Further Mathematics Notes - Core Pure

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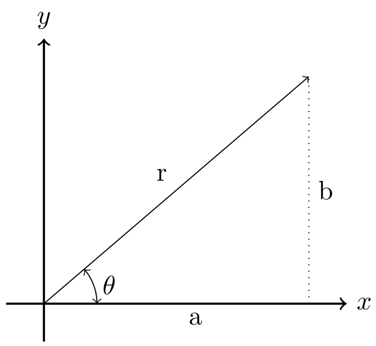
2024

Contents

1 Proof

2 Complex numbers

2.1 Forms expressing complex number



2.1.1 Cartesian form

$z = a + bi$
$\text{Re}(z) = a$, $\text{Im}(z) = b$
$|z| = \sqrt{a^2 + b^2}$

2.1.2 Polar form

$z = (r, \theta)$
$r = \sqrt{a^2 + b^2}$, $\theta = \arg(z)$
$a = r\cos\theta$, $b = r\sin\theta$, $\tan\theta = \dfrac{b}{a}$
$z = r\cos\theta + r\sin\theta i = r(\cos\theta + \sin\theta i) = r\,\text{cis}\,\theta$

2.1.3 Exponential / Euler form

$z = re^{i\theta}$

2.2 Calculations

- $z_1 z_2 = r_1 r_2 \,\text{cis}(\theta 1 + \theta 2) = r_1 r_2 e^{i((\theta 1 + \theta 2))}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \,\text{cis}(\theta 1 - \theta 2) = \dfrac{r_1}{r_2} e^{i((\theta 1 - \theta 2))}$
- $z^n = r^n \,\text{cis}(n\theta) = r^n e^{i\theta n}$
- $\sqrt[n]{z} = \sqrt[n]{r}\,\text{cis}(\dfrac{\theta + 2k\pi}{n}) = \sqrt[n]{r} e^{i\dfrac{\theta + 2k\pi}{n}} \quad (k = 0, 1, 2, \ldots, n-2, n-1)$
- $\sqrt{a + ib} = \pm(\sqrt{\dfrac{|z| + a}{2}} + i\dfrac{b}{|b|}\sqrt{\dfrac{|z| - a}{2}})$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg(\dfrac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$

3 Matrices

3.1 Finding determinants

3.2 Finding inverse matrices

2×2 matrices:
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3×3 matrices:

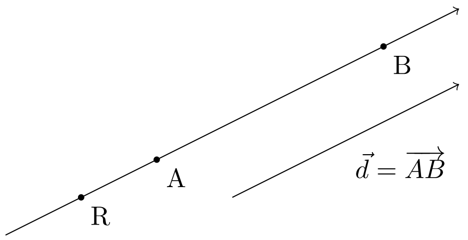
4 Further algebra and functions

4.1 Vieta's Law

5 Further calculus

6 Further vectors

6.1 Expressing linear equations



6.1.1 Vector form

$\vec{r} = \vec{a} + t\vec{d}$ or $(\vec{r} - \vec{a}) \times \vec{d} = 0$ / $(\vec{r} - \vec{a}) \times \vec{b} = 0$

6.1.2 Parametric form

$$\begin{cases} x = x_0 + tu \\ y = y_0 + tv \\ z = z_0 + tw \end{cases}$$

6.1.3 Cartesian form

$$\frac{x - x_0}{u} = \frac{y - y_0}{v} = \frac{z - z_0}{w} (= t)$$

6.2 Expressing planes

6.2.1 Vector form

$\vec{r} \cdot \vec{n} = \vec{r} \cdot \vec{a}$ ($\vec{n}$ = normal, $\vec{a}$ = a point on plane)

6.2.2 Parametric form

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$

6.2.3 Cartesian form

When $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$: $ax + by + cz = d$

6.3 Formulae

6.3.1 Dot and cross product

- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = x_1x_2 + y_1y_2 + z_1z_2$

  $\vec{a} \perp \vec{b}$: $\vec{a} \cdot \vec{b} = 0$

  $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

- $\vec{a} \times \vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$

  Calculating cross product using matrix: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

  $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{n}$, so $\vec{a} \times \vec{b} = \vec{n}$

  $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

  $\vec{a} \parallel \vec{b}$: $\vec{a} = \lambda\vec{b}$; $\vec{a} \times \vec{b} = \vec{0}$

6.4 Questions

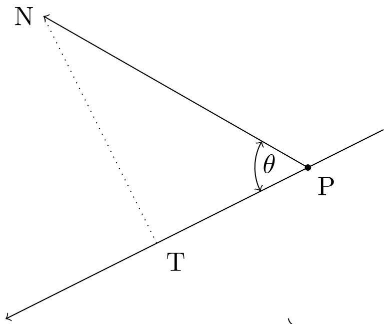
6.4.1 Angle between planes

1. Find the normal of the two planes
2. Use $\theta = \cos^{-1} \dfrac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$ to find the angle between planes
3. If $\theta > 90$ than the angle is $180 - \theta$

6.4.2 Angle between plane and line

$\phi = 90 - \cos^{-1} \dfrac{\vec{n} \cdot \vec{d}}{|\vec{n}||\vec{d}|}$ or $\phi = \sin^{-1} \dfrac{\vec{n} \cdot \vec{d}}{|\vec{n}||\vec{d}|}$

6.4.3 Finding distances between point and line



$$|\overrightarrow{PT}| = |\overrightarrow{PN}| \cos\theta = \frac{\overrightarrow{PN} \cdot \vec{d}}{|\vec{d}|}$$
$$|\overrightarrow{NT}| = \sqrt{|\overrightarrow{PN}|^2 - |\overrightarrow{PT}|^2}$$

6.4.4 Finding distances from point to plane

When $P = (x_0, y_0, z_0)$ and the plane has equation $ax + by + cz - d = 0$: distance $= \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

6.4.5 Finding distances between lines

distance $= \dfrac{|\overrightarrow{AB} \cdot \vec{n}|}{|\vec{n}|}$ ($\overrightarrow{AB}$ = any line that connects 2 lines together)

6.4.6 Finding intersections between line and plane

Write line in parametric form, substitute x, y and z into the equation for plane (Cartesian form)

7 Polar coordinates

8 Hyperbolic functions

9 Differential equations