Tutorial 3

Matrix inverse and solving linear systems

Question 1. Confirm the validity of the given statements for the following matrices and scalars.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 1 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix}, \quad \alpha = 4, \quad \beta = -7$$

1. $(A+B)+C = A+(B+C)$
2. $(AB)C = A(BC)$
3. $(\alpha+\beta)C = \alpha C + \beta C$
4. $\alpha(BC) = (\alpha B)C = B(\alpha C)$
5. $A(B-C) = AB - AC$
6. $(B+C)A = BA + CA$
7. $\alpha(\beta C) = (\alpha\beta)C$
8. $(A^\top)^\top = A$
9. $(AB)^\top = B^\top A^\top$
10. $(A+B)^\top = A^\top + B^\top$
11. $(\alpha C)^\top = \alpha C^\top$
12. $A(B+C) = AB + AC$

Solution.

1.

$$(A+B)+C = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 0 & -2 \end{pmatrix}, \quad A+(B+C) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 0 & -2 \end{pmatrix}$$

2.

$$(AB)C = \begin{pmatrix} 1 & 2 \\ 4 & -12 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 52 & 28 \end{pmatrix}, \quad A(BC) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -6 & -4 \\ 16 & 9 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 52 & 28 \end{pmatrix}$$

3.

$$(\alpha+\beta)C = -3 \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -12 & -3 \\ 9 & 6 \end{pmatrix}, \quad \alpha C + \beta C = \begin{pmatrix} 16 & 4 \\ -12 & -8 \end{pmatrix} + \begin{pmatrix} -28 & -7 \\ 21 & 14 \end{pmatrix} = \begin{pmatrix} -12 & -3 \\ 9 & 6 \end{pmatrix}$$

4.

$$\alpha(BC) = 4 \begin{pmatrix} -6 & -4 \\ 16 & 9 \end{pmatrix} = \begin{pmatrix} -24 & -16 \\ 64 & 36 \end{pmatrix}, \quad (\alpha B)C = \begin{pmatrix} 0 & 8 \\ 4 & -16 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -24 & -16 \\ 64 & 36 \end{pmatrix}$$

$$B(\alpha C) = \begin{pmatrix} 0 & 2 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 16 & 4 \\ -12 & -8 \end{pmatrix} = \begin{pmatrix} -24 & -16 \\ 64 & 36 \end{pmatrix}$$

5.

$$A(B-C) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -8 & 1 \\ 8 & -6 \end{pmatrix}, \quad AB-AC = \begin{pmatrix} 1 & 2 \\ 4 & -12 \end{pmatrix} - \begin{pmatrix} 9 & 1 \\ -4 & -6 \end{pmatrix} \begin{pmatrix} -8 & 1 \\ 8 & -6 \end{pmatrix}$$

6.

$$(B+C)A = \begin{pmatrix} 4 & 3 \\ -2 & -6 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 18 & 16 \\ -18 & -26 \end{pmatrix}$$

$$BA + CA = \begin{pmatrix} 4 & 8 \\ -5 & -15 \end{pmatrix} + \begin{pmatrix} 14 & 8 \\ -13 & -11 \end{pmatrix} = \begin{pmatrix} 18 & 16 \\ -18 & -26 \end{pmatrix}$$

7.

$$\alpha(\beta C) = 4\begin{pmatrix} -28 & -7 \\ 21 & 14 \end{pmatrix} = \begin{pmatrix} -112 & -28 \\ 84 & 56 \end{pmatrix}, \quad (\alpha\beta)C = -28\begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -112 & -28 \\ 84 & 56 \end{pmatrix}$$

8.

$$A^\top = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad (A^\top)^\top = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = A$$

9.

$$(AB)^\top = \begin{pmatrix} 1 & 2 \\ 4 & -12 \end{pmatrix}^\top = \begin{pmatrix} 1 & 4 \\ 2 & -12 \end{pmatrix}, \quad B^\top A^\top = \begin{pmatrix} 0 & 1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -12 \end{pmatrix}$$

10.

$$(A+B)^\top = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}^\top = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}, \quad A^\top + B^\top = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}$$

11.

$$(\alpha C)^\top = \begin{pmatrix} 16 & 4 \\ -12 & -8 \end{pmatrix}^\top = \begin{pmatrix} 16 & -12 \\ 4 & -8 \end{pmatrix}, \quad \alpha C^\top = 4\begin{pmatrix} 4 & -3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 16 & -12 \\ 4 & -8 \end{pmatrix}$$

12.

$$A(B+C) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ -2 & -6 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 0 & -18 \end{pmatrix}, \; AB+AC = \begin{pmatrix} 1 & 2 \\ 4 & -12 \end{pmatrix} + \begin{pmatrix} 9 & 1 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 0 & -18 \end{pmatrix}$$

Question 2. Find the inverse of the given matrix using the formula for computing the inverse of a 2×2 matrix.

1. $A = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$
2. $B = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$
3. $C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$
4. $D = \begin{pmatrix} 6 & 4 \\ -2 & -1 \end{pmatrix}$

Solution.

1. $A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix}$
2. $B^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$
3. $C^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$
4. $D = \begin{pmatrix} -\frac{1}{2} & -2 \\ 1 & 3 \end{pmatrix}$

Question 3. Find the inverse of the given matrix

1. $\begin{pmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{pmatrix}$

2. $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

Solution.

1. The determinant of the matrix is

$$\frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}(2e^x e^{-x} + 2e^x e^{-x}) = 1.$$

   The inverse of the matrix is

$$\begin{pmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^{-x} - e^x) \\ \frac{1}{2}(e^{-x} - e^x) & \frac{1}{2}(e^x + e^{-x}) \end{pmatrix}$$

2. The determinant of the matrix is

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

   The inverse of the matrix is

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Question 4. Verify whether the given equalities hold for the following matrices.

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

1. $(A^\top)^{-1} = (A^{-1})^\top$
2. $(A^{-1})^{-1} = A$
3. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
4. $(ABC)^\top = C^\top B^\top A^\top$

Solution.

1.

$$A^\top = \begin{pmatrix} 2 & 4 \\ -3 & 4 \end{pmatrix}, \quad (A^\top)^{-1} = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix}, \quad (A^{-1})^\top = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{pmatrix} = (A^\top)^{-1}$$

2.

$$A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix}, \quad (A^{-1})^{-1} = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix} = A$$

3.

$$ABC = \begin{pmatrix} -18 & -12 \\ 64 & 36 \end{pmatrix}, \quad (ABC)^{-1} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{8}{15} & -\frac{3}{20} \end{pmatrix}$$

$$C^{-1}B^{-1}A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{8}{15} & -\frac{3}{20} \end{pmatrix}$$

4.

$$(ABC)^\top = \begin{pmatrix} -18 & 64 \\ -12 & 36 \end{pmatrix}, \quad C^\top B^\top A^\top = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -18 & 64 \\ -12 & 36 \end{pmatrix}$$

Question 5. Find the matrix A with the given information

1. $(7A)^{-1} = \begin{pmatrix} -3 & 7 \\ 1 & -2 \end{pmatrix}$
2. $(5A^\top)^{-1} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$
3. $(I + 2A)^{-1} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$
4. $A^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$

For each matrix A, compute:

(a) A^3

(b) A^{-3}

(c) $A^2 - 2A + I$

Solution.

1. From the given condition we have

$$7A = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix} \Longrightarrow A = \frac{1}{7} \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$$

Then

$$A^3 = \frac{1}{7^3} \begin{pmatrix} 57 & 182 \\ 26 & 83 \end{pmatrix} = \frac{1}{343} \begin{pmatrix} 57 & 182 \\ 26 & 83 \end{pmatrix}$$

$$A^{-3} = 7^3 \begin{pmatrix} -3 & 7 \\ 1 & -2 \end{pmatrix}^3 = 343 \begin{pmatrix} -83 & 182 \\ 26 & -57 \end{pmatrix}$$

$$A^2 - 2A + I = \frac{1}{49} \begin{pmatrix} 32 & -63 \\ -9 & 23 \end{pmatrix}$$

2. From the given condition we have

$$5A^\top = \begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix} \Longrightarrow A = \frac{1}{5} \begin{pmatrix} -2 & 5 \\ -1 & 3 \end{pmatrix}$$

$$A^3 = \frac{1}{125} \begin{pmatrix} -3 & 10 \\ -2 & 7 \end{pmatrix}, \quad A^{-3} = 125 \begin{pmatrix} -7 & 10 \\ -2 & 3 \end{pmatrix}, \quad A^2 - 2A - I = \frac{1}{25} \begin{pmatrix} 44 & -45 \\ 9 & -1 \end{pmatrix}$$

3. From the given condition we have

$$I + 2A = \frac{1}{13}\begin{pmatrix} -5 & 2 \\ 4 & 1 \end{pmatrix} \Longrightarrow A = \frac{1}{13}\begin{pmatrix} -9 & 1 \\ 2 & -6 \end{pmatrix}$$

Then

$$A^3 = \frac{1}{13^3}\begin{pmatrix} -777 & 173 \\ 346 & -258 \end{pmatrix}, \quad A^{-3} = \frac{1}{64}\begin{pmatrix} -258 & -173 \\ -346 & -777 \end{pmatrix}, \quad A^2-2A-I = \frac{1}{169}\begin{pmatrix} 486 & -41 \\ -82 & 363 \end{pmatrix}$$

4. We have

$$A = \frac{1}{13}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}$$

Then

$$A^3 = \frac{1}{13^3}\begin{pmatrix} 89 & 36 \\ -108 & -19 \end{pmatrix}, \quad A^{-3} = \begin{pmatrix} -19 & -36 \\ 108 & 89 \end{pmatrix}, \quad A^2 - 2A + I = \frac{1}{169}\begin{pmatrix} 61 & -19 \\ 57 & 118 \end{pmatrix}$$

Question 6. Compute $p(A)$ for the given matrix A and the following polynomials.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$$

(a) $p(x) = x - 2$

(b) $p(x) = 2x^2 - x + 1$

(c) $p(x) = x^3 - 2x + 1$

Solution.

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 20 & 7 \\ 14 & 6 \end{pmatrix}, \quad \begin{pmatrix} 36 & 13 \\ 26 & 10 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}, \quad \begin{pmatrix} 7 & 0 \\ 20 & 2 \end{pmatrix}, \quad \begin{pmatrix} 5 & 0 \\ 20 & 0 \end{pmatrix}$$

Question 7. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

1. Find all values of a, b, c, d (if any) for which the matrices A and B commute.
2. Find all values of a, b, c, d (if any) for which the matrices A and C commute.

1. $AB = BC$ gives

$$\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix} \Longrightarrow a = b, \quad c = 0$$

2. $AC = CA$ gives

$$\begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \Longrightarrow b = 0, \quad d = a$$

Question 8. If a polynomial $p(x)$ can be factored as a product of lower degree polynomials, say

$$p(x) = p_1(x)p_2(x)$$

and if A is a square matrix, then it can be proved that

$$p(A) = p_1(A)p_2(A).$$

Let

$$p(x) = x^2 - 9, \quad p_1(x) = x + 3, \quad p_2(x) = x - 3$$

1. Verify this statement for the above polynomials and the following matrices

$$A_1 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$$

2. Prove that for any square matrix A, $p(A) = p_1(A)p_2(A)$

Solution.

1.

$$p(A_1) = \begin{pmatrix} 2 & 4 \\ 8 & -6 \end{pmatrix}, \ p_1(A_1) = \begin{pmatrix} 6 & 1 \\ 2 & 4 \end{pmatrix}, \ p_2(A_1) = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix},$$

$$p_1(A_1)p_2(A_1) = \begin{pmatrix} 2 & 4 \\ 8 & -6 \end{pmatrix} = p(A_1)$$

$$p(A_2) = \begin{pmatrix} -5 & 0 \\ 12 & -8 \end{pmatrix}, \ p_1(A_2) = \begin{pmatrix} 5 & 0 \\ 4 & 4 \end{pmatrix}, \ p_2(A_2) = \begin{pmatrix} -1 & 0 \\ 4 & -2 \end{pmatrix},$$

$$p_1(A_2)p_2(A_2) = \begin{pmatrix} -5 & 0 \\ 12 & -8 \end{pmatrix} = p(A_2)$$

2. Take any square matrix A, we have

$$p_1(A)p_2(A) = (A + 3I)(A - 3I) = A^2 - A(3I) + (3I)A - (3I)(3I)$$

Since $AI = IA = A$, $I^2 = I$, and by associative law of scalar and matrix multiplication, we have

$$p_1(A)p_2(A) = A^2 - 3A + 3A - 9I = A^2 - 9I.$$

Question 9.

1. Give an example of two 2×2 matrices such that

$$(A + B)(A - B) \neq A^2 - B^2$$

2. Find a valid formula for multiplying out $(A + B)(A - B)$.

3. Establish the conditions on matrices A and B, under which the equality

$$(A + B)(A - B) = A^2 - B^2$$

is valid.

4. Find a valid formula for $(A+B)^3$.
5. Find a valid formula for $(A-B)^3$.

Solution.

1.
$$A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$(A+B)(A-B) = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 0 & 16 \end{pmatrix}, \quad A^2 - B^2 = \begin{pmatrix} 1 & 10 \\ 0 & 16 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 10 \\ 0 & 16 \end{pmatrix}$$

2.
$$(A+B)(A-B) = A^2 - AB + BA - B^2$$

3. We have
$$(A+B)(A-B) = A^2 - B^2 \iff A^2 - AB + BA - B^2 = A^2 - B^2 \iff AB = BA.$$
Thus, $(A+B)(A-B) = A^2 - B^2$ is true if and only if A and B commute.

4.
$$(A+B)^3 = A^3 + A^2B + ABA + BA^2 + AB^2 + BAB + B^2A + B^3.$$

5.
$$(A-B)^3 = A^3 - A^2B - ABA - BA^2 + AB^2 + BAB + B^2A - B^3.$$

Question 10. Show that if a square matrix A satisfies the equation
$$A^2 + 2A + I = 0$$
then A must be invertible. Find the inverse of A in this case.

Solution.
$$A^2 + 2A + I = 0 \Longrightarrow -(A^2 + 2A) = I \Longrightarrow A(-A - 2I) = I$$
Similarly we have $(-A - 2I)A = I$. Thus A is invertible and $A^{-1} = -A - 2I$.

Question 11. Show that if $p(x)$ is a polynomial with a nonzero constant term, and if A is a square matrix for which $p(A) = 0$, then A is invertible.

Solution. Suppose
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$
where $a_0 \neq 0$ (i.e., the constant term is nonzero).

Substituting A into the polynomial we get
$$p(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0,$$
which gives
$$a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A = -a_0 I.$$
Factor A from the left:
$$A\left(a_n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I\right) = -a_0 I.$$

Since $a_0 \neq 0$, we can divide both sides by $-a_0$:

$$A\left(-\frac{1}{a_0}\left(a_nA^{n-1}+a_{n-1}A^{n-2}+\cdots+a_1I\right)\right)=I.$$

Let

$$B=-\frac{1}{a_0}\left(a_nA^{n-1}+a_{n-1}A^{n-2}+\cdots+a_1I\right).$$

Then we obtain:

$$AB=I.$$

Similarly, we can factor A from the right to show:

$$BA=I.$$

Thus A is invertible with $A^{-1}=B$.

Question 12. Is it possible for $A^3=I$ without A being invertible? Why?

Solution. Let

$$p(x)=x^3-1.$$

$A^3=I$ implies that $p(A)=0$. Since the constant term of $p(x)$ is $-1\neq 0$, from the last question, we know that A is invertible.

Question 13. Can a matrix with a row of zeros or a column of zeros have an inverse? Why?

Solution. Suppose that the ith row of $A \in \mathcal{M}_{n\times n}$ consists entirely of zeros, and assume that A is invertible. Let B denote the inverse of A, we have

$$AB=I.$$

Consider the ith row of the product AB– the ith row of AB is given by the linear combination of the rows of B, with the coefficients being the entries of the ith row of A. Explicitly, denoting the rows of B by $\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_n$, the ith row of AB is computed as:

$$0\cdot\boldsymbol{b}_1+0\cdot\boldsymbol{b}_2+\cdots+0\cdot\boldsymbol{b}_n=\mathbf{0}.$$

Thus, the ith row of AB consists entirely of zeros and implies that $AB\neq I$, a contradiction. Therefore, A is singular.

Question 14. Simplify the given expression assuming that A,B,C,D are invertible.

1. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$
2. $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

Solution.

1.

$$\begin{aligned}(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1} &= B^{-1}A^{-1}AC^{-1}CDD^{-1}=B^{-1}(A^{-1}A)(C^{-1}C)(DD^{-1})\\ &= B^{-1}.\end{aligned}$$

2.

$$\begin{aligned}(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1} &= CA^{-1}AC^{-1}CA^{-1}AD^{-1} = C(A^{-1}A)(C^{-1}C)(A^{-1}A)D^{-1} \\ &= CD^{-1}.\end{aligned}$$

Question 15. Assuming that all matrices are $n \times n$ and invertible, solve for D

1. $C^\top B^{-1}A^2BAC^{-1}DA^{-2}B^\top C^{-2} = C^\top$
2. $ABC^\top DBA^\top C = AB^\top$

Solution.

1.

$$\begin{aligned} D &= (C^\top B^{-1}A^2BAC^{-1})^{-1}C^\top(A^{-2}B^\top C^{-2})^{-1} \\ &= CA^{-1}B^{-1}A^{-2}B(C^\top)^{-1}C^\top C^2(B^\top)^{-1}A^2 \\ &= CA^{-1}B^{-1}A^{-2}BC^2(B^\top)^{-1}A^2 \end{aligned}$$

2.

$$\begin{aligned} D &= (ABC^\top)^{-1}AB^\top(BA^\top C)^{-1} \\ &= (C^\top)^{-1}B^{-1}A^{-1}AB^\top C^{-1}(A^\top)^{-1}B^{-1} \\ &= (C^\top)^{-1}B^{-1}B^\top C^{-1}(A^\top)^{-1}B^{-1} \end{aligned}$$

Question 16. Determine whether the given matrix is elementary

1. $\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$
2. $\begin{pmatrix} -5 & 1 \\ 1 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$
4. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
5. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix}$
7. $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
8. $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
9. $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Solution. Elementary matrices are 1, 3, 4, 5, 6 the corresponding elementary row operations are

1. $R_2 \to R_2 - 5R_1$
3. $R_2 \to \sqrt{3}R_2$
5. $R_1 \leftrightarrow R_2$
6. $R_2 \to R_2 + 9R_3$

Question 17. Find a row operation and the corresponding elementary matrix that will transform the given elementary matrix to the identity matrix.

1. $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} -7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

7. $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 0 & -\dfrac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Solution.

1. The row operation is: $R_1 \to R_1 + 3R_2$, the elementary matrix is: $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$
2. The row operation is: $R_1 \to -\frac{1}{7}R_1$, the elementary matrix is: $\begin{pmatrix} -\dfrac{1}{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3. The row operation is: $R_3 \to R_3 + 5R_1$, the elementary matrix is: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$
4. The row operation is: $R_3 \leftrightarrow R_1$, the elementary matrix is: $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
5. The row operation is: $R_2 \to R_2 - 3R_1$, the elementary matrix is: $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$
6. The row operation is: $R_3 \to \frac{1}{3}R_3$, the elementary matrix is: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{3} \end{pmatrix}$
7. The row operation is: $R_1 \leftrightarrow R_4$, the elementary matrix is: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
8. The row operation is: $R_1 \to \frac{1}{7}R_3 + R_1$, the elementary matrix is: $\begin{pmatrix} 1 & 0 & \dfrac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Question 18. In the following examples, an elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

1. $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{pmatrix}$

2. $E = \begin{pmatrix} -6 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{pmatrix}$

3. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix}$

4. $E = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix}$

5. $E = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

6. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Solution.

1. The row operation is $R_1 \leftrightarrow R_2$

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{pmatrix} = EA$$

2. The row operation is $R_1 \to -6R_1$

$$A \xrightarrow{R_1 \to -6R_1} \begin{pmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{pmatrix} = EA$$

3. The row operation is $R_3 \to -3R_2 + R_3$

$$A \xrightarrow{R_3 \to -3R_2 + R_3} \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{pmatrix} = EA$$

4. The row operation is $R_2 \to -4R_1 + R_2$

$$A \xrightarrow{R_2 \to -4R_1 + R_2} \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix} = EA$$

5. The row operation is $R_1 \to 4R_3 + R_1$

$$A \xrightarrow{R_1 \to 4R_3 + R_1} \begin{pmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = EA$$

6. The row operation is $R_2 \to 5R_2$

$$A \xrightarrow{R_2 \to 5R_2} \begin{pmatrix} 1 & 4 \\ 10 & 25 \\ 3 & 6 \end{pmatrix} = EA$$

Question 19. Use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{pmatrix},$$
$$F = \begin{pmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}$$

1. $EA = B$
2. $EA = C$
3. $EB = A$
4. $EC = A$
5. $EB = D$
6. $ED = B$
7. $EB = F$
8. $EF = B$

Solution.

1. To obtain B from A, we can perform a elementary row operation by changing row 1 and row 3, i.e $A \xrightarrow{R_1 \leftrightarrow R_3} B$, hence
$$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

2. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

3. $E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

4. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

5. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

7. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

8. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

Question 20. Determine the inverse matrix A^{-1} (if the inverse exists) for each of the given matrices using the matrix inversion algorithm. Identify the sequence of elementary matrices $E_1, E_2, \ldots$ associated with each row operation performed throughout the process.

1. $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$
2. $A = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & -5 \\ 3 & -16 \end{pmatrix}$
4. $A = \begin{pmatrix} 6 & 4 \\ -3 & -1 \end{pmatrix}$
5. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$
6. $A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$
7. $A = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & 9 \end{pmatrix}$
8. $A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{3}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

Solution.

1. We adjoin I_2 to the right side of A

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right).$$

Then we perform the elementary row operation $R_2 \to R_2 - 2R_1$

$$E_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right).$$

$R_2 \to -R_2$

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

$R_1 \to R_1 - 4R_2$

$$E_3 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, \quad \left(\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

We have

$$A^{-1} = \begin{pmatrix} -7 & 4 \\ 2 & -1 \end{pmatrix}$$

2. The matrix A is singular.

3. Inverse of A:
$$\begin{pmatrix} 16 & -5 \\ 3 & -1 \end{pmatrix}$$
Sequence of Elementary Matrices:
$$E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$

4. Inverse of A:
$$\begin{pmatrix} -\frac{1}{6} & -\frac{2}{3} \\ \frac{1}{2} & 1 \end{pmatrix}$$
Sequence of Elementary Matrices:
$$E_1 = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{pmatrix}$$

5. Inverse of A:
$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$
Sequence of Elementary Matrices:
$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$
$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & 0 & -9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

6. Inverse of A:
$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$
Sequence of Elementary Matrices:
$$E_1 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{5} & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$
$$E_8 = \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

7. Inverse of A:

$$\begin{pmatrix} -\frac{17}{90} & \frac{7}{36} & -\frac{19}{180} \\ \frac{11}{90} & \frac{5}{36} & \frac{7}{180} \\ -\frac{1}{9} & \frac{1}{18} & \frac{1}{18} \end{pmatrix}$$

Sequence of Elementary Matrices:

$$E_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{18} \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 1 & 0 & -\frac{19}{10} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{7}{10} \\ 0 & 0 & 1 \end{pmatrix}$$

8. The matrix A is singular.

Question 21. Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

1. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{pmatrix}$

4. $\begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{pmatrix}$

6. $\begin{pmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{pmatrix}$

7. $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{pmatrix}$

8. $\begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & -5 & 0 \\ 0 & -3 & 0 & 0 \\ 2 & 0 & 5 & -3 \end{pmatrix}$

Solution.

1. $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

2. $\begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

4. $\begin{pmatrix} \frac{17}{58} & \frac{19}{116} & -\frac{3}{116} \\ \frac{9}{58} & -\frac{7}{116} & -\frac{5}{116} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$

5. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{pmatrix}$

6. $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{pmatrix}$

7. $\begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{2} & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{pmatrix}$

8. $\begin{pmatrix} \frac{3}{10} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{5} & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 \end{pmatrix}$

Question 22. Find the inverse of the given matrix, where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$ and α are all nonzero constants.

1. $\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}$

2. $\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & \alpha_3 & 0 & 0 \\ \alpha_4 & 0 & 0 & 0 \end{pmatrix}$

4. $\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 1 & \alpha & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & \alpha \end{pmatrix}$

Solution.

1. $\begin{pmatrix} \frac{1}{\alpha_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_3} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha_4} \end{pmatrix}$

2.

$$\left(\begin{array}{cccc|cccc}\alpha & 1 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & \alpha & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) \xrightarrow[R_3\to\frac{1}{\alpha}R_3]{R_1\to\frac{1}{\alpha}R_1} \left(\begin{array}{cccc|cccc}1 & \dfrac{1}{\alpha} & 0 & 0 & \dfrac{1}{\alpha} & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & \dfrac{1}{\alpha} & 0 & 0 & \dfrac{1}{\alpha} & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) \xrightarrow[R_3\to R_3-\frac{1}{\alpha}R_4]{R_1\to R_1-\frac{1}{\alpha}R_2}$$

$$\left(\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & \dfrac{1}{\alpha} & -\dfrac{1}{\alpha} & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & \dfrac{1}{\alpha} & -\dfrac{1}{\alpha}\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right)$$

Thus the inverse of the matrix is $\begin{pmatrix}\dfrac{1}{\alpha} & -\dfrac{1}{\alpha} & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \dfrac{1}{\alpha} & -\dfrac{1}{\alpha}\\ 0 & 0 & 0 & 1\end{pmatrix}$.

3.

$$\left(\begin{array}{cccc|cccc}0 & 0 & 0 & \alpha_1 & 1 & 0 & 0 & 0\\ 0 & 0 & \alpha_2 & 0 & 0 & 1 & 0 & 0\\ 0 & \alpha_3 & 0 & 0 & 0 & 0 & 1 & 0\\ \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{array}\right) \xrightarrow[R_2\leftrightarrow R_3]{R_4\leftrightarrow R_1} \left(\begin{array}{cccc|cccc}\alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & \alpha_3 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & \alpha_2 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \alpha_1 & 1 & 0 & 0 & 0\end{array}\right)$$

$$\xrightarrow[R_3\to\frac{1}{\alpha_2}R_3,\ R_4\to\frac{1}{\alpha_1}R_4]{R_1\to\frac{1}{\alpha_4},\ R_2\to\frac{1}{\alpha_3}R_2} \left(\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{\alpha_4}\\ 0 & 1 & 0 & 0 & 0 & 0 & \dfrac{1}{\alpha_3} & 0\\ 0 & 0 & 1 & 0 & 0 & \dfrac{1}{\alpha_2} & 0 & 0\\ 0 & 0 & 0 & 1 & \dfrac{1}{\alpha_1} & 0 & 0 & 0\end{array}\right)$$

Thus the inverse of the matrix is $\begin{pmatrix}0 & 0 & 0 & \dfrac{1}{\alpha_4}\\ 0 & 0 & \dfrac{1}{\alpha_3} & 0\\ 0 & \dfrac{1}{\alpha_2} & 0 & 0\\ \dfrac{1}{\alpha_1} & 0 & 0 & 0\end{pmatrix}$.

4.

$$\left(\begin{array}{cccc|cccc}\alpha & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 1 & \alpha & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1\end{array}\right) \xrightarrow{R_1\to\frac{1}{\alpha}R_1} \left(\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & \dfrac{1}{\alpha} & 0 & 0 & 0\\ 1 & \alpha & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1\end{array}\right) \xrightarrow{R_2\to R_2-R_1}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & -\frac{1}{\alpha} & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{\alpha} R_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & 0 & \frac{1}{\alpha^2} & -\frac{1}{\alpha} & 1 & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{\alpha} R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_4 \to R_4 - R_3}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & \alpha & -\frac{1}{\alpha^3} & \frac{1}{\alpha^2} & -\frac{1}{\alpha} & 1 \end{array}\right) \xrightarrow{R_4 \to \frac{1}{\alpha} R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{\alpha^4} & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} \end{array}\right)$$

The inverse of the matrix is $\begin{pmatrix} \frac{1}{\alpha} & 0 & 0 & 0 \\ -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ -\frac{1}{\alpha^4} & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} \end{pmatrix}$

Question 23. Find all values of c, if any, for which the given matrix is invertible

1. $A = \begin{pmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix}$
2. $B = \begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix}$

Solution.

1. If $c = 0$, A contains a row of zeros and is singular. If $c \neq 0$, we apply row operations on A

$$\begin{pmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{c} R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & c-1 & c-1 \\ 0 & 0 & c-1 \end{pmatrix}$$

If $c = 1$, the reduced row echelon form of A becomes $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3$ and A is singular.

If $c \neq 0, 1$, we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & c-1 & c-1 \\ 0 & 0 & c-1 \end{pmatrix} \xrightarrow[R_3 \to \frac{1}{c-1} R_3]{R_2 \to \frac{1}{c-1} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 \to R_1 - R_2]{R_2 \to R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Thus, A is invertible if $c \neq 0, 1$.

2.

$$\begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix} \xrightarrow[R_1 \to R_2,\ R_2 \to R_3]{R_3 \to R_1} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ c & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - cR_1} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 1-c^2 & -c \end{pmatrix} \xrightarrow{R_3 \to R_3 - (1-c^2)R_2} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & -2c+c^3 \end{pmatrix}$$

If $-2c + c^3 = 0$, then $c = 0$ or $\pm\sqrt{2}$. In this case, the reduced row echelon form of B will contain a row of zeros, indicating that B is singular.
If $c \neq 0, \pm\sqrt{2}$

$$\begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & -2c+c^3 \end{pmatrix} \xrightarrow{R_3 \to \frac{1}{-2c+c^3} R_3} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - cR_3} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 \to R_1 - cR_2]{R_3 \to R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

In conclusion, B is invertible if $c \neq 0, \pm\sqrt{2}$

Question 24. Express the given matrix A and its inverse as products of elementary matrices

1. $A = \begin{pmatrix} -3 & 1 \\ 2 & 2 \end{pmatrix}$
2. $A = \begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$
4. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Solution.

1. We apply inversion algorithm to find inverse of A and record the elementary matrices corresponding to each row operation during the process

$$\left(\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to -\frac{1}{3}R_1} \left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{8}{3} & \frac{2}{3} & 1 \end{array}\right) \xrightarrow{R_2 \to \frac{3}{8}R_2}$$

$$\left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array}\right) \xrightarrow{R_1 \to R_1 + \frac{1}{3}R_2} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{12} & \frac{1}{8} \\ 0 & 1 & \frac{2}{7} & \frac{3}{7} \end{array}\right)$$

$$E_1 = \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{8} \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}$$

The inverse of each of the elementary matrices are

$$E_4^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{8}{3} \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$$

We have

$$E_4E_3E_2E_1A = I \Longrightarrow A = E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$$

And

$$A^{-1} = (E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1})^{-1} = E_1E_2E_3E_4.$$

2. $A^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix}$.

   A^{-1} can be expressed as a product of the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

   A can be expressed as a product of the following elementary matrices:

$$E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}.$$

3. $A^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$.

   A^{-1} can be expressed as a product of the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

   A can be expressed as a product of the following elementary matrices:

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. $A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

   A^{-1} can be expressed as a product of the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   A can be expressed as a product of the following elementary matrices:

$$E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 25. For each of the following pairs of matrices A and B, show that the matrices A and B are row equivalent by

(a) Find a sequence of elementary row operations that produce B from A.

(b) Find a sequence of elementary row operations that produce A from B.

(c) Use the results from 1 to find a matrix C such that $CA = B$.

(d) Use the results from 2 to find a matrix D such that $DB = A$.

1. $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$

Solution.

1. (a) We can use the following elementary row operations to transform A to B

$$A \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{pmatrix}$$

The three row operations correspond to the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

(b) By applying the inverse of each row operation, we can transform B back to A through the following sequence of row operations:

$$B \xrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{pmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{pmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{pmatrix}$$

The three row operations correspond to the following elementary matrices:

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$C = E_3 E_2 E_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

(d)

$$D = E_1^{-1}E_2^{-1}E_3^{-1} = C^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

2. (a) We can use the following elementary row operations to transform A to B

$$A \xrightarrow{R_2 \to R_2 - \frac{8}{9}R_3} \begin{pmatrix} 2 & 1 & 0 \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - \frac{4}{9}R_3} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 \to \frac{1}{9}R_3}$$

$$\begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ \frac{1}{3} & 0 & -\frac{1}{9} \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_1 \to 9R_1} B$$

The six row operations correspond to the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{8}{9} \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & -\frac{4}{9} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(b) By applying the inverse of each row operation, we can transform B back to A through the following sequence of row operations:

$$B \xrightarrow{R_1 \to \frac{1}{9}R_1} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 2R_1}$$

$$\begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ \frac{1}{3} & 0 & -\frac{1}{9} \end{pmatrix} \xrightarrow{R_3 \to 9R_3} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{R_1 \to R_1 + \frac{4}{9}R_3} \begin{pmatrix} 2 & 1 & 0 \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + \frac{8}{9}R_3} A$$

The six row operations correspond to the following elementary matrices:

$$E_6^{-1} = \begin{pmatrix} \frac{1}{9} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & \frac{4}{9} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{8}{9} \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$C = E_6E_5E_4E_3E_2E_1 = \begin{pmatrix} 9 & 0 & -4 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

(d)

$$D = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1} = C^{-1} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 8 \\ 2 & 0 & 9 \end{pmatrix}$$

Question 26. Show that

$$A = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$$

is singular for any values of a, b, c, d, e, f, g, h.

Solution. First, we note that if A an entire row of zeros, its reduced row echelon form cannot be the identity matrix, which implies that A is singular. We assume that $a \neq 0,\ h \neq 0$. Then we can perform the following row operations

$$A \xrightarrow{R_3 - \frac{d}{a}R_1} \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix} \xrightarrow{R_3 - \frac{e}{h}R_1} \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$$

We have reached a matrix with an entire row of zeros, and we can conclude that A is singular.

Question 27. Suppose that each of the following is the augmented matrix for a linear system. Solve the systems by by inverting the coefficient matrix (if the inverse existst) and apply the following theorem

Theorem 1 *Given $A \in \mathcal{M}_{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^n$, if A is invertible, then the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.*

1. $\left(\begin{array}{cc|c} 2 & 0 & -1 \\ 3 & 2 & 0 \end{array}\right)$

2. $\left(\begin{array}{cccc|c} 3 & 0 & 0 & 1 & -4 \\ 3 & 0 & 2 & 1 & 7 \\ -1 & 3 & 0 & -2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right)$

3. $\left(\begin{array}{ccc|c} 1 & 0 & 8 & 6 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 0 & 3 & 2 \\ 0 & 1 & 1 & -5 \end{array}\right)$

5. $\left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right)$

6. $\left(\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 13 & 13 \end{array}\right)$

7. $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 2 & 3 \end{array}\right)$

8. $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \end{array}\right)$

9. $\left(\begin{array}{ccc|c} 2 & 1 & -5 & -20 \\ 0 & 2 & 2 & 7 \\ 3 & 1 & -9 & -36 \end{array}\right)$

Solution.

1. The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{3}{4} & \frac{1}{2} \end{pmatrix}.$$

The unique solution

$$\boldsymbol{x} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{pmatrix}.$$

2. The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{5}{12} & -\frac{1}{12} & 0 & -\frac{1}{6} \\ -\frac{1}{36} & \frac{5}{36} & \frac{1}{3} & \frac{5}{18} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}.$$

The unique solution

$$\boldsymbol{x} = \begin{pmatrix} -\frac{29}{12} \\ \frac{97}{36} \\ \frac{11}{2} \\ \frac{13}{4} \end{pmatrix}.$$

3. The inverse of the coefficient matrix is

$$\begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

The unique solution

$$\boldsymbol{x} = \begin{pmatrix} -10 \\ -5 \\ 2 \end{pmatrix}$$

4. The inverse of the coefficient matrix is

$$\begin{pmatrix} 1 & \frac{7}{3} & 0 \\ 0 & -\frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$$

The unique solution

$$\boldsymbol{x} = \begin{pmatrix} \frac{38}{3} \\ -\frac{17}{3} \\ \frac{2}{3} \end{pmatrix}$$

5. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

and the system has no solutions.

6. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 4 & 0 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and the solution set is

$$\{\, (-4t - 13,\ t, -3) \mid t \in \mathbb{R} \,\}\,.$$

7. The inverse of the coefficient matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The unique solution

$$\boldsymbol{x} = \begin{pmatrix} -\frac{13}{2} \\ 1 \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$$

8. The inverse of the coefficient matrix is

$$\begin{pmatrix} -1 & 0 & -1 \\ 4 & 1 & 2 \\ -5 & -1 & -4 \end{pmatrix}$$

The unique solution

$$\boldsymbol{x} = \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$$

9. The inverse of the coefficient matrix is

$$\begin{pmatrix} 5 & -1 & -3 \\ -\frac{3}{2} & \frac{3}{4} & 1 \\ \frac{3}{2} & -\frac{1}{4} & -1 \end{pmatrix}$$

The unique solution

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ -\frac{3}{4} \\ \frac{17}{4} \end{pmatrix}$$

For Questions 28 – 30 do the following

Solve the given linear system by inverting the coefficient matrix (if it is invertible) and applying Theorem 1.

Question 28.

1.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

2.

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

3.

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z - 2w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

4.

$$\begin{aligned} -2y + 3z &= 1 \\ 3x + 6y - 3z &= -2 \\ 6x + 6y + 3z &= 5 \end{aligned}$$

5.

$$\begin{aligned} 5x_1 - 5x_2 - 15x_3 &= 40 \\ 4x_1 - 2x_2 - 6x_3 &= 19 \\ 3x_1 - 6x_2 - 17x_3 &= 41 \end{aligned}$$

6.

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

8.

$$\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right)$$

Inverse of the coefficient matrix is:

$$\begin{pmatrix} \frac{1}{4} & -\frac{9}{26} & \frac{7}{52} \\ \frac{1}{4} & -\frac{1}{26} & -\frac{5}{52} \\ \frac{1}{4} & \frac{5}{26} & -\frac{1}{52} \end{pmatrix}$$

Solution of the system is given by:

$$\boldsymbol{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

2. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right)$$

The coefficient matrix is singular. By Gauss–Jordan elimination, the reduced row echelon form of the matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution set is given by

$$\left\{ \left(-\frac{3t}{7} - \frac{1}{7},\ \frac{1}{7} - \frac{4t}{7},\ t\right) \;\middle|\; t \in \mathbb{R} \right\}$$

3. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The coefficient matrix is singular. The solution set is given by

$$\{\, (-1,\ 2t,\ t,\ 0) \;\mid\; t \in \mathbb{R} \,\}.$$

4. The coefficient matrix is singular. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The linear system has no solutions.

5. The augmented matrix is

$$\left(\begin{array}{ccc|c} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right)$$

Inverse of the coefficient matrix is:

$$\begin{pmatrix} -\frac{1}{5} & \frac{1}{2} & 0 \\ 5 & -4 & -3 \\ -\frac{9}{5} & \frac{3}{2} & 1 \end{pmatrix}$$

Solution of the system is given by:

$$\boldsymbol{x} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix}$$

6. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right)$$

Inverse of the coefficient matrix is:

$$\begin{pmatrix} 1 & -3 & -\frac{23}{4} \\ 0 & 1 & 2 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

The system only has the trivial solution

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

7. The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

Inverse of the coefficient matrix is:

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

The system only has the trivial solution

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

8. The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right)$$

Inverse of the coefficient matrix is:

$$\begin{pmatrix} \frac{11}{30} & \frac{1}{30} & \frac{3}{10} \\ \frac{1}{30} & \frac{11}{30} & \frac{3}{10} \\ -\frac{1}{10} & -\frac{1}{10} & \frac{1}{10} \end{pmatrix}$$

The system only has the trivial solution

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Question 29.

1.

$$
\begin{aligned}
-2x_1 + x_2 + 8x_3 &= 0\\
7x_1 - 2x_2 - 22x_3 &= 0\\
3x_1 - x_2 - 10x_3 &= 0
\end{aligned}
$$

2.

$$
\begin{aligned}
5x_1 - 2x_3 &= 0\\
-15x_1 - 16x_2 - 9x_3 &= 0\\
10x_1 + 12x_2 + 7x_3 &= 0
\end{aligned}
$$

3.

$$
\begin{aligned}
2x_1 - 3x_2 + 4x_3 - x_4 &= 0\\
7x_1 + x_2 - 8x_3 + 9x_4 &= 0\\
2x_1 + 8x_2 + x_3 - x_4 &= 0
\end{aligned}
$$

4.

$$
\begin{aligned}
v + 3w - 2x &= 0\\
2u + v - 4w + 3x &= 0\\
2u + 3v + 2w - x &= 0\\
-4u - 3v + 5w - 4x &= 0
\end{aligned}
$$

5.

$$
\begin{aligned}
2x + 2y + 4z &= 0\\
w - y - 3z &= 0\\
2w + 3x + y + z &= 0\\
-2w + x + 3y - 2z &= 0
\end{aligned}
$$

6.

$$
\begin{aligned}
2x_1 + 6x_2 + 13x_3 + x_4 &= 0\\
x_1 + 4x_2 + 10x_3 + x_4 &= 0\\
2x_1 + 8x_2 + 20x_3 + x_4 &= 0\\
3x_1 + 10x_2 + 21x_3 + 2x_4 &= 0
\end{aligned}
$$

7.

$$
\begin{aligned}
2x_1 - 6x_2 + 3x_3 - 21x_4 &= 0\\
4x_1 - 5x_2 + 2x_3 - 24x_4 &= 0\\
-x_1 + 3x_2 - x_3 + 10x_4 &= 0\\
-2x_1 + 3x_2 - x_3 + 13x_4 &= 0
\end{aligned}
$$

Solution.

1. The augmented matrix is

$$
\begin{pmatrix}
-2 & 1 & 8 & 0\\
7 & -2 & -22 & 0\\
3 & -1 & -10 & 0
\end{pmatrix}
$$

The coefficient matrix is singular. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$
\left(\begin{array}{ccc|c}
1 & 0 & -2 & 0\\
0 & 1 & 4 & 0\\
0 & 0 & 0 & 0
\end{array}\right)
$$

The solution set is given by

$$
\{ (2t,\ -4t,\ t) \ \mid\ t \in \mathbb{R} \}.
$$

2. The augmented matrix is
$$\begin{pmatrix} 5 & 0 & -2 & 0 \\ -15 & -16 & -9 & 0 \\ 10 & 12 & 7 & 0 \end{pmatrix}$$
The inverse of the coefficient matrix is
$$\begin{pmatrix} -\frac{1}{5} & -\frac{6}{5} & -\frac{8}{5} \\ \frac{3}{4} & \frac{11}{4} & \frac{15}{4} \\ -1 & -3 & -4 \end{pmatrix}$$
The solution of the system is
$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

3. The coefficient matrix is singular. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{46}{83} & 0 \\ 0 & 1 & 0 & -\frac{15}{83} & 0 \\ 0 & 0 & 1 & -\frac{55}{83} & 0 \end{array}\right)$$
The solution set is given by
$$\left\{ \left(-\frac{46t}{83}, \frac{15t}{83}, \frac{55t}{83}, t \right) \;\middle|\; t \in \mathbb{R} \right\}.$$

4. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is
$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
The solution set is given by
$$\left\{ \left(\frac{7s}{2} - \frac{5t}{2}, \; -3t + 2s, \; s, \; t \right) \;\middle|\; s, \; t \in \mathbb{R} \right\}.$$

5. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is
$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
The solution set is given by
$$\{ (t, \; -t, \; t, \; 0) \mid t \in \mathbb{R} \}.$$

6. The augmented matrix is

$$\begin{pmatrix} 2 & 6 & 13 & 1 & 0 \\ 1 & 4 & 10 & 1 & 0 \\ 2 & 8 & 20 & 1 & 0 \\ 3 & 10 & 21 & 2 & 0 \end{pmatrix}$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} 4 & 1 & -1 & -2 \\ -\frac{9}{4} & -\frac{7}{4} & \frac{1}{2} & \frac{7}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 2 & -1 & 0 \end{pmatrix}$$

The solution of the system is

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

7. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is given by

$$\{\, (3t,\ -2t,\ t,\ t) \ \mid\ t \in \mathbb{R} \,\}.$$

Question 30.

1.

$$\begin{aligned} x + y &= 1 \\ 4x + 3y &= 2 \end{aligned}$$

2.

$$\begin{aligned} -3x + y &= 1 \\ 4x + 2y &= 0 \end{aligned}$$

3.

$$\begin{aligned} 3x - 3y &= 3 \\ 4x - y - 3z &= 3 \\ -2x - 2y &= -2 \end{aligned}$$

4.

$$\begin{aligned} 2x - 4z &= 1 \\ 4x + 3y - 2z &= 0 \\ 2x + 2z &= 0 \end{aligned}$$

5.

$$\begin{aligned} x + 2y + z &= 1 \\ 2x + 3y + 2z &= 0 \\ x + y + z &= 2 \end{aligned}$$

6.

$$\begin{aligned} 2x - 4z &= 1 \\ 4x + 3y - 2z &= 0 \\ 2x + 2z &= 2 \end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 3 & 2 \end{array}\right)$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$$

The solution of the system is

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

2. The augmented matrix for the linear system is

$$\left(\begin{array}{cc|c} -3 & 1 & 1 \\ 4 & 2 & 0 \end{array}\right)$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} -\frac{1}{5} & \frac{1}{10} \\ \frac{2}{5} & \frac{3}{10} \end{pmatrix}$$

The solution of the system is

$$\boldsymbol{x} = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix}.$$

3. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 3 & -3 & 0 & 3 \\ 4 & -1 & -3 & 3 \\ -2 & -2 & 0 & -2 \end{array}\right)$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{6} & 0 & -\frac{1}{4} \\ -\frac{1}{6} & 0 & -\frac{1}{4} \\ \frac{5}{18} & -\frac{1}{3} & -\frac{1}{4} \end{pmatrix}$$

The solution of the system is

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{3} \end{pmatrix}.$$

4. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & 1 \\ 4 & 3 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right)$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{6} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix}$$

The solution of the system is

$$\boldsymbol{x} = \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ -\frac{1}{6} \end{pmatrix}$$

5. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The system is inconsistent.

6. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & 1 \\ 4 & 3 & -2 & 0 \\ 2 & 0 & 2 & 2 \end{array}\right)$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{6} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix}$$

The solution of the system is

$$\boldsymbol{x} = \begin{pmatrix} \frac{5}{6} \\ -1 \\ \frac{1}{6} \end{pmatrix}$$

Question 31. Show that the matrices

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

Solution. We have

$$AB = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}, \quad BA = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

Then

$$AB = BA \iff ae+bf = db+ec \iff b(d-f)-e(a-c)=0 \iff \begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

Question 32. Show that the matrices

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad B = \begin{pmatrix} d & 0 \\ e & f \end{pmatrix}.$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

Solution. We have

$$AB = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} ad & 0 \\ bd+ce & cf \end{pmatrix}, \quad BA = \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ad & 0 \\ ea+fb & cf \end{pmatrix}$$

Then

$$AB = BA \iff bd+ce = ea+fb \iff b(d-f)-e(a-c)=0 \iff \begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$