

Tutorial 4

Determinants

Homework

1. Provide detailed solutions to *two* selected parts from Q1 (parts 6–15).
2. Provide detailed solutions to Q4 and Q5.
3. Provide detailed solutions to *two* of the following questions: Q6–Q9.
4. Provide detailed solutions to *two* of the following questions: Q11–Q20.
5. Provide detailed solutions to Q23.
6. Provide a detailed solution to *one* selected part of Q25.
7. Provide a detailed solution to *one* selected part of Q28.
8. Provide detailed solutions to *three* selected parts from Q33.
9. Provide detailed solutions to Q34, part 4.
10. Provide detailed solutions to *three* selected parts from Q35.

Question 1. Compute the determinants of the following matrices using the standard determinant formula for 2×2 matrices and Sarrus' rule for 3×3 matrices.

1.
$$\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$$

2.
$$\begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$$

3.
$$\begin{pmatrix} 6 & -12 \\ -4 & 8 \end{pmatrix}$$

4.
$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

5.
$$\begin{pmatrix} 2 & 0 & 5 \\ -4 & 1 & 7 \\ 0 & 3 & -3 \end{pmatrix}$$

6.
$$\begin{pmatrix} 3 & -2 & 4 \\ 5 & 1 & -2 \\ -1 & 3 & 6 \end{pmatrix}$$

7.
$$\begin{pmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ -1 & 8 & 4 \end{pmatrix}$$

8.
$$\begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

9.
$$\begin{pmatrix} 3 & 1 & -2 \\ -1 & 4 & 5 \\ 3 & 1 & -2 \end{pmatrix}$$

10.
$$\begin{pmatrix} \alpha - 3 & 5 \\ -3 & \alpha - 2 \end{pmatrix}$$

11.
$$\begin{pmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{pmatrix}$$

12.
$$\begin{pmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{pmatrix}$$

13.
$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{pmatrix}$$

14.
$$\begin{pmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{pmatrix}$$

15.
$$\begin{pmatrix} \beta & -4 & 3 \\ 2 & 1 & \beta^2 \\ 4 & \beta - 1 & 2 \end{pmatrix}$$

Question 2. Find all the minors and cofactors of the matrix A .

1.
$$A = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{pmatrix}, \quad 2. \ A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{pmatrix}$$

Question 3. Let

$$A = \begin{pmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -1 & 1 & 4 \end{pmatrix}$$

Find

1. Minor M_{32} and cofactor C_{32}
2. Minor M_{41} and cofactor C_{41}
3. Minor M_{44} and cofactor C_{44}
4. Minor M_{24} and cofactor C_{24}

Question 4. For each of the given matrices A, B, C , find the specified minors and cofactors.

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 3 & -1 & 6 \\ 5 & -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & -3 & 1 \\ 1 & 4 & 2 & -1 \\ 3 & -2 & 4 & 0 \\ 4 & -1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 3 & 0 & 5 \\ 2 & 1 & -1 & 4 \\ 6 & -3 & 4 & 0 \\ -1 & 5 & 1 & -2 \end{pmatrix}$$

1. Minors $M_{12}, M_{21}, M_{23}, M_{31}$ and cofactors $C_{12}, C_{22}, C_{32}, C_{21}$ for matrix A .
2. Minors $M_{12}, M_{42}, M_{24}, M_{31}, M_{34}$ and cofactors $C_{12}, C_{24}, C_{31}, C_{42}, C_{33}$ for matrix B .
3. Minors $M_{11}, M_{41}, M_{23}, M_{32}, M_{44}$ and cofactors $C_{12}, C_{24}, C_{31}, C_{42}, C_{33}$ for matrix C .

Question 5. Find the minors M_{11}, M_{31}, M_{23} and cofactors C_{12}, C_{32}, C_{22} for the following matrix

$$A = \begin{pmatrix} \alpha + 1 & \alpha & \alpha - 7 \\ \alpha - 4 & \alpha + 5 & \alpha - 3 \\ \alpha - 1 & \alpha & \alpha + 2 \end{pmatrix}$$

Question 6. Compute the determinant of the given matrix. If the matrix is invertible, find its inverse using the formula for 2×2 matrices.

1. $\begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix}$
2. $\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix}$
3. $\begin{pmatrix} -5 & 7 \\ -7 & -2 \end{pmatrix}$
4. $\begin{pmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{pmatrix}$

Question 7. Find all values of λ for which $\det(A) = 0$.

$$1. A = \begin{pmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{pmatrix}$$

$$2. A = \begin{pmatrix} \lambda - 1 & 0 \\ 2 & \lambda + 1 \end{pmatrix}$$

$$3. A = \begin{pmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{pmatrix}$$

$$5. A = \begin{pmatrix} \lambda & 2 \\ 5 & \lambda + 3 \end{pmatrix}$$

$$6. A = \begin{pmatrix} 15 & \lambda - 4 \\ \lambda + 7 & -2 \end{pmatrix}$$

$$7. A = \begin{pmatrix} \lambda - 3 & 5 & -19 \\ 0 & \lambda - 1 & 6 \\ 0 & 0 & \lambda - 2 \end{pmatrix}$$

Question 8. Let

$$A = \begin{pmatrix} 3 & -2 & 4 \\ 5 & 1 & -2 \\ -1 & 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{pmatrix}.$$

Compute the determinant of matrices A, B by a cofactor expansion along

- | | |
|----------------------|---------------------|
| 1. the first row | 2. the second row |
| 3. the third row | 4. the first column |
| 5. the second column | 6. the third column |

Question 9. Compute the determinant of the following matrices by a cofactor expansion along a suitable row or column

1.
$$\begin{pmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{pmatrix}$$

2.
$$\begin{pmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{pmatrix}$$

3.
$$\begin{pmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{pmatrix}$$

4.
$$\begin{pmatrix} 5 & 2 & 1 & 0 \\ -1 & 3 & 5 & 2 \\ 4 & 1 & 0 & 2 \\ 0 & 2 & 3 & 0 \end{pmatrix}$$

5.
$$\begin{pmatrix} 0 & 5 & 4 & 0 \\ 4 & 1 & -2 & 7 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 5 \end{pmatrix}$$

6.
$$\begin{pmatrix} 2 & 1 & 9 & 7 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

7.
$$\begin{pmatrix} 3 & 3 & 0 & 5 \\ 2 & -2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{pmatrix}$$

8.
$$\begin{pmatrix} 0 & 4 & 1 & 3 & -2 \\ 2 & 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & -5 & 1 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

9.
$$\begin{pmatrix} 0 & 2 & 3 & 4 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 1 & 4 & 2 & 0 & -3 \\ 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & -2 & 0 \end{pmatrix}$$

Question 10. Evaluate the determinant of the given matrix by inspection.

1.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3.
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & 2 & 3 & 8 \end{pmatrix}$$

4.
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

5.
$$\begin{pmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

6.
$$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 40 & 10 & -1 & 0 \\ 100 & 200 & -23 & 3 \end{pmatrix}$$

Question 11. For each matrix, show that the value of the determinant is independent of θ .

1.
$$\begin{pmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{pmatrix}$$

2.
$$\begin{pmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ \sin \theta - \cos \theta & \sin \theta + \cos \theta & 1 \end{pmatrix}$$

Question 12. By inspection, what is the relationship between the following determinants?

$$d_1 = \begin{vmatrix} a & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}, \quad d_2 = \begin{vmatrix} a + \lambda & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}$$

Question 13. Show that

$$\det(A) = \frac{1}{2} \begin{vmatrix} \text{tr}(A) & 1 \\ \text{tr}(A^2) & \text{tr}(A) \end{vmatrix}$$

for every matrix $A \in \mathcal{M}_{2 \times 2}$.

Question 14. What can you say about $\det(A)$, where $A \in \mathcal{M}_{n \times n}$ has entries all equal to 1?

Question 15. What is the maximum number of zeros that a 3×3 matrix can have without having a zero determinant? Why?

Question 16. Explain why the determinant of a square matrix with integer entries must be an integer.

Question 17. Prove that the three points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Question 18. Prove that the equation of the line through the distinct points (a_1, b_1) and (a_2, b_2) can be written as

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

Question 19. Prove

Theorem 1 A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

Question 20. Prove Theorem 3 using Theorems 1 and 2.

Theorem 2 The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Theorem 3 If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Question 21. With the following examples, verify that $\det(A) = \det(A^\top)$

$$1. \quad A = \begin{pmatrix} -2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$2. \quad A = \begin{pmatrix} -6 & 1 \\ 2 & -2 \end{pmatrix}$$

$$3. \quad A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{pmatrix}$$

$$4. \quad A = \begin{pmatrix} 4 & 2 & -1 \\ 0 & 2 & -3 \\ -1 & 1 & 5 \end{pmatrix}$$

Question 22. Each of the following matrices is obtained from the identity matrix by performing a single elementary row operation. Identify the specific operation performed and find the determinant of each matrix using the properties of determinants in relation to row operations.

$$1. \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$2. \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$3. \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$$

$$4. \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$5. \quad \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$6. \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$7. \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$8. \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$9. \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$$10. \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 23. Find the determinant of the following matrices using some combination of row operations, column operations, and cofactor expansion.

1.
$$\begin{pmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ -5 & -1 & -12 \end{pmatrix}$$

2.
$$\begin{pmatrix} 18 & -9 & -14 \\ 6 & -3 & -5 \\ -3 & 1 & 2 \end{pmatrix}$$

3.
$$\begin{pmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{pmatrix}$$

4.
$$\begin{pmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{pmatrix}$$

5.
$$\begin{pmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{pmatrix}$$

6.
$$\begin{pmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

7.
$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{pmatrix}$$

8.
$$\begin{pmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & -1 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{pmatrix}$$

9.
$$\begin{pmatrix} 1 & -1 & 5 & 1 \\ -2 & 1 & -7 & 1 \\ -3 & 2 & -12 & -2 \\ 2 & -1 & 9 & 1 \end{pmatrix}$$

10.
$$\begin{pmatrix} 5 & 3 & -8 & 4 \\ \frac{15}{2} & \frac{1}{2} & -1 & -7 \\ -\frac{5}{2} & \frac{3}{2} & -4 & 1 \\ 10 & -3 & 8 & -8 \end{pmatrix}$$

11.
$$\begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ 2 & 4 & -3 & 1 & -4 \\ 2 & 6 & 4 & 8 & -4 \\ -3 & -8 & -1 & 1 & 0 \\ 1 & 3 & 3 & 10 & 1 \end{pmatrix}$$

12.
$$\begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$

Question 24. Prove that if a square matrix has two proportional rows or columns, then its determinant is zero.

Question 25. Compute the following determinants, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

1.
$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$$

2.
$$\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$$

3.
$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$$

4.
$$\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix}$$

5.
$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$$

6.
$$\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix}$$

7.
$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$$

8.
$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

Question 26. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Question 27. Verify the following two formulas and make a conjecture about a general result of which these results are special cases

1.
$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{13}a_{22}a_{31}$$

2.
$$\begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{14}a_{23}a_{32}a_{41}$$

Question 28. Confirm the following identities

1.
$$\begin{vmatrix} a_1 & b_1 & a_1 + b_1 + c_1 \\ a_2 & b_2 & a_2 + b_2 + c_2 \\ a_3 & b_3 & a_3 + b_3 + c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

2.
$$\begin{vmatrix} a_1 + b_1t & a_2 + b_2t & a_3 + b_3t \\ a_1t + b_1 & a_2t + b_2 & a_3t + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3.
$$\begin{vmatrix} a_1 + b_1 & a_1 - b_1 & c_1 \\ a_2 + b_2 & a_2 - b_2 & c_2 \\ a_3 + b_3 & a_3 - b_3 & c_3 \end{vmatrix} = -2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

4.
$$\begin{vmatrix} a_1 & b_1 + ta_1 & c_1 + rb_1 + sa_1 \\ a_2 & b_2 + ta_2 & c_2 + rb_2 + sa_2 \\ a_3 & b_3 + ta_3 & c_3 + rb_3 + sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Question 29. For the following matrices, show that $\det(A) = 0$ without explicitly computing the determinant.

$$1. A = \begin{pmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{pmatrix}$$

$$2. A = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$

Question 30. It can be proved that if a square matrix M can be partitioned into *block triangular form* as

$$M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where A and B are square matrices, then

$$\det(M) = \det(A) \det(B).$$

Use this result to compute the determinants of the following matrices:

$$1. M = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 8 & 6 & -9 \\ 2 & 5 & 0 & 4 & 7 & 5 \\ -1 & 3 & 2 & 6 & 9 & -2 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 8 & -4 \end{array} \right)$$

$$2. M = \left(\begin{array}{ccc|cc} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right)$$

Question 31. Given $A \in \mathbb{M}_{n \times n}$. Let B be the matrix that results when the rows of A are written in reverse order. Formulate a statement that describes the relationship between $\det(A)$ and $\det(B)$.

Question 32. Find the determinant of the following matrix.

$$\begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}$$

Question 33. Determine whether the given matrix is invertible by calculating its determinant. If the matrix is invertible, compute its inverse using the adjugate method.

$$1. A = \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}$$

$$2. A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$3. A = \begin{pmatrix} -12 & 7 & -27 \\ 4 & -1 & 2 \\ 3 & 2 & -8 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 31 & -20 & 106 \\ -11 & 7 & -37 \\ -9 & 6 & -32 \end{pmatrix}$$

$$5. A = \begin{pmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{pmatrix}$$

$$6. A = \begin{pmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{pmatrix}$$

$$7. A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{pmatrix}$$

$$8. A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{pmatrix}$$

$$9. A = \begin{pmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

$$10. A = \begin{pmatrix} -3 & 0 & 1 \\ 5 & 0 & 6 \\ 8 & 0 & 3 \end{pmatrix}$$

$$11. A = \begin{pmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{pmatrix}$$

$$12. A = \begin{pmatrix} \sqrt{2} & -\sqrt{7} & 0 \\ 3\sqrt{2} & -3\sqrt{7} & 0 \\ 5 & -9 & 0 \end{pmatrix}$$

$$13. A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -1 & 0 & -4 \end{pmatrix}$$

Question 34. Evaluate the determinant of the given matrix by

(a) cofactor expansion

(b) row operations

$$1. A = \begin{pmatrix} 2 & -2 & -2 & -2 \\ -2 & 2 & 3 & 0 \\ -2 & -2 & 2 & 0 \\ 1 & -1 & -3 & -1 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -3 & -3 & -1 & -1 \\ -1 & -1 & -3 & 2 \\ -1 & -2 & 2 & 1 \end{pmatrix}$$

$$3. A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix}$$

Question 35. Solve the following systems of equations using (if possible)

(a) the inverse of the coefficient matrix computed via the adjugate method

(b) Cramer's rule

1.

$$\begin{aligned} 7x_1 - 2x_2 &= 3 \\ 3x_1 + x_2 &= 5 \end{aligned}$$

2.

$$\begin{aligned} -9x - 4y &= 3 \\ -7x + 5y &= -10 \end{aligned}$$

3.

$$\begin{aligned} 2x + 3y &= 4 \\ 2x + 2y &= 4 \end{aligned}$$

4.

$$\begin{aligned} -10x - 7y &= -12 \\ 12x - 11y &= 5 \end{aligned}$$

5.

$$\begin{aligned} 5x - 5y &= 7 \\ 2x - 3y &= 6 \end{aligned}$$

6.

$$\begin{aligned} -x - 3y &= 4 \\ -8x + 4y &= 3 \end{aligned}$$

7.

$$\begin{aligned} 2x + 5y &= 4 \\ 4x + y &= 3 \end{aligned}$$

8.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

9.

$$\begin{aligned} -2x + y - 4z &= -8 \\ -4y + z &= 3 \\ 4x - z &= -8 \end{aligned}$$

10.

$$\begin{aligned} 4x + 5y &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

11.

$$\begin{aligned} 2x + 3y + 2z &= -2 \\ -x - 3y - 8z &= -2 \\ -3x + 2y - 7z &= 2 \end{aligned}$$

12.

$$\begin{aligned} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$$

13.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 4 \\ 2x_1 - x_2 &= -2 \\ 4x_1 - 3x_3 &= 0 \end{aligned}$$

14.

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 4 \\ -x_1 + 7x_2 - 3x_3 &= 1 \\ 2x_1 + 6x_2 - x_3 &= 5 \end{aligned}$$

15.

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

16.

$$\begin{aligned} -x_1 - 4x_2 + 2x_3 + x_4 &= -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 &= 14 \\ -x_1 + x_2 + 3x_3 + x_4 &= 11 \\ x_1 - 2x_2 + x_3 - 4x_4 &= -4 \end{aligned}$$

17.

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z - 2w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$