Tutorial 3

Matrix inverse and solving linear systems

Question 1. Confirm the validity of the given statements for the following matrices and scalars.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 1 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix}, \quad \alpha = 4, \quad \beta = -7$$

1. $(A+B)+C = A+(B+C)$
2. $(AB)C = A(BC)$
3. $(\alpha+\beta)C = \alpha C + \beta C$
4. $\alpha(BC) = (\alpha B)C = B(\alpha C)$
5. $A(B-C) = AB - AC$
6. $(B+C)A = BA + CA$
7. $\alpha(\beta C) = (\alpha\beta)C$
8. $(A^\top)^\top = A$
9. $(AB)^\top = B^\top A^\top$
10. $(A+B)^\top = A^\top + B^\top$
11. $(\alpha C)^\top = \alpha C^\top$
12. $A(B+C) = AB + AC$

Question 2. Find the inverse of the given matrix using the formula for computing the inverse of a 2×2 matrix.

1. $A = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$
2. $B = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$
3. $C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$
4. $D = \begin{pmatrix} 6 & 4 \\ -2 & -1 \end{pmatrix}$

Question 3. Find the inverse of the given matrix

1. $\begin{pmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{pmatrix}$
2. $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

Question 4. Verify whether the given equalities hold for the following matrices.

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

1. $(A^\top)^{-1} = (A^{-1})^\top$
2. $(A^{-1})^{-1} = A$
3. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
4. $(ABC)^\top = C^\top B^\top A^\top$

Question 5. Find the matrix A with the given information

1. $(7A)^{-1} = \begin{pmatrix} -3 & 7 \\ 1 & -2 \end{pmatrix}$
2. $(5A^\top)^{-1} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$
3. $(I + 2A)^{-1} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$
4. $A^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$

For each matrix A, compute:

(a) A^3

(b) A^{-3}

(c) $A^2 - 2A + I$

Question 6. Compute $p(A)$ for the given matrix A and the following polynomials.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$$

(a) $p(x) = x - 2$

(b) $p(x) = 2x^2 - x + 1$

(c) $p(x) = x^3 - 2x + 1$

Question 7. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

1. Find all values of a, b, c, d (if any) for which the matrices A and B commute.
2. Find all values of a, b, c, d (if any) for which the matrices A and C commute.

1. $AB = BC$ gives

$$\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix} \Longrightarrow a = b, \quad c = 0$$

2. $AC = CA$ gives

$$\begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \Longrightarrow b = 0, \quad d = a$$

Question 8. If a polynomial $p(x)$ can be factored as a product of lower degree polynomials, say

$$p(x) = p_1(x)p_2(x)$$

and if A is a square matrix, then it can be proved that

$$p(A) = p_1(A)p_2(A).$$

Let

$$p(x) = x^2 - 9, \quad p_1(x) = x + 3, \quad p_2(x) = x - 3$$

1. Verify this statement for the above polynomials and the following matrices

$$A_1 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$$

2. Prove that for any square matrix A, $p(A) = p_1(A)p_2(A)$

Question 9.

1. Give an example of two 2×2 matrices such that

$$(A+B)(A-B) \neq A^2 - B^2$$

2. Find a valid formula for multiplying out $(A+B)(A-B)$.
3. Establish the conditions on matrices A and B, under which the equality

$$(A+B)(A-B) = A^2 - B^2$$

   is valid.
4. Find a valid formula for $(A+B)^3$.
5. Find a valid formula for $(A-B)^3$.

Question 10. Show that if a square matrix A satisfies the equation

$$A^2 + 2A + I = 0$$

then A must be invertible. Find the inverse of A in this case.

Question 11. Show that if $p(x)$ is a polynomial with a nonzero constant term, and if A is a square matrix for which $p(A) = 0$, then A is invertible.

Question 12. Is it possible for $A^3 = I$ without A being invertible? Why?

Question 13. Can a matrix with a row of zeros or a column of zeros have an inverse? Why?

Question 14. Simplify the given expression assuming that A, B, C, D are invertible.

1. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$
2. $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

Question 15. Assuming that all matrices are $n \times n$ and invertible, solve for D

1. $C^\top B^{-1} A^2 B A C^{-1} D A^{-2} B^\top C^{-2} = C^\top$
2. $ABC^\top D B A^\top C = AB^\top$

Question 16. Determine whether the given matrix is elementary

1. $\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$
2. $\begin{pmatrix} -5 & 1 \\ 1 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$
4. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
5. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix}$
7. $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
8. $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
9. $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Question 17. Find a row operation and the corresponding elementary matrix that will transform the given elementary matrix to the identity matrix.

1. $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$
2. $\begin{pmatrix} -7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
5. $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
7. $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
8. $\begin{pmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Question 18. In the following examples, an elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

1. $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{pmatrix}$

2. $E = \begin{pmatrix} -6 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{pmatrix}$

3. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix}$

4. $E = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix}$

5. $E = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

6. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Question 19. Use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{pmatrix},$$
$$F = \begin{pmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}$$

1. $EA = B$
2. $EA = C$
3. $EB = A$
4. $EC = A$
5. $EB = D$
6. $ED = B$
7. $EB = F$
8. $EF = B$

Question 20. Determine the inverse matrix A^{-1} (if the inverse exists) for each of the given matrices using the matrix inversion algorithm. Identify the sequence of elementary matrices $E_1, E_2, \ldots$ associated with each row operation performed throughout the process.

1. $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$
2. $A = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & -5 \\ 3 & -16 \end{pmatrix}$
4. $A = \begin{pmatrix} 6 & 4 \\ -3 & -1 \end{pmatrix}$
5. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$
6. $A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$
7. $A = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & 9 \end{pmatrix}$
8. $A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{3}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

Question 21. Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

1. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
2. $\begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{pmatrix}$
4. $\begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}$
5. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{pmatrix}$
6. $\begin{pmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{pmatrix}$
7. $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{pmatrix}$
8. $\begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & -5 & 0 \\ 0 & -3 & 0 & 0 \\ 2 & 0 & 5 & -3 \end{pmatrix}$

Question 22. Find the inverse of the given matrix, where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$ and α are all nonzero constants.

1. $\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}$

2. $\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & \alpha_3 & 0 & 0 \\ \alpha_4 & 0 & 0 & 0 \end{pmatrix}$

4. $\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 1 & \alpha & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & \alpha \end{pmatrix}$

Question 23. Find all values of c, if any, for which the given matrix is invertible

1. $A = \begin{pmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix}$

2. $B = \begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix}$

Question 24. Express the given matrix A and its inverse as products of elementary matrices

1. $A = \begin{pmatrix} -3 & 1 \\ 2 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Question 25. For each of the following pairs of matrices A and B, show that the matrices A and B are row equivalent by

(a) Find a sequence of elementary row operations that produce B from A.

(b) Find a sequence of elementary row operations that produce A from B.

(c) Use the results from 1 to find a matrix C such that $CA = B$.

(d) Use the results from 2 to find a matrix D such that $DB = A$.

1. $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$

Question 26. Show that

$$A = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$$

is singular for any values of a, b, c, d, e, f, g, h.

Question 27. Suppose that each of the following is the augmented matrix for a linear system. Solve the systems by by inverting the coefficient matrix (if the inverse existst) and apply the following theorem

Theorem 1 *Given $A \in \mathcal{M}_{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^n$, if A is invertible, then the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.*

1. $\left(\begin{array}{cc|c} 2 & 0 & -1 \\ 3 & 2 & 0 \end{array}\right)$

2. $\left(\begin{array}{cccc|c} 3 & 0 & 0 & 1 & -4 \\ 3 & 0 & 2 & 1 & 7 \\ -1 & 3 & 0 & -2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right)$

3. $\left(\begin{array}{ccc|c} 1 & 0 & 8 & 6 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 0 & 3 & 2 \\ 0 & 1 & 1 & -5 \end{array}\right)$

5. $\left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right)$

6. $\left(\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 13 & 13 \end{array}\right)$

7. $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 2 & 3 \end{array}\right)$

8. $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \end{array}\right)$

9. $\left(\begin{array}{ccc|c} 2 & 1 & -5 & -20 \\ 0 & 2 & 2 & 7 \\ 3 & 1 & -9 & -36 \end{array}\right)$

For Questions 28 – 30 do the following
Solve the given linear system by inverting the coefficient matrix (if it is invertible) and applying Theorem 1.

Question 28.

1.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

2.

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

3.

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z - 2w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

4.

$$\begin{aligned} -2y + 3z &= 1 \\ 3x + 6y - 3z &= -2 \\ 6x + 6y + 3z &= 5 \end{aligned}$$

5.

$$\begin{aligned} 5x_1 - 5x_2 - 15x_3 &= 40 \\ 4x_1 - 2x_2 - 6x_3 &= 19 \\ 3x_1 - 6x_2 - 17x_3 &= 41 \end{aligned}$$

6.

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

8.

$$\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

Question 29.

1.

$$\begin{aligned} -2x_1 + x_2 + 8x_3 &= 0 \\ 7x_1 - 2x_2 - 22x_3 &= 0 \\ 3x_1 - x_2 - 10x_3 &= 0 \end{aligned}$$

2.

$$\begin{aligned} 5x_1 - 2x_3 &= 0 \\ -15x_1 - 16x_2 - 9x_3 &= 0 \\ 10x_1 + 12x_2 + 7x_3 &= 0 \end{aligned}$$

3.

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$$

4.

$$\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

5.

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

6.

$$\begin{aligned} 2x_1 + 6x_2 + 13x_3 + x_4 &= 0 \\ x_1 + 4x_2 + 10x_3 + x_4 &= 0 \\ 2x_1 + 8x_2 + 20x_3 + x_4 &= 0 \\ 3x_1 + 10x_2 + 21x_3 + 2x_4 &= 0 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 - 6x_2 + 3x_3 - 21x_4 &= 0 \\ 4x_1 - 5x_2 + 2x_3 - 24x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 10x_4 &= 0 \\ -2x_1 + 3x_2 - x_3 + 13x_4 &= 0 \end{aligned}$$

Question 30.

1.

$$\begin{aligned} x+y &= 1 \\ 4x+3y &= 2 \end{aligned}$$

2.

$$\begin{aligned} -3x+y &= 1 \\ 4x+2y &= 0 \end{aligned}$$

3.

$$\begin{aligned} 3x-3y &= 3 \\ 4x-y-3z &= 3 \\ -2x-2y &= -2 \end{aligned}$$

4.

$$\begin{aligned} 2x-4z &= 1 \\ 4x+3y-2z &= 0 \\ 2x+2z &= 0 \end{aligned}$$

5.

$$\begin{aligned} x+2y+z &= 1 \\ 2x+3y+2z &= 0 \\ x+y+z &= 2 \end{aligned}$$

6.

$$\begin{aligned} 2x-4z &= 1 \\ 4x+3y-2z &= 0 \\ 2x+2z &= 2 \end{aligned}$$

Question 31. Show that the matrices

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

Question 32. Show that the matrices

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad B = \begin{pmatrix} d & 0 \\ e & f \end{pmatrix}.$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$