Tutorial 6

Bases and matrix operators

Homework

1. Provide detailed solutions to Q2, part 2.
2. Provide a detailed solution to Q6.
3. Provide a detailed solution to Q7, part 2.
4. Provide a detailed solution to Q9, part 2.
5. Provide a detailed solution to *one* selected part of Q10.
6. Provide a detailed solution to Q19, part (b).
7. Provide a detailed solution to Q20, choosing either part (a) or part (b).
8. Provide a detailed solution to Q23.
9. Provide a detailed solution to Q28.

Question 1. Show that the following set of vectors forms a basis for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

1. $\left\{ \begin{pmatrix}2, & 1\end{pmatrix}, \begin{pmatrix}3, & 0\end{pmatrix} \right\}$
2. $\left\{ \begin{pmatrix}3, & 1, & -4\end{pmatrix}, \begin{pmatrix}2, & 5, & 6\end{pmatrix} \right\}, \begin{pmatrix}1, & 4, & 8\end{pmatrix}$

Solution.

- The determinant
$$\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = -3 \neq 0.$$
thus, the set of vectors forms a basis for $\mathbb{R}^2$.
- The determinant
$$\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = 26 \neq 0$$
thus, the set of vectors forms a basis for $\mathbb{R}^3$.

Question 2. Show that the following matrices form a basis for $\mathcal{M}_{2\times 2}$.

1. $\begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Solution.

1. We must show that the matrices are linearly independent and span $\mathcal{M}_{2\times 2}$. To prove linear independence we must show that the equation

$$\alpha_1 \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has only the trivial solution. Expanding both sides and equating corresponding entries, we obtain the following system of linear equations:

$$\begin{aligned} 3\alpha_1 + \alpha_4 &= 0 \\ 6\alpha_1 - \alpha_2 - 8\alpha_3 &= 0 \\ 3\alpha_1 - \alpha_2 - 12\alpha_3 - \alpha_4 &= 0 \\ -6\alpha_1 - 4\alpha_3 + 2\alpha_4 &= 0 \end{aligned}$$

The coefficient matrix of the linear system is

$$A = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{pmatrix}.$$

Computing the determinant, we get $\det(A) = 48 \neq 0$. Since the determinant is nonzero, the system has only the trivial solution, which confirms that the matrices are linearly independent.

To prove that the matrices span $\mathcal{M}_{2\times 2}$ we must show that every 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be expressed as

$$\beta_1 \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \beta_3 \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix} + \beta_4 \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This corresponds to a system of linear equations in the unknowns $\beta_1, \beta_2, \beta_3, \beta_4$ with the same coefficient matrix A as before. Since we have computed that $\det(A) \neq 0$, the system always has a unique solution for any given values a, b, c, d. Consequently, the given matrices span $\mathcal{M}_{2\times 2}$. Since the given set of matrices is both linearly independent and spans $\mathcal{M}_{2\times 2}$, it forms a basis for $\mathcal{M}_{2\times 2}$.

2. Using a similar approach as in Part 1, we consider the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Computing its determinant, we obtain $\det(A) = 1 \neq 0$.

Question 3. In each part, show that the set of vectors is not a basis for $\mathbb{R}^3$

1. $\{\, (2,\ -3,\ 1),\ (4,\ 1,\ 1),\ (0,\ -7,\ 1) \,\}$
2. $\{\, (1,\ 6,\ 4),\ (2,\ 4,\ -1),\ (-1,\ 2,\ 5) \,\}$

Solution.

1. The determinant
$$\begin{vmatrix} 2 & -3 & 1 \\ 4 & 1 & 1 \\ 0 & -7 & 1 \end{vmatrix} = 0.$$
Since the determinant is zero, the matrix is singular, implying that its column vectors are linearly dependent. Consequently, the given set of vectors does not form a basis for $\mathbb{R}^3$.

2. The determinant
$$\begin{vmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{vmatrix} = 0,$$
thus, the set of vectors is not a basis for $\mathbb{R}^3$.

Question 4. Show that the following matrices do not form a basis for $\mathcal{M}_{2\times 2}$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Solution.

$$-\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

demonstrates that the zero matrix can be expressed as a nontrivial linear combination of the given matrices. This implies that the matrices are linearly dependent and, therefore, do not form a basis for $\mathcal{M}_{2\times 2}$.

Question 5. Find the coordinate vector of $\boldsymbol{w}$ relative to the basis $S = \{\, \boldsymbol{u}_1,\ \boldsymbol{u}_2 \,\}$ for $\mathbb{R}^2$.

1. $\boldsymbol{u}_1 = \begin{pmatrix} 2, & -4 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 3, & 8 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 1, & 1 \end{pmatrix}$
2. $\boldsymbol{u}_1 = \begin{pmatrix} 1, & 1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 0, & 2 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} a, & b \end{pmatrix}$
3. $\boldsymbol{u}_1 = \begin{pmatrix} 1, & -1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1, & 1 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 1, & 0 \end{pmatrix}$
4. $\boldsymbol{u}_1 = \begin{pmatrix} 1, & -1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1, & 1 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 0, & 1 \end{pmatrix}$

Solution.

1. To find $[\boldsymbol{w}]_S$, we must find values $\alpha_1, \alpha_2, \alpha_3$ s.t.
$$\alpha_1 \boldsymbol{u}_1 + \alpha \boldsymbol{u}_2 = \boldsymbol{w} \Longrightarrow \alpha_1 \begin{pmatrix} 2, & -4 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3, & 8 \end{pmatrix} = \begin{pmatrix} 1, & 1 \end{pmatrix}$$
Equating corresponding components gives
$$\begin{aligned} 2\alpha_1 + 3\alpha_3 &= 1 \\ -4\alpha_1 + 8\alpha_2 &= 1 \end{aligned}$$
The augmented matrix of the system is
$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array}\right)$$

which has reduced row echelon form

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{array}\right)$$

We have

$$\alpha_1 = \frac{5}{28}, \quad \alpha_2 = \frac{3}{14}.$$

Therefore

$$[\boldsymbol{w}]_S = \begin{pmatrix} \frac{5}{28} \\ \frac{3}{14} \end{pmatrix}$$

2. To find $[\boldsymbol{w}]_S$, we must find values $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\alpha_1 \boldsymbol{u}_1 + \alpha \boldsymbol{u}_2 = \boldsymbol{w} \Longrightarrow \alpha_1 \begin{pmatrix} 1, & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0, & 2 \end{pmatrix} = \begin{pmatrix} a, & b \end{pmatrix}$$

Equating corresponding components gives

$$\begin{aligned} \alpha_1 + \alpha_2 &= a \\ 2\alpha_2 &= b \end{aligned}$$

We have

$$\alpha_1 = \frac{2a - b}{2}, \quad \alpha_2 = \frac{b}{2}.$$

Therefore

$$[\boldsymbol{w}]_S = \begin{pmatrix} \frac{2a-b}{2} \\ \frac{b}{2} \end{pmatrix}$$

3.

$$[\boldsymbol{w}]_S = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

4.

$$[\boldsymbol{w}]_S = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Question 6. Find the coordinate vector of $\boldsymbol{u}$ relative to the basis $S = \{ \boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \boldsymbol{v}_3 \}$ for $\mathbb{R}^3$

1. $\boldsymbol{v}_1 = \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 2, & 2, & 0 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 3, & 3, & 3 \end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix} 2, & -1, & 3 \end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix} 1, & 2, & 3 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} -4, & 5, & 6 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 7, & -8, & 9 \end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix} 5, & -12, & 3 \end{pmatrix}$

Solution.

1. $\boldsymbol{u} = 3\boldsymbol{v}_1 - 2\boldsymbol{v}_2 + \boldsymbol{v}_3$, and
$$[\boldsymbol{u}]_S = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

2. $\boldsymbol{u} = -2\boldsymbol{v}_1 + \boldsymbol{v}_3$, and
$$[\boldsymbol{u}]_S = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Question 7. For each case, first show that the set $S = \{ A_1, A_2, A_3, A_4 \}$ is a basis for $\mathcal{M}_{2\times2}$, then express A as a linear combination of the matrices in S, and then find the coordinate vector of A relative to S.

1. $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

2. $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

Solution.

1. To show S is a basis for $\mathcal{M}_{2\times2}$, we must show that S linearly independent and span $\mathcal{M}_{2\times2}$. To prove linear independence we must show that the equation
$$\alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
has only the trivial solution. Expanding both sides and equating corresponding entries, we obtain the following system of linear equations:
$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \end{aligned}$$
The coefficient matrix of the linear system is
$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$
Computing the determinant, we get $\det(M) = 1 \neq 0$. Since the determinant is nonzero, the system has only the trivial solution, which confirms that the matrices are linearly independent.

To prove that the matrices span $\mathcal{M}_{2\times2}$ we must show that every 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be expressed as
$$\beta_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \beta_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \beta_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This corresponds to a system of linear equations with coefficient matrix M. Since $\det(M) \neq 0$, the system always has a unique solution for any given values a, b, c, d, proving that the given matrices span $\mathcal{M}_{2\times 2}$. Since S is both linearly independent and spans $\mathcal{M}_{2\times 2}$, it forms a basis for $\mathcal{M}_{2\times 2}$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, we get the following system of linear equations in unknowns $\beta_1,\ \beta_2,\ \beta_3,\ \beta_4$

$$\begin{aligned} \beta_1 &= 1 \\ \beta_1 + \beta_2 &= 0 \\ \beta_1 + \beta_2 + \beta_3 &= 1 \\ \beta_1 + \beta_2 + \beta_3 + \beta_4 &= 0 \end{aligned}$$

Solving the system gives

$$\beta_1 = 1, \quad \beta_2 = -1, \quad \beta_3 = 1, \quad \beta_4 = -1$$

Thus, A can be expressed as

$$A = A_1 - A_2 + A_3 - A_4$$

Therefore, the coordinate vector of A relative to S is

$$[A]_S = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

2. The proof follows a similar argument as in Part 1. In this case, the coefficient matrix is

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which has determinant $\det(M) = -1$. Furthermore,

$$A = 1A_1 + 2A_2 + 3A_3 + 4A_4$$

and

$$[A]_S = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Question 8. In each part, let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by A, and let $\{\, \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3 \,\}$ be the standard basis for $\mathbb{R}^3$. Determine whether the set $\{\, T_A(\boldsymbol{e}_1),\ T_A(\boldsymbol{e}_2),\ T_A(\boldsymbol{e}_3) \,\}$ is linearly independent in $\mathbb{R}^3$

1. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}$

Solution. Since $T_A(\boldsymbol{e}_i)$ is equal to the ith column of A, the set $\{\, T_A(\boldsymbol{e}_1),\ T_A(\boldsymbol{e}_2),\ T_A(\boldsymbol{e}_3)\,\}$ is linearly independent in $\mathbb{R}^3$ iff $\det(A) \neq 0$

1. $\det(A) = 10 \neq 0$, the set $\{\, T_A(\boldsymbol{e}_1),\ T_A(\boldsymbol{e}_2),\ T_A(\boldsymbol{e}_3)\,\}$ is linearly independent in $\mathbb{R}^3$
2. $\det(A) = 0$, the set $\{\, T_A(\boldsymbol{e}_1),\ T_A(\boldsymbol{e}_2),\ T_A(\boldsymbol{e}_3)\,\}$ is linearly dependent in $\mathbb{R}^3$

Question 9. In each part, let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by A, and let $\boldsymbol{u} = \begin{pmatrix}1, & -2, & -1\end{pmatrix}$. Find the coordinate vector of $T_A(\boldsymbol{u})$ relative to the basis $S = \{\, \begin{pmatrix}1, & 1, & 0\end{pmatrix},\ \begin{pmatrix}0, & 1, & 1\end{pmatrix},\ \begin{pmatrix}1, & 1, & 1\end{pmatrix} \,\}$ for $\mathbb{R}^3$.

1. $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}$
2. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution.

1.
$$T_A(\boldsymbol{u}) = A\boldsymbol{u}^\top = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$$

   By solving for $\alpha_1, \alpha_2, \alpha_3$ in
$$\alpha_1\begin{pmatrix}1, & 1, & 0\end{pmatrix} + \alpha_2\begin{pmatrix}0, & 1, & 1\end{pmatrix} + \alpha_3\begin{pmatrix}1, & 1, & 1\end{pmatrix} = \begin{pmatrix}4, & -2, & 0\end{pmatrix},$$
   we get
$$T_A(\boldsymbol{u}) = -2\begin{pmatrix}1, & 1, & 0\end{pmatrix} - 6\begin{pmatrix}0, & 1, & 1\end{pmatrix} + 6\begin{pmatrix}1, & 1, & 1\end{pmatrix}$$
   Hence the coordinate vector is given by
$$[T_A(\boldsymbol{u})]_S = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix}$$

2.
$$T_A(\boldsymbol{u}) = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}, \quad [T_A(\boldsymbol{u})]_S = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

Question 10. Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

1.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

2.

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

3.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 5x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

4.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

5.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

6.

$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

Solution.

1. The reduced row echelon form of the augmented matrix is $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. The solution is given by
$$\begin{pmatrix} x_1, & x_2, & x_3 \end{pmatrix} = \begin{pmatrix} t, & 0, & t \end{pmatrix} = t\begin{pmatrix} 1, & 0, & 1 \end{pmatrix}$$
Thus the solution space is equal to $\text{span}(\{\begin{pmatrix} 1, & 0, & 1 \end{pmatrix}\})$ and has dimension 1

2. The reduced row echelon form of the augmented matrix is
$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{7} & 0 & 0 \\ 0 & 1 & \frac{3}{7} & 1 & 0 \end{array}\right).$$
The solution is given by
$$\begin{pmatrix} x_1, & x_2, & x_3, & x_4 \end{pmatrix} = \begin{pmatrix} -\frac{2t}{7}, & -s - \frac{3t}{7}, & t, & s \end{pmatrix} = t\begin{pmatrix} -\frac{2}{7}, & -\frac{3}{7}, & 1, & 0 \end{pmatrix} + s\begin{pmatrix} 0, & -1, & 0, & 1 \end{pmatrix}$$
A basis for the solution space is
$$\left\{ \begin{pmatrix} -\frac{2}{7}, & -\frac{3}{7}, & 1, & 0 \end{pmatrix}, \begin{pmatrix} 0, & -1, & 0, & 1 \end{pmatrix} \right\}$$
and the solution space has dimension 2.

3. The reduced row echelon form of the augmented matrix is
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

The system has only the trivial solution. The solution space is the zero vector space and has dimension zero.

4. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution is given by

$$\begin{pmatrix} x_1, & x_2, & x_3 \end{pmatrix} = \begin{pmatrix} 3t - s, & t, & s \end{pmatrix} = t \begin{pmatrix} 3, & 1, & 0 \end{pmatrix} + s \begin{pmatrix} -1, & 0, & 1 \end{pmatrix}$$

A basis for the solution space is

$$\left\{ \begin{pmatrix} 3, & 1, & 0 \end{pmatrix}, \begin{pmatrix} -1, & 0, & 1 \end{pmatrix} \right\}$$

and the solution space has dimension 2.

5. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The solution is given by

$$\begin{aligned} \begin{pmatrix} x_1, & x_2, & x_3, & x_4 \end{pmatrix} = \begin{pmatrix} 4t - 3s + r, & t, & s, & r \end{pmatrix} = t \begin{pmatrix} 4, & 1, & 0, & 0 \end{pmatrix} + s \begin{pmatrix} -3, & 0, & 1, & 0 \end{pmatrix} \\ + r \begin{pmatrix} 1, & 0, & 0, & 1 \end{pmatrix} \end{aligned}$$

A basis for the solution space is

$$\left\{ \begin{pmatrix} 4, & 1, & 0, & 0 \end{pmatrix}, \begin{pmatrix} -3, & 0, & 1, & 0 \end{pmatrix} \right\}, \begin{pmatrix} 1, & 0, & 0, & 1 \end{pmatrix}$$

and the solution space has dimension 3.

6. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution is given by

$$\begin{pmatrix} x_1, & x_2, & x_3 \end{pmatrix} = \begin{pmatrix} 4t, & -5t, & t \end{pmatrix} = t \begin{pmatrix} 4, & -5, & 1 \end{pmatrix}$$

A basis for the solution space is

$$\left\{ \begin{pmatrix} 4, & -5, & 1 \end{pmatrix} \right\}$$

and the solution space has dimension 1.

Question 11. In each part, find a basis for the given subspace of $\mathbb{R}^3$, and sate its dimension

1. The plane $3x - 2y + 5z = 0$.
2. The plane $x - y = 0$.

3. The line $x = 2t$, $y = -t$, $z = 4t$.
4. All vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a + c$.

Solution.

1. The points in the plane are of the form

$$\begin{pmatrix} x, & y, & z \end{pmatrix} = \begin{pmatrix} \frac{2t}{3} - \frac{5s}{3}, & t, & s \end{pmatrix} = t\begin{pmatrix} \frac{2}{3}, & 1, & 0 \end{pmatrix} + s\begin{pmatrix} -\frac{5}{3}, & 0, & 1 \end{pmatrix}$$

   A basis for the plane is:

$$\left\{ \begin{pmatrix} \frac{2}{3}, & 1, & 0 \end{pmatrix}, \begin{pmatrix} -\frac{5}{3}, & 0, & 1 \end{pmatrix} \right\}$$

   Since the basis consists of two vectors, the dimension is 2.

2. The points in the plane are of the form

$$\begin{pmatrix} x, & y, & z \end{pmatrix} = \begin{pmatrix} t, & t, & s \end{pmatrix} = t\begin{pmatrix} 1, & 1, & 0 \end{pmatrix} + s\begin{pmatrix} 0, & 0, & 1 \end{pmatrix}$$

   A basis for the plane is:

$$\left\{ \begin{pmatrix} 1, & 1, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 0, & 1 \end{pmatrix} \right\}$$

   Since the basis consists of two vectors, the dimension is 2.

3. The points on the line if of the form

$$\begin{pmatrix} x, & y, & z \end{pmatrix} = \begin{pmatrix} 2t, & -t, & 4t \end{pmatrix} = t\begin{pmatrix} 2, & -1, & 4 \end{pmatrix}$$

   A basis for the line is:

$$\left\{ \begin{pmatrix} 2, & -1, & 4 \end{pmatrix} \right\}$$

   The dimension is 1.

4. A vector of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$ where $b = a + c$ can be rewritten as:

$$\begin{pmatrix} a, & b, & c \end{pmatrix} = \begin{pmatrix} a, & a+c, & c \end{pmatrix} = a\begin{pmatrix} 1, & 1, & 0 \end{pmatrix} + c\begin{pmatrix} 0, & 1, & 1 \end{pmatrix}.$$

   Thus, a basis is:

$$\left\{ \begin{pmatrix} 1, & 1, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 1, & 1 \end{pmatrix} \right\}$$

   The dimension is 2.

Question 12. In each part, find a basis for the given subspace of $\mathbb{R}^4$, and state its dimension.

1. All vectors of the form $\begin{pmatrix} a, & b, & c, & 0 \end{pmatrix}$.
2. All vectors of the form $\begin{pmatrix} a, & b, & c, & d \end{pmatrix}$, where $d = a + b$ and $c = a - b$.

c) All vectors of the form $\begin{pmatrix} a, & b, & c, & d \end{pmatrix}$, where $a = b = c = d$.

Solution.

1. We can express any vector in this subspace as

$$\begin{pmatrix} a, & b, & c, & 0 \end{pmatrix} = a\begin{pmatrix} 1, & 0, & 0, & 0 \end{pmatrix} + b\begin{pmatrix} 0, & 1, & 0, & 0 \end{pmatrix} + c\begin{pmatrix} 0, & 0, & 1, & 0 \end{pmatrix}$$

   Therefore, a basis for this subspace is:

$$\left\{ \begin{pmatrix} 1, & 0, & 0, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 1, & 0, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 0, & 1, & 0 \end{pmatrix} \right\}$$

   The dimension of the subspace is 3.

2. We can express any vector in this subspace as

$$\begin{pmatrix} a, & b, & a-b, & a+b \end{pmatrix} = a\begin{pmatrix} 1, & 0, & 1, & 1 \end{pmatrix} + b\begin{pmatrix} 0, & 1, & -1, & 1 \end{pmatrix}$$

   Therefore, a basis for this subspace is:

$$\left\{ \begin{pmatrix} 1, & 0, & 1, & 1 \end{pmatrix}, \begin{pmatrix} 0, & 1, & -1, & 1 \end{pmatrix} \right\}$$

   The dimension of the subspace is 2.

3. We can express any vector in this subspace as

$$\begin{pmatrix} a, & a, & a, & a \end{pmatrix} = a\begin{pmatrix} 1, & 1, & 1, & 1 \end{pmatrix}$$

   Therefore, a basis for this subspace is:

$$\left\{ \begin{pmatrix} 1, & 1, & 1, & 1 \end{pmatrix} \right\}$$

   The dimension of the subspace is 1.

Question 13. Show that the matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for $\mathcal{M}_{2\times 2}$.

Solution. We must show that the matrices are linearly independent and span $\mathcal{M}_{2\times 2}$. To prove linear independence we must show that the equation

$$\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = O$$

has only the trivial solution. To prove that the matrices span $\mathcal{M}_{2\times 2}$ we must show that every 2×2 matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can be expressed as

$$\beta_1 M_1 + \beta_2 M_2 + \beta_3 M_3 + \beta_4 M_4 = B.$$

The matrix forms for the two equations are

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\beta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \beta_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \beta_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which can be rewritten as

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since the first equation has only the trivial solution

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

the matrices are linearly independent, and since the second equation has the solution

$$\beta_1 = a, \quad \beta_2 = b, \quad \beta_3 = c, \quad \beta_4 = d$$

the matrices space $\mathcal{M}_{2\times 2}$. This proves that the matrices M_1, M_2, M_3 and M_4 form a basis for $\mathcal{M}_{2\times 2}$.

More generally

Definition 1 *The mn different matrices whose entries are zero except for a single entry of* 1 *form a basis for $\mathcal{M}_{m\times n}$ called the standard basis for $\mathcal{M}_{M\times n}$.*

Question 14. Find the dimension of each of the following vector spaces.

1. The vector space of all diagonal $n \times n$ matrices.
2. The vector space of all symmetric $n \times n$ matrices.
3. The vector space of all upper triangular $n \times n$ matrices.
4. The vector space of all lower triangular $n \times n$ matrices.

Solution. Let E_{ij} denote the $n \times n$ matrix with a 1 in the $(i,i)-$entry and 0 elsewhere, for $1 \leq i, j \leq n$.

1. The vector space of all diagonal $n \times n$ matrices.

   A diagonal matrix A is of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}E_{11} + a_{22}E_{22} + \cdots + a_{nn}E_{nn}.$$

   Therefore, a basis for the vector space of diagonal matrices is given by the set:

$$\{\, E_{11},\ E_{22},\ \ldots,\ E_{nn} \,\}.$$

   The dimension of this vector space is n.

2. A symmetric matrix A satisfies $A^\top = A$, meaning that $a_{ij} = a_{ji}$ for all i, j. It has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{pmatrix} = \sum_{i=1}^{n} a_{ii}E_{ii} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{ij}(E_{ij} + E_{ji})$$

Therefore, a basis consists of n diagonal matrices E_{ii} and $\frac{n(n-1)}{2}$ symmetric matrices of the form $E_{ij}+E_{ji}$ for $i<j$ which correspond to the $\frac{n(n-1)}{2}$ entries above (or below) the diagonal. The dimension of this vector space is

$$\frac{n(n+1)}{2}.$$

3. An upper triangular matrix A satisfies $a_{ij}=0$ for all $i>j$, meaning it has the form

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}=\sum_{i=1}^{n}\sum_{j=i}^{n} a_{ij}E_{ij}$$

Therefore, a basis consists of $\frac{n(n+1)}{2}$ matrices E_{ij} for $i\le j$, and the dimension of this vector space is

$$\frac{n(n+1)}{2}.$$

4. A lower triangular matrix A satisfies $a_{ij}=0$ for all $i<j$, meaning it has the form

$$A=\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}=\sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij}E_{ij}$$

Therefore, a basis consists of $\frac{n(n+1)}{2}$ matrices E_{ij} for $i\ge j$, and the dimension of this vector space is

$$\frac{n(n+1)}{2}.$$

Question 15. Find a standard basis vector in $\mathbb{R}^3$ that can be added to the set $\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$ to produce a basis for $\mathbb{R}^3$.

1. $\boldsymbol{v}_1=\begin{pmatrix}-1, & 2, & 3\end{pmatrix}$, $\boldsymbol{v}_2=\begin{pmatrix}1, & 2, & -2\end{pmatrix}$.
2. $\boldsymbol{v}_1=\begin{pmatrix}1, & -1, & 0\end{pmatrix}$, $\boldsymbol{v}_2=\begin{pmatrix}3, & 1, & -2\end{pmatrix}$.

Solution.

1. We observe that both vectors share the same y-coordinate, indicating that they lie in a plane where the standard basis vector $\boldsymbol{e}_2$ is linearly independent from them. To confirm that $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{e}_2\}$ forms a basis, we compute the determinant of the matrix whose columns are these vectors:

$$\begin{vmatrix} -1 & 2 & 3 \\ 1 & 2 & -2 \\ 0 & 1 & 0 \end{vmatrix}=1$$

2. To determine a standard basis vector that can be added to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to form a basis of $\mathbb{R}^3$, we compute the following determinants:

$$\begin{vmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 2, \qquad \begin{vmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2, \qquad \begin{vmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = 4.$$

Since all of these determinants are nonzero, it follows that any of the standard basis vectors can be added to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to form a basis for $\mathbb{R}^3$.

Question 16. Find standard basis vectors for $\mathbb{R}^4$ that can be added to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to produce a basis for $\mathbb{R}^4$.

$$v_1 = \begin{pmatrix} 1, & -4, & 2, & -3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3, & 8, & -4, & 6 \end{pmatrix}$$

Solution. To determine the necessary standard basis vectors, it suffices to find two standard basis vectors that are not contained in $\text{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$. A vector belongs to $\text{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$ if and only if it can be written as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. To identify which standard basis vectors satisfy this condition, we construct the following augmented matrix, where the first two columns represent $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and the last four columns correspond to the four standard basis vectors

$$\left(\begin{array}{cc|cccc} 1 & -3 & 1 & 0 & 0 & 0 \\ -4 & 8 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 \end{array}\right).$$

Performing row reduction, we obtain the reduced row echelon form:

$$\left(\begin{array}{cc|cccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} \end{array}\right)$$

From the reduced matrix, we observe that $\boldsymbol{e}_1$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ while $\boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$ do not belong to $\text{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$. This means that any two of $\{\boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ can be added to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to produce a basis for $\mathbb{R}^4$.

Question 17. The vectors $\boldsymbol{v}_1 = \begin{pmatrix} 1, & -2, & 3 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} 0, & 5, & -3 \end{pmatrix}$ are linearly independent. Enlarge the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to a basis for $\mathbb{R}^3$.

Solution. Following a similar approach to Question 16, we seek a standard basis vector to add to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to form a basis for $\mathbb{R}^3$. To determine which standard basis vectors are not contained in $\text{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$, we construct the following augmented matrix, where the first two columns represent $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and the last three columns correspond to the three standard basis vectors

$$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ -2 & 5 & 0 & 1 & 0 \\ 3 & -3 & 0 & 0 & 1 \end{array}\right)$$

Applying row reduction, we obtain the reduced row echelon form:

$$\left(\begin{array}{cc|cccc} 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{9} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{9} \end{array}\right)$$

Since the last three columns remain pivot columns, this confirms that any standard basis vector can be added to $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to form a basis for $\mathbb{R}^3$.

Question 18. The vectors $\boldsymbol{v}_1 = \begin{pmatrix}1, & 0, & 0, & 0\end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix}1, & 1, & 0, & 0\end{pmatrix}$ are linearly independent. Enlarge the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to a basis for $\mathbb{R}^4$.

Solution. Observing that both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ have zero entries in the third and fourth components, we select the standard basis vectors $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$ to add to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. To confirm that the resulting set forms a basis for $\mathbb{R}^4$, we verify that the corresponding matrix is invertible by computing its determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

Since the determinant is nonzero, the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ forms a basis for $\mathbb{R}^4$.

Question 19. Consider the bases $B_1 = \{\boldsymbol{u}_1,\ \boldsymbol{u}_2\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ for $\mathbb{R}^2$, where

(a)

$$\boldsymbol{u}_1 = \begin{pmatrix}2\\2\end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix}4\\-1\end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix}1\\3\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}-1\\-1\end{pmatrix}$$

(b)

$$\boldsymbol{u}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix}2\\1\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}-3\\4\end{pmatrix}$$

1. Find the transition matrix from B_2 to B_1
2. Find the transition matrix from B_1 to B_2
3. Compute the coordinate vector $[\boldsymbol{w}]_{B_1}$, where

$$\boldsymbol{w} = \begin{pmatrix}3\\-5\end{pmatrix}$$

   and compute $[\boldsymbol{w}]_{B_2}$ using the transition matrix from B_1 to B_2
4. Compute $[\boldsymbol{w}]_{B_2}$ directly

Solution.

(a) Firstly, we express $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as linear combinations of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$

$$\boldsymbol{v}_1 = \frac{13}{10}\boldsymbol{u}_1 - \frac{2}{5}\boldsymbol{u}_2, \quad \boldsymbol{v}_2 = -\frac{1}{2}\boldsymbol{u}_1$$

And $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ as linear combinations of v_1 and $\boldsymbol{v}_2$

$$\boldsymbol{u}_1 = -2\boldsymbol{v}_2, \quad \boldsymbol{u}_2 = -\frac{5}{2}\boldsymbol{v}_1 - \frac{13}{2}\boldsymbol{v}_2.$$

We have

$$[\boldsymbol{v}_1]_{B_1} = \begin{pmatrix} \frac{13}{10} \\ -\frac{2}{5} \end{pmatrix}, \quad [\boldsymbol{v}_2]_{B_1} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}, \quad [\boldsymbol{u}_1]_{B_2} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad [\boldsymbol{u}_2]_{B_2} = \begin{pmatrix} -\frac{5}{2} \\ -\frac{13}{2} \end{pmatrix},$$

1.

$$P_{B_2 \to B_1} = \begin{pmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{pmatrix}$$

2.

$$P_{B_1 \to B_2} = \begin{pmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{pmatrix}$$

3.

$$[\boldsymbol{w}]_{B_1} = \begin{pmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{pmatrix}, \quad [\boldsymbol{w}]_{B_2} = P_{B_1 \to B_2}[\boldsymbol{w}]_{B_1} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$$

4. To find the coordinate vector for $\boldsymbol{w}$ relative to B_2, we solve for α_1, α_2 in the equation

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 = \boldsymbol{w},$$

This corresponds to the system of equations:

$$\begin{aligned} 2\alpha_1 - 3\alpha_2 &= 3 \\ \alpha_1 + 4\alpha_2 &= -5 \end{aligned}$$

We obtain $\alpha_1 = -4$, $\alpha_2 = -7$. Thus

$$[\boldsymbol{w}]_{B_2} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}.$$

(b)

$$P_{B_1 \to B_2} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix}, \quad P_{B_2 \to B_1} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}, \quad [\boldsymbol{w}]_{B_1} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \quad [\boldsymbol{w}]_{B_2} = \begin{pmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{pmatrix}$$

$$P_{B_1 \to B_2}[\boldsymbol{w}]_{B_1} = [\boldsymbol{w}]_{B_2}$$

Question 20. Consider the bases $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for $\mathbb{R}^3$, where

(a)

$$\boldsymbol{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

(b)

$$\boldsymbol{u}_1 = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

1. Find the transition matrix from B_2 to B_1
2. Find the transition matrix from B_1 to B_2
3. Compute the coordinate vector $[\boldsymbol{w}]_{B_1}$, where

$$\boldsymbol{w} = \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix}$$

   and compute $[\boldsymbol{w}]_{B_2}$ using the transition matrix from B_1 to B_2
4. Compute $[\boldsymbol{w}]_{B_2}$ directly

Solution.

(a)

$$P_{B_2 \to B_1} = \begin{pmatrix} \frac{35}{2} & \frac{19}{2} & -\frac{13}{2} \\ -\frac{19}{2} & -\frac{11}{2} & \frac{7}{2} \\ -13 & -7 & 5 \end{pmatrix}, \quad P_{B_1 \to B_2} = \begin{pmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{pmatrix},$$

$$[\boldsymbol{w}]_{B_1} = \begin{pmatrix} 9 \\ -9 \\ -5 \end{pmatrix}, \quad [\boldsymbol{w}]_{B_2} = \begin{pmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{pmatrix}.$$

(b)

$$P_{B_2 \to B_1} = \begin{pmatrix} 0 & -\frac{4}{3} & -\frac{17}{6} \\ \frac{3}{2} & \frac{3}{2} & 3 \\ -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}, \quad P_{B_1 \to B_2} = \begin{pmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix},$$

$$[\boldsymbol{w}]_{B_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad [\boldsymbol{w}]_{B_2} = \begin{pmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{pmatrix}.$$

Question 21. Let $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the bases for $\mathbb{R}^2$, where

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

An efficient way to compute the transition matrix $P_{B_1 \to B_2}$ is as follows

Step 1. Form the matrix $\left(B_2 \mid B_1\right)$

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form

Step 3. The resulting matrix will be $\left(I \mid P_{B_1 \to B_2}\right)$

Step 4. Extract the matrix $P_{B_1 \to B_2}$ from the right side of the matrix in Step 3

In diagram

$$\left(\text{new basis} \mid \text{old basis}\right) \xrightarrow{\text{row operations}} \left(I \mid \text{transition from old to new}\right) \tag{1}$$

1. Apply the above procedure to find the transition matrix $P_{B_2 \to B_1}$
2. Apply the above procedure to find the transition matrix $P_{B_1 \to B_2}$
3. Confirm that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another
4. Let $\boldsymbol{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find $[\boldsymbol{w}]_{B_1}$ and then use the matrix $P_{B_1 \to B_2}$ to compute $[\boldsymbol{w}]_{B_2}$ from $[\boldsymbol{w}]_{B_1}$
5. Let $\boldsymbol{w} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Find $[\boldsymbol{w}]_{B_2}$ and then use the matrix $P_{B_2 \to B_1}$ to compute $[\boldsymbol{w}]_{B_1}$ from $[\boldsymbol{w}]_{B_2}$

Solution.

1. Following the procedure, we first form the matrix

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right)$$

The reduced row echelon form of the matrix is

$$\left(\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right)$$

Thus the transition matrix

$$P_{B_2 \to B_1} = \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}$$

2. Following the procedure, we first form the matrix

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array}\right)$$

The reduced row echelon form of the matrix is

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right)$$

Thus the transition matrix

$$P_{B_1 \to B_2} = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$$

3.
$$P_{B_1 \to B_2} P_{B_2 \to B_1} = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4.
$$[\boldsymbol{w}]_{B_1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad [\boldsymbol{w}]_{B_2} = P_{B_1 \to B_2} [\boldsymbol{w}]_{B_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

5.
$$[\boldsymbol{w}]_{B_2} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad [\boldsymbol{w}]_{B_1} = P_{B_2 \to B_1} [\boldsymbol{w}]_{B_2} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Question 22. Let S be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis in which

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

1. Find the transition matrix $P_{B \to S}$ by inspection
2. Use Formula (1) to find the transition matrix $P_{S \to B}$
3. Confirm that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another
4. Let $\boldsymbol{w} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Find $[\boldsymbol{w}]_B$ and then use the matrix $P_{B \to S}$ to compute $[\boldsymbol{w}]_S$ from $[\boldsymbol{w}]_B$
5. Let $\boldsymbol{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Find $[\boldsymbol{w}]_S$ and then use the matrix $P_{S \to B}$ to compute $[\boldsymbol{w}]_B$ from $[\boldsymbol{w}]_S$

Solution.

1. Since the columns of $P_{B \to S}$ are given by the coordinate vectors of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ relative to the standard basis, which are themselves, we have

$$P_{B \to S} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

2. We first form the matrix

$$\left(\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right).$$

The reduced row echelon form of the matrix is

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right)$$

Therefore

$$P_{S\to B} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix}$$

3.

$$P_{S\to B}P_{B\to S} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4.

$$[\boldsymbol{w}]_B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad [\boldsymbol{w}]_S = P_{B\to S}[\boldsymbol{w}]_B = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

5.

$$[\boldsymbol{w}]_S = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \quad [\boldsymbol{w}]_B = P_{S\to B}[\boldsymbol{w}]_S = \begin{pmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{pmatrix}$$

Question 23. Let S be the standard basis for $\mathbb{R}^3$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be the basis in which

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}$$

1. Find the transition matrix $P_{B\to S}$ by inspection
2. Use Formula (1) to find the transition matrix $P_{S\to B}$
3. Confirm that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another
4. Let $\boldsymbol{w} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$. Find $[\boldsymbol{w}]_B$ and then use the matrix $P_{B\to S}$ to compute $[\boldsymbol{w}]_S$ from $[\boldsymbol{w}]_B$
5. Let $\boldsymbol{w} = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}$. Find $[\boldsymbol{w}]_S$ and then use the matrix $P_{S\to B}$ to compute $[\boldsymbol{w}]_B$ from $[\boldsymbol{w}]_S$

Solution.

1. $P_{B\to S} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

2. $P_{S\to B} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$

3. $P_{S\to B}P_{B\to S} = I$

4. $[\boldsymbol{w}]_B = \begin{pmatrix} -239 \\ 77 \\ 30 \end{pmatrix}$, $[\boldsymbol{w}]_S = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$

5. $[\boldsymbol{w}]_S = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}$, $[\boldsymbol{w}]_S = \begin{pmatrix} -200 \\ 64 \\ 25 \end{pmatrix}$

Question 24. Let $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis that results when the vectors in S are reflected about the line $y = x$.

1. Find the transition matrix $P_{B\to S}$
2. Show that $P_{B\to S}^{\top} = P_{S\to B}$

Solution.

1. The standard matrix for the reflection about the line $y = x$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where the first column is the image of $\boldsymbol{e}_1$ under the reflection operator and the second column is the image of $\boldsymbol{e}_2$ under the reflection operator. Thus $\boldsymbol{v}_1 = \begin{pmatrix} 0, & 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 1, & 0 \end{pmatrix}$ Then

$$P_{B\to S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. Since

$$\boldsymbol{e}_1 = \boldsymbol{v}_2, \quad \boldsymbol{e}_2 = \boldsymbol{v}_1,$$

we have $P_{S\to B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$P_{B\to S}^{\top} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = P_{S\to B}$$

Question 25. Let $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis that results when the vectors in S are reflected about the line that makes an angle θ with the positive $x-$axis.

1. Find the transition matrix $P_{B\to S}$
2. Show that $P_{B\to S}^{\top} = P_{S\to B}$

Solution.

1. The standard matrix for the reflection about the line that makes an angle θ with the positive $x-$axis is given by
$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$
Thus $\boldsymbol{v}_1 = \big(\cos 2\theta, \quad \sin 2\theta\big)$, $\boldsymbol{v}_2 = \big(\sin 2\theta, \quad -\cos 2\theta\big)$, and
$$P_{B\to S} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

2. To find $P_{S\to B}$, we first express the standard basis vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in terms of the basis vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. That is, we solve for the scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$ in the equations:
$$\boldsymbol{e}_1 = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2, \quad \boldsymbol{e}_2 = \beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2.$$
Using trigonometric identities, we obtain the relationships:
$$\boldsymbol{e}_1 = \cos 2\theta \boldsymbol{v}_1 + \sin 2\theta \boldsymbol{v}_2, \quad \boldsymbol{e}_2 = \sin 2\theta \boldsymbol{v}_1 - \cos 2\theta \boldsymbol{v}_2$$
Thus, the transition matrix from S to B is given by:
$$P_{S\to B} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$
Furthermore,
$$P_{B\to S}^{\top} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = P_{S\to B}.$$

Question 26. Find the domain and codomain of the transformation $T_A(\boldsymbol{x}) = A\boldsymbol{x}$

1. $A \in \mathcal{M}_{3\times 2}$
2. $A \in \mathcal{M}_{2\times 3}$
3. $A \in \mathcal{M}_{3\times 3}$
4. $A \in \mathcal{M}_{1\times 6}$
5. $A \in \mathcal{M}_{4\times 5}$
6. $A \in \mathcal{M}_{5\times 4}$
7. $A \in \mathcal{M}_{4\times 4}$
8. $A \in \mathcal{M}_{3\times 1}$

Solution.

1. $A\boldsymbol{x} \in \mathbb{R}^3$, thus $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ - domain is $\mathbb{R}^2$, codomain is $\mathbb{R}^3$

2. $T_A : \mathbb{R}^3 \to \mathbb{R}^2$
3. $T_A : \mathbb{R}^3 \to \mathbb{R}^3$
4. $T_A : \mathbb{R}^6 \to \mathbb{R}$
5. $T_A : \mathbb{R}^5 \to \mathbb{R}^4$
6. $T_A : \mathbb{R}^4 \to \mathbb{R}^5$
7. $T_A : \mathbb{R}^4 \to \mathbb{R}^4$
8. $T_A : \mathbb{R} \to \mathbb{R}^3$

Question 27. Find the domain and codomain of the transformation defined by the equations

1.

$$\begin{aligned} w_1 &= 4x_1 + 5x_2 \\ w_2 &= x_1 - 8x_2 \end{aligned}$$

2.

$$\begin{aligned} w_1 &= 5x_1 - 7x_2 \\ w_2 &= 6x_1 + x_2 \\ w_3 &= 2x_1 + 3x_2 \end{aligned}$$

3.

$$\begin{aligned} w_1 &= x_1 - 4x_2 + 8x_3 \\ w_2 &= -x_1 + 4x_2 + 2x_3 \\ w_3 &= -3x_1 + 2x_2 - 5x_3 \end{aligned}$$

4.

$$\begin{aligned} w_1 &= 2x_1 + 7x_2 - 4x_3 \\ w_2 &= 4x_1 - 3x_2 + 2x_3 \end{aligned}$$

Solution.

1. Domain: $\mathbb{R}^2$. Codomain: $\mathbb{R}^2$
2. Domain: $\mathbb{R}^2$. Codomain: $\mathbb{R}^3$
3. Domain: $\mathbb{R}^3$. Codomain: $\mathbb{R}^3$
4. Domain: $\mathbb{R}^3$. Codomain: $\mathbb{R}^2$

Question 28. Find the standard matrix for the transformation defined below

1.

$$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 \\ w_2 &= 3x_1 + 5x_2 - x_3 \end{aligned}$$

2.

$$\begin{aligned} w_1 &= 7x_1 + 2x_2 - 8x_3 \\ w_2 &= -x_2 + 5x_3 \\ w_3 &= 4x_1 + 7x_2 - x_3 \end{aligned}$$

3. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix}$

4. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7x_1 + 2x_2 - x_3 + x_4 \\ x_2 + x_3 \\ -x_1 \end{pmatrix}$

5. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

6. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{pmatrix}$

Solution.

1. $\begin{pmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{pmatrix}$
2. $\begin{pmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 3 \\ 1 & -1 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$
5. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
6. $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$

Question 29. Find $T_A(\boldsymbol{x})$.

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$
2. $A = \begin{pmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$
3. $A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
4. $A = \begin{pmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Solution.

1. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
2. $\begin{pmatrix} 3 \\ 13 \end{pmatrix}$
3. $\begin{pmatrix} -2x_1 + x_2 + 4x_3 \\ 3x_1 + 5x_2 + 7x_3 \\ 6x_1 - x_3 \end{pmatrix}$
4. $\begin{pmatrix} -x_1 + x_2 \\ 2x_1 + 4x_2 \\ 7x_1 + 8x_2 \end{pmatrix}$

Question 30. The images of the standard basis vectors for $\mathbb{R}^3$ are given for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Find the standard matrix for the transformation, and find $T(\boldsymbol{x})$.

1. $T(\boldsymbol{e}_1) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $T(\boldsymbol{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $T(\boldsymbol{e}_3) = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$
2. $T(\boldsymbol{e}_1) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $T(\boldsymbol{e}_2) = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$, $T(\boldsymbol{e}_3) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Solution.

1. $[T] = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 0 & -3 \\ 0 & 1 & -1 \end{pmatrix}$, $T(\boldsymbol{x}) = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$

2. $[T] = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$, $T(\boldsymbol{x}) = \begin{pmatrix} 1 \\ 1 \\ 11 \end{pmatrix}$

Question 31. Use matrix multiplication to find the reflection of $\begin{pmatrix} -1, & 2 \end{pmatrix}$ about the

1. $x-$axis
2. $y-$axis
3. line $y = x$

Solution.

1. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

2. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Question 32. Use matrix multiplication to find the reflection of $\begin{pmatrix} a, & b \end{pmatrix}$ about the

1. $x-$axis
2. $y-$axis
3. line $y = x$

Solution.

1. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$.

2. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}$

3. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$

Question 33. Use matrix multiplication to find the reflection of $\begin{pmatrix} 2, & -5, & 3 \end{pmatrix}$ about the

1. $xy-$plane
2. $xz-$plane
3. $yz-$plane

Solution.

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix}$

2. $\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\\ -5\\ 3\end{pmatrix} = \begin{pmatrix}2\\ 5\\ -3\end{pmatrix}$

3. $\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\\ -5\\ 3\end{pmatrix} = \begin{pmatrix}-2\\ -5\\ 3\end{pmatrix}$

Question 34. Use matrix multiplication to find the reflection of $\begin{pmatrix}a\\ b\\ c\end{pmatrix}$ about the

1. xy-plane
2. xz-plane
3. yz-plane

Solution.

1. $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}a\\ b\\ -c\end{pmatrix}$

2. $\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}a\\ -b\\ c\end{pmatrix}$

3. $\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}-a\\ b\\ c\end{pmatrix}$

Question 35. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix}2\\ -5\end{pmatrix}$ onto the

1. x-axis
2. y-axis

Solution.

1. $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}2\\ -5\end{pmatrix} = \begin{pmatrix}2\\ 0\end{pmatrix}$

2. $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}2\\ -5\end{pmatrix} = \begin{pmatrix}0\\ -5\end{pmatrix}$

Question 36. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix}a\\ b\end{pmatrix}$ onto the

1. x-axis
2. y-axis

Solution.

1. $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}a\\ 0\end{pmatrix}$

2. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$

Question 37. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ onto the

1. xy-plane
2. xz-plane
3. yz-plane

Solution.

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

Question 38. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ onto the

1. xy-plane
2. xz-plane
3. yz-plane

Solution.

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ c \end{pmatrix}$
3. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$

Question 39. Use matrix multiplication to find the image of the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ when it is rotated about the origin through an angle of

1. $\theta = 30°$
2. $\theta = -60°$
3. $\theta = 45°$
4. $\theta = 90°$

Solution. Recall
$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

1. $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{4+3\sqrt{3}}{2} \\ \frac{3-4\sqrt{3}}{2} \end{pmatrix} \approx \begin{pmatrix} 4.598 \\ -1.964 \end{pmatrix}$

2. $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{3-4\sqrt{3}}{2} \\ -\frac{3\sqrt{3}+4}{2} \end{pmatrix} \approx \begin{pmatrix} -1.964 \\ -4.598 \end{pmatrix}$

3. $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} \approx \begin{pmatrix} 4.950 \\ -0.7071 \end{pmatrix}$

4. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Question 40. Use matrix multiplication to find the image of the nonzero vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ when it is rotated about the origin through

1. a positive angle θ
2. a negative angle $-\theta$

Solution.

1. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1\cos\theta - v_2\sin\theta \\ v_1\sin\theta + v_2\cos\theta \end{pmatrix}$

2. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1\cos\theta + v_2\sin\theta \\ -v_1\sin\theta + v_2\cos\theta \end{pmatrix}$

Question 41. Use matrix multiplication to find the image of the vector $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ if it is rotated

1. 30° clockwise about the positive x-axis.
2. 30° counterclockwise about the positive y-axis.
3. 45° clockwise about the positive y-axis.
4. 90° counterclockwise about the positive z-axis.

Solution.

1. 30° clockwise corresponds to $-30°$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right) \\ 0 & \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 - \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 2 \\ 0.1340 \\ 2.2321 \end{pmatrix}$$

2.

$$\begin{pmatrix} \cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{6} & 0 & \cos\frac{\pi}{6} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+\sqrt{3} \\ -1 \\ -1+\sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 2.7321 \\ -1 \\ 0.7321 \end{pmatrix}$$

3. 45° clockwise corresponds to $-45°$

$$\begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & 0 & \sin\left(-\frac{\pi}{4}\right) \\ 0 & 1 & 0 \\ -\sin\left(-\frac{\pi}{4}\right) & 0 & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2\sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 0 \\ -1 \\ 2.8284 \end{pmatrix}$$

4.

$$\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Question 42. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with factor $\alpha = \frac{1}{2}$.
2. The dilation of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with factor $\alpha = 3$.

Solution.

1. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$
2. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

Question 43. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} a \\ b \end{pmatrix}$ with factor $\frac{1}{\alpha}$, where $\alpha > 1$.

2. The dilation of $\begin{pmatrix} a \\ b \end{pmatrix}$ with factor α, where $\alpha > 1$.

Solution.

1. $\begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{\alpha} \\ \frac{b}{\alpha} \end{pmatrix}$

2. $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}$

Question 44. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with factor $\frac{1}{4}$.

2. The dilation of $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with factor 2.

Solution.

1. $\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$

2. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$

Question 45. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with factor $\frac{1}{\alpha}$, where $\alpha > 1$.

2. The dilation of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with factor α, where $\alpha > 1$.

Solution.

1. $\begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a}{\alpha} \\ -\frac{b}{\alpha} \\ \frac{c}{\alpha} \end{pmatrix}$

2. $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \\ \alpha c \end{pmatrix}$

Question 46. Use matrix multiplication to find:

1. The compression of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the x-direction with factor $\frac{1}{2}$.
2. The compression of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the y-direction with factor $\frac{1}{2}$.

Solution.

1. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Question 47. Use matrix multiplication to find:

1. The expansion of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the x-direction with factor 3.
2. The expansion of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the y-direction with factor 3.

Solution.

1. $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

Question 48. Use matrix multiplication to find:

1. The compression of $\begin{pmatrix} a \\ b \end{pmatrix}$ in the x-direction with factor $\frac{1}{\alpha}$, where $\alpha > 1$.

2. The expansion of $\begin{pmatrix} a \\ b \end{pmatrix}$ in the y-direction with factor α, where $\alpha > 1$.

Solution.

1. $\begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{\alpha} \\ b \end{pmatrix}$
2. $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ \alpha b \end{pmatrix}$

Question 49. In each part, determine whether the operators T_1 and T_2 commute, i.e. whether $T_1 \circ T_2 = T_2 \circ T_1$.

1. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the line $y = x$, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the x-axis.
2. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the x-axis, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the line $y = x$.
3. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the x-axis, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the y-axis.
4. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation about the origin through an angle of $\frac{\pi}{4}$, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the y-axis.
5. $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is a dilation with factor α, and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is a contraction with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
6. $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation about the x-axis through an angle θ_1, and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation about the z-axis through an angle θ_2.

Solution.

1. $[T_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $[T_2] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

   $$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   T_1 and T_2 do not commute.
2. $[T_1] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $[T_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   $$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

   T_1 and T_2 do not commute.

3. $[T_1] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $[T_2] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

T_1 and T_2 commute.

4. $[T_1] = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$, $[T_2] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

T_1 and T_2 do not commute.

5. $[T_1] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$, $[T_2] = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix}$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

T_1 and T_2 commute.

6. $[T_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$, $[T_2] = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2\cos\theta_1 & \cos\theta_1\cos\theta_2 & -\sin\theta_1 \\ \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2 & \cos\theta_1 \end{pmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2\cos\theta_1 & \sin\theta_1\sin\theta_2 \\ \sin\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_1\cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

T_1 and T_2 do not commute.

Question 50. Find the standard matrix for the stated composition in $\mathbb{R}^2$.

1. A rotation of 90°, followed by a reflection about the line $y = x$.
2. An orthogonal projection onto the y-axis, followed by a contraction with factor $\frac{1}{2}$.
3. A reflection about the x-axis, followed by a dilation with factor 3, followed by a rotation about the origin of 60°.

4. A rotation about the origin of 60°, followed by an orthogonal projection onto the x-axis, followed by a reflection about the line $y = x$.

5. A dilation with factor 2, followed by a rotation about the origin of 45°, followed by a reflection about the y-axis.

6. A rotation about the origin of 15°, followed by a rotation about the origin of 105°, followed by a rotation about the origin of 60°.

Solution.

1.
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2.
$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

3.
$$\begin{aligned} \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix} \end{aligned}$$

4.
$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \end{aligned}$$

5.
$$\begin{aligned} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix} \end{aligned}$$

6. The total rotated angle is

$$15° + 105° + 60° = 180°.$$

Hence the transition matrix is given by

$$\begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Question 51. Find the standard matrix for the stated composition in $\mathbb{R}^3$.

1. A reflection about the yz-plane, followed by an orthogonal projection onto the xz-plane.
2. A rotation of 45° about the y-axis, followed by a dilation with factor $\sqrt{2}$.
3. An orthogonal projection onto the xy-plane, followed by a reflection about the yz-plane.
4. A rotation of 30° about the x-axis, followed by a rotation of 30° about the z-axis, followed by a contraction with factor $\frac{1}{4}$.
5. A reflection about the xy-plane, followed by a reflection about the xz-plane, followed by an orthogonal projection onto the yz-plane.
6. A rotation of 270° about the x-axis, followed by a rotation of 90° about the y-axis, followed by a rotation of 180° about the z-axis.

Solution.

1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.

$$\begin{aligned} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{4} & 0 & \sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{pmatrix} &= \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \end{aligned}$$

3.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4.

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ 0 & \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{8} & -\frac{1}{8} \\ 0 & \frac{1}{8} & \frac{\sqrt{3}}{8} \end{pmatrix}$$

5.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

6.

$$\begin{pmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{2} & 0 & \sin\frac{\pi}{2} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{2} & 0 & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} \\ 0 & \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Question 52. Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector in $\mathbb{R}^2$. Consider the linear transformations $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_1(\boldsymbol{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}, \quad T_2(\boldsymbol{x}) = \begin{pmatrix} 3x_1 \\ 2x_1 + 4x_2 \end{pmatrix}$$

1. Find the standard matrices for T_1 and T_2.
2. Find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$.
3. Find the standard matrices for $T_1 \circ T_2 \circ T_1$ and $T_1 \circ T_2 \circ T_2$.
4. Use the matrices obtained in part 2 to find formulas for $T_1(T_2(\boldsymbol{x}))$ and $T_2(T_1(\boldsymbol{x}))$

Solution.

1. $[T_1] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $[T_2] = \begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix}$

2.

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 5 & 4 \\ 1 & -4 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 3 & 3 \\ 6 & -2 \end{pmatrix}$$

3.

$$[T_1 \circ T_2 \circ T_1] = [T_1][T_2][T_1] = \begin{pmatrix} 9 & 1 \\ -3 & 5 \end{pmatrix}, \quad [T_1 \circ T_2 \circ T_2] = [T_1][T_2][T_2] = \begin{pmatrix} 23 & 16 \\ -5 & -16 \end{pmatrix}$$

4.

$$T_1(T_2(\boldsymbol{x})) = \begin{pmatrix} 5x_1 + 4x_2 \\ x_1 - 4x_2 \end{pmatrix}, \quad T_2(T_1(\boldsymbol{x})) = \begin{pmatrix} 3x_1 + 3x_2 \\ 6x_1 - 2x_2 \end{pmatrix}$$

Question 53. Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be a vector in $\mathbb{R}^3$. Consider the linear transformations $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T_1(\boldsymbol{x}) = \begin{pmatrix} 4x_1 \\ -2x_1 + x_2 \\ -x_1 - 3x_3 \end{pmatrix}, \quad T_2(\boldsymbol{x}) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_3 \\ 4x_1 - x_3 \end{pmatrix}$$

1. Find the standard matrices for T_1 and T_2.
2. Find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$.
3. Find the standard matrices for $T_1 \circ T_2 \circ T_1$ and $T_1 \circ T_2 \circ T_2$.
4. Use the matrices obtained in part 2 to find formulas for $T_1(T_2(\boldsymbol{x}))$ and $T_2(T_1(\boldsymbol{x}))$

Solution.

1. $[T_1] = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & -3 \end{pmatrix}$, $[T_2] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 4 & 0 & -1 \end{pmatrix}$

2.

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 4 & 8 & 0 \\ -2 & -4 & 2 \\ -13 & -2 & 3 \end{pmatrix}, \quad [T_2 \circ T_1] = [T_2][T_1] = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & -6 \\ 17 & 0 & 3 \end{pmatrix}$$

3.

$$[T_1 \circ T_2 \circ T_1] = [T_1][T_2][T_1] = \begin{pmatrix} 0 & 8 & 0 \\ -2 & -4 & -6 \\ -51 & -2 & -9 \end{pmatrix}, \quad [T_1 \circ T_2 \circ T_2] = [T_1][T_2][T_2] = \begin{pmatrix} 4 & 8 & 16 \\ 6 & -4 & -10 \\ -1 & -26 & -7 \end{pmatrix}$$

4.

$$T_1(T_2(\boldsymbol{x})) = \begin{pmatrix} 4x_1 + 8x_2 \\ -2x_1 - 4x_2 + 2x_3 \\ -13x_1 - 2x_2 + 3x_3 \end{pmatrix}, \quad T_2(T_1(\boldsymbol{x})) = \begin{pmatrix} 2x_2 \\ -2x_1 - 6x_3 \\ 17x_1 + 3x_3 \end{pmatrix}$$