Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Sequents, Semantics, and Propositional Natural Deduction — Conjunction, Implication, Theorems

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Introduction

Recap: Sequents

- We want to know when one logical formula follows logically from another.

- Suppose we know that "p is true' (technically: it is *interpreted* as true), and we know that $p \rightarrow q$ holds as well. Then we can logically conclude that q also holds!

- We can express this with *sequents*: $p, p \rightarrow q \models q$

- These conclusions can be arbitrarily complicated, however! I.e., it might not be obvious that the conclusion follows from the premises.

- Two ways to show validity of sequents: semantics (validity by meaning) and syntactic proof system (validity by following deductive rules; natural deduction introduced soon).

Sequents

Introduction

In general, a sequent is of the following form with X a set of formulae and A a single formula:

$$X \models A$$

- Read it: A follows from X; or X entails A.

- For example, "q follows from p and $p \rightarrow q$"

- We write down:
  but that just abbreviates:

$$p, p \rightarrow q \models q$$
$$\underbrace{\{p, p \rightarrow q\}}_{X} \models \underbrace{q}_{A}$$

- Also $X, Y \models A$ abbreviates $X \cup Y \models A$,

Semantically Valid Sequents

Definition:
$X \models A$ means the sequent is *valid*. This is the case if and only if:

- *A is true for every interpretation for which all the formulae in X are true.* Or, equivalently:
- *There is no interpretation that makes X true, but not A.*

How to check/test/prove $X \models A$? Create the truth tables!

- Create a table t_X for all formulae in X (all need to be true)
- Create another table t_A for A and check validity criterion.

Checking Validity, Example 2

$$\text{Show } \overbrace{(p \vee q) \rightarrow r, p}^{X} \models \overbrace{(p \rightarrow r) \wedge (q \rightarrow r)}^{A}$$

Table t_X for premises:

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	X
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Table t_A for conclusion:

p	q	r	$p \rightarrow r$	$q \rightarrow r$	A
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Recall the definition: The sequent is valid if all interpretations that make *X* true also make *A* true!

Checking Validity, Example 2

Show $\overbrace{(p \vee q) \to r, p}^{X} \models \overbrace{(p \to r) \wedge (q \to r)}^{A}$

Table t_X for premises:

p	q	r	$p \vee q$	$(p \vee q) \to r$	X
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Table t_A for conclusion:

p	q	r	$p \to r$	$q \to r$	A
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Only two interpretations exist that make all $x \in X$ true:

1. $I_1(p) = I_1(r) = 1, I_1(q) = 0$ 2. $I_2(p) = I_2(q) = I_2(r) = 1$

Both of them make A true! Thus, $X \models A$.

Natural Deduction

Natural Deduction and Derivations

- Natural deduction is pure syntax manipulation and acts as *proof system* with a formal notion of proof as a mathematical entity (cf. informal proof in ordinary math).

- Natural Deduction exploits *derivations* (or formal proofs).

- A derivation is a finite sequence of formulae, which are derived from each other based on elementary formula manipulations ("1-step inference rules")

- Syntax (proof system) vs. semantics is arguably the most important distinction in formal logic.

Syntax of Sequents

- From now on, we write $X \vdash A$ rather than $X \models A$.
- $X \vdash A$ means A syntactically follows from X, i.e., you can formally prove the conclusion A using X as assumptions (within the system of natural deduction).
- $X \models A$ means A semantically follows from X.

Conjunction

Introduction
oo
Sequents
oooo
Natural Deduction
ooo
Conjunction
o●ooooo
Implication
oooooooooo
Theorems
oo
Summary
oo

The 1-Step Rules: And-Elimination

What are the 1-step rules for dealing with conjunction?

Elimination rule:

$$\frac{A \wedge B}{A} \wedge E \qquad\qquad \frac{A \wedge B}{B} \wedge E$$

Which reads: If we derived $A \wedge B$, we can derive both A and B.

The 1-Step Rules: And-Introduction

What are the 1-step rules for dealing with conjunction?

Introduction rule:

$$\frac{A \qquad B}{A \wedge B} \wedge I$$

Which reads: If we derived A and we derived B, we can derive $A \wedge B$.

Proof Syntax / Notation: Overview

- How to write down proofs?
- There are many different notations that describe the same thing
- We introduce two:
    - Tree-like representation of the applied rules
      (just since it's another standard)
    - list-like representation (only use that one!)

Proof Syntax / Notation: Tree- and List-like Representations

- Assume we want to prove $p \wedge q \vdash q \wedge p$
- In the tree-like format:

$$\frac{\dfrac{p \wedge q}{q} \wedge E \quad \dfrac{p \wedge q}{p} \wedge E}{q \wedge p} \wedge I$$

- Leaves are assumptions, root is conclusion
- Advantages: Makes the proof structure obvious
- In exercises, etc: *Do not use it, unless we ask you to!*

Proof Syntax / Notation: Tree- and List-like Representations

- Assume we want to prove $p \wedge q \vdash q \wedge p$

- In the list format:

$$
\begin{array}{llllll}
\alpha_1 & (1) & p \wedge q & \text{A} & \equiv & p \wedge q \vdash p \wedge q \\
\alpha_1 & (2) & q & 1 \wedge E & \equiv & p \wedge q \vdash q \\
\alpha_1 & (3) & p & 1 \wedge E & \equiv & p \wedge q \vdash p \\
\alpha_1 & (4) & q \wedge p & 2,3 \wedge I & \equiv & p \wedge q \vdash q \wedge p
\end{array}
$$

  column 1: assumption number   *column 2:* line number
  column 3: derivation   *column 4:* how it was derived

- **Note:** Each line represents a sequent! (Sequence of sequents.)

The 1-Step Rules (Based on Sequents):   Derivation Rules

- Derivation Rules as considered so far:

$$\frac{A \wedge B}{A} \wedge E \qquad\qquad \frac{A \wedge B}{B} \wedge E \qquad\qquad \frac{A \qquad B}{A \wedge B} \wedge I$$

- Re-written in terms of sequents:

$$\frac{X \vdash A \wedge B}{X \vdash A} \wedge E \qquad \frac{X \vdash A \wedge B}{X \vdash B} \wedge E \qquad \frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

$\rightarrow$ I.e., now we see how premises accumulate!

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$p, q \vdash p \wedge q$

$$
\begin{array}{llll}
\alpha_1 & (1) \quad p & \text{A} & \equiv \quad p \vdash p \text{ (by assumption)} \\
\alpha_2 & (2) \quad q & \text{A} & \equiv \quad q \vdash q \text{ (by assumption)} \\
\alpha_1, \alpha_2 & (3) \quad p \wedge q \quad 1,2 \wedge I & \equiv \quad p, q \vdash p \wedge q \quad (\wedge I)
\end{array}
$$

$$
\boxed{\dfrac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I}
$$

Implication

The 1-Step Rules: Implication-Elimination and -Introduction

- **Elimination rule:**

$$\frac{A \to B \qquad A}{B} \to E$$

- **Introduction rule:**

if we can derive   then we can derive
B using A: $A \to B$ and discharge A:

$$
\begin{array}{c}
[A] \\
\vdots \\
B \\
\hline
A \to B
\end{array} \to I
\qquad \equiv \qquad
\begin{array}{c}
A \\
\vdots \\
B
\end{array}
\qquad + \qquad
\begin{array}{c}
[A] \\
\vdots \\
B \\
\hline
A \to B
\end{array} \to I
$$

The 1-Step Rules: Implication-Elimination and -Introduction (Based on Sequents)

- Derivation Rules as considered so far:

$$\frac{A \rightarrow B \qquad A}{B} \rightarrow E$$

$$\begin{array}{c} [A] \\ \vdots \\ B \\ \hline A \rightarrow B \end{array} \rightarrow I$$

- Re-written in terms of sequents:

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

$$\underbrace{\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I}$$

Has side effect of

removing the assumption A

- We say that A gets *discharged*, and annotate that in the proof.

The 1-Step Rules: Deduction Equivalence

$$\underbrace{X \vdash A \rightarrow B \quad \text{iff} \quad X, A \vdash B}$$

deduction equivalence
(or deduction theorem)

Why does this hold?

- If $X, A \vdash B$, then $X \vdash A \rightarrow B$:

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

- If $X \vdash A \rightarrow B$, then $X, A \vdash B$:

$$\frac{X \vdash A \rightarrow B \qquad A \vdash A}{X, A \vdash B} \rightarrow E$$

(That's the $\rightarrow E$ rule with Y substituted by A)

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$$p \rightarrow q \vdash (p \wedge r) \rightarrow q$$

α_1	(1)	$p \rightarrow q$	A
α_2	(2)	$p \wedge r$	A
α_2	(3)	p	2 $\wedge E$
α_1, α_2	(4)	q	1,3 $\rightarrow E$
α_1	(5)	$(p \wedge r) \rightarrow q$	4[α_2] $\rightarrow I$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

α_1, α_2	(n-1)	q
α_1	(n)	$(p \wedge r) \rightarrow q$	(n-1)[α_2] $\rightarrow I$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption α_2.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1 (cont'd)

The proof of $p \rightarrow q \vdash (p \wedge r) \rightarrow q$ in a tree-like structure:

$$\cfrac{p \rightarrow q \quad \cfrac{\cfrac{[p \wedge r]^{(1)}}{p} \wedge E}{} }{\cfrac{q}{(p \wedge r) \rightarrow q} \rightarrow I(1)} \rightarrow E$$

Here, we denote discharged assumptions by $[\ldots]^{(n)}$, where we number each assumption so that they can be distinguished from each other, i.e., so that we know which rule discharged which assumption(s).

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \wedge q) \to r \vdash p \to (q \to r)$$

α_1	(1)	$(p \wedge q) \to r$	A
α_2	(2)	p	A
α_3	(3)	q	A
α_2, α_3	(4)	$p \wedge q$	2,3 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(5)	r	1,4 $\to E$
α_1, α_2	(6)	$q \to r$	5[α_3] $\to I$
α_1	(7)	$p \to (q \to r)$	6[α_2] $\to I$

$$\frac{X, A \vdash B}{X \vdash A \to B}\to I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B}\to E$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \to r$	(n-2)[α_3] $\to I$
α_1	(n)	$p \to (q \to r)$	(n-1)[α_2] $\to I$

Vacuous Discharge: Discharging Non-existent Assumptions

- We can "discharge" assumptions that are not there; this happens if the conclusion does not depend on its assumption.

$p \vdash q \to p$

$$
\begin{array}{llll}
\alpha_1 & (1) & p & \text{A} \\
\cancel{\alpha_2} & (2) & q & \text{A} \\
\alpha_1 & (2) & q \to p & 1[]{\to}I
\end{array}
\qquad\qquad
\boxed{\dfrac{X, A \vdash B}{X \vdash A \to B}{\to}I}
$$

$$
\begin{array}{llll}
\alpha_1, \cancel{\alpha_2} & \text{(n-1)} & p & \\
\alpha_1 & \text{(n)} & q \to p & \text{(n-1)}[\cancel{\alpha_2}] \to I
\end{array}
$$

- We call such a discharge a *vacuous discharge*.
- I.e., whenever we "would remove" some assumption α from a set of assumptions X, but $\alpha \notin X$, we write $i[]{\to}I$ instead of $i[\alpha]{\to}I$

Excursion: $\vdash$ vs. $\rightarrow$: An Often Asked Question in Previous Courses

- $\vdash$ and $\rightarrow$ seem to be of a very related nature:
  E.g., compare $A, B \vdash C$ with $A \wedge B \rightarrow C$
- So what's the difference?
- $\rightarrow$ is a logical connective, whereas $\vdash$ is not.
- $\vdash$ is a relation between formulae and cannot be used within a formula.
- They are linked by the deduction theorem: $X, A \vdash B$ if and only if $X \vdash A \rightarrow B$. In particular: $A \vdash B$ if and only if $\vdash A \rightarrow B$

Theorems

Unconditionally True Formulas

- Sequents that do not depend on anything are called *theorems*!
- Thus, A is a theorem if "$\vdash A$", e.g., $\vdash p \rightarrow (q \rightarrow p)$.
- Another (slightly more complex) example:

$$
\begin{array}{llll}
\alpha_1 & (1) & p & A \\
\alpha_2 & (2) & q & A \\
\alpha_1, \alpha_2 & (3) & p \wedge q & 1,2 \wedge I \\
\alpha_1 & (4) & q \rightarrow (p \wedge q) & 3[\alpha_2] \rightarrow I \\
& (5) & p \rightarrow (q \rightarrow (p \wedge q)) & 4[\alpha_1] \rightarrow I
\end{array}
$$

- Thus, we get $\vdash p \rightarrow (q \rightarrow (p \wedge q))$, so its formula is a theorem.
- Note that A in $\vdash A$ is a tautology!

Summary

Content of this Lecture

- Sequents and their semantics
    - What does $X \models A$ mean?
- The most important basics of Natural Deduction!
    - How can proofs be written?
    - What does $X \vdash A$ mean?
    - Every logical connective comes with two 1-step rules:
      Introduction and *Elimination*
    - What's a theorem?
- $\rightarrow$ The Logic Notes sections:
    - *3. More about propositional logic*: *Truth Tables*
    - *2. Propositional natural deduction*: *Conjunction*
    - *2. Propositional natural deduction*: *Implication*
    - *2. Propositional natural deduction*: *Counting assumptions*
      (except *Contraction*, which you should look up!)