9x_1 - 36x_2 - 5x_3 + 3x_4 - 15x_5 - 29x_6 &= 0 \\ 3x_1 + 12x_2 + 2x_3 + 6x_5 + 11x_6 &= 0 \\ 6x_1 + 24x_2 + 3x_3 - 3x_4 + 10x_5 + 20x_6 &= 0 \end{aligned}$$

Solution.

1. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The system only has the trivial solution $(0,\ 0,\ 0, 0)$.

2. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\, (-s - t,\ s,\ -t,\ 0,\ t) \ \mid\ s,\ t \in \mathbb{R} \,\}.$$

3. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\, (t,\ -2s-t,\ -3s,\ s,\ t) \ \mid\ s,\ t \in \mathbb{R} \,\}.$$

4. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccccc|c} 1 & 4 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\, (-4r+2s-t,\ r,\ -3s+2t,\ s,\ -2t, t) \ \mid\ r,\ s,\ t \in \mathbb{R} \,\}.$$

Question 18. The matrices below represent augmented matrices of linear systems, where $*$ denotes an arbitrary real number. For each case, determine whether the system is consistent. If it is consistent, establish whether the solution is unique. If the solution is not unique, analyze the structure of the solution set.

1. $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right)$

2. $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{array}\right)$

3. $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{array}\right)$

5. $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right)$

6. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{array}\right)$

7. $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{array}\right)$

8. $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right)$

Solution.

1. We change the representation of the given matrix to as follows

$$\left(\begin{array}{ccc|c} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \end{array}\right),$$

where a_{ij}'s represent real numbers. We can apply Gauss-Jordan elimination method on the matrix and we have

$$\xrightarrow{R_1 \to R_1 - a_{12}R_2} \left(\begin{array}{ccc|c} 1 & 0 & a_{13} - a_{12}a_{23} & a_{14} - a_{12}a_{24} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \end{array}\right)$$

$$\xrightarrow[R_2 \to R_2 - a_{23}R_3]{R_1 \to R_1 - (a_{13} - a_{12}a_{23})R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a_{14} - a_{12}a_{24} - (a_{13} - a_{12}a_{23})a_{34} \\ 0 & 1 & 0 & a_{24} - a_{23}a_{34} \\ 0 & 0 & 1 & a_{34} \end{array}\right)$$

We can see that the system has a unique solution

$$(a_{14} - a_{12}a_{24} - (a_{13} - a_{12}a_{23})a_{34},\ a_{24} - a_{23}a_{34},\ a_{34})$$

2. By letting $a_{34} = 1$, it follows from 1 that the system has a unique solution

$$(a_{14} - a_{12}a_{24} - a_{13} + a_{12}a_{23},\ a_{24} - a_{23},\ 1)$$

3. We change the representation of the given matrix to as follows

$$\left(\begin{array}{ccc|c} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \end{array}\right),$$

where a_{ij}'s represent real numbers. We can apply Gauss-Jordan elimination method to the matrix and we have

$$\xrightarrow{R_1 \to R_1 - a_{12}R_2} \left(\begin{array}{ccc|c} 1 & 0 & a_{13} - a_{12}a_{23} & a_{14} - a_{12}a_{24} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We can see that the system has infinitely many solutions and the solution set is given by

$$\{\, (a_{14} - a_{12}a_{24} - (a_{13} - a_{12}a_{23})t,\ a_{24} - a_{23}t,\ t) \mid t \in \mathbb{R} \,\}.$$

4. We change the representation of the given matrix to as follows

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a_{14} \\ a_{21} & 1 & 0 & a_{24} \\ a_{31} & a_{32} & 1 & a_{34} \end{array}\right),$$

where a_{ij}'s represent real numbers. We can apply Gauss-Jordan elimination method to the matrix and we have

$$\xrightarrow[R_2 \to R_2 - a_{21}R_1]{R_3 \to R_3 - a_{31}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & a_{24} - a_{21}a_{14} \\ 0 & a_{32} & 1 & a_{34} - a_{31}a_{14} \end{array}\right) \xrightarrow{R_3 \to R_3 - a_{32}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & a_{24} - a_{21}a_{14} \\ 0 & 0 & 1 & a_{34} - a_{31}a_{14} - a_{32}(a_{24} - a_{21}a_{14}) \end{array}\right)$$

We can see that the system has a unique solution

$$(a_{14},\ a_{24} - a_{21}a_{14},\ a_{34} - a_{31}a_{14} - a_{32}(a_{24} - a_{21}a_{14}))$$

5. The last row of the matrix corresponds to

$$0 = 1,$$

a contradiction. Thus the corresponding linear system is inconsistent.

6. We change the representation of the given matrix to as follows

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & a_{32} & a_{33} & a_{34} \end{array}\right),$$

where a_{ij}'s represent real numbers. We can apply Gauss-Jordan elimination method to the matrix and we have

$$\xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{32} & a_{33} & a_{34} \end{array}\right),$$

where the second row corresponds to

$$0 = 1,$$

a contradiction. Thus the corresponding linear system is inconsistent.

7. We change the representation of the given matrix to as follows

$$\left(\begin{array}{ccc|c} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 1 & a_{34} \end{array}\right),$$

where a_{ij}'s represent real numbers. We can apply Gauss-Jordan elimination method to the matrix and we have

$$\xrightarrow[R_1 \to R_1 - a_{13} R_3]{R_2 \to R_2 - a_{23} R_3} \left(\begin{array}{ccc|c} 1 & a_{12} & 0 & a_{14} - a_{13}a_{34} \\ 0 & 0 & 0 & -a_{23}a_{34} \\ 0 & 0 & 1 & a_{34} \end{array}\right).$$

Thus, if $a_{23}a_{34} \neq 0$, the system is inconsistent. Otherwise, the solution set of the system is given by

$$\{\, (a_{14} - a_{13}a_{34} - a_{12}t,\ t,\ a_{34}) \;\mid\; t \in \mathbb{R} \,\}.$$

8. We change the representation of the given matrix to as follows

$$\left(\begin{array}{ccc|c} 1 & a_{12} & a_{13} & a_{14} \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right),$$

where a_{ij}'s represent real numbers. We can apply Gauss-Jordan elimination method to the matrix and we have

$$\xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & a_{12} & a_{13} & a_{14} \\ 0 & -a_{12} & -a_{13} & 1 - 1a_{14} \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -a_{12} & -a_{13} & 1 - a_{14} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

If $a_{12} = a_{13} = 0$, the system is inconsistent.

If $a_{12} = 0$ and $a_{13} \neq 0$, the solution set is given by

$$\left\{ \left(1,\ t,\ \frac{a_{14} - 1}{a_{13}})\right) \;\middle|\; t \in \mathbb{R} \right\}.$$

If $a_{12} \neq 0$, the solution set is given by

$$\left\{ \left(1,\ \frac{-1 + a_{14} - a_{13}t)}{a_{12}}\right) \;\middle|\; t \in \mathbb{R} \right\}.$$

Question 19. For each of the following linear systems, determine the value of α that results in the system having no solutions, a unique solution, or infinitely many solutions.

1.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 2x - y + 5z &= 2 \\ 4x + y + (\alpha^2 - 14)z &= \alpha + 2 \end{aligned}$$

2.

$$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - (\alpha^2 - 3)z &= \alpha \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 2 & -1 & 5 & 2 \\ 4 & 1 & \alpha^2 - 14 & \alpha + 2 \end{array}\right)$$

Applying Gauss-Jordan elimination

$$\xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -5 & 11 & -6 \\ 0 & -7 & \alpha^2 - 2 & \alpha - 14 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -\dfrac{11}{5} & \dfrac{6}{5} \\ 0 & -7 & \alpha^2 - 2 & \alpha - 14 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -\dfrac{11}{5} & \dfrac{6}{5} \\ 0 & 0 & \alpha^2 - \dfrac{87}{5} & \alpha - \dfrac{28}{5} \end{array}\right)$$

The third row corresponds to the equation:

$$\left(\alpha^2 - \frac{87}{5}\right) z = \alpha - \frac{28}{5}.$$

- *Case 1: No solutions*
  If
  $$\alpha^2 - \frac{87}{5} = 0 \quad \text{i.e.,} \quad \alpha = \pm\sqrt{\frac{87}{5}},$$
  then
  $$\alpha - \frac{28}{5} \neq 0,$$
  and the system is inconsistent.
- *Case 2: Unique solution*
  If
  $$\alpha^2 - \frac{87}{5} \neq 0, \quad \text{i.e.} \quad \alpha \neq \pm\sqrt{\frac{87}{5}}$$
  the system has a *unique solution.*

- *Case 3: Infinitely many solutions*
  For the system to have infinitely many solutions, we need
  $$\alpha^2 - \frac{87}{5} = 0, \quad \alpha - \frac{28}{5} = 0,$$
  a contradiction. Thus the system will not have infinitely many solutions no matter what value α takes.

2. The augmented matrix for the linear system is
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -2 & 3 & 1 \\ 1 & 2 & 3-\alpha^2 & \alpha \end{array}\right)$$
Applying Gauss-Jordan elimination
$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -6 & 1 & -3 \\ 0 & 0 & 2-\alpha^2 & \alpha-2 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{6}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{2} \\ 0 & 0 & 2-\alpha^2 & \alpha-2 \end{array}\right)$$
$$\xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 1 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{2} \\ 0 & 0 & 2-\alpha^2 & \alpha-2 \end{array}\right)$$
The third row corresponds to the equation:
$$(2-\alpha^2)z = \alpha - 2.$$

- *Case 1: No solution*
  If
  $$2 - \alpha^2 = 0 \quad \text{i.e.,} \quad \alpha = \pm\sqrt{2},$$
  then
  $$\alpha - 2 \neq 0$$
  and the system is inconsistent.
- *Case 2: Unique solution*
  If
  $$2 - \alpha^2 \neq 0 \quad \text{i.e.,} \quad \alpha \neq \pm\sqrt{2},$$
  the system has a *unique solution.*
- *Case 3: Infinitely many solutions*
  For the system to have infinitely many solutions, we need
  $$2 - \alpha^2 = 0, \quad \alpha - 2 = 0,$$
  a contradiction. Thus the system will not have infinitely many solutions no matter what value α takes.

Question 20. What conditions must the parameters a, b, c satisfy for the following linear system to be consistent?

1.

$$\begin{aligned} x+3y-z &= a\\ x+y+2z &= b\\ 2y-3z &= c \end{aligned}$$

2.

$$\begin{aligned} x+3y+z &= a\\ -x-2y+z &= b\\ 3x+7y-z &= c \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & a\\ 1 & 1 & 2 & b\\ 0 & 2 & -3 & c \end{array}\right)$$

Applying Gauss-Jordan elimination

$$\xrightarrow{R_2\to R_2-R_1}\left(\begin{array}{ccc|c} 1 & 3 & -1 & a\\ 0 & -2 & 3 & b-a\\ 0 & 2 & -3 & c \end{array}\right)\xrightarrow{R_2\to-\frac{1}{2}R_2}\left(\begin{array}{ccc|c} 1 & 3 & -1 & a\\ 0 & 1 & -\dfrac{3}{2} & \dfrac{a-b}{2}\\ 0 & 2 & -3 & c \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-2R_2}\left(\begin{array}{ccc|c} 1 & 3 & -1 & a\\ 0 & 1 & -\dfrac{3}{2} & \dfrac{a-b}{2}\\ 0 & 0 & 0 & b+c-a \end{array}\right)$$

For the system to be consistent, we must have

$$b+c-a=0.$$

2. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & a\\ -1 & -2 & 1 & b\\ 3 & 7 & -1 & c \end{array}\right)$$

Applying Gauss-Jordan elimination

$$\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2+R_1}\left(\begin{array}{ccc|c} 1 & 3 & 1 & a\\ 0 & 1 & 2 & a+b\\ 0 & -2 & -4 & c-3a \end{array}\right)\xrightarrow{R_3\to R_3+2R_2}\left(\begin{array}{ccc|c} 1 & 3 & 1 & a\\ 0 & 1 & 2 & a+b\\ 0 & 0 & 0 & c-a+2b \end{array}\right)$$

For the system to be consistent, we must have

$$2b+c-a=0.$$

Question 21. Solve the following systems, where a, b, c are constants.

1.

$$\begin{aligned} 2x+y &= a\\ 3x+6y &= b \end{aligned}$$

2.

$$\begin{aligned} x_1+x_2+x_3 &= a\\ 2x_1+2x_3 &= b\\ 3x_2+3x_3 &= c \end{aligned}$$

Solution.

1. Add $-6\times$ the first equation to the second equation, we get

$$\begin{aligned} 2x + y &= a \\ -9x &= b - 6a \end{aligned}$$

Then

$$x = \frac{6a-b}{9}, \quad y = a - \frac{2(6a-b)}{9} = \frac{2b-3a}{9}.$$

2. The augmented matrix of the linear system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 2 & 0 & 2 & b \\ 0 & 3 & 3 & c \end{array}\right).$$

Applying Gauss-Jordan elimination

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & 0 & b-2a \\ 0 & 3 & 3 & c \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & \dfrac{2a-b}{2} \\ 0 & 3 & 3 & c \end{array}\right) \xrightarrow{R_3 \to R_3 - 3R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & \dfrac{2a-b}{2} \\ 0 & 0 & 3 & c - \dfrac{6a-3b}{2} \end{array}\right) \xrightarrow{R_3 \to \frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & \dfrac{2a-b}{2} \\ 0 & 0 & 1 & \dfrac{c}{3} - \dfrac{2a-b}{2} \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & \dfrac{b}{2} \\ 0 & 1 & 0 & \dfrac{2a-b}{2} \\ 0 & 0 & 1 & \dfrac{c}{3} - \dfrac{2a-b}{2} \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a - \dfrac{c}{3} \\ 0 & 1 & 0 & a - \dfrac{b}{2} \\ 0 & 0 & 1 & \dfrac{c}{3} - a + \dfrac{b}{2} \end{array}\right)$$

The system has a unique solution

$$\left(a - \frac{c}{3},\ a - \frac{b}{2},\ -a + \frac{b}{2} + \frac{c}{3}\right)$$

Question 22. Show that the following nonlinear system for the unknowns u, v, w, where $0 \leq u, v, w \leq 2\pi$, has 18 solutions.

$$\begin{aligned} \sin u + 2\cos v + 3\tan w &= 0 \\ 2\sin u + 5\cos v + 3\tan w &= 0 \\ -\sin u - 5\cos v + 5\tan w &= 0 \end{aligned}$$

Solution. Let

$$x = \sin u, \quad y = \cos v, \quad z = \tan w. \tag{1}$$

Substituting these variables, the given nonlinear system in u, v, w is transformed into the following linear system in the unknowns x, y, z.

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 5y + 3z &= 0 \\ -x - 5y + 5z &= 0 \end{aligned}$$

The augmented matrix of the linear system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{array}\right).$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system has a unique solution

$$x = y = z = 0.$$

Substituting to Equation 1 we get

$$\sin u = 0, \quad \cos v = 0, \quad \tan w = 0.$$

Since

$$0 \le u \le 2\pi, \quad ,0 \le v \le 2\pi, \quad 0 \le w \le 2\pi,$$

we have

$$u = 0,\ \pi,\ 2\pi \quad v = \frac{\pi}{2}, \frac{3\pi}{2}, \quad w = 0,\ \pi,\ 2\pi,$$

in total we have

$$3 \times 2 \times 3 = 18$$

solution.

Question 23. Solve the following nonlinear system for the unknowns u, v, w, where $0 \le u, v, w \le 2\pi$.

$$\begin{aligned} 2\sin u - \cos v + 3\tan w &= 3 \\ 4\sin u + 2\cos v - 2\tan w &= 2 \\ 6\sin u - 3\cos v + \tan w &= 9 \end{aligned}$$

Solution. Let

$$x = \sin u, \quad y = \cos v, \quad z = \tan w. \tag{2}$$

Substituting these variables, the given nonlinear system in u, v, w is transformed into the following linear system in the unknowns x, y, z.

$$\begin{aligned} 2x - y + 3z &= 3 \\ 4x + 2y - 2z &= 2 \\ 6x - 3y + z &= 9 \end{aligned}$$

The augmented matrix of the linear system is

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right).$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system has a unique solution

$$x = 1, \quad y = -1, \quad z = 0.$$

Substituting to Equation 2 we get

$$\sin u = 1, \quad \cos v = -1, \quad \tan w = 0.$$

Since

$$0 \le u \le 2\pi, \quad ,0 \le v \le 2\pi, \quad 0 \le w \le 2\pi,$$

we have

$$u = \frac{\pi}{2}, \quad v = \pi, \quad w = 0,\ \pi,\ 2\pi.$$

Thus the given nonlinear system has three solutions:

$$\left(\frac{\pi}{2},\ \pi,\ 0\right), \quad \left(\frac{\pi}{2},\ \pi,\ \pi\right), \quad \left(\frac{\pi}{2},\ \pi,\ 2\pi\right).$$

Question 24. Solve the following nonlinear system for the unknowns x, y, z.

1.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

2.

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ \frac{1}{x} + \frac{9}{y} + \frac{10}{z} &= 5 \end{aligned}$$

Solution.

1. Let

$$u = x^2, \quad v = y^2, \quad w = z^2.$$

Substituting these variables, the given nonlinear system in x, y, z is transformed into the following linear system in the unknowns u, v, w

$$\begin{aligned} u + v + w &= 6 \\ u - v + w &= 2 \\ 2u + v - w &= 3 \end{aligned}$$

The augmented matrix of the linear system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 3 \end{array}\right).$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{7}{3} \end{array}\right).$$

The linear system has a unique solution

$$x = \frac{5}{3}, \quad y = 2, \quad z = \frac{7}{3}.$$

Consequently, the solutions for the nonlinear system are given by

$$\left(\pm\frac{\sqrt{15}}{3}, \ \pm\sqrt{2}, \ \pm\frac{\sqrt{21}}{3}\right)$$

2. Let

$$u = \frac{1}{x}, \quad v = \frac{1}{y}, \quad w = \frac{1}{z}.$$

Substituting these variables, the given nonlinear system in x, y, z is transformed into the following linear system in the unknowns u, v, w

$$\begin{aligned} u + 2v - 4w &= 1 \\ 2u + 3v + 8w &= 0 \\ u + 9v + 10w &= 5 \end{aligned}$$

The augmented matrix of the linear system is

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 0 \\ 1 & 9 & 10 & 5 \end{array}\right).$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{7}{9} \\ 0 & 1 & 0 & \frac{46}{63} \\ 0 & 0 & 1 & -\frac{5}{63} \end{array}\right).$$

The linear system has a unique solution

$$x = -\frac{7}{9}, \quad y = \frac{46}{63}, \quad z = -\frac{5}{63}.$$

Consequently, the unique solution for the nonlinear system is given by

$$\left(-\frac{9}{7}, \frac{63}{46}, -\frac{63}{5}\right)$$

For Questions 25 – 27 do the following

Suppose that each of the following is the augmented matrix for a linear system. The matrices are all in reduced row echelon form. For each case, determine whether the system is consistent. If it is consistent, find the complete solution set.

Question 25.

1. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$

2. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right)$

3. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

5. $\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 4 \\ 0 & 1 & 3 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$

6. $\left(\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

7. $\left(\begin{array}{ccc|c} 1 & 5 & 5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

8. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$

9. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Solution.

1. The last row corresponds to
$$1 = 0,$$
a contradiction. Thus, the system has no solutions.

2. The system has a unique solution
$$\left(2,\ 0,\ -\frac{2}{3}\right)$$

3. The last row corresponds to
$$1 = 0,$$
a contradiction. Thus, the system has no solutions.

4. The solution set for the system is
$$\{\,(-3-2t,\ 2+t,\ t)\ \mid\ t \in \mathbb{R}\,\}.$$

5. The solution set for the system is
$$\left\{\left(4+\frac{1}{3}t,\ \frac{4}{3}-3t,\ t\right)\ \middle|\ t \in \mathbb{R}\right\}.$$

6. The solution set for the system is

$$\{\,(-3+2t,\ t,\ 2)\ \mid\ t\in\mathbb{R}\,\}\,.$$

7. The solution set for the system is

$$\{\,(-1-5s-5t,\ s,\ t)\ \mid\ s,\ t\in\mathbb{R}\,\}\,.$$

8. The system has a unique solution $(2,\ 0,\ 1)$.

9. The last row corresponds to

$$1=0,$$

   a contradiction. Thus, the system has no solutions.

Question 26.

$$1.\ \left(\begin{array}{cccc|c}1&0&-2&5&3\\0&1&-1&2&2\end{array}\right),\quad 2.\ \left(\begin{array}{cccc|c}1&3&-3&0&1\\0&0&0&1&4\end{array}\right),\quad 3.\ \left(\begin{array}{cccc|c}1&0&0&-3&1\\0&1&0&-1&7\end{array}\right)$$

Solution.

1. The solution set is given by

$$\{\,(3+2s-5t,\ 2+s-2t,\ s,\ t)\ \mid\ s,\ t\in\mathbb{R}\,\}\,.$$

2. The solution set is given by

$$\{\,(1-3s+3t,\ s,\ t,\ 4)\ \mid\ s,\ t\in\mathbb{R}\,\}\,.$$

3. The solution set is given by

$$\{\,(1+3t,\ 7+t,\ s,\ t)\ \mid\ s,\ t\in\mathbb{R}\,\}\,.$$

Question 27.

$$1.\ \left(\begin{array}{cccc|c}1&0&\frac{2}{3}&0&-1\\0&1&-3&0&1\\0&0&0&1&\frac{4}{5}\end{array}\right),\quad 2.\ \left(\begin{array}{cccc|c}1&0&0&-3&1\\0&1&0&-1&7\\0&0&1&0&0\end{array}\right),\quad 3.\ \left(\begin{array}{cccc|c}1&0&0&-3&1\\0&1&0&-1&7\\0&0&1&2&-1\end{array}\right)$$

Solution.

1. The solution set is given by

$$\left\{\left(-1-\frac{2}{3}t,\ 1+3t,\ t,\ \frac{4}{5}\right)\ \middle|\ t\in\mathbb{R}\right\}.$$

2. The solution set is given by

$$\{\,(1+3t,\ 7+t,\ 0,\ t)\ \mid\ t\in\mathbb{R}\,\}.$$

3. The solution set is given by

$$\{\,(1+3t,\ 7+t,\ -1-2t,\ t)\ \mid\ t\in\mathbb{R}\,\}.$$

Question 28. Suppose that each of the following is the augmented matrix for a linear system. The matrices are all in row echelon form. Find the reduced row echelon form of each matrix, and determine whether the corresponding system is consistent. If it is consistent, find the complete solution set.

1. $\left(\begin{array}{cccccc|c} 1 & -2 & 0 & 3 & 5 & -1 & 1 \\ 0 & 0 & 1 & 4 & 23 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

2. $\left(\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 1 & 8 \\ 0 & 1 & 3 & -2 & 6 & -11 \\ 0 & 0 & 0 & 1 & -3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

3. $\left(\begin{array}{ccccc|c} 1 & -5 & 2 & 3 & -2 & -4 \\ 0 & 1 & -1 & -3 & -7 & -2 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

4. $\left(\begin{array}{ccccc|c} 1 & -7 & -3 & -2 & -1 & -5 \\ 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right)$

5. $\left(\begin{array}{cccccc|c} 1 & -3 & 6 & 0 & -2 & 4 & -3 \\ 0 & 0 & 1 & -2 & 8 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

6. $\left(\begin{array}{cccccc|c} 1 & 4 & -8 & -1 & 2 & -3 & -4 \\ 0 & 1 & -7 & 2 & -9 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Solution.

1. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccccc|c} 1 & -2 & 0 & 3 & 5 & 0 & 17 \\ 0 & 0 & 1 & 4 & 23 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\,(17+2r-3s-5t,\ r,\ -9-4s-23t,\ s,\ t,\ 16)\ \mid\ r,\ s,\ t\in\mathbb{R}\,\}.$$

2. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -13 & 0 & 7 & -38 \\ 0 & 1 & 3 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\,(-38+13s-7t,\ 7-3s,\ s,\ 9+3t,\ t)\ \mid\ s,\ t\in\mathbb{R}\,\}.$$

3. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & -13 & 46 \\ 0 & 1 & -1 & 0 & -1 & 13 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\,(46+3s+13t,\ 13+s+t,\ s,\ 5-2t,\ t)\ \mid\ s,\ t\in\mathbb{R}\,\}.$$

4. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccccc|c} 1 & -7 & 0 & 0 & 12 & 0 \\ 0 & 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

The system has no solutions.

5. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccccc|c} 1 & -3 & 0 & 12 & -50 & 0 & -13 \\ 0 & 0 & 1 & -2 & 8 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\,(-13+3r-12s+50t,\ r,\ 3+2s-8t,\ s,\ t,\ -2)\ \mid\ r,\ s,\ t\in\mathbb{R}\,\}.$$

6. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 20 & -9 & 0 & 153 & -68 \\ 0 & 1 & -7 & 2 & 0 & -37 & 15 \\ 0 & 0 & 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\,(-68-20r+9s-153t,\ 15+7r-2s+37t,\ r,\ s,\ 2+4t,\ t)\ \mid\ r,\ s,\ t\in\mathbb{R}\,\}.$$

For Questions 29 – 32 do the following
Solve the following linear systems using Gauss–Jordan elimination or Gaussian elimination. In each case, indicate whether the system is consistent or inconsistent. Give the complete solution set if the system is consistent.

Question 29.

1.

$$\begin{aligned} 2x-3y &= 5 \\ -x+y &= -3 \end{aligned}$$

2.

$$\begin{aligned} 2x-2y &= 1 \\ 3x &= 1 \end{aligned}$$

3.

$$\begin{aligned} 2x-z &= 4 \\ x+4y+z &= 2 \\ 4x+y-z &= 1 \end{aligned}$$

4.

$$\begin{aligned} -3x+y+z &= 2 \\ -4z &= 0 \\ -4x+2y-3z &= 1 \end{aligned}$$

5.

$$\begin{aligned} 2x_1-x_3 &= 4 \\ x_1+4x_2+x_3 &= 2 \end{aligned}$$

6.

$$\begin{aligned} 4x_1+x_2-4x_3 &= 1 \\ 4x_1-4x_2+2x_3 &= -2 \end{aligned}$$

7.

$$\begin{aligned} 2x_1+4x_2+2x_3+2x_4 &= -2 \\ 4x_1-2x_2-3x_3-2x_4 &= 2 \\ x_1+3x_2+3x_3-3x_4 &= -4 \end{aligned}$$

8.

$$\begin{aligned} 3x_1-3x_3+4x_4 &= -3 \\ -4x_1+2x_2-2x_3-4x_4 &= 4 \\ 4x_2-3x_3+2x_4 &= -3 \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{cc|c} 2 & -3 & 5 \\ -1 & 1 & -3 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right)$$

The system has a unique solution $(4,\ 1)$.

2. The augmented matrix for the linear system is

$$\left(\begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 0 & 1 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{6} \end{array}\right)$$

The system has a unique solution $\left(\frac{1}{3},\ -\frac{1}{6}\right)$.

3. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 4 \\ 1 & 4 & 1 & 2 \\ 4 & 1 & -1 & 1 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{18}{5} \\ 0 & 1 & 0 & \frac{21}{5} \\ 0 & 0 & 1 & -\frac{56}{5} \end{array}\right)$$

The system has a unique solution $\left(-\frac{18}{5},\ -\frac{21}{5},\ -\frac{56}{5}\right)$.

4. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} -3 & 1 & 1 & 2 \\ 0 & 0 & -4 & 0 \\ -4 & 2 & -3 & 1 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system has a unique solution $\left(-\frac{3}{2},\ -\frac{5}{2},\ 0\right)$.

5. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 4 \\ 1 & 4 & 1 & 2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 2 \\ 0 & 1 & \frac{3}{8} & 0 \end{array}\right)$$

The solution set is

$$\left\{ \left(2 + \frac{t}{2},\ -\frac{3t}{8},\ t\right) \;\middle|\; t \in \mathbb{R} \right\}.$$

6. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 4 & 1 & -4 & 1 \\ 4 & -4 & 2 & -2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{7}{10} & \frac{1}{10} \\ 0 & 1 & -\frac{6}{5} & \frac{3}{5} \end{array}\right)$$

The solution set is

$$\left\{ \left(\frac{1}{10} + \frac{7t}{10}, \frac{3}{5} + \frac{6t}{5}, t \right) \;\middle|\; t \in \mathbb{R} \right\}.$$

7. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 2 & 4 & 2 & 2 & -2 \\ 4 & -2 & -3 & -2 & 2 \\ 1 & 3 & 3 & -3 & -4 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{21}{13} & -\frac{7}{13} \\ 0 & 1 & 0 & \frac{40}{13} & \frac{9}{13} \\ 0 & 0 & 1 & -\frac{46}{13} & -\frac{24}{13} \end{array}\right)$$

The solution set is

$$\left\{ \left(-\frac{7}{13} + \frac{21t}{13}, \frac{9}{13} - \frac{40t}{13}, -\frac{24}{13} + \frac{46t}{13}, t \right) \;\middle|\; t \in \mathbb{R} \right\}.$$

8. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 3 & 0 & -3 & 4 & -3 \\ -4 & 2 & -2 & -4 & 4 \\ 0 & 4 & -3 & 2 & -3 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{34}{27} & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{4}{9} & -1 \\ 0 & 0 & 1 & -\frac{2}{27} & -\frac{1}{3} \end{array}\right)$$

The solution set is

$$\left\{ \left(-\frac{4}{3} - \frac{34t}{27}, -1 - \frac{4t}{9}, -\frac{1}{3} + \frac{2t}{27}, t \right) \;\middle|\; t \in \mathbb{R} \right\}.$$

Question 30.

1.

$$\begin{aligned} x+y &= 1 \\ 4x+3y &= 2 \end{aligned}$$

2.

$$\begin{aligned} -3x+y &= 1 \\ 4x+2y &= 0 \end{aligned}$$

3.

$$\begin{aligned} 3x-3y &= 3 \\ 4x-y-3z &= 3 \\ -2x-2y &= -2 \end{aligned}$$

4.

$$\begin{aligned} 2x-4z &= 1 \\ 4x+3y-2z &= 0 \\ 2x+2z &= 0 \end{aligned}$$

5.

$$\begin{aligned} -3x_2-x_3 &= 2 \\ x_1+x_3 &= -2 \end{aligned}$$

6.

$$\begin{aligned} x+2y+z &= 1 \\ 2x+3y+2z &= 0 \\ x+y+z &= 2 \end{aligned}$$

7.

$$\begin{aligned} 3x-2z &= -3 \\ -2x+z &= -2 \\ -z &= 2 \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 3 & 2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right)$$

The system has a unique solution $(-1,\ 2)$.

2. The augmented matrix for the linear system is

$$\left(\begin{array}{cc|c} -3 & 1 & 1 \\ 4 & 2 & 0 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{2}{5} \end{array}\right)$$

The system has a unique solution $\left(-\frac{1}{5},\ \frac{2}{5}\right)$.

3. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 3 & -3 & 0 & 3 \\ 4 & -1 & -3 & 3 \\ -2 & -2 & 0 & -2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

The system has a unique solution $\left(1,\ 0,\ \frac{1}{3}\right)$.

4. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & 1 \\ 4 & 3 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \end{array}\right)$$

The system has a unique solution $\left(\frac{1}{6},\ -\frac{1}{3},\ -\frac{1}{6}\right)$.

5. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 0 & -3 & -1 & 2 \\ 1 & 0 & 1 & -2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \end{array}\right)$$

The solution set is given by

$$\left\{ \left(-2-t,\ -\frac{2}{3}-\frac{t}{3},\ t\right) \;\middle|\; t \in \mathbb{R} \right\}.$$

6. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The system is inconsistent.

7. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 3 & 0 & -2 & -3 \\ -2 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The system is inconsistent.

Question 31.

1.

$$\begin{aligned} 3x_1 + 2x_2 + 3x_3 &= -3 \\ x_1 + 2x_2 - x_3 &= -2 \end{aligned}$$

2.

$$\begin{aligned} -x_1 + 3x_3 + x_4 &= 2 \\ 2x_1 + 3x_2 - 3x_3 + x_4 &= 2 \\ 2x_1 - 2x_2 - 2x_3 - x_4 &= -2 \end{aligned}$$

3.

$$\begin{aligned} 3x_1 - x_2 + 3x_3 + 3x_4 &= -3 \\ x_1 - x_2 + x_3 + x_4 &= 3 \\ -3x_1 + 3x_2 - x_3 + 2x_4 &= 1 \end{aligned}$$

4.

$$\begin{aligned} 3x_1 - 3x_2 + x_3 + 3x_4 &= -3 \\ x_1 + x_2 - x_3 - 2x_4 &= 3 \\ 4x_1 - 2x_2 + x_4 &= 0 \end{aligned}$$

5.

$$\begin{aligned} -3x_1 + 2x_2 - x_3 - 2x_4 &= 2 \\ x_1 - x_2 - 3x_4 &= 3 \\ 4x_1 - 3x_2 + x_3 - x_4 &= 1 \end{aligned}$$

6.

$$\begin{aligned} 3x_1 + x_2 + 7x_3 + 2x_4 &= 13 \\ 2x_1 - 4x_2 + 14x_3 - x_4 &= -10 \\ 5x_1 + 11x_2 - 7x_3 + 8x_4 &= 59 \\ 2x_1 + 5x_2 - 4x_3 - 3x_4 &= 39 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 4x_4 &= 9 \\ x_1 - 2x_3 + 7x_4 &= 11 \\ 3x_1 - 3x_2 + x_3 + 5x_4 &= 8 \\ 2x_1 + x_2 + 4x_3 + 4x_4 &= 10 \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 3 & 2 & 3 & -3 \\ 1 & 2 & -1 & -2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{4} \end{array}\right)$$

The solution set is

$$\left\{ \left(-\frac{1}{2} - 2t,\ -\frac{3}{4} + \frac{3t}{2},\ t\right) \ \middle|\ t \in \mathbb{R} \right\}.$$

2. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} -1 & 0 & 3 & 1 & 2 \\ 2 & 3 & -3 & 1 & 2 \\ 2 & -2 & -2 & -1 & -2 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 1 \end{array}\right)$$

The solution set is

$$\left\{ \left(1 - \frac{t}{2},\ 1 - \frac{t}{2},\ 1 - \frac{t}{2},\ t\right) \ \middle|\ t \in \mathbb{R} \right\}.$$

3. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 3 & -1 & 3 & 3 & -3 \\ 1 & -1 & 1 & 1 & 3 \\ -3 & 3 & -1 & 2 & 1 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{3}{2} & -8 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & \frac{5}{2} & 5 \end{array}\right)$$

The solution set is

$$\left\{ \left(-8 + \frac{3t}{2},\ -6,\ 5 - \frac{5t}{2},\ t\right) \ \middle|\ t \in \mathbb{R} \right\}.$$

4. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 3 & -3 & 1 & 3 & -3 \\ 1 & 1 & -1 & -2 & 3 \\ 4 & -2 & 0 & 1 & 0 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & -\frac{1}{2} & 1 \\ 0 & 1 & -\frac{2}{3} & -\frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is

$$\left\{ \left(1+\frac{s}{3}+\frac{t}{2},\ 2+\frac{2s}{3}+\frac{3t}{2},\ s,\ t\right) \ \middle|\ s,\ t\in\mathbb{R} \right\}.$$

5. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} -3 & 2 & -1 & -2 & 2 \\ 1 & -1 & 0 & -3 & 3 \\ 4 & -3 & 1 & -1 & 1 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 8 & -8 \\ 0 & 1 & 1 & 11 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is

$$\{ (-8-s-8t,\ -11-s-11t,\ s,\ t) \ |\ s,\ t\in\mathbb{R} \}.$$

6. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 3 & 1 & 7 & 2 & 13 \\ 2 & -4 & 14 & -1 & -10 \\ 5 & 11 & -7 & 8 & 59 \\ 2 & 5 & -4 & -3 & 39 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is

$$\{ (4-3t,\ 5+2t,\ t,\ -2) \ |\ t\in\mathbb{R} \}.$$

7. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 2 & -1 & 3 & 4 & 9 \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

The system has a unique solution $(-1,\ 0,\ 1,\ 2)$

Question 32.

1.

$$\begin{aligned} -5x_1 - 2x_2 + 2x_3 &= 16 \\ 3x_1 + x_2 - x_3 &= -9 \\ 2x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

2.

$$\begin{aligned} 3x_1 - 3x_2 - 2x_3 &= 23 \\ -6x_1 + 4x_2 + 3x_3 &= -40 \\ -2x_1 + x_2 + x_3 &= -12 \end{aligned}$$

3.

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= -54 \\ -x_1 + x_2 - 2x_3 &= 20 \\ 5x_1 - 4x_2 + 8x_3 &= -83 \end{aligned}$$

4.

$$\begin{aligned} 4x_1 - 2x_2 - 7x_3 &= 5 \\ -6x_1 + 5x_2 + 10x_3 &= -11 \\ -2x_1 + 3x_2 + 4x_3 &= -3 \\ -3x_1 + 2x_2 + 5x_3 &= -5 \end{aligned}$$

5.

$$\begin{aligned} -2x_1 + 3x_2 - 4x_3 + x_4 &= -17 \\ 8x_1 - 5x_2 + 2x_3 - 4x_4 &= 47 \\ -5x_1 + 9x_2 - 13x_3 + 3x_4 &= -44 \\ -4x_1 + 3x_2 - 2x_3 + 2x_4 &= -25 \end{aligned}$$

6.

$$\begin{aligned} 5x_1 - x_2 - 9x_3 - 2x_4 &= 26 \\ 4x_1 - x_2 - 7x_3 - 2x_4 &= 21 \\ -2x_1 + 4x_3 + x_4 &= -12 \\ -3x_1 + 2x_2 + 4x_3 + 2x_4 &= -11 \end{aligned}$$

7.

$$\begin{aligned} 6x_1 - 12x_2 - 5x_3 + 16x_4 - 2x_5 &= -53 \\ -3x_1 + 6x_2 + 3x_3 - 9x_4 + x_5 &= 29 \\ -4x_1 + 8x_2 + 3x_3 - 10x_4 + x_5 &= 33 \end{aligned}$$

8.

$$\begin{aligned} 5x_1 - 5x_2 - 15x_3 - 3x_4 &= -34 \\ -2x_1 + 2x_2 + 6x_3 + x_4 &= 12 \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} -5 & -2 & 2 & 16 \\ 3 & 1 & -1 & -9 \\ 2 & 2 & -1 & -4 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right).$$

The system has a unique solution $(-2,\ 3,\ 6)$.

2. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 3 & -3 & -2 & 23 \\ -6 & 4 & 3 & -40 \\ -2 & 1 & 1 & -12 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

The system has a unique solution $(5,\ -4,\ 2)$.

3. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 3 & -2 & 4 & -54 \\ -1 & 1 & -2 & 20 \\ 5 & -4 & 8 & -83 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

The system has no solutions.

4. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 4 & -2 & -7 & 5 \\ -6 & 5 & 10 & -11 \\ -2 & 3 & 4 & -3 \\ -3 & 2 & 5 & -5 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system has a unique solution $(6,\ -1,\ 3)$.

5. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} -2 & 3 & -4 & 1 & -17 \\ 8 & -5 & 2 & -4 & 47 \\ -5 & 9 & -13 & 3 & -44 \\ -4 & 3 & -2 & 2 & -25 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 7 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The solution set is

$$\{\, (7+t,\ -3+2t,\ t,\ 6) \ \mid\ t \in \mathbb{R} \,\}.$$

6. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 5 & -1 & -9 & -2 & 26 \\ 4 & -1 & -7 & -2 & 21 \\ -2 & 0 & 4 & 1 & -12 \\ -3 & 2 & 4 & 2 & -11 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

The system is inconsistent.

7. The augmented matrix for the linear system is

$$\left(\begin{array}{ccccc|c} 6 & -12 & -5 & 16 & -2 & -53 \\ -3 & 6 & 3 & -9 & 1 & 29 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right).$$

The solution set is

$$\{\, (-4 + 2s - t,\ s,\ 5 + 2t,\ t,\ 2) \mid s,\ t \in \mathbb{R} \,\}.$$

8. The augmented matrix for the linear system is

$$\left(\begin{array}{cccc|c} 5 & -5 & -15 & -3 & -34 \\ -2 & 2 & 6 & 1 & 12 \end{array}\right)$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -1 & -3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right).$$

The solution set is

$$\{\, (-2 + s + 3t,\ s,\ t,\ 8) \mid s,\ t \in \mathbb{R} \,\}.$$

Question 33.

Find the reduced row echelon form of the given matrix

1. $\begin{pmatrix}1&0&2\\0&1&3\end{pmatrix}$

2. $\begin{pmatrix}1&2&0\\0&0&1\end{pmatrix}$

3. $\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix}$

4. $\begin{pmatrix}1&2&0\\0&0&2\\0&0&0\end{pmatrix}$

5. $\begin{pmatrix}1&2&0&-1\\0&0&1&-2\\0&0&0&0\end{pmatrix}$

6. $\begin{pmatrix}1&0&-3&4\\0&1&1&5\\0&0&0&0\end{pmatrix}$

7. $\begin{pmatrix}1&0&0&4&-1\\0&0&1&5&2\\0&1&0&0&-1\end{pmatrix}$

8. $\begin{pmatrix}1&1&0&4&\frac{2}{3}\\0&1&1&5&6\\0&0&0&1&\frac{1}{3}\end{pmatrix}$

9. $\begin{pmatrix}-2&2&-1&2\\0&3&3&-3\\1&-4&2&2\end{pmatrix}$

10. $\begin{pmatrix}4&-3&-4&-2\\-4&2&1&-4\\-1&-3&1&-4\end{pmatrix}$

11. $\begin{pmatrix}-4&1&4\\3&4&-3\end{pmatrix}$

12. $\begin{pmatrix}-4&-2&-1\\-2&-3&0\end{pmatrix}$

Solution.

1. $\begin{pmatrix}1&0&2\\0&1&3\end{pmatrix}$

2. $\begin{pmatrix}1&2&0\\0&0&1\end{pmatrix}$

3. $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

4. $\begin{pmatrix}1&2&0\\0&0&1\\0&0&0\end{pmatrix}$

5. $\begin{pmatrix}1&2&0&-1\\0&0&1&-2\\0&0&0&0\end{pmatrix}$

6. $\begin{pmatrix}1&0&-3&4\\0&1&1&5\\0&0&0&0\end{pmatrix}$

7. $\begin{pmatrix}1&0&0&4&-1\\0&1&0&0&-1\\0&0&1&5&2\end{pmatrix}$

8. $\begin{pmatrix}1&0&-1&0&-5\\0&1&1&0&\frac{13}{3}\\0&0&0&1&\frac{1}{3}\end{pmatrix}$

9. $\begin{pmatrix}1&0&0&-2\\0&1&0&-1\\0&0&1&0\end{pmatrix}$

10. $\begin{pmatrix}1&0&0&2\\0&1&0&\frac{6}{5}\\0&0&1&\frac{8}{5}\end{pmatrix}$

11. $\begin{pmatrix}1&0&-1\\0&1&0\end{pmatrix}$

12. $\begin{pmatrix}1&0&\frac{3}{8}\\0&1&-\frac{1}{4}\end{pmatrix}$

Question 34.

Consider each of the following matrices as the augmented matrix of a linear system:

$$1.\ \left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 2 & 3 & -2 & b \\ -1 & -1 & 1 & c \end{array}\right), \quad 2.\ \left(\begin{array}{cc|c} a & 1 & 1 \\ 2 & a-1 & 1 \end{array}\right), \quad 3.\ \left(\begin{array}{ccc|c} -2 & 3 & 1 & a \\ 1 & 1 & -1 & b \\ 0 & 5 & -1 & c \end{array}\right)$$

For each case,

(a) Determine the values of a, b, c for which this linear system is *inconsistent*.

(b) Determine the values of a, b, c for which this linear system is *consistent*.

(c) If this system is consistent, determine whether it has a unique solution or infinitely many solutions. Express the solution(s) in terms of a, b, c.

(d) Choose a specific set of values for a, b, c that ensures the system is consistent and solve the system.

Solution.

1. Applying Gauss-Jordan elimination, we have

$$\xrightarrow[R_2\to R_2-2R_1]{R_3\to R_3+R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & 0 & b-2a \\ 0 & 1 & 0 & c+a \end{array}\right) \xrightarrow{R_2\to -R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & 1 & 0 & 2a-b \\ 0 & 1 & 0 & c+a \end{array}\right) \xrightarrow{R_3\to R_3-R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & 1 & 0 & 2a-b \\ 0 & 0 & 0 & c-a+b \end{array}\right) \xrightarrow{R_1\to R_1-2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -3a+b \\ 0 & 1 & 0 & 2a-b \\ 0 & 0 & 0 & c-a+b \end{array}\right)$$

(a) If

$$c-a+b\neq 0$$

then the system is inconsistent.

(b) If

$$c-a+b=0$$

the system is consistent.

(c) In the case $c-a+b=0$, the system has infinitely many solutions given by

$$\{\,(-3a+b+t,\ 2a-b,\ t)\ \mid\ t\in\mathbb{R}\,\}.$$

No matter what values $a,\ b,\ c$ take, the system will not have a unique solution.

d) Let

$$a=1,\ b=1,\ c=0,$$

which satisfies $c-a+b=0$. Then the solution set for the system becomes

$$\{\,(-2+t,\ 1,\ t)\ \mid\ t\in\mathbb{R}\,\}.$$

2. Applying Gauss-Jordan elimination, we have

$$\xrightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{cc|c} 2 & a-1 & 1 \\ a & 1 & 1 \end{array}\right) \xrightarrow{R_1\to\frac{1}{2}R_1} \left(\begin{array}{cc|c} 1 & \frac{a-1}{2} & \frac{1}{2} \\ a & 1 & 1 \end{array}\right) \xrightarrow{R_2\to R_2-aR_1} \left(\begin{array}{cc|c} 1 & \frac{a-1}{2} & \frac{1}{2} \\ 0 & \frac{2+a-a^2}{2} & 1-\frac{a}{2} \end{array}\right)$$

(a) The system is inconsistent if

$$\frac{2+a-a^2}{2} = 0, \quad \text{which implies} \quad a = -1,\ 2$$

and

$$1 - \frac{a}{2} \neq 0, \quad \text{which implies} \quad a \neq 2.$$

Thus, we conclude that when

$$a = -1.$$

the system is inconsistent.

(b) The system is consistent if $a \neq -1$.

(c) The system has infinitely many solutions if

$$\frac{2+a-a^2}{2} = 0, \quad \text{which implies} \quad a = -1,\ 2$$

and

$$1 - \frac{a}{2} = 0, \quad \text{which implies} \quad a = 2.$$

Thus when $a = 2$, the system has infinitely many solutions. In this case, the augmented matrix becomes

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right).$$

The solution set is given by

$$\left\{ \left(\frac{1}{2} - \frac{t}{2},\ t\right) \;\middle|\; t \in \mathbb{R} \right\}.$$

When $a \neq -1, 2$, we can continue with Gauss-Jordan elimination to get

$$\xrightarrow{R_2 \to \frac{2}{2+a-a^2} R_2} \left(\begin{array}{cc|c} 1 & \frac{a-1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{a+1} \end{array}\right) \xrightarrow{R_1 \to R_1 - \frac{a-1}{2} R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{a+1} \\ 0 & 1 & \frac{1}{a+1} \end{array}\right).$$

The system has a unique solution $\left(\frac{1}{a+1},\ \frac{1}{a+1}\right)$

(d) Let $a = 0$, then the system has a unique solution $(1,\ 1)$.

3. Applying Gauss-Jordan elimination, we have

$$\xrightarrow{R_1 \to -\frac{1}{2} R_1} \left(\begin{array}{ccc|c} 1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{a}{2} \\ 1 & 1 & -1 & b \\ 0 & 5 & -1 & c \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{a}{2} \\ 0 & \frac{5}{2} & -\frac{1}{2} & b + \frac{a}{2} \\ 0 & 5 & -1 & c \end{array}\right) \xrightarrow{R_2 \to \frac{2}{5} R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{a}{2} \\ 0 & 1 & -\frac{1}{5} & \frac{2b}{5} + \frac{a}{5} \\ 0 & 5 & -1 & c \end{array}\right) \xrightarrow{R_3 \to R_3 - 5R_2} \left(\begin{array}{ccc|c} 1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{a}{2} \\ 0 & 1 & -\frac{1}{5} & \frac{2b}{5} + \frac{a}{5} \\ 0 & 0 & 0 & c - 2b - a \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + \frac{3}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{5} & \frac{3b}{5} - \frac{a}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2b}{5} + \frac{a}{5} \\ 0 & 0 & 0 & c - 2b - a \end{array}\right)$$

(a) The system is inconsistent when

$$c - 2b - a \neq 0.$$

(b) The system is consistent when

$$c - 2b - a = 0.$$

(c) When $c - 2b - a = 0$, the system has infinitely many solutions and the solution set is given by

$$\left\{ \left(\frac{3b}{5} - \frac{a}{5} + \frac{4t}{5},\ \frac{2b}{5} + \frac{a}{5} + \frac{t}{5},\ t \right) \;\middle|\; t \in \mathbb{R} \right\}.$$

No matter what values $a,\ b,\ c$ take, the system will not have a unique solution.

(d) Let

$$a = 0,\ b = 1,\ c = 2,$$

then

$$c - 2b - a = 0.$$

In this case, the system has solution set

$$\left\{ \left(\frac{3}{5} + \frac{4t}{5},\ \frac{2}{5} + \frac{t}{5},\ t \right) \;\middle|\; t \in \mathbb{R} \right\}.$$

Question 35. Suppose $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are two distinct solutions of the linear system

$$A\boldsymbol{x} = \boldsymbol{b},$$

where $A \in \mathcal{M}_{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$.

1. Show that for any $\alpha \in \mathbb{R}$, $\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1)$ is also a solution of the linear system.
2. Show that if $\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{x}_1 + \beta(\boldsymbol{x}_2 - \boldsymbol{x}_1)$, where $\alpha,\ \beta \in \mathbb{R}$, then $\alpha = \beta$.
3. Prove that if a linear system has two distinct solutions, then it must have infinitely many solutions.

Solution.

1. Since $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are solutions of

$$A\boldsymbol{x} = \boldsymbol{b},$$

we have

$$A\boldsymbol{x}_1 = \boldsymbol{b}, \quad A\boldsymbol{x}_2 = \boldsymbol{b}.$$

Take any $\alpha \in \mathbb{R}$, then

$$A\left(\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1)\right) = A\boldsymbol{x}_1 + \alpha A(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{b} + \alpha(A\boldsymbol{x}_2 - A\boldsymbol{x}_1) = \boldsymbol{b} + \boldsymbol{0} = \boldsymbol{b}.$$

This shows that $\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1)$ is also a solution of the linear system.

2. Since $\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{x}_1 + \beta(\boldsymbol{x}_2 - \boldsymbol{x}_1)$, we have

$$\alpha\boldsymbol{x}_2 + (1-\alpha)\boldsymbol{x}_1 = \beta\boldsymbol{x}_2 + (1-\beta)\boldsymbol{x}_1 \Longrightarrow (\alpha-\beta)\boldsymbol{x}_2 = (\alpha-\beta)\boldsymbol{x}_1.$$

   Since $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, we must have $\alpha - \beta = 0$, which implies $\alpha = \beta$.

3. Suppose a linear system

$$A\boldsymbol{x} = \boldsymbol{b}$$

   has two distinct solutions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. From 1 we know that $\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1)$ is also a solution of the linear system for any $\alpha \in \mathbb{R}$. From 2 we know that for different values of α, we can get different solutions $\boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1)$. There are infinitely many different values for α, by varying α, we get infinitely many solutions for the linear system.