IDENTITY

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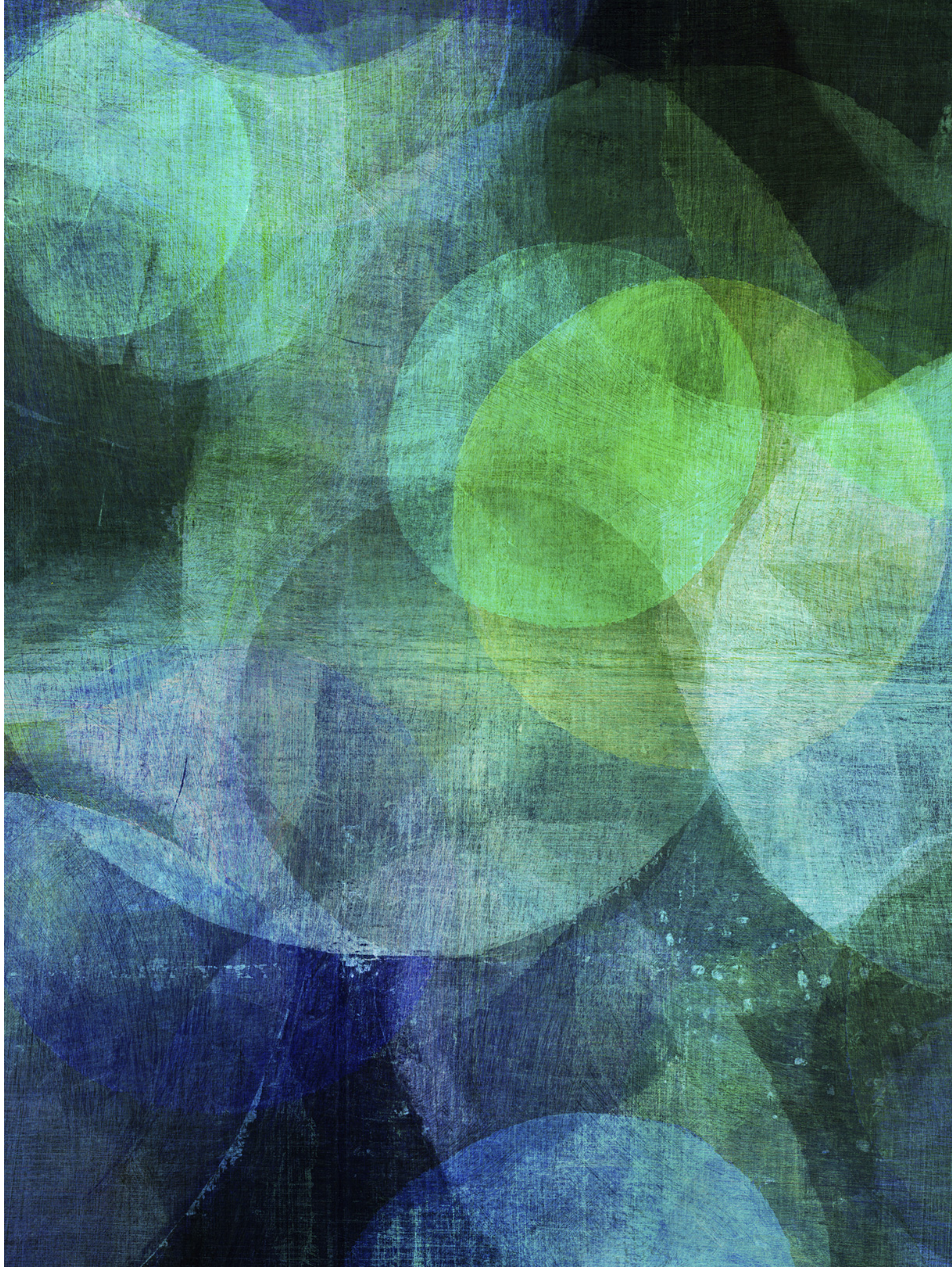


IDENTITY AND DENOTATION

- A = B means two (possibly different) expressions A and B actually denote (or refer to) the same thing.
  - "=" is the binary relation everything has with itself.
  - $I(=)$ is thus given by $\{(x, x) \mid x \in D\}$, or $x = y$ if and only if x is y.
  - Note: $A \neq B$ abbreviates $\neg(A = B)$.
- What rules of inference govern identity (aka. equality)?

RULES FOR IDENTITY: INTRODUCTION

$$\frac{}{t = t} = I$$

- Everything is equal to itself.
  - (If it sounds too obvious, remember that logic makes everything trivial, i.e., transparent, decomposing complex knowledge into trivial pieces or steps.)
  - (If you have any objection, you may become a continental philosopher. Or logical pluralism is common these days, so you can disagree and find your own logic.)

RULES FOR IDENTITY: ELIMINATION

$$\frac{Ft \qquad\qquad\qquad t = u}{Fu} = E$$

- The same things have the same property. (If two things have the same properties, are they same? Cf. Leibniz' Identity of Indiscernibles: $\forall F(Fa \leftrightarrow Fb) \rightarrow a = b$.)

1.	$t = u$	
2.	Ft	
3.	Fu	1,2 $= E$

- This elim. rule should be less obvious than the introduction rule.

RULES FOR IDENTITY: EXAMPLE

$$Fa \vdash \exists x((x = a) \land Fx)$$

α_1	(1)	Fa	A
	(2)	$a = a$	$= I$
α_1	(3)	$(a = a) \land Fa$	1,2 $\quad \land I$
α_1	(4)	$\exists x((x = a) \land Fx)$	3 $\quad \exists I$

UNIQUENESS

We can express "there exists exactly one…" (which is crucial in math) as follows:

$$\exists x \forall y (Fy \leftrightarrow x = y)$$

The forward direction ($\rightarrow$) says, e.g., that:

There exists some ball x such that for all balls, if a ball y is red, then y is x itself. This is the "at most one" condition because <u>there might not be any red balls at all</u>.

The backward direction ($\leftarrow$) says, e.g., that:

There exists some ball x such that for all balls, if a ball y is the same as x, then y must be red. This is the "at least one" condition because <u>in particular x is red</u>.

WHY IS IDENTITY USEFUL?

- Without identity we can formalise assertions such as "Ted is running":

$$Rt$$

  and such assertions as that "there exists a faster runner than Ted":

$$\exists x\ Fxt$$

- However, some things can only be expressed with identity, for example:

  1. "The only person running is Ted" — $Rt \wedge \forall x\ (Rx \rightarrow x = t)$
     - In other words, Ted is running and if x is running, then x is Ted.
  2. "Everyone not Ted is running" — $\forall x\ (x \neq t \rightarrow\ \ Rx)$
     - Note that it makes no claim about whether Ted is running.
  3. "Ted is the fastest" — $\forall x\ (x \neq t \rightarrow Ftx)$. "Faster" plus identity gives "fastest".

WHY IS IDENTITY USEFUL? (CONT'D)

Without identity we cannot express there exists two or more distinct things of the same kind. Note that:

$$\exists x \exists y(Fx \wedge Fy)$$

does not assert the existence of two instances of F because it could be the case that $x = y$.

With identity we may express the idea as: $\exists x \exists y\ ((Fx \wedge Fy) \wedge x \neq y)$

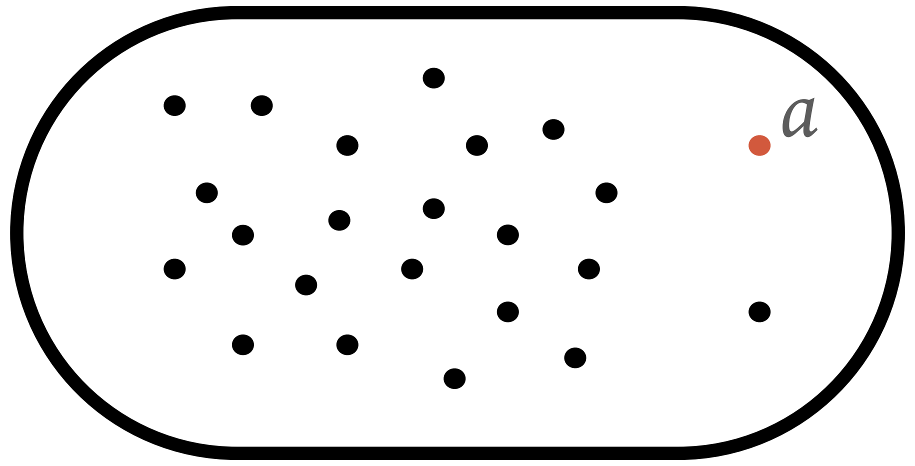
A shorter formula expressing the same thing: $\forall x \exists y\ (Fy \wedge x \neq y)$

(Note: ordinary first-order logic assumes there exists at least one thing.)

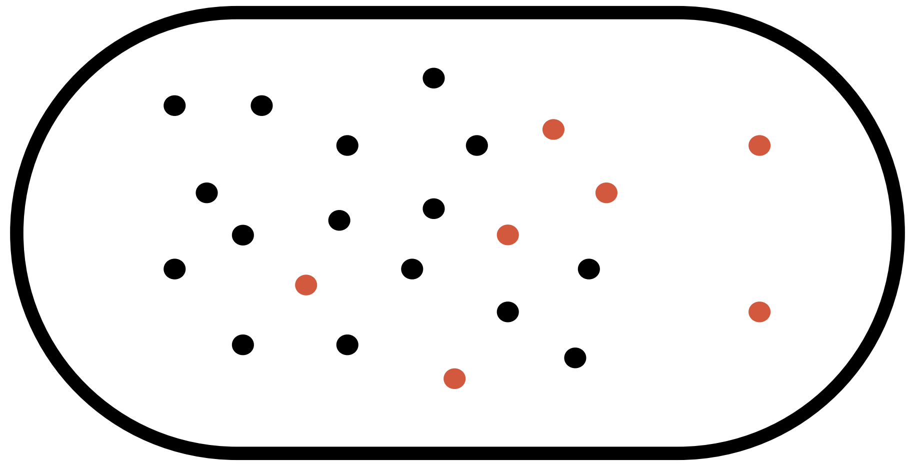
WHY IS IDENTITY USEFUL?

$$\forall x \exists y\, (Fy \wedge x \neq y)$$

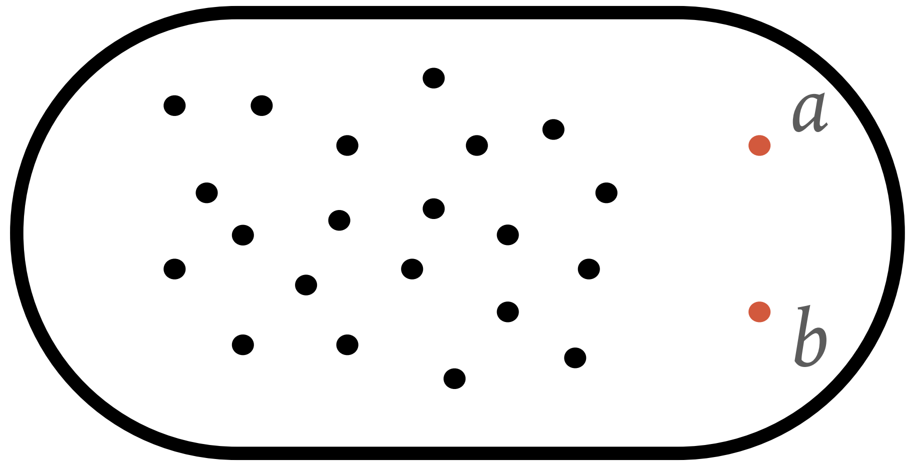
It may not be immediately evident why this shorter formula asserts the existence of at least two *F*s. Take *F* as "is red", let's draw out some cases:



This does not satisfy the formula because: for the dot "a" there is no other dot that is red even though for all other dots the existential condition is satisfied.



This clearly satisfies the formula.



Notice the limiting case where there are only two red dots "a" and "b". They can refer to each other to satisfy the existential condition!

NUMERICALLY DEFINITE QUANTIFIERS

- In the same manner we can say "there are at least three things":

$$\forall x \forall y \exists z (x \neq z \wedge y \neq z)$$

- We can say "there are at least n + *1* things" as follows:

$$\forall x_1 \ldots \forall x_n \exists y (x_1 \neq y \wedge \ldots \wedge x_n \neq y)$$

- On the other hand: we can express "there are at most n things" as follows:

$$\exists x_1 \ldots \exists x_n \forall y (x_1 = y \vee \ldots \vee x_n = y)$$

- Obviously, there are exactly n things if there are at least n and at most n things.

LIMITATIONS OF IDENTITY IN FIRST ORDER LOGIC

- We are able to express the following "arithmetical" argument

I) "There are at most 70 guests at the party"

II) "At least 60 of them are students"

III) "Exactly 15 of the party guests are logicians"

IV) "Therefore, at least 5 students are logicians"

- This argument can be represented in a purely logical language with equality, and does not require any arithmetic. To do things like $70 - 60 = 10$, we need arithmetic.

Peano Axioms for natural numbers

PA1 $\forall x(\neg(s(x) = 0))$
PA2 $\forall x \forall y(s(x) = s(y) \to x = y)$
PA3 $\forall x(x + 0 = x)$
PA4 $\forall x \forall y(x + s(y) = s(x + y))$
PA5 $\forall x(x \cdot 0 = 0)$
PA6 $\forall x \forall y(x \cdot s(y) = x \cdot y + x)$
PA7 $[A(0) \wedge \forall x(A(x) \to A(s(x)))] \to \forall x A(x)$

https://www.chegg.com/homework-help/questions-and-answers/relation-u-greater-equal-v-natural-numbers-n-defined-formula-ez-u-z-v--give-formal-proof-u-q64888301

QUESTIONS AND REMARKS

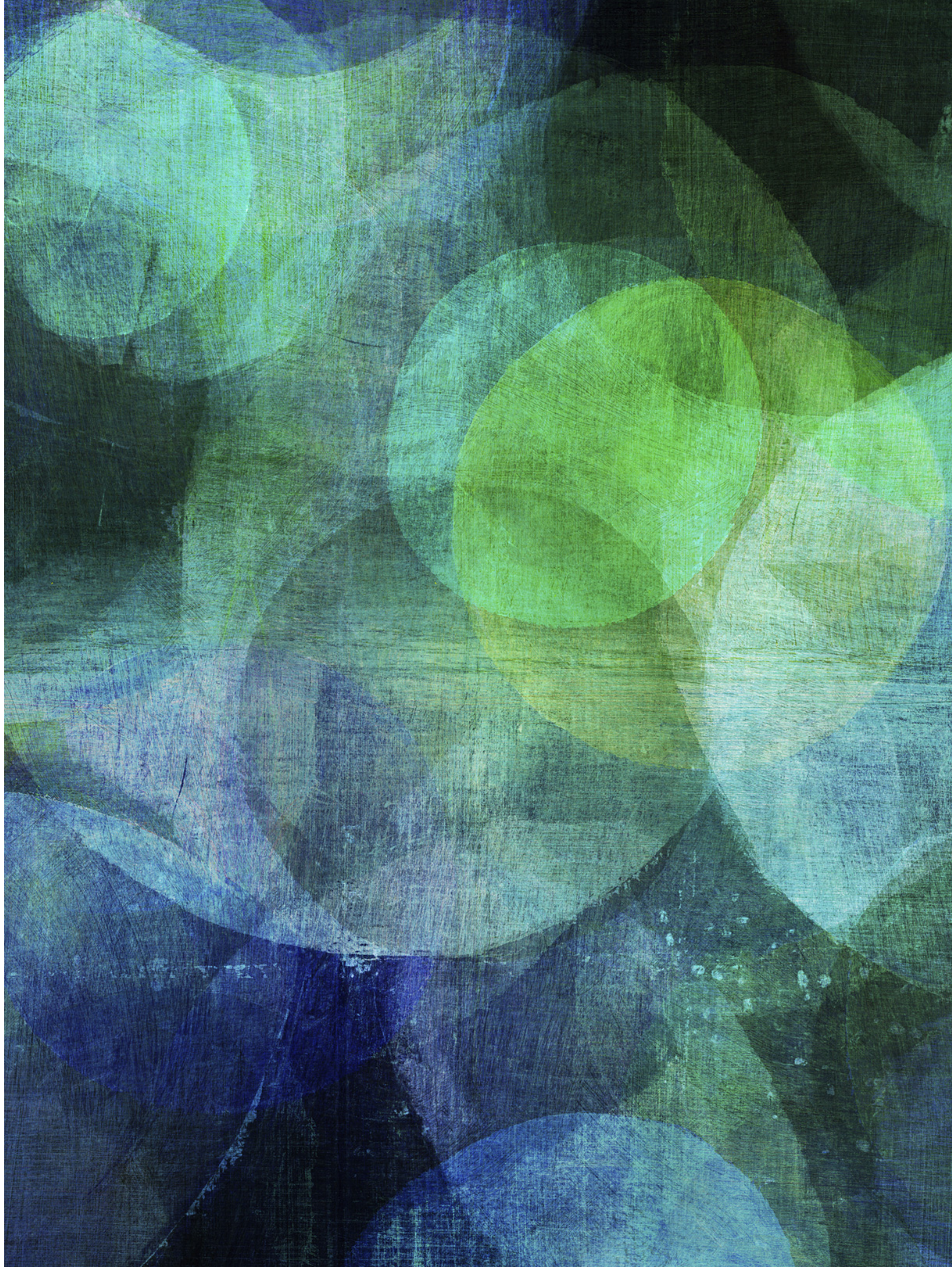
- Is equality/identity a logical vocabulary?
  - What part of language is logical (and what part is not)?
  - How can we mathematically define the concept of being a logical vocabulary?
- Do we already have all logical vocabularies in our logic? Anything missing?
  - If so, can we mathematically prove that all logical connectives are in our logic?
- Lots of questions on identity, such as:
  - Our body changes everyday due to metabolism; are “we today” identical to “us tomorrow”? What does it mean we are the same across time? 1 and 1 are located in different points of space. Why can they be identical? What does it mean?

HIGHER REMARKS

- First-order logic does not allow us to express “there are finitely many things” or “there are countably infinite things”, etc., which is the limitation of first-order logic.
  - There are related to deeper theorems such as the compactness theorem and the Löwenheim-Skolem theorem.
  - They allow us to prove the existence of infinitary numbers in models of first-order arithmetic and the existence of countable models of uncountable real numbers (Skolem’s paradox).
- Some mathematicians don’t like logic because logic allows us to prove these strange things in an even more rigorous manner than ordinary mathematics.
- Logic is interesting, at least to me, because it challenges human reason.

APPENDIX

IDENTITY AND PHILOSOPHY



THE MEANING OF IDENTITY

- $2^{10} = 1024$.
    - They are different expressions, but mean / denote / refer to the same thing (although they have different cognitive values).
    - Remark: meaning is denotation in denotational semantics.
- Hesperus (the Evening Star) = Phosphorus (the Morning Star); Frege's example.
    - Because both of them are Venus (although they have different cognitive values).



PHILOSOPHY OF LANGUAGE

- Frege distinguished between ***sense*** (i.e., modes of presentation) and ***reference*** (i.e., denotations; what expressions *refer* to). Hesperos and Phosporos have the same reference (denotation), but different senses (modes of presentation).
  - This sort of subtle differences in language are studied in Intensional Logic, which we do not learn in this course, but is highly interesting, both mathematically and philosophically. See, e.g., https://plato.stanford.edu/entries/logic-intensional/
  - Frege is an origin of contemporary logic, and worked on both logical foundations of mathematics and logical analyses of natural language.
- Sense is intensional meaning. Reference is extensional meaning.
  - Cf. intensional vs. extensional definitions in math.

HISTORY

- "Ancient Egyptians and Greeks had already discovered Venus, but thought it to be two separate heavenly bodies: the morning star (visible in the morning, also Phosporos) and the evening star (visible in the evening, also Hesperos). It was later discovered by the Greeks that the two were two occurrences of the same object."
  - From: Rick Nouwen, Foundations of Semantics V: Intensionality, 2011, available at: http://www.gist.ugent.be/file/220)
- Intensionality is intensively studied in formal linguistics and formal semantics. Montague, known for Montague semantics in natural language semantics, was a student of Tarski. (He was killed at 40, and the killer(s) are still unknown.)
  - In his PhD thesis he has shown that set theory is not finitely axiomatisable.

INTENSIONALITY OF COMPUTATION

- Ordinary extensional logic is only concerned with extensional meaning, i.e., reference, but intensional logic is concerned with sense as well as reference.
  - Semantics of programs (or programming language as a whole) must be intensional, since modes of presentation make huge differences in the meaning of programs (e.g. their complexity). 2^{10} and 1024 have different computational senses.
  - The meaning of programs is not extensional functions but intensional processes.
- Intensional semantics of computation is one of the most important challenges in theoretical computer science, concerned with fundamental questions such as when two programs are equal to each other, what algorithms are as mathematical objects.
  - No one has ever succeeded in mathematically defining what exactly algorithms are.

THE INTENSIONALITY PARADOX

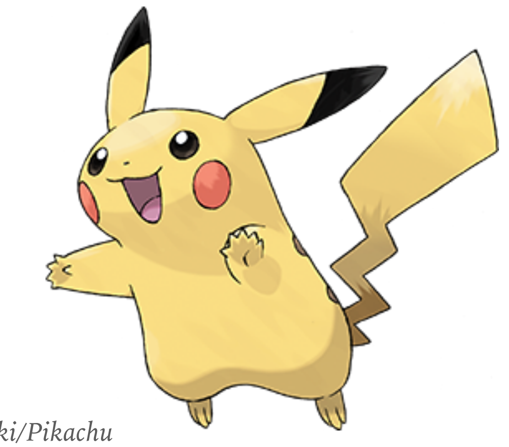
- Let Fx be "John knows what x is", and assume F(MorningStar).
  - We have: MorningStar=EveningStar.
- Then we have F(EveningStar) via substitution, which is paradoxical:
  - There can be someone who knows what MorningStar is, but knows nothing about EveningStar (e.g., since he/she is always sleeping in the evening).
  - Substitution does not always work in intensional contexts.
- It may be called the intensionality paradox.

THE KNOWABILITY PARADOX

- Likewise: assume $\vdash A$ and $\vdash B$, and K(A) meaning "Pascal knows A". Then we can prove: $\vdash A \leftrightarrow B$. Equivalent propositions are mutually substitutable; hence K(B).
  - If you know one truth/theorem, then you know all truths/theorems. No way!
  - It's called the knowability paradox (cf. Hintikka's "scandal of deduction": "since tautologies carry no information at all, no logical inference can yield an increase of information" from D'Agostino-Floridi, Synthese, 2009).

THEORY OF DEFINITE DESCRIPTIONS

- A definite description is a description that specifies exactly one entity: e.g., "the present king of France" (Russell's favourite).
  - Is "the present king of France is hairy" true (when no such entity exists)?
- Russell's analysis of its logical form: $\exists x(Fx \land \forall y(Fy \rightarrow x = y) \land Hx)$, which consists of three parts: existence (of x s.t. Fx), uniqueness (of x s.t. Fx), and property (hairy).
  - Then it's false; any such sentence is false. Frege-Strawson argued it does not have a truth value. How about "Pikachu (the mouse-like electrical Pokemon) is yellow"?
    - It should be true. Both Russell and Frege-Strawson are wrong then? See also: G. Priest, *Towards Non-Being: The Logic and Metaphysics of Intentionality*, OUP.



https://en.wikipedia.org/wiki/Pikachu

THEORY OF DEFINITE DESCRIPTIONS (CONT'D)

- How about "the present king of France is not hairy"? Is it true?
  - An analysis of its logical form: $\neg \exists x(Fx \wedge \forall y(Fy \rightarrow x = y) \wedge Hx)$, which is true.
  - Another analysis: $\exists x(Fx \wedge \forall y(Fy \rightarrow x = y) \wedge \neg Hx)$, which is false.
    - The scope of negation is ambiguous.
  - How about "the present king of France is bald or not bald"? Is it true?
- For more on Russell's theory of descriptions, see John's logic notes.
  - It's a classic topic in analytic philosophy, i.e., philosophy via logical analysis.

REMARKS ON INTEN*S*IONALITY AND INTEN*T*IONALITY

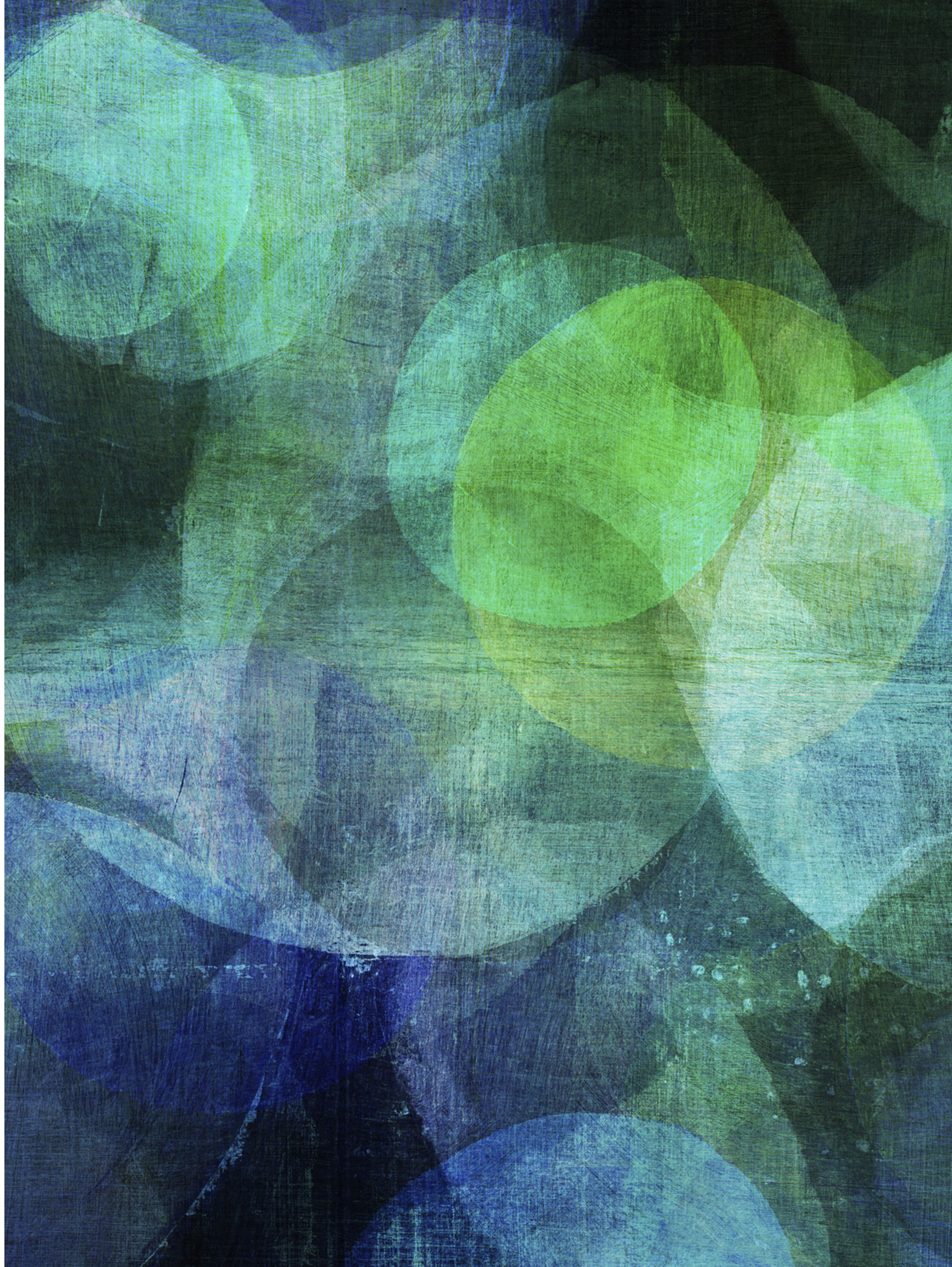
- Intensionality is one of the deepest issues across logic, computer science, linguistics, and philosophy. Broadly, it even connects with philosophy of inten*t*ionality in the continental tradition, such as Brentano and Husserl's phenomenology.

  - ANU's School of Philosophy is steadily ranked within top 10 in the world, and well known for philosophy of mind, in which intentionality is essential. One of the greatest contemporary philosophers worked at the ANU: i.e., David Chalmers.

  - There are different arguments to show the impossibility of AI (in the sense of genuine AGI). Searle's Chinese Room Argument is one of them, concerned with the intentionality of mind (which, Searle argues, any AI cannot have).

  - Another argument to show the impossibility of AI is Lucas-Penrose's Gödelian argument. Gödel's second incompleteness theorem is concerned with the intensionality of the so-called provability predicate in logical systems.

RESTRICTED QUANTIFIERS



RESTRICTED QUANTIFIERS

- Restricted quantifiers: “every A is B” and “some A is B”.
  - “Some footballers are hairy”: $\exists(x : Fx)Hx$.
- When included in the deduction system quantifiers were unrestricted: “everything is A” or “something is A”.
- We now extend the deduction system to encompass restricted quantifiers.

DEDUCTIVE RULES FOR RESTRICTED QUANTIFIERS

- Formulae with restricted quantifier can be turned into unrestricted formulae via the following equivalences:

  - $\forall(v : A)\ B \equiv \forall v(A \to B)$ and $\exists(v : A)\ B \equiv \exists v(A \land B)$.

- To get introduction rules and elimination rules for restricted quantifiers we can start with the equivalent unrestricted version and proceed with the usual deductive rules.

INTRODUCTION RULE FOR RESTRICTED UNIVERSAL QUANTIFIER

- We can derive the introduction rule for restricted universal quantifier (with the side-condition as usual) in the following way:

$$\dfrac{\dfrac{X, A \vdash B}{X \vdash A \to B}\ \to I}{X \vdash \forall v\,(A \to B)}\ \forall I \qquad \Longrightarrow \qquad \dfrac{X, A \vdash B}{X \vdash \forall (v : A)\ \ B}\ \forall I_R$$

ELIMINATION RULE FOR RESTRICTED UNIVERSAL QUANTIFIER

- The elimination rule for restricted universal quantifier takes the formula $\forall(v : A)\ B$ to a specific instance of B.
- We can start with the equivalent formula $\forall v(A \rightarrow B)$ and proceed with the usual deductive rules:

$$\dfrac{\dfrac{\forall v(A \rightarrow B)}{A_v^t \rightarrow B_v^t}\ \forall E \qquad A_v^t}{B_v^t} \rightarrow E$$

ELIMINATION RULE FOR RESTRICTED UNIVERSAL QUANTIFIER

- With restricted quantifier we can collapse the two step into one:

$$\dfrac{\dfrac{\forall v(A \rightarrow B)}{A_v^t \rightarrow B_v^t}\,\forall E \qquad A_v^t}{B_v^t}\rightarrow E \qquad \Longrightarrow \qquad \dfrac{\forall(v : A)\ B \qquad A_v^t}{B_v^t}\,\forall E_R$$

- In particular this rule allows us to prove $\forall(x : Fx)\ Gx, Fa \vdash Ga$ in one step.

INTRODUCTION RULE FOR RESTRICTED EXISTENTIAL QUANTIFIER

- Likewise: through the equivalence $\exists(v : A)\ B \equiv \exists v(A \wedge B)$, we have:

$$\dfrac{\dfrac{A_v^t \qquad B_v^t}{A_v^t \wedge B_v^t}\ \wedge I}{\exists v(A \wedge B)}\ \exists I \quad \Rightarrow \quad \dfrac{A_v^t \qquad B_v^t}{\exists(v : A)\ B}\ \exists I_R$$

ELIMINATION RULE FOR RESTRICTED EXISTENTIAL QUANTIFIER

- The elimination rule for restricted existential quantifier:

$$\dfrac{X \vdash \exists v(A \land B) \qquad\qquad Y, A \land B \vdash C}{X, Y \vdash C}\ \exists E \quad \Longrightarrow \quad \dfrac{X \vdash \exists(v : A)\ B \qquad\qquad Y, A, B \vdash C}{X, Y \vdash C}\ \exists E_R$$

EXAMPLE

Here is a sample proof with restricted quantifiers. Note: when you are confused about restricted quantifiers, consider what they stand for in terms of ordinary quantifiers.

$$\forall(x : Fx)Gx \vdash \forall(x : \neg Gx)\neg Fx$$

α_1	(1)	$\forall(x : Fx)\ Gx$	A
α_2	(2)	$\neg Ga$	A
α_3	(3)	Fa	A
α_1, α_3	(4)	Ga	1,3 $\quad \forall E_R$
α_1, α_2	(5)	$\neg Fa$	2,4[α_3] $\quad RAA$
α_1	(6)	$\forall(x : \neg Gx)\neg Fx$	5[α_2] $\quad \forall I_R$

Appendix

RESTRICTED QUANTIFIER EXAMPLE

Classic syllogisms such as the following can be expressed via restricted quantifiers:

All F is G;

Some H is F;

Therefore some H is G

FORMALLY

Formally, we can prove the following sequent in the following manner:

$$\forall(x : Fx)Gx, \exists(x : Hx)Fx \vdash \exists(x : Hx)Gx$$

α_1	(1)	$\forall(x : Fx)\ Gx$	A
α_2	(2)	$\exists(x : Hx)\ Fx$	A
α_3	(3)	Fa	A
α_1, α_3	(4)	Ga	1,3 $\quad \forall E_R$
α_4	(5)	Ha	A
$\alpha_1, \alpha_3, \alpha_4$	(6)	$\exists(x : Hx)\ Gx$	4,5 $\quad \exists I_R$
α_1, α_2	(7)	$\exists(x : Hx)\ Gx$	2,6 $[\alpha_3, \alpha_4] \quad \exists E_R$

EXAMPLE: COMPLEX PROOF WITH IDENTITY

$$\forall x \exists y(Fy \wedge (x \neq y)) \vdash \exists x \exists y((Fx \wedge Fy) \wedge (x \neq y))$$

α_1	(1)	$\forall x \exists y(Fy \wedge (x \neq y))$	A
α_1	(2)	$\exists y(Fy \wedge (a \neq y))$	$1\ \forall E$
α_2	(3)	$Fb \wedge (a \neq b)$	A
α_2	(4)	Fb	$3 \wedge E$
α_1	(5)	$\exists y(Fy \wedge b \neq y)$	$1 \forall E$
α_3	(6)	$Fc \wedge (b \neq c)$	A
α_3	(7)	Fc	$6 \wedge E$
α_3	(8)	$b \neq c$	$6 \wedge E$
α_2, α_3	(9)	$Fb \wedge Fc$	$4{,}7 \wedge I$
α_2, α_3	(10)	$(Fb \wedge Fc) \wedge (b \neq c)$	$8{,}9 \wedge I$
α_2, α_3	(11)	$\exists y((Fb \wedge Fy) \wedge (b \neq y))$	$10 \exists I$
α_2, α_3	(12)	$\exists x \exists y((Fx \wedge Fy) \wedge (x \neq y))$	$11 \exists I$
α_1, α_2	(13)	$\exists x \exists y((Fx \wedge Fy) \wedge (x \neq y))$	$5{,}12[\alpha_3] \exists E$
α_1	(14)	$\exists x \exists y((Fx \wedge Fy) \wedge (x \neq y))$	$2{,}3[\alpha_2] \exists E$

$\blacksquare$

EXAMPLE: MIXING RESTRICTED AND UNRESTRICTED QUANTIFIERS

We can also mix restricted and unrestricted quantifiers:

$$\forall(x : Fx)\ \exists y\ Rxy, \quad \forall x \forall(y : Rxy)\ (Ryx \wedge \forall(z : Ryz)\ Rxz) \vdash \forall(x : Fx)\ Rxx$$

α_1	(1)	$\forall(x : Fx)\ \exists y\ Rxy$	A
α_2	(2)	$\forall x \forall(y : Rxy)(Ryx \wedge \forall(z : Ryz)\ Rxz)$	A
α_3	(3)	Fa	A
α_1, α_3	(4)	$\exists y\ Ray$	1,3 $\forall E_R$
α_4	(5)	Rab	A
α_2	(6)	$\forall(y : Ray)(Rya \wedge \forall(z : Ryz)\ Raz)$	2 $\forall E$
α_2, α_4	(7)	$Rba \wedge \forall(z : Rbz)\ Raz$	5,6 $\forall E_R$
α_2, α_4	(8)	Rba	7 $\wedge E$
α_2, α_4	(9)	$\forall(z : Rbz)\ Raz$	7 $\wedge E$
α_2, α_4	(10)	Raa	8,9$\forall E$
$\alpha_1, \alpha_2, \alpha_3$	(11)	Raa	4,10[α_4] $\exists E$
α_1, α_2	(12)	$\forall(x : Fx)\ Rxx$	11[α_2] $\forall I_R$