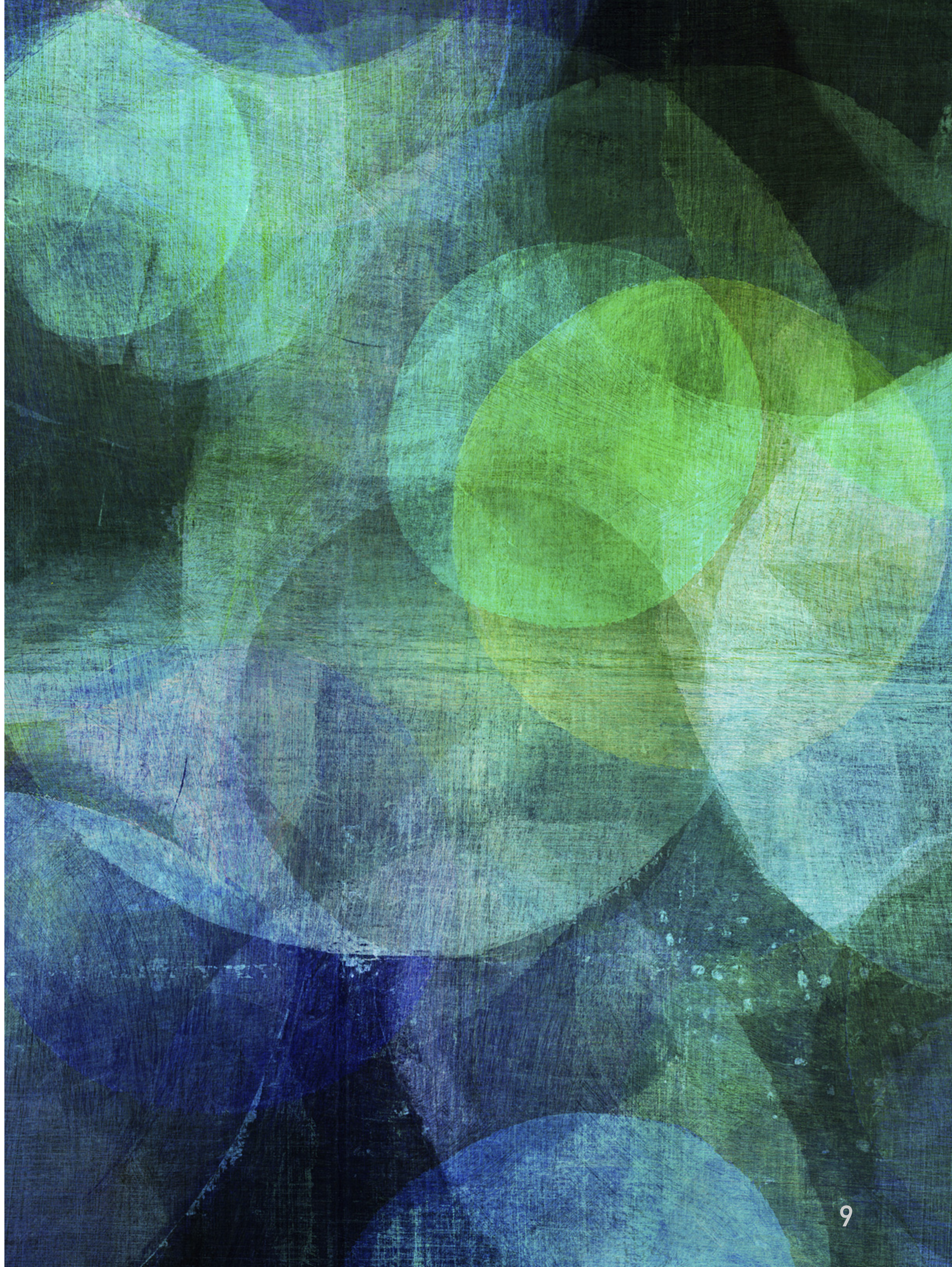
PARADOXES OF IMPLICATION



STRANGE VALID SEQUENTS

In the logic we learned, some strange sequents are valid:

- $p \vdash q \to p$
  "I have \$300, therefore if it rains on Tuesday then I have \$300."
- $\neg p \vdash p \to q$
  "I do not have \$300, therefore if I have \$300 it rains on Tuesday."
- $p, \neg p \vdash q$
  "The cat is dead and the cat is alive, therefore I am a chicken."

PARADOXES OF MATERIAL IMPLICATION

- All of the previous sequents are semantically valid and are syntactically provable in the logic we learned so far.
    - However, these arguments do not make much sense.
    - Especially, conclusions are ***irrelevant*** to premises.
- They are known as **Paradoxes of Material Implication.**
    - Note that $\neg A$ is equivalent to $A \rightarrow \bot$, which can, e.g., be verified via truth tables.
- **Relevant logic** saves us from the paradoxes of implication.

PARADOXES OF MATERIAL IMPLICATION (CONT'D)

- A possible perspective on paradoxes of implication: assumptions are not used in any ***relevant*** manner for deriving the conclusion.
- Consider a proof of $p, \neg p \vdash q$. RAA in line 4 seemingly suggests the assumption $\neg q$ is what caused the contradiction. But $\neg q$ is ***irrelevant*** to p and $\neg p$.

$$p, \neg p \vdash q$$

$$\begin{array}{rlll} \alpha_1 & (1) & p & A \\ \alpha_2 & (2) & \neg p & A \\ \alpha_3 & (3) & \neg q & A \\ \alpha_1, \alpha_2 & (4) & \neg\neg q & 1{,}2[\alpha_3]\ RAA \\ \alpha_1, \alpha_2 & (5) & q & 4\ \neg\neg E \end{array}$$

- Relevant logic allows us to prohibit this sort of ***irrelevant reasoning***.

THE LOGIC OF MATH VS. THE LOGIC OF NATURAL LANGUAGE

- Note that these arguments are allowed in the logic of mathematics.
  - Classical logic, the logic you learned, is basically the logic of mathematics. Frege, Russell, and Whitehead (origins of formal logic) all tried to formalise mathematics.
- Is the logic of natural language the same as the logic of mathematics?
  - No obvious answer to this.
- Natural language is seemingly richer and contextual than the logic of mathematics, so they may be different from each other; if so, what is the logic of natural language?
  - Relevant logic sheds light on relevance aspects of natural language reasoning.

SAVING LOGIC FROM PARADOXES

- How to save logic from paradoxes of implications?
- In relevant logic, we reconsider the way assumptions are combined with each other.

THE WAYS ASSUMPTIONS ARE COMBINED

- In the proof of the following sequent, the assumptions α_1 and α_2:
  - are put together by $\wedge I$;
  - are pulled apart by $\rightarrow I$.
- Any of $\wedge I$ and $\rightarrow I$ is no problem on its own, but the way they combine assumptions are subtly different in the following sense.

$$q \vdash p \rightarrow q$$

α_1	(1)	q	A
α_2	(2)	p	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$
α_1, α_2	(4)	q	3 $\wedge E$
α_1	(5)	$p \rightarrow q$	4[α_2] $\rightarrow I$

THE WAYS ASSUMPTIONS ARE COMBINED (CONT'D)

- In the $\wedge$**I** rule:

$$\frac{X \vdash A \qquad\qquad Y \vdash B}{X, Y \vdash A \wedge B} \;\wedge\mathrm{I}$$

  - The two assumptions X, Y are just collected together and there is no requirement that they interact with each other to yield the conclusion $A \wedge B$.

- Yet in the $\rightarrow$**E** rule:

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \;\rightarrow\mathrm{E}$$

  - The assumption Y is "applied" to X so that they yield the conclusion B while A disappears after the application. The same thing happens in $\rightarrow$**I**.

- To distinguish the two ways, we introduce a new symbol ";" (semicolon) besides ",".

RELEVANT LOGIC RULES

- We modify the rules by introducing the semicolon to distinguish the different ways assumptions are combined. Here are ND rules for conjunction and implication:

Classical Logic

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E$$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

Relevant Logic

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X; Y \vdash B} \to E$$

$$\frac{X; A \vdash B}{X \vdash A \to B} \to I$$

SAVING LOGIC FROM PARADOX

- With these new rules we can no longer go from line (4) to line (5) as the new $\to I$ rule requires ";" between the assumptions:

$$\frac{X; A \vdash B}{X \vdash A \to B} \to I$$

- We can still prove something meaningful via the modified rules: e.g., $p \to (q \to r) \vdash (p \land q) \to r$ (as below)

$$q \vdash p \to q$$

α_1	(1)	q	A
α_2	(2)	p	A
α_1, α_2	(3)	$p \land q$	1,2 $\land I$
α_1, α_2	(4)	q	3 $\land E$
~~α_1~~	~~(5)~~	~~$p \to q$~~	~~$4[\alpha_2]$ $\to I$~~

RELEVANT AND FUZZY LOGICS AS SUBSTRUCTURAL LOGICS

- Weakening below is not valid in relevant logic: it does not allow us to add irrelevant assumptions, but it usually allows to reduce duplications, i.e., contraction below.
- (Note: Contraction is not valid in certain fuzzy logic: it does not allow us to reduce two precious resources to just one resource, but it does allow to increase resources.)

$$\frac{X \vdash B}{X; A \vdash B}\ \textit{weakening} \qquad \frac{X; A; A \vdash B}{X; A \vdash B}\ \textit{contraction}$$

- X is not regarded as a set; it matters how many times the same formula appears in X.
- Full proof theory is given in an additional chapter of the logic notes (see 8: Reference), but in this course, you don't have to understand all of them in detail.

CONTRACTION IS ALLOWED IN RELEVANT LOGIC (BUT NOT IN FUZZY LOGIC)

$$p \to (q \to r) \vdash (p \land q) \to r$$

$$
\begin{array}{rlll}
\alpha_1 & (1) & p \to (q \to r) & A \\
\alpha_2 & (2) & p \land q & A \\
\alpha_2 & (3) & p & 2 \quad \land E \\
\alpha_2 & (4) & q & 2 \quad \land E \\
\alpha_1;\alpha_2 & (5) & q \to r & 1,3 \quad \to E \\
\alpha_1;\alpha_2;\alpha_2 & (6) & r & 4,5 \quad \to E \\
\alpha_1;\alpha_2 & (7) & r & 6 \text{ contraction} \\
\alpha_1 & (8) & (p \land q) \to r & 7[\alpha_2] \quad \to I
\end{array}
$$

RESOURCE SENSITIVITY IN SUBSTRUCTURAL LOGIC

- You have three dollars *and* you have three dollars, i.e., you have six dollars!
  - This is a resource-sensitive interpretation of substructural conjunction.
- Having two Tim Tams is different from having one Tim Tam.
  - This world is resource-sensitive (as well as computer science, in which substructural logic has played essential roles, esp. in semantics of computation).

$\neq$

https://www.arnotts.com/products/tim-tam

- John told me that there are two possible readings of "A;B": "A and B" (B. Meyer) or "A is compatible with B" (in the sense that "not (A implies not B)"; S. Read).

SAVING LOGIC FROM ANOTHER PARADOX

- Which step is wrong with the classical proof of the following sequent?

Relevant Logic

$$\frac{X; A \vdash \bot}{X \vdash \neg A}\ \neg I$$

$$\frac{X \vdash A \qquad X \vdash \neg A}{X; Y \vdash \bot}\ \neg E$$

$$\frac{X \vdash \neg\neg A}{X \vdash A}\ \neg\neg E$$

$$p, \neg p \vdash q$$

α_1	(1)	p	A
α_2	(2)	$\neg p$	A
α_3	(3)	$\neg q$	A
~~α_1, α_2~~	~~(4)~~	~~$\neg\neg q$~~	~~1,2[α_3] RAA~~
α_1, α_2	(5)	q	4 $\neg\neg E$

- We can get $p; \neg p \vdash \bot$, but to derive $\neg\neg q$, we need $p; \neg p; \neg q \vdash \bot$, which is not allowed because we cannot add irrelevant assumptions without the weakening rule.
- This means contradictions don't entail explosion (i.e., it is not possible to derive an arbitrary formula from contradictions); it's ***paraconsistent*** (contradiction-tolerant) logic.

MODELLING RELEVANT LOGIC

- We have the following three-valued semantics for relevant logic:
  - We consider **i** to be a "confused" value between true and false. In a way, it can be considered to be both true and false; it's paraconsistent.

$\wedge$	T	i	F
T	T	i	F
i	i	i	F
F	F	F	F

$\vee$	T	i	F
T	T	T	T
i	T	i	i
F	T	i	F

$\rightarrow$	T	i	F
T	T	F	F
i	T	i	F
F	T	T	T

A	$\neg A$
T	F
i	i
F	T

MODELLING RELEVANT LOGIC

- The semicolon and comma are interpreted as follows:

,	T	i	F
T	T	i	F
i	i	i	F
F	F	F	F

;	T	i	F
T	T	T	F
i	T	i	F
F	F	F	F

- In the three-valued relevant logic, $X \vdash A$ is valid iff in any assignment of values, the value of A is at least as good as the value of X. Note: **T** better than **i** better than **F**.
  - Concretely: the **T** ⊢ **i** case is invalid, and the **i** ⊢ **F** case is invalid; in contrast, the **i** ⊢ **T** case is valid; the **F** ⊢ **i** case is valid.
- Tautologies (i.e., valid formulae) are those to which it is impossible to assign **F**.

INVALID SEQUENTS

- Consider the sequent:

$$(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$$

- Classical logic make $(p \wedge q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ be equivalent with each other.
  - If we assign **T** to p and **i** to both q and r, it is not valid in the three-valued semantics for relevant logic because the premise of the sequent is **i**, while the conclusion is **F**.

$$\begin{array}{ccccccccccc} \mathbf{T} & \mathbf{i} & \mathbf{i} & \mathbf{i} & \mathbf{i} & & \mathbf{T} & \mathbf{F} & \mathbf{i} & \mathbf{i} & \mathbf{i} \\ (p & \wedge & q) & \rightarrow & r & \vdash & p & \rightarrow & (q & \rightarrow & r) \end{array}$$

- The converse sequent $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$ is valid.

Appendix

QUESTIONS

- Is relevant logic closer to natural language than classical logic?
  - Vacuous discharge is not allowed in relevant logic since it cannot discharge irrelevant assumptions.
- Can we do ordinary mathematics (e.g., arithmetic) with relevant logic?
- How many logics exist in the world?

INVALID SEQUENTS

- The following are not valid (and thus not provable because of soundness):

$$p \vdash q \to p$$

$$(p \lor q) \land \neg p \vdash q$$

$$\neg(p \to q) \vdash p \land \neg q$$

$$p \land q \vdash p \leftrightarrow q$$

$$(p \land q) \to r \vdash p \to (q \to r)$$