

Tutorial 7

Fundamental spaces and decompositions

Homework

1. Provide detailed solutions to *one* of the following questions: Q1, Q2, Q3, Q4, Q5, Q8.
2. Provide detailed solutions to *one* of the following questions: Q10, Q11.
3. Provide detailed solutions to *one* of the following questions: Q19, Q20.
4. Provide detailed solutions to *three* selected parts from Q23 (parts 2–14).
5. Provide detailed solutions to *two* selected parts from Q40 (parts 1–4).
6. Provide detailed solutions to *two* selected parts from any of the following: Q42 (parts 1–8), Q43 (parts 1–8), Q44 (parts 1–9).

Question 1. Determine whether \mathbf{b} is in the column space of A , and if so, express \mathbf{b} as a linear combination of the column vectors of A

1. $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$
2. $A = \begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$
4. $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 5 \\ 7 \end{pmatrix}$

Solution. \mathbf{b} is in the column space of A iff $A\mathbf{x} = \mathbf{b}$ for some vector \mathbf{x} . Therefore, for each case, we solve for \mathbf{x} in the linear system $A\mathbf{x} = \mathbf{b}$.

1. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

The system does not have a solution and \mathbf{b} does not belong to the column space of A .

2. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array} \right).$$

The system has a unique solution $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$. We have

$$\mathbf{b} = \begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

3. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The system does not have a solution and \mathbf{b} does not belong to the column space of A .

4. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -26 \\ 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

The system has a unique solution $\begin{pmatrix} -26 \\ 13 \\ -7 \\ 4 \end{pmatrix}$. We have

$$\mathbf{b} = -26 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 13 \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} - 7 \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

Question 2. Suppose that $x_1 = 3, x_2 = 0, x_3 = -1, x_4 = 5$ is a solution of a nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ and that the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is given by the formulas

$$x_1 = 5r - 2s, \quad x_2 = s, \quad x_3 = s + t, \quad x_4 = t$$

1. Find a vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.
2. Find a vector form of the general solution of $A\mathbf{x} = \mathbf{b}$.

Solution.

1. A vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

2. A vector form of the general solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 5 \end{pmatrix} + r \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Question 3. Suppose that $x_1 = -1, x_2 = 2, x_3 = 4, x_4 = -3$ is a solution of a nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ and that the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

1. Find a vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.
2. Find a vector form of the general solution of $A\mathbf{x} = \mathbf{b}$.

Solution.

1. A vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

2. A vector form of the general solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ -3 \end{pmatrix} + r \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Question 4. Find the vector form of the general solution of the linear system $A\mathbf{x} = \mathbf{b}$, and then use that result to find the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

- 1.
- 2.

$$\begin{aligned} x_1 - 3x_3 &= 1 \\ 2x_1 - 6x_2 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 5 \\ x_1 + x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 3 \end{aligned}$$

1. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 0 \end{array} \right)$$

The general solution of the system is

$$x_1 = 3t + 1, \quad x_2 = t, \quad x_3 = t.$$

A vector form of the general solution of $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

A vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

2. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution of the system is

$$x_1 = -t - 2, \quad x_2 = 7 - t, \quad x_3 = t.$$

A vector form of the general solution of $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

A vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Question 5. Find bases for the null space and row space of A .

$$1. A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{pmatrix}$$

Solution.

1. The reduced row echelon form of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is

$$\left(\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16t \\ 19t \\ t \end{pmatrix} = t \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

Thus, a basis for the null space of A is given by

$$\left\{ \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix} \right\}$$

The nonzero rows of the reduced row echelon form correspond to a basis for the row space of A :

$$\{(1, 0, -16), (0, 1, -19)\}$$

2. The reduced row echelon form of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

A vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{t}{2} \\ s \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

Thus, a basis for the null space of A is given by

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

A basis for the row space of A is given by

$$\left\{ \left(1, 0, -\frac{1}{2} \right) \right\}$$

3. The reduced row echelon form of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -\frac{2}{7} & 0 \\ 0 & 1 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

A vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{2t}{7} - s \\ \frac{4t}{7} - s \\ s \\ t \end{pmatrix} = t \begin{pmatrix} \frac{2}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

Thus, a basis for the null space of A is given by

$$\left\{ \begin{pmatrix} \frac{2}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

A basis for the row space of A is given by

$$\left\{ \left(1, 0, 1, -\frac{2}{7} \right), \left(0, 1, 1, \frac{4}{7} \right) \right\}$$

4. The reduced row echelon form of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -r - 2s - t \\ -r - s - 2t \\ r \\ s \\ t \end{pmatrix} = r \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad r, s, t \in \mathbb{R}$$

Thus, a basis for the null space of A is given by

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

A basis for the row space of A is given by

$$\{(1, 0, 1, 2, 1), (0, 1, 1, 1, 2)\}$$

Question 6. By inspection, find a basis for the row space and for the column space of the given matrix

$$1. A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution.

1. A basis for the row space of A is

$$\{(1, 0, 2), (0, 0, 1)\}$$

A basis for the column space of A consist of the following vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

2. A basis for the row space of A is

$$\{(1, -3, 0, 0), (0, 1, 0, 0)\}$$

A basis for the column space of A consist of the following vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

3. A basis for the row space of A is

$$\{(1, 2, 4, 5), (0, 1, -3, 0), (0, 0, 1, -3), (0, 0, 0, 1)\}$$

A basis for the column space of A consist of the following vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

4. A basis for the row space of A is

$$\{(1, 2, -1, 5), (0, 1, 4, 3), (0, 0, 1, -7), (0, 0, 0, 1)\}$$

A basis for the column space of A consist of the following vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 3 \\ -7 \\ 1 \\ 1 \end{pmatrix}$$

Question 7. Construct a matrix whose null space consists of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 4 \end{pmatrix}$$

Solution. Since the null space of A and the row space of A are orthogonal complements, it suffices to find a basis for the orthogonal complement of $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$.

Equivalently, we seek a basis for the space of vectors \mathbf{x} , that satisfy the homogeneous system

$$\mathbf{v}_1^\top \cdot \mathbf{x} = 0, \quad \mathbf{v}_2^\top \cdot \mathbf{x} = 0.$$

Thus, we aim to find a basis for the solution space of the system $A\mathbf{x} = \mathbf{0}$, where the rows of A are given by \mathbf{v}_1^\top and \mathbf{v}_2^\top . The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -4 & 0 & 0 \end{array} \right)$$

Expressing the general solution in vector form, we obtain

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t + s \\ 4s \\ s \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \quad t, s \in \mathbb{R}.$$

Thus a basis for the solution space of $A\mathbf{x} = \mathbf{0}$ consists of vectors

$$\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ 1 \\ 0 \end{pmatrix}.$$

Consequently, a possible choice for the matrix A is given by

$$\begin{pmatrix} -2 & 0 & 0 & 1 \\ 1 & 4 & 1 & 0 \end{pmatrix}$$

Question 8. Find the rank and nullity of the matrix A by reducing it to row echelon form.

1. $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix}$

1. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank}(A) = 1, \text{ nullity}(A) = 3.$$

2. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank}(A) = 2, \text{ nullity}(A) = 3.$$

3. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank}(A) = 3, \text{ nullity}(A) = 2.$$

4. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank}(A) = 3, \text{nullity}(A) = 1.$$

Question 9. The matrix R is the reduced row echelon form of the matrix A .

- (a) By inspection of the matrix R , find the rank and nullity of A .
- (b) Find the number of leading variables and the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ without solving the system.

$$1. A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & -6 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 2 & -1 & -3 \\ -2 & 1 & 3 \\ -4 & 2 & 6 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution. Recall that we have discussed in the lecture

- $\text{rank}(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$
 - $\text{nullity}(A)$ = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$
1. $\text{rank}(A) = 3, \text{nullity}(A) = 0$
 2. $\text{rank}(A) = 2, \text{nullity}(A) = 1$
 3. $\text{rank}(A) = 1, \text{nullity}(A) = 2$
 4. $\text{rank}(A) = 3, \text{nullity}(A) = 1$

Question 10. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by the formula

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{pmatrix}.$$

1. Find the rank of the standard matrix for T .
2. Find the nullity of the standard matrix for T .

Solution. The standard matrix for T is given by

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$$

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We have $\text{rank}(A) = 2$, $\text{nullity}(A) = 0$.

Question 11. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be the linear transformation defined by the formula

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{pmatrix}$$

1. Find the rank of the standard matrix for T .
2. Find the nullity of the standard matrix for T .

Solution. The standard matrix for T is given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

We have $\text{rank}(A) = 3$, $\text{nullity}(A) = 2$.

Question 12. Discuss how the rank of A varies with t .

$$1. A = \begin{pmatrix} 1 & 1 & t \\ 1 & 1 & 1 \\ t & 1 & 1 \end{pmatrix} \qquad 2. A = \begin{pmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{pmatrix}$$

Solution.

1. If $t = 1$, $\text{rank}(A) = 1$. If $t \neq 1$, apply row reduction

$$A \xrightarrow[R_3 \rightarrow R_3 - tR_1]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & t \\ 0 & 0 & 1-t \\ 0 & 1-t & 1-t^2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & t \\ 0 & 1-t & 1-t^2 \\ 0 & 0 & 1-t \end{pmatrix} \xrightarrow[R_3 \rightarrow \frac{1}{t-1}R_3]{R_2 \rightarrow \frac{1}{t-1}R_2} \begin{pmatrix} 1 & 1 & t \\ 0 & 1 & t+1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, when $t \neq 1$, $\text{rank}(A) = 3$

- 2.

$$A \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} -1 & -3 & t \\ 3 & 6 & -2 \\ t & 3 & -1 \end{pmatrix} \xrightarrow[R_1 \rightarrow -R_1]{R_2 \rightarrow R_2 + 3R_1, R_3 \rightarrow R_3 + tR_1} \begin{pmatrix} 1 & 3 & -t \\ 0 & -3 & -2+3t \\ 0 & 3-3t & -1+t^2 \end{pmatrix}$$

$$\xrightarrow[R_3 \rightarrow R_3 + (1-t)R_2]{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & -2+2t \\ 0 & -3 & -2+3t \\ 0 & 0 & t^2 - 1 + (1-t)(3t-2) \end{pmatrix}$$

$$\text{rank}(A) = \begin{cases} 3, & \text{if } t \neq 1, \frac{3}{2}, \\ 2, & \text{if } t = 1 \text{ or } \frac{3}{2}. \end{cases}$$

Question 13. Are there values of r and s for which

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{pmatrix}$$

has rank 1? Has rank 2? If so, find those values.

Solution. Since the first and fourth rows of A are linearly independent, the rank of A must be at least 2. Therefore, there do not exist values of r and s for which $\text{rank}(A) = 1$.

For A to have rank 2, the second and third rows must be contained within the vector space

$$\text{span}(\{(1, 0, 0), (0, 0, 3)\})$$

This condition requires that the second and third rows be expressible as linear combinations of these two basis vectors. That is, there must exist scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1(1, 0, 0) + \alpha_2(0, 0, 3) = (0, r-2, 2), \quad \beta_1(1, 0, 0) + \beta_2(0, 0, 3) = (0, s-1, r+2)$$

By comparing components, we obtain the system of equations:

$$\alpha_1 = 0, \quad r-2 = 0, \quad 3\alpha_2 = 2, \quad \beta_1 = 0, \quad s-1 = 0, \quad 3\beta_2 = r+2.$$

Solving for r and s we find

$$r = 2, \quad s = 1.$$

Thus, we conclude that $\text{rank}(A) = 2$ iff $r = 2, s = 1$.

Question 14.

1. If A is a 3×5 matrix, then the rank of A is at most ____.

2. If A is a 3×5 matrix, then the nullity of A is at most ____.
3. If A is a 3×5 matrix, then the rank of A^\top is at most ____.
4. If A is a 5×3 matrix, then the nullity of A^\top is at most ____.

Solution.

1. If A is a 3×5 matrix, then the rank of A is at most **5**.
2. If A is a 3×5 matrix, then the nullity of A is at most **5**.
3. If A is a 3×5 matrix, then the rank of A^\top is at most **3**.
4. If A is a 5×3 matrix, then the nullity of A^\top is at most **5**.

Question 15.

1. If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ____.
2. If A is a 3×5 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ____.
3. If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ____.
4. If A is a 5×3 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ____.

Solution.

1. If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most **5**.
2. If A is a 3×5 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most **5**.
3. If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most **3**.
4. If A is a 5×3 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most **3**.

Question 16. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Show that A has rank 2 if and only if one or more of the following determinants is nonzero:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Solution. A has rank 2 if and only if its two rows are linearly independent. That is, the second row of A cannot be written as a scalar multiple of the first row. This means the following system of equations in the unknown x has no solutions

$$x \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Expanding component-wise, this is equivalent to the system

$$a_{11}x = a_{21}$$

$$a_{12}x = a_{22}$$

$$a_{13}x = a_{23}$$

Then the system has no solutions if there does not exist a single scalar x satisfying all three equations simultaneously. Note that at least one of a_{11}, a_{12}, a_{13} is nonzero. The system having no solutions is then equivalent to requiring that at least one of the following inequalities hold:

$$a_{22}a_{11} \neq a_{21}a_{12}, \quad a_{11}a_{23} \neq a_{21}a_{13}, \quad a_{12}a_{23} \neq a_{22}a_{13}.$$

Rewriting in determinant form, this means that at least one of the following determinants must be nonzero:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Question 17. Determine whether the matrix operator $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by the equations is bijective.

1.

$$\begin{aligned} w_1 &= x_1 + 2x_2 \\ w_2 &= -x_1 + x_2 \end{aligned}$$

2.

$$\begin{aligned} w_1 &= 4x_1 - 6x_2 \\ w_2 &= -2x_1 + 3x_2 \end{aligned}$$

3.

$$\begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$$

4.

$$\begin{aligned} w_1 &= x_1 - 3x_2 + 4x_3 \\ w_2 &= -x_1 + x_2 + x_3 \\ w_3 &= -2x_2 + 5x_3 \end{aligned}$$

Solution.

1. The standard matrix for T is

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$$

since $\det(A) = 3 \neq 0$, T is bijective.

2. The standard matrix for T is

$$A = \begin{pmatrix} 4 & -6 \\ -2 & 3 \end{pmatrix}$$

since $\det(A) = 0$, T is not bijective.

3. The standard matrix for T is

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

since $\det(A) = -1 \neq 0$, T is bijective.

4. The standard matrix for T is

$$A = \begin{pmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{pmatrix}$$

since $\det(A) = 0$, T is not bijective.

Question 18. Determine whether multiplication by A is an injective matrix transformation.

1. $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -4 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

4. $A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$

Solution. The matrix transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by multiplication by a matrix A is injective if and only if for all distinct vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$, we have

$$A\mathbf{v}_1 \neq A\mathbf{v}_2.$$

This condition is equivalent to requiring that the only solution to

$$A\mathbf{v}_1 = A\mathbf{v}_2$$

is when $\mathbf{v}_1 = \mathbf{v}_2$. Subtracting both sides, we obtain

$$A(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}.$$

Setting $\mathbf{x} = \mathbf{v}_1 - \mathbf{v}_2$, we conclude that A is injective if and only if the homogeneous equation

$$A\mathbf{x} = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$. In other words,

$$\text{nullity}(A) = 0.$$

Thus, a matrix transformation given by multiplication by A is injective if and only if $\text{nullity}(A) = 0$.

1. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\text{nullity}(A) = 0$ and the matrix transformation is injective.

2. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

nullity(A) = 1 and the matrix transformation is not injective.

3. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

nullity(A) = 1 and the matrix transformation is not injective.

4. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

nullity(A) = 0 and the matrix transformation is injective.

Question 19. Let T_A be multiplication by the matrix A . Find:

- (a) a basis for the range of T_A .
- (b) a basis for the kernel of T_A .
- (d) the rank and nullity of A .

1. $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 20 & 0 & 0 \end{pmatrix}$

Solution. Recall that the range of T_A , denoted $R(T_A)$, is the column space of A .

1. The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & \frac{14}{11} \\ 0 & 1 & -\frac{19}{11} \\ 0 & 0 & 0 \end{pmatrix}.$$

Observing R , we note that the first two columns are pivot columns, indicating that they form a basis for the column space of R . Consequently, the first two columns of A form a basis for the column space of A .

- (a) A basis for $R(T_A)$ consists of the following vectors:

$$\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix}$$

- (b) From R , we can deduce that a vector form for the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} -\frac{14t}{11} \\ \frac{19t}{11} \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{14}{11} \\ \frac{19}{11} \\ 1 \end{pmatrix}$$

Thus a basis for the kernel of T_A consists of the vector

$$\begin{pmatrix} -\frac{14}{11} \\ \frac{19}{11} \\ 1 \end{pmatrix}$$

- (c) $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$.

2. The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observing R , we note that the first and the last columns are pivot columns, indicating that they form a basis for the column space of R . Consequently, the first and the last columns of A form a basis for the column space of A .

- (a) A basis for $\text{R}(T_A)$ consists of the following vectors:

$$\begin{pmatrix} 2 \\ 4 \\ 20 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

- (b) From R , we can deduce that a vector form for the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus a basis for the kernel of T_A consists of the vector

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- (c) $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$.

Question 20. Let $T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be multiplication by A . Find:

- (a) a basis for the kernel of T_A .
- (b) a basis for the range of T_A that consists of column vectors of A .

$$1. A = \begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -2 & 4 & 2 & 2 \\ -1 & 8 & 3 & 5 \end{pmatrix}$$

Solution.

1. The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & 0 & -\frac{10}{7} \\ 0 & 1 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- (a) From R , we can deduce that a vector form for the general solution of the homogeneous system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} \frac{10t}{7} \\ \frac{2t}{7} \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} \frac{10}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{pmatrix}$$

Thus a basis for the kernel of T_A consists of the vector

$$\begin{pmatrix} \frac{10}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{pmatrix}$$

- (b) Observing R , we note that the first three columns are pivot columns, indicating that they form a basis for the column space of R . Consequently, the first three columns of A form a basis for the column space of A . A basis for $\mathcal{R}(T_A)$ consists of the following vectors:

$$\begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$$

2. The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- (a) From R , we can deduce that a vector form for the general solution of the homogeneous system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} \frac{s}{3} - \frac{t}{3} \\ -\frac{s}{3} - \frac{2t}{3} \\ s \\ t \end{pmatrix} = s \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$$

Thus a basis for the kernel of T_A consists of the vectors

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$$

- (b) Observing R , we note that the first two columns are pivot columns, indicating that they form a basis for the column space of R . Consequently, the first two columns of A form a basis for the column space of A . A basis for $\mathcal{R}(T_A)$ consists of the following vectors:

$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}.$$

Question 21. Let A be an $n \times n$ matrix such that $\det(A) = 0$, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be multiplication by A .

1. What can you say about the range of the matrix operator T ? Give an example that illustrates your conclusion.
2. What can you say about the number of vectors that T maps into $\mathbf{0}$?

What if $\det(A) \neq 0$?

Solution.

1. Since $\det(A) = 0$, the range of T is a subspace of \mathbb{R}^n . T is not surjective, and there exist vectors in \mathbb{R}^n that are not in the image of A .

Example: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

The determinant of A is 0. The column space of A is spanned by a single vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This means that A maps all vectors in \mathbb{R}^2 to a one-dimensional subspace of \mathbb{R}^2 , confirming that T is not surjective.

2. Since $\det(A) = 0$, the null space of A is nontrivial. This means that there exist nonzero vectors $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} = \mathbf{0}.$$

In other words, T is not injective, and there are infinitely many vectors that T maps to $\mathbf{0}$.

Example: For the matrix A given above, we solve $A\mathbf{x} = \mathbf{0}$ and we get a basis for the kernel of T that consists of the following vector

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Any scalar multiple of this vector is mapped to $\mathbf{0}$ by T .

If $\det(A) \neq 0$, then $R(T) = \mathbb{R}^n$ and only the zero vector $\mathbf{0}$ is mapped to $\mathbf{0}$ by T .

Question 22. Confirm by multiplication that \mathbf{x} is an eigenvector of A , and find the corresponding eigenvalue.

$$1. A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution.

$$1. A\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \text{ the corresponding eigenvalue is } -1.$$

$$2. A\mathbf{x} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}, \text{ the corresponding eigenvalue is } 5.$$

$$3. A\mathbf{x} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \text{ the corresponding eigenvalue is } 4.$$

$$4. A\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ the corresponding eigenvalue is } 0.$$

Question 23. Find the characteristic equation, the eigenvalues, and the bases for the eigenspaces of the matrix A .

$$1. A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

$$2. A = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$5. A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$6. A = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$$

$$7. A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$8. A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$$

$$9. A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$10. A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

$$11. A = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$$

$$12. A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$13. A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$14. A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

Solution.

1. The characteristic equation of A is given by

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0$$

Solving for λ , we obtain the eigenvalues:

$$\lambda_1 = 5, \quad \lambda_2 = -1$$

To determine a basis for the eigenspace corresponding to $\lambda_1 = 5$, we solve the homogeneous system

$$(5I - A)\mathbf{x} = \mathbf{0},$$

where

$$5I - A = \begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix}.$$

Reducing the system, we find the general solution:

$$\mathbf{x} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, a basis for the eigenspace corresponding to $\lambda_1 = 5$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Similarly, to determine a basis for the eigenspace corresponding to $\lambda_2 = -1$, we solve the homogeneous system

$$\begin{pmatrix} -2 & -4 \\ -2 & -4 \end{pmatrix} \mathbf{x} = \mathbf{0}.$$

We obtain the general solution:

$$\mathbf{x} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Thus, a basis for the eigenspace corresponding to $\lambda_2 = -1$ is

$$\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}.$$

2. Characteristic equation: $\lambda^2 + 3 = 0$

$$\begin{aligned} \lambda_1 &= \sqrt{3}i, \text{ basis vector: } \begin{pmatrix} \frac{7}{-2 - \sqrt{3}i} \\ 1 \end{pmatrix} \\ \lambda_2 &= -\sqrt{3}i, \text{ basis vector: } \begin{pmatrix} \frac{7}{-2 + \sqrt{3}i} \\ 1 \end{pmatrix} \end{aligned}$$

3. Characteristic equation: $\lambda^2 - 2\lambda + 1 = 0$

$$\lambda = 1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4. Characteristic equation: $\lambda^2 - 2\lambda + 1 = 0$

$$\lambda = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

5. Characteristic equation: $\lambda^2 - 4\lambda + 3 = 0$

$$\begin{aligned} \lambda_1 &= 1, \text{ basis vector: } \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \lambda_2 &= 3, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

6. Characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

$$\lambda = 2, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

7. Characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

$$\lambda = 2, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

8. Characteristic equation: $\lambda^2 + 3 = 0$

$$\lambda_1 = \sqrt{3}i, \text{ basis vector: } \begin{pmatrix} -\frac{2}{1-\sqrt{3}i} \\ 1 \end{pmatrix}$$

$$\lambda_2 = -\sqrt{3}i, \text{ basis vector: } \begin{pmatrix} -\frac{2}{1+\sqrt{3}i} \\ 1 \end{pmatrix}$$

9. Characteristic equation: $-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2, \text{ basis vector: } \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 3, \text{ basis vector: } \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

10. Characteristic equation: $-\lambda^3 + 5\lambda^2 = 0$

$$\lambda_1 = 0, \text{ basis vectors: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 5, \text{ basis vector: } \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

11. Characteristic equation: $-\lambda^3 + \lambda^2 + 16\lambda + 20 = 0$

$$\lambda_1 = 5, \text{ basis vector: } \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

12. Characteristic equation: $-\lambda^3 + 3\lambda + 2 = 0$

$$\lambda_1 = 2, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \text{ basis vectors: } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

13. Characteristic equation: $-\lambda^3 + 9\lambda^2 - 27\lambda + 27 = 0$

$$\lambda = 3, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$14. \text{ Characteristic equation: } -\lambda^3 + 12\lambda + 16 = 0$$

$$\lambda_1 = -2, \text{ basis vector: } \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4, \text{ basis vectors: } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Question 24. Find the characteristic equation of the matrix by inspection.

$$1. A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{pmatrix} \qquad 2. A = \begin{pmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

Solution.

1. $(\lambda - 3)(\lambda - 7)(\lambda - 1) = 0$
2. $(\lambda - 9)(\lambda + 1)(\lambda - 3)(\lambda - 7) = 0$

Question 25. Find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on \mathbb{R}^2 . Use geometric reasoning to help finding the answers.

1. Reflection about the line $y = x$.
2. Orthogonal projection onto the x-axis.
3. Rotation about the origin through a positive angle of 90° .
4. Contraction with factor α ($0 < \alpha < 1$).
5. Shear in the x -direction by a factor α ($\alpha \neq 0$).
6. Reflection about the y -axis.
7. Rotation about the origin through a positive angle of 180° .
8. Dilation with factor α ($\alpha > 1$).
9. Expansion in the y -direction with factor α ($\alpha > 1$).
10. Shear in the y -direction by a factor α ($\alpha \neq 0$).

Solution.

1. The standard matrix for the reflection about the line $y = x$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To determine the eigenvectors of A , we note that an eigenvector must remain on the same line after reflection. This occurs only if the vector lies either along the line $y = x$ or along the line perpendicular to $y = x$, which is $y = -x$. Thus, the only possible eigenvectors are those parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Furthermore, since the reflection does not change the magnitude of the vectors, the possible eigenvalues are only $1, -1$.

In conclusion

- The eigenspace corresponding to $\lambda_1 = 1$ consists of all scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so a basis for this eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

- The eigenspace corresponding to $\lambda_2 = -1$ consists of all scalar multiples of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so a basis for this eigenspace is

$$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

2. The standard matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

An eigenvector must remain on the same line after projection – they are either on the x -axis or y -axis

$$\lambda_1 = 0, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

3. The standard matrix is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda_1 = i, \text{ basis vector: } \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\lambda_2 = -i, \text{ basis vector: } \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

4. The standard matrix is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

Any vector in \mathbb{R}^2 serves as an eigenvector since contraction scales all vectors without altering their directions.

$$\lambda = \alpha, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

5. The standard matrix is

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

The only vectors that remain collinear after a shearing transformation in the x -direction are those that initially lie along the x -axis.

$$\lambda = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6. The standard matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The vectors that remain collinear after reflection about the y -axis are those that lie either on the y -axis or the x -axis.

$$\lambda = 1, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = -1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

7. The standard matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

All vectors in \mathbb{R}^2 remain collinear after a 180° rotation, with their directions reversed.

$$\lambda = -1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

8. The standard matrix is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

Any vector in \mathbb{R}^2 serves as an eigenvector since dilation scales all vectors without altering their directions.

$$\lambda = \alpha, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

9. The standard matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

The eigenvectors consist of those lying on the x -axis, which remain unchanged, and those on the y -axis, which are scaled by a factor of α .

$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = \alpha, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

10. The standard matrix is

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

The eigenvectors consist of those lie on the y -axis that remain unchanged.

$$\lambda = 1, \text{ basis vectors: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Question 26. Find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on \mathbb{R}^3 . Use geometric reasoning to find the answers.

1. Reflection about the xy -plane.
2. Orthogonal projection onto the xz -plane.
3. Counterclockwise rotation about the positive x -axis through an angle of 90° .
4. Contraction with factor α ($0 \leq \alpha < 1$).
5. Reflection about the xz -plane.
6. Orthogonal projection onto the yz -plane.
7. Counterclockwise rotation about the positive y -axis through an angle of 180° .
8. Dilation with factor α ($\alpha > 1$).

Solution.

1. The standard matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigenvectors consist of those lying on the xy -plane, which remain unchanged, and those on the z -axis, whose directions are reversed.

$$\lambda_1 = 1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1, \text{ basis vectors: } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2. The standard matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvectors consist of those lying on the y -axis, which are mapped to $\mathbf{0}$ and those on the xz -plane which remain unchanged.

$$\lambda_1 = 0, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3. The standard matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = i, \text{ basis vector: } \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

$$\lambda_3 = -i, \text{ basis vector: } \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$$

4. The standard matrix is

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

Any vector in \mathbb{R}^3 serves as an eigenvector since contraction scales all vectors without altering their directions.

$$\lambda = \alpha, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

5. The standard matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvectors consist of those lying on the xz -plane, which remain unchanged, and those on the y -axis, whose directions are reversed.

$$\lambda_1 = 1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \text{ basis vectors: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

6. The standard matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvectors consist of those lying on the x -axis, which are mapped to $\mathbf{0}$ and those on the yz -plane which remain unchanged.

$$\lambda_1 = 0, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 1, \text{ basis vectors: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

7. The standard matrix is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Vectors along the y -axis remain unchanged, while those in the xz -plane have their directions reversed after the rotation.

$$\begin{aligned} \lambda_1 &= 1, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \lambda_2 &= -1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

8. The standard matrix is

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

Any vector in \mathbb{R}^2 serves as an eigenvector since dilation scales all vectors without altering their directions.

$$\lambda = \alpha, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Question 27. Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.

1. $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$
2. $p(\lambda) = \lambda^4 - \lambda^3 + 7$

Solution.

1. We have

$$\det(\lambda I - A) = \lambda^3 - 2\lambda^2 + \lambda + 5$$

Setting $\lambda = 0$, we obtain

$$\det(-A) = 5.$$

Since the highest power of λ in the characteristic equation corresponds to the size of A , we conclude that $A \in \mathcal{M}_{3 \times 3}$.

Using the determinant property $\det(-A) = (-1)^n \det(A)$ for an $n \times n$ matrix, we have

$$\det(-A) = (-1)^3 \det(A) = -\det(A).$$

Substituting $\det(-A) = 5$, we solve for $\det(A)$:

$$-\det(A) = 5 \implies \det(A) = -5.$$

2. $(-1)^4 \det(A) = 7, \det(A) = 7$

Question 28. Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

1. What is the size of A ?
2. Is A invertible?
3. How many eigenspaces does A have?

Solution.

1. The size of A is determined by the degree of its characteristic polynomial. Since the polynomial is of degree

$$1 + 2 + 3 = 6,$$

we conclude that A is a 6×6 matrix.

2. A matrix is invertible if and only if zero is not an eigenvalue. Substituting $\lambda = 0$ into the polynomial we get

$$p(0) = (-1) \times 9 \times (-64) \neq 0,$$

thus A is invertible.

3. The number of eigenspaces of A corresponds to the number of distinct eigenvalues. From the characteristic polynomial, we see that A has the three distinct eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = 4.$$

Therefore, A has three eigenspaces.

Question 29. The eigenvectors that we have been studying are sometimes called *right eigenvectors* to distinguish them from *left eigenvectors*, which are $n \times 1$ column vectors \mathbf{x} that satisfy the equation

$$\mathbf{x}^\top A = \mu \mathbf{x}^\top$$

for some scalar μ . For a given matrix A , how are the right eigenvectors and their corresponding eigenvalues related to the left eigenvectors and their corresponding eigenvalues?

Solution. We begin by considering the transpose of the expression $\mathbf{x}^\top A$:

$$(\mathbf{x}^\top A)^\top = A^\top \mathbf{x}.$$

Additionally, we have

$$(\mu \mathbf{x}^\top)^\top = \mu \mathbf{x}.$$

Thus, if \mathbf{x} is a left eigenvector of A corresponding to eigenvalue μ , we have

$$A^\top \mathbf{x} = \mu \mathbf{x}.$$

This shows that the left eigenvectors of A are the right eigenvectors of A^\top , associated with the same eigenvalue.

Question 30. Prove that the characteristic equation of a 2×2 matrix A can be expressed as

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0,$$

where $\text{tr}(A)$ is the trace of A .

Solution. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. Then

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a + d)\lambda + (ad - bc)$$

Since $\text{tr}(A) = a + d$, $\det(A) = ad - bc$. The characteristic equation of A is given by

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0.$$

Question 31. Use the result from the Question 30 to show that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the solutions of the characteristic equation of A are

$$\lambda = \frac{1}{2} \left((a + d) \pm \sqrt{(a - d)^2 + 4bc} \right).$$

Use this result to show that A has

- (a) two distinct real eigenvalues if $(a - d)^2 + 4bc > 0$.
- (b) two repeated real eigenvalues if $(a - d)^2 + 4bc = 0$.
- (c) complex conjugate eigenvalues if $(a - d)^2 + 4bc < 0$.

Solution. Since the characteristic equation of A is given by

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0.$$

Using the quadratic formula, the solutions to the characteristic equation are:

$$\begin{aligned} \lambda &= \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2} = \frac{1}{2} \left((a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)} \right) \\ &= \frac{1}{2} \left((a + d) \pm \sqrt{(a - d)^2 + 4bc} \right) \end{aligned}$$

Consequently, A has two eigenvalues

$$\lambda_1 = \frac{1}{2} \left[(a + d) + \sqrt{(a - d)^2 + 4bc} \right]$$

and

$$\lambda_2 = \frac{1}{2} \left[(a + d) - \sqrt{(a - d)^2 + 4bc} \right].$$

- (a) If the discriminant of the quadratic equation satisfies

$$(a - d)^2 + 4bc > 0,$$

then the square root term is a positive real number. Since we are adding and subtracting a positive quantity to $(a + d)$, we obtain two distinct real eigenvalues λ_1 and λ_2 .

(b) If

$$(a - d)^2 + 4bc = 0,$$

the quadratic equation has two repeated roots:

$$\lambda_1 = \lambda_2 = \frac{a + d}{2}.$$

Thus, A has two repeated real eigenvalues.

(c) If

$$(a - d)^2 + 4bc < 0,$$

then the square root term is an imaginary number. This results in two complex conjugate eigenvalues of the form:

$$\lambda_1 = \frac{a + d}{2} + i \frac{\sqrt{|(a - d)^2 + 4bc|}}{2}.$$

and

$$\lambda_2 = \frac{a + d}{2} - i \frac{\sqrt{|(a - d)^2 + 4bc|}}{2}.$$

Thus, A has complex conjugate eigenvalues.

Question 32. Let A be the matrix in from the Question 31. Show that if $b \neq 0$, then

$$\mathbf{x}_1 = \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -b \\ a - \lambda_2 \end{pmatrix}$$

are eigenvectors of A that correspond, respectively, to the eigenvalues

$$\lambda_1 = \frac{1}{2} \left[(a + d) + \sqrt{(a - d)^2 + 4bc} \right]$$

and

$$\lambda_2 = \frac{1}{2} \left[(a + d) - \sqrt{(a - d)^2 + 4bc} \right].$$

Solution. Computing $A\mathbf{x}_1$:

$$A\mathbf{x}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix} = \begin{pmatrix} -ab + ba - b\lambda_1 \\ -bc + da - d\lambda_1 \end{pmatrix} = \begin{pmatrix} -b\lambda_1 \\ -bc + da - d\lambda_1 \end{pmatrix}.$$

Using the characteristic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0,$$

we substitute λ_1 :

$$\lambda_1^2 - (a + d)\lambda_1 + (ad - bc) = 0 \implies ad - bc = -\lambda_1^2 + (a + d)\lambda_1$$

Then

$$A\mathbf{x}_1 = \begin{pmatrix} -b\lambda_1 \\ -\lambda_1^2 + (a + d)\lambda_1 - d\lambda_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix} = \lambda_1 \mathbf{x}_1.$$

This confirms that \mathbf{x}_1 is an eigenvector of A corresponding to λ_1 .

Similarly, for \mathbf{x}_2 , we follow the same process and verify:

$$A\mathbf{x}_2 = \lambda_2\mathbf{x}_2.$$

Question 33. Use the result of Question 30 to prove that if

$$p(\lambda) = \lambda^2 + c_1\lambda + c_2$$

is the characteristic polynomial of a 2×2 matrix, then

$$p(A) = A^2 + c_1A + c_2I = O.$$

(Stated informally, A satisfies its characteristic equation. This result is true as well for $n \times n$ matrices.)

Solution. From Question 30 we know that

$$p(\lambda) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A).$$

Thus

$$c_1 = -\operatorname{tr}(A), \quad c_2 = \det(A)$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\begin{aligned} A^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{pmatrix} \\ \operatorname{tr}(A)A &= (a + d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + ad & ab + db \\ ac + dc & ad + d^2 \end{pmatrix} \\ \det(A)I &= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}. \end{aligned}$$

And

$$\begin{aligned} p(A) &= A^2 - \operatorname{tr}(A)A + \det(A)I \\ &= \begin{pmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - db \\ ca + dc - ac - dc & cb + d^2 - ad - d^2 + ad - bc \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Question 34. Prove: If a, b, c, d are integers such that $a + b = c + d$, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has integer eigenvalues.

Solution. Using the result from Question 31, the eigenvalues of A are given by

$$\lambda = \frac{1}{2} \left((a + d) \pm \sqrt{(a - d)^2 + 4bc} \right).$$

Since $a + b = c + d$, we have $a - d = c - b$ and

$$\begin{aligned}\lambda &= \frac{1}{2} \left((a + d) \pm \sqrt{(c - b)^2 + 4bc} \right) = \frac{1}{2} \left((a + d) \pm \sqrt{(c + b)^2} \right) \\ &= \frac{(a + d) \pm |c + b|}{2}.\end{aligned}$$

Consequently, the two eigenvalues are given by

$$\begin{aligned}\lambda_1 &= \frac{a + d + c + b}{2} = \frac{2(a + b)}{2} = a + b \\ \lambda_2 &= \frac{a + d - (c + b)}{2} = \frac{c + d - b + d - c - b}{2} = \frac{2(d - b)}{2} = d - b\end{aligned}$$

Since a, b, c, d are integers, λ_1 and λ_2 are also integers.

Question 35. Prove: If λ is an eigenvalue of an invertible matrix A and \mathbf{x} is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and \mathbf{x} is a corresponding eigenvector.

Solution. Since λ is an eigenvalue of A and \mathbf{x} is a corresponding eigenvector, we have

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Since A is invertible, we can multiply both sides of the equation by A^{-1} from the left:

$$A^{-1}(A\mathbf{x}) = A^{-1}(\lambda\mathbf{x}),$$

which gives

$$I\mathbf{x} = \lambda A^{-1}\mathbf{x}.$$

Thus,

$$A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}.$$

This shows that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with the same eigenvector \mathbf{x} .

Question 36. Prove: If λ is an eigenvalue of A , \mathbf{x} is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$ and \mathbf{x} is a corresponding eigenvector.

Solution.

$$(A - sI)\mathbf{x} = A\mathbf{x} - sI\mathbf{x} = A\mathbf{x} - s\mathbf{x} = \lambda\mathbf{x} - s\mathbf{x} = (\lambda - s)\mathbf{x}$$

Question 37. Prove: If λ is an eigenvalue of A and \mathbf{x} is a corresponding eigenvector, then $s\lambda$ is an eigenvalue of sA for every scalar s and \mathbf{x} is a corresponding eigenvector.

Solution.

$$(sA)\mathbf{x} = s(A\mathbf{x}) = s(\lambda\mathbf{x}) = (s\lambda)\mathbf{x}$$

Question 38. Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$$

and then use Questions 35 and 36 to find the eigenvalues and bases for the eigenspaces of

1. A^{-1}
2. $A - 3I$
3. $A + 2I$

Solution. The characteristic equation of A is

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

A has three eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. The bases for the eigenspaces are as follows

$$\begin{aligned} \lambda_1 = 1, \text{ basis vector: } & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ \lambda_2 = 2, \text{ basis vector: } & \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} \\ \lambda_3 = 3, \text{ basis vector: } & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

1. Eigenvalues and bases for eigenspaces for A^{-1}

$$\begin{aligned} \lambda_1 = 1, \text{ basis vector: } & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ \lambda_2 = \frac{1}{2}, \text{ basis vector: } & \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} \\ \lambda_3 = \frac{1}{3}, \text{ basis vector: } & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

2. Eigenvalues and bases for eigenspaces for $A - 3I$

$$\begin{aligned} \lambda_1 = -2, \text{ basis vector: } & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ \lambda_2 = -1, \text{ basis vector: } & \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} \\ \lambda_3 = 0, \text{ basis vector: } & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

3. Eigenvalues and bases for eigenspaces for $A + 2I$

$$\lambda_1 = 3, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4, \text{ basis vector: } \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 5, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Question 39. Show that A and B are not similar matrices.

$$1. A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution.

1. We can compute that $\det(A) = -1$, $\det(B) = -2$. Since the determinant is invariant under similarity transformations, and given that $\det(A) \neq \det(B)$, it follows that A and B cannot be not similar matrices.
2. $\det(A) = 18$, $\det(B) = 14$
3. $\det(A) = 1$, $\det(B) = 0$
4. $\text{rank}(A) = 1$, $\text{rank}(B) = 2$. Since rank is invariant under similarity transformations, and given that $\text{rank}(A) \neq \text{rank}(B)$, it follows that A and B cannot be not similar matrices.

Question 40.

A Procedure for Diagonalizing an $n \times n$ Matrix

Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n , then it is not.

Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$$

whose column vectors are the n basis vectors you obtained in Step 1.

Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ that correspond to the successive columns of P .

Following the above procedure to find a matrix P that diagonalizes A , and check your work by computing $P^{-1}AP$.

$$1. A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$$

$$2. A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution.

1. The eigenvalues and bases for eigenspaces for A are:

$$\lambda_1 = 1, \quad \text{basis vector : } \begin{pmatrix} 1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \quad \text{basis vector : } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

We can take

$$P = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{pmatrix}$$

And

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. The eigenvalues and bases for eigenspaces for A are:

$$\lambda_1 = 1, \quad \text{basis vector : } \begin{pmatrix} 4 \\ \frac{5}{4} \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2, \quad \text{basis vector : } \begin{pmatrix} 3 \\ \frac{4}{3} \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{4}{5} & \frac{3}{4} \\ 1 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

3.

$$\lambda_1 = 2, \text{ basis vector : } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 3, \text{ basis vectors : } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

4.

$$\lambda_1 = 0, \text{ basis vector : } \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1, \text{ basis vector : } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 2, \text{ basis vector : } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Question 41. Let

$$1. A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

- Find the eigenvalues of A .
- For each eigenvalue λ , find the rank of the matrix $\lambda I - A$.
- Is A diagonalizable? Justify your conclusion.

Solution.

- The characteristic equation of A is

$$-\lambda^3 + 11\lambda^2 - 39\lambda + 45 = 0$$

- The eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 5$

(b)

$$3I - A = \begin{pmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{pmatrix}$$

The reduced row echelon form of $3I - A$ is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $\text{rank}(3I - A) = 1$.

$$5I - A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

The reduced row echelon form of $5I - A$ is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $\text{rank}(5I - A) = 2$.

- (c) A is diagonalizable. A basis for the eigenspace corresponding to $\lambda_1 = 3$ is given by the vectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

A basis for the eigenspace corresponding to $\lambda_2 = 5$ is given by the vector

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Since we have three linearly independent eigenvectors, we construct the matrix

$$P = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then, the diagonalization of A is given by

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

2. The characteristic equation of A is

$$-\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$$

- (a) The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$
 (b)

$$2I - A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

The reduced row echelon form of $2I - A$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $\text{rank}(2I - A) = 2$.

$$3I - A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

The reduced row echelon form of $3I - A$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $\text{rank}(3I - A) = 2$.

- (c) A is not diagonalizable. A basis for the eigenspace corresponding to $\lambda_1 = 2$ is given by the vectors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A basis for the eigenspace corresponding to $\lambda_2 = 3$ is given by the vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Since A has only two linearly independent eigenvectors, A is not diagonalizable, which is fewer than the required three for a 3×3 matrix, it is not diagonalizable.

For Questions 42 – 44 Find an LU-decomposition of the coefficient matrix - try both Doolittle decomposition and Crout decomposition. Then solve the system using the decomposition.

Question 42.

$$1. \begin{pmatrix} 2 & 8 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

$$2. \begin{pmatrix} -5 & -10 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 \\ 19 \end{pmatrix}$$

$$3. \begin{pmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$$

$$4. \begin{pmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -33 \\ 7 \\ -1 \end{pmatrix}$$

$$5. \begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$6. \begin{pmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -22 \\ 3 \end{pmatrix}$$

$$7. \begin{pmatrix} -1 & -3 & 2 \\ -6 & -19 & 10 \\ 3 & 9 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9 \\ -59 \\ 28 \end{pmatrix}$$

$$8. \begin{pmatrix} -4 & 0 & 1 \\ 8 & 2 & -1 \\ -8 & -6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 15 \\ -46 \end{pmatrix}$$

Solution.

1. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 8 \\ 0 & 3 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

2.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 \\ 6 & 1 \\ -\frac{5}{5} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -5 & -10 \\ 0 & -7 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -10 \\ 7 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} -5 & 0 \\ 6 & -7 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

3.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -4 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

4.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & 12 & -6 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -33 \\ -4 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 11 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

5.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

6. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{1}{4} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & -6 & -3 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 3 \\ -24 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -\frac{25}{2} \\ -7 \\ \frac{1}{2} \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ -6 \\ \frac{1}{2} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -\frac{25}{2} \\ -7 \\ \frac{1}{2} \end{pmatrix}.$$

7. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & -3 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -9 \\ -5 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -6 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 9 \\ 5 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

8. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -4 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -5 \\ 5 \\ -21 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \frac{1}{5} \\ \frac{23}{5} \\ \frac{21}{5} \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -4 & 0 & 0 \\ 8 & 2 & 0 \\ -8 & -6 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \frac{5}{4} \\ \frac{5}{2} \\ -\frac{21}{5} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \frac{1}{5} \\ \frac{23}{5} \\ -\frac{21}{5} \end{pmatrix}.$$

Question 43.

$$\begin{aligned}
1. \quad & \begin{pmatrix} -2 & -1 & -1 & -2 \\ 2 & 4 & 3 & 1 \\ 4 & 2 & -2 & -1 \\ -1 & 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -10 \\ 8 \end{pmatrix} & 2. \quad & \begin{pmatrix} 2 & -2 & 0 & 3 \\ -2 & 3 & 2 & -3 \\ 4 & -5 & -4 & 10 \\ -4 & 7 & 10 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 10 \\ 2 \end{pmatrix} \\
3. \quad & \begin{pmatrix} 4 & 3 & 4 & -2 \\ 0 & -1 & 3 & 3 \\ 1 & 2 & -3 & 1 \\ 0 & -4 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -20 \\ 1 \\ 5 \\ 14 \end{pmatrix} & 4. \quad & \begin{pmatrix} -5 & 2 & -2 & 2 \\ -10 & 0 & -8 & 7 \\ 20 & -4 & 10 & -7 \\ -20 & 12 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 25 \\ -49 \\ 52 \end{pmatrix} \\
5. \quad & \begin{pmatrix} -1 & 1 & 0 & -3 \\ 0 & 4 & -4 & -4 \\ 0 & -2 & 4 & 0 \\ 4 & -3 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -8 \\ -13 \end{pmatrix} & 6. \quad & \begin{pmatrix} 1 & -2 & -3 & -3 \\ 3 & -10 & -12 & -5 \\ 5 & -22 & -26 & 0 \\ 2 & 12 & 0 & -18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -89 \\ -199 \\ 66 \end{pmatrix} \\
7. \quad & \begin{pmatrix} 1 & 0 & -1 & -4 \\ -2 & 1 & -2 & -2 \\ 4 & -3 & 4 & -2 \\ -2 & -4 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \\ 2 \\ 24 \end{pmatrix} & 8. \quad & \begin{pmatrix} 1 & 1 & -5 & -1 \\ 2 & -1 & -5 & 3 \\ -4 & -16 & 38 & 23 \\ -5 & -20 & 52 & 35 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -25 \\ 122 \\ 158 \end{pmatrix}
\end{aligned}$$

Solution.

1. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{25}{24} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & -1 & -1 & -2 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & -\frac{97}{24} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 3 \\ 14 \\ -4 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 4 \\ 1 \\ 0 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 0 & -4 & 0 \\ -1 & \frac{1}{2} & \frac{25}{6} & -\frac{97}{24} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -\frac{3}{2} \\ \frac{14}{3} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 4 \\ 1 \\ 0 \end{pmatrix}.$$

2. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -2 & 3 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -2 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 11 \\ 2 \\ -10 \\ -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ 3 \\ -1 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -1 & -2 & 0 \\ -4 & 3 & 4 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & 0 & \frac{3}{2} \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \frac{11}{2} \\ 2 \\ 5 \\ -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ 3 \\ -1 \end{pmatrix}.$$

3. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{5}{4} & 1 & 0 \\ 0 & 4 & 56 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & 3 & 4 & -2 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & -\frac{1}{4} & \frac{21}{4} \\ 0 & 0 & 0 & -310 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -20 \\ 1 \\ \frac{45}{4} \\ -620 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ -4 \\ -3 \\ 2 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & \frac{5}{4} & -\frac{1}{4} & 0 \\ 0 & -4 & -14 & -310 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{3}{4} & 1 & -\frac{1}{2} \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & -21 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -5 \\ -1 \\ -45 \\ 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ -4 \\ -3 \\ 2 \end{pmatrix}.$$

4. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -1 & 1 & 0 \\ 4 & -1 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -5 & 2 & -2 & 2 \\ 0 & -4 & -4 & 3 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 16 \\ -7 \\ 8 \\ -3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 3 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -5 & 0 & 0 & 0 \\ -10 & -4 & 0 & 0 \\ 20 & 4 & -2 & 0 \\ -20 & 4 & 4 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{2}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -\frac{16}{5} \\ \frac{7}{4} \\ -4 \\ 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 3 \end{pmatrix}.$$

5. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ -4 & \frac{1}{4} & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & 0 & -3 \\ 0 & 4 & -4 & -4 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -12 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 3 \\ -4 \\ -10 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -9 \\ -6 \\ -5 \\ 0 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 4 & 1 & 0 & -12 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ -1 \\ -5 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -9 \\ -6 \\ -5 \\ 0 \end{pmatrix}.$$

6. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & 3 & 1 & 0 \\ 2 & -4 & 3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & -4 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -5 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -19 \\ -32 \\ -8 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 0 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 5 & -12 & -2 & 0 \\ 2 & 16 & -6 & -5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & 1 & \frac{3}{4} & -1 \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -19 \\ 8 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 0 \end{pmatrix}.$$

7.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -3 & 1 & 0 \\ -2 & -4 & \frac{21}{4} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & 34 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 11 \\ 26 \\ 36 \\ -39 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ -\frac{53}{17} \\ -\frac{75}{17} \\ \frac{39}{34} \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -3 & -4 & 0 \\ -2 & -4 & -21 & 34 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 11 \\ 26 \\ -9 \\ -\frac{39}{34} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ -\frac{53}{17} \\ -\frac{75}{17} \\ \frac{39}{34} \end{pmatrix}.$$

8.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & 4 & 1 & 0 \\ -5 & 5 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -5 & -1 \\ 0 & -3 & 5 & 5 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -19 \\ 13 \\ -6 \\ -8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 4 \\ -2 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ -4 & -12 & -2 & 0 \\ -5 & -15 & 2 & 4 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -5 & -1 \\ 0 & 1 & -\frac{5}{3} & -\frac{5}{3} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -19 \\ -\frac{13}{3} \\ 3 \\ -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 4 \\ -2 \end{pmatrix}.$$

Question 44.

1.

$$\begin{aligned} -x_1 + 5x_2 - 7x_3 &= 14 \\ -3x_1 + 8x_2 - 4x_3 &= 28 \\ x_1 - 3x_2 + 2x_3 &= -10 \end{aligned}$$

2.

$$\begin{aligned} -12x_1 - 24x_2 - 16x_3 &= 8 \\ -3x_1 - 5x_2 - 2x_3 &= 4 \\ 6x_1 + 9x_2 + x_3 &= -11 \end{aligned}$$

3.

$$\begin{aligned} -4x_1 + 14x_2 - 7x_3 &= 107 \\ -x_1 + 2x_2 - x_3 &= 17 \\ 3x_1 - 9x_2 + 7x_3 &= -83 \end{aligned}$$

4.

$$\begin{aligned} 20x_1 - 11x_2 - 29x_3 &= -92 \\ -4x_1 + 2x_2 + 5x_3 &= 15 \\ -8x_1 + 4x_2 + 14x_3 &= 50 \end{aligned}$$

5.

$$\begin{aligned} 4x_1 - 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 2 \\ x_2 - 5x_3 &= 2 \end{aligned}$$

6.

$$\begin{aligned} -9x_1 + 12x_2 - 9x_3 &= 45 \\ -9x_1 + 3x_2 - 6x_3 &= 24 \\ -3x_1 + 3x_2 - 3x_3 &= 12 \end{aligned}$$

7.

$$\begin{aligned} 3x_1 + 27x_2 - 29x_3 &= 28 \\ -x_1 - 4x_2 + 2x_3 + 5x_4 &= -5 \\ -3x_2 - 3x_3 + 13x_4 &= -1 \\ -5x_1 - 23x_2 + 15x_3 + 21x_4 &= -28 \end{aligned}$$

8.

$$\begin{aligned} 4x_1 - 12x_2 - 29x_3 - 12x_4 &= -149 \\ 3x_2 - 4x_3 + x_4 &= -11 \\ x_1 - 3x_2 - x_3 + x_4 &= -18 \\ 3x_1 + 6x_2 - 28x_3 + 4x_4 &= -122 \end{aligned}$$

9.

$$\begin{aligned} -16x_1 + 15x_2 - 21x_3 - 26x_4 &= -86 \\ 20x_2 - 20x_3 - 10x_4 &= -100 \\ 4x_1 + 2x_3 + 4x_4 &= 2 \\ 16x_1 + 5x_2 + 3x_3 + 14x_4 &= -16 \end{aligned}$$

Solution.

1. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -\frac{2}{7} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 5 & -7 \\ 0 & -7 & 17 \\ 0 & 0 & -\frac{1}{7} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 14 \\ -14 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -3 & -7 & 0 \\ 1 & 2 & -\frac{1}{7} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -5 & 7 \\ 0 & 1 & -\frac{17}{7} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -14 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}.$$

2. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ -\frac{1}{2} & -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -12 & -24 & -16 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -12 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -3 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & \frac{4}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -\frac{2}{3} \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

3. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ -\frac{3}{4} & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -4 & 14 & -7 \\ 0 & -\frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{5}{2} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 107 \\ -\frac{39}{4} \\ -\frac{25}{2} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 4 \\ -5 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -4 & 0 & 0 \\ -1 & -\frac{3}{2} & 0 \\ 3 & \frac{3}{2} & \frac{5}{2} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{7}{2} & \frac{7}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -\frac{107}{4} \\ \frac{13}{2} \\ -5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 4 \\ -5 \end{pmatrix}.$$

4. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 20 & -11 & -29 \\ 0 & -\frac{1}{5} & -\frac{4}{5} \\ 0 & 0 & 4 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -92 \\ -\frac{17}{5} \\ 20 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 20 & 0 & 0 \\ -4 & -\frac{1}{5} & 0 \\ -8 & -\frac{2}{5} & 4 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{11}{20} & -\frac{29}{20} \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -\frac{23}{5} \\ 17 \\ 5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$$

5. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{5} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & -2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -\frac{2}{5} \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 5 & 0 \\ 0 & 1 & -5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ -\frac{2}{5} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -\frac{2}{5} \end{pmatrix}.$$

6.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{3} & \frac{1}{9} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -9 & 12 & -9 \\ 0 & -9 & 3 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 45 \\ -21 \\ -\frac{2}{3} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -9 & 0 & 0 \\ -9 & -9 & 0 \\ -3 & -1 & -\frac{1}{3} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{4}{3} & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -5 \\ \frac{7}{3} \\ 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}.$$

7.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 \\ 0 & -\frac{3}{5} & 1 & 0 \\ -\frac{5}{3} & \frac{22}{5} & -\frac{1}{19} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 27 & -29 & 0 \\ 0 & 5 & -\frac{23}{3} & 5 \\ 0 & 0 & -\frac{38}{5} & 16 \\ 0 & 0 & 0 & -\frac{3}{19} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 28 \\ \frac{13}{3} \\ \frac{8}{5} \\ -\frac{6}{19} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 2 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 5 & 0 & 0 \\ 0 & -3 & -\frac{38}{5} & 0 \\ -5 & 22 & \frac{2}{5} & -\frac{3}{19} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 9 & -\frac{29}{3} & 0 \\ 0 & 1 & -\frac{23}{15} & 1 \\ 0 & 0 & 1 & -\frac{40}{19} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \frac{28}{3} \\ \frac{13}{15} \\ -\frac{4}{19} \\ 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 2 \end{pmatrix}.$$

8.
 - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & 1 & 0 \\ \frac{3}{4} & 5 & \frac{11}{5} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & -12 & -29 & -12 \\ 0 & 3 & -4 & 1 \\ 0 & 0 & \frac{25}{4} & 4 \\ 0 & 0 & 0 & -\frac{4}{5} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -149 \\ -11 \\ \frac{77}{4} \\ \frac{12}{5} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 5 \\ -3 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & \frac{25}{4} & 0 \\ 3 & 15 & \frac{55}{4} & -\frac{4}{5} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -3 & -\frac{29}{4} & -3 \\ 0 & 1 & -\frac{4}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{16}{25} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -\frac{149}{4} \\ -\frac{11}{3} \\ \frac{77}{25} \\ -3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 5 \\ -3 \end{pmatrix}.$$

9. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{4} & \frac{3}{16} & 1 & 0 \\ -1 & 1 & 4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -16 & 15 & -21 & -26 \\ 0 & 20 & -20 & -10 \\ 0 & 0 & \frac{1}{2} & -\frac{5}{8} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -86 \\ -100 \\ \frac{77}{4} \\ -79 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 243 \\ -243 \\ -159 \\ -158 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -16 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 4 & \frac{15}{4} & \frac{1}{2} & 0 \\ 16 & 20 & 2 & \frac{1}{2} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{15}{16} & \frac{21}{16} & \frac{13}{8} \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \frac{43}{8} \\ -5 \\ \frac{77}{2} \\ -158 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 243 \\ -243 \\ -159 \\ -158 \end{pmatrix}.$$

Question 45. Determine A^{-1} using LU decomposition and verify the correctness of the following identity:

$$A = LU \implies A^{-1} = (LU)^{-1} = U^{-1}L^{-1}.$$

Try both Doolittle decomposition and Crout decomposition.

1. $\begin{pmatrix} 5 & -2 & -4 \\ 15 & -10 & -7 \\ -10 & -8 & 25 \end{pmatrix}$

2. $\begin{pmatrix} -1 & 2 & 5 \\ -2 & 0 & 12 \\ 1 & -6 & -1 \end{pmatrix}$

3. $\begin{pmatrix} -1 & 5 & -2 \\ -2 & 8 & -8 \\ -1 & 1 & -12 \end{pmatrix}$

4. $\begin{pmatrix} -2 & 1 & 3 \\ 2 & 2 & 2 \\ 0 & -4 & -3 \end{pmatrix}$

5. $\begin{pmatrix} 4 & -4 & 5 \\ 12 & -16 & 14 \\ -16 & 36 & -14 \end{pmatrix}$

6. $\begin{pmatrix} 2 & -5 & 1 \\ -2 & 4 & -2 \\ -2 & 7 & 2 \end{pmatrix}$

7. $\begin{pmatrix} -1 & -4 & 5 \\ 0 & 2 & -4 \\ -5 & -22 & -26 \end{pmatrix}$

8. $\begin{pmatrix} -3 & -5 & -3 \\ -12 & -19 & -10 \\ 0 & 0 & 1 \end{pmatrix}$

$$9. \begin{pmatrix} 3 & 3 & 0 & -5 \\ 15 & 16 & 0 & -23 \\ -3 & 2 & 4 & 10 \\ 0 & 2 & -8 & 12 \end{pmatrix}$$

$$10. \begin{pmatrix} 1 & -5 & 4 & -2 \\ 4 & -17 & 12 & -8 \\ -2 & -2 & 11 & 9 \\ 5 & -37 & 36 & -8 \end{pmatrix}$$

$$11. \begin{pmatrix} 1 & 1 & -4 & -2 \\ 3 & 2 & -10 & -2 \\ 2 & -1 & -1 & 13 \\ -1 & -4 & 8 & 5 \end{pmatrix}$$

$$12. \begin{pmatrix} 3 & 2 & -3 & 5 \\ -6 & -9 & 6 & -11 \\ -6 & 6 & 9 & -10 \\ -15 & 10 & 12 & -16 \end{pmatrix}$$

Solution.

$$1. A^{-1} = \begin{pmatrix} \frac{153}{20} & -\frac{41}{20} & \frac{13}{20} \\ \frac{61}{8} & -\frac{17}{8} & \frac{5}{8} \\ \frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 5 & -2 & -4 \\ 0 & -4 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 11 & -3 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{5} & -\frac{1}{10} & \frac{13}{20} \\ 0 & -\frac{1}{4} & \frac{5}{8} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{153}{20} & -\frac{41}{20} & \frac{13}{20} \\ \frac{61}{8} & -\frac{17}{8} & \frac{5}{8} \\ \frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} 5 & 0 & 0 \\ 15 & -4 & 0 \\ -10 & -12 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ \frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & \frac{2}{5} & \frac{13}{10} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{153}{20} & -\frac{41}{20} & \frac{13}{20} \\ \frac{61}{8} & -\frac{17}{8} & \frac{5}{8} \\ \frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$2. A^{-1} = \begin{pmatrix} 9 & -\frac{7}{2} & 3 \\ \frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 2 & 5 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} -1 & -\frac{1}{2} & 3 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} 9 & -\frac{7}{2} & 3 \\ \frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -4 & 0 \\ 1 & -4 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} 9 & -\frac{7}{2} & 3 \\ \frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$3. A^{-1} = \begin{pmatrix} 22 & -\frac{29}{2} & 6 \\ 4 & -\frac{5}{2} & 1 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 5 & -2 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} -1 & -\frac{5}{2} & 6 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} 22 & -\frac{29}{2} & 6 \\ 4 & -\frac{5}{2} & 1 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -2 & 0 \\ -1 & -4 & -2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -5 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 5 & -12 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} 22 & -\frac{29}{2} & 6 \\ 4 & -\frac{5}{2} & 1 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

$$4. A^{-1} = \begin{pmatrix} \frac{1}{11} & \frac{9}{22} & \frac{2}{11} \\ -\frac{3}{11} & -\frac{3}{11} & -\frac{5}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -\frac{4}{3} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 1 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{11}{3} \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{4}{3} & \frac{4}{3} & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{6} & \frac{2}{11} \\ 0 & \frac{1}{3} & -\frac{5}{11} \\ 0 & 0 & \frac{3}{11} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -\frac{1}{11} & \frac{9}{22} & \frac{2}{11} \\ -\frac{3}{11} & -\frac{3}{11} & -\frac{5}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & -4 & \frac{11}{3} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{2}{3} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -\frac{1}{11} & \frac{9}{22} & \frac{2}{11} \\ -\frac{3}{11} & -\frac{3}{11} & -\frac{5}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}$$

$$5. A^{-1} = \begin{pmatrix} \frac{35}{2} & -\frac{31}{4} & -\frac{3}{2} \\ 7 & 3 & 1 \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{4} \\ -11 & 5 & 1 \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & -4 & 5 \\ 0 & -4 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & 5 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{3}{2} \\ 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{35}{2} & -\frac{31}{4} & -\frac{3}{2} \\ 7 & 3 & 1 \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{4} \\ -11 & 5 & 1 \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 \\ 12 & -4 & 0 \\ -16 & 20 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -11 & 5 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 1 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{35}{2} & -\frac{31}{4} & -\frac{3}{2} \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{4} \\ -11 & 5 & 1 \end{pmatrix}$$

$$6. A^{-1} = \begin{pmatrix} -11 & -\frac{17}{2} & -3 \\ -4 & -3 & -1 \\ 3 & 2 & 1 \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -5 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -11 & -\frac{17}{2} & -3 \\ -4 & -3 & -1 \\ 3 & 2 & 1 \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -2 & -1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & \frac{5}{2} & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -11 & -\frac{17}{2} & -3 \\ -4 & -3 & -1 \\ 3 & 2 & 1 \end{pmatrix}$$

$$7. A^{-1} = \begin{pmatrix} -\frac{14}{11} & -\frac{107}{55} & \frac{3}{55} \\ \frac{2}{11} & \frac{51}{110} & -\frac{2}{55} \\ \frac{1}{11} & -\frac{1}{55} & -\frac{1}{55} \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & -4 & 5 \\ 0 & 2 & -4 \\ 0 & 0 & -55 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 1 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} -1 & -2 & \frac{3}{55} \\ 0 & \frac{1}{2} & -\frac{2}{55} \\ 0 & 0 & -\frac{1}{55} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -\frac{14}{11} & -\frac{107}{55} & \frac{3}{55} \\ \frac{2}{11} & \frac{51}{110} & -\frac{2}{55} \\ \frac{1}{11} & -\frac{1}{55} & -\frac{1}{55} \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ -5 & -2 & -55 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{11} & -\frac{1}{55} & -\frac{1}{55} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -4 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -\frac{14}{11} & -\frac{107}{55} & \frac{3}{55} \\ \frac{2}{11} & \frac{51}{110} & -\frac{2}{55} \\ \frac{1}{11} & -\frac{1}{55} & -\frac{1}{55} \end{pmatrix}$$

$$8. A^{-1} = \begin{pmatrix} \frac{19}{3} & -\frac{5}{3} & \frac{7}{3} \\ -4 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & -5 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{5}{3} & \frac{7}{3} \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{19}{3} & -\frac{5}{3} & \frac{7}{3} \\ -4 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

- Crout decomposition

$$L = \begin{pmatrix} -3 & 0 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{5}{3} & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -\frac{5}{3} & \frac{7}{3} \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{19}{3} & -\frac{5}{3} & \frac{7}{3} \\ -4 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$9. A^{-1} = \begin{pmatrix} -\frac{325}{3} & 21 & -\frac{11}{3} & -\frac{11}{6} \\ 57 & -11 & 2 & 1 \\ -\frac{129}{4} & \frac{25}{4} & -1 & -\frac{5}{8} \\ -31 & 6 & -1 & -\frac{1}{2} \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -1 & 5 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 3 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 26 & -5 & 1 & 0 \\ 62 & -12 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{3} & -1 & 0 & -\frac{11}{6} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & -\frac{5}{8} \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

- Crout decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 15 & 1 & 0 & 0 \\ -3 & 5 & 4 & 0 \\ 0 & 2 & -8 & -2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ \frac{13}{2} & -\frac{5}{4} & \frac{1}{4} & 0 \\ -31 & 6 & -1 & -\frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -1 & 0 & \frac{11}{3} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

$$10. A^{-1} = \begin{pmatrix} \frac{68}{9} & \frac{1}{3} & \frac{8}{9} & -\frac{11}{9} \\ \frac{142}{9} & -\frac{7}{3} & \frac{4}{9} & -\frac{10}{9} \\ \frac{77}{6} & -2 & \frac{1}{3} & -\frac{5}{6} \\ -\frac{21}{2} & 2 & 0 & \frac{1}{2} \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & -4 & 1 & 0 \\ 5 & -4 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -5 & 4 & -2 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -14 & 4 & 1 & 0 \\ -21 & 4 & 0 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & \frac{5}{3} & \frac{8}{9} & -\frac{11}{9} \\ 0 & \frac{1}{3} & \frac{4}{9} & -\frac{10}{9} \\ 0 & 0 & \frac{1}{3} & -\frac{5}{6} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

- Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ -2 & -12 & 3 & 0 \\ 5 & -12 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -5 & 4 & -2 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{4}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{14}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ -\frac{21}{2} & 2 & 0 & \frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 5 & \frac{8}{3} & -\frac{22}{9} \\ 0 & 1 & \frac{4}{3} & -\frac{20}{9} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

$$11. A^{-1} = \begin{pmatrix} -276 & 103 & -22 & -12 \\ -127 & 47 & -10 & -6 \\ -113 & 42 & -9 & -5 \\ 24 & -9 & 2 & 1 \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -1 & 3 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -4 & -2 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 7 & -3 & 1 & 0 \\ 24 & -9 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 1 & 2 & -12 \\ 0 & -1 & 2 & -6 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

- Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -1 & -3 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -4 & -2 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 7 & -3 & 1 & 0 \\ 24 & -9 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -1 & 2 & -12 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

$$12. A^{-1} = \begin{pmatrix} -\frac{26}{9} & -\frac{14}{15} & \frac{2}{45} & -\frac{13}{45} \\ -\frac{5}{3} & -\frac{3}{5} & -\frac{1}{15} & -\frac{1}{15} \\ \frac{56}{9} & 2 & \frac{5}{9} & \frac{2}{9} \\ \frac{19}{3} & 2 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -5 & -4 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 & -3 & 5 \\ 0 & -5 & 0 & -1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 2 & 1 & 0 \\ 19 & 6 & 1 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{15} & \frac{1}{3} & -\frac{13}{45} \\ 0 & -\frac{1}{5} & 0 & -\frac{1}{15} \\ 0 & 0 & \frac{1}{3} & \frac{2}{9} \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

- Crout decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ -6 & 10 & 3 & 0 \\ -15 & 20 & -3 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{2}{3} & -1 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ -\frac{2}{5} & -\frac{1}{5} & 0 & 0 \\ 2 & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{19}{3} & 2 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} & 1 & -\frac{13}{15} \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

Question 46. Determine $\det(A)$ using LU decomposition and verify the correctness of the following identity:

$$A = LU \implies \det(A) = \det(L) \det(U).$$

Try both Doolittle decomposition and Crout decomposition.

1. $A = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -4 \\ -4 & -5 & -4 \end{pmatrix}$

2. $A = \begin{pmatrix} -2 & 4 & 3 \\ -4 & 13 & 6 \\ 8 & -36 & -11 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 17 & 0 \\ 0 & -20 & 13 \end{pmatrix}$

4. $A = \begin{pmatrix} -3 & -1 & 4 \\ -15 & -4 & 23 \\ 3 & 2 & -6 \end{pmatrix}$

$$5. A = \begin{pmatrix} -2 & -2 & 4 \\ 2 & 3 & -9 \\ -2 & -7 & 25 \end{pmatrix}$$

$$6. A = \begin{pmatrix} -2 & -4 & 0 \\ 0 & -3 & -3 \\ 0 & 9 & 6 \end{pmatrix}$$

$$7. A = \begin{pmatrix} -2 & 4 & 2 & 0 \\ 6 & -8 & -2 & -1 \\ 8 & -24 & -12 & -3 \\ -8 & 0 & -16 & 13 \end{pmatrix}$$

$$8. A = \begin{pmatrix} 2 & -2 & 4 & -1 \\ -6 & 11 & -9 & 0 \\ 8 & -33 & 2 & 8 \\ 4 & -4 & 11 & -8 \end{pmatrix}$$

$$9. A = \begin{pmatrix} 4 & 0 & -4 & 1 \\ 8 & -3 & -3 & 6 \\ -4 & -12 & 23 & 11 \\ -16 & 12 & -2 & -7 \end{pmatrix}$$

$$10. A = \begin{pmatrix} 3 & -4 & 3 & 5 \\ -15 & 24 & -10 & -30 \\ -12 & 20 & -4 & -20 \\ -6 & 28 & 25 & -23 \end{pmatrix}$$

Solution.

1. $\det(A) = 0$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & -5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 0$$

- Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ -4 & -5 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 0, \quad \det(U) = 1$$

In both cases, we have $\det(A) = \det(L) \det(U)$.

2. $\det(A) = -10$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = -10$$

- Crout decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 \\ -4 & 5 & 0 \\ 8 & -20 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = -10, \quad \det(U) = 1$$

3. $\det(A) = 5$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 5$$

- Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 0 & -20 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 5, \quad \det(U) = 1$$

4. $\det(A) = 15$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & -1 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -5 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 15$$

- Crout decomposition

$$L = \begin{pmatrix} -3 & 0 & 0 \\ -15 & 1 & 0 \\ 3 & 1 & -5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{4}{3} \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 15, \quad \det(U) = 1$$

5. $\det(A) = 8$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & -2 & 4 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 8$$

- Crout decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -5 & -4 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 8, \quad \det(U) = 1$$

6. $\det(A) = -18$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & -4 & 0 \\ 0 & -3 & -3 \\ 0 & 0 & -3 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = -18$$

- Crout decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 9 & -3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = -18, \quad \det(U) = 1$$

7. $\det(A) = 32$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 \\ 4 & -4 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 4 & 2 & 0 \\ 0 & 4 & 4 & -1 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 32$$

- Crout decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ 8 & -8 & 4 & 0 \\ -8 & -16 & -8 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 32, \quad \det(U) = 1$$

8. $\det(A) = 30$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 4 & -5 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -2 & 4 & -1 \\ 0 & 5 & 3 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 30$$

- Crout decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -6 & 5 & 0 & 0 \\ 8 & -25 & 1 & 0 \\ 4 & 0 & 3 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & 2 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 30, \quad \det(U) = 1$$

9. $\det(A) = 60$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ -4 & -4 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & 0 & -4 & 1 \\ 0 & -3 & 5 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 60$$

- Crout decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 8 & -3 & 0 & 0 \\ -4 & -12 & -1 & 0 \\ -16 & 12 & 2 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -1 & \frac{1}{4} \\ 0 & 1 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 60, \quad \det(U) = 1$$

10. $\det(A) = 72$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ -4 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & -4 & 3 & 5 \\ 0 & 4 & 5 & -5 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 72$$

- Crount decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -15 & 4 & 0 & 0 \\ -12 & 4 & 3 & 0 \\ -6 & 20 & 6 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{4}{3} & 1 & \frac{5}{3} \\ 0 & 1 & \frac{5}{4} & -\frac{5}{4} \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 72, \quad \det(U) = 1$$