Algebra and Discrete Mathematics
ADM

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Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces and matrix transformations
- Fundamental spaces and decompositions
- Eulerian tours
- Hamiltonian cycles
- Midterm
- Paths and spanning trees
- Trees and networks
- Matching

Recommended reading

- Anton, Howard, and Chris Rorres. Elementary linear algebra: applications version. John Wiley & Sons, 2013.
  - Sections 1.8, 4.1 – 4.6, 4.9, 4.10
  - Accessible online (free copy)
  - Alternative download link

Lecture outline

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
- Dimension
- Change of basis
- Matrix operators
- Proofs and principles

Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
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- Proofs and principles

Definition

Definition

Let V be a nonempty set on which two operations are defined:

- Addition: $V \times V \to V$, $\boldsymbol{v} + \boldsymbol{u}$ is called the *sum* of $\boldsymbol{v}$ and $\boldsymbol{u}$
- Scalar multiplication: $\mathbb{R} \times V \to V$, $\alpha \boldsymbol{v}$ is called the *scalar multiple* of $\boldsymbol{v}$ by α

V, (together with the associated addition and scalar multiplication), is called a *vector space (over* $\mathbb{R}$*)* if $\forall \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w} \in V$ (*vectors*) and $\forall \alpha, \beta \in \mathbb{R}$ (*scalars*):

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
2. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
3. $\exists \mathbf{0} \in V$, called *zero vector* or *additive identity*, s.t. $\mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$
4. $\exists -\boldsymbol{u} \in V$, called *negative* or *additive inverse* of $\boldsymbol{u}$ s.t. $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$
5. $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha \boldsymbol{u} + \alpha \boldsymbol{v}$
6. $(\alpha + \beta)\boldsymbol{u} = \alpha \boldsymbol{u} + \beta \boldsymbol{u}$
7. $\alpha(\beta \boldsymbol{u}) = (\alpha\beta)\boldsymbol{u}$
8. $1\boldsymbol{u} = \boldsymbol{u}$

Show that a set with two operations is a vector space

- Identify the set V and elements of V that will become vectors
- Identify the addition and and scalar multiplication operations on V
- Verify that $\boldsymbol{v} + \boldsymbol{u} \in V$ (*closure under addition*) and $\alpha \boldsymbol{u}$ V (*closure under scalar multiplication*) for all $\boldsymbol{v}, \boldsymbol{u} \in V$, $\alpha \in \mathbb{R}$
- Verify that all axioms are satisfied

Remark

Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on $\mathbb{R}^n$.

Vector space – example

Example (The zero vector space)

Let $V = \{\mathbf{0}\}$, define

$$\mathbf{0} + \mathbf{0} = \mathbf{0}, \quad \alpha\mathbf{0} = \mathbf{0}, \ \alpha \in \mathbb{R}$$

It is easy to check all axioms are satisfied and V together with the addition and scalar multiplication defined above, $(V, +, \cdot)$, is a vector space.

Vector space – example

Example

Consider $\mathbb{R}^n$ with the usual operations of addition and scalar multiplication

$$\boldsymbol{u} + \boldsymbol{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

$$\alpha \boldsymbol{u} = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$$

zero vector

$$\mathbf{0} = (0, 0, \ldots, 0)$$

$(\mathbb{R}^n, +, \cdot)$ is a vector space.

Vector space – example

Example

Let V consist of objects of the form

$$\boldsymbol{u} = (u_1, u_2, \dots),$$

where $u_1, u_2, \dots$ is an infinite sequence of real numbers.
We define two infinite sequences to be equal if their corresponding components are equal and we define addition and scalar multiplication component-wise by

$$\boldsymbol{u} + \boldsymbol{v} = (u_1, u_2, \dots) + (v_1, v_2, \dots) = (u_1 + v_1, u_2 + v_2, \dots)$$

$$\alpha \boldsymbol{u} = (\alpha u_1, \alpha u_2, \dots)$$

$(V, +, \cdot)$ is a vector space, denoted by $\mathbb{R}^\infty$

Vector space – example

Example (The vector space of 2×2 matrices)

Consider $\mathcal{M}_{2\times 2}$ together with matrix addition and multiplication by a scalar

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{pmatrix}$$

$$\alpha \boldsymbol{u} = \alpha \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha u_{11} & \alpha u_{12} \\ \alpha u_{21} & \alpha u_{22} \end{pmatrix}$$

$\mathcal{M}_{2\times 2}$ is closed under addition and scalar multiplication. The additive identity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

Additive inverse of $\boldsymbol{u}$

$$\begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$$

Example (The vector space of 2×2 matrices)

Consider $\mathcal{M}_{2\times 2}$ together with matrix addition and multiplication by a scalar

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{pmatrix}$$

$$\alpha \boldsymbol{u} = \alpha \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha u_{11} & \alpha u_{12} \\ \alpha u_{21} & \alpha u_{22} \end{pmatrix}$$

$$\begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$1\boldsymbol{u} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$(\mathcal{M}_{2\times 2}, +, \cdot)$ is a vector space

Vector space – example

Example (The vector space of $m \times n$ matrices)

- Consider $\mathcal{M}_{m\times n}$ with matrix addition and scalar multiplication.
- Similar to $\mathcal{M}_{2\times 2}$, we can show that $(\mathcal{M}_{m\times n}, +, \cdot)$ is also a vector space.
- The additive inverse is the zero matrix.

Vector space – example

Example (The vector space of real-valued functions)

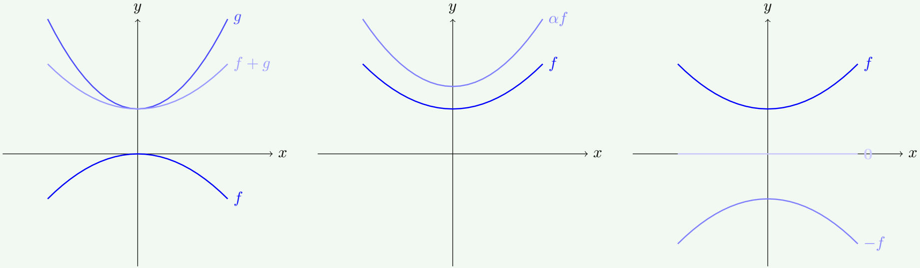
- Let $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$ be the set of real-valued functions that are defined at each $x \in \mathbb{R}$
- Define addition and scalar multiplication as follows: for any $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, and any $\alpha \in \mathbb{R}$,
$$(f + g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha(f(x))$$
- It is easy to see that $f + g$ and αf are real-valued functions defined at each $x \in \mathbb{R}$ - closure under addition and scalar multiplication
- Additive identity: the function $\mathbf{0}$ that outputs 0 for every $x \in \mathbb{R}$
- Additive inverse: the additive inverse of f is the function defined by
$$\begin{aligned} -f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto -f(x) \end{aligned}$$

Vector space – example

Example (The vector space of real-valued functions)

- $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$
- Validity of the axioms follow from the properties of real numbers
- For example, Axiom 1

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$



Vector space – example

Example (Not a vector space)

- Consider $\mathbb{R}^2$ with addition and scalar multiplication defined as follows:
- For any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}$ $\boldsymbol{v} = \begin{pmatrix} v_1, & v_2 \end{pmatrix} \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 + v_1, & u_2 + v_2 \end{pmatrix}, \quad \alpha \otimes \boldsymbol{u} = \begin{pmatrix} \alpha u_1, & 0 \end{pmatrix}.$$

- For example

$$\begin{pmatrix} 2, & 4 \end{pmatrix} + \begin{pmatrix} -3, & 5 \end{pmatrix} = \begin{pmatrix} -1, & 9 \end{pmatrix}, \quad 7 \otimes \begin{pmatrix} 2, & 4 \end{pmatrix} = \begin{pmatrix} 14, & 0 \end{pmatrix}$$

- Not a vector space because for $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}$ with $u_2 \neq 0$

$$1 \otimes \boldsymbol{u} = (u_1, 0) \neq \boldsymbol{u}.$$

Vector space – example

Example (A special vector space)

- Consider $\mathbb{R}_{>0}$, the set of positive real numbers
- Define addition and scalar multiplication as follows: for any $u, v \in \mathbb{R}_{>0}$ and any $\alpha \in \mathbb{R}$

$$u \oplus v = uv, \quad \alpha \otimes u = u^{\alpha}$$

- The additive identity is the number 1

$$u \oplus 1 = u1 = u$$

- The additive inverse of an element u is its reciprocal

$$u \oplus \frac{1}{u} = u\frac{1}{u} = 1$$

because $u > 0$, $\frac{1}{u} > 0$ is an element of $\mathbb{R}_{>0}$

Vector space – example

Example (A special vector space)

- Consider $\mathbb{R}_{>0}$, the set of positive real numbers
- Define addition and scalar multiplication as follows: for any $u, v \in \mathbb{R}_{>0}$ and any $\alpha \in \mathbb{R}$

$$u \oplus v = uv, \quad \alpha \otimes u = u^{\alpha}$$

- Other axioms are also satisfied
- For example, Axiom 6

$$(\alpha + \beta) \otimes u = u^{\alpha+\beta} = u^{\alpha}u^{\beta} = \alpha \otimes u + \beta \otimes u$$

- $(\mathbb{R}_{>0}, \oplus, \otimes)$ is a vector space

Some properties of vectors

Theorem 1

Let V be a vector space, for any $\boldsymbol{u} \in V$ and $\alpha \in \mathbb{R}$

(a) $0\boldsymbol{u} = \mathbf{0}$

(b) $\alpha\mathbf{0} = \mathbf{0}$

(c) $(-1)\boldsymbol{u} = -\boldsymbol{u}$

(d) *If $\alpha\boldsymbol{u} = \mathbf{0}$, then $\alpha = 0$ or $\boldsymbol{u} = \mathbf{0}$*

Remark

- Whenever we discover a new theorem about general vector spaces, we will at the same time be discovering a theorem about $\mathbb{R}^n$, matrices, etc.
- For example, consider the vector space $\mathbb{R}_{>0}$ with the two operations defined in the previous example

$$0\boldsymbol{u} = \boldsymbol{0}$$

translates to for $u \in \mathbb{R}_{>0}$

$$u^0 = 1$$

Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
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Subspace – definition

Definition

A subset W of a vector space V is called a *subspace* of V if W is a vector space under the addition and scalar multiplication defined on V.

- To show W is a vector space, certain properties are "inherited" from V
- e.g. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
- It remains to show
    - Closure of W under addition
    - Closure of W under scalar multiplication
    - Additive identity $\in W$
    - Existence of additive inverse

Theorem

Let V be a vector space, a nonempty set $W \subseteq V$ is a subspace of V iff for any $\boldsymbol{u}, \boldsymbol{v} \in W$, $\alpha \in \mathbb{R}$

1. $\boldsymbol{u} + \boldsymbol{v} \in W$
2. $\alpha \boldsymbol{u} \in W$

Subspace – example

Example

- Let V be any vector space and let $W = \{\mathbf{0}\}$
- W is closed under addition and scalar multiplication

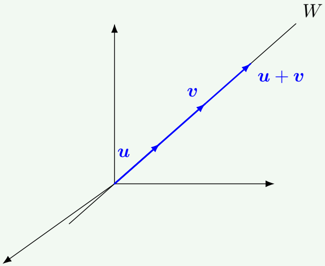
$$\mathbf{0} + \mathbf{0} = \mathbf{0}, \quad \alpha\mathbf{0} = \mathbf{0}$$

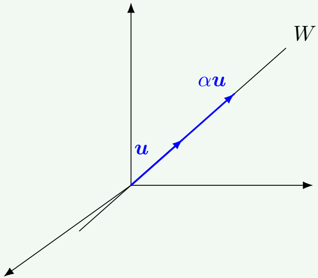
- W is called the *zero subspace* of V

Subspace – example

Example (Lines through the origin are subsapces of $\mathbb{R}^2$ and $\mathbb{R}^3$)

- $W =$ a line through the origin of either $\mathbb{R}^2$ or $\mathbb{R}^3$
- Adding two vectors on the line or multiplying a vector on the line by scalar gives another vector on the line - closed under addition and scalar multiplication

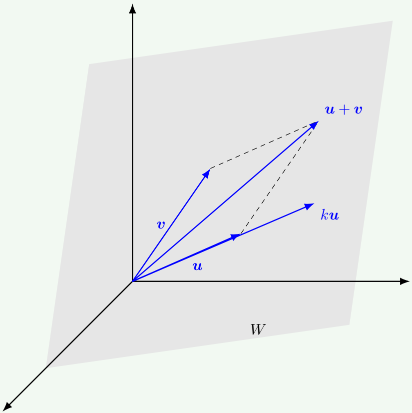




Subspace – example

Example (Planes through the origin are subsapces of $\mathbb{R}^3$)

- W = a plane through the origin of $\mathbb{R}^3$
- Adding two vectors on the line or multiplying a vector on the line by scalar gives another vector in the same plane - closed under addition and scalar multiplication



Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

Summary of what we have discussed

Subspaces pf $\mathbb{R}^2$	Subspaces of $\mathbb{R}^3$
$\{\mathbf{0}\}$	$\{\mathbf{0}\}$
Lines through the origin	Lines through the origin
$\mathbb{R}^2$	Planes through the origin
	$\mathbb{R}^3$

Subspace – example

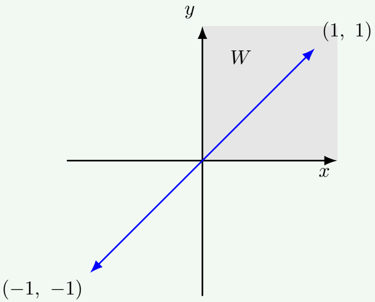
Example (A subset of $\mathbb{R}^2$ that is not a subspace)

- Let

$$W = \{ (x, y) \mid x \geq 0,\ y \geq 0 \} \subseteq \mathbb{R}^2$$

- W is not a subspace of $\mathbb{R}^2$
- W is not closed under scalar multiplication

$$(-1)\begin{pmatrix}1, & 1\end{pmatrix} = \begin{pmatrix}-1, & -1\end{pmatrix} \notin W$$



Linear combination

Definition

$\boldsymbol{u} \in V$ is a *linear combination of* vectors $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \cdots,\ \boldsymbol{v}_r \in V$ if

$$\boldsymbol{u} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_r \boldsymbol{v}_r,$$

where $\alpha_1, \alpha_2 \ldots, \alpha_r \in \mathbb{R}$ are called the *coefficients* of the linear combination.

Linear combinations – example

Example

- Consider $\boldsymbol{u} = (1, 2, -1)$, $\boldsymbol{v} = (6, 4, 2)$ from $\mathbb{R}^3$
- $\boldsymbol{w} = (9, 2, 7)$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$:
  Suppose
  $$\boldsymbol{w} = \alpha_1 \boldsymbol{u} + \alpha_2 \boldsymbol{v}$$
  $$\left(9, \quad 2, \quad 7\right) = \left(\alpha_1 + 6\alpha_2, \quad 2\alpha_1 + 4\alpha_2, \quad -\alpha_1 + 2\alpha_2\right)$$
  gives
  $$\begin{aligned} \alpha_1 + 6\alpha_2 &= 9 \\ 2\alpha_1 + 4\alpha_2 &= 2 \\ -\alpha_1 + 2\alpha_2 &= 7 \end{aligned}$$
  Solving the linear system gives $\alpha_1 = -3$, $\alpha_2 = 2$
  $$\boldsymbol{w} = -3\boldsymbol{u} + 2\boldsymbol{v}.$$

Linear combinations – example

Example

- Consider $\boldsymbol{u} = (1, 2, -1)$, $\boldsymbol{v} = (6, 4, 2)$ from $\mathbb{R}^3$
- $\boldsymbol{w}' = (4, -1, 8)$ is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$:
  Suppose
  $$\boldsymbol{w}' = \alpha_1 \boldsymbol{u} + \alpha_2 \boldsymbol{v}$$
  $$(4, \quad -1, \quad 8) = (\alpha_1 + 6\alpha_2, \quad 2\alpha_1 + 4\alpha_2, \quad -\alpha_1 + 2\alpha_2)$$
  gives
  $$\begin{array}{rcl} \alpha_1 + 6\alpha_2 & = & 4 \\ 2\alpha_1 + 4\alpha_2 & = & -1 \\ -\alpha_1 + 2\alpha_2 & = & 8 \end{array}$$
  The linear system is inconsistent

Subspace from a set of vectors

Theorem

Let $S = \{ \boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \cdots, \ \boldsymbol{v}_r \} \subseteq V$*, let* W *be the set of all possible linear combinations of vectors in* S*. Then*

- W *is a subspace of* V
- W *is the "smallest" subspace of* V *that contain* S *– any other subsapce of* V *containing* S *contains* W

Definition

Let $S = \{ \boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \cdots, \ \boldsymbol{v}_r \} \subseteq V$, the subspace W of V that consists of all linear combinations of the vectors in S is called the subspace of V *generated* by S, and we say the vectors $\boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \cdots, \ \boldsymbol{v}_r$ *span* W. We write

$$W = \operatorname{span}\{\boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \cdots, \ \boldsymbol{v}_r\}, \quad W = \operatorname{span}(S), \quad W = \langle S \rangle.$$

Subspace generated by S – example

Example (The standard unit vectors span $\mathbb{R}^n$)

- The unit vectors

$$\boldsymbol{e}_1 := (1, 0, 0, \cdots, 0), \quad \boldsymbol{e}_2 := (0, 1, 0, \cdots, 0), \quad \ldots, \quad \boldsymbol{e}_n := (0, 0, 0, \cdots, 1)$$

  are called the *standard unit vectors* of $\mathbb{R}^n$.
- Any $\boldsymbol{v} = (v_1, \ v_2, \ \ldots, \ v_n) \in \mathbb{R}^n$ can be expressed as

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n$$

- $\mathbb{R}^n = \operatorname{span}\{\boldsymbol{e}_1, \ \boldsymbol{e}_2, \ \cdots, \ \boldsymbol{e}_n\}$
- For example,

$$\boldsymbol{e}_1 = (1, 0, 0), \quad \boldsymbol{e}_2 = (0, 1, 0), \quad \boldsymbol{e}_3 = (0, 0, 1)$$

  span $\mathbb{R}^3$
- e.g. $\boldsymbol{v} = (2, 3, -2)$

$$\boldsymbol{v} = 2\boldsymbol{v}_1 + 3\boldsymbol{v}_2 + (-2)\boldsymbol{e}_3$$

Test for spanning

Example

- Determine if the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ span $\mathbb{R}^3$

$$\boldsymbol{v}_1 = \begin{pmatrix}1, & 1, & 2\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}1, & 0, & 1\end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix}2, & 1, & 3\end{pmatrix}$$

- We need to show every vector $\boldsymbol{u} \in \mathbb{R}^3$ can be expressed as a linear combination of $\boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \boldsymbol{v}_3$

$$\begin{pmatrix}u_1, & u_2, & u_3\end{pmatrix} = \alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3$$

$$\begin{pmatrix}u_1, & u_2, & u_3\end{pmatrix} = \begin{pmatrix}\alpha_1 + \alpha_2 + 2\alpha_3, & \alpha_1 + \alpha_3, & 2\alpha_1 + \alpha_2 + 3\alpha_3\end{pmatrix}$$

or

$$\begin{aligned} \alpha_1 + \alpha_2 + 2\alpha_3 &= u_1 \\ \alpha_1 + \alpha_3 &= u_2 \\ 2\alpha_1 + \alpha_2 + 3\alpha_3 &= u_3 \end{aligned}$$

Test for spanning

Example

$$
\begin{aligned}
\alpha_1 + \alpha_2 + 2\alpha_3 &= u_1 \\
\alpha_1 + \alpha_3 &= u_2 \\
2\alpha_1 + \alpha_2 + 3\alpha_3 &= u_3
\end{aligned}
$$

- We need to show the linear system is consistent for all values of u_1, u_2, u_3, which is true iff the coefficient matrix is invertible

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \quad \det(A) = 0$$

- The vectors $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3$ do not span $\mathbb{R}^3$

Test for spanning – $\mathbb{R}^n$

From the previous examples, we have the following result for a special case

Theorem

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ *spans* $\mathbb{R}^n$ *iff the determinant* $\begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{vmatrix} \neq 0.$

Solution spaces of homogeneous systems

Theorem

The solution set of a homogeneous linear system $A\boldsymbol{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of $\mathbb{R}^n$.

Definition

W is called the *solution space* of the system.

Solution space – example

Example

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

General solution

$$x = 2s - 3t, \quad y = s, \quad z = t$$

it follows that

$$x - 2y + 2z = 0$$

Corresponds to a plane through the origin.

Solution space – example

Example

$$\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

General solution

$$x = -35t, \quad y = -t, \quad z = t$$

Corresponds to a line through the origin. The parametric equation for the line is

$$x = -35t, \quad y = -t, \quad z = t, \quad t \in \mathbb{R}.$$

Solution space – example

Example

$$\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Has a unique solution $\mathbf{0}$, corresponding to the zero subspace of $\mathbb{R}^3$

Solution space – example

Example

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solution space $= \mathbb{R}^3$

Vector spaces and matrix transformations

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Linearly independent vectors

Definition

$S = \{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \ldots, \boldsymbol{v}_r\} \subseteq V$ is said to be *linearly independent* if

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_r \boldsymbol{v}_r = \mathbf{0} \Longrightarrow \alpha_1 = 0,\ \alpha_2 = 0,\ \cdots,\ \alpha_r = 0$$

Otherwise, S is said to be *linearly dependent*.

Linearly independent vectors – $\mathbb{R}^n$

Theorem

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \subseteq \mathbb{R}^n$ *is linearly independent iff the determinant* $\begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{vmatrix} \neq 0.$

Proof.

S is linearly independent iff the linear system

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_r \boldsymbol{v}_r = \boldsymbol{0}$$

has only the trivial solution. The coefficient matrix A has columns given by $\boldsymbol{v}_i^\top$.
$\det(A) = \det(A^\top)$ □

Linearly dependent vectors in $\mathbb{R}^n$

Theorem

$S = \{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \ldots, \boldsymbol{v}_r\} \subseteq \mathbb{R}^n$. *If* $r > n$, *then* S *is linearly dependent.*

Proof.

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_r \boldsymbol{v}_r = \mathbf{0}$$

corresponds to the homogeneous system

$$\begin{array}{ccccccccc}
v_{11}\alpha_1 & + & v_{21}\alpha_2 & + & \cdots & + & v_{r1}\alpha_r & = & 0 \\
v_{12}\alpha_1 & + & v_{22}\alpha_2 & + & \cdots & + & v_{r2}\alpha_r & = & 0 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
v_{1n}\alpha_1 & + & v_{2n}\alpha_2 & + & \cdots & + & v_{rn}\alpha_r & = & 0
\end{array}$$

which has more equations than unknowns $\square$

Linearly independent vectors – example

Example

$$\boldsymbol{v}_1 = \begin{pmatrix}1, & -2, & 3\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}5, & 6, & -1\end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix}3, & 2, & 1\end{pmatrix}$$

$$\alpha_1\begin{pmatrix}1, & -2, & 3\end{pmatrix} + \alpha_2\begin{pmatrix}5, & 6, & -1\end{pmatrix} + \alpha_3\begin{pmatrix}3, & 2, & 1\end{pmatrix} = \begin{pmatrix}0, & 0, & 0\end{pmatrix}$$

$$\begin{aligned} \alpha_1 + 5\alpha_2 + 3\alpha_3 &= 0 \\ -2\alpha_1 + 6\alpha_2 + 2\alpha_3 &= 0 \\ 3\alpha_1 - \alpha_2 + \alpha_3 &= 0 \end{aligned}$$

The determinant of the coefficient matrix

$$\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 0 \Longrightarrow \text{linearly dependent}$$

Solution set is $\left\{ \begin{pmatrix} -\frac{1}{2}t, & -\frac{1}{2}t, & t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

Linearly independent vectors – example

Example

- Determine whether $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3 \in \mathbb{R}^3$ are linearly independent

$$\boldsymbol{v}_1 = \begin{pmatrix}1, & -2, & 3\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}5, & 6, & -1\end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix}3, & 2, & 1\end{pmatrix}$$

$$\begin{vmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

- In fact, $\boldsymbol{v}_3 - \frac{1}{2}\boldsymbol{v}_1 - \frac{1}{2}\boldsymbol{v}_2 = \mathbf{0}$

Linearly independent vectors – example

Example

- Determine whether $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3 \in \mathbb{R}^4$ are linearly independent

$$\boldsymbol{v}_1 = \begin{pmatrix} 1, & 2, & 2, & -1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 4, & 9, & 9, & -4 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 5, & 8, & 9, & -5 \end{pmatrix}$$

$$\alpha_1 \begin{pmatrix} 1, & 2, & 2, & -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 4, & 9, & 9, & -4 \end{pmatrix} + \alpha_3 \begin{pmatrix} 5, & 8, & 9, & -5 \end{pmatrix} = \begin{pmatrix} 0, & 0, & 0, & 0 \end{pmatrix}$$

$$\begin{aligned} \alpha_1 + 4\alpha_2 + 5\alpha_3 &= 0 \\ 2\alpha_1 + 9\alpha_2 + 8\alpha_3 &= 0 \\ 2\alpha_1 + 9\alpha_2 + 9\alpha_3 &= 0 \\ -\alpha_1 - 4\alpha_2 - 5\alpha_3 &= 0 \end{aligned}$$

- The system has only the trivial solution $\Longrightarrow$ the vectors are linearly independent

Linearly independent vectors

Theorem

$S = \{\boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \ldots, \boldsymbol{v}_r\} \subseteq V$, $r \geq 2$, S is linearly independent iff no vector in S can be expressed as a linear combination of the others.

Example

- $S = \{\begin{pmatrix}2, & 3\end{pmatrix}, \begin{pmatrix}1, & 0\end{pmatrix}, \begin{pmatrix}0, & 1\end{pmatrix}\}$, linearly dependent
- $S = \{\begin{pmatrix}1, & 0\end{pmatrix}, \begin{pmatrix}0, & 1\end{pmatrix}\}$, linearly independent

Special cases

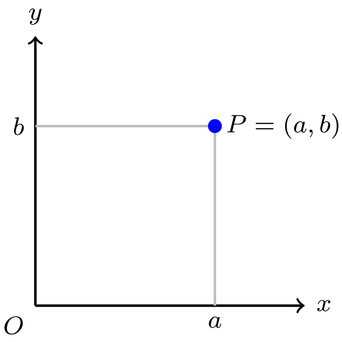
- A finite set that contains $\mathbf{0}$ is linearly dependent
- A set with exactly one vector is linearly independent iff that vector is not $\mathbf{0}$
- A set with exactly two vectors is linearly independent iff neither vector is a scalar multiple of the other

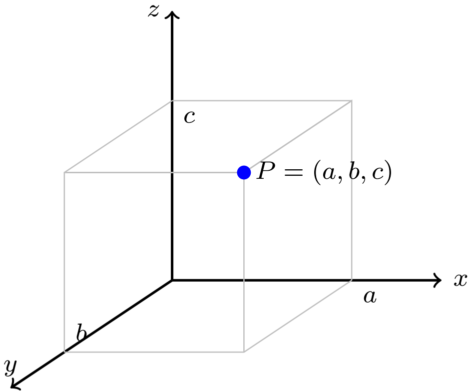
Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
- Dimension
- Change of basis
- Matrix operators
- Proofs and principles

Rectangular coordinate systems

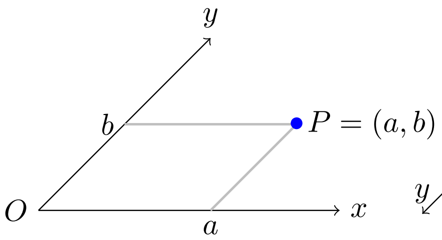
- It is common to use rectangular coordinate systems to create a one-to-one correspondence between points in $2-$D space and ordered pairs of real numbers and between points in $3-$D space and ordered triple of real numbers

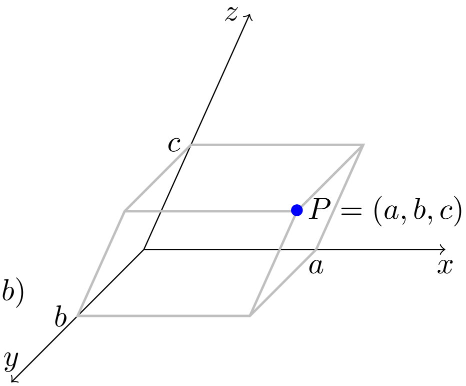




Non-rectangular coordinate system

- Although rectangular coordinate systems are common, they are not essential.

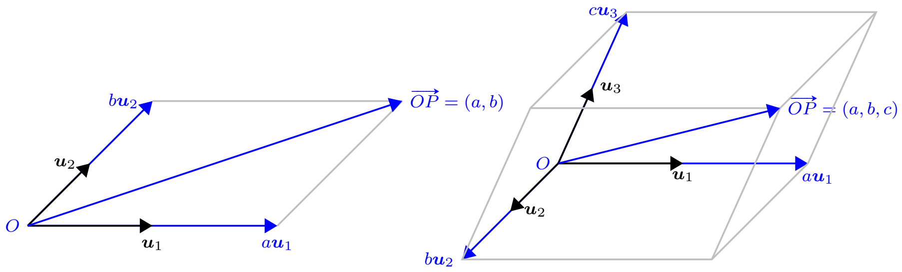




Coordinate systems

- We can specify a coordinate system using vectors rather than coordinate axes
- Here, we have re-created the coordinate system from the previous slide by using unit vectors to identify the positive directions and then attaching coordinates to a point P using the scalar coefficients in the equations

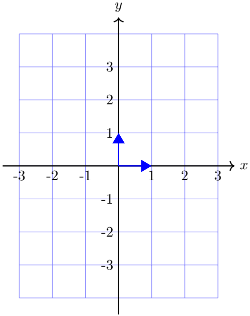
$$\overrightarrow{OP} = a\boldsymbol{u}_1 + b\boldsymbol{u}_2, \quad \overrightarrow{OP} = a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3$$



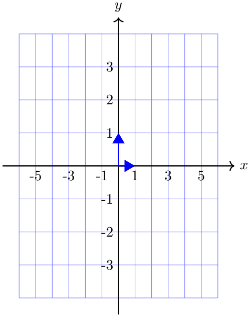
Units of measurement

- Units of measurement are essential ingredients of any coordinate system
- In geometry problems one tries to use the same unit of measurement on all axes to avoid distorting the shapes of figures.
- This is less important in applications where coordinates represent physical quantities with diverse units (for example, time in seconds on one axis and temperature in degrees Celsius on another axis).
- To allow for this level of generality, we will relax the requirement that unit vectors be used to identify the positive directions and require only that those vectors be linearly independent.
- We will refer to these as the “basis vectors” for the coordinate system.

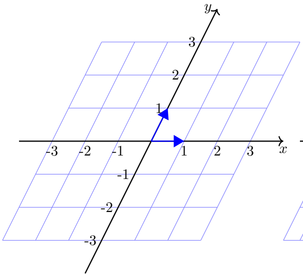
Units of measurement



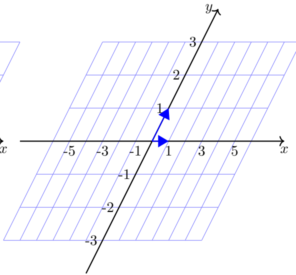
Equal spacing
Perpendicular axes



Unequal spacing
Perpendicular axes



Equal spacing
Skew axes



Unequal spacing
Skew axes

The directions of the basis vectors establish the positive directions, the lengths of the basis vectors establish the spacing between the integer points on the axes

Dimension of vector space

- We will now extend the concept of “basis vectors” and “coordinate systems” to general vector spaces
- A vector space V is said to be *finite-dimensional* if there is a finite set of vectors in V that spans V and is said to be *infinite-dimensional* if no such set exists

Basis

Definition

If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a *basis* for V if

(a) S spans V

(b) S is linearly independent

Basis – example

Example

- We have discussed that the standard unit vectors of $\mathbb{R}^n$

$$\boldsymbol{e}_1 := (1, 0, 0, \cdots, 0), \quad \boldsymbol{e}_2 := (0, 1, 0, \cdots, 0), \quad \ldots, \quad \boldsymbol{e}_n := (0, 0, 0, \cdots, 1)$$

  span $\mathbb{R}^n$
- They are also linearly independent: the matrix $\begin{pmatrix} \boldsymbol{e}_1^\top & \boldsymbol{e}_2^\top & \cdots & \boldsymbol{e}_n^\top \end{pmatrix} = I_n$
- Thus, they form a basis for $\mathbb{R}^n$ – *standard basis for* $\mathbb{R}^n$.

Basis for $\mathbb{R}^n$

Theorem

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ *is a basis for* $\mathbb{R}^n$ *iff the determinant* $\begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{vmatrix} \neq 0.$

Example (Another basis for $\mathbb{R}^3$)

$\boldsymbol{v}_1 = \begin{pmatrix} 1, & 2, & 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 2, & 9, & 0 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 3, & 3, & 4 \end{pmatrix}$ The determinant

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 9 & 0 \\ 3 & 3 & 4 \end{vmatrix} = -1 \neq 0$$

Thus $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\mathbb{R}^3$.

Uniqueness of basis representation

Theorem

If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ *is a basis for a vector space* V*, then every vector* $\boldsymbol{v} \in V$ *has a unique representation as a linear combination of vectors in* S*:*

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_v \boldsymbol{v}_n.$$

Coordinates

Definition

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a basis for V. Suppose $\boldsymbol{v} \in V$

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_v \boldsymbol{v}_n,$$

then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the *coordinates* of $\boldsymbol{v}$ relative to the basis S. The vector $\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots, \alpha_n \end{pmatrix}^\top$ is called the *coordinate vector of* $\boldsymbol{v}$ *relative to* S, denoted by

$$[\boldsymbol{v}]_S = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Remark

The order of vectors matters for coordinate vectors – we assume the underlying basis is ordered without saying so explicitly

Coordinates – example

Example

- Let S be the standard basis for $\mathbb{R}^n$
- The coordinate vector $[\boldsymbol{v}]_S$ and the vector $\boldsymbol{v}$ are the same

Coordinates – example

Example

- We have discussed that $\boldsymbol{v}_1 = \begin{pmatrix} 1, & 2, & 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 2, & 9, & 0 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 3, & 3, & 4 \end{pmatrix}$ form a basis for $\mathbb{R}^3$
- Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$
- $\boldsymbol{v} = \begin{pmatrix} 5, & -1, & 9 \end{pmatrix}$
- Solve for $\alpha_1, \alpha_2, \alpha_3$

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3$$

  we get

$$\alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = 2$$

- Then

$$[\boldsymbol{v}]_S = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Vector spaces and matrix transformations

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Bases

Theorem

If B_1, B_2 *are bases of a vector space* V, *then* $|B_1| = |B_2|$

Dimension

Definition

The *dimension* of a vector space V, denoted $\dim(V)$, is given by the cardinality of B, $|B|$, where B is a basis of V. The zero vector space is defined to have dimension zero.

Example

- $\dim(\mathbb{R}^n) = n$ - the standard basis has n vectors
- $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$, linearly independent, then $\operatorname{span}(S)$ has dimension r

Dimension – example

Example (Dimension of a solution space)

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Solution is of the form

$$\begin{aligned} &\begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \end{pmatrix} = \begin{pmatrix} -3r - 4s - 2t, & r - 2s, & s, & t, & 0 \end{pmatrix} \\ &= r\begin{pmatrix} -3, & 1, & 0, & 0, & 0, & 0 \end{pmatrix} + s\begin{pmatrix} -4, & 0, & -2, & 1, & 0, & 0 \end{pmatrix} + t\begin{pmatrix} -2, & 0, & 0, & 0, & 1, & 0 \end{pmatrix} \end{aligned}$$

$\Longrightarrow$ the vectors

$\begin{pmatrix} -3, & 1, & 0, & 0, & 0, & 0 \end{pmatrix}$, $\begin{pmatrix} -4, & 0, & -2, & 1, & 0, & 0 \end{pmatrix}$, $\begin{pmatrix} -2, & 0, & 0, & 0, & 1, & 0 \end{pmatrix}$

span the solution space. They are also linearly independent (verify), thus the solution space has dimension 3.

Remark

- It can be shown that for any homogeneous linear system, the method of the last example always produces a basis for the solution space of the system.
- We omit the formal proof.

Vector spaces and matrix transformations

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The change-of-basis problem

The change-of-basis problem

$\boldsymbol{v} \in V$, $\dim(V) < \infty$. Let B_1 and B_2 be two bases for V. What is the relation between $[\boldsymbol{v}]_{B_1}$ and $[\boldsymbol{v}]_{B_2}$?

- B_1 old basis
- B_2 new basis
- Our objective is to find a relationship between the old and new coordinates of a fixed vector $\boldsymbol{v} \in V$

Two dimensional vector spaces

- Suppose $\dim(V) = 2$ $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}, \quad B_2 = \{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ and

$$[\boldsymbol{w}_1]_{B_1} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad [\boldsymbol{w}_2]_{B_1} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\text{i.e.} \quad \boldsymbol{w}_1 = a\boldsymbol{u}_1 + b\boldsymbol{u}_2, \quad \boldsymbol{w}_2 = c\boldsymbol{u}_1 + d\boldsymbol{u}_2$$

- Suppose $[\boldsymbol{v}]_{B_2} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, so

$$\boldsymbol{v} = k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2 = k_1(a\boldsymbol{u}_1 + b\boldsymbol{u}_2) + k_2(c\boldsymbol{u}_1 + d\boldsymbol{u}_2) = (k_1a + k_2c)\boldsymbol{u}_1 + (k_1b + k_2d)\boldsymbol{u}_2$$

$$[\boldsymbol{v}]_{B_1} = \begin{pmatrix} k_1a + k_2c \\ k_1b + k_2d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} [\boldsymbol{v}]_{B_2}$$

$$P := \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \big([\boldsymbol{w}_1]_{B_1} \quad [\boldsymbol{w}_2]_{B_1}\big)$$

Change of basis and transition matrices

- $\boldsymbol{v} \in V$, $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, $B_2 = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ basis for V
- Then

$$[\boldsymbol{v}]_{B_1} = P[\boldsymbol{v}]_{B_2}$$

  where

$$P = \begin{pmatrix} [\boldsymbol{w}_1]_{B_1} & [\boldsymbol{w}_2]_{B_1} & \cdots & [\boldsymbol{w}_n]_{B_1} \end{pmatrix}$$

- P: *transition matrix* from B_2 to B_1, denoted $P_{B_2 \to B_1}$

$$P_{B_2 \to B_1} = \begin{pmatrix} [\boldsymbol{w}_1]_{B_1} & [\boldsymbol{w}_2]_{B_1} & \cdots & [\boldsymbol{w}_n]_{B_1} \end{pmatrix}$$

$$P_{B_1 \to B_2} = \begin{pmatrix} [\boldsymbol{u}_1]_{B_2} & [\boldsymbol{u}_2]_{B_2} & \cdots & [\boldsymbol{u}_n]_{B_2} \end{pmatrix}$$

- The coordinate vector

$$[\boldsymbol{v}]_{B_1} = P_{B_2 \to B_1}[\boldsymbol{v}]_{B_2}, \quad [\boldsymbol{v}]_{B_2} = P_{B_1 \to B_2}[\boldsymbol{v}]_{B_1}$$

Change of basis and transition matrices – example

Example

- $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $B_2 = \{\boldsymbol{w}_1, \boldsymbol{w}_2\}$, basis for $\mathbb{R}^2$

$$\boldsymbol{u}_1 = \begin{pmatrix}1, & 0\end{pmatrix}, \; \boldsymbol{u}_2 = \begin{pmatrix}0, & 1\end{pmatrix}, \; \boldsymbol{w}_1 = \begin{pmatrix}1, & 1\end{pmatrix}, \; \boldsymbol{w}_2 = \begin{pmatrix}2, & 1\end{pmatrix}$$

- Transition matrix $P_{B_2 \to B_1}$

$$[\boldsymbol{w}_1]_{B_1} = \begin{pmatrix}1\\1\end{pmatrix}, \quad [\boldsymbol{w}_2]_{B_1} = \begin{pmatrix}2\\1\end{pmatrix} \Longrightarrow P_{B_2 \to B_1} = \begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}$$

- Transition matrix $P_{B_1 \to B_2}$

$$[\boldsymbol{u}_1]_{B_2} = \begin{pmatrix}-1\\1\end{pmatrix}, \quad [\boldsymbol{u}_2]_{B_2} = \begin{pmatrix}2\\-1\end{pmatrix} \Longrightarrow P_{B_1 \to B_2} = \begin{pmatrix}-1 & 2\\1 & -1\end{pmatrix}$$

- Suppose $[\boldsymbol{v}]_{B_2} = \begin{pmatrix}-3\\5\end{pmatrix}$, $[\boldsymbol{v}]_{B_1} = P_{B_2 \to B_1}[\boldsymbol{v}]_{B_2} = \begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}\begin{pmatrix}-3\\5\end{pmatrix} = \begin{pmatrix}7\\2\end{pmatrix}$

Invertibility of transition matrices

$$P_{B_2 \to B_1} P_{B_1 \to B_2} = P_{B_1 \to B_1} = I$$

Using the previous example

$$P_{B_2 \to B_1} P_{B_1 \to B_2} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem

The transition matrix $P_{B_1 \to B_2}$ is invertible and $P^{-1}_{B_1 \to B_2} = P_{B_2 \to B_1}$.

Vector spaces and matrix transformations

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Matrix transformations

- Consider the linear system $A\boldsymbol{x} = \boldsymbol{w}$
- We can view it as a transformation that maps a vector $\boldsymbol{x} \in \mathbb{R}^n$ into the vector $\boldsymbol{w} \in \mathbb{R}^m$
- We call this a *matrix transformation* (or *matrix operator* when $m = n$), denoted

$$\begin{aligned} T_A : \mathbb{R}^n &\to \mathbb{R}^m \\ \boldsymbol{x} &\mapsto \boldsymbol{w} \end{aligned}$$

- We call the transformation T_A *multiplication by* A

Example

- *Zero transformation*:

$$T_O(\boldsymbol{x}) = O\boldsymbol{x} = \boldsymbol{0}$$

- *Identity operator*:

$$T_I(\boldsymbol{x}) = I\boldsymbol{x} = \boldsymbol{x}$$

Properties of matrix transformations

Theorem

For any $A \in \mathcal{M}_{m\times n}$, $\alpha \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $T_A : \mathbb{R}^n \to \mathbb{R}^m$ *satisfies*

- $T_A(\mathbf{0}) = \mathbf{0}$
- $T_A(\alpha\boldsymbol{u}) = \alpha T_A(\boldsymbol{u})$ *(Homogeneity property)*
- $T_A(\boldsymbol{u}+\boldsymbol{v}) = T_A(\boldsymbol{u}) + T_A(\boldsymbol{v})$ *(Additivity property)*
- $T_A(\boldsymbol{u}-\boldsymbol{v}) = T_A(\boldsymbol{u}) - T_A(\boldsymbol{v})$

Remark

When we deal with matrix transformations, vectors are assumed to be column vectors unless explicitly stated otherwise.

Standard matrix

Theorem

$T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^n \to \mathbb{R}^m$. *If* $T_A(\boldsymbol{x}) = T_B(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathbb{R}^n$, *then* $A = B$.

Note

- Every $A \in \mathcal{M}_{m\times n}$ produces exactly one matrix transformation (multiplication by A)
- Every matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ arises from exactly one $A \in \mathcal{M}_{m\times n}$ – *standard matrix* for the transformation

Find standard matrix

- The standard matrix for $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$A = \begin{pmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \end{pmatrix}$$

- $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, . . ., $\boldsymbol{e}_n$: standard basis

Operators on $\mathbb{R}^2$ and $\mathbb{R}^3$

- There are many ways to transform the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$
- Some can be accomplished by using a *matrix operator* T_A, $A \in \mathcal{M}_{2\times 2}$ or $\mathcal{M}_{3\times 3}$
- e.g. rotations about the origin, reflections about lines and planes through the origin, and projections onto lines and planes through the origin

Reflection operators

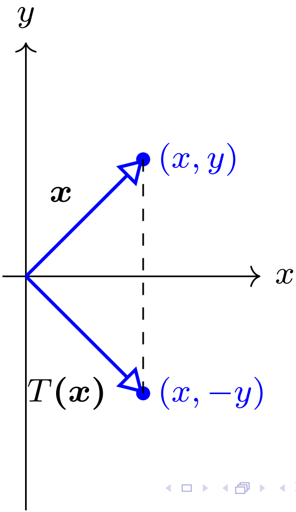
- Map each point into its symmetric image about a fixed line or a fixed plane that contains the origin

Reflection operators on $\mathbb{R}^2$ – reflection about the $x-$axis

- $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$
- Images of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

$$\begin{aligned} T(\boldsymbol{e}_1) &= T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ T(\boldsymbol{e}_2) &= T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

- Standard matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



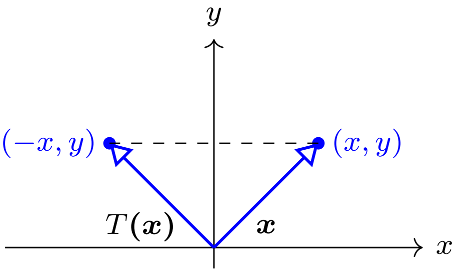
Reflection operators on $\mathbb{R}^2$ – reflection about the $y-$axis

- $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$
- Images of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

$$T(\boldsymbol{e}_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$T(\boldsymbol{e}_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Standard matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



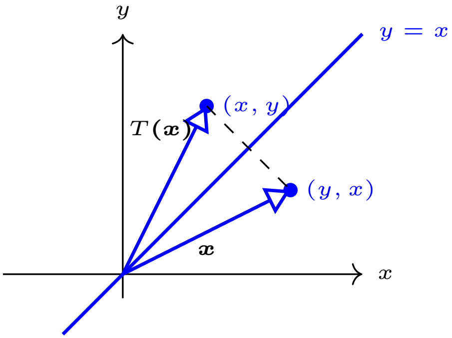
Reflection operators on $\mathbb{R}^2$ – reflection about the line $y = x$

- $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$
- Images of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

$$T(\boldsymbol{e}_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T(\boldsymbol{e}_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Standard matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



Reflection operators on $\mathbb{R}^3$

Reflection about the $xy-$plane, $xz-$plane and $yz-$plane

$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\y\\-z\end{pmatrix}$	$T(\boldsymbol{e}_1)=\begin{pmatrix}1\\0\\0\end{pmatrix},\ T(\boldsymbol{e}_2)=\begin{pmatrix}0\\1\\0\end{pmatrix},\ T(\boldsymbol{e}_3)=\begin{pmatrix}0\\0\\-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$
$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\-y\\z\end{pmatrix}$	$T(\boldsymbol{e}_1)=\begin{pmatrix}1\\0\\0\end{pmatrix},\ T(\boldsymbol{e}_2)=\begin{pmatrix}0\\-1\\0\end{pmatrix},\ T(\boldsymbol{e}_3)=\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$
$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-x\\y\\z\end{pmatrix}$	$T(\boldsymbol{e}_1)=\begin{pmatrix}-1\\0\\0\end{pmatrix},\ T(\boldsymbol{e}_2)=\begin{pmatrix}0\\1\\0\end{pmatrix},\ T(\boldsymbol{e}_3)=\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Projection operators

- Projection operators/orthogonal projection operators: Map each point into its orthogonal projection onto a fixed line or plane through the origin

Projection operators on $\mathbb{R}^2$

- Orthogonal projection onto the $x-$axis

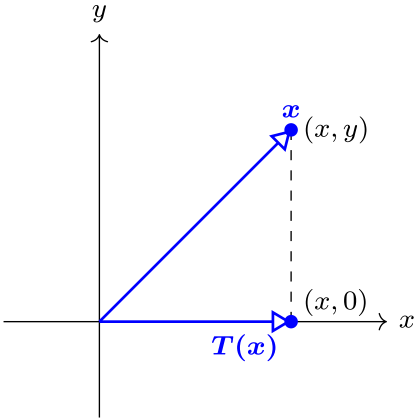
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

- Images of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

$$\begin{aligned} T(\boldsymbol{e}_1) &= T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ T(\boldsymbol{e}_2) &= T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

- Standard matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



Projection operators on $\mathbb{R}^2$

- Orthogonal projection onto the $y-$axis

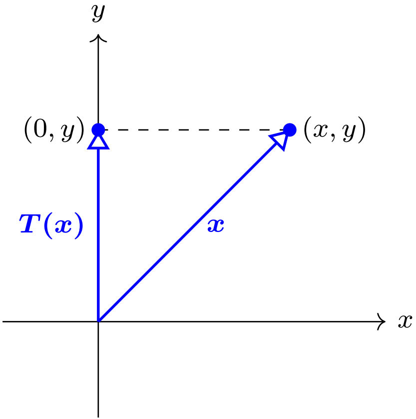
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

- Images of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

$$\begin{aligned} T(\boldsymbol{e}_1) &= T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ T(\boldsymbol{e}_2) &= T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

- Standard matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



Projection operators on $\mathbb{R}^3$

Orthogonal projection onto the $xy-$plane, $xz-$plane and $yz-$plane

$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$	$T(\boldsymbol{e}_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; T(\boldsymbol{e}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; T(\boldsymbol{e}_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$	$T(\boldsymbol{e}_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; T(\boldsymbol{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; T(\boldsymbol{e}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$	$T(\boldsymbol{e}_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; T(\boldsymbol{e}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; T(\boldsymbol{e}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

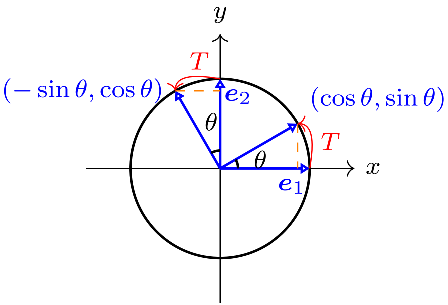
Rotation operators

- Move points along arcs of circles centered at the origin

Rotation operators

- Consider the rotation operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ that moves points *counterclockwise* about the origin through a positive angle θ

$$T(\boldsymbol{e}_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad T(\boldsymbol{e}_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



- Standard matrix - *rotation matrix*

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotation equations:

- $\boldsymbol{x} = (x, \quad y)$, $\boldsymbol{w} = R_\theta \boldsymbol{x}^\top$, $\longrightarrow$

$$\begin{aligned} w_1 &= x\cos\theta - y\sin\theta \\ w_2 &= x\sin\theta + y\cos\theta \end{aligned}$$

Rotation operators

- The rotation matrix for a clockwise rotation of θ, or a rotation of $-\theta$, radians has rotation matrix
$$R_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Rotation operator on $\mathbb{R}^2$ – example

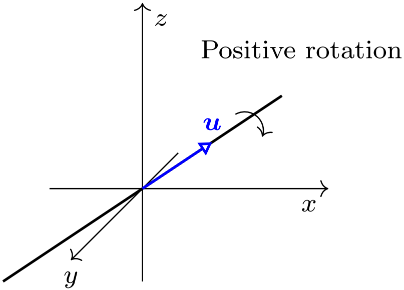
Example

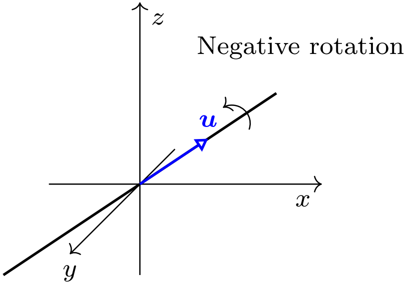
Find the image of $\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ under a rotation of $\dfrac{\pi}{6}$ radians $(= 30^\circ)$ about the origin

$$R_{\frac{\pi}{6}}\boldsymbol{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{3}-1}{2} \\ \dfrac{1+\sqrt{3}}{2} \end{pmatrix} \approx \begin{pmatrix} 0.37 \\ 1.37 \end{pmatrix}$$

Rotation operators on $\mathbb{R}^3$

- A rotation of vectors in $\mathbb{R}^3$ is commonly described in relation to a line through the origin called the *axis of rotation* and unit vector $\boldsymbol{u}$ along that line
- Positive angle: counterclockwise looking toward the origin along the positive coordinate axis
- *right-hand-rule*: cup the fingers of right hand so they curl in the direction of the rotation, thumb points in the direction of $\boldsymbol{u}$ corresponds to positive angle





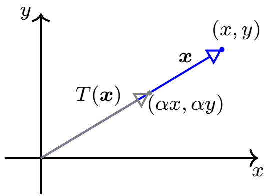
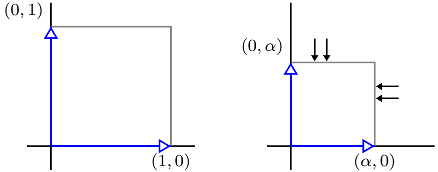
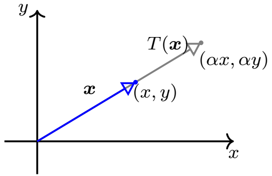
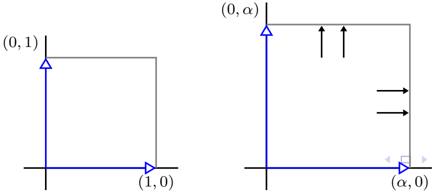
Rotation operators on $\mathbb{R}^3$

Operator	Rotation equations	Standard matrix
Counterclockwise rotation about the positive $x-$axis through an angle θ	$\begin{aligned} w_1 &= x \\ w_2 &= y\cos\theta - z\sin\theta \\ w_3 &= y\sin\theta + z\cos\theta \end{aligned}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive $y-$axis through an angle θ	$\begin{aligned} w_1 &= x\cos\theta + z\sin\theta \\ w_2 &= y \\ w_3 &= -x\sin\theta + z\cos\theta \end{aligned}$	$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive $z-$axis through an angle θ	$\begin{aligned} w_1 &= x\cos\theta - y\sin\theta \\ w_2 &= x\sin\theta + y\cos\theta \\ w_3 &= z \end{aligned}$	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Dilations and contractions

- $\alpha \in \mathbb{R}$, $\alpha \geq 0$
- $T(\boldsymbol{x}) = \alpha \boldsymbol{x}$ on $\mathbb{R}^2$ or $\mathbb{R}^3$ has the effect of increasing or decreasing the length of each vector by a factor of α
- *Contraction* with factor α: $0 \leq \alpha < 1$
- *Dilation* with factor α: $\alpha > 1$
- Identity operator: $\alpha = 1$

Dilations and contractions on $\mathbb{R}^2$ – $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$

Illustration	Effect on the unit square	Standard matrix
		$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$
		$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$

Dilations and contractions on $\mathbb{R}^3$

- Similarly, for dilation/contraction with factor α on $\mathbb{R}^3$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$$

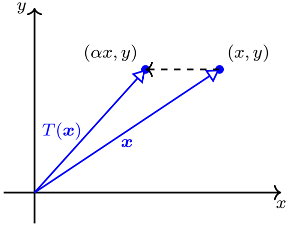
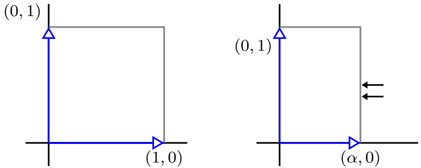
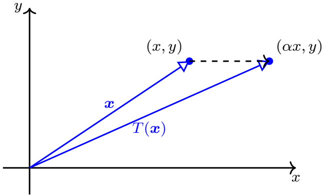
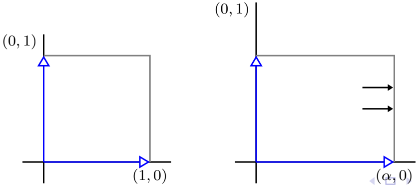
- The standard matrix is

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

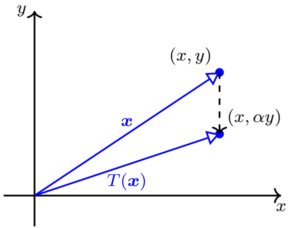
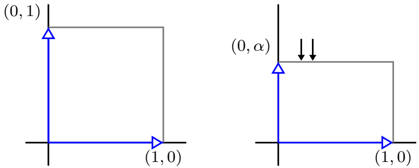
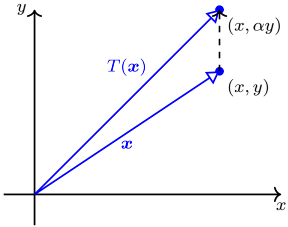
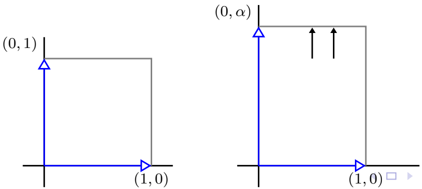
Expansions and compressions

- Dilation or contraction: all coordinates are multiplied by a nonnegative factor
- *Compression/expansion*: only one coordinate is multiplied by α

Expansions and compressions in the $x-$direction – $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ y \end{pmatrix}$

<table>
<tr><th>Illustration</th><th>Effect on the unit square</th><th>Standard matrix</th></tr>
<tr><td>
</td><td>
</td><td>$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$</td></tr>
<tr><td>
</td><td>
</td><td>$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$</td></tr>
</table>

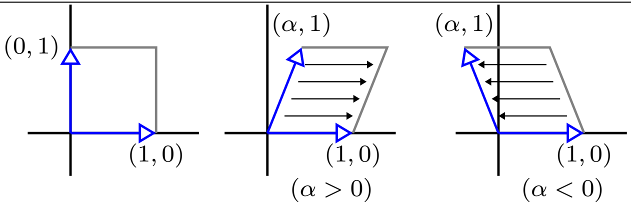
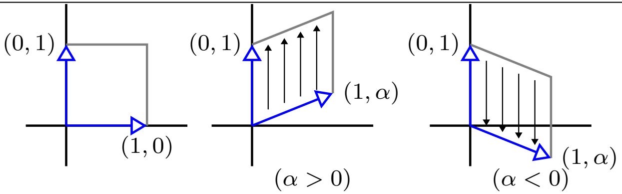
Expansions and compressions in the $y-$direction – $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \alpha y \end{pmatrix}$

Illustration	Effect on the unit square	Standard matrix
		$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$
		$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$

Shears

- $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y \end{pmatrix}$
- Translates a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the $xy-$plane parallel to the $x-$axis by an amount αy that is proportional to the $y-$coordinate of the point.
- Points on the $x-$axis are fixed $(y = 0)$, the translation distance increases as we progress away from the $x-$axis
- *Shear in the $x-$direction by a factor α*
- Similarly, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \alpha x \end{pmatrix}$ - *Shear in the $y-$direction by a factor α*

Shears

Operator	Effect on the unit square	Standard matrix
$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y \end{pmatrix}$		$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$
$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \alpha x \end{pmatrix}$		$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$

Shears – example

Example

- $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$: shear in the $x-$direction by a factor 2
- $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$: shear in the $x-$direction by a factor -2
- $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: dilation with factor 2
- $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$: expansion in the $x-$direction with factor 2

Orthogonal projections onto lines through the origin

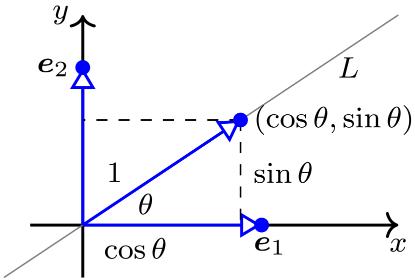
- Orthogonal projections onto the coordinate axes in $\mathbb{R}^2$ are special cases of the operator that maps each point into its orthogonal projection onto a line L through the origin that makes an angle θ with the positive $x-$axis

Orthogonal projection on a line

- Find the orthogonal projections of the vectors $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ on the line L that makes an angle θ with the positive $x-$axis in $\mathbb{R}^2$
- First we find the orthogonal projection of $\boldsymbol{e}_1$ onto $\boldsymbol{a} := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$$\begin{aligned} \operatorname{proj}_{\boldsymbol{a}} \boldsymbol{e}_1 &= \frac{\boldsymbol{e}_1 \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \frac{\cos\theta + 0}{1}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{pmatrix} \end{aligned}$$

- We note that for any other vector, $\boldsymbol{u}$ on the line L, $\boldsymbol{u} = \alpha\boldsymbol{a}$ for some $\alpha \in \mathbb{R}$



- Similarly

$$\begin{aligned} \operatorname{proj}_{\boldsymbol{a}} \boldsymbol{e}_2 &= \frac{\boldsymbol{e}_1 \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} \\ &= \begin{pmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{pmatrix} \end{aligned}$$

Orthogonal projection onto a line through the origin

- projection onto a line L through the origin that makes an angle θ with the positive $x-$axis
- The images of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are

$$T(\boldsymbol{e}_1) = \begin{pmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{pmatrix}, \quad T(\boldsymbol{e}_2) = \begin{pmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{pmatrix}$$

- Standard matrix is

$$P_\theta := \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \dfrac{1}{2}\sin 2\theta \\ \dfrac{1}{2}\sin 2\theta & \sin^2\theta \end{pmatrix}$$

Orthogonal projection onto a line through the origin

Example

- Find the orthogonal projection of the vector $\boldsymbol{x} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ onto the line through the origin that makes an angle of $\dfrac{\pi}{6}(= 30^\circ)$ with the positive $x-$axis
- The standard matrix is

$$P_{\pi/6} = \begin{pmatrix} \cos^2(\pi/6) & \sin(\pi/6)\cos(\pi/6) \\ \sin(\pi/6)\cos(\pi/6) & \sin^2(\pi/6) \end{pmatrix} = \begin{pmatrix} \dfrac{3}{4} & \dfrac{\sqrt{3}}{4} \\ \dfrac{\sqrt{3}}{4} & \dfrac{1}{4} \end{pmatrix}$$

$$P_{\pi/6}\boldsymbol{x} = \begin{pmatrix} \dfrac{3}{4} & \dfrac{\sqrt{3}}{4} \\ \dfrac{\sqrt{3}}{4} & \dfrac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \dfrac{3+5\sqrt{3}}{4} \\ \dfrac{\sqrt{3}+5}{4} \end{pmatrix} \approx \begin{pmatrix} 2.91 \\ 1.68 \end{pmatrix}$$

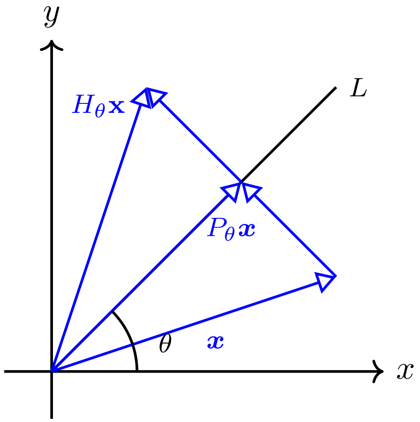
Reflection about lines through the origin

- $H_\theta : \mathbb{R}^2 \to \mathbb{R}^2$
- Maps each point into its reflection about a line L through the origin that makes an angle θ with the positive $x-$axis
- From the figure, we can see

$$P_\theta \boldsymbol{x} - \boldsymbol{x} = \frac{1}{2}(H_\theta \boldsymbol{x} - \boldsymbol{x}) \Longrightarrow H_\theta \boldsymbol{x} = (2P_\theta - I)\boldsymbol{x}$$

- It follows from the theorem that

$$H_\theta = 2P_\theta - I = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$



Theorem

$T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^n \to \mathbb{R}^m$. *If* $T_A(\boldsymbol{x}) = T_B(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathbb{R}^n$, *then* $A = B$.

Reflection about lines through the origin

Example

- Find the reflection of the vector $\boldsymbol{x} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ about the line through the origin that makes an angle $\pi/6$ $(= 30^\circ)$ with the positive $x-$axis
- The standard matrix

$$H_{\pi/6} = \begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{pmatrix}$$

$$H_{\pi/6}\boldsymbol{x}^\top = \begin{pmatrix} \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \dfrac{1+5\sqrt{3}}{2} \\ \dfrac{\sqrt{3}-5}{2} \end{pmatrix} \approx \begin{pmatrix} 4.83 \\ -1.63 \end{pmatrix}$$

Compositions of matrix transformations

- $T_A : \mathbb{R}^n \to \mathbb{R}^k$, $T_B : \mathbb{R}^k \to \mathbb{R}^m$
- The composition of T_B and T_A is a matrix transformation

$$\begin{aligned} T_B \circ T_A : \mathbb{R}^n &\to \mathbb{R}^m \\ \boldsymbol{x} &\mapsto (BA)\boldsymbol{x} \end{aligned}$$

- Because

$$T_B \circ T_A(\boldsymbol{x}) = T_B(T_A(\boldsymbol{x})) = B(A\boldsymbol{x}) = (BA)\boldsymbol{x}$$

- Thus

$$T_B \circ T_A = T_{BA}$$

Compositions of matrix transformations

- Consider

$$T_A : \mathbb{R}^n \to \mathbb{R}^k, \quad T_B : \mathbb{R}^k \to \mathbb{R}^\ell, \quad T_C : \mathbb{R}^\ell \to \mathbb{R}^m$$

- Then

$$T_C \circ T_B \circ T_A : \mathbb{R}^n \to \mathbb{R}^m$$

- and

$$(T_C \circ T_B \circ T_A)(\boldsymbol{x}) = T_C(T_B(T_A(\boldsymbol{x}))) = (CBA)\boldsymbol{x}$$

- i.e.

$$T_C \circ T_B \circ T_A = T_{CBA}$$

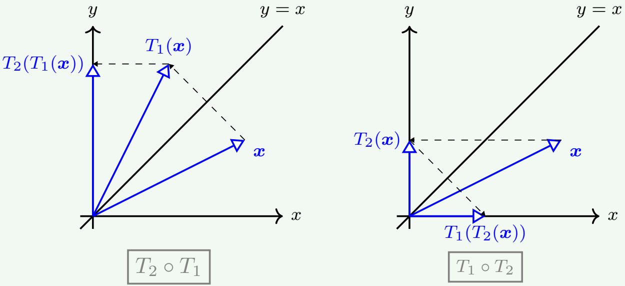
Compositions of matrix transformations

- Let us denote the standard matrix for T by $[T]$
- Then

$$[T_2 \circ T_1] = [T_2][T_1], \quad [T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$$

Composition is not commutative

Example (Composition is not commutative)



- T_1: reflection about the line $y = x$; T_2: orthogonal projection onto the $y-$axis

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Composition of rotations is commutative

Example

- Rotation in $\mathbb{R}^2$
- T_1: rotate vectors about the origin through the angle θ_1
- T_2: rotate vectors about the origin through the angle θ_2
- Standard matrix

$$[T_1] = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \quad [T_2] = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

- With trigonometric identities, it can be verified that

$$[T_2 \circ T_1] = [T_1 \circ T_2] = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}$$

Vector spaces and matrix transformations

- Real Vector Space
- Subspaces
- Linear independence
- Coordinates and basis
- Dimension
- Change of basis
- Matrix operators
- Proofs and principles

Some properties of vectors

Theorem 1

Let V be a vector space, for any $\boldsymbol{u} \in V$ and $\alpha \in \mathbb{R}$

(a) $0\boldsymbol{u} = \mathbf{0}$

(b) $\alpha\mathbf{0} = \mathbf{0}$

(c) $(-1)\boldsymbol{u} = -\boldsymbol{u}$

(d) *If $\alpha\boldsymbol{u} = \mathbf{0}$, then $\alpha = 0$ or $\boldsymbol{u} = \mathbf{0}$*

Proof.

We will show the proof of part (c)

$$
\begin{aligned}
\boldsymbol{u} + (-1)\boldsymbol{u} &= 1\boldsymbol{u} + (-1)\boldsymbol{u} \quad \text{(Axiom 8)} \\
&= (1 + (-1))\boldsymbol{u} \quad \text{(Axiom 6)} \\
&= 0\boldsymbol{u} \quad \text{(property of real numbers)} \\
&= \mathbf{0} \quad \text{(part (a))}
\end{aligned}
$$

Subspace

Theorem

Let V be a vector space, a nonempty set $W \subseteq V$ is a subspace of V iff for any $\boldsymbol{u}, \boldsymbol{v} \in W$, $\alpha \in \mathbb{R}$

1. $\boldsymbol{u} + \boldsymbol{v} \in W$
2. $\alpha \boldsymbol{u} \in W$

Proof.

$\Longrightarrow$ by definition.
$\Longleftarrow$ By the previous discussion, we just need to prove the existence of additive identity and additive inverse. By Axiom 2 and Theorem 2 (a), (c)

$$0\boldsymbol{u} = \mathbf{0} \in W, \quad -\boldsymbol{u} \in W$$

$\square$

Subspace from a set of vectors

Theorem

Let $S = \{ \boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \cdots, \ \boldsymbol{v}_r \} \subseteq V$, let W be the set of all possible linear combinations of vectors in S. Then

- *W is a subspace of V*
- *W is the "smallest" subspace of V that contain S – any other subsapce of V containing S contains W*

Proof.

For any $\boldsymbol{u} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_r \boldsymbol{v}_r, \ \boldsymbol{w} = \beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \cdots + \beta_r \boldsymbol{v}_r \in W$

$$\boldsymbol{u} + \boldsymbol{w} = (\alpha_1 + \beta_1)\boldsymbol{v}_1 + (\alpha_2 + \beta_2)\boldsymbol{v}_2 + \cdots + (\alpha_r + \beta_r)\boldsymbol{v}_r \in W$$

$$\gamma \boldsymbol{u} = (\gamma \alpha_1)\boldsymbol{v}_1 + (\gamma \alpha_2)\boldsymbol{v}_2 + \cdots + (\gamma \alpha_r)\boldsymbol{v}_r \in W$$

W is closed under vector addition and scalar multiplication

Subspace from a set of vectors

Theorem

Let $S = \{\, \boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \cdots,\ \boldsymbol{v}_r \,\} \subseteq V$*, let* W *be the set of all possible linear combinations of vectors in* S*. Then*

- W *is a subspace of* V
- W *is the "smallest" subspace of* V *that contain* S *– any other subsapce of* V *containing* S *contains* W

Proof.

Let W' be any subspace of V containing S. Since W' is closed under vector addition and scalar multiplication, it contains all linear combinations of vectors in S and hence contains W. □

Solution spaces of homogeneous systems

Theorem

The solution set of a homogeneous linear system $A\boldsymbol{x} = \mathbf{0}$ *of* m *equations in* n *unknowns is a subspace of* $\mathbb{R}^n$*.*

Proof.

- Let W be the solution set
- $\mathbf{0} \in W$, W is not empty
- For any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in W$

$$A(\boldsymbol{x}_1 + \boldsymbol{x}_2) = A\boldsymbol{x}_1 + A\boldsymbol{x}_2 = \mathbf{0} \Longrightarrow \boldsymbol{x}_1 + \boldsymbol{x}_2 \in W$$

$$A(\alpha\boldsymbol{x}_1) = \alpha_1(A\boldsymbol{x}_1) = \mathbf{0} \Longrightarrow \alpha\boldsymbol{x}_1 \in W$$

□

Definition

W is called the *solution space* of the system.

Test for spanning – $\mathbb{R}^n$

From the previous examples, we have the following result for a special case

Theorem

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ *spans* $\mathbb{R}^n$ *iff the determinant* $\begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{vmatrix} \neq 0.$

Proof.

S spans $\mathbb{R}^n$ iff the following equation has solutions for $\alpha_1, \alpha_2, \dots, \alpha_n$

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n = \boldsymbol{u}.$$

for any vector $\boldsymbol{u} \in \mathbb{R}^n$. This equation corresponds to a linear system in the unknowns α_i's with coefficient matrix A, whose columns are $\boldsymbol{v}_j$'s. Consequently, the equation has solutions for all $\boldsymbol{u}$ iff $\det(A) \neq 0$. The result follows from that $\det(A) = \det(A^\top)$.

Linearly independent vectors

Theorem

$S = \{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \dots, \boldsymbol{v}_r\} \subseteq V$, S is linearly independent iff no vector in S can be expressed as a linear combination of the others.

Proof.

We will prove S is linearly dependent iff at least one vector in S can be expressed as a linear combination of the others.
$\Longrightarrow$ Consider the equation

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_r \boldsymbol{v}_r = \boldsymbol{0}$$

If S is linearly dependent, then WLOG, suppose $\alpha_1 \neq 0$, and

$$\boldsymbol{v}_1 = \left(-\frac{\alpha_2}{\alpha_1}\right) \boldsymbol{v}_2 + \cdots + \left(-\frac{\alpha_r}{\alpha_1}\right)$$

Linearly independent vectors

Theorem

$S = \{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \ldots, \boldsymbol{v}_r\} \subseteq V$, $r \geq 2$, S is linearly independent iff no vector in S can be expressed as a linear combination of the others.

Proof.

We will prove S is linearly dependent iff at least one vector in S can be expressed as a linear combination of the others.

$\Longleftarrow$ Suppose $\boldsymbol{v}_1 = \beta_2 \boldsymbol{v}_2 + \cdots + \beta_2 \boldsymbol{v}_r$, then

$$-\boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \cdots + \beta_2 \boldsymbol{v}_r = \mathbf{0}.$$

□

Example

- $S = \{\begin{pmatrix}2, & 3\end{pmatrix}, \begin{pmatrix}1, & 0\end{pmatrix}, \begin{pmatrix}0, & 1\end{pmatrix}\}$, linearly dependent
- $S = \{\begin{pmatrix}1, & 0\end{pmatrix}, \begin{pmatrix}0, & 1\end{pmatrix}\}$, linearly independent

Special cases

Theorem

- *A finite set that contains $\mathbf{0}$ is linearly dependent*
- *A set with exactly one vector is linearly independent iff that vector is not $\mathbf{0}$*
- *A set with exactly two vectors is linearly independent iff neither vector is a scalar multiple of the other*

Proof.

We prove the first part. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r, \mathbf{0}\}$

$$0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_r + 1\mathbf{0} = \mathbf{0}$$

$\square$

Basis for $\mathbb{R}^n$

Theorem

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ *is a basis for* $\mathbb{R}^n$ *iff the determinant* $\begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{vmatrix} \neq 0.$

Proof.

We have proved that

Theorem

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \subseteq \mathbb{R}^n$ *is linearly independent iff the determinant* $\neq 0$

Theorem

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ *spans* $\mathbb{R}^n$ *iff the determinant* $\neq 0$

Uniqueness of basis representation

Theorem

If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a basis for a vector space V, then every vector $\boldsymbol{v} \in V$ has a unique representation as a linear combination of vectors in S:

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_v \boldsymbol{v}_n.$$

Proof.

Suppose

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n = \beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \cdots + \beta_n \boldsymbol{v}_n,$$

then

$$(\alpha_1 - \beta_1)\boldsymbol{v}_1 + (\alpha_2 - \beta_2)\boldsymbol{v}_2 + \cdots + (\alpha_n - \beta_n)\boldsymbol{v}_n = \mathbf{0}$$

S is linearly independent $\Longrightarrow \alpha_1 - \beta_1 = 0,\ \alpha_2 - \beta_2 = 0,\ \ldots,\ \alpha_n - \beta_n = 0$ $\square$

Lemma

Let S_1, S_2 be subsets of V. If $V = \operatorname{span}(S_1)$ and vectors in S_2 are linearly independent, then $|S_1| \geq |S_2|$.

Proof.

Suppose $S_1 = \{\, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{r_1} \,\}$ and $S_2 = \{\, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{r_2} \,\}$. Since $V = \operatorname{span}(S_1)$,

$$\boldsymbol{w}_1 = \sum_{j=1}^{r_1} \alpha_j \boldsymbol{v}_j$$

At least one of $a_j \neq 0$ as vectors in S_2 are linearly independent – $\mathbf{0} \notin S_2$. WLOG, assume $a_1 \neq 0$, then

$$\boldsymbol{v}_1 = -\sum_{j=2}^{r_1} \frac{a_j}{a_1} \boldsymbol{v}_{\boldsymbol{j}} + \frac{1}{a_1} \boldsymbol{w}_1 \implies V = \operatorname{span}(\{\, \boldsymbol{w}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{r_1} \,\})$$

Lemma

Let S_1, S_2 be subsets of V. If $V = \mathrm{span}(S_1)$ and vectors in S_2 are linearly independent, then $|S_1| \geq |S_2|$.

Proof.

$V = \mathrm{span}(\{\, \boldsymbol{w}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{r_1} \,\})$, then, we can write

$$\boldsymbol{w}_2 = \beta_1 \boldsymbol{w}_1 + \sum_{j=2}^{r_1} \beta_j \boldsymbol{v}_j,$$

at least one of $\beta_j \neq 0$ for $2 \leq j \leq r_1$, otherwise $\boldsymbol{w}_2$ is a linear combination of $\boldsymbol{w}_1$. Suppose $\beta_2 \neq 0$, We have

$$\boldsymbol{v}_2 = -\frac{\beta_1}{\beta_2}\boldsymbol{w}_1 - \sum_{j=3}^{r_1} \frac{\beta_j}{\beta_2}\boldsymbol{v}_j + \frac{1}{\beta_2}\boldsymbol{w}_2 \Longrightarrow V = \mathrm{span}(\{\, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{r_1} \,\})$$

Bases

Lemma

Let S_1, S_2 *be subsets of* V. *If* $V = \mathrm{span}(S_1)$ *and vectors in* S_2 *are linearly independent, then* $|S_1| \geq |S_2|$.

Proof.

We can continue in this manner, if $r_1 < r_2$, we will deduce that $\{ \boldsymbol{w}_1, \boldsymbol{w}_2 \dots, \boldsymbol{w}_{r_1} \}$ spans V and $\boldsymbol{w}_{r_1+1}$ can be written as a linear combination of $\{ \boldsymbol{w}_1, \boldsymbol{w}_2 \dots, \boldsymbol{w}_{r_1} \}$, a contradiction. □

Theorem

If B_1, B_2 are bases of a vector space V, then $|B_1| = |B_2|$

Proof.

By the previous lemma,

$$|B_1| \geq |B_2|, |B_2| \geq |B_1| \implies |B_1| = |B_2|.$$

$\square$

Standard matrix

Theorem

$T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^n \to \mathbb{R}^m$. *If* $T_A(\boldsymbol{x}) = T_B(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathbb{R}^n$, *then* $A = B$.

Proof.

Consider standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$

$$A\boldsymbol{e}_j = B\boldsymbol{e}_j, \quad j = 1, 2, \ldots, n$$

$A\boldsymbol{e}_j$ (resp. $B\boldsymbol{e}_j$) is the jth column of A (resp. B) □

Note

- Every $A \in \mathcal{M}_{m\times n}$ produces exactly one matrix transformation (multiplication by A)
- Every matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ arises from exactly one $A \in \mathcal{M}_{m\times n}$ – *standard matrix* for the transformation