A Level Further Mathematics Notes

Xingzhi Lu

2129570@concordcollege.org.uk

Contents

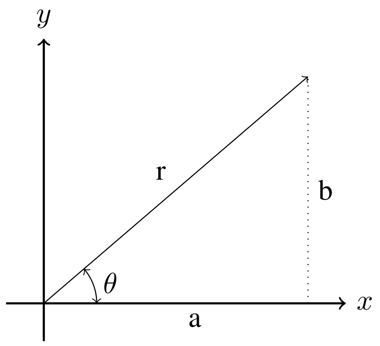
Part I

Core Pure 1

Chapter 1

Complex numbers

1.1 Cartesian form for expressing complex numbers



- $z = a + bi$
- $\mathrm{Re}(z) = a, \mathrm{Im}(z) = b$
- $|z| = \sqrt{a^2 + b^2}$

1.2 Multiplying complex numbers

- $i^2 = -1$
- Multiply out the terms as if they are polynomials

1.3 Conjugation of complex numbers

- $|z| = |z^*|$
- $z \cdot z^* = |z|^2$
- If $z = z*$ then z is a real number
- If $z = -z*$ then z is a pure imaginary number

1.3.1 Roots of quadratic equations

If $a + bi$ is the root of a quadratic equation, then $a - bi$ is also a root of the same quadratic equation.

Chapter 2

Argand diagrams

2.1 Modulus and argument

For $z = x + yi$:

Modulus $|z| = \sqrt{x^2 + y^2}$

Argument $\theta = \arg z = \arctan\left(\frac{y}{x}\right)$

2.1.1 Modulus-argument form

For complex number z with $|z| = r$ and $\arg z = \theta$:

$$z = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$$

2.1.2 Calculations with arguments

- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

2.1.3 Calculations with modulus

- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$ (same as triangular inequalities)

2.2 Loci of different expressions

2.2.1 Circle

- $|z| = a$: circle with centre O, radius a
- $|z - z_0| = a$: circle with centre z_0, radius a
- $|z - z_0| < a$: circle with centre z_0, radius a, shaded in
- $|z - z_0| > a$: circle with centre z_0, radius a, outside of circle shaded

2.2.2 Perpendicular bisector

- $|z - z_1| = |z - z_2|$: z on perpendicular bisector of $z_1 z_2$
- $|z - z_1| < |z - z_2|$ / $|z - z_1| > |z - z_2|$: on the left / right side of perpendicular bisector, bisector = dotted line

2.2.3 Argument angle

- $\arg z = \theta$: ray starting from origin, angle with x-axis = θ
- $\arg(z - z_0) = \theta$: ray starting from z_0, angle with horizontal = θ

2.2.4 Ellipse

- $|z - z_1| + |z - z_2| = 2a$: ellipse with focus at z_1 and z_2, longest diameter = $2a$

Chapter 3

Series

3.1 Summation of squares

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

3.1.1 Proof (without mathematical induction)

3.2 Summation of cubes

$$\sum_{r=1}^{n} r^3 = \frac{(n(n+1))^2}{4}$$

3.2.1 Proof (without mathematical induction)

Chapter 4

Roots of polynomials

4.1 Vieta's Law

For $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$:

- $\sum x_i = -\dfrac{a_{n-1}}{a_n}$
- $\sum x_i x_j = \dfrac{a_{n-2}}{a_n}$
- $\sum x_i x_j x_k = -\dfrac{a_{n-3}}{a_n}$
- $\displaystyle\sum_{1\le i_1<i_2<\cdots<i_k\le n} \left(\prod_{j=1}^{k} r_{i_j}\right) = (-1)^k \frac{a_{n-k}}{a_n}$
- $\prod x_i = (-1)^n \dfrac{a_0}{a_n}$

Chapter 5

Volumes of revolution

5.1 Finding volumes of revolution

Volume = $\pi \int y^2 \, dx$ or $\pi \int x^2 \, dy$

Chapter 6

Matrices

6.1 Basic calculations

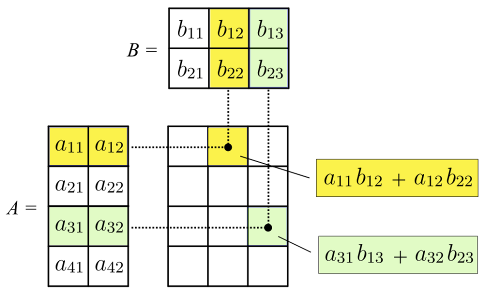
6.1.1 Addition / subtraction

$$(\mathbf{A}+\mathbf{B})_{i,j} = \mathbf{A}_{i,j} + \mathbf{B}_{i,j}, \quad 1 \le i \le m, \quad 1 \le j \le n$$
$$(\mathbf{A}-\mathbf{B})_{i,j} = \mathbf{A}_{i,j} - \mathbf{B}_{i,j}, \quad 1 \le i \le m, \quad 1 \le j \le n$$

6.1.2 Scalar multiplication

$$(c\mathbf{A})_{i,j} = c \cdot A_{i,j}$$

6.1.3 Matrix multiplication



6.1.4 Transposition

$$(\mathbf{A}^{\mathrm{T}})_{i,j} = A_{j,i}$$

6.2 Special matrices

Square matrix: The number of rows and columns are the same

Zero matrix: All of the elements are zero

Identity matrix: A square matrix in which all the elements on the leading diagonal are 1 and the remaining elements are 0, denoted by $\mathbf{I}_k$ for $k \times k$ identity matrix

6.3 Determinants

6.3.1 2×2 matrices

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

6.3.2 3×3 matrices

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

6.3.3 Singular matrices

- Singular matrices are square matrices with a determinant of 0
- It does not have an inverse
- If $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

6.3.4 Properties of determinants

- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{B})\det(\mathbf{A}) = \det(\mathbf{BA})$
- $\det(k\mathbf{A}) = k^n \det(\mathbf{A})$ ($\mathbf{A}$ is a $n \times n$ matrix)

6.4 Inverse matrices

6.4.1 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

6.4.2 3×3 matrices

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}^{\mathrm{T}} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} A & D & G \\ B & E & H \\ C & F & I \end{bmatrix}$$

$$\begin{aligned} A &= (ei - fh), & D &= -(bi - ch), & G &= (bf - ce), \\ B &= -(di - fg), & E &= (ai - cg), & H &= -(af - cd), \\ C &= (dh - eg), & F &= -(ah - bg), & I &= (ae - bd). \end{aligned}$$

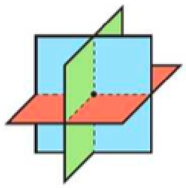
6.4.3 Solving equations with matrices

If $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{v}$ then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\mathbf{v}$

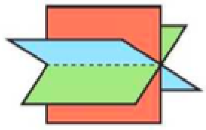
Consistent system of linear equations: there is at least one set of values that satisfies all the equations simultaneously

Inconsistent: such set of values does not exist

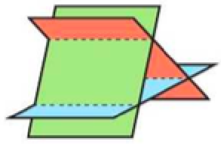
6.4.4 Possible outcomes of solutions



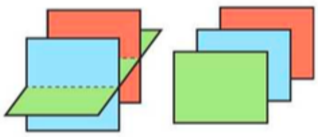
The planes meet at a **point**. The system of equations is **consistent** and has **one solution** represented by this point. This is the only case when the corresponding matrix is **non-singular**.



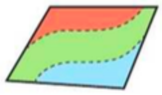
The planes form a **sheaf**. The system of equations is **consistent** and has **infinitely many solutions** represented by the line of intersection of the three planes.



The planes form a **prism**. The system of equations is **inconsistent** and has **no solutions**.



Two or more of the planes are parallel and non-identical. The system of equations is **inconsistent** and has **no solutions**.



All three equations represent the same plane. In this case the system of equations is **consistent** and has **infinitely many solutions**.

Chapter 7

Linear transformations

7.1 Linear transformations in 2D

7.1.1 Properties

If $L(\vec{v})$ is linear:

1. $L(\vec{v})$ should always map the origin onto itself
2. $L(\vec{v})$ can be represented by a matrix
3. $L(\vec{v_1} + \vec{v_2}) = L(\vec{v_1}) + L(\vec{v_2})$ (closure in addition)
4. $L(\lambda\vec{v_1}) = \lambda L(\vec{v_1})$ (closure in scalar multiplication)

7.1.2 Invariant points and lines

Invariant points: Points which are mapped onto themselves under the given transformation

Invariant lines: Lines which map onto themselves

7.1.3 Reflection

Reflection in y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, invariant points: points on the y-axis; invariant lines: $x = 0$, $y = k$

Reflection in x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, invariant points: points on the x-axis; invariant lines: $y = 0$, $x = k$

Reflection in line $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, invariant points: points on $y = x$; invariant lines: $y = x$, $y = -x + k$

Reflection in line $y = -x$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, invariant points: points on $y = -x$; invariant lines: $y = -x$, $y = x + k$

7.1.4 Rotation

Rotation through angle θ anticlockwise about the origin $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Invariant points: Only $(0, 0)$

Invariant lines: When $\theta = 180$ any line passing through the origin is an invariant line, otherwise no invariant lines

7.1.5 Enlargement / stretches

Transformation matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ = a stretch of scale factor a parallel to the x-axis and scale factor b parallel to the y-axis

Invariant lines x- and y-axes for all stretches

- Stretch parallel to the x-axes: any line parallel to the x-axes
- Stretch parallel to the y-axes: any line parallel to the y-axes

Invariant points The origin is always an invariant point

- Stretch parallel to the x-axes: points on the y-axes
- Stretch parallel to the y-axes: points on the x-axes

Change in area $\det(\mathbf{M})$ = area scale factor

7.2 Linear transformations in 3D

Reflection in plane $x = 0$ $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Reflection in plane $y = 0$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Reflection in plane $z = 0$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Rotation angle θ anticlockwise about the x-axis $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

Rotation angle θ anticlockwise about the y-axis $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Rotation angle θ anticlockwise about the z-axis $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Chapter 8

Proof by induction

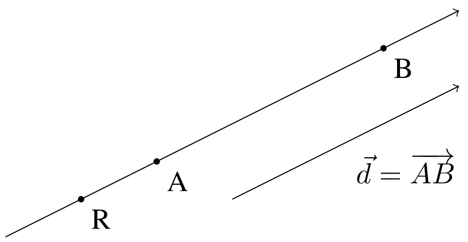
8.1 Proof by mathematical induction

1. Prove the general statement is true for $n = 0$ / $n = 1$ / the smallest possible value of n
2. Assume the general statement is true for $n = k$
3. Show that the general statement is true for $n = k + 1$
4. The general statement is then true for all positive integers n

Chapter 9

Vectors

9.1 Expressing linear equations



9.1.1 Vector form

$\vec{r} = \vec{a} + t\vec{d}$ or $(\vec{r} - \vec{a}) \times \vec{d} = 0$ / $(\vec{r} - \vec{a}) \times \vec{b} = 0$

9.1.2 Parametric form

$$\begin{cases} x = x_0 + tu \\ y = y_0 + tv \\ z = z_0 + tw \end{cases}$$

9.1.3 Cartesian form

$$\frac{x - x_0}{u} = \frac{y - y_0}{v} = \frac{z - z_0}{w} (= t)$$

9.2 Expressing planes

9.2.1 Vector form

$\vec{r} \cdot \vec{n} = \vec{r} \cdot \vec{a}$ ($\vec{n}$ = normal, $\vec{a}$ = a point on plane)

9.2.2 Parametric form

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$

9.2.3 Cartesian form

When $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$: $ax + by + cz = d$

9.3 Formulae

9.3.1 Dot and cross product

- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = x_1x_2 + y_1y_2 + z_1z_2$

  $\vec{a} \perp \vec{b}$: $\vec{a} \cdot \vec{b} = 0$

  $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

- $\vec{a} \times \vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$

  Calculating cross product using matrix: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

  $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{n}$, so $\vec{a} \times \vec{b} = \vec{n}$

  $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

  $\vec{a} \parallel \vec{b}$: $\vec{a} = \lambda\vec{b}$; $\vec{a} \times \vec{b} = \vec{0}$

9.4 Questions

9.4.1 Angle between planes

1. Find the normal of the two planes
2. Use $\theta = \cos^{-1} \dfrac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$ to find the angle between planes
3. If $\theta > 90$ than the angle is $180 - \theta$

9.4.2 Angle between plane and line

$\phi = 90 - \cos^{-1} \dfrac{\vec{n} \cdot \vec{d}}{|\vec{n}||\vec{d}|}$ or $\phi = \sin^{-1} \dfrac{\vec{n} \cdot \vec{d}}{|\vec{n}||\vec{d}|}$

9.4.3 Finding distances between point and line

For point $\mathbf{x}_0$ to line $\mathbf{x}_1\mathbf{x}_2$: $d = \dfrac{|(\mathbf{x}_2 - \mathbf{x}_1) \times (\mathbf{x}_1 - \mathbf{x}_0)|}{|\mathbf{x}_2 - \mathbf{x}_1|} = \dfrac{|(\mathbf{x}_0 - \mathbf{x}_1) \times (\mathbf{x}_0 - \mathbf{x}_2)|}{|\mathbf{x}_2 - \mathbf{x}_1|}$

9.4.4 Finding distances from point to plane

When $P = (x_0, y_0, z_0)$ and the plane has equation $ax + by + cz - d = 0$: distance $= \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

9.4.5 Finding distances between lines

distance $= \dfrac{|\overrightarrow{AB} \cdot \vec{n}|}{|\vec{n}|}$ ($\overrightarrow{AB}$ = any line that connects 2 lines together)

9.4.6 Finding intersections between line and plane

Write line in parametric form, substitute x, y and z into the equation for plane (Cartesian form)

Part II

Core Pure 2

Chapter 1

Complex numbers

1.1 More forms of expressing complex number

1.1.1 Polar form

- $z = (r, \theta)$
- $r = \sqrt{a^2 + b^2}$, $\theta = \arg(z)$
- $a = r\cos\theta$, $b = r\sin\theta$, $\tan\theta = \dfrac{b}{a}$
- $z = r\cos\theta + r\sin\theta i = r(\cos\theta + \sin\theta i) = r\operatorname{cis}\theta$

1.1.2 Exponential / Euler form

$$z = re^{i\theta}$$

1.1.3 When to use which form

Addition / subtraction: Cartesian form (see CP1 chapter 1)

Multiple / division / power / root: Polar / Euler form

1.2 Calculations of complex numbers

1.2.1 Multiplication and division

- $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) = r_1 r_2 e^{i(\theta_1+\theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2) = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$
-
- : $(k = 0, 1, 2, \ldots, n-2, n-1)$
    - ⋆ They form vertices of a regular n-gon in the Argand diagram with its centre at the origin
- $\sqrt{a+ib} = \pm\left(\sqrt{\dfrac{|z|+a}{2}} + i\dfrac{b}{|b|}\sqrt{\dfrac{|z|-a}{2}}\right)$

⋆ Complex numbers' sizes cannot be compared

1.2.2 Exponential calculations

- De Moivre's theorem: for any integer n: $z^n = r^n\operatorname{cis}(n\theta) = r^n e^{in\theta}$

1.2.3 nth root of complex numbers

- $\sqrt[n]{z} = \sqrt[n]{r}\operatorname{cis}\left(\dfrac{\theta + 2k\pi}{n}\right) = \sqrt[n]{r}e^{i\left(\frac{\theta+2k\pi}{n}\right)}$, where $k \in \mathbb{Z}$ ($k = 0$ to $k = n-1$ gives a complete cycle)
- The nth roots of any complex number a lie at the vertices of a regular n-gon with its centre at the origin

1.2.4 Roots of unity

- If z_1 is one root of the equation $z^n = s$, and $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the nth roots of unity, then the roots of $z^n = s$ are given by $z_1, z_1\omega, z_1\omega^2, \ldots, z_1\omega^{n-1}$
- For equation $z^n = 1$: $\omega = e^{i\frac{\pi}{n}}$, $\omega^n = 1$

1.3 Trigonometric identities

1.3.1 Properties for z

- $z = \operatorname{cis}\theta = e^{i\theta} = \cos\theta + i\sin\theta$, $\dfrac{1}{z} = e^{-i\theta} = \operatorname{cis}(-\theta)$
- $z + \dfrac{1}{z} = 2\cos\theta$
- $\left(z + \dfrac{1}{z}\right)^n = \left(e^{i\theta} + e^{-i\theta}\right)^n = 2^n\cos^n\theta$
- $z - \dfrac{1}{z} = 2i\sin\theta$
- $\left(z - \dfrac{1}{z}\right)^n = \left(e^{i\theta} - e^{-i\theta}\right)^n = 2^n i^n \sin^n\theta$
- $z^n + \dfrac{1}{z^n} = 2\cos(n\theta)$
- $z^n - \dfrac{1}{z^n} = 2i\sin(n\theta)$

1.3.2 Finding $\sin(n\theta)$ or $\cos(n\theta)$

1. Use $\left(z + \dfrac{1}{z}\right)$ for cos and $\left(z + \dfrac{1}{z}\right)$ for sin
2. Use the nth power of $z + \dfrac{1}{z}$ or $z - \dfrac{1}{z}$ (n is the same as coefficient)
3. Binomial expansion
4. Merge the terms into $z^n + \dfrac{1}{z^n}$ or $z^n - \dfrac{1}{z^n}$
5. Use properties above to convert then into trig functions

1.3.3 Find $\sin^n\theta$

n is odd:

- $n = 4k+1$: $\sin^n\theta = \dfrac{1}{2^{n-1}}\left[\binom{n}{0}\sin n\theta - \binom{n}{1}\sin(n-2)\theta + \binom{n}{2}\sin(n-4)\theta - \cdots + \binom{n}{\frac{n-1}{2}}\sin\theta\right]$
- $n = 4k+3$: $\sin^n\theta = \dfrac{1}{2^{n-1}}\left[-\binom{n}{0}\sin n\theta + \binom{n}{1}\sin(n-2)\theta - \binom{n}{2}\sin(n-4)\theta + \cdots - \binom{n}{\frac{n-1}{2}}\sin\theta\right]$

n is even:

- $n = 4k$: $\sin^n \theta = \dfrac{1}{2^{n-1}} \left[\binom{n}{0} \cos n\theta - \binom{n}{1} \cos(n-2)\theta + \binom{n}{1} \cos(n-4)\theta - \cdots + \binom{n}{\frac{n-1}{2}} \cos\theta \right]$
- $n = 4k+2$: $\sin^n \theta = \dfrac{1}{2^{n-1}} \left[-\binom{n}{0} \cos n\theta + \binom{n}{1} \cos(n-2)\theta - \binom{n}{1} \cos(n-4)\theta + \cdots - \binom{n}{\frac{n-1}{2}} \cos\theta \right]$

1.3.4 Find $\cos^n \theta$

- $\cos^n \theta = \dfrac{1}{2^{n-1}} \displaystyle\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n}{i} \cos(n-2i)\theta$
- $\displaystyle\int \cos^n \theta \, d\theta = \dfrac{1}{2^{n-1}} \sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n}{i} \frac{\sin(n-2i)\theta}{n-2i}$

1.4 Sum of geometric series

For $w, z \in \mathbb{C}$:

- $\displaystyle\sum_{r=0}^{n-1} wz^r = \frac{w(z^n - 1)}{z - 1}$
- $\displaystyle\sum_{r=0}^{n-1} wz^r = \frac{w}{1-z}, \; |z| < 1$

Chapter 2

Series

2.1 Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^{(r)}(0)x^r}{r!} + \ldots$$

This series is valid provided that $f(0), f'(0), f''(0), \ldots, f^{(r)}(0), \ldots$ all have **finite values**

2.2 Series expansion of compound functions

- $e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^r}{r!} + ...$ for all x
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^{r+1}\frac{x^r}{r!} + ...$ for $-1 < x \leq 1$
- $\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots$ for all x
- $\cos x = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$ for all x
- $\arctan x = x - \frac{x^3}{x} + \frac{x^5}{5} - \cdots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots$ for $-1 < x \leq 1$

2.3 Proving series properties

1. Use Taylor and Maclaurin Series
2. Use basic formulae for expansion
3. Use geometric series

2.4 Testing for convergence

2.4.1 nth term test

- $\lim_{n\to\infty} a_n \neq 0 \to \sum_{n=1}^{\infty} a_n$ diverges
- $\sum_{n=1}^{\infty} a_n$ converges $\to \lim_{n\to\infty} a_n = 0$

2.4.2 Integral test

- If a_n decrease and $a_n > 0$, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,\mathrm{dx}$ has the same properties of convergence or divergence

2.4.3 Comparison test

Suppose $b_n < a_n$ for all n:

- $\sum_{n=1}^{\infty} b_n$ diverges $\rightarrow \sum_{n=1}^{\infty} a_n$ diverges
-
- $\sum_{n=1}^{\infty} a_n$ converges $\rightarrow \sum_{n=1}^{\infty} b_n$ converges
- Compare a_n with p-series and geometrical series

  ***p*-series** $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^p$ is divergent if $p \leq 1$ and convergent if $p > 1$

2.4.4 Root test

- $\lim_{n\to\infty} \sqrt[n]{a_n} < 1 \rightarrow a_n$ converge
-
- $\lim_{n\to\infty} \sqrt[n]{a_n} > 1 \rightarrow a_n$ diverge

2.4.5 Ratio test

- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$: convergent
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$: divergent
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$: not sure

2.5 Summation of series

- Try to break down a_n into the form $a_n = b_{n+1} - b_n$

Chapter 3

Methods in calculus

3.1 Integrating to other functions

- $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{dx} = \arcsin\left(\frac{x}{a}\right) + c$
- $\displaystyle\int \frac{1}{a^2 + x^2}\,\mathrm{dx} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$
- $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,\mathrm{dx} = \operatorname{arcosh}\left(\frac{x}{a}\right) + c = \ln\left(x + \sqrt{x^2 - a^2}\right) + c \;\; (x > a)$
- $\displaystyle\int \frac{1}{\sqrt{a^2 + x^2}}\,\mathrm{dx} = \operatorname{arsinh}\left(\frac{x}{a}\right) + c = \ln\left(x + \sqrt{x^2 + a^2}\right) + c$
- $\displaystyle\int \frac{1}{a^2 - x^2}\,\mathrm{dx} = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$
- $\displaystyle\int \frac{1}{x^2 - a^2}\,\mathrm{dx} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

3.2 Trig substitution for square roots

- $\sqrt{a(x+b)^2 - c}$

  Substitution $x = \sqrt{\dfrac{c}{a}}\sec\theta - b$

- $\sqrt{a(x+b)^2 + c}$

  Substitution $x = \sqrt{\dfrac{c}{a}}\tan\theta - b$

- $\sqrt{-a(x+b)^2 - c}$

  Substitution $x = \sqrt{\dfrac{c}{a}}\cos\theta - b$ or $\sqrt{\dfrac{c}{a}}\sin\theta - b$

3.3 Integration techniques

3.3.1 Improper integrals with infinite range

3.3.2 Improper integrals with integrand undefined at a value in the range

3.3.3 Integrate using partial functions

3.3.4 Trigonometric integration

sin and cos

sec and tan

$$\int \sec^n x \, \mathrm{dx} = \int \sec^{n-2} \sec^2 x \, \mathrm{dx} = \int \sec^{n-2} x \, \mathrm{d} \tan x$$

3.4 Mean value of a function

Mean of $f(x)$ over the interval $[a, b]$ = $\dfrac{1}{b-a} \displaystyle\int_a^b f(x)\, dx$

Chapter 4

Volumes of revolution

Chapter 5

Polar coordinates

5.1 Polar form

- $P = (r, \theta)$
- $r = f(\theta)$

5.2 Converting between different forms of coordinates

5.2.1 Polar to Cartesian form

- $x = r\cos\theta$
- $y = r\sin\theta$

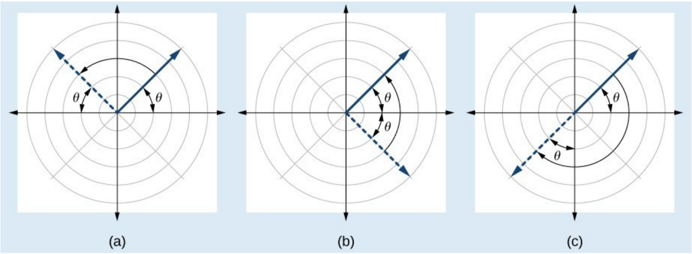
5.2.2 Cartesian to polar form

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \dfrac{y}{x}$

5.3 Sketching curves for polar equations

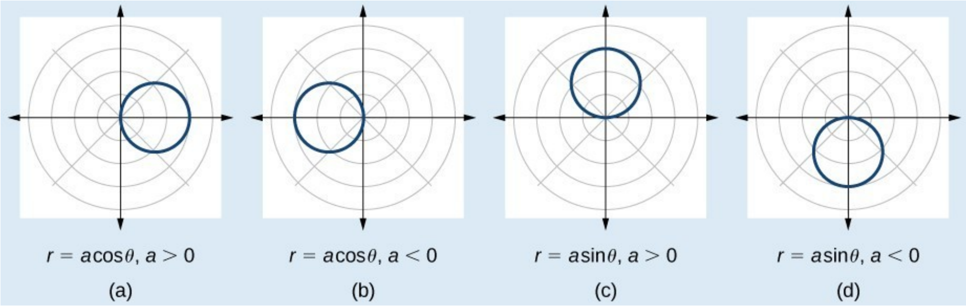
1. Plot points
2. Consider symmetry
3. Convert to Cartesian form

5.3.1 Testing for symmetry



- Symmetry about $\theta = 0$ / x-axis: $f(\theta) = f(-\theta)$
- Symmetry about $\theta = \dfrac{\pi}{2}$ / y-axis: $f(\theta) = f(\pi - \theta)$
- Symmetry about pole / origin: $f(\theta) = f(\pi + \theta)$ / replacing (r, θ) with $(-r, \theta)$ gives an equivalent equation

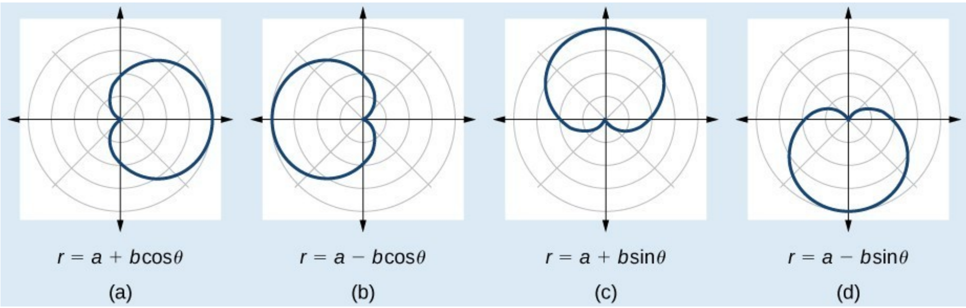
5.3.2 Circle polar form



5.3.3 Polar form of lines

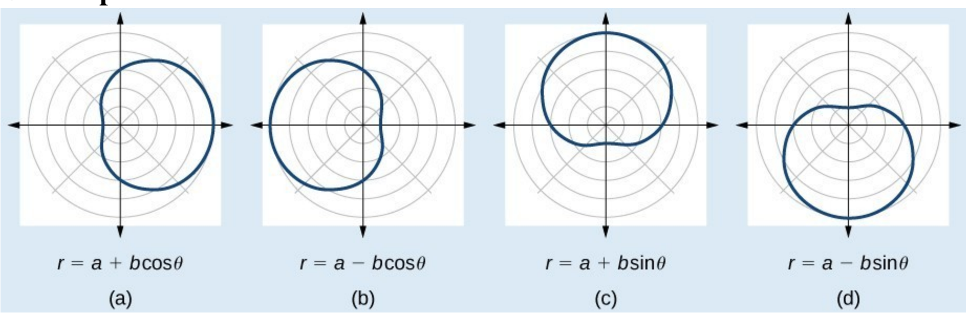
- $\theta = a$: ray from origin
- $r \cos\theta = a$: equivalent to $x = a$
- $r \sin\theta = a$: equivalent to $y = a$

5.3.4 Cardioids shape

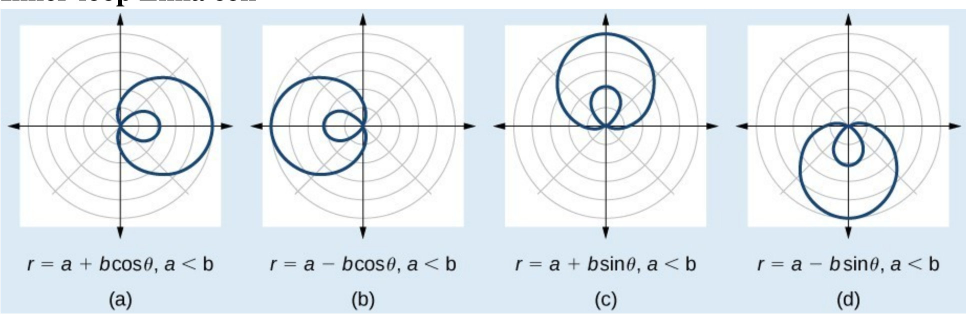


5.3.5 Lima con shape

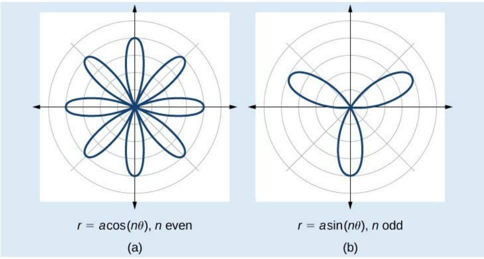
One-loop Lima con



Inner-loop Lima con

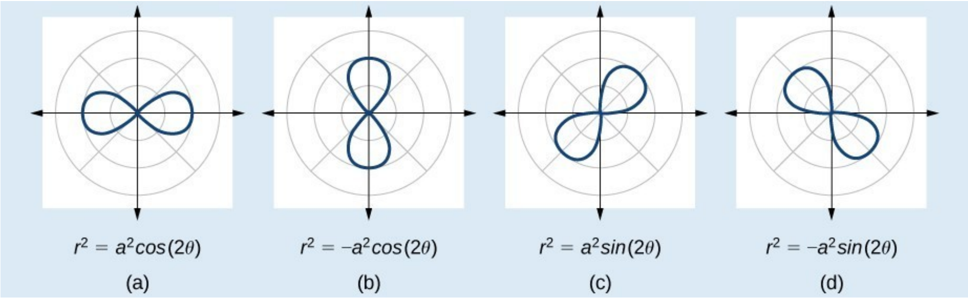


5.3.6 Rose shape

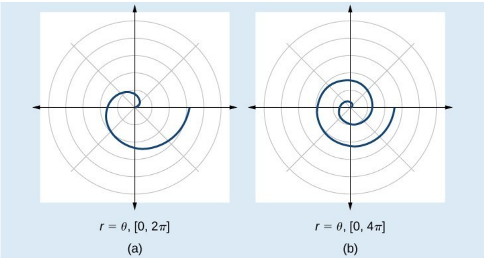


If n is even the curve has $2n$ petals, if n is odd the curve has n petals

5.3.7 Lemniscates shape



5.3.8 Spiral shape



5.4 Finding area enclosed

Area = $\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$

Chapter 6

Hyperbolic functions

6.1 Hyperbolic function definitions

6.1.1 $\sinh x$

Definition $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Domain $x \in \mathbb{R}$

Range $\sinh x \in \mathbb{R}$

Asymptotes $x \to +\infty, y \to \dfrac{e^x}{2}; x \to -\infty, y \to -\dfrac{e^{-x}}{2}$

x-intercept $(0, 0)$

y-intercept $(0, 0)$

6.1.2 $\cosh x$

Definition $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Domain $x \in \mathbb{R}$

Range $\cosh x \geq 1$

Asymptotes $x \to +\infty, y \to \dfrac{e^x}{2}; x \to -\infty, y \to \dfrac{e^{-x}}{2}$

x-intercept No

y-intercept $(0, 1)$

6.1.3 $\tanh x$

Definition $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Domain $x \in \mathbf{R}$

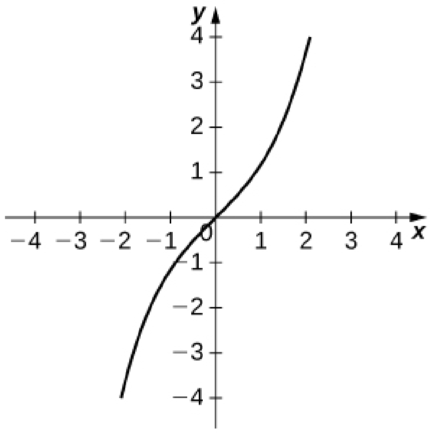
Range $-1 < \tanh x < 1$

Asymptotes $x \to +\infty, y \to 1; x \to -\infty, y \to -1$

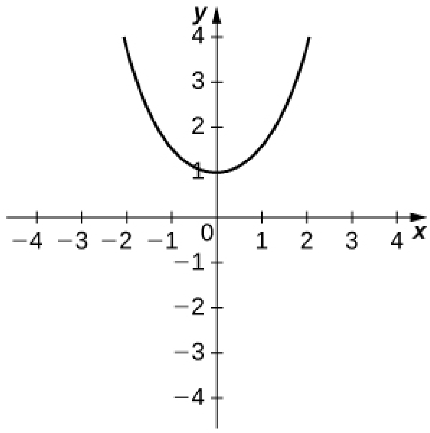
x-intercept $(0, 0)$

y-intercept $(0, 0)$

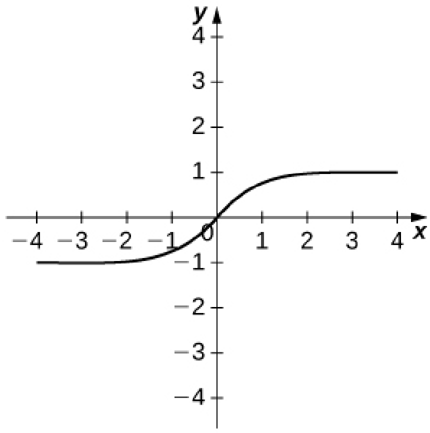
6.1.4 Function graphs



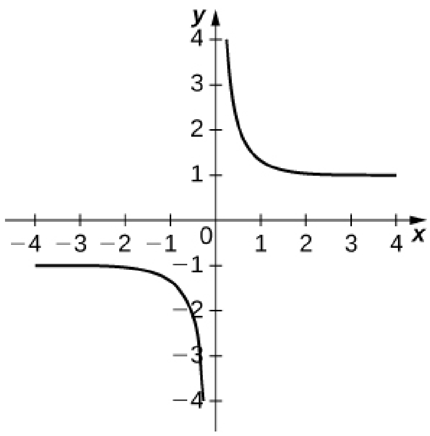
$y = \sinh x$
(a)



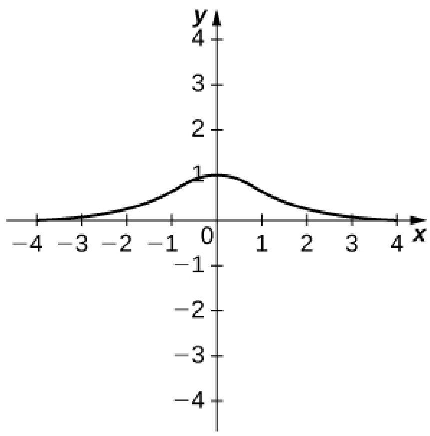
$y = \cosh x$
(b)



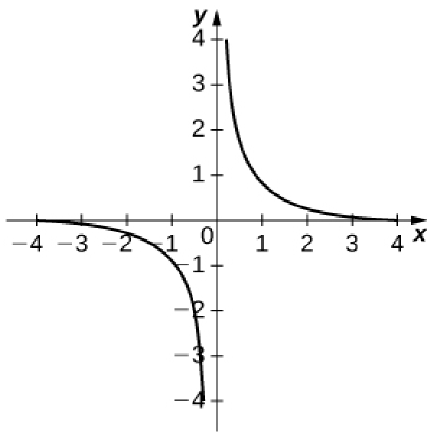
$y = \tanh x$
(c)



$y = \coth x$
(d)



$y = \operatorname{sech} x$
(e)



$y = \operatorname{csch} x$
(f)

6.2 Inverse hyperbolic functions

- $\operatorname{arsinh} x = \ln\left[x + \sqrt{x^2+1}\right]$
- $\operatorname{arcosh} x = \ln\left[x + \sqrt{x^2-1}\right] \quad (x \geq 1)$
- $\operatorname{artanh} x = \ln\left[\dfrac{1+x}{1-x}\right] \quad (-1 < x < 1)$

6.2.1 Proof

Example 6.1

Show that $\operatorname{arsinh} x = \ln\left[x + \sqrt{x^2+1}\right]$

Solution

Let $y = \operatorname{arsinh} x$

$$\begin{aligned}
\sinh y &= x \\
\frac{e^y - e^{-y}}{2} &= x \\
e^y - e{-}y &= 2x \\
e^{2y} - 1 &= 2xe^y \\
(e^y - x)^2 &= x^2 + 1 \\
e^y &= x \pm \sqrt{x^2+1}
\end{aligned}$$

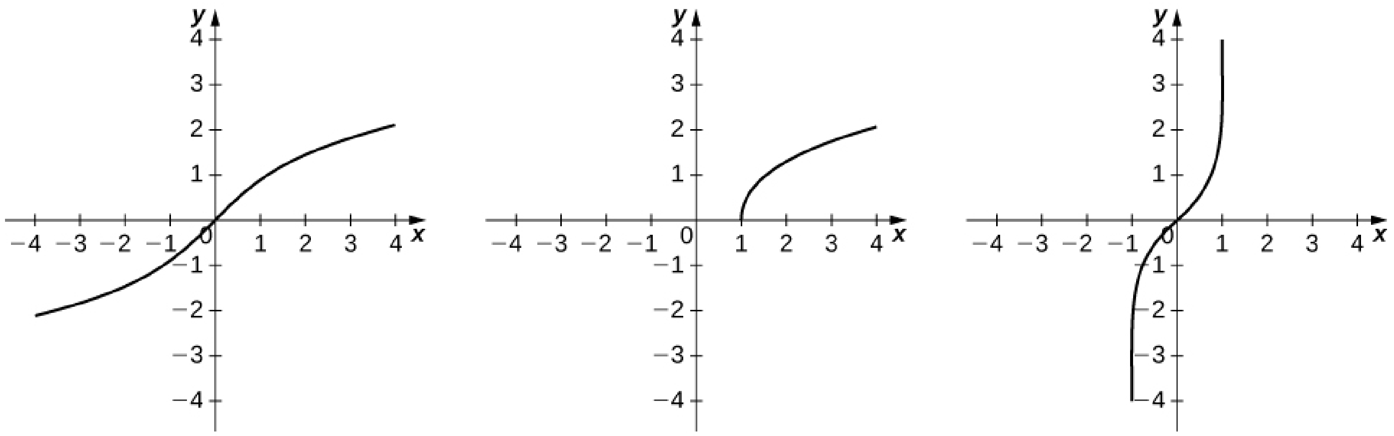
$e^y = x + \sqrt{x^2+1}$ since $\sqrt{x^2+1} > 0$ so it makes e^y negative which is impossible

Hence $y = \ln\left[x + \sqrt{x^2+1}\right]$ so $\operatorname{arsinh} x = \ln\left[x + \sqrt{x^2+1}\right]$

☞ **Remark**

Prove these identities by finding value of e^y in quadratic equations, think about domain when deciding the sign before the square root

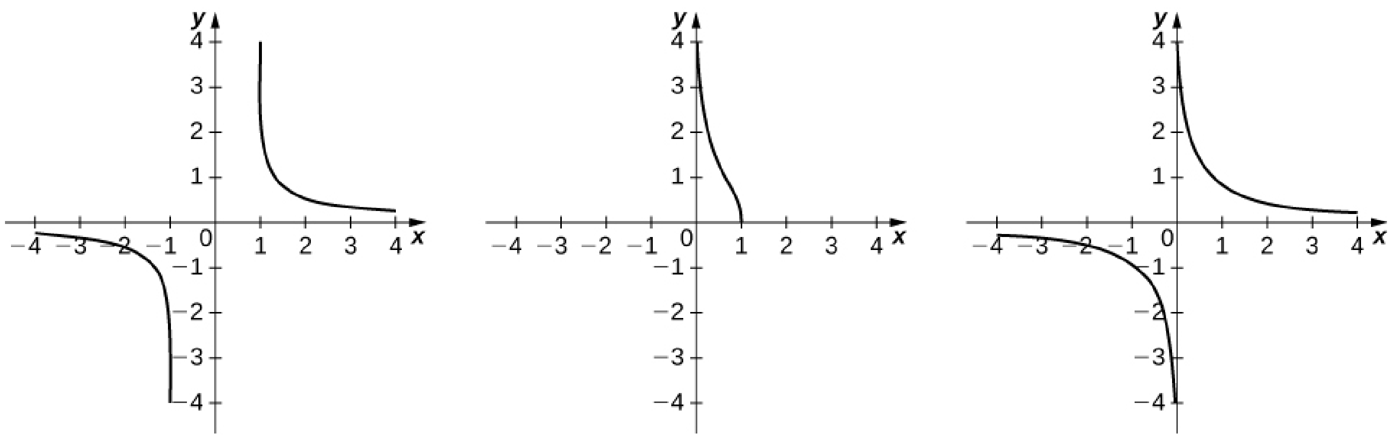
6.2.2 Graphs



$y = \sinh^{-1} x$
(a)

$y = \cosh^{-1} x$
(b)

$y = \tanh^{-1} x$
(c)



$y = \coth^{-1} x$
(d)

$y = \operatorname{sech}^{-1} x$
(e)

$y = \operatorname{csch}^{-1} x$
(f)

6.3 Identities of hyperbolic functions

Similar to trigonometric identities:

- $\tanh x = \dfrac{\sinh x}{\cosh x}$
- $\cosh^2 x - \sinh^2 x = 1$
- $\tanh^2 x + \operatorname{sech}^2 x = 1$
- $\coth^2 x - \operatorname{csch}^2 x = 1$

6.3.1 Addition

- $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
- $\tanh(x+y) = \dfrac{\sinh(x+y)}{\cosh(x+y)} = \dfrac{\sinh x \cosh y + \sinh y \cosh x}{\cosh x \cosh y + \sinh x \sinh y} = \dfrac{\frac{\sinh x}{\cosh x} + \frac{\sinh y}{\cosh y}}{1 + \frac{\sinh x \sinh y}{\cosh x \cosh y}} = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

6.3.2 Double angle

- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 + 1$
- $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

6.3.3 Power descending

- $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
- $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

6.4 Calculus with hyperbolic functions

6.4.1 Differentiation

$$(\sinh x)' = \cosh x$$
$$(\cosh x)' = \sinh x$$
$$(\tanh x)' = 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$(\operatorname{csch} x)' = -\coth x \operatorname{csch} x$$
$$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$$
$$(\coth x)' = -\operatorname{csch}^2 x$$
$$(\operatorname{arsinh} x)' = \frac{1}{\sqrt{1+x^2}}$$
$$(\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}}$$
$$(\operatorname{artanh} x)' = \frac{1}{1-x^2}$$

6.4.2 Integration

$$\int \sinh x \,\mathrm{dx} = \cosh x + c$$

$$\int \cosh x \,\mathrm{dx} = \sinh x + c$$

$$\int \tanh x \,\mathrm{dx} = \ln|\cosh x| + c$$

$$\int \coth x \,\mathrm{dx} = \ln|\sinh x| + c$$

$$\int \operatorname{sech} x \,\mathrm{dx} = \ln|-\operatorname{sech} x + \tanh x| + c = \ln\left|\tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right| + c$$

$$\int \operatorname{csch} x \,\mathrm{dx} = -\ln|\operatorname{csch} x + \coth x| + c$$

$$\int \sinh^2 x \,\mathrm{dx} = \int \frac{\cosh 2x - 1}{2}\,\mathrm{dx} = \frac{\sinh 2x - 2x}{4} + c$$

$$\int \cosh^2 x \,\mathrm{dx} = \int \frac{\cosh 2x + 1}{2}\,\mathrm{dx} = \frac{\sinh 2x + 2x}{4} + c$$

$$\int \tanh^2 x \,\mathrm{dx} = \int 1 - \operatorname{sech}^2 x \,\mathrm{dx} = x - \tanh x + c$$

$$\int \coth^2 x \,\mathrm{dx} = \int 1 + \operatorname{csch}^2 x \,\mathrm{dx} = x - \coth x + c$$

$$\int \sinh x \cosh x \,\mathrm{dx} = \int \frac{\sinh 2x}{2}\,\mathrm{dx} = \frac{\cosh 2x}{4} + c$$

$$\int \tanh x \operatorname{sech} x \,\mathrm{dx} = -\operatorname{sech} x + c$$

$$\int \coth x \operatorname{csch} x \,\mathrm{dx} = -\operatorname{csch} x + c$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,\mathrm{dx} = \operatorname{arcosh}\left(\frac{x}{a}\right) + c = \ln\left(x + \sqrt{x^2 - a^2}\right) + c \ \ (x > a)$$

$$\int \frac{1}{\sqrt{a^2 + x^2}}\,\mathrm{dx} = \operatorname{arsinh}\left(\frac{x}{a}\right) + c = \ln\left(x + \sqrt{x^2 + a^2}\right) + c$$

$$\int \frac{1}{a^2 - x^2}\,\mathrm{dx} = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + c$$

Chapter 7

Methods in differential equations

7.1 Types of differential equations

7.1.1 Order

First order $\frac{\mathrm{d}y}{\mathrm{d}x}$

Second order $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

nth order $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$

7.1.2 Homogeneous / non-homogeneous

Homogeneous differential equations

Non-homogeneous differential equations

7.2 First order differential equations

7.2.1 Separation

- Try to separate the equation into $f(y)\,\mathrm{dy} = g(x)\,\mathrm{dx}$
- Integrate both sides

7.2.2 The integrating factor

A first order differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ can be solved by multiplying every term by the integrating factor $I = e^{\int P(x)dx}$

Example 7.1

Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = e^x$$

Solution

The integrating factor is $e^{\int(-4)dx} = e^{-4x}$

$$
\begin{aligned}
e^{-4x}\frac{\mathrm{d}y}{\mathrm{d}x} - 4e^{-4x}y &= e^{x}e^{-4x} \\
\frac{\mathrm{d}}{\mathrm{dx}}(e^{-4x}y) &= e^{-3x} \\
e^{-4x}y &= \int e^{-4x}\,dx \\
&= -\frac{1}{3}e^{-3x} + c \\
y &= -\frac{1}{3}e^{x} + ce^{4x}
\end{aligned}
$$

7.2.3 u-substitution

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

1. Divide by y^n
2. u-sub: $u = y^{1-n}$
3. Change into first order linear differential differential equation

7.3 Second order homogeneous differential equations

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

1. Use u-sub: $u = \ln x$, $x = e^u$
2. Convert into $\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} = f(y)$

7.4 Second order non-homogeneous differential equations

Chapter 8

Modelling in differential equations

Part III

Further Pure 1

Chapter 1

Further vectors

1.1 Finding areas of shapes

- Area of triangle $\triangle ABC$ = $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$
- Area of parallelogram $ABCD$ = $|(b-a) \times (d-a)| = |(a \times b) + (b \times d) + (d \times a)|$ (A, B, C, D have position vector a, b, c, d respectively)

1.2 Scalar triple product

- Volume of parallelepiped = $a \cdot (b \times c)$ (a, b, c = 3 different sides)
- Volume of tetrahedron $ABCD$ = $\frac{1}{6}|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})|$

1.3 Straight lines

1.3.1 Vector equation of line

- $(\vec{r} - \vec{a}) \times \vec{b} = 0$
- $\vec{a}$ = position vector of a point on line, $\vec{b}$ = directional vector

1.3.2 Direction cosines

For straight line $(r-a) \times b = 0$, where $a = x\vec{i} + y\vec{j} + z\vec{k}$ and the line makes angle α, β and γ with the positive x-, y- and z-axes respectively:

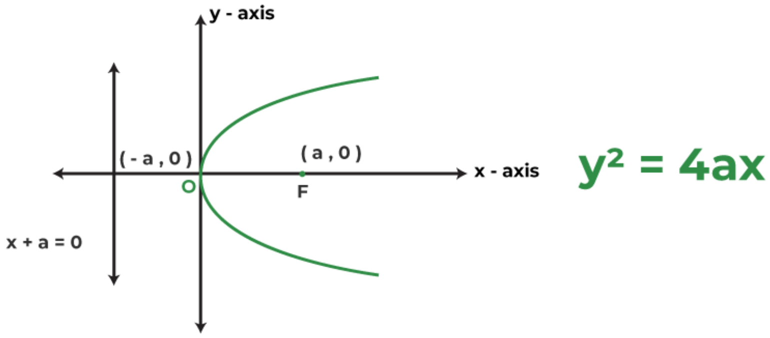
- $l = \cos\alpha = \frac{x}{|a|}$
- $m = \cos\beta = \frac{y}{|a|}$
- $n = \cos\gamma = \frac{z}{|a|}$
- $l^2 + m^2 + n^2 = 1$

Chapter 2

Conic sections 1

2.1 Parabola

2.1.1 Graph



- Symmetrix about the x-axis
- Focus at $(a, 0)$
- Vertex at $(0, 0)$

2.1.2 Definition

- The locus of points that are the **same distance** from a fixed point, S, called the **focus**, and a fixed straight line called the **directrix**
- $\dfrac{\text{distance to foci}}{\text{distance to directrix}} = e = 1$

2.1.3 Cartesian equation

- $y^2 = 4ax$ $(a > 0)$

2.1.4 Parametric equation

- $x = at^2$
- $y = 2at$
- $t \in \mathbb{R}$

2.1.5 Eccentricity

- $e = 1$

2.1.6 Directrix

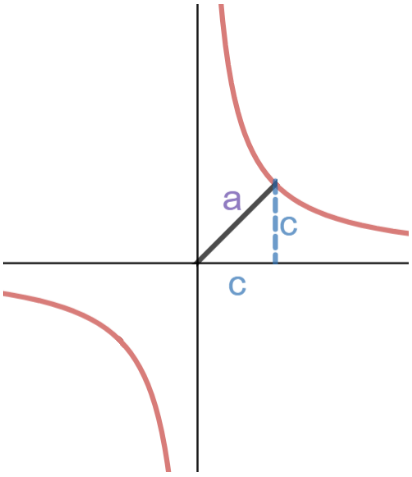
- The directrix has equation $x + a = 0$

2.1.7 Tangents and normals

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{t} = \dfrac{2a}{y}$
- Equation of tangent: $ty = x + at^2$ at $P(at^2, 2at)$
- Equation of normal: $y + tx = 2at + at^3$ at $P(at^2, 2at)$

2.2 Rectangular hyperbolas

2.2.1 Graph



- Asymptotes at $x = 0$ and $y = 0$ (x and y-axis)

2.2.2 Definition

- The locus of points that are the **same distance** from a fixed point, S, called the **focus**, and a fixed straight line called the **directrix**
- $\dfrac{\text{distance to foci}}{\text{distance to directrix}} = e = 1$

2.2.3 Cartesian equation

- $xy = c^2$ $(c > 0)$

2.2.4 Parametric equation

- $x = ct$
- $y = \dfrac{c}{t}$
- $t \neq 0, t \in \mathbb{R}$

2.2.5 Eccentricity

- $e = \sqrt{2}$

2.2.6 Directrix

- $x + y = \pm c\sqrt{2}$

2.2.7 Tangents and normals

- Equation of tangent: $x + t^2 y = 2ct$ at $P\left(ct, \frac{c}{t}\right)$ or $y \times y_0 = 4p\left(\frac{x + x_0}{2}\right)$ at (y_0, x_0)
- Equation of normal: $t^3 x - ty = c\left(t^4 - 1\right)$ at $P\left(ct, \frac{c}{t}\right)$

2.3 Reciprocal equations

2.3.1 $y = \frac{k}{x}$

- $x = t$
- $y = \frac{k}{t}$
- Asymptotes: $y = 0$, $x = 0$

2.3.2 $y = ax + \frac{b}{x}$

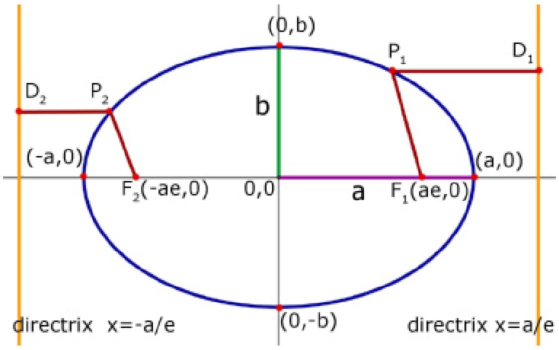
- Asymptotes: $y = ax$, $x = 0$

Chapter 3

Conic sections 2

3.1 Ellipse

3.1.1 Graph



3.1.2 Definition

- $PF_1 + PF_2 = 2a$
- $\dfrac{\text{distance from point } P \text{ to a focus}}{\text{distance from the same point to the corresponding directrix}} = e = \dfrac{c}{a}$

3.1.3 Cartesian equation

Centre $(0, 0)$ $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Centre (h, k) $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

3.1.4 Parametric equation

Centre $(0, 0)$ $x = a\cos t, y = b\sin t\ (0 \le t < 2\pi)$

Centre (h, k) $x = h + a\cos t, y = k + b\sin t\ (0 \le t < 2\pi)$

3.1.5 Eccentricity

- $e = \dfrac{c}{a} = \sqrt{1 - \dfrac{b^2}{a^2}}$
- $0 < e < 1$

3.1.6 Directrix

- $x = h \pm \dfrac{a^2}{c} = h \pm \dfrac{a}{e}$
- $(h-c,0)$ corresponds to $x = h - \dfrac{a}{e}$, $(h+c,0)$ corresponds to $x = h + \dfrac{a}{e}$

3.1.7 Tangents and normals

- Equation of tangent: $bx\cos t + ay\sin t = ab$ at $P(a\cos t, b\sin t)$
- Equation of normal: $ax\sin t - by\cos t = (a^2-b^2)\cos t\sin t$ at $P(a\cos t, b\sin t)$

3.1.8 Chord length

For chord AB with gradient m:

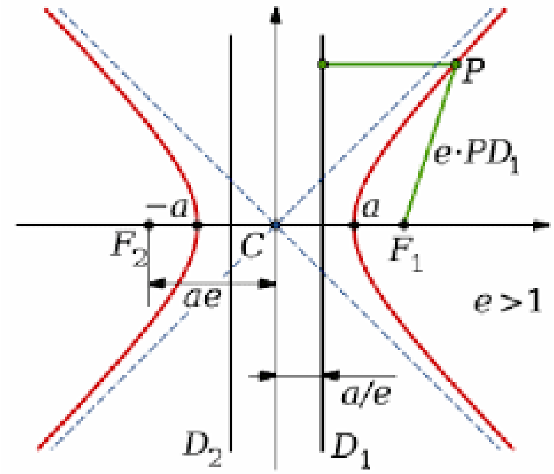
$$\begin{aligned}|AB| &= \sqrt{(1+m^2)(x_1-x_2)^2} \\ &= \sqrt{1+m^2} \times \sqrt{(x_1+x_2)^2 - 4x_1x_2}\end{aligned}$$

3.1.9 Circumference

$$\begin{aligned}C &= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{\mathrm{dx}}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{dy}}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\ &= 4a\int_0^{\frac{\pi}{2}} \sqrt{1-e^2\cos^2\theta}\,\mathrm{d}\theta \\ &\approx \pi\left[3(a+b) - \sqrt{(3a+b)(3b+a)}\right]\end{aligned}$$

3.2 Hyperbola

3.2.1 Graph



- Asymptotes at $y = \pm\dfrac{b}{a}x$

3.2.2 Definition

- $|PF_1 - PF_2| = 2a$

3.2.3 Cartesian equation

Centre $(0,0)$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(b^2 = c^2 - a^2)$

Centre (h,k) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

3.2.4 Parametric equation

Using sec and tan

Centre $(0,0)$ $x = a\sec t,\ y = b\tan t$

Centre (h,k) $x = h + a\sec t,\ y = k + b\tan t$

Domain $t \neq \frac{\pi}{2} + k\pi$

Using cosh and sinh

Centre $(0,0)$ $x = \pm a\cosh t,\ y = b\sinh t$

Centre (h,k) $x = h \pm a\cosh t,\ y = k + b\sinh t$

Domain $t \in \mathbb{R}$

3.2.5 Eccentricity

- $e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}$
- $e > 1$

3.2.6 Directrix

- $x = h \pm \frac{a^2}{c} = h \pm \frac{a}{e}$
- $(h-c,0)$ corresponds to $x = h + \frac{a}{e}$, $(h+c,0)$ corresponds to $x = h - \frac{a}{e}$

3.2.7 Tangents and normals

Tangent equations

- $ay\sinh t + ab = bx\cosh t$ at $P(a\cosh t, b\sinh t)$
- $bx\sec\theta - ay\tan\theta = ab$ at $P(a\sec\theta, b\tan\theta)$

Normal equations

- $ax\sinh t + by\cosh t = (a^2+b^2)\sinh t\cosh t$ at $P(a\cosh t, b\sinh t)$
- $by + ax\sin\theta = (a^2+b^2)\tan\theta$ at $P(a\sec\theta, b\tan\theta)$

3.3 Finding the type of the quadratic curve

For curve $ax^2 + bxy + cy^2 + dx + ey + f = 0$:

- Find the determinant: $\Delta = b^2 - 4ac$
- $\Delta < 0$: ellipse if $a \neq c$ or $b \neq 0$, circle if $a = c$ and $b = 0$
- $\Delta = 0$: if $a = c = 0$ then a straight line, otherwise parabola
- $\Delta > 0$: $a = -c$ and $b = 0$: intersecting lines, otherwise hyperbola

Chapter 4

Inequalities

4.1 Triangle inequalities

$$|a| - |b| \leq |a \pm b| \leq |a| + |b|$$

4.2 Modulus inequalities

4.2.1 Common methods

- $|f(x)| \geq g(x)$: $f(x) \geq g(x)$ or $f(x) \leq -g(x)$
- $|f(x)| \leq g(x)$: $-g(x) \leq f(x) \leq g(x)$ or square both sides (solution only exists when $g(x) \geq 0$ so no solution is omitted)
- $|f(x)| \leq |g(x)|$: $f(x)^2 <\leq g(x)^2$
- $|f(x)| + |g(x)| \leq m(x)$: solve each interval separately

4.2.2 Things to consider

1. Sketching graphs
2. Finding critical values
3. Rearranging the inequality

Chapter 5

The t-formulae

5.1 t-substitution formulae

When $t = \tan\left(\frac{\theta}{2}\right)$:

- $\sin\theta = \frac{2t}{1+t^2}$
- $\cos\theta = \frac{1-t^2}{1+t^2}$
- $\tan\theta = \frac{2t}{1-t^2}$

5.2 Uses

- Trigometric equations
- Integration / differentiation

Chapter 6

Taylor series

6.1 Taylor series

$$f(x+a) = f(a) + f'(a)x + \frac{f''(a)x^2}{2!} + \cdots + \frac{{}^{(r)}(a)x^r}{r!} + \ldots$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(r)}(a)(x-a)^r}{r!} + \ldots$$

6.2 Finding limits

Given $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$:

- $\lim_{x \to a} cf(x) = cL$ (c is a constant)
- $\lim_{x \to a} (f(x) + g(x)) = L + M$
- $\lim_{x \to a} f(x)g(x) = LM$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

6.3 Series solution of differential equations

Chapter 7

Methods in calculus

Chapter 8

Numerical methods

Chapter 9

Redicible differential equations

Part IV

Further Mechanics 1

Chapter 1

Momentum and impulse

1.1 Equations

Momentum: $p = mv$ (unit = kg m s^{-1})

Impulse: $I = \Delta mv = mv - mu = Ft$ (unit = N s)

1.2 Conservation of momentum

- Momentum is always conserved in any interaction where no external forces act
- Elastic collision: $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$
- Sticking together: $m_1u_1 + m_2u_2 = (m_1 + m_2)v_{1+2}$
- Explosion: $m_1v_1 + m_2v_2 = 0$

1.3 Momentum as a vector

Calculate each direction independently

Chapter 2

Work, energy and power

2.1 Work and energy

Work done: $w = Fd =$ force $\times$ distance moved in the direction of the force $=$ change in kinetic energy

Kinetic energy: K.E. $= \frac{1}{2}mv^2$

Potential energy: P.E. $= mgh$

* You must choose a **zero** level of potential energy before calculating a particle's potential energy

2.2 Conservation of mechanical energy

When no external forces (other than gravity) do work on a particle during its motion, the sum of the particle's **kinetic energy and (gravitational and elastic) potential energy** remains constant

2.3 Work-energy principle

The change in the total energy of a particle is equal to the work done on the particle

2.4 Power

Definition: Power is the rate of doing work

Equation: Power $= Fv =$ driving force produced by the engine $\times$ velocity

Chapter 3

Elastic strings and springs and elastic energy

3.1 Hooke's law

- Tension produced $\propto x \to T = kx$, where k is a constant
- k depends on the unstretched length of the string or spring (l) and the **modulus of elasticity of the string or spring** (λ, unit = N)
- Hence $T = \dfrac{\lambda x}{l}$

* Can also be applied if the string or spring is compressed

3.2 Elastic energy

- Work done in stretching an elastic string or spring of modulus of elasticity λ from its natural length l to a length of $(l + x)$ = $\dfrac{\lambda x^2}{2l}$
- Elastic potential energy stored = amount of energy used to stretch the spring to a length of $(l + x)$ = $\dfrac{1}{2}kx^2$ = $\dfrac{\lambda x^2}{2l}$

* Can also be applied when an elastic string or spring is compressed

Chapter 4

Elastic collisions in one dimension

4.1 Newton's law of restitution

Particles colliding: $e = \frac{\text{speed of separation of particles}}{\text{speed of approach of particles}} = \frac{v_B - v_A}{u_B - u_A}$

Direct collision with a smooth plane: $e = \frac{\text{speed of rebound}}{\text{speed of approach}}$

Range of e: $0 \leq e \leq 1$

4.2 Loss of kinetic energy

Loss of kinetic energy due to impact = $(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2) - (\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2)$

4.3 Problems with modelling

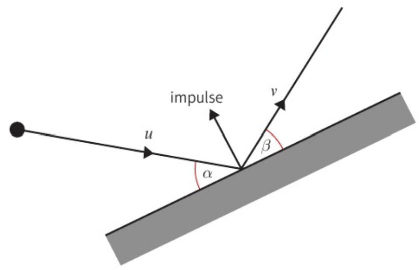
Bouncing for an infinite number of times: the ball should stop bouncing after a finite number of bounces

Spheres modelled as particles: air resistance is ignored

Chapter 5

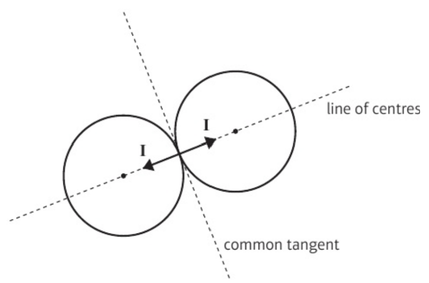
Elastic collisions in two dimensions

5.1 Oblique impact with a fixed surface



- The component of the velocity of the sphere parallel to the surface is unchanged: $v \cos \beta = u \cos \alpha$
- The component of the velocity of the sphere perpendicular to the surface can be found with Newton's law of restitution: $v \sin \beta = eu \sin \alpha$
- Hence $\tan \beta = e \tan \alpha$, since $0 \leq e \leq 1$, $\beta \leq \alpha$
- Loss of kinetic energy $= \dfrac{1}{2}mu^2 - \dfrac{1}{2}mv^2$

5.2 Oblique impact of smooth spheres



- Impulse affecting each sphere acts along the line of centres
- The components of the velocities of the spheres **perpendicular** to the line of centres are unchanged in the impact
- The principle of conservation of momentum and Newton's law of restitution applies **parallel** to the line of centres