Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Propositional Logic
— Recap on Proof Strategies

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Introduction

Recap: Sequents, Validity, Proof Systems

- You know how to prove $X \vdash A$ via ND
- You know how to prove $X \vdash A$ via ST
- ND is useful to show $X \vdash A$ is valid.
- To show *invalidity*, ST is more convenient.

So, when to use which?

- If we don't tell whether it's valid or invalid:
  - If you think the sequent is invalid: ST is most useful
  - If you think the sequent is valid: Choose what you are stronger in!
  - If you don't know either way: Use ST and *let it tell you*!

Strategies: Overview

Tips for Semantic Tableau and Natural Deduction

Semantic Tableau:

- Always apply non-branching rules first.
- In case of invalid sequents you could follow down branches leading to an open branch more quickly – which requires “seeing” which interpretation proves invalidity.

Natural Deduction:

- See the next slide.
- We also sometimes need the fall-back strategy: Assume negation of final derivation and exploit contradiction.

Natural Deduction: Overview

How to show $X \vdash A$? Depends on A!

A could be . . .

$\underline{B \wedge C}$:	$\underline{B \vee C}$:	$\underline{B \rightarrow C}$:	$\underline{\neg B}$:	p (atom)
derive B	assume B	assume B	assume B	
derive C	assume C	derive C	derive contradiction	
$\wedge I$	$\vee E$	$\rightarrow I$	use negation rules	

Note:

- $X \vdash A$ can also refer to sub steps!
- Usually, you will need $\vee E$ if $B \vee C \in X$, not if $B \vee C = A$.

Examples for Natural Deduction

$X \vdash A$, A is an Implication

$$p \to ((q \wedge r) \to s) \vdash q \to ((p \wedge r) \to s)$$

α_1	(1)	$p \to ((q \wedge r) \to s)$	A
α_2	(2)	q	A
α_3	(3)	$p \wedge r$	A
α_3	(4)	p	3 $\wedge E$
α_1, α_3	(5)	$(q \wedge r) \to s$	1,4 $\to E$
α_3	(6)	r	3 $\wedge E$
α_2, α_3	(7)	$q \wedge r$	2,6 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(8)	s	5,7 $\to E$
α_1, α_2	(9)	$(p \wedge r) \to s$	8[α_3] $\to I$
α_1	(10)	$q \to ((p \wedge r) \to s)$	9[α_2] $\to I$

$X \vdash A$, A is a Negation

$$\boxed{\dfrac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\ RAA}$$

$\neg(p \vee q) \vdash \neg p$

α_1	(1)	$\neg(p \vee q)$	A
α_2	(2)	p	A
α_2	(3)	$p \vee q$	2 $\vee I$
α_1	(4)	$\neg p$	1,3[α_2] RAA

$X \vdash A$, A is a Disjunction (here: in one of the Substeps)

$$\neg p \wedge \neg q \vdash \neg(p \vee q)$$

$$\frac{X \vdash A \vee B \qquad Y, A \vdash C \qquad Z, B \vdash C}{X, Y, Z \vdash C} \vee E$$

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B} RAA$$

α_1	(1)	$\neg p \wedge \neg q$	A
α_2	(2)	$p \vee q$	A
α_3	(3)	p	A
α_4	(4)	q	A
α_1	(5)	$\neg p$	1 $\wedge E$
α_3	(6)	$\neg(\neg p \wedge \neg q)$	3,5$[\alpha_1]$ RAA
α_1	(7)	$\neg q$	1 $\wedge E$
α_4	(8)	$\neg(\neg p \wedge \neg q)$	4,7$[\alpha_1]$ RAA
α_2	(9)	$\neg(\neg p \wedge \neg q)$	2,6$[\alpha_3]$,8$[\alpha_4]$ $\vee E$
α_1	(10)	$\neg(p \vee q)$	1,9$[\alpha_2]$ RAA

$X \vdash A$, A is an Atom (Or: all Other Strategies Fail)

If everything else fails, assume $\neg A$ and derive A: $\dfrac{X, \neg A \vdash A}{X \vdash A}$

Closely related is the sequent: $\neg p \to p \vdash p$
(p is so true that it's even implied by its own negation!)

Example: $(p \to q) \to p \vdash p$

$$\boxed{\dfrac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\; RAA}$$

α_1	(1)	$(p \to q) \to p$	A
α_2	(2)	$\neg p$	A
α_3	(3)	p	A
α_2, α_3	(4)	$\neg\neg q$	2,3[] RAA
α_2, α_3	(5)	q	4 $\neg\neg E$
α_2	(6)	$p \to q$	5[α_3] $\to I$
α_1, α_2	(7)	p	1,6 $\to E$
α_1	(8)	$\neg\neg p$	2,7[α_2] RAA
α_1	(9)	p	8 $\neg\neg E$

Examples for Semantic Tableau

Previous Example, shown with Semantic Tableau

We now show $(p \to q) \to p \vdash p$ via Semantic Tableau.

$$\frac{\textbf{T: } A \to B}{\textbf{F: } A \mid \textbf{T: } B} \qquad \frac{\textbf{F: } A \to B}{\textbf{T: } A\, ,\ \textbf{F: } B}$$

$$(p \to q) \to p \vdash p$$

(1) **T:** $(p \to q) \to p$ ✓
(2) **F:** p

(3)	**F:**	$p \to q$	✓	from (1)	(4)	**T:**	p	↯	from (1)
(5)	**T:**	p	↯	from (3)					
(6)	**F:**	q		from (3)					

The primary strategy (that often suffices to create small trees) is:

- *Always* apply rules first that don't branch!
- Don't forget branches!! And mark lines that are "done".

Summary

Content of this Lecture

- Today, we did a recap on how to prove various kinds of sequents via Natural Deduction and Semantic Tableau