A Level Pure Mathematics Notes

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2024

Contents

1 Proof

1.1 Methods

- Proof by deduction
- Proof by exhaustion
- Disproof by counter example
- Proof by contradiction

1.2 Proof by contradiction

1.2.1 Steps

1. Assume that the first statement is false
2. Use logical steps / contradiction from knowledge to show that the assumption is false
3. Conclude that the assumption is false so the original statement must be true

1.2.2 Irrationality of $\sqrt{2}$

Assumption: $\sqrt{2}$ is a rational number

Then $\sqrt{2} = \frac{a}{b}$ for some integers a and b

Also assume that a and b has no common factors so the fraction is in the simplest form

So $2 = \frac{a^2}{b^2}$, $a^2 = 2b^2$

So a^2 must be even, so a is also even

If a is even, then it can be expressed in the form $a = 2n$, where n is an integer

Substitute $a = 2n$: $(2n)^2 = 2b^2$

So $4n^2 = 2b^2$

So $b^2 = 2n^2$, hence b^2 must be even and b is also even

If a and b are both even, they will have a common factor or 2

This contradicts that a and b has no common factors, so $\sqrt{2}$ is an irrational number

1.2.3 Infinity of primes

Assumption: there is a finite number of prime numbers

List all the prime numbers that exist: $p_1, p_2, p_3, \ldots, p_n$

Consider the number $N = p_1 \times p_2 \times p_3 \times \cdots \times p_n + 1$

When N is divided by any of $p_1, p_2, p_3, \ldots, p_n$ a remainder of 1 is produced so none of them is a factor of N

Therefore N must be prime or have a prime factor not in the list of all the prime numbers that exist

This contradicts the assumption that there is a finite number of prime numbers

Therefore there must be an infinite number of prime numbers

2 Algebra and functions

2.1 Expressing solutions with set notations

Examples:

- $x > a$ and $x < b$ can be expressed as $\{x : x > a\} \cap \{x : x < b\}$
- $x < c$ or $x > d$ can be expressed as $\{x : x > c\} \cup \{x : x < d\}$

2.2 Sketching graphs

2.2.1 Quadratic / cubic / quartic

Find:

- Roots (may only be one or none)
- y-intercept
- Turning point
- Shape

2.2.2 Reciprocal graphs

Find:

- Horizontal asymptotes (by long division)
- Vertical asymptotes (where denominator = 0)

3 Coordinate geometry in (x, y) plane

3.1 Parametric equations

3.1.1 Convert to Cartesian form

- Express t in terms of x, then substitute $t = f(x)$ into $y = g(t)$
- Find the range of x by using the original parametric equation
- Find the range of y using original equation / considering the domain of x

3.1.2 Sketching curve

Sketch at regular intervals of t

4 Sequences and series

4.1 Binomial expansion

4.1.1 Expanding $(1+x)^n$

When $|x| < 1$:
$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

4.1.2 Expanding $(a+bx)^n$

$(a+bx)^n = (a(1+\frac{b}{a}x))^n = a^n(1+\frac{b}{a}x)^n$
Valid for $|\frac{b}{a}x| < 1$ or $|x| < \frac{a}{b}$

4.2 Divergent / convergent series

$\sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + \cdots + u_n$
If $\lim_{n\to\infty} S_n$ exists, $\sum_{i=1}^{n} u_i$ converges
If $\lim_{n\to\infty} S_n$ does not exist, $\sum_{i=1}^{n} u_i$ diverges

4.3 Geometric series

Sum of first n terms: $S_n = \frac{a(1-r^n)}{1-r}$

Sum to infinity: When $|r| < 1$ (convergent series): $S_\infty = \frac{a}{1-r}$

4.4 Recurrence relations

Increasing sequence: $u_{n+1} > u_n$ for all $n \in \mathbf{N}$

Decreasing sequence: $u_{n+1} < u_n$ for all $n \in \mathbf{N}$

Periodic sequence: If there is an integer k such that $u_{n+k} = u_n$ for all $n \in \mathbf{N}$, $k =$ the order of the sequence

5 Trigonometry

5.1 Radian calculations

Arc length: $s = r\theta$

Area of sector: $A = \frac{1}{2}r^2\theta$

5.2 Trigonometry formulae

5.2.1 Addition / subtraction

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$
- $\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$
- $\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$
- $\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

5.2.2 Double angle

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
- $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

5.2.3 Power descending

(Derive from double angle)

- $\sin A \cos A = \dfrac{\sin 2A}{2}$
- $\sin^2 A = \dfrac{1 - \cos 2A}{2}$
- $\cos^2 A = \dfrac{1 + \cos 2A}{2}$

5.2.4 Half angle

- $\sin \dfrac{A}{2} = \pm\sqrt{\dfrac{1 - \cos A}{2}}$
- $\cos \dfrac{A}{2} = \pm\sqrt{\dfrac{1 + \cos A}{2}}$
- $\tan \dfrac{A}{2} = \pm\sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \dfrac{1 - \cos A}{\sin A} = \dfrac{\sin A}{1 + \cos A}$

5.2.5 Small angle estimation

When θ is small:

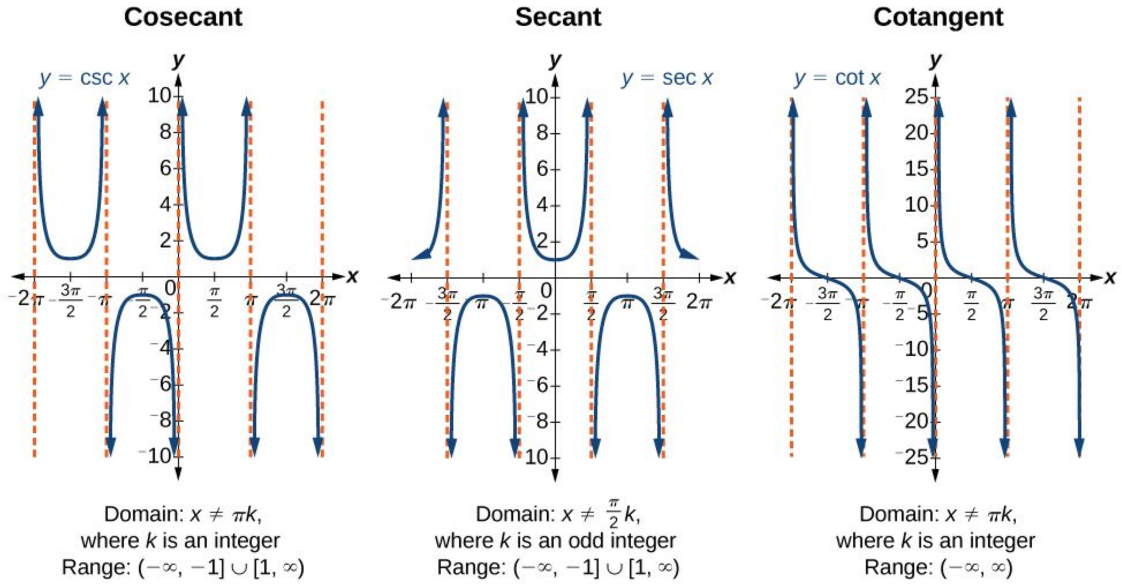
- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$
- $\tan\theta \approx \theta$

5.3 Identities

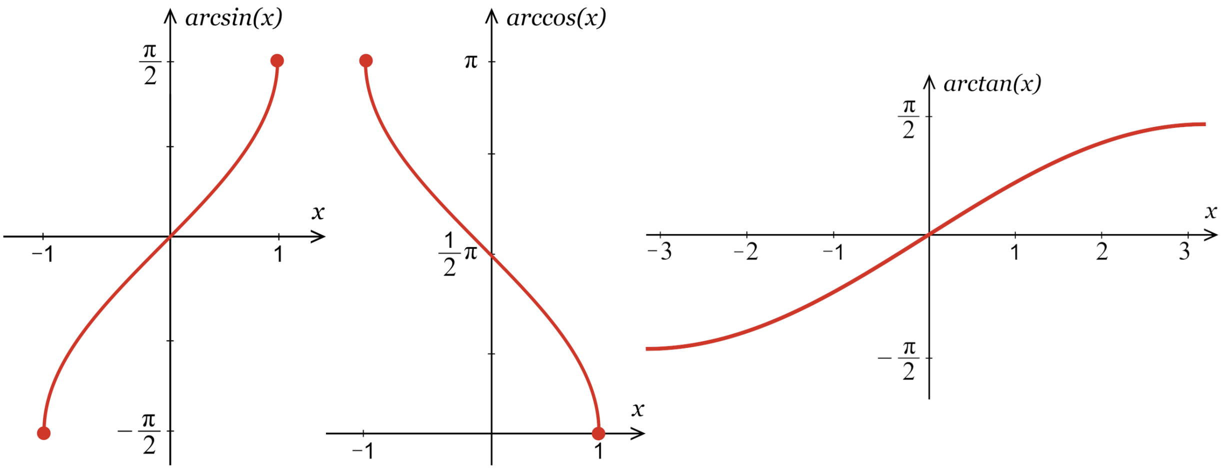
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \text{cosec}^2\,\theta$

5.4 Graphs

5.4.1 secant, cosecant and cotangent



5.4.2 arcsin, arccos, arctan



6 Exponentials and logarithms

6.1 Sketching graphs

Find the y-intercept of the graph

6.2 e^x function

$(e^x)' = e^x$

$(e^{kx})' = ke^{kx}$ (gradient directly proportional to y value)

6.3 Logarithm

$a^x = n$: $\log_a n = x$ ($a \neq 1$ and $a > 0$, $x \geq 0$)

6.3.1 Laws

The multiplication law: $\log_a x + \log_a y = \log_a xy$

The division law: $\log_a x - \log_a y = \log_a(\frac{x}{y})$

The power law: $\log_a x^k = k \log_a x$

Change base formula : $\log_a b = \frac{\log_c b}{\log_c a}$

6.3.2 Logarithms in non-linear form

Exponential

- $y = ab^x \rightarrow \ln y = x \ln b + \ln a$
- x-axis $= x$, y-axis $= \ln y$, gradient $= \ln b$, y-intercept $= \ln a$

Power

- $y = ax^b \rightarrow \ln y = b \ln x + \ln a$
- x-axis $= \ln x$, y-axis $= \ln y$, gradient $= b$, y-intercept $= \ln a$

Logarithmic

- $y = a \ln x \rightarrow$ kept the same
- x-axis $= \ln x$, y-axis $= y$, gradient $= a$

7 Differentiation

$f'(x) = \lim_{n \to 0} \frac{f(x+h) - f(x)}{h}$

7.1 Formulae

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$\tan kx$	$k \sec^2 kx$
$\sec kx$	$k \sec kx \tan kx$
$\cot kx$	$-k \operatorname{cosec}^2 kx$
$\operatorname{cosec} kx$	$-k \operatorname{cosec} kx \cot kx$

7.2 Rules

Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Product rule: $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$

Quotient rule: $\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

7.3 Tangent and normal

For curve $y = f(x)$:

Tangent at $(a, f(a))$: $y - f(a) = f'(a)(x - a)$

Normal at $(a, f(a))$: $y - f(a) = -\frac{1}{f'(a)}(x - a)$

8 Integration

8.1 Definition

$\int_a^b f(x)\,dx = \lim\limits_{\delta x \to 0} \sum_{x=a}^{b} f(x)\delta x$

8.2 Formulae

Formula sheet:

f(x)	$\int f(x)\,dx$ $(+c)$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert$
$\cot kx$	$\frac{1}{k}\ln\lvert\sin kx\rvert$
$\operatorname{cosec} kx$	$-\frac{1}{k}\ln\lvert\operatorname{cosec} kx + \cot kx\rvert,\ \frac{1}{k}\ln\lvert\tan(\frac{1}{2}kx)\rvert$
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert,\ \frac{1}{k}\ln\lvert\tan(\frac{1}{2}kx + \frac{1}{4}\pi)\rvert$

Integrating $\sin^2 x$ or $\cos^2 x$ (power descending):

- $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}(x - \frac{\sin 2x}{2}) + c$
- $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}(x + \frac{\sin 2x}{2}) + c$

By part:

- $\int \ln x\,dx = x\ln x + x + c$

8.3 Techniques

8.3.1 U-sub

- $\int f'(ax+b)\,dx = \frac{f(ax+b)}{a} + c$

  Substitution: $u = ax + b$

- $\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$

  Substitution: $u = f(x)$

8.3.2 By part

- $\int u\,dv = uv - \int v\,du$
- LIPET rule: leftmost $= u$

  L: logarithmic

  I: inverse trigonometry

  P: polynomial

  E: exponential

  T: trigonometry

9 Numerical methods

9.1 Locating roots

9.1.1 Method

If a function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root, x, which satisfies $a < x < b$

9.1.2 How change of sign can fail

- When the interval is too large sign may not change as there may be an even number of roots
- If the function is not continuous, sign may change but there may be an asymptote e.g. reciprocal

9.2 Iteration

- Iterative formula can be found by rewriting $f(x) = 0$ into $x = g(x)$, then $x_{n+1} = g(x_n)$
- Find the value of x_1, x_2, x_3, etc.
- If converge (get closer to the root from the same direction / alternate above and below the root) then solution can be found

9.3 The Newton-Raphson method

- $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
- Find approximation to x decimal places
- Show accurate to x decimal places: use change in sign to show

⋆ If any value of x_i is at a turning point then the method will fail as $f'(x) = 0$ which results in division by zero

9.4 The trapezium rule

$\int_a^b y\, dx \approx \dfrac{1}{2}(y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n)$ where $h = \dfrac{b-a}{n}$ and $y_i = f(a + ih)$

10 Vectors

10.1 Angle with x-, y-, and z-axis (direction cosines)

If $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$ makes an angle θ_x with the positive x-axis then $\cos\theta_x = \frac{x}{|a|}$, similarly for angles θ_y and θ_z

10.2 Vectors in equations

10.2.1 2D

If $\vec{a}$ and $\vec{b}$ are 2 non-parallel vectors and $p\vec{a} + q\vec{b} = r\vec{a} + s\vec{b}$ then $p = r$ and $q = s$

10.2.2 3D

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors in 3 dimensions which do not all lie on the same plane (parallel to the same plane) then you can compare their coefficients on both sides of an equation