

Boundary Scattering in 1+1 Dimensions as an Aharonov-Bohm Effect

Surya Ganguli, Petr Hořava and Anthony Ndirango

*Center for Theoretical Physics and Department of Physics, University of California
Berkeley, CA 94720-7300, USA*

and

*Theoretical Physics Group, Lawrence Berkeley National Laboratory
Berkeley, CA 94720-8162, USA*

`sganguli, horava, andirang@socrates.berkeley.edu`

Abstract

The boundary scattering problem in $1+1$ dimensional CFT is relevant to a multitude of areas of physics, ranging from the Kondo effect in condensed matter theory to tachyon condensation in string theory. Invoking a correspondence between CFT on $1+1$ dimensional manifolds with boundaries and Chern-Simons gauge theory on $2+1$ dimensional \mathbb{Z}_2 orbifolds, we show that the $1+1$ dimensional conformal boundary scattering problem can be reformulated as an Aharonov-Bohm effect experienced by chiral edge states moving on a $1+1$ dimensional boundary of the corresponding $2+1$ dimensional Chern-Simons system. The secretly topological origin of this physics leads to a new and simple derivation of the scattering of a massless scalar field on the line interacting with a sinusoidal boundary potential.

January 2005

1. Introduction

Conformal field theory (CFT) in $1 + 1$ dimensions on a manifold with boundary [1] serves as a unifying framework for many important problems in physics. Examples in both condensed matter and string theory include quantum impurities in metals (the Kondo effect [2,3]), proton-monopole scattering [4], quantum wire junctions [5], D-branes [6-8], and tachyon condensation [9]. In all of these problems the scattering amplitude of a bulk excitation in the presence of various boundary conditions is of physical interest. In this letter we prove a conjecture first stated in [10] and show that this amplitude is essentially an Aharonov-Bohm phase by exploiting a connection to Chern-Simons (CS) theory. Thus all of the physical phenomena above are secretly topological in origin.

In section 2 we briefly review the connection between $2 + 1$ dimensional CS theory and CFT in $1 + 1$ dimensions [11,12], and an extension of this relation connecting boundary CFT to CS theory on a \mathbb{Z}_2 orbifold [10]. In particular we explain the $2 + 1$ dimensional origin of the bulk one point function in the presence of a boundary. In section 3 we state the formulation of Kondo scattering in terms of boundary CFT, and show that the scattering amplitude is precisely an Aharonov-Bohm phase when interpreted using the results of section 2. In section 4 we use similar techniques applied to the boundary CFT of a massless scalar field interacting with a sinusoidal boundary potential, which enables us to give a new and simpler topological derivation of some results in [13]. We conclude in section 5.

2. Boundary CFT and CS Theory on Orbifolds

2.1. The Chiral Picture

CS theory with a compact, simple gauge group \mathcal{G} on a three manifold M has the action

$$S[A] = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \quad (2.1)$$

The observables of the theory are products of Wilson lines

$$W_R(C) = \text{Tr}_R P \exp \int_C A, \quad (2.2)$$

where R belongs to the finite set of integrable representations of the Kac-Moody algebra \mathfrak{g} at level k , and C is a closed line in M . In the case where $M = \Sigma \times \mathbf{R}$ with Wilson lines in representations R_i piercing Σ at points z_i , one can canonically quantize the theory [11,12] to obtain a finite-dimensional quantum Hilbert space $\mathcal{H}_{\Sigma, R_i}$. The connection to CFT lies precisely in the fact that $\mathcal{H}_{\Sigma, R_i}$ is also the space of conformal blocks of a $2D$ rational CFT, namely a level k WZW model associated with group \mathcal{G} [11,12]. Thus the chiral correlators

of primary fields in the CFT can be thought of as elements of the CS quantum Hilbert space.

Furthermore, one can use the path integral formulation of CS theory to pick out specific elements of $\mathcal{H}_{\Sigma, R_i}$. Suppose M is now a three manifold with boundary Σ and appropriate Wilson line insertions ending on the boundary. To define the path integral on M , one must fix the gauge field A over the boundary Σ . The path integral over A in the bulk is then a function (or more precisely a section of a line bundle) over the space of gauge equivalence classes of A on the boundary Σ . However $\mathcal{H}_{\Sigma, R_i}$ itself can be thought of as the space of such sections, and so the path integral on M picks out a distinguished element of this space.

In the important case where M is a solid torus with boundary $\Sigma = T^2$, \mathcal{H}_{Σ} is spanned by the basis $|\Psi_{T^2}^{R_i}\rangle$ where again R_i is an integrable representation of \mathfrak{g} at level k . In particular, $|\Psi_{T^2}^{R_i}\rangle$ can be obtained by evaluating the path integral on M with a single Wilson loop in the representation R_i inserted along the unique noncontractible cycle of M . Since $SL(2; \mathbb{Z})$ acts as the group of homotopy classes of diffeomorphisms of T^2 , it induces an action on \mathcal{H}_{T^2} . In the following, we will need the matrix elements of the generator

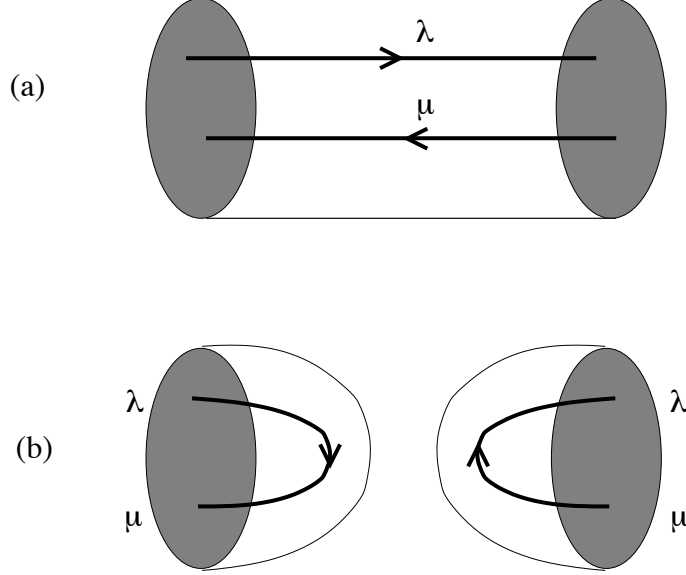
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2; \mathbb{Z}) \quad (2.3)$$

in the basis $|\Psi_{T^2}^{R_i}\rangle$. These elements can be read off from the transformation properties of the characters of \mathfrak{g} at level k under the modular transformation $\tau \rightarrow -1/\tau$ where τ is the complex structure modulus of T^2 . When $\mathfrak{g} = S\mathfrak{U}(2)$, there are $k+1$ integrable representations with highest weight λ where $0 \leq \lambda \leq k$, and in this basis, the matrix elements of the corresponding modular S-matrix are given by,

$$S_{\lambda\mu} \equiv \langle \Psi_{T^2}^{\lambda} | S | \Psi_{T^2}^{\mu} \rangle = \frac{1}{k+2} \sin \frac{\pi(\lambda+1)(\mu+1)}{k+2}. \quad (2.4)$$

2.2. Left Movers and Right Movers

So far we have reviewed the three dimensional origin of chiral correlators in the CFT. Now, to explain the origin of boundary CFT, we must first see how both chiral halves consisting of left-movers and right-movers come from three dimensions in the case without boundary. This issue was addressed in [12,14-16]. The key idea is to consider a thickening $M = \Sigma \times [0, 1]$ of the closed Riemann surface Σ . One can then show [12] that the gauge invariant degrees of freedom of CS theory live on the two disjoint boundary components of M , and correspond to the left-movers and right-movers of the full CFT. The insertion of a left-moving primary field in representation R_i at point z_i in the CFT corresponds, in CS theory, to the insertion of an oriented Wilson line in the representation R_i travelling from



1. Elements of the space of two point conformal blocks on Σ . (a) $M = \Sigma \times I$ with Wilson line insertions. The shaded boundary represents two disjoint copies of Σ . (b) A standard basis of the space of conformal blocks.

the left to right boundary components, along $z_i \times I$. For the insertion of a right-moving primary, the orientation of the Wilson line is reversed, as in figure (1.a).

Since M has two boundary components, the path integral of CS theory on M yields an element of $\mathcal{H}_{\Sigma, R_i} \otimes \mathcal{H}_{\Sigma, R_i}^*$ where R_i represents the full collection of left and right moving Wilson line representations. To go from an element of this space to a specific CFT correlation function, one must choose a basis of conformal blocks, or correlators, and expand in this basis. For example, consider figure (1) in the case when $\Sigma = S^2$. In order for the space of conformal blocks to have nonzero dimension, we must have $\mu = \lambda^*$ where λ^* is the conjugate representation of λ . In this case the dimension of $\mathcal{H}_{S^2, \{\lambda, \lambda^*\}}$ is one, corresponding to the fact that there is only one two point function on the sphere, namely

$$\langle \mathcal{O}^\lambda(z_1) \bar{\mathcal{O}}^{\lambda^*}(\bar{z}_2) \rangle = \frac{1}{(z_1 - z_2)^{2h_\lambda}}, \quad (2.5)$$

where $\mathcal{O}^\lambda(z)$ is a left moving WZW primary field with conformal weight h_λ .

Now let $|\Psi_M\rangle$ be the CS path integral over the three manifold in figure (1.a) with $\Sigma = S^2$ and $\mu = \lambda^*$. Similarly let $|\Psi_{S^2}^{\{\lambda, \lambda^*\}}\rangle \otimes |\Psi_{-S^2}^{\{\lambda, \lambda^*\}}\rangle$ be the path integral in (1.b), and choose c^2 times this to be our standard basis for $\mathcal{H}_{S^2, \{\lambda, \lambda^*\}} \otimes \mathcal{H}_{S^2, \{\lambda, \lambda^*\}}^*$, where c is an arbitrary normalization constant. According to the prescription, we require that $|\Psi_M\rangle$ correspond to the correlator (2.5), and so its expansion coefficient in the standard basis should be one:

$$c^2 \langle \Psi_M | \Psi_{S^2}^{\{\lambda, \lambda^*\}} \rangle \otimes | \Psi_{-S^2}^{\{\lambda, \lambda^*\}} \rangle \equiv 1. \quad (2.6)$$

According to the axioms of topological quantum field theory [17], the inner product in (2.6) can be computed by gluing together the two three manifolds in figure (1). After gluing the left (right) side of figure (1.b) to the right (left) side of (1.a) along the common S^2 , the inner product reduces to the path integral of CS theory on a three sphere with the unknot in the representation λ . This knot invariant is given by $S_{0\lambda}$ where 0 denotes the vacuum representation [11]. Thus we obtain the correct normalization $c = \frac{1}{\sqrt{S_{0\lambda}}}$. Indeed one can readily check that $\frac{1}{\sqrt{S_{0\lambda}}}|\Psi_{S^2}^{\{\lambda, \lambda^*\}}\rangle$ has unit norm in $\mathcal{H}_{S^2, \{\lambda, \lambda^*\}}$. This is the CS analogue of canonical normalization for the chiral primary field $\mathcal{O}^\lambda(z)$.

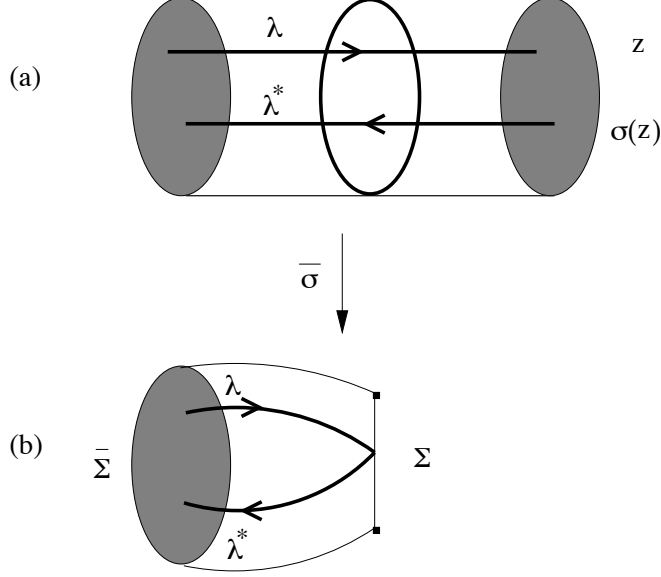
2.3. CS Orbifolds and Boundary CFT's

We have seen that the left and right-movers in the full CFT on a closed surface Σ appear in the CS picture at the two ends of a thickening, $\Sigma \times I$ of Σ . Now given that open strings, or boundary CFT's can be obtained via worldsheet orbifolds [6], it is natural to try to seek the three dimensional origin of boundary CFT's in orbifolds of the CS picture. This was carried out in detail in [10], and here we summarize the results.

Suppose we wish to calculate the bulk one-point function on a Riemann surface Σ with boundary. Let $\bar{\Sigma}$ be the oriented double of Σ . By definition $\bar{\Sigma}$ is a closed, oriented Riemann surface with an orientation reversing involution σ such that $\Sigma = \bar{\Sigma}/\sigma$. The fixed point locus of σ projects to the boundary of Σ . For example, when Σ is the disk D_2 , $\bar{\Sigma}$ is the sphere S^2 , and σ is a reflection across a great circle of S^2 . Now the key idea is to consider the thickening $\bar{\Sigma} \times I$, and to extend the action of σ to a \mathbb{Z}_2 involution $\bar{\sigma}$ on $\bar{\Sigma} \times I$ which acts as $(z, t) \mapsto (\sigma(z), 1 - t)$. CS theory on the three dimensional orbifold $M_\Sigma \equiv (\bar{\Sigma} \times I)/\bar{\sigma}$ will correspond to boundary CFT.

Now consider the insertion of a left-moving bulk primary field in the representation λ at point z on Σ . This point has two pre-images on $\bar{\Sigma}$: z and $\sigma(z)$. Following the logic in the previous subsection, we wish to insert Wilson lines on $\bar{\Sigma} \times I$ corresponding to the primary field insertion on Σ , but now we must respect the $\bar{\sigma}$ involution. Thus in addition to inserting a Wilson line in the representation λ oriented from left to right along $z \times I$, we must also insert another with the representation λ^* oriented from right to left along $\sigma(z) \times I$ as in figure (2.a). Alternatively one can work directly on the singular orbifold projection M_Σ as in figure (2.b).

Thus we see that the CS orbifold couples the left and right-moving degrees of freedom together in a manner similar to the way in which the boundary in CFT reflects the two into each other. However when doing CS theory on orbifolds, special attention must be given to the fixed-point locus of $\bar{\sigma}$ in $\bar{\Sigma} \times I$ which gives rise to singularities in the orbifold M_Σ . It was shown in [10] that in order to define the path integral, extra data, namely a choice of holonomies of the CS gauge field around the singular locus, needs to be given. Once this choice is made, the singular locus, which now serves merely as an effective source

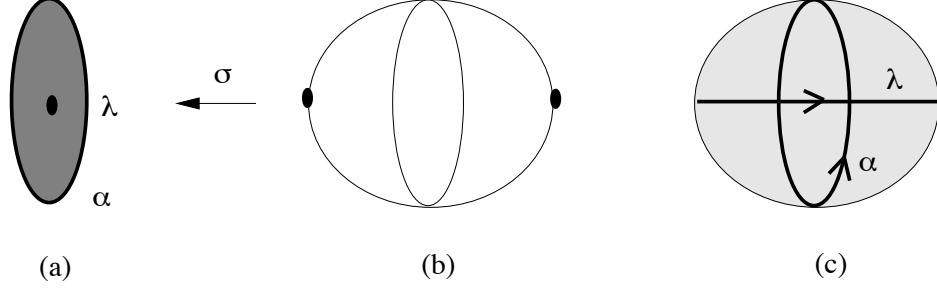


- 2.** (a) $\bar{\Sigma} \times I$ with $\bar{\sigma}$ invariant Wilson line insertions. The fixed point locus of $\bar{\sigma}$ carries an additional Wilson line wrapping the usual insertions. (b) The projection M_Σ . The circular Wilson line in (a) projects to the singular locus or boundary of Σ on the right.

of gauge field curvature, can be traded for a link of Wilson lines mimicking this source. Thus in figure (2.a), an extra set of Wilson line(s) must circle the two required by the bulk operator insertion. The representation(s) carried by these Wilson lines encode the boundary condition chosen in the CFT.

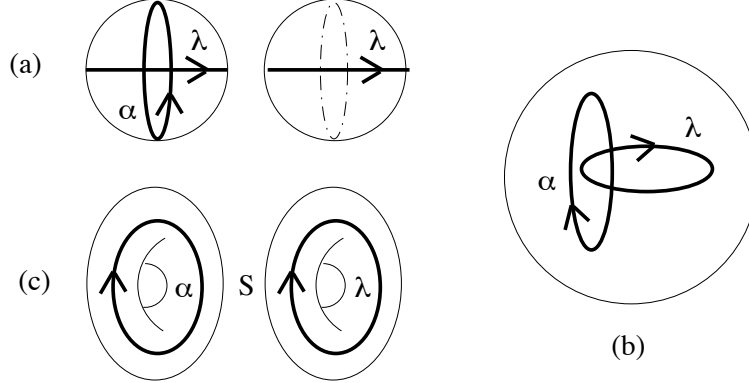
For the case when $\Sigma = D_2$, one can visualize M_Σ as a solid ball with boundary S^2 obtained from a continuous deformation of figure (2.b) [18,19]. One simply contracts all intervals $z \times I$ to a single point, for those $z \in \bar{\Sigma}$ lying on the fixed point locus of σ . The manifolds Σ , $\bar{\Sigma}$, and M_Σ are shown in figure (3), when $\Sigma = D_2$. More generally it can be shown that M_Σ obeys two important properties: its boundary ∂M_Σ is $\bar{\Sigma}$, and for every pair of images z and $\sigma(z)$ in $\bar{\Sigma}$, there exists a connecting path through the bulk M_Σ such that no two connecting paths for different z 's intersect. These two properties are apparent in figure (3). In general, bulk primary field insertions on Σ in the boundary CFT correspond to Wilson line insertions along the associated connecting path in M_Σ .

With the above preliminaries, we are now ready to compute the bulk one-point function on the disk. It is simply given by the path integral of CS theory on the solid ball in figure (3.c), and is therefore an element $|\Psi\rangle$ of the one dimensional vector space $\mathcal{H}_{S^2, \{\lambda, \lambda^*\}}$. We must specify the boundary condition by specifying the representation of the Wilson line circling the bulk Wilson line insertion in figure (3.c). For WZW models, one can show using the boundary state formalism [20], that symmetry preserving boundary conditions (Cardy states) are in one to one correspondence with the integrable representations of $\hat{\mathcal{G}}$



3. (a) A disk D_2 with Cardy boundary condition α and bulk primary field insertion in the representation λ . (b) Its oriented double is S^2 . The insertion point has two preimages, and the great circle projects to the boundary. (c) M_{D_2} is the solid ball, with horizontal paths connecting image pairs on the boundary. Both bulk and boundary in CFT become linked Wilson lines in three dimensions.

at level k , or equivalently with the set of allowed Wilson line representations in CS theory. Thus we simply choose the Wilson line to be in a representation α corresponding to the boundary condition we are interested in.



4. There are two ways to get the link in S^3 (b). One can either glue together two solid balls along their boundary S^2 as in (a) or glue together two solid tori via the modular diffeomorphism S as in (c).

To compute the actual one-point function, we calculate the inner product of $|\Psi\rangle$ with the normalized basis element $\frac{1}{\sqrt{S_{0\lambda}}}|\Psi_{S^2}^{\{\lambda, \lambda^*\}}\rangle$. This involves gluing the two spheres in figure (4.a) together to obtain a link of Wilson lines in S^3 as shown in (4.b). The inner product is then given by the link invariant $S_{\alpha\lambda}$ [11]. One can understand the appearance of $S_{\alpha\lambda}$ by considering an alternative surgery to obtain the link invariant. As shown in figure (4.c) one can obtain an S^3 with a link inside by gluing together two tori with Wilson lines inside via the gluing map S in (2.3). As explained in section 2.1, the path integrals on the tori in (4.c) yield the vectors $|\Psi_{T^2}^\lambda\rangle$ and $|\Psi_{T^2}^\alpha\rangle$ in the space \mathcal{H}_{T^2} . The result of gluing them together via the map S then computes the matrix element of S in this basis, which

is just the link invariant $S_{\alpha\lambda}$. Finally, we learn that the properly normalized one point function $\langle \mathcal{O}^\lambda(z, \bar{z}) \rangle_\alpha$ of a primary field in the λ representation, in the presence of a Cardy boundary condition corresponding to the α representation is $\frac{S_{\alpha\lambda}}{\sqrt{S_{0\lambda}}}$ times the standard one point function $\langle \mathcal{O}^\lambda(z, \bar{z}) \rangle_0$, recovering the results of the boundary state formalism. [20].

3. Boundary Scattering and Aharonov-Bohm: The Kondo Effect

The Kondo effect [2-3] concerns the interaction of conduction electrons with magnetic spin impurities in metals. In the case where there is a single impurity of spin s at the origin, there is a δ -function interaction with the conduction electrons. Hence in a partial wave expansion, only the S-wave interacts with the spin, and the problem is essentially one dimensional. The Hamiltonian lives on the half-line $r \geq 0$. In general there may be k flavors of electrons interacting with the spin s impurity. A renormalization group analysis of the interaction Hamiltonian [2-3] shows that when $2s \leq k$ there exists a nontrivial interacting boundary CFT that represents the infra-red fixed point of the system. The physics of the impurity-electron interactions renormalizes to a universality class of boundary conditions on the electron density determined by the spin s .

In modern language, this fixed point is given by $S\mathcal{U}(2)_k$ WZW theory. The boundary condition is the Cardy state corresponding to the weight $2s$ representation of $SU(2)$. Furthermore, the single particle scattering amplitude $S^{(1 \rightarrow 1)}$ for the conduction electrons can be written in terms of one-point functions of the boundary CFT:

$$S^{(1 \rightarrow 1)} = \frac{\langle \mathcal{O}^\lambda(z, \bar{z}) \rangle_\alpha}{\langle \mathcal{O}^\lambda(z, \bar{z}) \rangle_0}. \quad (3.1)$$

To make contact with the Kondo problem we choose $\lambda = 1$ corresponding to the weight of the spin-1/2 representation of the conduction electrons, and $\alpha = 2s$ corresponding to the weight of the spin- s representation of the impurity.

Now using the results at the end of section 2.3, we find simply that the above ratio of one-point functions yields

$$S^{(1 \rightarrow 1)} = \frac{S_{\alpha\lambda}}{S_{0\lambda}}. \quad (3.2)$$

Topologically, this is the ratio of the link invariant in figure (4.b) to the knot invariant of an unknot. The boundary condition appears in the numerator, where it essentially manifests itself as a charged particle traversing the knot in the denominator. Physically the one particle boundary scattering amplitude when viewed from the CS perspective probes the Aharonov-Bohm phase picked up by this charged particle as it traverses the unknot.

This viewpoint highlights the general structure of boundary scattering amplitudes in CS theory. The boundary dependence of the scattering of quanta is isolated in a bulk

one-point insertion on a disk as in figure (3.a). In CS language the disk puffs up into a solid ball, while the field insertion elongates into a Wilson line propagating across the bulk, as in figure (3.c). The boundary is promoted to a charged particle linking the Wilson line. Finally the phase that the scattered quanta pick up in the physical theory is none other than the Aharanov-Bohm phase that the boundary charged particle picks up as it encircles the propagating Wilson line.

4. Application to a $c = 1$ Boundary CFT

To illustrate the generality of the connection between boundary scattering and the Aharanov-Bohm effect, we now study a $c = 1$ CFT consisting of a free field interacting with a dynamical, sinusoidal potential which was analyzed in [13]. A Wick rotation of this same theory leads to a hyperbolic potential which was used to model tachyon condensation in string theory [9]. The Lagrangian is given by

$$L = \frac{1}{8\pi} \int_0^R d\sigma (\partial_\mu X)^2 - \frac{1}{2} (g e^{iX(0)/\sqrt{2}} + \bar{g} e^{-iX(0)/\sqrt{2}}) . \quad (4.1)$$

At the self-dual radius this is $SU(2)_1$ WZW theory with (chiral) currents $J^\pm(z) = e^{\pm i\sqrt{2}X(z)}$ and $J^3 = i\partial X(z)$. The zero modes of these currents, J^\pm and J^3 are global $SU(2)_L$ rotations, acting on the Hilbert space. The key result in [13] is that the boundary state $|B\rangle$ corresponding to the interaction (4.1) is simply a global $SU(2)$ rotation of the left-movers relative to the right-movers in the original Neumann boundary state $|N\rangle$. So if we define

$$U(g) \equiv e^{i\pi(gJ^+ + \bar{g}J^-)} \quad (4.2)$$

then we have

$$|B\rangle = U(g)|N\rangle. \quad (4.3)$$

Furthermore, in the prescription for calculating scattering amplitudes, right-moving insertions on the upper half plane (conformally equivalent to the disk) are mapped to $U(g)$ rotated left-moving insertions on the mirror image of that point, in the lower half-plane [13]. For g real, $U(g) = e^{i\pi g J^1}$, and its action on the currents is given explicitly by

$$\begin{array}{ccc} \begin{array}{c} 0 \\ J^1 \\ J^2 \\ J^3 \end{array} & \mapsto & \begin{array}{ccc} 1 & 0 & 0 \\ \cos(2\pi g) & \sin(2\pi g) & -\sin(2\pi g) \\ 0 & \cos(2\pi g) & \sin(2\pi g) \end{array} \begin{array}{c} 1 \\ J^1 \\ J^2 \\ J^3 \end{array} \end{array} \quad (4.4)$$

In (4.3) and (4.4) both the original boundary state $|N\rangle$ and the bulk insertion ∂X are acted on by the same rotation $U(g)$. This means that from the point of view of CS theory in figure (4.b), both Wilson lines corresponding to the boundary condition and

bulk insertion should be equally rotated before calculating the Aharanov-Bohm phase associated to the scattering. Thus in the calculation of the link invariant in (4.b) via the surgery (4.c), we first rotate the two vectors in \mathcal{H}_{T^2} before gluing them together via the modular S matrix. For $S\mathcal{P}(2)_1$, \mathcal{H}_{T^2} is spanned by $|\Psi_{T^2}^0\rangle$ and $|\Psi_{T^2}^1\rangle$ where 0 and 1 are the two integrable representations of $S\mathcal{P}(2)_1$. In the free field language, the representation 1 corresponds either to the Neumann boundary condition or the ∂X operator insertion, whereas 0 corresponds to a trivial boundary condition or the identity operator.[†] Applying the logic in (3.2), the single particle scattering amplitude is then given by the ratio

$$S^{(1 \rightarrow 1)} = \frac{\langle \Psi_{T^2}^1 | \tilde{U}(g)^\dagger S \tilde{U}(g) | \Psi_{T^2}^1 \rangle}{\langle \Psi_{T^2}^0 | S | \Psi_{T^2}^1 \rangle}. \quad (4.5)$$

The denominator corresponds to turning off the boundary interaction, but retaining the bulk insertion. We have yet to specify the 2×2 matrix $\tilde{U}(g)$ that is the analogue of $U(g)$ in (4.3) and (4.4). A natural choice is simply to pick the 2 dimensional representation of the group element in (4.2), namely

$$\tilde{U}(g) = \begin{pmatrix} \cos(\pi g) & i \sin(\pi g) \\ i \sin(\pi g) & \cos(\pi g) \end{pmatrix}. \quad (4.6)$$

From equation (2.4), the explicit form of the modular matrix S at level $k = 1$ is given by

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (4.7)$$

Plugging (4.6) and (4.7) into (4.5) we find the scattering amplitude

$$S^{(1 \rightarrow 1)} = \cos 2\pi g \quad (4.8)$$

which is indeed the result found in [13]. Thus even in this case, it is correct to interpret this amplitude as an Aharanov-Bohm effect in three dimensions. The rotations acting on the vectors in figure (4.c) can be reinterpreted as a linear combination of Wilson lines running around the tori via fusion. The combinations of Wilson lines in the two tori link each other in S^3 after gluing via the modular S matrix and the combination of phases contributes to the scattering amplitude in (4.8).

One may feel slightly uneasy about the “natural” choice made in (4.6). However, it has been shown recently [21] that all real conformal boundary conditions in $S\mathcal{P}(2)_1$ are in one to one correspondence with group elements $U \in SU(2)$. The associated boundary state

[†] The reader may wonder what happened to the Dirichlet boundary state. It is actually another $SU(2)$ rotation of the Neumann boundary state [21].

is simply $U|N\rangle$, the U rotation of the Neumann state. Therefore the space of boundary conditions has a group structure, and if the CS formula (4.5) is to be consistent for this more generic case, the map from the abstract group element U to a 2×2 matrix \tilde{U} must be a group homomorphism, or a representation of $SU(2)$. Then the identification of $e^{i\pi g J^1}$ with (4.6) is chosen to enforce (4.8). One can fix the rest of the map by considering other boundary conditions. Then (4.5) is immediately generalized to a universal Aharonov-Bohm phase for boundary scattering off any conformal boundary condition in $SU(2)_1$.

5. Discussion

From the early days of Kaluza-Klein theory, to the present day study of M -theory, experience has taught us that a higher dimensional viewpoint can unify and shed insight on various disparate lower dimensional phenomena. Certainly CS theory has been able to do this for CFT on closed Riemann surfaces. When the CFT correlators are given by CS path integrals, then the factorization and sewing axiom's of CFT turn out to be simple consequences of knot theory in 3 dimensions. By exploiting an orbifolding of this picture, we elucidated the three dimensional origin of boundary scattering. Bulk insertions on a surface Σ with boundary become Wilson lines propagating from one side of a connecting three manifold M_Σ to another. Boundaries lift to chiral edge states moving along the orbifold singular locus that encircles these particles. And the scattering amplitude is the Aharonov-Bohm phase picked up by these chiral edge states.

It would be interesting to see how general this picture is and search for other examples of this correspondence. Furthermore, the literature has shown that it pays to take the bulk CS physics in the connecting three manifold seriously. For example in [22] it was shown how interactions between the propagating Wilson lines and non-perturbative instanton processes in $U(1)^n$ CS theory yield the Narain lattice of toroidal compactification in the CFT. Thus non-local effects in the bulk of M_Σ , such as chiral edge states, and instanton gases, can affect physics at the boundaries which carry the left and right-movers of the CFT. Also there have been long-standing puzzles about unitarity violations of the boundary scattering amplitude in the Kondo effect, and the existence of solitonic sectors which restore unitarity. Perhaps the answers to these 1+1 dimensional puzzles will be found in the third dimension.

Acknowledgement

This work has been supported in part by NSF grant PHY-0244900 and by the Berkeley Center for Theoretical Physics.

References

- [1] J. Cardy, “Conformal Invariance and Surface Critical Behavior,” *Nucl. Phys.* **B240** (1984) 514.
- [2] I. Affleck, “Conformal Field Theory Approach to Quantum Impurity Problems,” *cond-mat/9311054*.
- [3] I. Affleck and A.W.W. Ludwig, “Exact Conformal Field Theory Results on the Multi-channel Kondo Effect: Single-fermion Green’s Function, Self-energy, and resistivity,” *Phys. Rev.* **B48** (1993) 7297.
- [4] C. G. Callan and S. R. Das, “Boundary Conditions on the Monopole-Dirac Equation,” *Phys. Rev. Lett.* **51** (1983) 1155.
- [5] C. L. Kane and M. P. A. Fisher, “Transport in a One-channel Luttinger Liquid,” *Phys. Rev. Lett.* **68** (1992) 1220.
- [6] P. Horava, “Strings On World-Sheet Orbifolds,” *Nucl. Phys.* **B327** (1989) 461, “Background Duality Of Open String Models,” *Phys. Lett.* **B321** (1989) 251.
- [7] J. Dai, R. G. Leigh, and J. Polchinski, “New Connections Between String Theories,” *Mod. Phys. Lett.* **A4** (1989) 2073.
- [8] J. Polchinski, “Combinatorics of Boundaries In String Theory,” *Phys. Rev.* **D50** (1994) 6041, *hep-th/9407031*.
- [9] A. Sen, “Rolling Tachyon,” *JHEP* **0207** (2002) 65, *hep-th/0203211*.
- [10] P. Hořava, “Chern-Simons Gauge Theory on Orbifolds: Open Strings from Three Dimensions,” *J. Geom. Phys.* **21** (1996) 1, *hep-th/9404101*.
- [11] E. Witten, “Quantum Field Theory and the Jones Polynomial,” *Commun. Math. Phys.* **121** (1989) 351.
- [12] S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg, “Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory,” *Nucl. Phys.* **B326** (1989) 108.
- [13] C.G. Callan, I.R. Klebanov, A.W.W. Ludwig and J.M. Maldacena, “Exact Solution of a Boundary Conformal Field Theory,” *Nucl. Phys.* **B422** (1994) 417, *hep-th/9402113*; see also: J. Polchinski and L. Thorlacius, “Free Fermion Representation of a Boundary Conformal Field Theory,” *Phys. Rev.* **D50** (1994) 622, *hep-th/9404008*.
- [14] G. Moore and N. Seiberg, “Taming the Conformal Zoo,” *Phys. Lett.* **B220** (89) 422
- [15] Ya.I. Kogan, “The Off-Shell Closed Strings as Topological Open Membranes. Dynamical Transmutation of World-Sheet Dimension,” *Phys. Lett.* **B231** (89) 377.
- [16] S. Carlip and Ya. Kogan, “Quantum Geometrodynamics of the Open Topological Membrane and String Moduli Space,” *Phys. Rev. Lett.* **64** (90) 1487.
- [17] R. Dijkgraaf and E. Witten, “Topological Gauge Theories and Group Cohomology,” *Commun. Math. Phys.* **129** (90) 393.
- [18] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, “Conformal Boundary Conditions and Three-dimensional Topological Field Theory,” *Phys. Rev. Lett.* **84** (2000) 1659, *hep-th/9909140*.

- [19] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, “Correlation Functions and Boundary Conditions in RCFT and Three-dimensional Topology,” *Compos.Math.* **131** (2002) 189, hep-th/9912239.
- [20] J.L. Cardy, “Boundary Conditions, Fusion Rules, and the Verlinde Formula,” *Nucl. Phys.* **B324** (89) 581.
- [21] M.R. Gaberdiel, A. Recknagel and G. M. T. Watts, “The Conformal Boundary States for SU(2) at Level 1,” *Nucl. Phys.* **B626** (2002) 344, hep-th/0108102.
- [22] P. C. Ferreira, I. I. Kogan and B. Tekin, “Toroidal Compactification in String Theory from Chern-Simons Theory,” *Nucl. Phys.* **B589** (2000) 167, hep-th/0004078.