Further Mathematics Notes - Core Pure

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2024

Contents

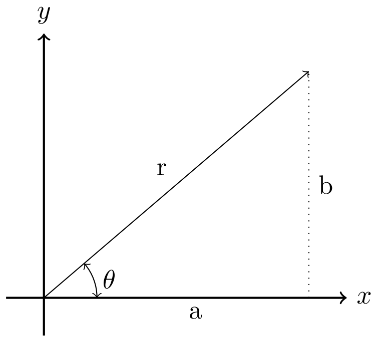
1 Proof

1.1 Proof by mathematical induction

1. Prove the general statement is true for $n = 1$
2. Assume the general statement is true for $n = k$
3. Show that the general statement is true for $n = k + 1$
4. The general statement is then true for all positive integers n

2 Complex numbers

2.1 Forms expressing complex number



2.1.1 Cartesian form

$z = a + bi$
$\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$
$|z| = \sqrt{a^2 + b^2}$

2.1.2 Polar form

$z = (r, \theta)$
$r = \sqrt{a^2 + b^2}$, $\theta = \arg(z)$
$a = r\cos\theta$, $b = r\sin\theta$, $\tan\theta = \dfrac{b}{a}$
$z = r\cos\theta + r\sin\theta i = r(\cos\theta + \sin\theta i) = r \operatorname{cis}\theta$

2.1.3 Exponential / Euler form

$z = re^{i\theta}$

2.1.4 When to use which form

Addition / subtraction: Cartesian form

Multiple / division / power / root: Polar / Euler form

2.2 Calculations

- $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) = r_1 r_2 e^{i((\theta_1+\theta_2))}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2) = \dfrac{r_1}{r_2}e^{i((\theta_1-\theta_2))}$
- $z^n = r^n \operatorname{cis}(n\theta) = r^n e^{i\theta n}$
- De Moivre's theorem: $\sqrt[n]{z} = \sqrt[n]{r}\operatorname{cis}(\dfrac{\theta + 2k\pi}{n}) = \sqrt[n]{r}e^{i(\dfrac{\theta+2k\pi}{n})}$ $(k = 0, 1, 2, \ldots, n-2, n-1)$
- $\sqrt{a+ib} = \pm(\sqrt{\dfrac{|z|+a}{2}} + i\dfrac{b}{|b|}\sqrt{\dfrac{|z|-a}{2}})$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg(\dfrac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$

⋆ Complex numbers' sizes cannot be compared

2.3 Conjugate

- $|z| = |z^*|$
- $z \cdot z^* = |z|^2$
- If $z = z*$ then z is a real number
- If $z = -z*$ then z is a pure imaginary number

2.4 Deducing multiple angle formulae by complex number

2.4.1 Properties for z

- $z = \operatorname{cis}\theta = e^{i\theta} = \cos\theta + i\sin\theta$, $\frac{1}{z} = e^{-i\theta} = \operatorname{cis}(-\theta)$
- $z + \frac{1}{z} = 2\cos\theta$
- $(z + \frac{1}{z})^n = (e^{i\theta} + e^{-i\theta})^n = 2^n \cos^n\theta$
- $z - \frac{1}{z} = 2i\sin\theta$
- $(z - \frac{1}{z})^n = (e^{i\theta} - e^{-i\theta})^n = 2^n i^n \sin^n\theta$
- $z^n + \frac{1}{z^n} = 2\cos(n\theta)$
- $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

2.4.2 Finding $\sin(n\theta)$ or $\cos(n\theta)$

1. Use $(z + \frac{1}{z})$ for cos and $(z + \frac{1}{z})$ for sin
2. Use the nth power of $z + \frac{1}{z}$ or $z - \frac{1}{z}$ (n is the same as coefficient)
3. Binomial expansion
4. Merge the terms into $z^n + \frac{1}{z^n}$ or $z^n - \frac{1}{z^n}$
5. Use properties above to convert then into trig functions

2.4.3 Find $\sin^n\theta$

n is odd:

- $n = 4k+1$: $\sin^n\theta = \frac{1}{2^{n-1}}[\binom{n}{0}\sin n\theta - \binom{n}{1}\sin(n-2)\theta + \binom{n}{1}\sin(n-4)\theta - \cdots + \binom{n}{\frac{n-1}{2}}\sin\theta]$
- $n = 4k+3$: $\sin^n\theta = \frac{1}{2^{n-1}}[-\binom{n}{0}\sin n\theta + \binom{n}{1}\sin(n-2)\theta - \binom{n}{1}\sin(n-4)\theta + \cdots - \binom{n}{\frac{n-1}{2}}\sin\theta]$

n is even:

- $n = 4k$: $\sin^n\theta = \frac{1}{2^{n-1}}[-\binom{n}{0}\cos n\theta + \binom{n}{1}\cos(n-2)\theta - \binom{n}{1}\cos(n-4)\theta + \cdots - \binom{n}{\frac{n-1}{2}}\cos\theta]$
- $n = 4k+2$: $\sin^n\theta = \frac{1}{2^{n-1}}[-\binom{n}{0}\cos n\theta + \binom{n}{1}\cos(n-2)\theta - \binom{n}{1}\cos(n-4)\theta + \cdots - \binom{n}{\frac{n-1}{2}}\cos\theta]$

2.4.4 Find $\cos^n\theta$

- $\cos^n\theta = \frac{1}{2^{n-1}}\sum$
- $\int \cos^n\theta \, d\theta = \frac{1}{2^{n-1}} \sum_{i=0}^{\lceil\frac{n-1}{2}\rceil} \binom{n}{i} \frac{\sin(n-2i)\theta}{den}$

2.5 Argand diagrams

2.5.1 Circle

- $|z| = a$: circle with centre O, radius a
- $|z - z_0| = a$: circle with centre z_0, radius a
- $|z - z_0| < a$: circle with centre z_0, radius a, shaded in
- $|z - z_0| > a$: circle with centre z_0, radius a, outside of circle shaded

2.5.2 Perpendicular bisector

- $|z - z_1| = |z - z_2|$: z on perpendicular bisector of $z_1 z_2$
- $|z - z_1| < |z - z_2|$ / $|z - z_1| > |z - z_2|$: on the left / right side of perpendicular bisector, bisector = dotted line

2.5.3 Argument angle

- $\arg z = \theta$: ray starting from origin, angle with x-axis = θ
- $\arg(z - z_0) = \theta$: ray starting from z_0, angle with horizontal = θ

2.5.4 Ellipse

- $|z - z_1| + |z - z_2| = 2a$: ellipse with focus at z_1 and z_2, longest diameter = $2a$

3 Matrices

3.1 Finding determinants

3.2 Finding inverse matrices

2×2 matrices:
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3×3 matrices:

4 Further algebra and functions

4.1 Vieta's Law

- $x_1 + x_2 + \cdots + x_n = -\dfrac{a_{n-1}}{a_n}$
- $(r_1 r_2 + r_1 r_3 + \cdots + r_1 r_n) + (r_2 r_3 + r_2 r_3 + \cdots + r_2 r_n) + \cdots + r_{n-1} r_n = \dfrac{a_{n-2}}{a_n}$
- $r_1 r_2 \ldots r_n = (-1)^n \dfrac{a_0}{a_n}$

4.2 Summation formulae

4.2.1 Summation of squares

$\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$

4.2.2 Summation of cubes

$\sum_{r=1}^{n} r^3 = \dfrac{(n(n+1))^2}{4}$

4.3 Maclaurin series

5 Further calculus

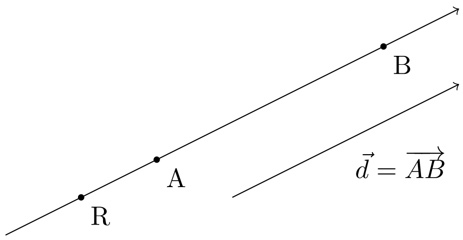
5.1 Volumes of revolution

Volume $= \pi \int y^2 \, dx$ or $\pi \int x^2 \, dy$

5.2 Inverse trig differentiation and integration

6 Further vectors

6.1 Expressing linear equations



6.1.1 Vector form

$\vec{r} = \vec{a} + t\vec{d}$ or $(\vec{r} - \vec{a}) \times \vec{d} = 0$ / $(\vec{r} - \vec{a}) \times \vec{b} = 0$

6.1.2 Parametric form

$$\begin{cases} x = x_0 + tu \\ y = y_0 + tv \\ z = z_0 + tw \end{cases}$$

6.1.3 Cartesian form

$$\frac{x - x_0}{u} = \frac{y - y_0}{v} = \frac{z - z_0}{w} (= t)$$

6.2 Expressing planes

6.2.1 Vector form

$\vec{r} \cdot \vec{n} = \vec{r} \cdot \vec{a}$ ($\vec{n}$ = normal, $\vec{a}$ = a point on plane)

6.2.2 Parametric form

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$

6.2.3 Cartesian form

When $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$: $ax + by + cz = d$

6.3 Formulae

6.3.1 Dot and cross product

- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = x_1x_2 + y_1y_2 + z_1z_2$

  $\vec{a} \perp \vec{b}$: $\vec{a} \cdot \vec{b} = 0$

  $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

- $\vec{a} \times \vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$

  Calculating cross product using matrix: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

  $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{n}$, so $\vec{a} \times \vec{b} = \vec{n}$

  $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

  $\vec{a} \parallel \vec{b}$: $\vec{a} = \lambda\vec{b}$; $\vec{a} \times \vec{b} = \vec{0}$

6.4 Questions

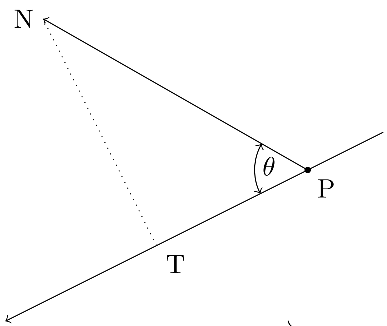
6.4.1 Angle between planes

1. Find the normal of the two planes
2. Use $\theta = \cos^{-1} \dfrac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$ to find the angle between planes
3. If $\theta > 90$ than the angle is $180 - \theta$

6.4.2 Angle between plane and line

$\phi = 90 - \cos^{-1} \dfrac{\vec{n} \cdot \vec{d}}{|\vec{n}||\vec{d}|}$ or $\phi = \sin^{-1} \dfrac{\vec{n} \cdot \vec{d}}{|\vec{n}||\vec{d}|}$

6.4.3 Finding distances between point and line



$|\overrightarrow{PT}| = |\overrightarrow{PN}| \cos\theta = \dfrac{\overrightarrow{PN} \cdot \vec{d}}{|\vec{d}|}$

$|\overrightarrow{NT}| = \sqrt{|\overrightarrow{PN}|^2 - |\overrightarrow{PT}|^2}$

6.4.4 Finding distances from point to plane

When $P = (x_0, y_0, z_0)$ and the plane has equation $ax + by + cz - d = 0$: distance $= \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

6.4.5 Finding distances between lines

distance $= \dfrac{|\overrightarrow{AB} \cdot \vec{n}|}{|\vec{n}|}$ ($\overrightarrow{AB}$ = any line that connects 2 lines together)

6.4.6 Finding intersections between line and plane

Write line in parametric form, substitute x, y and z into the equation for plane (Cartesian form)

7 Polar coordinates

7.1 Polar form

- $P = (r, \theta)$
- $r = f(\theta)$

7.2 Converting between different forms of coordinates

7.2.1 Polar to Cartesian form

- $x = r\cos\theta$
- $y = r\sin\theta$

7.2.2 Cartesian to polar form

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \dfrac{y}{x}$

7.3 Sketching curves for polar equations

1. Plot points
2. Consider symmetry
3. Convert to Cartesian form

7.3.1 Circle polar form

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7.3.2 Polar form of lines

7.3.3 Cardioids

7.3.4 Lima con

7.3.5 Rose

7.3.6 Lemniscates

7.3.7 Spiral

7.4 Testing for symmetry

7.5 Finding area enclosed

Area $= \dfrac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$

8 Hyperbolic functions

9 Differential equations