

Tutorial 7

Fundamental spaces and decompositions

Homework

1. Provide detailed solutions to *one* of the following questions: Q1, Q2, Q3, Q4, Q5, Q8.
2. Provide detailed solutions to *one* of the following questions: Q10, Q11.
3. Provide detailed solutions to *one* of the following questions: Q19, Q20.
4. Provide detailed solutions to *three* selected parts from Q23 (parts 2–14).
5. Provide detailed solutions to *two* selected parts from Q40 (parts 1–4).
6. Provide detailed solutions to *two* selected parts from any of the following: Q42 (parts 1–8), Q43 (parts 1–8), Q44 (parts 1–9).

Question 1. Determine whether \mathbf{b} is in the column space of A , and if so, express \mathbf{b} as a linear combination of the column vectors of A

$$\begin{array}{ll} 1. A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}, & \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \\ 2. A = \begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, & \mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \\ 3. A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, & \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \\ 4. A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}, & \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 5 \\ 7 \end{pmatrix} \end{array}$$

Question 2. Suppose that $x_1 = 3, x_2 = 0, x_3 = -1, x_4 = 5$ is a solution of a nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ and that the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is given by the formulas

$$x_1 = 5r - 2s, \quad x_2 = s, \quad x_3 = s + t, \quad x_4 = t$$

1. Find a vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.
2. Find a vector form of the general solution of $A\mathbf{x} = \mathbf{b}$.

Question 3. Suppose that $x_1 = -1, x_2 = 2, x_3 = 4, x_4 = -3$ is a solution of a nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ and that the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

1. Find a vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.
2. Find a vector form of the general solution of $A\mathbf{x} = \mathbf{b}$.

Question 4. Find the vector form of the general solution of the linear system $A\mathbf{x} = \mathbf{b}$, and then use that result to find the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

1.

$$\begin{aligned}x_1 - 3x_3 &= 1 \\2x_1 - 6x_2 &= 2\end{aligned}$$

2.

$$\begin{aligned}x_1 + x_2 + 2x_3 &= 5 \\x_1 + x_3 &= -2 \\2x_1 + x_2 + 3x_3 &= 3\end{aligned}$$

1. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 0 \end{array} \right)$$

The general solution of the system is

$$x_1 = 3t + 1, \quad x_2 = t, \quad x_3 = t.$$

A vector form of the general solution of $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

A vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

2. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The general solution of the system is

$$x_1 = -t - 2, \quad x_2 = 7 - t, \quad x_3 = t.$$

A vector form of the general solution of $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

A vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Question 5. Find bases for the null space and row space of A .

$$1. A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{pmatrix}$$

Question 6. By inspection, find a basis for the row space and for the column space of the given matrix

$$1. A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Question 7. Construct a matrix whose null space consists of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 4 \end{pmatrix}$$

Question 8. Find the rank and nullity of the matrix A by reducing it to row echelon form.

$$1. A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix}$$

1. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 1$, $\text{nullity}(A) = 3$.

2. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 2$, $\text{nullity}(A) = 3$.

3. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 3$, $\text{nullity}(A) = 2$.

4. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.

Question 9. The matrix R is the reduced row echelon form of the matrix A .

(a) By inspection of the matrix R , find the rank and nullity of A .

(b) Find the number of leading variables and the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ without solving the system.

1. $A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & -6 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

3. $A = \begin{pmatrix} 2 & -1 & -3 \\ -2 & 1 & 3 \\ -4 & 2 & 6 \end{pmatrix}$ $R = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Question 10. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by the formula

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{pmatrix}.$$

1. Find the rank of the standard matrix for T .
2. Find the nullity of the standard matrix for T .

Question 11. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be the linear transformation defined by the formula

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{pmatrix}$$

1. Find the rank of the standard matrix for T .
2. Find the nullity of the standard matrix for T .

Question 12. Discuss how the rank of A varies with t .

$$\text{1. } A = \begin{pmatrix} 1 & 1 & t \\ 1 & 1 & 1 \\ t & 1 & 1 \end{pmatrix} \quad \text{2. } A = \begin{pmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{pmatrix}$$

Question 13. Are there values of r and s for which

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{pmatrix}$$

has rank 1? Has rank 2? If so, find those values.

Question 14.

1. If A is a 3×5 matrix, then the rank of A is at most ____.
2. If A is a 3×5 matrix, then the nullity of A is at most ____.

3. If A is a 3×5 matrix, then the rank of A^\top is at most ____.
4. If A is a 5×3 matrix, then the nullity of A^\top is at most ____.

Question 15.

1. If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ____.
2. If A is a 3×5 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ____.
3. If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ____.
4. If A is a 5×3 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ____.

Question 16. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Show that A has rank 2 if and only if one or more of the following determinants is nonzero:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Question 17. Determine whether the matrix operator $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by the equations is bijective.

1.

$$\begin{aligned} w_1 &= x_1 + 2x_2 \\ w_2 &= -x_1 + x_2 \end{aligned}$$

2.

$$\begin{aligned} w_1 &= 4x_1 - 6x_2 \\ w_2 &= -2x_1 + 3x_2 \end{aligned}$$

3.

$$\begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$$

4.

$$\begin{aligned} w_1 &= x_1 - 3x_2 + 4x_3 \\ w_2 &= -x_1 + x_2 + x_3 \\ w_3 &= -2x_2 + 5x_3 \end{aligned}$$

Question 18. Determine whether multiplication by A is an injective matrix transformation.

$$1. A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -4 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$$

Question 19. Let T_A be multiplication by the matrix A . Find:

- (a) a basis for the range of T_A .
- (b) a basis for the kernel of T_A .
- (d) the rank and nullity of A .

$$1. A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 20 & 0 & 0 \end{pmatrix}$$

Question 20. Let $T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be multiplication by A . Find:

- (a) a basis for the kernel of T_A .
- (b) a basis for the range of T_A that consists of column vectors of A .

$$1. A = \begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -2 & 4 & 2 & 2 \\ -1 & 8 & 3 & 5 \end{pmatrix}$$

Question 21. Let A be an $n \times n$ matrix such that $\det(A) = 0$, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be multiplication by A .

1. What can you say about the range of the matrix operator T ? Give an example that illustrates your conclusion.
2. What can you say about the number of vectors that T maps into $\mathbf{0}$?

What if $\det(A) \neq 0$?

Question 22. Confirm by multiplication that \mathbf{x} is an eigenvector of A , and find the corresponding eigenvalue.

$$1. A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Question 23. Find the characteristic equation, the eigenvalues, and the bases for the eigenspaces of the matrix A .

$$1. A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

$$2. A = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$5. A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$6. A = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$$

$$7. A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$8. A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$$

$$9. A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$10. A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

$$11. A = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$$

$$12. A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$13. A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$14. A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

Question 24. Find the characteristic equation of the matrix by inspection.

$$1. A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

Question 25. Find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on \mathbb{R}^2 . Use geometric reasoning to help finding the answers.

1. Reflection about the line $y = x$.
2. Orthogonal projection onto the x-axis.
3. Rotation about the origin through a positive angle of 90° .
4. Contraction with factor α ($0 < \alpha < 1$).
5. Shear in the x -direction by a factor α ($\alpha \neq 0$).
6. Reflection about the y -axis.
7. Rotation about the origin through a positive angle of 180° .
8. Dilation with factor α ($\alpha > 1$).
9. Expansion in the y -direction with factor α ($\alpha > 1$).
10. Shear in the y -direction by a factor α ($\alpha \neq 0$).

Question 26. Find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on \mathbb{R}^3 . Use geometric reasoning to find the answers.

1. Reflection about the xy -plane.
2. Orthogonal projection onto the xz -plane.
3. Counterclockwise rotation about the positive x -axis through an angle of 90° .
4. Contraction with factor α ($0 \leq \alpha < 1$).
5. Reflection about the xz -plane.
6. Orthogonal projection onto the yz -plane.
7. Counterclockwise rotation about the positive y -axis through an angle of 180° .
8. Dilation with factor α ($\alpha > 1$).

Question 27. Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.

1. $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$
2. $p(\lambda) = \lambda^4 - \lambda^3 + 7$

Question 28. Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

1. What is the size of A ?

2. Is A invertible?
3. How many eigenspaces does A have?

Question 29. The eigenvectors that we have been studying are sometimes called *right eigenvectors* to distinguish them from *left eigenvectors*, which are $n \times 1$ column vectors \mathbf{x} that satisfy the equation

$$\mathbf{x}^\top A = \mu \mathbf{x}^\top$$

for some scalar μ . For a given matrix A , how are the right eigenvectors and their corresponding eigenvalues related to the left eigenvectors and their corresponding eigenvalues?

Question 30. Prove that the characteristic equation of a 2×2 matrix A can be expressed as

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0,$$

where $\text{tr}(A)$ is the trace of A .

Question 31. Use the result from the Question 30 to show that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the solutions of the characteristic equation of A are

$$\lambda = \frac{1}{2} \left((a+d) \pm \sqrt{(a-d)^2 + 4bc} \right).$$

Use this result to show that A has

- (a) two distinct real eigenvalues if $(a-d)^2 + 4bc > 0$.
- (b) two repeated real eigenvalues if $(a-d)^2 + 4bc = 0$.
- (c) complex conjugate eigenvalues if $(a-d)^2 + 4bc < 0$.

Question 32. Let A be the matrix in from the Question 31. Show that if $b \neq 0$, then

$$\mathbf{x}_1 = \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -b \\ a - \lambda_2 \end{pmatrix}$$

are eigenvectors of A that correspond, respectively, to the eigenvalues

$$\lambda_1 = \frac{1}{2} \left[(a+d) + \sqrt{(a-d)^2 + 4bc} \right]$$

and

$$\lambda_2 = \frac{1}{2} \left[(a+d) - \sqrt{(a-d)^2 + 4bc} \right].$$

Question 33. Use the result of Question 30 to prove that if

$$p(\lambda) = \lambda^2 + c_1\lambda + c_2$$

is the characteristic polynomial of a 2×2 matrix, then

$$p(A) = A^2 + c_1A + c_2I = O.$$

(Stated informally, A satisfies its characteristic equation. This result is true as well for $n \times n$ matrices.)

Question 34. Prove: If a, b, c, d are integers such that $a + b = c + d$, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has integer eigenvalues.

Question 35. Prove: If λ is an eigenvalue of an invertible matrix A and \mathbf{x} is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and \mathbf{x} is a corresponding eigenvector.

Question 36. Prove: If λ is an eigenvalue of A , \mathbf{x} is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$ and \mathbf{x} is a corresponding eigenvector.

Question 37. Prove: If λ is an eigenvalue of A and \mathbf{x} is a corresponding eigenvector, then $s\lambda$ is an eigenvalue of sA for every scalar s and \mathbf{x} is a corresponding eigenvector.

Question 38. Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$$

and then use Questions 35 and 36 to find the eigenvalues and bases for the eigenspaces of

1. A^{-1}
2. $A - 3I$
3. $A + 2I$

Question 39. Show that A and B are not similar matrices.

$$1. A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Question 40.**A Procedure for Diagonalizing an $n \times n$ Matrix**

Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n , then it is not.

Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$$

whose column vectors are the n basis vectors you obtained in Step 1.

Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ that correspond to the successive columns of P .

Following the above procedure to find a matrix P that diagonalizes A , and check your work by computing $P^{-1}AP$.

$$1. A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$$

$$2. A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Question 41. Let

$$1. A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

- (a) Find the eigenvalues of A .

(b) For each eigenvalue λ , find the rank of the matrix $\lambda I - A$.

(c) Is A diagonalizable? Justify your conclusion.

For Questions 42 – 44 Find an LU-decomposition of the coefficient matrix - try both Doolittle decomposition and Crout decomposition. Then solve the system using the decomposition.

Question 42.

$$1. \begin{pmatrix} 2 & 8 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

$$2. \begin{pmatrix} -5 & -10 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 \\ 19 \end{pmatrix}$$

$$3. \begin{pmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$$

$$4. \begin{pmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -33 \\ 7 \\ -1 \end{pmatrix}$$

$$5. \begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$6. \begin{pmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -22 \\ 3 \end{pmatrix}$$

$$7. \begin{pmatrix} -1 & -3 & 2 \\ -6 & -19 & 10 \\ 3 & 9 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9 \\ -59 \\ 28 \end{pmatrix}$$

$$8. \begin{pmatrix} -4 & 0 & 1 \\ 8 & 2 & -1 \\ -8 & -6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 15 \\ -46 \end{pmatrix}$$

Question 43.

$$1. \begin{pmatrix} -2 & -1 & -1 & -2 \\ 2 & 4 & 3 & 1 \\ 4 & 2 & -2 & -1 \\ -1 & 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -10 \\ 8 \end{pmatrix}$$

$$2. \begin{pmatrix} 2 & -2 & 0 & 3 \\ -2 & 3 & 2 & -3 \\ 4 & -5 & -4 & 10 \\ -4 & 7 & 10 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 10 \\ 2 \end{pmatrix}$$

$$3. \begin{pmatrix} 4 & 3 & 4 & -2 \\ 0 & -1 & 3 & 3 \\ 1 & 2 & -3 & 1 \\ 0 & -4 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -20 \\ 1 \\ 5 \\ 14 \end{pmatrix}$$

$$4. \begin{pmatrix} -5 & 2 & -2 & 2 \\ -10 & 0 & -8 & 7 \\ 20 & -4 & 10 & -7 \\ -20 & 12 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 25 \\ -49 \\ 52 \end{pmatrix}$$

$$5. \begin{pmatrix} -1 & 1 & 0 & -3 \\ 0 & 4 & -4 & -4 \\ 0 & -2 & 4 & 0 \\ 4 & -3 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -8 \\ -13 \end{pmatrix}$$

$$6. \begin{pmatrix} 1 & -2 & -3 & -3 \\ 3 & -10 & -12 & -5 \\ 5 & -22 & -26 & 0 \\ 2 & 12 & 0 & -18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -89 \\ -199 \\ 66 \end{pmatrix}$$

$$7. \begin{pmatrix} 1 & 0 & -1 & -4 \\ -2 & 1 & -2 & -2 \\ 4 & -3 & 4 & -2 \\ -2 & -4 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \\ 2 \\ 24 \end{pmatrix}$$

$$8. \begin{pmatrix} 1 & 1 & -5 & -1 \\ 2 & -1 & -5 & 3 \\ -4 & -16 & 38 & 23 \\ -5 & -20 & 52 & 35 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -25 \\ 122 \\ 158 \end{pmatrix}$$

Question 44.

1.

$$\begin{aligned} -x_1 + 5x_2 - 7x_3 &= 14 \\ -3x_1 + 8x_2 - 4x_3 &= 28 \\ x_1 - 3x_2 + 2x_3 &= -10 \end{aligned}$$

2.

$$\begin{aligned} -12x_1 - 24x_2 - 16x_3 &= 8 \\ -3x_1 - 5x_2 - 2x_3 &= 4 \\ 6x_1 + 9x_2 + x_3 &= -11 \end{aligned}$$

3.

$$\begin{aligned} -4x_1 + 14x_2 - 7x_3 &= 107 \\ -x_1 + 2x_2 - x_3 &= 17 \\ 3x_1 - 9x_2 + 7x_3 &= -83 \end{aligned}$$

4.

$$\begin{aligned} 20x_1 - 11x_2 - 29x_3 &= -92 \\ -4x_1 + 2x_2 + 5x_3 &= 15 \\ -8x_1 + 4x_2 + 14x_3 &= 50 \end{aligned}$$

5.

$$\begin{aligned} 4x_1 - 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 2 \\ x_2 - 5x_3 &= 2 \end{aligned}$$

6.

$$\begin{aligned} -9x_1 + 12x_2 - 9x_3 &= 45 \\ -9x_1 + 3x_2 - 6x_3 &= 24 \\ -3x_1 + 3x_2 - 3x_3 &= 12 \end{aligned}$$

7.

$$\begin{aligned} 3x_1 + 27x_2 - 29x_3 &= 28 \\ -x_1 - 4x_2 + 2x_3 + 5x_4 &= -5 \\ -3x_2 - 3x_3 + 13x_4 &= -1 \\ -5x_1 - 23x_2 + 15x_3 + 21x_4 &= -28 \end{aligned}$$

8.

$$\begin{aligned} 4x_1 - 12x_2 - 29x_3 - 12x_4 &= -149 \\ 3x_2 - 4x_3 + x_4 &= -11 \\ x_1 - 3x_2 - x_3 + x_4 &= -18 \\ 3x_1 + 6x_2 - 28x_3 + 4x_4 &= -122 \end{aligned}$$

9.

$$\begin{aligned} -16x_1 + 15x_2 - 21x_3 - 26x_4 &= -86 \\ 20x_2 - 20x_3 - 10x_4 &= -100 \\ 4x_1 + 2x_3 + 4x_4 &= 2 \\ 16x_1 + 5x_2 + 3x_3 + 14x_4 &= -16 \end{aligned}$$

Question 45. Determine A^{-1} using LU decomposition and verify the correctness of the following identity:

$$A = LU \implies A^{-1} = (LU)^{-1} = U^{-1}L^{-1}.$$

Try both Doolittle decomposition and Crout decomposition.

1.
$$\begin{pmatrix} 5 & -2 & -4 \\ 15 & -10 & -7 \\ -10 & -8 & 25 \end{pmatrix}$$

2.
$$\begin{pmatrix} -1 & 2 & 5 \\ -2 & 0 & 12 \\ 1 & -6 & -1 \end{pmatrix}$$

3.
$$\begin{pmatrix} -1 & 5 & -2 \\ -2 & 8 & -8 \\ -1 & 1 & -12 \end{pmatrix}$$

4.
$$\begin{pmatrix} -2 & 1 & 3 \\ 2 & 2 & 2 \\ 0 & -4 & -3 \end{pmatrix}$$

5.
$$\begin{pmatrix} 4 & -4 & 5 \\ 12 & -16 & 14 \\ -16 & 36 & -14 \end{pmatrix}$$

6.
$$\begin{pmatrix} 2 & -5 & 1 \\ -2 & 4 & -2 \\ -2 & 7 & 2 \end{pmatrix}$$

7.
$$\begin{pmatrix} -1 & -4 & 5 \\ 0 & 2 & -4 \\ -5 & -22 & -26 \end{pmatrix}$$

8.
$$\begin{pmatrix} -3 & -5 & -3 \\ -12 & -19 & -10 \\ 0 & 0 & 1 \end{pmatrix}$$

9.
$$\begin{pmatrix} 3 & 3 & 0 & -5 \\ 15 & 16 & 0 & -23 \\ -3 & 2 & 4 & 10 \\ 0 & 2 & -8 & 12 \end{pmatrix}$$

10.
$$\begin{pmatrix} 1 & -5 & 4 & -2 \\ 4 & -17 & 12 & -8 \\ -2 & -2 & 11 & 9 \\ 5 & -37 & 36 & -8 \end{pmatrix}$$

11.
$$\begin{pmatrix} 1 & 1 & -4 & -2 \\ 3 & 2 & -10 & -2 \\ 2 & -1 & -1 & 13 \\ -1 & -4 & 8 & 5 \end{pmatrix}$$

12.
$$\begin{pmatrix} 3 & 2 & -3 & 5 \\ -6 & -9 & 6 & -11 \\ -6 & 6 & 9 & -10 \\ -15 & 10 & 12 & -16 \end{pmatrix}$$

Question 46. Determine $\det(A)$ using LU decomposition and verify the correctness of the following identity:

$$A = LU \implies \det(A) = \det(L) \det(U).$$

Try both Doolittle decomposition and Crout decomposition.

1.
$$A = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -4 \\ -4 & -5 & -4 \end{pmatrix}$$

2.
$$A = \begin{pmatrix} -2 & 4 & 3 \\ -4 & 13 & 6 \\ 8 & -36 & -11 \end{pmatrix}$$

3.
$$A = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 17 & 0 \\ 0 & -20 & 13 \end{pmatrix}$$

4.
$$A = \begin{pmatrix} -3 & -1 & 4 \\ -15 & -4 & 23 \\ 3 & 2 & -6 \end{pmatrix}$$

5.
$$A = \begin{pmatrix} -2 & -2 & 4 \\ 2 & 3 & -9 \\ -2 & -7 & 25 \end{pmatrix}$$

6.
$$A = \begin{pmatrix} -2 & -4 & 0 \\ 0 & -3 & -3 \\ 0 & 9 & 6 \end{pmatrix}$$

7.
$$A = \begin{pmatrix} -2 & 4 & 2 & 0 \\ 6 & -8 & -2 & -1 \\ 8 & -24 & -12 & -3 \\ -8 & 0 & -16 & 13 \end{pmatrix}$$

8.
$$A = \begin{pmatrix} 2 & -2 & 4 & -1 \\ -6 & 11 & -9 & 0 \\ 8 & -33 & 2 & 8 \\ 4 & -4 & 11 & -8 \end{pmatrix}$$

9.
$$A = \begin{pmatrix} 4 & 0 & -4 & 1 \\ 8 & -3 & -3 & 6 \\ -4 & -12 & 23 & 11 \\ -16 & 12 & -2 & -7 \end{pmatrix}$$

10.
$$A = \begin{pmatrix} 3 & -4 & 3 & 5 \\ -15 & 24 & -10 & -30 \\ -12 & 20 & -4 & -20 \\ -6 & 28 & 25 & -23 \end{pmatrix}$$