

Tutorial 3

Matrix inverse and solving linear systems

Homework

1. Provide detailed solutions to Q2 and Q6.
2. Provide detailed solutions to *three* of the following questions: Q5, Q7–Q11, Q14, Q15.
3. Provide detailed solutions to *one* of the following questions: Q16–Q18.
4. Provide detailed solutions to Q19.
5. Provide detailed solutions to *two* selected parts from Q20 (parts 2–8).
6. Provide detailed solutions to *two* selected parts from Q21.
7. Provide detailed solutions to *two* selected parts from Q24.
8. Provide detailed solutions to *two* selected parts chosen from: Q28, Q29, Q30.

Question 1. Confirm the validity of the given statements for the following matrices and scalars.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 1 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix}, \quad \alpha = 4, \quad \beta = -7$$

- | | |
|---|---|
| 1. $(A + B) + C = A + (B + C)$ | 2. $(AB)C = A(BC)$ |
| 3. $(\alpha + \beta)C = \alpha C + \beta C$ | 4. $\alpha(BC) = (\alpha B)C = B(\alpha C)$ |
| 5. $A(B - C) = AB - AC$ | 6. $(B + C)A = BA + CA$ |
| 7. $\alpha(\beta C) = (\alpha\beta)C$ | 8. $(A^\top)^\top = A$ |
| 9. $(AB)^\top = B^\top A^\top$ | 10. $(A + B)^\top = A^\top + B^\top$ |
| 11. $(\alpha C)^\top = \alpha C^\top$ | 12. $A(B + C) = AB + AC$ |

Solution.

1.

$$(A+B)+C = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 0 & -2 \end{pmatrix}, \quad A+(B+C) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 0 & -2 \end{pmatrix}$$

2.

$$(AB)C = \begin{pmatrix} 1 & 2 \\ 4 & -12 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 52 & 28 \end{pmatrix}, \quad A(BC) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -6 & -4 \\ 16 & 9 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 52 & 28 \end{pmatrix}$$

3.

$$(\alpha + \beta)C = -3 \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -12 & -3 \\ 9 & 6 \end{pmatrix}, \quad \alpha C + \beta C = \begin{pmatrix} 16 & 4 \\ -12 & -8 \end{pmatrix} + \begin{pmatrix} -28 & -7 \\ 21 & 14 \end{pmatrix} = \begin{pmatrix} -12 & -3 \\ 9 & 6 \end{pmatrix}$$

4.

$$\alpha(BC) = 4 \begin{pmatrix} -6 & -4 \\ 16 & 9 \end{pmatrix} = \begin{pmatrix} -24 & -16 \\ 64 & 36 \end{pmatrix}, \quad (\alpha B)C = \begin{pmatrix} 0 & 8 \\ 4 & -16 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -24 & -16 \\ 64 & 36 \end{pmatrix}$$

$$B(\alpha C) = \begin{pmatrix} 0 & 2 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 16 & 4 \\ -12 & -8 \end{pmatrix} = \begin{pmatrix} -24 & -16 \\ 64 & 36 \end{pmatrix}$$

5.

$$A(B-C) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -8 & 1 \\ 8 & -6 \end{pmatrix}, \quad AB-AC = \begin{pmatrix} 1 & 2 \\ 4 & -12 \end{pmatrix} - \begin{pmatrix} 9 & 1 \\ -4 & -6 \end{pmatrix} \begin{pmatrix} -8 & 1 \\ 8 & -6 \end{pmatrix}$$

6.

$$(B+C)A = \begin{pmatrix} 4 & 3 \\ -2 & -6 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 18 & 16 \\ -18 & -26 \end{pmatrix}$$

$$BA+CA = \begin{pmatrix} 4 & 8 \\ -5 & -15 \end{pmatrix} + \begin{pmatrix} 14 & 8 \\ -13 & -11 \end{pmatrix} = \begin{pmatrix} 18 & 16 \\ -18 & -26 \end{pmatrix}$$

7.

$$\alpha(\beta C) = 4 \begin{pmatrix} -28 & -7 \\ 21 & 14 \end{pmatrix} = \begin{pmatrix} -112 & -28 \\ 84 & 56 \end{pmatrix}, \quad (\alpha\beta)C = -28 \begin{pmatrix} 4 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -112 & -28 \\ 84 & 56 \end{pmatrix}$$

8.

$$A^\top = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad (A^\top)^\top = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = A$$

9.

$$(AB)^\top = \begin{pmatrix} 1 & 2 \\ 4 & -12 \end{pmatrix}^\top = \begin{pmatrix} 1 & 4 \\ 2 & -12 \end{pmatrix}, \quad B^\top A^\top = \begin{pmatrix} 0 & 1 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -12 \end{pmatrix}$$

10.

$$(A+B)^\top = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}^\top = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}, \quad A^\top + B^\top = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}$$

11.

$$(\alpha C)^\top = \begin{pmatrix} 16 & 4 \\ -12 & -8 \end{pmatrix}^\top = \begin{pmatrix} 16 & -12 \\ 4 & -8 \end{pmatrix}, \quad \alpha C^\top = 4 \begin{pmatrix} 4 & -3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 16 & -12 \\ 4 & -8 \end{pmatrix}$$

12.

$$A(B+C) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -2 & -6 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 0 & -18 \end{pmatrix}, \quad AB+AC = \begin{pmatrix} 1 & 2 \\ 4 & -12 \end{pmatrix} + \begin{pmatrix} 9 & 1 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 0 & -18 \end{pmatrix}$$

Question 2. Find the inverse of the given matrix using the formula for computing the inverse of a 2×2 matrix.

1. $A = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$

2. $B = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

3. $C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

4. $D = \begin{pmatrix} 6 & 4 \\ -2 & -1 \end{pmatrix}$

Solution.

$$1. A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix}$$

$$2. B^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$3. C^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$4. D^{-1} = \begin{pmatrix} -\frac{1}{2} & -2 \\ 1 & 3 \end{pmatrix}$$

Question 3. Find the inverse of the given matrix

$$1. \begin{pmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{pmatrix}$$

$$2. \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Solution.

1. The determinant of the matrix is

$$\frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}(2e^x e^{-x} + 2e^x e^{-x}) = 1.$$

The inverse of the matrix is

$$\begin{pmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^{-x} - e^x) \\ \frac{1}{2}(e^{-x} - e^x) & \frac{1}{2}(e^x + e^{-x}) \end{pmatrix}$$

2. The determinant of the matrix is

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

The inverse of the matrix is

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Question 4. Verify whether the given equalities hold for the following matrices.

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$1. (A^\top)^{-1} = (A^{-1})^\top$$

$$2. (A^{-1})^{-1} = A$$

$$3. (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$4. (ABC)^\top = C^\top B^\top A^\top$$

Solution.

1.

$$A^T = \begin{pmatrix} 2 & 4 \\ -3 & 4 \end{pmatrix}, \quad (A^T)^{-1} = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix}, \quad (A^{-1})^T = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{pmatrix} = (A^T)^{-1}$$

2.

$$A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix}, \quad (A^{-1})^{-1} = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix} = A$$

3.

$$ABC = \begin{pmatrix} -18 & -12 \\ 64 & 36 \end{pmatrix}, \quad (ABC)^{-1} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{8}{15} & -\frac{3}{20} \end{pmatrix}$$

$$C^{-1}B^{-1}A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{8}{15} & -\frac{3}{20} \end{pmatrix}$$

4.

$$(ABC)^T = \begin{pmatrix} -18 & 64 \\ -12 & 36 \end{pmatrix}, \quad C^T B^T A^T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -18 & 64 \\ -12 & 36 \end{pmatrix}$$

Question 5. Find the matrix A with the given information

$$\begin{array}{ll} 1. (7A)^{-1} = \begin{pmatrix} -3 & 7 \\ 1 & -2 \end{pmatrix} & 2. (5A^T)^{-1} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix} \\ 3. (I + 2A)^{-1} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} & 4. A^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} \end{array}$$

For each matrix A , compute:

- (a) A^3
- (b) A^{-3}
- (c) $A^2 - 2A + I$

Solution.

1. From the given condition we have

$$7A = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix} \implies A = \frac{1}{7} \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$$

Then

$$\begin{aligned} A^3 &= \frac{1}{7^3} \begin{pmatrix} 57 & 182 \\ 26 & 83 \end{pmatrix} = \frac{1}{343} \begin{pmatrix} 57 & 182 \\ 26 & 83 \end{pmatrix} \\ A^{-3} &= 7^3 \begin{pmatrix} -3 & 7 \\ 1 & -2 \end{pmatrix}^3 = 343 \begin{pmatrix} -83 & 182 \\ 26 & -57 \end{pmatrix} \\ A^2 - 2A + I &= \frac{1}{49} \begin{pmatrix} 32 & -63 \\ -9 & 23 \end{pmatrix} \end{aligned}$$

2. From the given condition we have

$$\begin{aligned} 5A^\top &= \begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix} \Rightarrow A = \frac{1}{5} \begin{pmatrix} -2 & 5 \\ -1 & 3 \end{pmatrix} \\ A^3 &= \frac{1}{125} \begin{pmatrix} -3 & 10 \\ -2 & 7 \end{pmatrix}, \quad A^{-3} = 125 \begin{pmatrix} -7 & 10 \\ -2 & 3 \end{pmatrix}, \quad A^2 - 2A - I = \frac{1}{25} \begin{pmatrix} 44 & -45 \\ 9 & -1 \end{pmatrix} \end{aligned}$$

3. From the given condition we have

$$I + 2A = \frac{1}{13} \begin{pmatrix} -5 & 2 \\ 4 & 1 \end{pmatrix} \Rightarrow A = \frac{1}{13} \begin{pmatrix} -9 & 1 \\ 2 & -6 \end{pmatrix}$$

Then

$$A^3 = \frac{1}{13^3} \begin{pmatrix} -777 & 173 \\ 346 & -258 \end{pmatrix}, \quad A^{-3} = \frac{1}{64} \begin{pmatrix} -258 & -173 \\ -346 & -777 \end{pmatrix}, \quad A^2 - 2A - I = \frac{1}{169} \begin{pmatrix} 486 & -41 \\ -82 & 363 \end{pmatrix}$$

4. We have

$$A = \frac{1}{13} \begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}$$

Then

$$A^3 = \frac{1}{13^3} \begin{pmatrix} 89 & 36 \\ -108 & -19 \end{pmatrix}, \quad A^{-3} = \begin{pmatrix} -19 & -36 \\ 108 & 89 \end{pmatrix}, \quad A^2 - 2A + I = \frac{1}{169} \begin{pmatrix} 61 & -19 \\ 57 & 118 \end{pmatrix}$$

Question 6. Compute $p(A)$ for the given matrix A and the following polynomials.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$$

(a) $p(x) = x - 2$

(b) $p(x) = 2x^2 - x + 1$

(c) $p(x) = x^3 - 2x + 1$

Solution.

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 20 & 7 \\ 14 & 6 \end{pmatrix}, \quad \begin{pmatrix} 36 & 13 \\ 26 & 10 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}, \quad \begin{pmatrix} 7 & 0 \\ 20 & 2 \end{pmatrix}, \quad \begin{pmatrix} 5 & 0 \\ 20 & 0 \end{pmatrix}$$

Question 7. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

1. Find all values of a, b, c, d (if any) for which the matrices A and B commute.
2. Find all values of a, b, c, d (if any) for which the matrices A and C commute.
1. $AB = BC$ gives

$$\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix} \implies a = b, \quad c = 0$$

2. $AC = CA$ gives

$$\begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \implies b = 0, \quad d = a$$

Question 8. If a polynomial $p(x)$ can be factored as a product of lower degree polynomials, say

$$p(x) = p_1(x)p_2(x)$$

and if A is a square matrix, then it can be proved that

$$p(A) = p_1(A)p_2(A).$$

Let

$$p(x) = x^2 - 9, \quad p_1(x) = x + 3, \quad p_2(x) = x - 3$$

1. Verify this statement for the above polynomials and the following matrices

$$A_1 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$$

2. Prove that for any square matrix A , $p(A) = p_1(A)p_2(A)$

Solution.

- 1.

$$p(A_1) = \begin{pmatrix} 2 & 4 \\ 8 & -6 \end{pmatrix}, \quad p_1(A_1) = \begin{pmatrix} 6 & 1 \\ 2 & 4 \end{pmatrix}, \quad p_2(A_1) = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix},$$

$$p_1(A_1)p_2(A_1) = \begin{pmatrix} 2 & 4 \\ 8 & -6 \end{pmatrix} = p(A_1)$$

$$p(A_2) = \begin{pmatrix} -5 & 0 \\ 12 & -8 \end{pmatrix}, \quad p_1(A_2) = \begin{pmatrix} 5 & 0 \\ 4 & 4 \end{pmatrix}, \quad p_2(A_2) = \begin{pmatrix} -1 & 0 \\ 4 & -2 \end{pmatrix},$$

$$p_1(A_2)p_2(A_2) = \begin{pmatrix} -5 & 0 \\ 12 & -8 \end{pmatrix} = p(A_2)$$

2. Take any square matrix A , we have

$$p_1(A)p_2(A) = (A + 3I)(A - 3I) = A^2 - A(3I) + (3I)A - (3I)(3I)$$

Since $AI = IA = A$, $I^2 = I$, and by associative law of scalar and matrix multiplication, we have

$$p_1(A)p_2(A) = A^2 - 3A + 3A - 9I = A^2 - 9I.$$

Question 9.

1. Give an example of two 2×2 matrices such that

$$(A + B)(A - B) \neq A^2 - B^2$$

2. Find a valid formula for multiplying out $(A + B)(A - B)$.
3. Establish the conditions on matrices A and B , under which the equality

$$(A + B)(A - B) = A^2 - B^2$$

is valid.

4. Find a valid formula for $(A + B)^3$.
5. Find a valid formula for $(A - B)^3$.

Solution.

- 1.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A+B)(A-B) = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 0 & 16 \end{pmatrix}, \quad A^2 - B^2 = \begin{pmatrix} 1 & 10 \\ 0 & 16 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 10 \\ 0 & 16 \end{pmatrix}$$

- 2.

$$(A + B)(A - B) = A^2 - AB + BA - B^2$$

3. We have

$$(A + B)(A - B) = A^2 - B^2 \iff A^2 - AB + BA - B^2 = A^2 - B^2 \iff AB = BA.$$

Thus, $(A + B)(A - B) = A^2 - B^2$ is true if and only if A and B commute.

- 4.

$$(A + B)^3 = A^3 + A^2B + ABA + BA^2 + AB^2 + BAB + B^2A + B^3.$$

- 5.

$$(A - B)^3 = A^3 - A^2B - ABA - BA^2 + AB^2 + BAB + B^2A - B^3.$$

Question 10. Show that if a square matrix A satisfies the equation

$$A^2 + 2A + I = 0$$

then A must be invertible. Find the inverse of A in this case.

Solution.

$$A^2 + 2A + I = 0 \implies -(A^2 + 2A) = I \implies A(-A - 2I) = I$$

Similarly we have $(-A - 2I)A = I$. Thus A is invertible and $A^{-1} = -A - 2I$.

Question 11. Show that if $p(x)$ is a polynomial with a nonzero constant term, and if A is a square matrix for which $p(A) = 0$, then A is invertible.

Solution. Suppose

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0 \neq 0$ (i.e., the constant term is nonzero).

Substituting A into the polynomial we get

$$p(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0,$$

which gives

$$a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A = -a_0 I.$$

Factor A from the left:

$$A (a_n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I) = -a_0 I.$$

Since $a_0 \neq 0$, we can divide both sides by $-a_0$:

$$A \left(-\frac{1}{a_0} (a_n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I) \right) = I.$$

Let

$$B = -\frac{1}{a_0} (a_n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I).$$

Then we obtain:

$$AB = I.$$

Similarly, we can factor A from the right to show:

$$BA = I.$$

Thus A is invertible with $A^{-1} = B$.

Question 12. Is it possible for $A^3 = I$ without A being invertible? Why?

Solution. Let

$$p(x) = x^3 - 1.$$

$A^3 = I$ implies that $p(A) = 0$. Since the constant term of $p(x)$ is $-1 \neq 0$, from the last question, we know that A is invertible.

Question 13. Can a matrix with a row of zeros or a column of zeros have an inverse? Why?

Solution. Suppose that the i th row of $A \in \mathcal{M}_{n \times n}$ consists entirely of zeros, and assume that A is invertible. Let B denote the inverse of A , we have

$$AB = I.$$

Consider the i th row of the product AB —the i th row of AB is given by the linear combination of the rows of B , with the coefficients being the entries of the i th row of A . Explicitly, denoting the rows of B by $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$, the i th row of AB is computed as:

$$0 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 + \cdots + 0 \cdot \mathbf{b}_n = \mathbf{0}.$$

Thus, the i th row of AB consists entirely of zeros and implies that $AB \neq I$, a contradiction. Therefore, A is singular.

Question 14. Simplify the given expression assuming that A, B, C, D are invertible.

1. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$
2. $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

Solution.

1.

$$\begin{aligned}(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1} &= B^{-1}A^{-1}AC^{-1}CDD^{-1} = B^{-1}(A^{-1}A)(C^{-1}C)(DD^{-1}) \\ &= B^{-1}.\end{aligned}$$

2.

$$\begin{aligned}(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1} &= CA^{-1}AC^{-1}CA^{-1}AD^{-1} = C(A^{-1}A)(C^{-1}C)(A^{-1}A)D^{-1} \\ &= CD^{-1}.\end{aligned}$$

Question 15. Assuming that all matrices are $n \times n$ and invertible, solve for D

1. $C^{\top}B^{-1}A^2BAC^{-1}DA^{-2}B^{\top}C^{-2} = C^{\top}$
2. $ABC^{\top}DBA^{\top}C = AB^{\top}$

Solution.

1.

$$\begin{aligned}D &= (C^{\top}B^{-1}A^2BAC^{-1})^{-1}C^{\top}(A^{-2}B^{\top}C^{-2})^{-1} \\ &= CA^{-1}B^{-1}A^{-2}B(C^{\top})^{-1}C^{\top}C^2(B^{\top})^{-1}A^2 \\ &= CA^{-1}B^{-1}A^{-2}BC^2(B^{\top})^{-1}A^2\end{aligned}$$

2.

$$\begin{aligned}D &= (ABC^{\top})^{-1}AB^{\top}(BA^{\top}C)^{-1} \\ &= (C^{\top})^{-1}B^{-1}A^{-1}AB^{\top}C^{-1}(A^{\top})^{-1}B^{-1} \\ &= (C^{\top})^{-1}B^{-1}B^{\top}C^{-1}(A^{\top})^{-1}B^{-1}\end{aligned}$$

Question 16. Determine whether the given matrix is elementary

1. $\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$

2. $\begin{pmatrix} -5 & 1 \\ 1 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix}$

7. $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

8. $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

9. $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Solution. Elementary matrices are 1, 3, 4, 5, 6 the corresponding elementary row operations are

1. $R_2 \rightarrow R_2 - 5R_1$

3. $R_2 \rightarrow \sqrt{3}R_2$

5. $R_1 \leftrightarrow R_2$

6. $R_2 \rightarrow R_2 + 9R_3$

Question 17. Find a row operation and the corresponding elementary matrix that will transform the given elementary matrix to the identity matrix.

1. $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$	2. $\begin{pmatrix} -7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	5. $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$	6. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
7. $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	8. $\begin{pmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	

Solution.

1. The row operation is: $R_1 \rightarrow R_1 + 3R_2$, the elementary matrix is: $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

2. The row operation is: $R_1 \rightarrow -\frac{1}{7}R_1$, the elementary matrix is: $\begin{pmatrix} -\frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. The row operation is: $R_3 \rightarrow R_3 + 5R_1$, the elementary matrix is: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$

4. The row operation is: $R_3 \leftrightarrow R_1$, the elementary matrix is: $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

5. The row operation is: $R_2 \rightarrow R_2 - 3R_1$, the elementary matrix is: $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

6. The row operation is: $R_3 \rightarrow \frac{1}{3}R_3$, the elementary matrix is: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$

7. The row operation is: $R_1 \leftrightarrow R_4$, the elementary matrix is:
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

8. The row operation is: $R_1 \rightarrow \frac{1}{7}R_3 + R_1$, the elementary matrix is:
$$\begin{pmatrix} 1 & 0 & \frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Question 18. In the following examples, an elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A .

1. $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{pmatrix}$

2. $E = \begin{pmatrix} -6 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{pmatrix}$

3. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix}$

4. $E = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix}$

5. $E = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

6. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Solution.

1. The row operation is $R_1 \leftrightarrow R_2$

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{pmatrix} = EA$$

2. The row operation is $R_1 \rightarrow -6R_1$

$$A \xrightarrow{R_1 \rightarrow -6R_1} \begin{pmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{pmatrix} = EA$$

3. The row operation is $R_3 \rightarrow -3R_2 + R_3$

$$A \xrightarrow{R_3 \rightarrow -3R_2 + R_3} \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{pmatrix} = EA$$

4. The row operation is $R_2 \rightarrow -4R_1 + R_2$

$$A \xrightarrow{R_2 \rightarrow -4R_1 + R_2} \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix} = EA$$

5. The row operation is $R_1 \rightarrow 4R_3 + R_1$

$$A \xrightarrow{R_1 \rightarrow 4R_3 + R_1} \begin{pmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = EA$$

6. The row operation is $R_2 \rightarrow 5R_2$

$$A \xrightarrow{R_2 \rightarrow 5R_2} \begin{pmatrix} 1 & 4 \\ 10 & 25 \\ 3 & 6 \end{pmatrix} = EA$$

Question 19. Use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{pmatrix},$$

$$F = \begin{pmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}$$

1. $EA = B$
3. $EB = A$
5. $EB = D$
7. $EB = F$

2. $EA = C$
4. $EC = A$
6. $ED = B$
8. $EF = B$

Solution.

1. To obtain B from A , we can perform a elementary row operation by changing row 1 and row 3, i.e $A \xrightarrow{R_1 \leftrightarrow R_3} B$, hence

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$2. E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$3. E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$4. E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$5. E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$6. E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$7. E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$8. E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 20. Determine the inverse matrix A^{-1} (if the inverse exists) for each of the given matrices using the matrix inversion algorithm. Identify the sequence of elementary matrices E_1, E_2, \dots associated with each row operation performed throughout the process.

$$1. A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & -5 \\ 3 & -16 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 6 & 4 \\ -3 & -1 \end{pmatrix}$$

$$5. A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

$$6. A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$$

$$7. A = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & 9 \end{pmatrix}$$

$$8. A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{3}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$$

Solution.

1. We adjoin I_2 to the right side of A

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array} \right).$$

Then we perform the elementary row operation $R_2 \rightarrow R_2 - 2R_1$

$$E_1 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -2 & 1 & -2 & 1 \end{array} \right).$$

$R_2 \rightarrow -R_2$

$$E_2 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 \end{array} \right).$$

$R_1 \rightarrow R_1 - 4R_2$

$$E_3 = \left(\begin{array}{cc|cc} 1 & -4 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array} \right).$$

We have

$$A^{-1} = \begin{pmatrix} -7 & 4 \\ 2 & -1 \end{pmatrix}$$

2. The matrix A is singular.

3. Inverse of A :

$$\begin{pmatrix} 16 & -5 \\ 3 & -1 \end{pmatrix}$$

Sequence of Elementary Matrices:

$$E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$

4. Inverse of A :

$$\begin{pmatrix} -\frac{1}{6} & -\frac{2}{3} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Sequence of Elementary Matrices:

$$E_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{6} & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{pmatrix}$$

5. Inverse of A :

$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Sequence of Elementary Matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & 0 & -9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

6. Inverse of A :

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

Sequence of Elementary Matrices:

$$E_1 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{5} & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$E_8 = \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

7. Inverse of A :

$$\begin{pmatrix} -\frac{17}{90} & \frac{7}{36} & -\frac{19}{180} \\ \frac{11}{90} & \frac{5}{36} & \frac{7}{180} \\ -\frac{1}{9} & \frac{1}{18} & \frac{1}{18} \end{pmatrix}$$

Sequence of Elementary Matrices:

$$E_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{18} \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 1 & 0 & -\frac{19}{10} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{7}{10} \\ 0 & 0 & 1 \end{pmatrix}$$

8. The matrix A is singular.

Question 21. Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

$$1. \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$2. \begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$3. \begin{pmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{pmatrix}$$

$$4. \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}$$

$$5. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{pmatrix}$$

$$6. \begin{pmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{pmatrix}$$

$$7. \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{pmatrix}$$

$$8. \begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & -5 & 0 \\ 0 & -3 & 0 & 0 \\ 2 & 0 & 5 & -3 \end{pmatrix}$$

Solution.

$$1. \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$2. \begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$3. \begin{pmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$4. \begin{pmatrix} \frac{17}{58} & \frac{19}{116} & -\frac{3}{116} \\ \frac{9}{58} & -\frac{7}{116} & -\frac{5}{116} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$5. \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{pmatrix}$$

$$6. \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{pmatrix}$$

$$7. \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{2} & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{pmatrix}$$

$$8. \begin{pmatrix} \frac{3}{10} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{5} & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 \end{pmatrix}$$

Question 22. Find the inverse of the given matrix, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α are all nonzero constants.

$$1. \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}$$

$$3. \begin{pmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & \alpha_3 & 0 & 0 \\ \alpha_4 & 0 & 0 & 0 \end{pmatrix}$$

$$2. \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$4. \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 1 & \alpha & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & \alpha \end{pmatrix}$$

Solution.

$$1. \begin{pmatrix} \frac{1}{\alpha_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_3} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha_4} \end{pmatrix}$$

2.

$$\left(\begin{array}{cccc|cccc} \alpha & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow[R_3 \rightarrow \frac{1}{\alpha} R_3]{R_1 \rightarrow \frac{1}{\alpha} R_1} \left(\begin{array}{cccc|cccc} 1 & \frac{1}{\alpha} & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{\alpha} & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow[R_3 \rightarrow R_3 - \frac{1}{\alpha} R_4]{R_1 \rightarrow R_1 - \frac{1}{\alpha} R_2}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & -\frac{1}{\alpha} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{\alpha} & -\frac{1}{\alpha} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

Thus the inverse of the matrix is $\begin{pmatrix} \frac{1}{\alpha} & -\frac{1}{\alpha} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & -\frac{1}{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

3.

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & \alpha_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & \alpha_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow[R_2 \leftrightarrow R_3]{R_4 \leftrightarrow R_1} \left(\begin{array}{cccc|cccc} \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \alpha_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow[R_3 \rightarrow \frac{1}{\alpha_2} R_3, R_4 \rightarrow \frac{1}{\alpha_1} R_4]{R_1 \rightarrow \frac{1}{\alpha_4}, R_2 \rightarrow \frac{1}{\alpha_3} R_2}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{\alpha_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{\alpha_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{\alpha_1} & 0 & 0 & 0 \end{array} \right)$$

Thus the inverse of the matrix is

$$\left(\begin{array}{cccc} 0 & 0 & 0 & \frac{1}{\alpha_4} \\ 0 & 0 & \frac{1}{\alpha_3} & 0 \\ 0 & \frac{1}{\alpha_2} & 0 & 0 \\ \frac{1}{\alpha_1} & 0 & 0 & 0 \end{array} \right).$$

4.

$$\begin{aligned} & \left(\begin{array}{cccc|cccc} \alpha & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \alpha & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1 \rightarrow \frac{1}{\alpha} R_1} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 1 & \alpha & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 - R_1} \\ & \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & -\frac{1}{\alpha} & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_2 \rightarrow \frac{1}{\alpha} R_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 - R_2} \\ & \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & 0 & \frac{1}{\alpha^2} & -\frac{1}{\alpha} & 1 & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_3 \rightarrow \frac{1}{\alpha} R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_4 \rightarrow R_4 - R_3} \\ & \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & \alpha & -\frac{1}{\alpha^3} & \frac{1}{\alpha^2} & -\frac{1}{\alpha} & 1 \end{array} \right) \xrightarrow{R_4 \rightarrow \frac{1}{\alpha} R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{\alpha^4} & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} \end{array} \right) \end{aligned}$$

The inverse of the matrix is

$$\begin{pmatrix} \frac{1}{\alpha} & 0 & 0 & 0 \\ -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ -\frac{1}{\alpha^4} & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} \end{pmatrix}$$

Question 23. Find all values of c , if any, for which the given matrix is invertible

$$1. A = \begin{pmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix} \quad 2. B = \begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix}$$

Solution.

1. If $c = 0$, A contains a row of zeros and is singular. If $c \neq 0$, we apply row operations on A

$$\begin{pmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{c}R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & c-1 & c-1 \\ 0 & 0 & c-1 \end{pmatrix}$$

If $c = 1$, the reduced row echelon form of A becomes $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3$ and A is singular.

If $c \neq 0, 1$, we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & c-1 & c-1 \\ 0 & 0 & c-1 \end{pmatrix} \xrightarrow[R_3 \rightarrow \frac{1}{c-1}R_3]{R_2 \rightarrow \frac{1}{c-1}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 \rightarrow R_1 - R_2]{R_2 \rightarrow R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Thus, A is invertible if $c \neq 0, 1$.

2.

$$\begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix} \xrightarrow[R_1 \rightarrow R_2, R_2 \rightarrow R_3]{R_3 \rightarrow R_1} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ c & 1 & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - cR_1} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 1 - c^2 & -c \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - (1 - c^2)R_2} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & -2c + c^3 \end{pmatrix}$$

If $-2c + c^3 = 0$, then $c = 0$ or $\pm\sqrt{2}$. In this case, the reduced row echelon form of B will contain a row of zeros, indicating that B is singular.

If $c \neq 0, \pm\sqrt{2}$

$$\begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & -2c + c^3 \end{pmatrix} \xrightarrow{R_3 \rightarrow \frac{1}{-2c + c^3}R_3} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - cR_3} \begin{pmatrix} 1 & c & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 \rightarrow R_1 - cR_2]{R_3 \rightarrow R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

In conclusion, B is invertible if $c \neq 0, \pm\sqrt{2}$

Question 24. Express the given matrix A and its inverse as products of elementary matrices

$$1. A = \begin{pmatrix} -3 & 1 \\ 2 & 2 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix}$$

$$3. A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$4. A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution.

1. We apply inversion algorithm to find inverse of A and record the elementary matrices corresponding to each row operation during the process

$$\begin{aligned} \left(\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array} \right) &\xrightarrow{R_1 \rightarrow -\frac{1}{3}R_1} \left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 2 & 2 & 0 & 1 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{8}{3} & \frac{2}{3} & 1 \end{array} \right) \xrightarrow{R_2 \rightarrow \frac{3}{8}R_2} \\ &\left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array} \right) \xrightarrow{R_1 \rightarrow R_1 + \frac{1}{3}R_2} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{12} & \frac{1}{8} \\ 0 & 1 & \frac{2}{7} & \frac{3}{7} \end{array} \right) \\ E_1 &= \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{8} \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

The inverse of each of the elementary matrices are

$$E_4^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{8}{3} \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$$

We have

$$E_4 E_3 E_2 E_1 A = I \implies A = E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1}$$

And

$$A^{-1} = (E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1})^{-1} = E_1 E_2 E_3 E_4.$$

$$2. A^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix}.$$

A^{-1} can be expressed as a product of the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

A can be expressed as a product of the following elementary matrices:

$$E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}.$$

$$3. A^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

A^{-1} can be expressed as a product of the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

A can be expressed as a product of the following elementary matrices:

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$4. A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

A^{-1} can be expressed as a product of the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A can be expressed as a product of the following elementary matrices:

$$E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 25. For each of the following pairs of matrices A and B , show that the matrices A and B are row equivalent by

- Find a sequence of elementary row operations that produce B from A .
- Find a sequence of elementary row operations that produce A from B .
- Use the results from 1 to find a matrix C such that $CA = B$.
- Use the results from 2 to find a matrix D such that $DB = A$.

$$1. A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{pmatrix}$$

$$2. A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

Solution.

1. (a) We can use the following elementary row operations to transform A to B

$$A \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{pmatrix}$$

The three row operations correspond to the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

- (b) By applying the inverse of each row operation, we can transform B back to A through the following sequence of row operations:

$$B \xrightarrow{R_3 \rightarrow R_3 + R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{pmatrix}$$

The three row operations correspond to the following elementary matrices:

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$C = E_3 E_2 E_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

(d)

$$D = E_1^{-1} E_2^{-1} E_3^{-1} = C^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

2. (a) We can use the following elementary row operations to transform A to B

$$A \xrightarrow{R_2 \rightarrow R_2 - \frac{8}{9}R_3} \begin{pmatrix} 2 & 1 & 0 \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 - \frac{4}{9}R_3} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 \rightarrow \frac{1}{9}R_3} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 1 & 0 & -\frac{1}{9} \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 1 & 0 & -\frac{1}{9} \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_1 \rightarrow 9R_1} B$$

The six row operations correspond to the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{8}{9} \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & -\frac{4}{9} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

- (b) By applying the inverse of each row operation, we can transform B back to A through the following sequence of row operations:

$$B \xrightarrow{R_1 \rightarrow \frac{1}{9}R_1} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 + 2R_1} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_1}$$

$$\begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ \frac{1}{3} & 0 & -\frac{1}{9} \end{pmatrix} \xrightarrow{R_3 \rightarrow 9R_3} \begin{pmatrix} \frac{2}{3} & 1 & \frac{4}{9} \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + \frac{4}{9}R_3} \begin{pmatrix} 2 & 1 & 0 \\ -\frac{11}{3} & 1 & \frac{8}{9} \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 + \frac{8}{9}R_3} A$$

The six row operations correspond to the following elementary matrices:

$$E_6^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{9} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & \frac{4}{9} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{8}{9} \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$C = E_6 E_5 E_4 E_3 E_2 E_1 = \begin{pmatrix} 9 & 0 & -4 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

(d)

$$D = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} = C^{-1} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 8 \\ 2 & 0 & 9 \end{pmatrix}$$

Question 26. Show that

$$A = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$$

is singular for any values of a, b, c, d, e, f, g, h .

Solution. First, we note that if A an entire row of zeros, its reduced row echelon form cannot be the identity matrix, which implies that A is singular. We assume that $a \neq 0$, $h \neq 0$. Then we can perform the following row operations

$$A \xrightarrow{R_3 - \frac{d}{a}R_1} \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix} \xrightarrow{R_3 - \frac{e}{h}R_1} \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$$

We have reached a matrix with an entire row of zeros, and we can conclude that A is singular.

Question 27. Suppose that each of the following is the augmented matrix for a linear system. Solve the systems by by inverting the coefficient matrix (if the inverse exists) and apply the following theorem

Theorem 1 Given $A \in \mathcal{M}_{n \times n}$, $\mathbf{b} \in \mathbb{R}^n$, if A is invertible, then the system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

1. $\left(\begin{array}{cc|c} 2 & 0 & -1 \\ 3 & 2 & 0 \end{array} \right)$
2. $\left(\begin{array}{cccc|c} 3 & 0 & 0 & 1 & -4 \\ 3 & 0 & 2 & 1 & 7 \\ -1 & 3 & 0 & -2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{array} \right)$
3. $\left(\begin{array}{ccc|c} 1 & 0 & 8 & 6 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right)$
4. $\left(\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 0 & 3 & 2 \\ 0 & 1 & 1 & -5 \end{array} \right)$
5. $\left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{array} \right)$
6. $\left(\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 13 & 13 \end{array} \right)$
7. $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 2 & 3 \end{array} \right)$
8. $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \end{array} \right)$
9. $\left(\begin{array}{ccc|c} 2 & 1 & -5 & -20 \\ 0 & 2 & 2 & 7 \\ 3 & 1 & -9 & -36 \end{array} \right)$

Solution.

1. The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{3}{4} & \frac{1}{2} \end{pmatrix}.$$

The unique solution

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{pmatrix}.$$

2. The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{5}{12} & -\frac{1}{12} & 0 & -\frac{1}{6} \\ -\frac{1}{36} & \frac{5}{36} & \frac{1}{3} & \frac{5}{18} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}.$$

The unique solution

$$\mathbf{x} = \begin{pmatrix} -\frac{29}{12} \\ \frac{97}{36} \\ \frac{11}{2} \\ \frac{13}{4} \end{pmatrix}.$$

3. The inverse of the coefficient matrix is

$$\begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

The unique solution

$$\mathbf{x} = \begin{pmatrix} -10 \\ -5 \\ 2 \end{pmatrix}$$

4. The inverse of the coefficient matrix is

$$\begin{pmatrix} 1 & \frac{7}{3} & 0 \\ 0 & -\frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$$

The unique solution

$$\mathbf{x} = \begin{pmatrix} \frac{38}{3} \\ -\frac{17}{3} \\ \frac{2}{3} \end{pmatrix}$$

5. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

and the system has no solutions.

6. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 4 & 0 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

and the solution set is

$$\{ (-4t - 13, t, -3) \mid t \in \mathbb{R} \}.$$

7. The inverse of the coefficient matrix is

$$\left(\begin{array}{cccc} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{array} \right)$$

The unique solution

$$\mathbf{x} = \begin{pmatrix} \frac{13}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$$

8. The inverse of the coefficient matrix is

$$\left(\begin{array}{ccc} -1 & 0 & -1 \\ 4 & 1 & 2 \\ -5 & -1 & -4 \end{array} \right)$$

The unique solution

$$\mathbf{x} = \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$$

9. The inverse of the coefficient matrix is

$$\left(\begin{array}{ccc} 5 & -1 & -3 \\ -\frac{3}{2} & \frac{3}{4} & 1 \\ \frac{3}{2} & -\frac{1}{4} & -1 \end{array} \right)$$

The unique solution

$$\mathbf{x} = \begin{pmatrix} 1 \\ -\frac{3}{4} \\ \frac{17}{4} \end{pmatrix}$$

For Questions 28 – 30 do the following

Solve the given linear system by inverting the coefficient matrix (if it is invertible) and applying Theorem 1.

Question 28.

1.

$$\begin{aligned}x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10\end{aligned}$$

2.

$$\begin{aligned}2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1\end{aligned}$$

3.

$$\begin{aligned}x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z - 2w &= 1 \\ 3x - 3w &= -3\end{aligned}$$

4.

$$\begin{aligned}-2y + 3z &= 1 \\ 3x + 6y - 3z &= -2 \\ 6x + 6y + 3z &= 5\end{aligned}$$

5.

$$\begin{aligned}5x_1 - 5x_2 - 15x_3 &= 40 \\ 4x_1 - 2x_2 - 6x_3 &= 19 \\ 3x_1 - 6x_2 - 17x_3 &= 41\end{aligned}$$

6.

$$\begin{aligned}x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0\end{aligned}$$

7.

$$\begin{aligned}2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0\end{aligned}$$

8.

$$\begin{aligned}2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0\end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right)$$

Inverse of the coefficient matrix is:

$$\left(\begin{array}{ccc} \frac{1}{4} & -\frac{9}{26} & \frac{7}{52} \\ \frac{1}{4} & -\frac{1}{26} & -\frac{5}{52} \\ \frac{1}{4} & \frac{5}{26} & -\frac{1}{52} \end{array} \right)$$

Solution of the system is given by:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

2. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array} \right)$$

The coefficient matrix is singular. By Gauss–Jordan elimination, the reduced row echelon form of the matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array} \right).$$

The solution set is given by

$$\left\{ \left(-\frac{3t}{7} - \frac{1}{7}, \frac{1}{7} - \frac{4t}{7}, t \right) \mid t \in \mathbb{R} \right\}$$

3. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The coefficient matrix is singular. The solution set is given by

$$\{ (-1, 2t, t, 0) \mid t \in \mathbb{R} \}.$$

4. The coefficient matrix is singular. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

The linear system has no solutions.

5. The augmented matrix is

$$\left(\begin{array}{ccc|c} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array} \right)$$

Inverse of the coefficient matrix is:

$$\left(\begin{array}{ccc} -\frac{1}{5} & \frac{1}{2} & 0 \\ 5 & -4 & -3 \\ -\frac{9}{5} & \frac{3}{2} & 1 \end{array} \right)$$

Solution of the system is given by:

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 5 \\ -\frac{5}{2} \end{pmatrix}$$

6. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right)$$

Inverse of the coefficient matrix is:

$$\begin{pmatrix} 1 & -3 & -\frac{23}{4} \\ 0 & 1 & 2 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

The system only has the trivial solution

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

7. The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

Inverse of the coefficient matrix is:

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

The system only has the trivial solution

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

8. The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{array} \right)$$

Inverse of the coefficient matrix is:

$$\begin{pmatrix} \frac{11}{30} & \frac{1}{30} & \frac{3}{10} \\ \frac{1}{30} & \frac{11}{30} & \frac{3}{10} \\ -\frac{1}{10} & -\frac{1}{10} & \frac{1}{10} \end{pmatrix}$$

The system only has the trivial solution

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Question 29.

1.

$$\begin{aligned} -2x_1 + x_2 + 8x_3 &= 0 \\ 7x_1 - 2x_2 - 22x_3 &= 0 \\ 3x_1 - x_2 - 10x_3 &= 0 \end{aligned}$$

2.

$$\begin{aligned} 5x_1 - 2x_3 &= 0 \\ -15x_1 - 16x_2 - 9x_3 &= 0 \\ 10x_1 + 12x_2 + 7x_3 &= 0 \end{aligned}$$

3.

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$$

4.

$$\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

5.

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

6.

$$\begin{aligned} 2x_1 + 6x_2 + 13x_3 + x_4 &= 0 \\ x_1 + 4x_2 + 10x_3 + x_4 &= 0 \\ 2x_1 + 8x_2 + 20x_3 + x_4 &= 0 \\ 3x_1 + 10x_2 + 21x_3 + 2x_4 &= 0 \end{aligned}$$

7.

$$\begin{aligned} 2x_1 - 6x_2 + 3x_3 - 21x_4 &= 0 \\ 4x_1 - 5x_2 + 2x_3 - 24x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 10x_4 &= 0 \\ -2x_1 + 3x_2 - x_3 + 13x_4 &= 0 \end{aligned}$$

Solution.

1. The augmented matrix is

$$\begin{pmatrix} -2 & 1 & 8 & 0 \\ 7 & -2 & -22 & 0 \\ 3 & -1 & -10 & 0 \end{pmatrix}$$

The coefficient matrix is singular. By Gauss–Jordan elimination, the reduced row echelon

form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The solution set is given by

$$\{ (2t, -4t, t) \mid t \in \mathbb{R} \}.$$

2. The augmented matrix is

$$\left(\begin{array}{cccc} 5 & 0 & -2 & 0 \\ -15 & -16 & -9 & 0 \\ 10 & 12 & 7 & 0 \end{array} \right)$$

The inverse of the coefficient matrix is

$$\left(\begin{array}{ccc} \frac{1}{5} & \frac{6}{5} & \frac{8}{5} \\ -\frac{3}{4} & \frac{11}{4} & \frac{15}{4} \\ -1 & -3 & -4 \end{array} \right)$$

The solution of the system is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

3. The coefficient matrix is singular. By Gauss–Jordan elimination, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{46}{83} & 0 \\ 0 & 1 & 0 & -\frac{15}{83} & 0 \\ 0 & 0 & 1 & -\frac{55}{83} & 0 \end{array} \right)$$

The solution set is given by

$$\left\{ \left(-\frac{46t}{83}, \frac{15t}{83}, \frac{55t}{83}, t \right) \mid t \in \mathbb{R} \right\}.$$

4. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The solution set is given by

$$\left\{ \left(\frac{7s}{2} - \frac{5t}{2}, -3t + 2s, s, t \right) \mid s, t \in \mathbb{R} \right\}.$$

5. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The solution set is given by

$$\{ (t, -t, t, 0) \mid t \in \mathbb{R} \}.$$

6. The augmented matrix is

$$\left(\begin{array}{ccccc} 2 & 6 & 13 & 1 & 0 \\ 1 & 4 & 10 & 1 & 0 \\ 2 & 8 & 20 & 1 & 0 \\ 3 & 10 & 21 & 2 & 0 \end{array} \right)$$

The inverse of the coefficient matrix is

$$\left(\begin{array}{cccc} 4 & 1 & -1 & -2 \\ 9 & 7 & 1 & 7 \\ -\frac{9}{4} & -\frac{7}{4} & \frac{1}{2} & \frac{7}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 2 & -1 & 0 \end{array} \right)$$

The solution of the system is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

7. The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The solution set is given by

$$\{ (3t, -2t, t, t) \mid t \in \mathbb{R} \}.$$

Question 30.

1.

$$\begin{aligned}x + y &= 1 \\4x + 3y &= 2\end{aligned}$$

2.

$$\begin{aligned}-3x + y &= 1 \\4x + 2y &= 0\end{aligned}$$

3.

$$\begin{aligned}3x - 3y &= 3 \\4x - y - 3z &= 3 \\-2x - 2y &= -2\end{aligned}$$

4.

$$\begin{aligned}2x - 4z &= 1 \\4x + 3y - 2z &= 0 \\2x + 2z &= 0\end{aligned}$$

5.

$$\begin{aligned}x + 2y + z &= 1 \\2x + 3y + 2z &= 0 \\x + y + z &= 2\end{aligned}$$

6.

$$\begin{aligned}2x - 4z &= 1 \\4x + 3y - 2z &= 0 \\2x + 2z &= 2\end{aligned}$$

Solution.

1. The augmented matrix for the linear system is

$$\left(\begin{array}{cc|c}1 & 1 & 1 \\4 & 3 & 2\end{array}\right)$$

The inverse of the coefficient matrix is

$$\left(\begin{array}{cc} -3 & 1 \\ 4 & -1 \end{array}\right)$$

The solution of the system is

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

2. The augmented matrix for the linear system is

$$\left(\begin{array}{cc|c}-3 & 1 & 1 \\4 & 2 & 0\end{array}\right)$$

The inverse of the coefficient matrix is

$$\left(\begin{array}{cc} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{array}\right)$$

The solution of the system is

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix}.$$

3. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 3 & -3 & 0 & 3 \\ 4 & -1 & -3 & 3 \\ -2 & -2 & 0 & -2 \end{array} \right)$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{6} & 0 & -\frac{1}{4} \\ -\frac{1}{6} & 0 & -\frac{1}{4} \\ \frac{5}{18} & -\frac{1}{3} & -\frac{1}{4} \end{pmatrix}$$

The solution of the system is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{3} \end{pmatrix}.$$

4. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & 1 \\ 4 & 3 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{array} \right)$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{6} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix}$$

The solution of the system is

$$\mathbf{x} = \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ -\frac{1}{6} \end{pmatrix}$$

5. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 1 & 1 & 2 \end{array} \right)$$

The coefficient matrix is singular. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

The system is inconsistent.

6. The augmented matrix for the linear system is

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & 1 \\ 4 & 3 & -2 & 0 \\ 2 & 0 & 2 & 2 \end{array}\right)$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{6} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix}$$

The solution of the system is

$$\mathbf{x} = \begin{pmatrix} \frac{5}{6} \\ -1 \\ \frac{1}{6} \end{pmatrix}$$

Question 31. Show that the matrices

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a - c \\ e & d - f \end{vmatrix} = 0$$

Solution. We have

$$AB = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ae + bf \\ 0 & cf \end{pmatrix}, \quad BA = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} da & db + ec \\ 0 & fc \end{pmatrix}$$

Then

$$AB = BA \iff ae + bf = db + ec \iff b(d - f) - e(a - c) = 0 \iff \begin{vmatrix} b & a - c \\ e & d - f \end{vmatrix} = 0$$

Question 32. Show that the matrices

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad B = \begin{pmatrix} d & 0 \\ e & f \end{pmatrix}.$$

commute if and only if

$$\begin{vmatrix} b & a - c \\ e & d - f \end{vmatrix} = 0$$

Solution. We have

$$AB = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} ad & 0 \\ bd + ce & cf \end{pmatrix}, \quad BA = \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ad & 0 \\ ea + fb & cf \end{pmatrix}$$

Then

$$AB = BA \iff bd + ce = ea + fb \iff b(d - f) - e(a - c) = 0 \iff \begin{vmatrix} b & a - c \\ e & d - f \end{vmatrix} = 0$$