

# Algebra and Discrete Mathematics (ADM)

## Tutorial 6 Matrix operators

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# Matrix transformations

- Consider  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  with standard matrix

$$[T] = \begin{pmatrix} -1 & -1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$$

- Find the image of  $\mathbf{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$T(\mathbf{x}) = \begin{pmatrix} -1 & -1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 14 \\ 7 \end{pmatrix}$$

# Matrix transformations

- Consider  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  with standard matrix

$$[T] = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

- Find the image of  $\mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

$$T(\mathbf{x}) = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ 16 \end{pmatrix}$$

## Reflection operators on $\mathbb{R}^2$

- Reflection about the  $x$ -axis,  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection about the  $y$ -axis,  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection about the line  $y = x$ ,  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Reflection about the line  $y = \sqrt{3}x$ 
  - $y = \sqrt{3}x$  makes an angle  $\pi/3$  ( $= 60^\circ$ ) with positive  $x$ -axis

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

## Reflection operators on $\mathbb{R}^3$

- Reflection about the  $xy$ -plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- Reflection about the  $xz$ -plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Reflection about the  $yz$ -plane

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Projection operators on $\mathbb{R}^2$

- Orthogonal projection onto the  $x$ -axis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- Orthogonal projection onto the  $y$ -axis

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

## Projection operators on $\mathbb{R}^3$

- Orthogonal projection onto the  $xy$ -plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Orthogonal projection onto the  $xz$ -plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Orthogonal projection onto the  $yz$ -plane

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Rotation operators on $\mathbb{R}^2$

- Moves points *counterclockwise* about the origin through a positive angle  $\theta$
- *Rotation matrix*

$$R_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- Clockwise about the origin through an angle  $\theta$

$$R_{-\theta} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$



## Rotation operators on $\mathbb{R}^3$

Operator	Rotation equations	Standard matrix
Counterclockwise rotation about the positive $x$ -axis through an angle $\theta$	$\begin{aligned} w_1 &= x \\ w_2 &= y \cos \theta - z \sin \theta \\ w_3 &= y \sin \theta + z \cos \theta \end{aligned}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$
Counterclockwise rotation about the positive $y$ -axis through an angle $\theta$	$\begin{aligned} w_1 &= x \cos \theta + z \sin \theta \\ w_2 &= y \\ w_3 &= -x \sin \theta + z \cos \theta \end{aligned}$	$\begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$
Counterclockwise rotation about the positive $z$ -axis through an angle $\theta$	$\begin{aligned} w_1 &= x \cos \theta - y \sin \theta \\ w_2 &= x \sin \theta + y \cos \theta \\ w_3 &= z \end{aligned}$	$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

## Dilations and contractions

- Dilation/contraction with factor  $\alpha$  on  $\mathbb{R}^2$ ,  $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

- Dilation/contraction with factor  $\alpha$  on  $\mathbb{R}^3$ ,  $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

## Expansions and compressions on $\mathbb{R}^2$

- In the  $x$ -direction –  $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ y \end{pmatrix}$

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

- In the  $y$ -direction –  $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \alpha y \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

## Shears on $\mathbb{R}^2$

- Shear in the  $x$ -direction by a factor  $\alpha$ ,  $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y \end{pmatrix}$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

- Shear in the  $y$ -direction by a factor  $\alpha$ ,  $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \alpha x \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

## Composition of matrix transformations

- Consider a square  $ABCD$  with vertices

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- Perform the following transformations
  - $T_1$ : shear in the  $x$ -direction by a factor 2
  - $T_2$ : reflection about the  $x$ -axis
  - $T_3$ : reflection about the  $y$ -axis

$$[T_1] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad [T_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [T_3] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[T_3][T_2][T_1] \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} -3 & -5 & -9 & -7 \\ -1 & -1 & -3 & -3 \end{pmatrix}$$

# Midterm

- Room -1.42 (DIGILAB)
- Online, AIS
- Time: next Friday (14th Nov) 9am, 90 minutes
- 50 marks
- 14 multiple choice or fill in the blank questions, 3 marks each
- 4 statement-based multiple-choice questions, 2 marks each
- Covers: Lectures 1 – 5, Tutorials 1 – 5
- Vyberte ľubovoľný počet možných odpovedí. Správna nemusí byť žiadna, ale tiež môžu byť správne všetky. - Select any number of possible answers. None of them have to be correct, but it is also possible that all of them are correct
- Vyberte iba jednu z nasledujúcich možných odpovedí. - Select only one of the following possible answers