Tutorial 8

Eulerian tours

Homework

1. Using either *Fleury's Algorithm* or *Hierholzer's Algorithm*, find an Eulerian circuit in any *two* of the following graphs: Q4–5, Q4–6, Q6–1, Q6–2, Q6–4, Q7–9, Q10–3, Q11–3, Q12–3, Q13–3, Q14–3.
2. For *one* of the graphs in Question 9 (parts 2–10), find an optimal Eulerization and provide a clear explanation of why your solution is optimal.

Question 1. Let G be a graph with vertex set $V(G) = \{a, b, c, d, e\}$ and edge set $E(G) = \{ab, ae, bc, cd, de, ea, eb\}$.

1. Draw G.
2. Is G connected?
3. Is G simple?
4. List the degrees of every vertex.
5. Find all edges incident to b.
6. List all the neighbors of a.
7. Find a walk, trail, and path in G, each of which has length 3.
8. Find a closed walk, circuit, and cycle in G, each of which starts at e.
9. Is G Eulerian, semi-Eulerian, or neither? Explain your answer.

Question 2. Which of the following scenarios could be modeled using an Eulerian circuit? Explain your answer.

1. A photographer wishes to visit each of the seven bridges in a city, take photos, then return to his hotel.
2. Salem Public Works must repave all the streets in the downtown area.
3. Frank's Flowers needs to deliver bouquets to 6 customers throughout the city, starting and ending at the flower shop.
4. Richmond Water Authority must read all the water meters throughout the town. One worker is tasked with this job.
5. Sam works in sales for a Fortune 500 company. He spends each day visiting his clients around southwest Virginia and must plan his route to avoid backtracking as much as possible.

Question 3. Recall the following theorem from the lecture

Theorem 1 *A graph G is Eulerian if and only if*

- *G is connected and*
- *every vertex has an even degree*

A graph G is semi-Eulerian if and only if

- *G is connected and*
- *exactly two vertices have odd degree*

Explain why a graph cannot be both Eulerian and semi-Eulerian.

Question 4. For each of the graphs in Figure 1

(a) Find the degree of each vertex.

(b) Use the results from 1 to determine if the graph is Eulerian, semi-Eulerian, or neither.

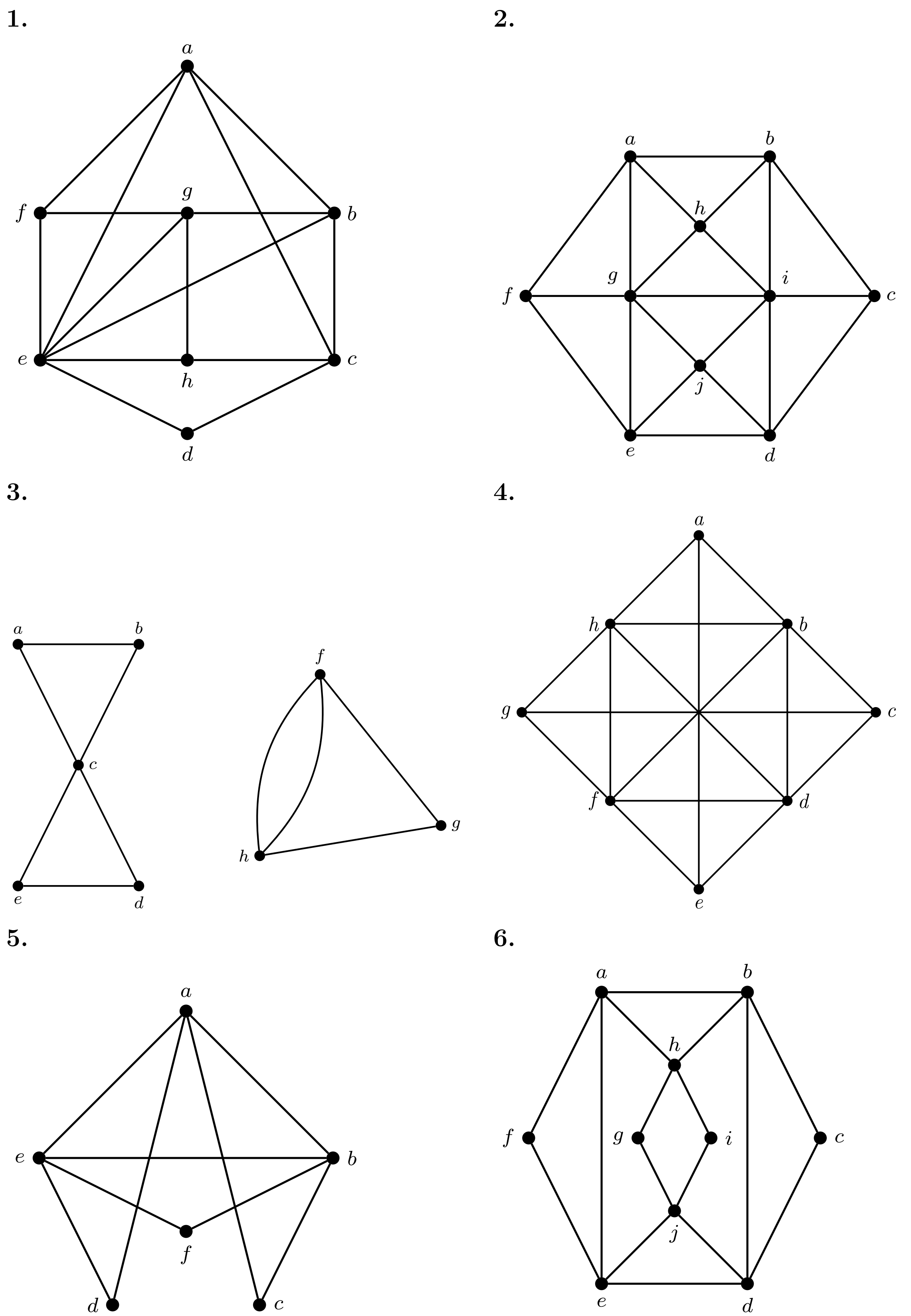
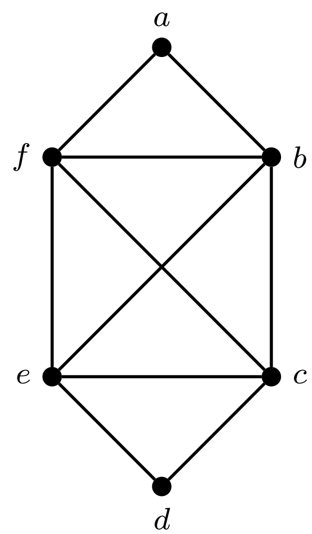


Figure 1

Question 5. For those graphs in Figure 1 that have one, find an Eulerian circuit or Eulerian trail.

Question 6. Find an Eulerian circuit or Eulerian trail for each of the graphs below

1.



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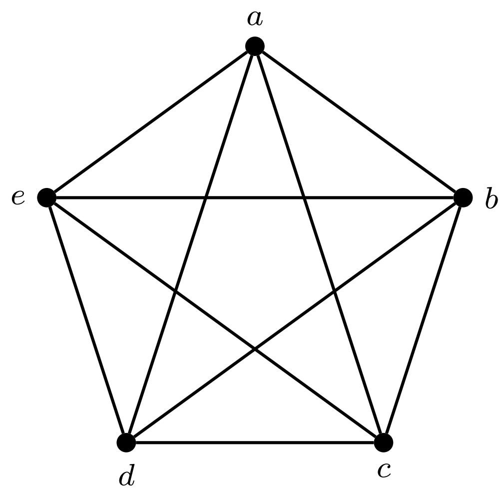

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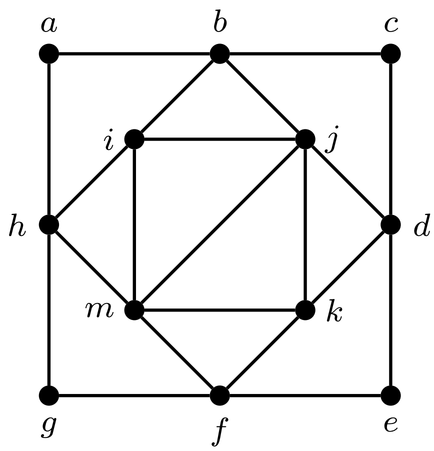

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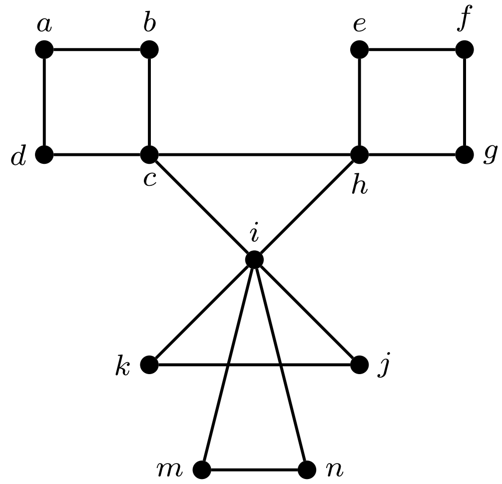


Question 7. For each of the following graphs G defined by its edge set, construct its graphical representation. Determine whether G admits an Eulerian circuit or an Eulerian path. If yes, find it using Fleury's algorithm or Hierholzer's algorithm.

1. $E(G) = \{ab, bc, cd, da, ac\}$
2. $E(G) = \{ab, bc, cd, da, bd\}$
3. $E(G) = \{ab, bc, cd, da, ac, bd, be\}$
4. $E(G) = \{mn, no, op, pm, mo, np, pq\}$
5. $E(G) = \{pq, qr, rs, st, tp, pr, qs\}$
6. $E(G) = \{ab, bc, cd, de, ea, bd, ce\}$

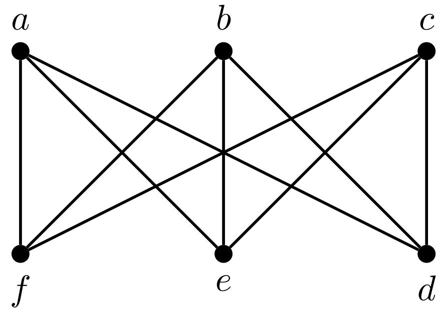
7. $E(G) = \{ab, bc, cd, de, ef, fg, gh, hi, ij, jk, kl, la, ag, bh, ci, dj, ek, fl\}$
8. $E(G) = \{ab, bc, cd, de, ef, fa, ag, bh, ci, dj, ek, fl, gh, hi, ij, jk, kl, la, aj, bi, ch, dg\}$
9. $E(G) = \{ab, bc, cd, de, ef, fg, gh, ha, ac, bd, ce, df, eg, fh, ga, hb, ad, be, cf, dg, eh, fa, gb, hc\}$
10. $E(G) = \{ab, bc, cd, de, ef, fa, gh, hi, ij, jg, kl, lm, mn, nk, ag, gk, bh, hl, ci, im, dj\}$

Question 8. For each of the following graph types, determine whether an Eulerian circuit or path is possible and justify your conclusions.

1. A complete graph.
2. A tree - a connected graph that contains no cycles or circuits
3. A graph containing at least one bridge, where a *bridge* is an edge whose removal increases the number of connected components in the graph.

Question 9. Find an optimal Eulerization and semi-Eulerization for each of the graphs below

1.



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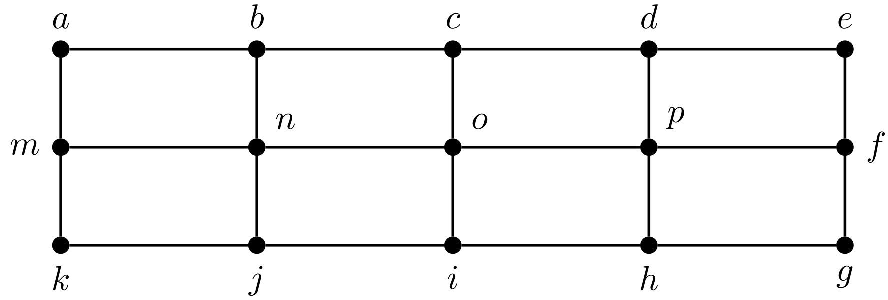

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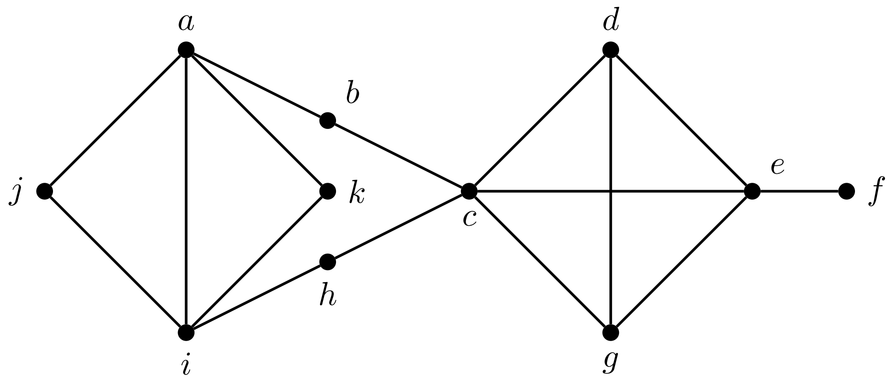

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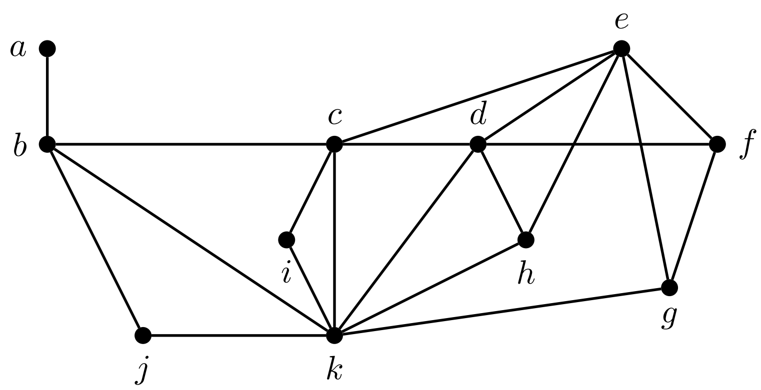

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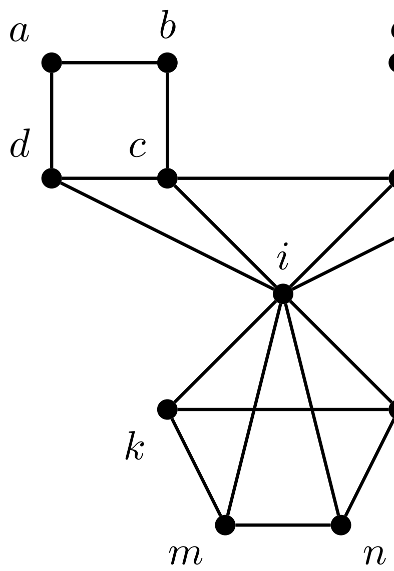

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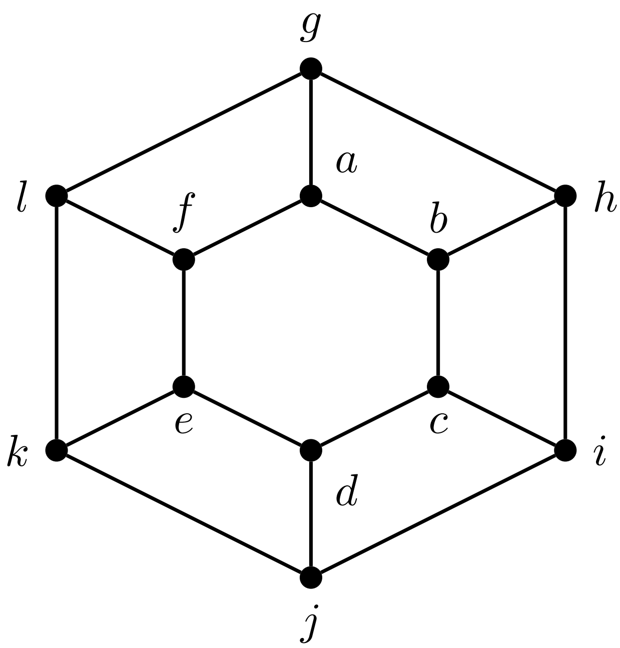

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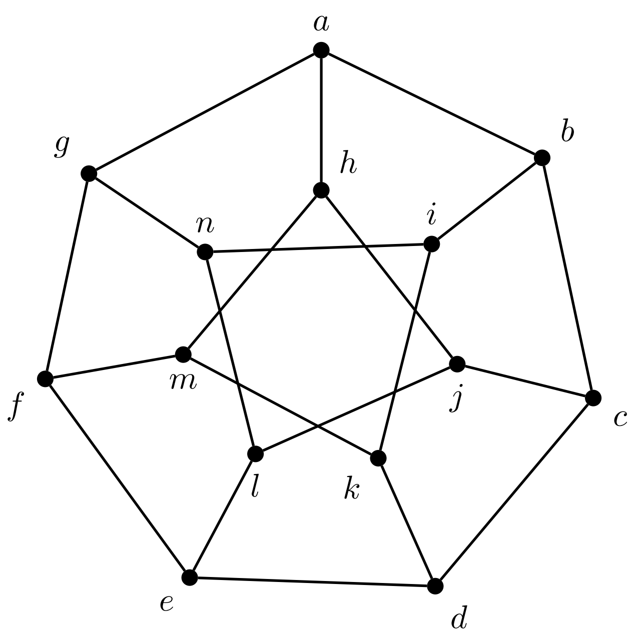

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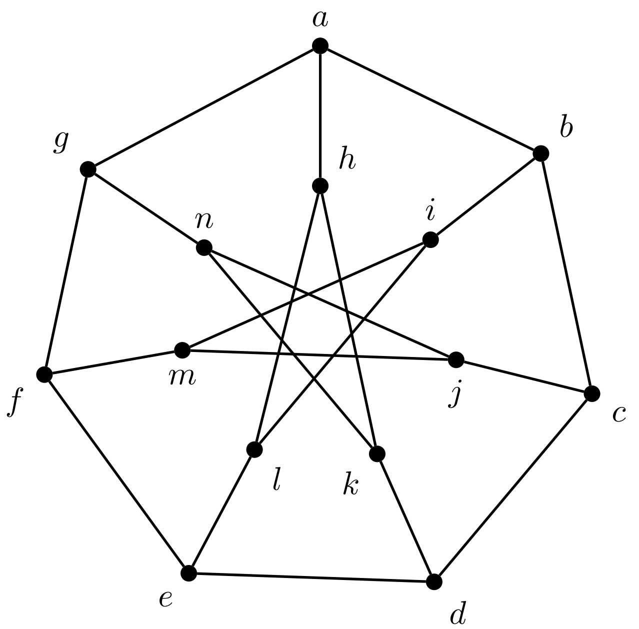

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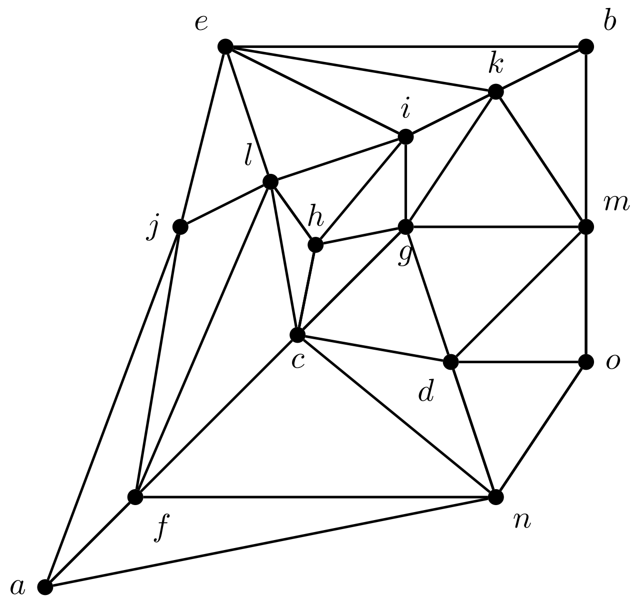

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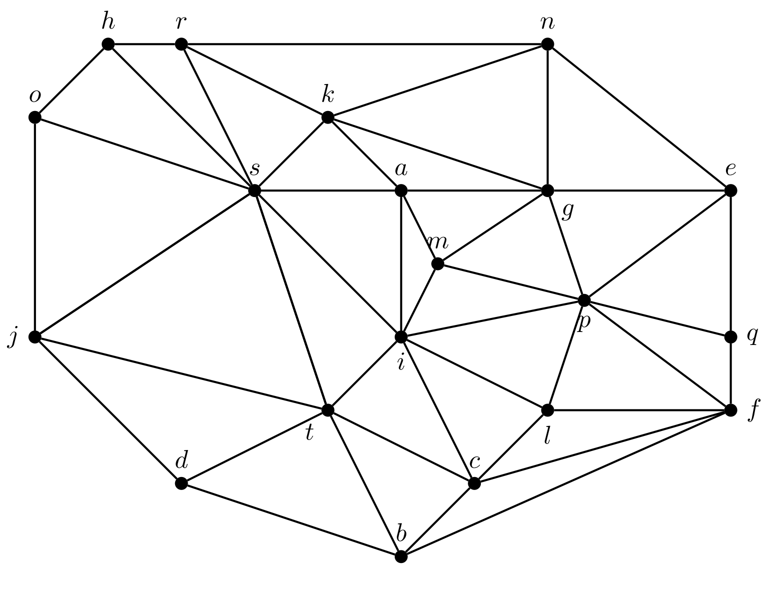


Use the following map for questions 10 and 11

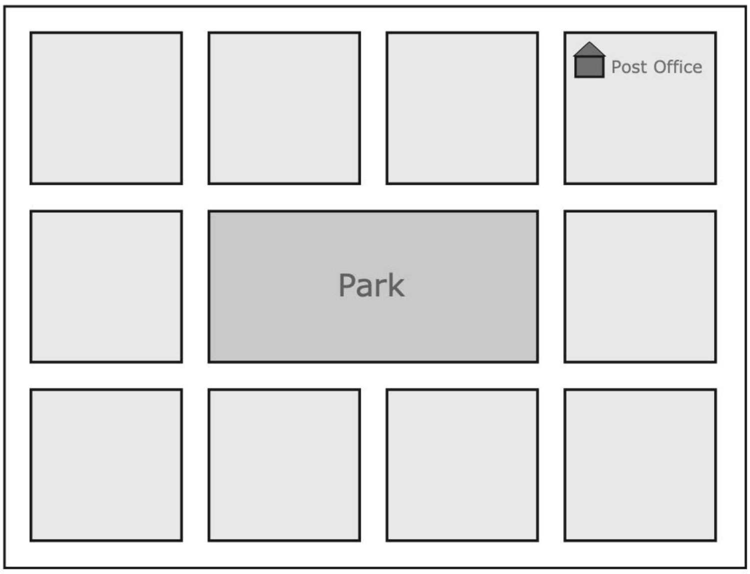
Question 10. Sarah is planning the route for the neighborhood watch night patrol for the community shown in the map above. The person on patrol must walk along each street at least once, including the perimeter of the park.

1. Model this scenario as a graph
2. Determine if the graph is Eulerian or semi-Eulerian or neither. Eulerize the graph if it is not Eulerian
3. Find an Eulerian circuit starting and ending at the post office

Question 11. A postal worker is delivering mail along her route shown in the map above. She must walk down both sides of the street if there are houses on both sides and does not need to walk the streets that only border the park (since no houses are in the park).

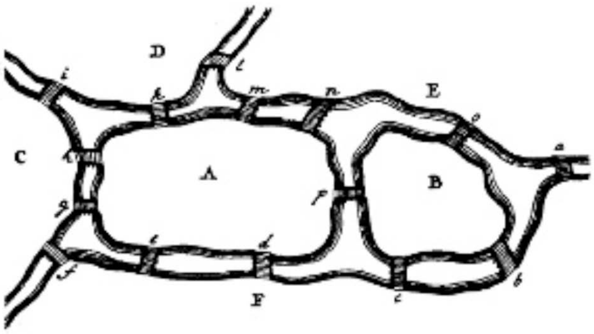
1. Model this scenario as a graph (Hint: make use of multi-edges).
2. Determine if the graph is Eulerian or semi-Eulerian or neither. Eulerize the graph if it is not Eulerian.
3. Find an Eulerian circuit starting and ending at the post office

Question 12. Repeat Question 10 with the map below.



Question 13. Repeat Question 11 with the map above.

Question 14. The image below appeared in Euler's original paper on Königsberg bridge problem[1]

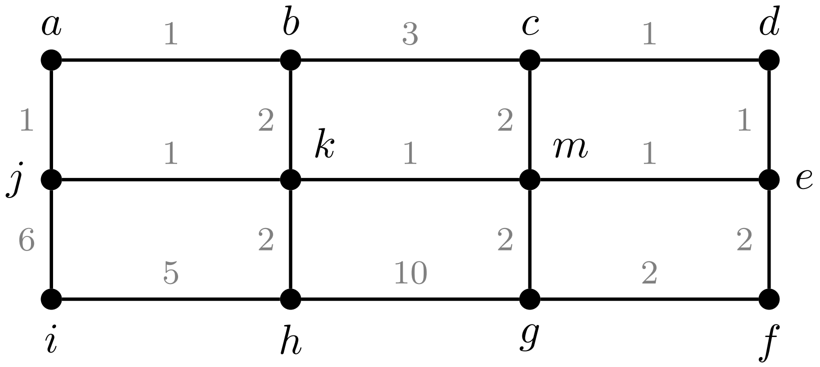


1. Model the map as a graph (Hint: make use of multi-edges).
2. Determine if the graph is Eulerian or semi-Eulerian or neither. Eulerize the graph if it is not Eulerian.
3. Find an Eulerian circuit for either the original graph or its Eulerization.

Question 15. Find an optimal Eulerization and semi-Eulerization for each of the weighted graphs below.

[1]Euler, Leonhard. *Solutio problematis ad geometriam situs pertinentis.* Commentarii academiae scientiarum Petropolitanae (1741): 128-140.

1.



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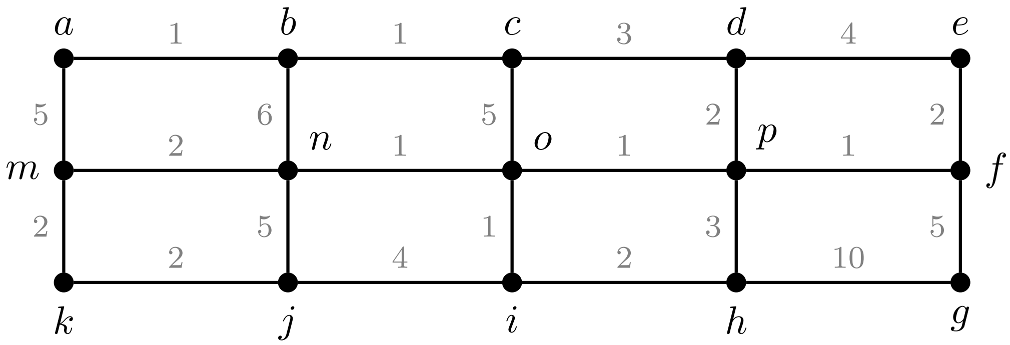


Question 16. Fleury's Algorithm can be used to find an Eulerian circuit or an Eulerian trail, yet Hierholzer's is written in such a way as to only work for Eulerian circuits. Modify Hierholzer's Algorithm so it can be used to find an Eulerian trail. (Hint: think about how Fleury's accounts for the two odd vertices.)

Question 17. A connected graph G has 8 vertices and 15 edges. You know three vertices have degree 5 and three vertices have degree 3. What are the possible degrees for the remaining two vertices?

Enrichment Questions

Question 18. Implement an algorithm in a programming language of your choice (e.g., Python, Mathematica, or C++) to determine whether a given graph contains an Eulerian trail or an Eulerian circuit. If such a trail or circuit exists, the algorithm should output the corresponding sequence of edges; otherwise, it should indicate that no Eulerian trail/circuit exists.

1. Use Fleury's algorithm or Hierholzer's algorithm.
2. Test the implementation on different example graphs and determine whether each graph has an Eulerian path or circuit.

Question 19. *Application of Eulerian circuits: a delivery service case study* Consider a scenario in which a delivery service seeks to optimize its routes by minimizing travel distance while ensuring that each street is traversed exactly once. The city streets are represented as a graph, where intersections correspond to vertices and streets between them correspond to edges. Your task is to:

1. Determine whether a route exists that satisfies this requirement.
2. If such a route exists, construct it. If no such route is possible, propose an alternative strategy that allows the delivery driver to traverse all streets while minimizing the number of repeated edges.

You may approach this problem in one of the following ways:

- Formulate a general solution applicable to any city layout.
- Apply the solution to your hometown (or a region with at least 20 vertices).
- Solve the problem specifically for Bratislava, ensuring that at least 20 vertices are considered.

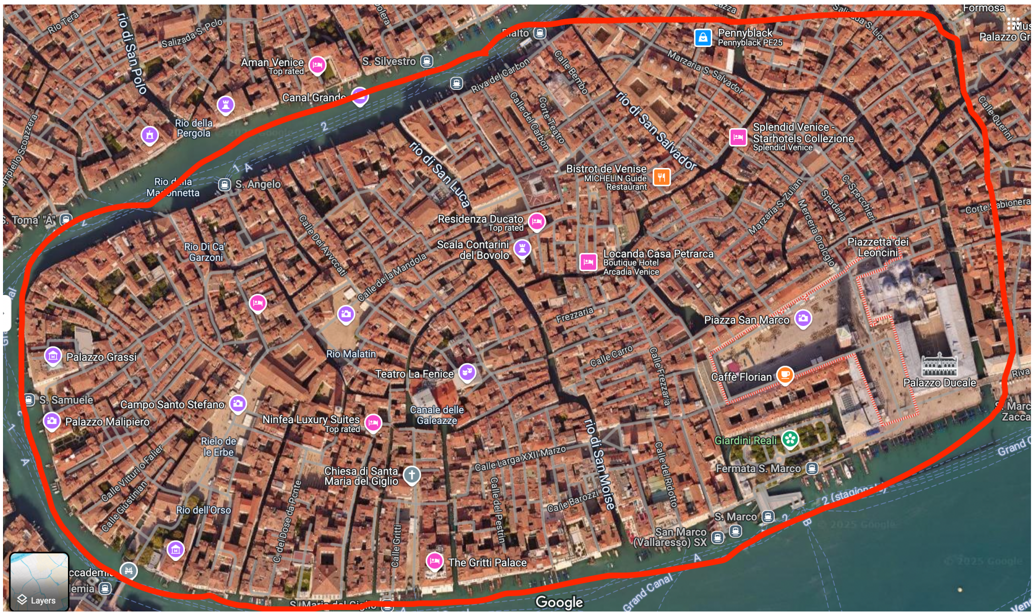
For practical implementation, use Google Maps as a reference.

Question 20. Identify and describe three real-world scenarios, distinct from those mentioned below, that can be modeled using Eulerian paths or circuits. For each case, describe why this model is appropriate, and propose a viable solution. Additionally, provide a simplified implementation of the model with at least 10 vertices.

Potential Applications:

1. Optimization of waste collection routes in urban areas.
2. Scheduling and maintenance of railway infrastructure.
3. Analysis and optimization of electrical energy distribution in power grids.

Question 21. The image represents a map of part of Venice, with the area of interest marked by a red outline. A standard Google Map is used; if you require more details, you can refer to the original source.



1. Model the map as a graph, where vertices represent landmasses (islands and riverbanks) and edges represent bridges connecting them. Include all bridges in your graph.
2. Determine if the graph is Eulerian or semi-Eulerian or neither. Eulerize the graph if it is not Eulerian
3. Find an Eulerian circuit either in the original graph or in its Eulerized version.

If you prefer not to solve the task manually, implement a program to do so.

Question 22. You have been hired to create a route for the bridge inspector for the city of Pittsburgh, Pennsylvania, which is known for its bridges. The city needs the bridges to be visually inspected on the first day of every month and so needs the route to be as short as possible in order for the inspector to complete his tour in one day.

1. Use a high quality map to create a graph similar to the one that arose from Königsberg.
2. Add edge weights that correspond to distance or time and find an optimal exhaustive route (a tour through the graph that visits each bridge at least once).
3. Write a detailed report for the Pittsburgh city manager outlining your methodology, results, and recommendations for the bridge inspector.

Question 23. Pick a city neighborhood. Using a quality map, model the neighborhood as a graph and assign weights to the edges. You may use any logical metric in assigning weights (such as time or distance). Find an optimal route for a street sweeper that must visit each street in the neighborhood.