

Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Propositional Logic
— Recap on Proof Strategies

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Introduction

Recap: Sequents, Validity, Proof Systems

- You know how to prove $X \vdash A$ via ND
- You know how to prove $X \vdash A$ via ST
- ND is useful to show $X \vdash A$ is valid.
- To show *invalidity*, ST is more convenient.

So, when to use which?

- If we don't tell whether it's valid or invalid:
 - If you think the sequent is invalid: ST is most useful
 - If you think the sequent is valid: Choose what you are stronger in!
 - If you don't know either way: Use ST and *let it tell you!*

Strategies: Overview

Tips for Semantic Tableau and Natural Deduction

Semantic Tableau:

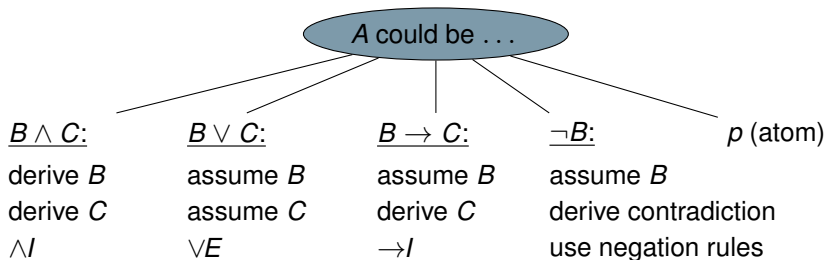
- Always apply non-branching rules first.
- In case of invalid sequents you could follow down branches leading to an open branch more quickly – which requires “seeing” which interpretation proves invalidity.

Natural Deduction:

- See the next slide.
- We also sometimes need the fall-back strategy: Assume negation of final derivation and exploit contradiction.

Natural Deduction: Overview

How to show $X \vdash A$? Depends on A !



Note:

- $X \vdash A$ can also refer to sub steps!
- Usually, you will need $\vee E$ if $B \vee C \in X$, not if $B \vee C = A$.

Examples for Natural Deduction

$X \vdash A$, A is an Implication

$$p \rightarrow ((q \wedge r) \rightarrow s) \vdash q \rightarrow ((p \wedge r) \rightarrow s)$$

α_1	(1)	$p \rightarrow ((q \wedge r) \rightarrow s)$	A
α_2	(2)	q	A
α_3	(3)	$p \wedge r$	A
α_3	(4)	p	3 $\wedge E$
α_1, α_3	(5)	$(q \wedge r) \rightarrow s$	1,4 $\rightarrow E$
α_3	(6)	r	3 $\wedge E$
α_2, α_3	(7)	$q \wedge r$	2,6 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(8)	s	5,7 $\rightarrow E$
α_1, α_2	(9)	$(p \wedge r) \rightarrow s$	8[α_3] $\rightarrow I$
α_1	(10)	$q \rightarrow ((p \wedge r) \rightarrow s)$	9[α_2] $\rightarrow I$

$X \vdash A$, A is a Negation

$\neg(p \vee q) \vdash \neg p$

$\frac{X, B \vdash A \quad Y, B \vdash \neg A}{X, Y \vdash \neg B} \text{RAA}$
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α_1	(1)	$\neg(p \vee q)$	A
α_2	(2)	p	A
α_2	(3)	$p \vee q$	2 $\vee I$
α_1	(4)	$\neg p$	1,3[α_2] RAA

$X \vdash A$, A is a Disjunction (here: in one of the Substeps)

$\neg p \wedge \neg q \vdash \neg(p \vee q)$

α_1 (1) $\neg p \wedge \neg q$ A

α_2 (2) $p \vee q$ A

α_3 (3) p A

α_4 (4) q A

α_1 (5) $\neg p$ 1 $\wedge E$

α_3 (6) $\neg(\neg p \wedge \neg q)$ 3,5[α_1] RAA

α_1 (7) $\neg q$ 1 $\wedge E$

α_4 (8) $\neg(\neg p \wedge \neg q)$ 4,7[α_1] RAA

α_2 (9) $\neg(\neg p \wedge \neg q)$ 2,6[α_3],8[α_4] $\vee E$

α_1 (10) $\neg(p \vee q)$ 1,9[α_2] RAA

$X \vdash A \vee B$	$Y, A \vdash C$	$Z, B \vdash C$	$\vee E$
$X, Y, Z \vdash C$			

$X, B \vdash A$	$Y, B \vdash \neg A$	RAA
$X, Y \vdash \neg B$		

$X \vdash A$, A is an Atom (Or: all Other Strategies Fail)

If everything else fails, assume $\neg A$ and derive A :
$$\frac{X, \neg A \vdash A}{X \vdash A}$$

Closely related is the sequent: $\neg p \rightarrow p \vdash p$

(p is so true that it's even implied by its own negation!)

Example: $(p \rightarrow q) \rightarrow p \vdash p$

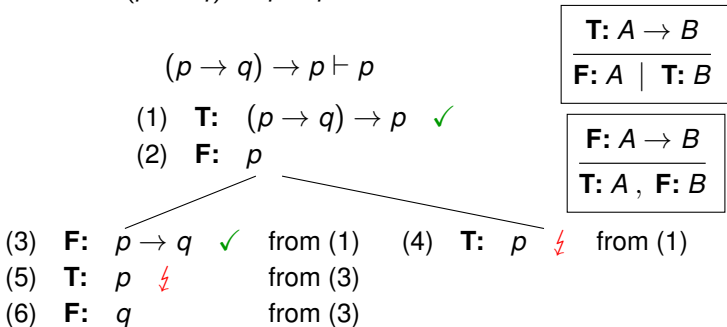
$$\frac{X, B \vdash A \quad Y, B \vdash \neg A}{X, Y \vdash \neg B} \text{RAA}$$

α_1	(1)	$(p \rightarrow q) \rightarrow p$	A
α_2	(2)	$\neg p$	A
α_3	(3)	p	A
α_2, α_3	(4)	$\neg \neg q$	2,3[] RAA
α_2, α_3	(5)	q	4 $\neg \neg E$
α_2	(6)	$p \rightarrow q$	5[α_3] $\rightarrow I$
α_1, α_2	(7)	p	1,6 $\rightarrow E$
α_1	(8)	$\neg \neg p$	2,7[α_2] RAA
α_1	(9)	p	8 $\neg \neg E$

Examples for Semantic Tableau

Previous Example, shown with Semantic Tableau

We now show $(p \rightarrow q) \rightarrow p \vdash p$ via Semantic Tableau.



The primary strategy (that often suffices to create small trees) is:

- *Always* apply rules first that don't branch!
- Don't forget branches!! And mark lines that are "done".

Summary

Content of this Lecture

- Today, we did a recap on how to prove various kinds of sequents via Natural Deduction and Semantic Tableau