Fundamental Mathematics

Content

Differential equation

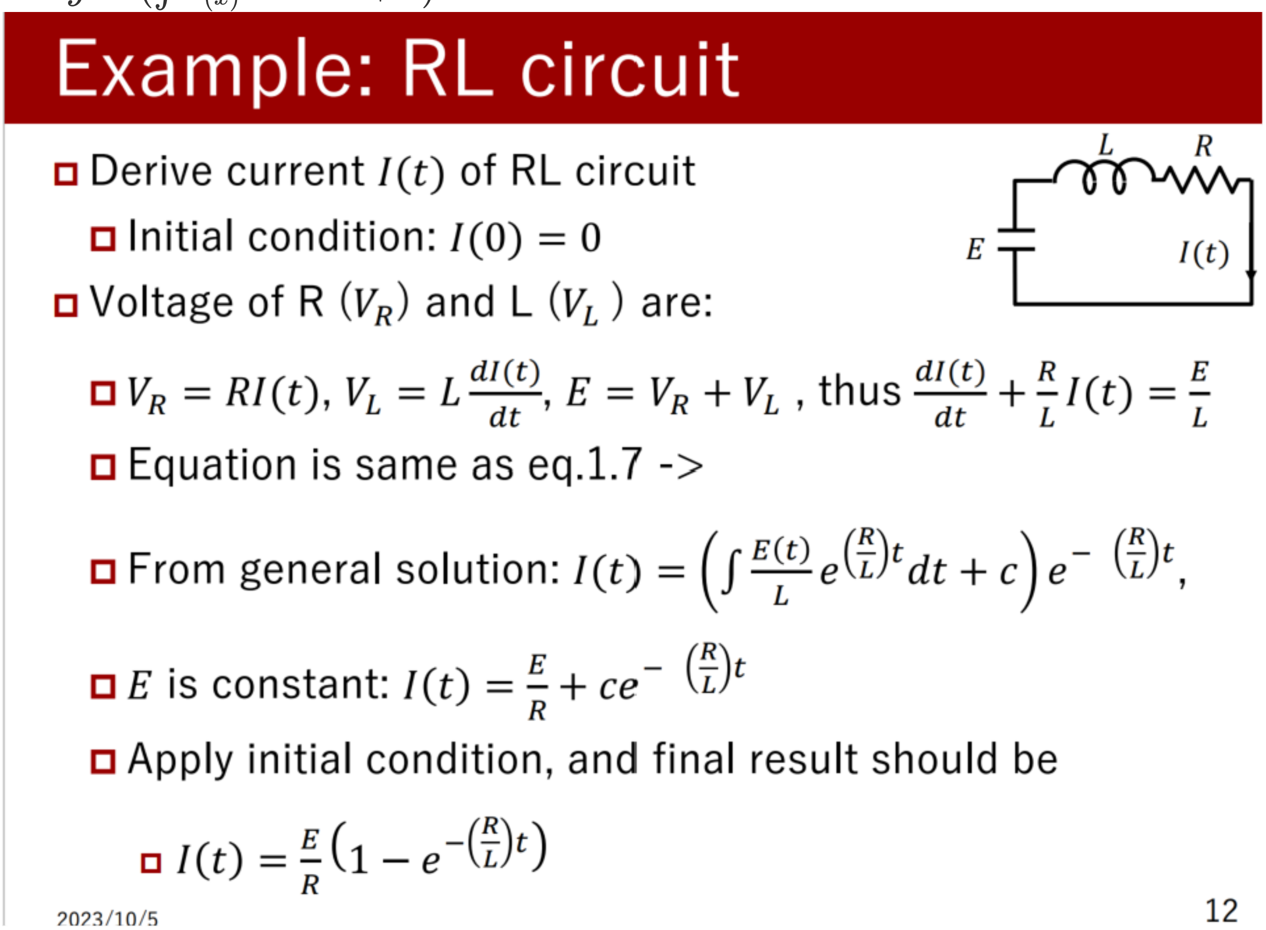
Linear differential equation

Differential equation defined by **linear polynomial** in the unknown function and its derivatives

- $y' + ay = 0$ `{ a is constant , eq.1.1}`
    - If $y_{(x)}$ satisfy above, $y_{(x)}$ is called as **solution**
    - Solve above equation to obtain $y_{(x)}$
    - Before that, y is called as **unknown function**
- Mission: solve function $y' + ay = 0$ `{ a is constant , eq.1.1}`
    - Use nature of exponential function
        - $(e^{ax})' = ae^{ax}$
    - Multiple e^{ax} to `{eq.1.1: y^\'^ + ay = 0}`
        - $e^{ax}y' + ae^{ax}y = 0$
    - Recall the differential for products
        - $(g_{(x)}h_{(x)})' = g_{(x)}h_{(x)}' + g_{(x)}'h_{(x)}$
    - `{eq.1.1: y^\'^ + ay = 0}` should be
        - $(e^{ax}y)' = 0 \rightarrow e^{ax}y = c$ `{c is arbitrary constant}`
        - Solution w/o constant: **a general solution**
        - If c has some specific value → **a particular solution**
- Shape of function depends arbitrary constant `{Initial value problem}`
    - We may don't know the arbitrary constant itself
    - We may know the value(y_0) on specific point(x_0)
        - y_0 : Initial value or initial condition
        - $y' + ay = 0, y_0 = y(x_0) = ce^{-ax_0}$
            - $c = y_0e^{ax_0}$
    - General form of `{eq.1.1: y^\'^ + ay = 0}` should be

- $y = y_0 e^{-a(x-x_0)}$

Homogeneous differential

- Think about following equation
  - $y' + ay = r_{(x)}$ `{eq.1.7}`
  - A differential equation is **homogeneous** when
    - $f_{(x,y)} dy = -g(x,y)dx \rightarrow f_{(x,y)} + g_{(x,y)} \frac{dx}{dy} = 0$
    - If $r_{(x)} = 0$, `{eq.1.7 }` is homogeneous
  - If not, a differential equation is **inhomogeneous**
    - If $r_{(x)} \neq 0$, `{eq.1.7}` is inhomogeneous
  - A general solution of inhomogeneous function `{eq.1.7}` is
    - $y = (\int r_{(x)} e^{ax} dx + c)e^{-ax}$
    - Multiple e^{ax} to `{eq.1.7}`
      - $(y' + ay)e^{ax} = r_{(x)} e^{ax} \rightarrow (ye^{ax})' = r_x e^{ax}$
    - Take integral
      - $ye^{ax} = \int r_{(x)} e^{(ax)} dx + c$ `{c is constant}`
      - $y = (\int r_{(x)} e^{(ax)} dx + c)e^{-ax}$
    - 

Quiz

- Solve a general form of following equations

- $y^{'} - y = 0$
  $a = -1, y = ce^{x}$ `{c is arbitrary constatn}`
- $y^{'} + y = 0$
  $a = 1, y = ce^{-x}$ `{c is arbitrary constatn}`

* Solve a initial value problem of following equations
  $c = y_0 e^{ax_0}, y = y_0 e^{-a(x-x_0)}$
  - $y^{'} + y = 0, y_{(0)} = 2$
    $a = 1, y = ce^{-x}$
    $c = 2, y = 2e^{-x}$
  - $y^{'} - 2y = 0, y_{(1)} = 2$
    $a = -2, y = ce^{2x}$
    $c = \frac{2}{e^2}, y = 2e^{2(x-1)}$
* Generalize
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Homogeneous differential eq.

- Similarly, variable coeff. diff. equation
  - $y' + f(x)y = r(x)$ (eq.1.14)
- Homogeneous case
  - $y' + f(x)y = 0$(eq.1.15)
  - Assume $F(x)$ as primitive function of $f(x)$
    - $(e^{F(x)})' = e^{F(x)}F'(x) = f(x)e^{F(x)}$
  - Multiply $e^{F(x)}$ to eq.1.15
    - $e^{F(x)}y' + e^{F(x)}f(x)y = \underbrace{e^{F(x)}y' + (e^{F(x)})'y = (e^{F(x)}y)'}_{\text{Leibniz product rule}} = 0$
  - Thus: $e^{F(x)}y = c$ (c is constant)
- General solution for homogeneous eq. $y' + f(x)y = 0$
  - $y = ce^{-F(x)}$

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- Inhomogeneous case
  - $y' + f(x)y = r(x)$ (eq.1.14)
  - Multiply $e^{F(x)}$ to eq.1.14
    - $e^{F(x)}y' + e^{F(x)}f(x)y = (e^{F(x)}y)' = e^{F(x)}r(x)$
  - Take integral
    - $e^{F(x)}y = \left(\int r(x)e^{F(x)}dx + c\right)$
- General solution for inhomogeneous eq. $y' + f(x)y = r(x)$
  - $y = \left(\int r(x)e^{F(x)}dx + c\right)e^{-F(x)}$

Solution of differential equation

- Constant coefficient 1st order differential equation
  - Homogeneous: $y^{'} + ay = 0 \leftrightarrow y = ce^{-ax}$ `{c is constant}`
  - Inhomogeneous:
    - $y^{'} + ay = r_{(x)} \leftrightarrow y = (\int r_{(x)}e^{ax}dx + c)e^{-ax}$
  - Variable coefficient 1st order differential equation
    - Homogeneous: $y^{'} + f_{(x)}y = 0 \leftrightarrow y = ce^{-F_{(x)}}$
    - Inhomogeneous: $y^{'} + f_{(x)}y = r_{(x)} \leftrightarrow y = (\int r_{(x)}e^{F_{(x)}}dx + c)e^{-F_{(x)}}$

> - Variation of constants
> - Method of indeterminate coefficient

Variation of constants

Can solve linear (inhomogeneous) differential equation

- Difficulty to solve high order equation
- Equation becomes complex for high order equation

- Strategy
  i. Change given inhomogeneous equation to homogeneous
  ii. Solve general solution for the homogeneous equation
  iii. Replace constant c to function $u(x)$
  iv. Substitute $u(x)$ to given inhomogeneous equation
     - Calculate general solution of $u(x)$
  v. Substitute $u(x)$ to solution of homogeneous equation
- Example: get general solution of : $y^{'} + f_{(x)}y = r_{(x)}$ `{eq.1.7}`
  i. Change given inhomogeneous equation to homogeneous

- $y' + f_{(x)}y = 0$

ii. Solve general solution for the homogeneous equation
- Use this relationship: $(e^{F(x)})' = e^{F_{(x)}} F'_{(x)} = e^{F_{(x)}} f_{(x)}$
- $e^{F_{(x)}} y' + e^{F_{(x)}} f_{(x)} y = e^{F(x)} y' + e^{F_{(x)}} (e^{F(x)})' y = 0$
- thus, $(e^{F(x)}y)' = 0 \rightarrow e^{F(x)}y = c$ `{c is constant}`

iii. Replace constant c to function $u(x)$
- $y = ce^{-F(x)} \rightarrow y = u_{(x)}e^{-F(x)} \rightarrow u_{(x)} = ye^{+F(x)}$

iv. Substitute $u(x)$ to given inhomogeneous equation
- $(u_{(x)}e^{-F(x)})' + f_{(x)}u_{(x)}e^{-F(x)} = r_{(x)}$
- $u'_{(x)}e^{-F(x)} - f_{(x)}u_{(x)}e^{-F(x)} + f_{(x)}u_{(x)}e^{-F(x)} = r_{(x)}$
- $u'_{(x)} = r_{(x)}e^{F(x)} \rightarrow u_{(x)} = \int r_{(x)}e^{F_{(x)}}dx + C$ `{c is constant}`

v. Substitute $u(x)$ to solution of homogeneous equation
- $y = u_{(x)}e^{-F(x)} = (\int r_{(x)}e^{F_{(x)}}dx + C)e^{-F(x)}$

Method of indeterminate coefficient

With some assumptions, we can easily solve differential equation

Guess the candidate of particular solution

If the right side of an equation is
- n-order polynormal: candidate should be n-polynormal
- sine function: candidate should be in sine
- exponential: candidate should be in exponential

- Example1: get general solution of: $y' + 3y = x^2 - 1$ `{eq.1.23}`
  - Assume particular solution is $y_p = \alpha x^2 + \beta x + \gamma$
    - α, β, γ are constant. Substitute y_p to `{eq.1.23}`
    - $y'_p + 3y_p = (2\alpha x + \beta) + 3(\alpha x^2 + \beta x + \gamma) = x^2 - 1$
    - This equation should satisfy following conditions
      - $x^2 : 2\alpha = 1, x : 2\alpha + 3\beta = 0, x^0 : \beta + 3\gamma = -1$, thus
    - $y_p = \frac{1}{3}x^2 - \frac{2}{9}x - \frac{7}{27}$
  - If one particular solution is clear, general solution can be easily solved.
  - Example: get general solution of : $y' + f_{(x)}y = r_{(x)}$
    - Assume particular solution y_p, general solution y, and its difference $y_h = y - y_p$ is
    - $y'_h + f_{(x)}y_h = (y - y_p)' + f_{(x)}(y - y_p) = y' + f_{(x)}y - (y'_p + f_{(x)}y_p) = r_{(x)} - r_{(x)} = 0$
    - This is homogeneous:$y_h = ce^{-F_{(x)}}$
    - $\mathbf{y = y_p + y_h = y_p + ce^{-F_{(x)}}}$ `{c is constant}`
      - we can use this as theorem
    - general solution of `{eq.1.23}` :$y = \frac{1}{3}x^2 - \frac{2}{9}x - \frac{7}{27} + ce^{-3x}$

- Example2: get general solution of :$y^{'}+2y=\cos(x)$ `{eq.1.25}`
  - Assume particular solution is $y_p=\alpha\cos x+\beta\sin x$
    - α,β are constant. Substitute y_p to `{eq.1.25}`
    - $y_p^{'}+2y_p=-\alpha\sin x+\beta\cos x+2(\alpha\cos c+\beta\sin x)=\cos x$
    - This equation should satisfy following conditions
    - $\cos x:2\alpha+\beta=1,\sin x:-\alpha+2\beta=0$, thus
  - $y_p=\frac{2}{5}\cos x+\frac{1}{5}\sin x$
  - $y=\frac{2}{5}\cos x+\frac{1}{5}\sin x+ce^{-2x}$
- Example3: get general solution of : $y^{'}-y=2e^{2x}$ `{eq.1.28}`
  - Assume particular solution is $y_p=\alpha e^{2x}$
    - α is constant. Substitute y_p to `{eq.1.28}`
    - $y_p^{'}-y_p=2\alpha e^{2x}-\alpha e^{2x}=\alpha e^{2x}=2e^{2x}$, thus
    - $y_p=2e^{2x}$
    - $y=2e^{2x}+ce^{x}$
    - However, this is not true for all of solution
- Example: get general solution of : $y^{'}-2y=2e^{2x}$ `{eq.1.29}`
  - Assume particular solution is $y_p=\alpha e^{2x}$
    - α is constant. Substitute y_p to `{eq.1.29}`
    - $y_p^{'}-y_p=2\alpha e^{2x}-2\alpha e^{2x}=0\rightarrow$ wrong assumption
  - Assume particular solution is $y_p=\alpha xe^{2x}$
    - α is constant. Substitute y_p to `{eq.1.29}`
    - $y_p^{'}-y_p=(\alpha e^{2x}+2\alpha xe^{2x})-2\alpha xe^{2x}=2e^{2x}$
    - $y_p=2xe^{2x}$
    - $y=(2x+c)e^{2x}$
  - If general solution is $\mathbf{y}^{'}+\mathbf{ay}=\mathbf{ke}^{-\mathbf{ax}}$, particular solution should be $y_p=kxe^{-ax}$
- **Exerceis(1)**
  - Solve general solutions for following equations
    - by Variation of constants method
      - $y^{'}-xy=x$

        a. $f_{(x)}=-x,F_{(x)}=-\frac{1}{2}x^2,r_{(x)}=x$
        b. $u_{(x)}=\int r_{(x)}e^{F_{(x)}}dx+c,$
        $y=u_{(x)}e^{-F_{(x)}}=(\int xe^{-\frac{1}{2}x^2}dx+c_1)e^{\frac{1}{2}x^2}=-1+c_2e^{\frac{x}{2}}$
      - `{c1, c2 is constant}`
      - $y^{'}+\frac{y}{x}=x^2+2x$

        a. $f_{(x)}=\frac{1}{x},F_{(x)}=\ln x,$
        $r_{(x)}=x^2+2x$
        b. $y=\frac{\int(x^2+2x)xdx+c_1}{x}=\frac{x^3}{4}+\frac{2x^2}{3}+\frac{c_2}{x}$
    - by Method of indeterminate coefficient

- $2y^{'} + 3y = 3x^2 + x$
  - Assuming the particular solution: $y_p = \alpha x^2 + \beta x + \gamma$
  - Substitute: $2(2\alpha x + \beta) + 3(\alpha x^2 + \beta x + \gamma) = 3x^2 + x$
    $3\alpha = 3, 4\alpha + 3\beta = 1, 2\beta + 3\gamma = 0$
    $\alpha = 1, \beta = -1, \gamma = \frac{2}{3}$
  - $f_{(x)} = \frac{3}{2}, F_{(x)} = \frac{3}{2}x$
  - particular solution: $y_p = x^2 - x + \frac{2}{3}$
  - general solution: $y_p = x^2 - x + \frac{2}{3} + ce^{-\frac{3}{2}x}$ `{c: const}`
- $y^{'} + 4y = 3e^{-x}$
  - $y_p = \alpha e^{-x}$
  - $-\alpha e^{-x} + 4\alpha e^{-x} = 3e^{-x}, \alpha = 1$
  - $f_{(x)} = 4, F_{(x)} = 4x$
  - $y_p = e^{-x}, y = e^{-x} + ce^{-4x}$

Euler's formula

- The trigonometric functions (sin cos) and complex exponential function satisfy following relationship
  - $e^{ix} = \cos x + i \sin x$
  - *e*: base of natural logarithm, *i*(or *j*): imaginary unit
- Euler’s formula is useful for circuit analysis, cause
  - Easy to take integral, differential
    - $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}, \int e^{\lambda x}dx = \frac{1}{\lambda}e^{\lambda x} + c$ `{c: constant}`
    - Phasor: expression of sine func. in complex exponent
      - $A \cos wx = \Re A \cos wx + iA \sin wx$
      - Calculate circuit in complex exponent, then convert to original sine functions

2nd order differential equation

- Introduce 2nd order differential equation
  - $y^{''} + ay^{'} + by = r_{(x)}$ `{a, b are constants, eq.3.1}`
    - If $r_{(x)} = 0$, `{eq.3.1}` is homogeneous
    - If $r_{(x)} \neq 0$, `{eq.3.1}` is inhomogeneous
- Inhomogeneous form is very tough for hand calculation
  - If $r(x)$ is constant, sine, or exponential we can use method of indeterminate coefficient
    - In physics, circuits, we can use this assumption

Characteristic equation

- If $r_{(x)} = 0$ and $y_{(x)} = ce^{\lambda x}$ (c, λ : constant), `eq.3.1` is
  - $y^{''} + ay^{'} + by = (\lambda^2 + a\lambda + b)ce^{\lambda x} = 0, ce^{\lambda x} \neq 0$ thus
    - $\lambda^2 + a\lambda + b = 0 :$ characteristic equation

- Solution and $\lambda = \frac{-a \pm \sqrt{a^2-4b}}{2}$ changes depend on discriminant function $(a^2 - 4b)$
  - $a^2 - 4b > 0 : \lambda_1, \lambda_2$ in real. Solutions: $c_1 e^{\lambda x}, c_2 e^{\lambda x}$
  - $a^2 - 4b = 0 : \lambda = -\frac{a}{2}$. Solutions: $c_1 e^{\lambda x}, c_2 x e^{\lambda x}$
  - $a^2 - 4b < 0 : \lambda_1, \lambda_2$ in imaginary value.
    - $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$
    - Solutions: $c_1 e^{\lambda x}, c_2 e^{\lambda x}$

Linearity of solution

- Use linearity of solution
- Theorem: if $y_{(x)}$ and $w_{(x)}$ are the solution of linear equation `{eq.3.1}` , sum $c_1 y_{(x)} + c_2 w_{(x)}$ is also the solution
- Proof: since $y_{(x)}$ and $w_{(x)}$ are the solution, it should satisfy
  - $y^{''} + ay^{'} + by = 0, w^{''} + aw^{'} + bw = 0,$
  - Multiply const c_1 and c_2 and get its sum
    - $c_1 y^{''} + c_1 a y^{'} + c_1 b y + c_2 w^{''} + c_2 a w^{'} + c_2 b w = 0$
  - $(c_1 y + c_2 w)^{''} + a(c_1 y + c_2 w)^{'} + b(c_1 y + c_2 w) = 0$
  - So, $c_1 y_{(x)} + c_2 w_{(x)}$ is also the solution
- Solution is the sum of exponents, comes from characteristic equation

General solution

- Theorem: General solution of 2nd order homogeneous differential equation is
  - $a^2 - 4b = 0 : y_{(x)} = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}, \lambda_1\text{: multiple root of char. eq}$
  - $a^2 - 4b \neq 0 : y_{(x)} = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \lambda_1, \lambda_2\text{: roots of char. eq}$
- Proof: if $y(x)$ is the solution of `{eq.3.1}` , multiply $e^{-\lambda x}$
  - $e^{-\lambda x} y^{''} + e^{-\lambda x} a y^{'} + e^{-\lambda x} b y = 0$
  - $(e^{-\lambda x} y)^{''} + (a + 2\lambda)(e^{-\lambda x} y)^{'} + (\lambda^2 + a\lambda + b) e^{-\lambda x} y = 0$
    - If we assume $\lambda 1$ is root of char. eq., $\lambda_1^2 + a\lambda_1 + b = 0$, thus
    - $(e^{-\lambda x} y)^{''} + (a + 2\lambda)(e^{-\lambda x} y)^{'} = 0$
  - $u^{''} + (a + 2\lambda_1) u^{'} = 0$, when $e^{-\lambda x} y_{(x)} = u_{(x)}$
    - Case $a^2 - 4b = 0 : \lambda = -\frac{a}{2}, \text{thus} u^{''} = 0$
      - $u(x) = c_1 + c_2 x, \text{ thus } y_{(x)} = c_1 e_{\lambda_1 x} + c_2 x e^{\lambda_1 x}$
      - Case $a^2 - 4b \neq 0 :$
        - $v^{'} + (a + 2\lambda_1) v = 0, \text{when} v = u^{'}$, solve this then
        - $v = C e^{-(a + 2\lambda_1) x}$, C is constant. Then integrate this
          - $u_{(x)} = c_1 - \frac{C}{a + 2\lambda_1} e^{-(a + 2\lambda_1) x}$, thus

- $y = c_1 e^{\lambda_1 x} - c_2 e^{\lambda_2 x}$, since (a + λ_1) si the solution λ_2. $(c_2 = \frac{C}{a+2\lambda_1}, \lambda_2 = a + \lambda_1)$
  - Transform from/to sine to/from exponent
    - Case $(a^2 - 4b < 0)$:
    - $y_{(x)} = c_1 e^{(A+iB)x} + c_2 e^{(A-iB)x}$
    - $y_{(x)} = c_1 e^{Ax}(\cos Bx + i \sin Bx) + c_2 e^{Ax}(\cos Bx - i \sin Bx)$
      - $= (c_1 + c_2)e^{Ax} \cos Bx + i(c_1 - c_2)e^{Ax} \sin Bx$
      - $= d_1 e^{Ax} \cos Bx + d_2 e^{Ax} \sin Bx$

Exercise(2)

- Solve characteristic equation and general solutions for following equations
  - by Method of indeterminate coefficient
    - $y^{''} + 2y^{'} + y = 0$
      - since $r_{(x)}$ = 0, assume $y = ce^{\lambda x}.c, \lambda :$ unknown
      - $(\lambda^2 + 2\lambda + 1)ce^{\lambda x} = 0$
      - characteristic. eq $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \lambda = -1$
      - general solution: $y = c_1 e^{-x} + c_2 x e^{-x}$
    - $y^{''} + 2y^{'} + 3y = 0$
      - since $r_{(x)}$ = 0, assume $y = ce^{\lambda x}.c, \lambda :$ unknown
      - $(\lambda^2 + 2\lambda + 3)ce^{\lambda x} = 0$
      - characteristic. eq $\lambda^2 + 2\lambda + 3 = 0 \rightarrow \lambda = -1 \pm \sqrt{2}i$
      - general solution: $y = c_1 e^{-(1+\sqrt{2}i)x} + c_2 e^{-(1-\sqrt{2}i)x}$
    - $y^{''} - 4y^{'} - 5y = 0$
      - characteristic. eq $\lambda^2 - 4\lambda - 5 = 0 \rightarrow \lambda = -1, 5$
      - general solution: $y = c_1 e^{5x} + c_2 e^{-x}$ `{c1, c2: constant}`

2nd order differential equation

- Introduce 2nd order differential equation
  - $y^{''} + ay^{'} + by = r_{(x)}$ (a, b are constants) `{eq.2.12}`
    - If r(x) = 0, `{eq.2.12}` is homogeneous `{eq.2.2}`
    - If r(x) ≠ 0, `{eq.2.12}` is inhomogeneous `
- Inhomogeneous form is very tough for hand calculation
  - If $r(x)$ is constant, sine, or exponential we can use **method of indeterminate coefficient**
  - Variation of constants
  - Method of indeterminate coefficient

Structure of solution for inhomogeneous equation

- Theorem:
  - Assume solution $u_{(x)}$ for $u^{''}+au^{'}+bu=0$ and particular solution $y_p(x)$ for $y^{''}+ay^{'}+by=r_{(x)}$
  - General solution for $y^{''}+ay^{'}+by=r(x)$ is $y(x)=y_p(x)+u(x)$
- Proof:
  - Calculate differential for $y(x)+u(x)$
    - 1st order diff: $(y(x)+u(x))^{'}=y^{'}(x)+u^{'}(x)$
    - 2nd order diff: $(y(x)+u(x))^{''}=y^{''}(x)+u^{''}(x)$
    - y(x) is general solution; u(x) is solution for homogeneous
  - y(x) + u(x) is also the solution for `{eq.2.12}`
    - $(y+u)^{''}+a(y+u)^{'}+b(y+u)=y^{''}+ay^{'}by+u^{''}+au^{'}+bu=r(x)$
  - Next, assume $y_1(x)-y_2(x)$ is solution for inhomogeneous equation `{eq.2.1}`
    - $(y_1-y_2)^{''}+a(y_1-y_2)^{'}+b(y_1-y_2)=(y_1^{''}+ay_1^{'}+by_1)-(y_2^{''}+ay_2^{'}+by_2)=r(x)-r(x)=0$
- $y(x)=y_p(x)+u(x)$
  - General solution for inhomogeneous equation y(x) is sum of particular solution for inhomogeneous $y_p(x)$ and general solution for homogenesis u(x)
- General solution for inhomogeneous equation
  - $y(x)=y_p(x)+u(x)$
  - Need particular solution for inhomogeneous eq. $u(x)$
- We can calculate solution for inhomogeneous eq. with sum assumption
  - Method of indeterminate coefficient
  - Variation of constants need to calculate array

Method of indeterminate coefficient(exponent)

- Solve general solution $y(x)$ of : $y^{''}+y^{'}+2y=e^{2x}$
  - Get general solution $Y_0(x)$ for homogeneous equation
    - $y^{''}+y^{'}+2y=0$
    - Its characteristic equation:
      - $\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0, \lambda=-1,-2$
    - $y_0(x)=c_1e^{-x}+c_2e^{-2x}$
  - Get particular solution $y_p(x)$ for inhomogeneous equation
    - Assume $y_p(x)=Ae^{2x}$, (A is const., e^{2x} is right side)
      - $4Ae^{2x}+3\cdot 2Ae^{2x}+2\cdot Ae^{2x}=e^{2x}, A=\frac{1}{12}$
    - $y(x)=y_0(x)+y_p(x)=c_1e^{-x}+c_2e^{-2x}+\frac{1}{12}e^{2x}$
- Solve general solution $y(x)$ of : $y^{''}+3y^{'}+2y=e^{-x}$
  - Get general solution $Y_0(x)$ for homogeneous equation
    - $y_0(x)=c_1e^{-x}+c_2e^{-2x}$

- Get particular solution $y_p(x)$ for inhomogeneous equation
  - Assume $y_p(x) = Ae^{-x}$, (A is const., e^{-x} is right side)
    - $Ae^{-x} - 3Ae^{-x} + 2 \cdot Ae^{-x} = 0??$
    - Assume $y_p(x) = Axe^{-x}$, (A is const., e^{-x} is right side)
      - $(-Ae^{-x} - Ae^{-x} + Axe^{-x}) + 3(Ae^{-x} - Axe^{-x}) + 2Axe^{-x} = e^{-x} \rightarrow A = 1$
  - $y(x) = y_0(x) + y_p(x) = c_1e^{-x} + c^2e^{-2x} + xe^{-x}$

- Solve particular solution of : $y^{''} + 3y^{'} + 2y = \cos x$
  - Ex1: Assume particular solution is $y_p = \alpha \cos x + \beta \sin x$
    - α , β are sonstant. Substitute y_p to equation
    - $\alpha = \frac{1}{10}, \beta = \frac{3}{10}, thusy_p = \frac{1}{10} \cos x + \frac{3}{10} \sin x$
  - Ex2: Solve it in imaginary space, then take real part
    - Assume target solution is $u^{''} + 3u^{'} + 2u = e^{ix}$ `{according to Euler formula:}`
    - > $e^{ix} = \cos x + i \sin x$
    - Assume particular solution is $u_p = Ae^{ix}$ `{A is const}`
    - $A = \frac{1}{10} - \frac{3}{10}i, u_p = (\frac{1}{10} \cos x + \frac{3}{10} \sin x)$
    - $y_p = \mathbb{R}\{u_p\} = \frac{1}{10} \cos x + \frac{3}{10} \sin x$

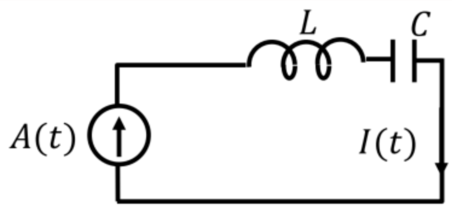
- Solve particular solution of : $y^{''} + 3y^{'} + 2y = x^2$
  - Assume particular solution is $y_p = \alpha x^2 + \beta x + \gamma$
    - α, β, γ, are sonstant. Substitute y_p to equation
    - $2\alpha x^2 + (6\alpha + 2\beta)x + (2\alpha + 3\beta + 2\gamma) = x^2$
  - This equation should satisfy following conditions
    - $x^2 : 2\alpha = 1, x : 6\alpha + 2\beta = 0, x^0 : 2\alpha + 3\beta + 2\gamma = 0$, thus
    - $y_p = \frac{1}{2}x^2 - \frac{3}{2}x - \frac{7}{4}$

- Solve particular solution of : $y^{''} + y^{'} = x^2$
  - Get general solution $y_0(x)$ for homogeneous equation
    - Characteristic equation: $\lambda(\lambda + 1) = 0$
    - $y_0(x) = c_1 + c_2e^{-x}$
  - Particular solution: cannot fix coefficient cx^0
  - Assume particular solution is $y_p = \alpha x^3 + \beta x^2 + \lambda x$
    - α, β, γ are constant. Substitute y_p to equation
    - $3\alpha x^2 + (6\alpha + 2\beta)x + (2\beta + \gamma) = x^2$
    - This equation should satisfy following conditions
      - $x^2 : 3\alpha = 1, x : 6\alpha + 2\beta = 0, x^0 : 2\beta + \gamma = 0$, thus
    - $y_p = \frac{1}{3}x^2 - x + 2, y = \frac{1}{3}x^2 - x + 2 + c_1 + c_2e^{-x}$

Initial condition

- Use initial condition to calculate particular solution
  - $y'' + ay' + by = r(x)$, use y(0) = A, y'(0) = B. (A,B: const)
  - If one particular solution y_p is known, general solution y(x):
    - $y(x) = c_1\phi(x) + c_2\psi(x) + y_p$ φ(x) and ψ(x): shape of basic functions
    - Calculate c_1 and c_2 using initial conditions
  - Generally, solve next simultaneous equation
    - $\begin{bmatrix} \phi(0) & \psi(0) \\ \phi'(0) & \psi'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} A - y_p(0) \\ B - y_p(0) \end{bmatrix}$
- Solve particular solution $y(x)$
  - $y'' + 3y' + 2y = e^{2x}, y(0) = 0, y' = 1$
  - Get general solution $y(x) = c_1e^{-x} + c_2e^{-2x} + \frac{1}{12}e^{2x}$
  - $y(0) = c_1 + c_2 + \frac{1}{12} = 0$
  - $y'(0) = -c_1 - 2c_2 + \frac{1}{6} = 1, \text{thus} c_1 = \frac{2}{3}, c_2 = -\frac{3}{4}$
  - $y(x) = \frac{2}{3}e^{-x} - \frac{3}{4}e^{-2x} + \frac{1}{12}e^{2x}$

Example 1: LC circuit



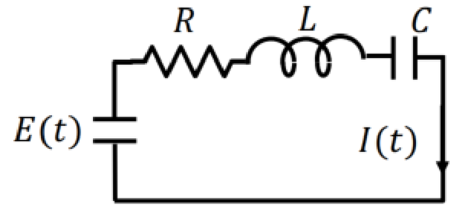
- Derive current $I(t)$ of LC circuit
  - Initial conditions:
    - For $t = 0$, $A(0) = I(0) = 2$,
    - Otherwise, $A(t) = 0$ (no impact on circuit)
    - $I'(0) = 0$
- Voltage of L (V_L) C (V_C) are:

$(I(t) = Q'(t))$

  - $V_L = L\frac{dI(t)}{dt}$, $V_C = \frac{Q(t)}{C}$, $LI'(t) + \frac{Q(t)}{C} = 0$
  - For the current $I(t)$, $I''(t) + \frac{I(t)}{LC} = 0$

- $I''(t) + \frac{I(t)}{LC} = 0$ ($E'(t) = 0$), assume $I(t) = ce^{\lambda t}$
  - Characteristic equation: $\lambda^2 + \frac{1}{LC} = 0$, $\lambda = \pm\sqrt{\frac{1}{LC}}i = \pm\omega i$,
- General solution $I_g(t)$:
  - $I_g(t) = \left(c_1 e^{\omega i t} + c_2 e^{-\omega i t}\right) = (d_1 \cos \omega t + d_2 \sin \omega t)$
- Select θ which satisfy $\cos\theta = \frac{d_2}{\sqrt{d_1^2+d_2^2}}$, $\sin\theta = \frac{d_1}{\sqrt{d_1^2+d_2^2}}$
  - $I_g(t) = \sqrt{d_1^2 + d_2^2}\sin(\omega t + \theta)$,
- $I_p(0) = \sqrt{d_1^2 + d_2^2}\sin(\theta) = 2$, $I_p{}'(0) = \sqrt{d_1^2 + d_2^2}\cos(\theta) = 0$
  - $I_p(t) = 2\sin(\omega t + \pi/2)$ (it will oscillate, “resonance”)
  - Frequency $f = 1/(2\pi\sqrt{LC})$

Example 2: RLC circuit



- Derive current $I(t)$ of RLC circuit
  - Initial condition: $I(0) = 0$
- Voltage of R (V_R) L (V_L) C (V_C) are:
  - $V_R = RI(t)$, $V_L = L\frac{dI(t)}{dt}$, $V_C = \frac{Q(t)}{C}$, $\mathrm{L}I'(t) + RI(t) + \frac{Q(t)}{C} = E(t)$
  - For the current $I(t)$, $I''(t) + \frac{R}{L}I'(t) + \frac{I(t)}{LC} = \frac{E\prime(t)}{L}$ $(I(t) = Q'(t))$
  - (You will learn this in electric circuit class)
- Solve this equation
  - For $E(t) = V$ (V is constant)
  - For $E(t) = Vt$ (V is constant)

- $I''(t) + \frac{R}{L}I'(t) + \frac{I(t)}{LC} = 0$ $(E'(t) = 0)$, assume $I(t) = ce^{\lambda t}$
  - Characteristic equation:$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$
  - $\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$, three candidates of solution
  - $R^2 > \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{R^2-4L/C}/2L} + c_2 e^{-t\sqrt{R^2-4L/C}/2L}\right)$
  - $R^2 = \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}(c_1 + c_2 t)$
  - $R^2 < \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{4L/C-R^2}/2L} + c_2 e^{-t\sqrt{4L/C-R^2}/2L}\right)$

- $I''(t) + \frac{R}{L} I'(t) + \frac{I(t)}{LC} = V$ assume particular solutions are $I_{p1}(t)$, $I_{p2}(t)$, $I_{p3}(t)$
  - General solutions are
  - $R^2 > \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}} \left(c_1 e^{t\sqrt{R^2 - 4L/C}/2L} + c_2 e^{-t\sqrt{R^2 - 4L/C}/2L} \right) + I_{p1}(t)$
  - $R^2 = \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}} (c_1 + c_2 t) + I_{p2}(t)$
  - $R^2 < \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}} \left(c_1 e^{t\sqrt{4L/C - R^2}/2L} + c_2 e^{-t\sqrt{4L/C - R^2}/2L} \right) + I_{p3}(t)$

Exercise

- $y'' + 3y' + 2y = \cos x$
  - $\lambda = -1, -2$
    $y_0 = c_1 e^{-x} + c_2 e^{-2x}$ (c_1, c_2 : constants)
  - asssume particular solution $y_p = \alpha \cos x + \beta \sin x$ (α, β : constant)
  - $(-\alpha \cos x - \beta \sin x) + 3(-\alpha \sin x + \beta \cos x) + 2(\alpha \cos x + \beta \sin x) = \cos x$
  - $(-\alpha + 3\beta + 2\alpha) \cos x + (-\beta - 2\alpha + 2\beta) \sin x = \cos x$
  - $\alpha = \frac{1}{10}, \beta = \frac{3}{10}$
  - $y_p = \frac{1}{10} \cos x + \frac{3}{10} \sin x$
  - $y = \frac{1}{10} \cos x + \frac{3}{10} \sin x + c_1 e^{-x} + c_2 e^{-2x}$
- $y'' - 2y' + 3y = x^2$
  - $\lambda = 1 \pm \sqrt{2} i$
  - $y_0 = c_1 e^{1+\sqrt{2}i} + c_2 e^{1-\sqrt{2}i}$ (c_1, c_2: const)
  - Assume particualr solution $y_p = \alpha x^2 + \beta x + \gamma$ (α, β, γ : const)
  - $2\alpha - (2\alpha x + \beta) + 3(\alpha x^2 + \beta) = x^2$
  - $\alpha = \frac{1}{3}, \beta = \frac{4}{9}, \gamma = \frac{2}{27}$
  - $y = y_0 + y_p = c_1 e^{1+\sqrt{2}i} + c_2 e^{1-\sqrt{2}i} + \frac{1}{3} x^2 + \frac{4}{9} x + \frac{2}{27}$
- $y'' - 2y' - 3y = e^x$
  - $\lambda = -1, 3$
  - $y_0 = c_1 e^{(3x)} + c_2 e^{(-x)}$ (c_1, c_2: const)

- Assume particular solution $y_p = Ae^x$ (A: const)
- $A = -\frac{1}{4}$
- $y = c_1 e^{(3x)} + c_2 e^{(-x)} + -\frac{1}{4}e^x$

- $y^{''} - 2y^{'} - 3y = e^{-x}$
  - $\lambda = -1, 3$
  - $y_0 = c_1 e^{(3x)} + c_2 e^{(-x)}$ (c_1, c_2: const)
  - Assume particular solution $y_p = Axe^{-x}$ (A: const)
  - $\mathbf{y_p^{'} = A(1-x)e^{-x}}$
  - $\mathbf{y_p^{''} = A(x-2)e^{-x}}$
  - $A = -\frac{1}{4}$
  - $y = y_0 + y_p = c_1 e^{(3x)} + c_2 e^{(-x)} - \frac{1}{4}e^{-x}$

Solve particular solution

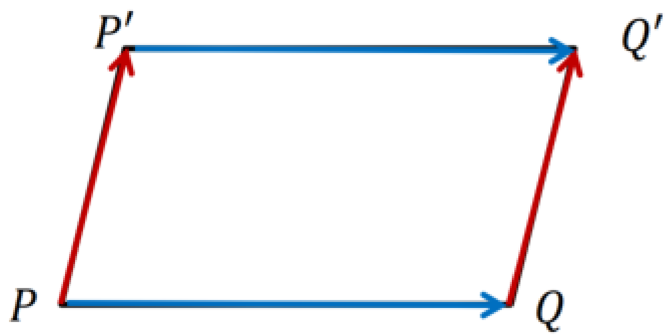
- $y^{''} + 3y^{'} + 2y = \cos x$
  - $\lambda = -1, -2$
    $y_0 = c_1 e^{-x} + c_2 e^{-2x}$ (c_1, c_2 : constants)
  - asssume particular solution $y_p = \alpha \cos x + \beta \sin x$ (α, β : constant)
  - $(-\alpha \cos x - \beta \sin x) + 3(-\alpha \sin x + \beta \cos x) + 2(\alpha \cos x + \beta \sin x) = \cos x$
  - $(-\alpha + 3\beta + 2\alpha) \cos x + (-\beta - 2\alpha + 2\beta) \sin x = \cos x$
  - $\alpha = \frac{1}{10}, \beta = \frac{3}{10}$
  - $y_p = \frac{1}{10} \cos x + \frac{3}{10} \sin x$
  - $y = \frac{1}{10} \cos x + \frac{3}{10} \sin x + c_1 e^{-x} + c_2 e^{-2x}$
  - $y^{'}(x) = -c_1 e^{-x} - 2c_2 e^{-2x} - \frac{1}{10} \sin x + \frac{3}{10} \cos x$
  - $y(\pi) = 0, y^{'}(\pi) = 1$
  - $c_1 = \frac{3}{2}, c_2 = \frac{3}{2}e^{\pi}$
  - $y(x) = \frac{1}{10} \cos x + \frac{3}{10} \sin x + \frac{3}{2}e^{-x+\pi} + \frac{3}{2}e^{-2(x-\pi)}$

Array and vector

Motivation

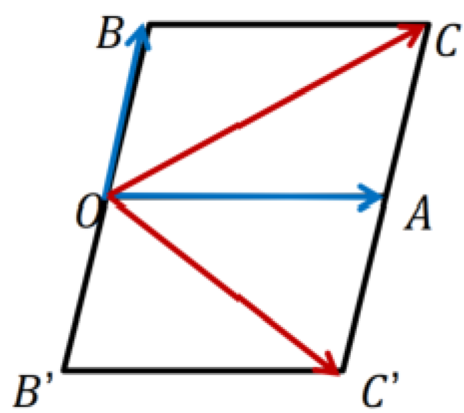
- div **D** = ρ
  - $\iint \mathbf{D} \cdot d\mathbf{S} = \iiint \rho dV$ (Gauss's eq of electricfield)
- div **B** = ρ
  - $\iint \mathbf{B} \cdot d\mathbf{S} = \iiint div\mathbf{B} dV$ (Gauss's eq of magneticfield)
  - $rot\mathbf{H} = i + \frac{\delta D}{\delta t} : \oint \mathbf{H} \cdot d\mathbf{r} = \iint (i + \frac{\delta \mathbf{D}}{\delta t}) \cdot \mathbf{S}$ `Ampele's law`
  - $div\mathbf{E} = -\frac{\delta B}{\delta t} : \oint \mathbf{E} \cdot d\mathbf{r} = -\frac{\delta}{\delta t} \iint \mathbf{B} \cdot d\mathbf{S}$ `Faraday's law`

Scalar and Vector

- Scalar: Value (only)
- Vector: Value (length) and its direction
- Vector from point P to Q is: $\vec{PQ}$
  - P: start point, Q: end point
  - if $\vec{P'Q'}$ is equal to $\vec{PQ}$, $\vec{PQ}$ and $\vec{P'Q'}$ is in the same class
  - If two points are the same, it is zero vector $\vec{PP}, \vec{QQ}$
- To show the vector, we use **bold and italic**
- Vector: $\boldsymbol{a} = \vec{PQ}$
- Zero vector $\boldsymbol{0}$ = $\vec{PP}$
- 

Add, sub, extension

- Assume $\boldsymbol{a} = \vec{OA}, \boldsymbol{b} = \vec{OB}, \boldsymbol{c} = \vec{OC}$, where O, A, B, C composes parallelogram
  - Define: $-\boldsymbol{a} = -\vec{OA} = \vec{AO}$
  - Define: $\boldsymbol{a} + \boldsymbol{b} = \vec{OA} + \vec{OB} = \vec{OC}$
  - Define: $\boldsymbol{a} - \boldsymbol{b} = \vec{OA} - \vec{OB} = \vec{OC'}$
- For real value λ, its product to the vector $\boldsymbol{a}$ is
  - $\boldsymbol{a}\lambda = \lambda\boldsymbol{a}$
- If the three points P, Q, R are on the same line: $\vec{PQ} = \lambda\vec{PR}$
- If the two vectors are in parallel: $\boldsymbol{a}\lambda = \boldsymbol{b}$
  - Geometric vector space
    - Vector space: more general and abstract



Vector space

- L is called vector space if element of L satisfy following definition and notation
  - Addition: result of $\boldsymbol{a} + \boldsymbol{b}$ is unique ($\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{L}$)
  - Scalar multiply: result of $\boldsymbol{a}\lambda$ is unique ($\boldsymbol{a} \in \lambda \in$ R)
- Both satisfy following:

- Association law: $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$
- Exchange law: $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
- Identity element: $\boldsymbol{a} + \boldsymbol{o} = \boldsymbol{a}$
- inverse element: $\boldsymbol{a} + \boldsymbol{-a} = \boldsymbol{o}$

Component

- Vector $\boldsymbol{a}$ is also defined by its components $[a_1, ..., a_n]$
  - n: its dimension
- For the xyz-coordinate system, $\boldsymbol{a} = [a_x, a_y, a_z]$
  - This also satisfy the rules of vector space
- Or, using unit vector $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ for xyz-coord system
  - $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, where $a_1 = |a_x|$, $a_2 = |a_y|$, $a_3 = |a_z|$
- Length: $|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$, unit vector $\boldsymbol{u} = \boldsymbol{a}/|\boldsymbol{a}|$

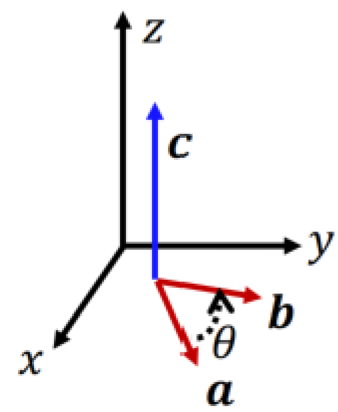
Inner product

- For two vectors $\boldsymbol{a} = \vec{OA}, \boldsymbol{b} = \vec{OB}, \boldsymbol{a} \cdot \boldsymbol{b} = c = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$ is called as inner product in scaler value $\theta = \angle AOB$
- Inner products has following characteristics
  - $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
  - $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
  - $\lambda\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \lambda\boldsymbol{b} = \lambda(\boldsymbol{a} \cdot \boldsymbol{b})$
- For unit vector $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$
  - $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$
  - $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = 0$

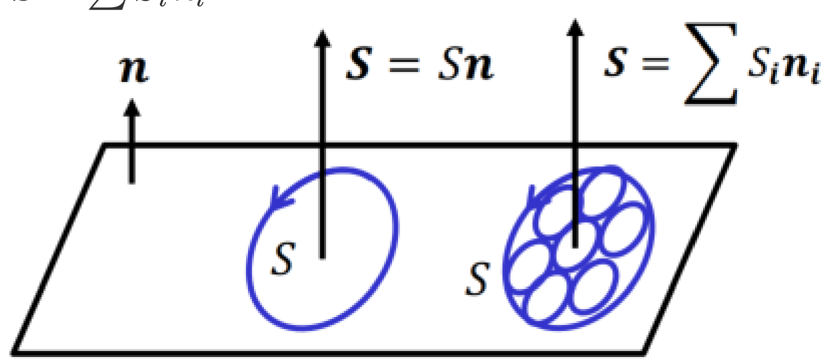
Outer product

- Assume right-hand side coordinate system
- For $\boldsymbol{a} = \vec{OA}, \boldsymbol{b} = \vec{OB}, \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$: outer product
  - $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$
  - Angle of $\boldsymbol{c}$: perpendicular to the surface of $|\boldsymbol{a}||\boldsymbol{b}|$
  - If $\boldsymbol{a}$ and $\boldsymbol{b}$ are in parallel ($\sin\theta = 0$), $\boldsymbol{a}$ or $\boldsymbol{b} = \boldsymbol{o}$, $c = \boldsymbol{o}$
- Theorem:
  - $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{o}$
  - $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
  - $\lambda\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \lambda\boldsymbol{b} = \lambda(\boldsymbol{a} \times \boldsymbol{b})$
  - $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = ***0***$

- $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{k}, \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{i}, \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{j}$

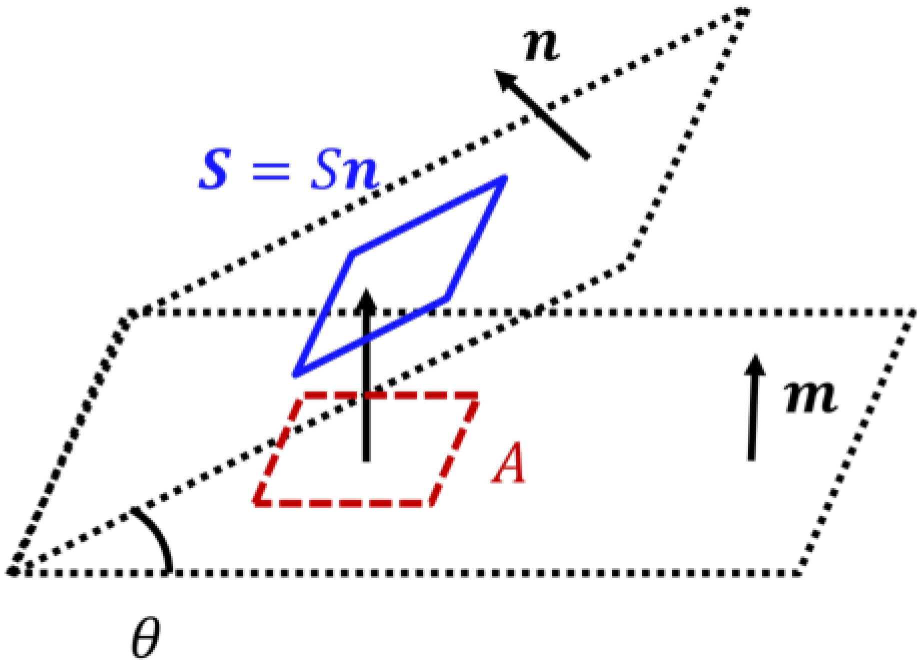


Vector area

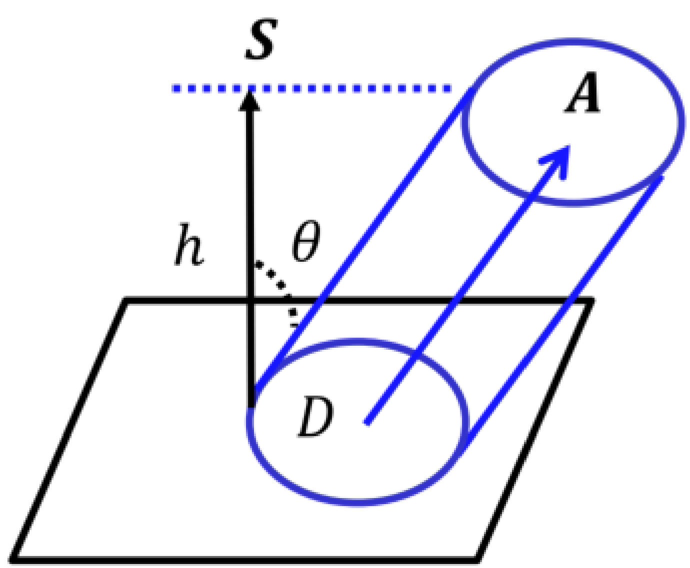
- Vector area: vector combining an area quality w/ dimension
- Assume surface S on signed area in two dimension system
  - Vector $\boldsymbol{S}$ can be expressed with its unit vector $\boldsymbol{n}$
    - $\boldsymbol{S}$ = S $\boldsymbol{n}$
  - Rotation of vector $\boldsymbol{n}$ express the sign
    - anticlockwise (right-hand screw) : plus
    - clockwise (left-hand screw): minus
  - If S is subset of S_i, the vector area $\boldsymbol{S}$ can be
    - $\boldsymbol{S} = \sum S_i \boldsymbol{n}_i$
    - 

Projection

- Area vector is used to calculate surface integral
  - Treat flus of a vector filed through a surface
  - Projection area A on plane $\boldsymbol{S}$ can be calculated by dot product with target plane unit normal $\boldsymbol{m}$
    - $A = \boldsymbol{S} \cdot \boldsymbol{m}$
  - If the two surface has same xy and angle θ for z-coordinate
  - $A = |\boldsymbol{S}| \cos \theta$
  -



Volume

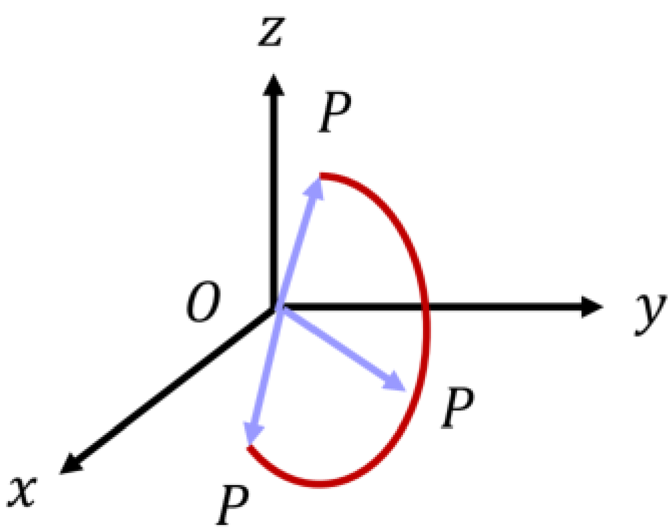


- Volume V can be calculated by area vector
  - Calculate volume V of tilted cylinder
    - Bottom plane: D
    - Area vector: $\boldsymbol{S}$
    - Direction: $\boldsymbol{A}$
    - Assume its angle: θ
  - Hight $h = |\boldsymbol{A}| \cos \theta$
  - Volume $V = h|\boldsymbol{S}| = |\boldsymbol{A}||\boldsymbol{S}| \cos \theta = |\boldsymbol{A}||\boldsymbol{S}|$
- Volume $V = |\boldsymbol{A}||\boldsymbol{S}|$ xpress the amount of flow $\boldsymbol{A}$ which punctulate the plane D

Vector analysis

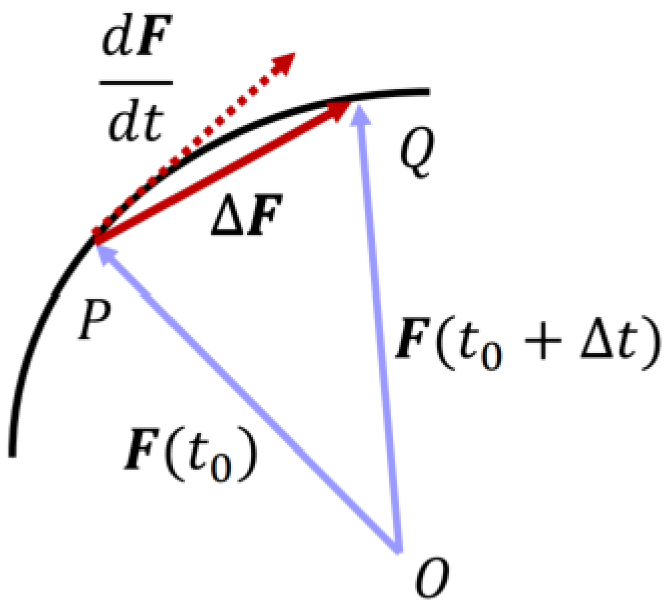
Derivation for vector function

- Vector function $\boldsymbol{F}$(t): vector $\boldsymbol{F}$ is a function of scalar t
  - if vector $\boldsymbol{F}$ is continuous to the t: $\boldsymbol{F}$ is continuous
- Assume vector $\boldsymbol{F}(t) = \vec{OP}$, where O is origin(fixed point)
  - Point P draw a curved line
  -



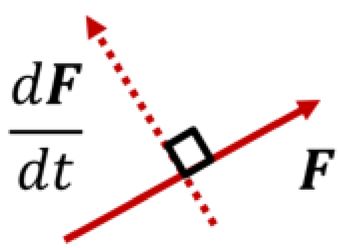
Characeristics

- A limit: if vector $\boldsymbol{A}$ satisfy $\lim_{n\to\infty}|\boldsymbol{A}_n - \boldsymbol{A}| = 0$ for $\boldsymbol{A}_0$ ... $\boldsymbol{A}_n$
  - $\lim_{n\to\infty}\boldsymbol{A}_n = \boldsymbol{A}$, and $\boldsymbol{A}$ is a limit of $\boldsymbol{A}_0$ ... $\boldsymbol{A}_n$
- A limit: if vector function $\boldsymbol{F}$(t) has const. vector $\boldsymbol{A}$, and it satisfy $\lim_{t\to t_0}|\boldsymbol{F}(t) - \boldsymbol{A}| = 0$ for $t \to t_0$
  - $\lim_{t\to t_0}\boldsymbol{F}(t) = \boldsymbol{A}$ and $\boldsymbol{A}$ is a limit of $\boldsymbol{F}$(t) for $t \to t_0$
  - For $\boldsymbol{F}(t) = F_1(t)\boldsymbol{i} + F_2(t)\boldsymbol{j} + F_3(t)\boldsymbol{k}$, $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$
    - $\lim_{t\to t_0}\boldsymbol{F}_1(t) = A_1, \lim_{t\to t_0}\boldsymbol{F}_2(t) = A_2, \lim_{t\to t_0}\boldsymbol{F}_3(t) = A_3$
- Continuity: if vector function $\boldsymbol{F}$(t) satisfy $\lim_{t\to t_0}\boldsymbol{F}(t) = \boldsymbol{F}(t_0)$ for $t \to t_0$, $\boldsymbol{F}$(t) is continuous.
- Derivative: if $\lim_{\Delta t\to 0}\frac{\Delta \boldsymbol{F}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta \boldsymbol{F}(t_0+\Delta t) - \boldsymbol{F}(t_0)}{\Delta t}$ is available, this is called as differential coefficient $\boldsymbol{F}'(t_0)$
  - For each t, the vector function $\boldsymbol{F}'(t_0)$ or $\frac{d\boldsymbol{F}}{dt}$ is called as derivative or derivative vector
  - Similary, derivative can be taken as $\boldsymbol{F}'(t_0)$ and $\boldsymbol{F}^{(n)}(t_0)$
- Geometric meaning
  - Assume $\vec{OP} = \boldsymbol{F}(t), \vec{OQ} = \boldsymbol{F}(t + \Delta)$
    -



- $\Delta\boldsymbol{F} = \boldsymbol{F}(t+\Delta t) - \boldsymbol{F}(t) = \vec{PQ}$
- Take Δt → 0 then $\Delta\boldsymbol{F}$ becomes tangent

Theorems for derivation

- Vector function $\boldsymbol{F}$(t) and $\boldsymbol{G}$(t), scalar function f(t), satisfy followings
  - (sum) : $\frac{d}{dt}(\boldsymbol{F}+\boldsymbol{G}) = \frac{d}{dt}\boldsymbol{F} + \frac{d}{dt}\boldsymbol{G}$
  - (scalar prod): $\frac{d}{dt}(f\boldsymbol{F}) = \frac{df}{dt}\boldsymbol{F} + f\frac{d}{dt}\boldsymbol{F}$
  - (inner prod): $\frac{d}{dt}(\boldsymbol{F}\cdot\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt}\cdot\boldsymbol{G} + \frac{d\boldsymbol{G}}{dt}\cdot\boldsymbol{F}$
  - (outer prod): $\frac{d}{dt}(\boldsymbol{F}\times\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt}\times\boldsymbol{G} + \frac{d\boldsymbol{G}}{dt}\times\boldsymbol{F}$
  - For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$, $\frac{d\boldsymbol{F}}{dt} = \frac{dF_1}{dt}\boldsymbol{i} + \frac{dF_2}{dt}\boldsymbol{j} + \frac{dF_3}{dt}\boldsymbol{k}$
  - If $\boldsymbol{F}$ is constant, $\frac{d\boldsymbol{F}}{dt}$ is $\boldsymbol{o}$, or perpendicular s.t. $\boldsymbol{F}\cdot\frac{d\boldsymbol{F}}{dt} = 0$
  - 

High order derivatives, partial difference

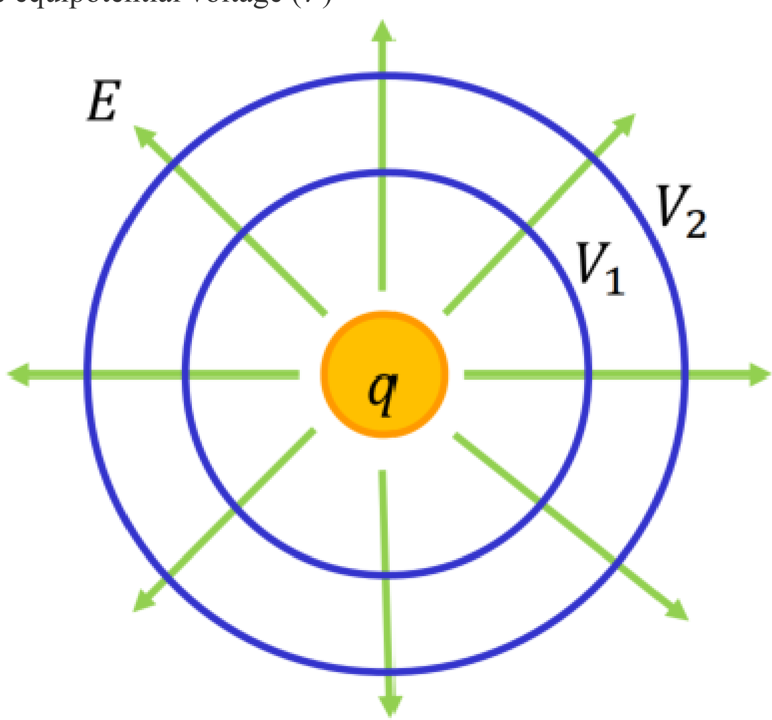
- High order derivatives can defined as similar to 1st order
  - $\frac{d^2\boldsymbol{F}}{dt^2}, \frac{d^3\boldsymbol{F}}{dt^3}, \ldots \frac{d^n\boldsymbol{F}}{dt^n}$
  - For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$, $\frac{d^n\boldsymbol{F}}{dt^n} = \frac{d^nF_1}{dt^n}\boldsymbol{i} + \frac{d^nF_2}{dt^n}\boldsymbol{j} + \frac{d^nF_3}{dt^n}\boldsymbol{k}$
- Partial difference also defined like derivation
  - $\boldsymbol{A} = \boldsymbol{A}(u,v)$, $\frac{\delta\boldsymbol{A}}{\delta u}, \frac{\delta\boldsymbol{A}}{\delta v}, \frac{\delta^2\boldsymbol{A}}{\delta v^2}, \frac{\delta^2\boldsymbol{A}}{\delta v\delta u}, \frac{\delta^2\boldsymbol{A}}{\delta u\delta v}, \frac{\delta^2\boldsymbol{A}}{\delta u^2}$
- Total difference of $\boldsymbol{A}(u,v)$ can be defined as

- $\delta \boldsymbol{A}(u, v) = \frac{\delta \boldsymbol{A}}{\delta v} du + \frac{\delta \boldsymbol{A}}{\delta u} dv$
- It approx. small delta of $\delta \boldsymbol{A}$ by small delta of du, dv
  - For $\boldsymbol{A} = A_1 \boldsymbol{i} + A_2 \boldsymbol{j} + A_3 \boldsymbol{k}$, $\delta \boldsymbol{A} = \delta A_1 \boldsymbol{i} + \delta A_2 \boldsymbol{j} + \delta A_3 \boldsymbol{k}$

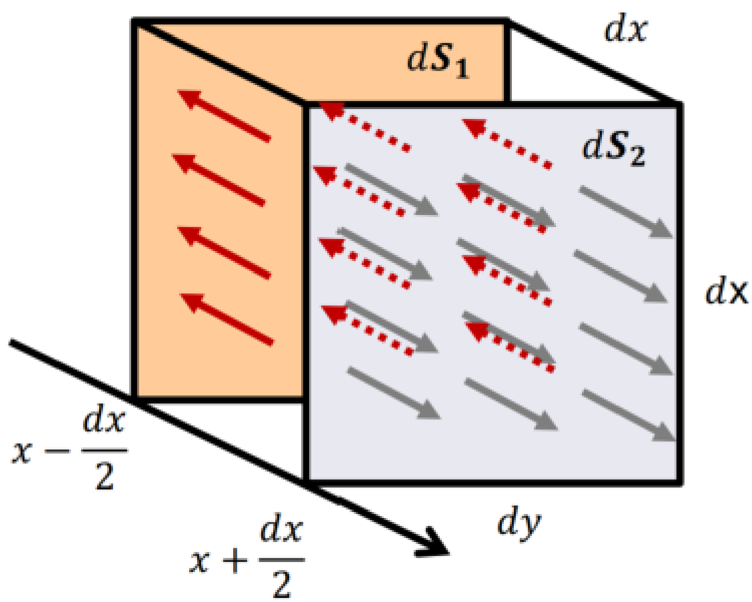
Gradient of scalar

- Scalar function: $f(x, y, z)$ can be defined in unique
  - This field is called scalar field f
    - Distribution of temperature, mass, voltage
- Vector function: $\boldsymbol{F}(x, y, z)$ can be defined in unique
  - This field is called vector field $\boldsymbol{F}$
    - Electric field, magnetic field, gravity field
- Gradient of scalar: grad $f = \nabla f = \frac{\partial f}{\partial x} \boldsymbol{i} + \frac{\partial f}{\partial y} \boldsymbol{j} + \frac{\partial f}{\partial z} \boldsymbol{k}$
  - ∇ Hamilton operator
    - $\nabla(f + g) = \nabla f + \nabla g$, $\nabla \lambda f = \lambda \nabla f$, $\nabla(fg) = g \nabla f + f \nabla g$
    - $\nabla \phi(f) = \frac{d\phi}{df} \nabla f$, where φ(f) is a funtion of f

Equipotential surface

- If group of points $P(x, y, z)$ satisfy $f(x, y, z) = c$ (c: const), P is called equipotential surface
  - In the case of $f(x, y, z) = x^2 + y^2 + z^2$
    - Surface of sphere
- In electro-magnetics, electron (q) create divergence of electric lines (electric field: E), and electric line create equipotential voltage (V)
  - 

Divergence of vector

- For vector $\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}$
  - div $\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \boldsymbol{F}$ is called as divergence
- Vector $\boldsymbol{F}$, $\boldsymbol{G}$ scalar f satisfy following conditions
  - div($\boldsymbol{F}$ + $\boldsymbol{G}$) = div($\boldsymbol{F}$) + div($\boldsymbol{G}$)
  - div(f$\boldsymbol{G}$) = grad(f) · $\boldsymbol{G}$ + f div$\boldsymbol{G}$
  - $\text{div grad}(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
- Physical meaning
  - div$\boldsymbol{F} > 0$: something spout (flow out)
  - div$\boldsymbol{F} < 0$: something swallowed (flow in)
- div $\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \boldsymbol{F}$
- Assume flow $\boldsymbol{F}$ of small box $dxdydz$
  - Assume flow $\boldsymbol{F}$ of area $d\boldsymbol{S}_1 = (-dydz, 0{,}0)$ at $x - \frac{dx}{2}$
  - Assume flow $\boldsymbol{F}$ of area $d\boldsymbol{S}_2 = (+dydz, 0{,}0)$ at $x + \frac{dx}{2}$
    - $\boldsymbol{F} \cdot d\boldsymbol{S} = \boldsymbol{F} \cdot d\boldsymbol{S}_2 + \boldsymbol{F} \cdot d\boldsymbol{S}_1$
    - = $F_1(x + \frac{dx}{2}, y, z)(dydz, 0, 0) + F_1(x - \frac{dx}{2}, y, z)(-dydz, 0, 0)$
    - = $\frac{\partial F_1}{\partial x} dxdydz$
    - 

    - Diff. flow in (the red arrow) and out (the gray arrow)

Rotation of vector

- For vector $\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}$, $\text{rot}\boldsymbol{F} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\boldsymbol{i} + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\boldsymbol{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\boldsymbol{k} = \nabla \times \boldsymbol{F}$ is called as rotation

- rot $\boldsymbol{F}$ = (rot$_1$ $\boldsymbol{F}$)$\boldsymbol{i}$ + (rot$_2$ $\boldsymbol{F}$)$\boldsymbol{j}$ + (rot$_3$ $\boldsymbol{F}$)$\boldsymbol{k}$
- Vector $\boldsymbol{F}$, $\boldsymbol{G}$, scalar f satisfy following conditions
  - rot($\boldsymbol{F}$ + $\boldsymbol{G}$) = rot($\boldsymbol{F}$) + rot($\boldsymbol{G}$)
  - rot(f$\boldsymbol{G}$) = grad(f)×G + f∇×G
- Physical meaning
  - rot $\boldsymbol{F}$ > 0: right-hand side (screw) rotation
  - rot $\boldsymbol{F}$ < 0: left-hand side (screw) rotation

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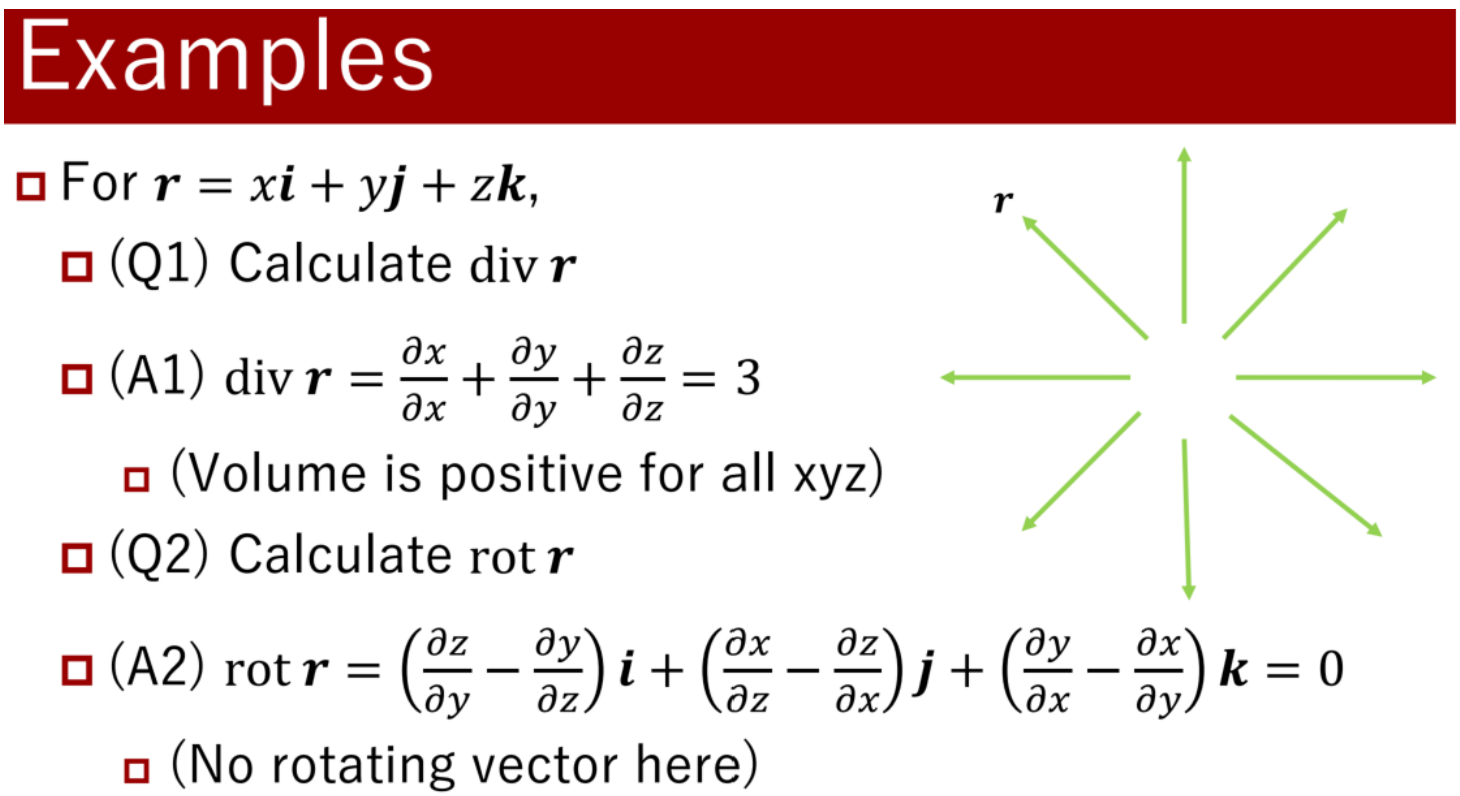
Physical meaning of rotation

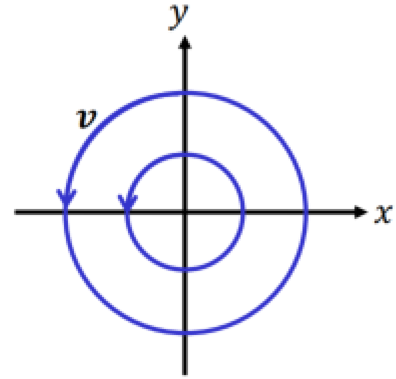
- Focus 3rd term ($\boldsymbol{k}$) of rotation
  - If $\frac{\partial F_2}{\partial x} > 0$, it generates right-hand side rotation
  - If $-\frac{\partial F_1}{\partial y} > 0$, it generates right-hand side rotation
  - If $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} > 0$ means right-hand side rotation is available

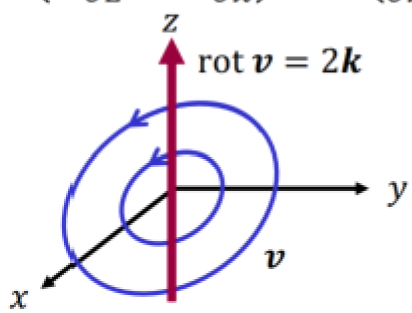
Examples



Examples

- For $\boldsymbol{v} = -y\boldsymbol{i} + x\boldsymbol{j}$,
  - (Q1) Calculate div $\boldsymbol{v}$
  - (A1) div $\boldsymbol{v} = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0$
  - This equation is $x^2 + y^2 = c$ (c: const.)
    - No flow in/out, rotation
  - (Q2) Calculate rot $\boldsymbol{v}$
  - (A2) rot $\boldsymbol{v} = \left(\frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial(-y)}{\partial z} - \frac{\partial 0}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\boldsymbol{k} = 2\boldsymbol{k}$





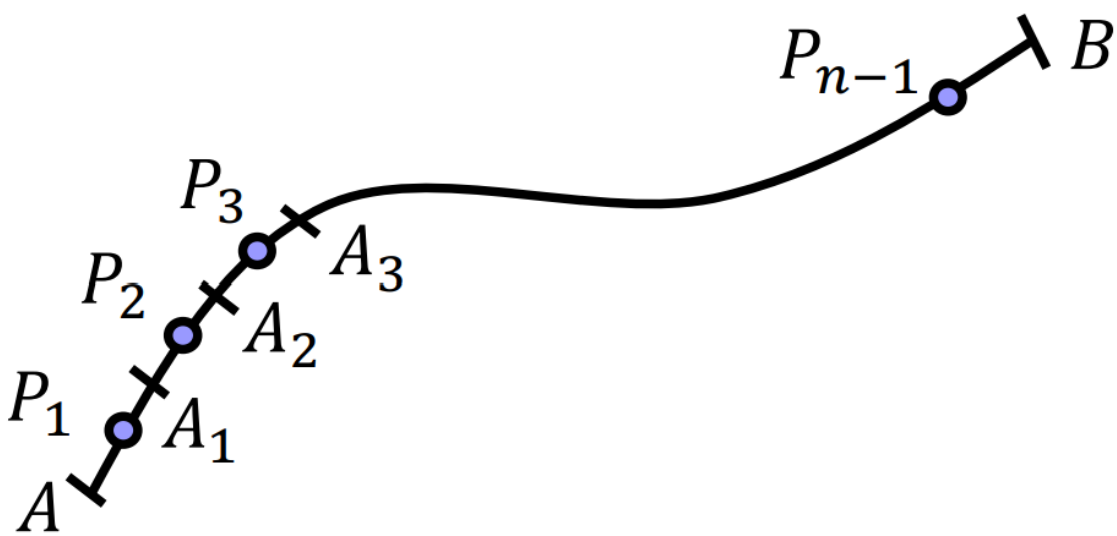
Exercise

- Assume **a**, **b** is constant vector, |**r**(t)| = r(t), calculate its derivation
  - $r\mathbf{r} + (\mathbf{a}\cdot\mathbf{r})\mathbf{b}$
    - $(r\mathbf{r} + (\mathbf{a}\cdot\mathbf{r})b)^{'}$
    - $= r^{'}\mathbf{r} + r\mathbf{r}^{'} + (\mathbf{a}\cdot\mathbf{r}^{'})\mathbf{b}$
  - $\frac{\mathbf{r}}{r^2}$
    - $= \frac{\mathbf{r}^{'}}{r^2} - \frac{\mathbf{r}}{r^3}$
- Calculate gradient for following functions
  - $f = xz^3 - x^2y$, calculate ∇f at point P(1,-2,2)
    - $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
    - $= (z^3 - 2xy)\mathbf{i} + (-x^2)\mathbf{j} + (3xz^2)\mathbf{k}$
    - $= (8+4)\mathbf{i} + (-1)\mathbf{j} + 12\mathbf{k} = 12\mathbf{i} - 1\mathbf{j} + 12\mathbf{k}$
  - $f = x^2y^2 - 2xz^3$, calculate ∇f at point P(1,-2,1)
    - $\nabla f = (2xy^2 - 2z^3)\mathbf{i} + (2x^2y)\mathbf{j} + (-6xz^2)\mathbf{k}$
    - $6\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}$
- Calculate divergence of following functions

- $x^2y\mathbf{i} - 2y^2z^2\mathbf{j} + 3z^3x^3\mathbf{k}$
    - $div\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
    - $= 2xy - 4yz^2 + 9z^2x^3$
- Calculate rotation of following functions
  - $x^2\mathbf{i} - 2xz\mathbf{j} + y^2z\mathbf{k}$
  - rot(f) = $(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z})\mathbf{i} + (\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x})\mathbf{j} + (\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})\mathbf{k}$
  - $= (2yz - (-2x))\mathbf{i} + (0 - 0)\mathbf{j} + ((-2z) - 0)\mathbf{k}$
  - $= 2(yz + x)\mathbf{i} - 2z\mathbf{k}$

Integral of Vector

Curvilinear Vector

- Assume a smooth curve C from point A to B, and scalar function f(P) = f(x, y, z) is continuous in curve C
  - Think curve C can divide into several arcs Δs_1 ... Δs_2
    - Points A_n divide a curve, these weight are points P_n
    - Assume limit of $n \to \infty$, $\Delta s_i \to 0$; **curvilinear inntegral**
    - $\lim_{n\to\infty, \Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i = \int_C f(P)ds = \int_C f(x, y, z)ds$
  - Point D on curve C is function of the length (s) of arc arc(AD)
  - 

- Point D on curve C is function of the length(s) of arc $\hat{AD}$
  - (Any) point D can be expressed as function of length s
    - $\mathbf{r} = \mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$
    - $\int_C f(x, y, z)ds = \int_A^B f(x(s), y(s), z(s))ds$
- If we use general parameter t to express the curve C;
  - $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
  - $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt$
  - $\int_C f(x, y, z)ds = \int_\alpha^\beta f(x(s), y(s), z(s))\sqrt{(\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2)}dt$

- where A, B of curve C are point α = t β = t

Experssions of curvilinear integral

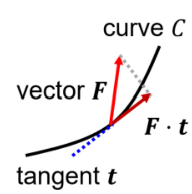
- Several expressions are available for curvilinear integral
  - $\int_C fds = \int_A^B fds = \int_{AB} fds$
  - $\int_{AB} fds = -\int_{BA} fds$
- If point P is on the curve C, $\int_{AB} fds = \int_{AP} fds + \int_{PB} fds$
- If the curve C is a closed curve, $\oint_C fds = \oint_{AB} fds$

Example of curvilinear integral

- Calculate curvilinear integral of f(x, y, z) = $y^2z + z^2x + x^2y$
  - Route 1: O(0, 0, 0) → Q(3, 0, 0) → R(3, 1, 0) → P(3, 1, 2)
    - $\int_{R1} fds = \int_O^Q fds + \int_Q^R fds + \int_R^P fds$
    - $= \int_0^3 f(x,0,0)dx + \int_0^1 f(3,y,0)dy + \int_0^2 f(3,1,z)dz = \frac{65}{2}$
  - Route 2: $\vec{OP}$
    - $\vec{OP} = \mathbf{r} = 3t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}(0 \leq t \leq 1)$
    - $ds = \sqrt{(sdt)^2 + (1dt)^2 + (2dt)^2} = \sqrt{14}dt$
    - $\int_{R2} fds = \int_0^{\sqrt{14}} (y^2z + z^2x + x^2y)ds = \int_0^1 (2t^3 + 12t^3 + 9t^3)\sqrt{3^2 + 1^2 + 2^2}dt = \frac{23\sqrt{14}}{4}$

Curvilinear integral for vector

- Assume a smooth curve C from point A to B, and vector function **F**(P) = **F**(x, y, z) is continuous in curve C
  - **r**(s) is a position vector from origin O to the point P on C
  - Assume $\mathbf{t} = \frac{d\mathbf{r}}{ds}$ is a tangent of curve C at point P
    - Curvilinear integral for the vector **F**: $\int_C \mathbf{F} \cdot \mathbf{t}ds$
    - Assume function of C: **r**(s) = x(s)**i** + y(s)**j** + z(s)**k**, **F** = F_1**i** + F_2**j** + F_3**k**
      - $\int_C \mathbf{F} \cdot \mathbf{t}ds = \int_C (\frac{F_1dx}{ds} + \frac{F_1dy}{ds} + \frac{F_1dz}{ds})$
    - Scalar $\mathbf{F} \cdot \mathbf{t}$ is a tangent component of the vector **F**



Characteristics of curvilinear integral for vector

- Curvilinear integral for vector has following characteristics
  - For scalar field $f(x, y, z)$ and vector field $\mathbf{F}(x, y, z)$
    - $\int_C f(x,y,z)d\mathbf{r} = \mathbf{i}\int_C fdx + \mathbf{j}\int_C fdy + \mathbf{k}\int_C fdz$
    - $\int_C \mathbf{F}(x,y,z)d\mathbf{r} = \mathbf{i}\int_C \mathbf{F}_1dx + \mathbf{j}\int_C \mathbf{F}_2dy + \mathbf{k}\int_C \mathbf{F}_3dz$

- $\int_C \mathbf{F} \times d\mathbf{r} = \int_C \mathbf{F} \times \mathbf{r} ds = \mathbf{i} \int_C (F_2 dz - F_3 dy) + \mathbf{j} \int_C (F_3 dz - F_1 dy) + \mathbf{k} \int_C (F_1 dz - F_2 dy)$

Exercise

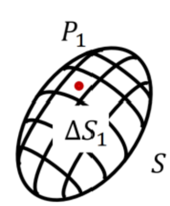
- Calculate curvilinear integral $\int_C y d\mathbf{r}$
  - C: x = a cos(t), y = a sin(t), z = ht, (0 ≤ t ≤ 2π)
- Solution
  - $\int_C y d\mathbf{r} = \int_C a \sin t (\mathbf{i} dx + \mathbf{j} dy + \mathbf{k} dz)$
    $-\mathbf{i} \int_0^{2\pi} a^2 \sin^2 t dt + \mathbf{j} \int_0^{2\pi} a^2 \sin t \cos t dt + \mathbf{k} \int_0^{2\pi} ah \sin t dt$
    $= -\pi a^2 \mathbf{i}$

Potential

- If scalar function φ(x, y, z) is aviable for **F**(x, y, z) = gradφ ; φ is called as potential or scalar potential of **F**
- Potential has following characteristics:
  - Assume vector field **F**(x, y, z) has potential φ
    - $\int_A^B \mathbf{F} \cdot d\mathbf{r} = -\int_A^B \nabla\varphi \cdot d\mathbf{r} = \varphi(A) - \varphi(B)$
  - If curve C is a closed curve
    - $\oint_C F \cdot d\mathbf{r} = -\oint_C \nabla\varphi \cdot d\mathbf{r} = 0$

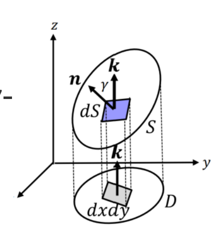
Surface integral for scalar

- Assume smooth curved surface S
  - Scalar function $f(P) = f(x, y, z)$ is continuous in S
    - Assume S can be divided into small area $\Delta S_1 \ldots \Delta S_n$, and any point of $P_1 \ldots P_n$
    - If $\lim\limits_{n\to\infty, \Delta S_i \to 0} \sum_{i=1}^n f(P_i)\Delta S_i$ is available, this is called surface integral for scalar $\int_S f(x, y, z) dS$
  - If f(P) = 1, $\int_S f(x, y, z) dS$ is area of S.
- for the curved surface, outside is the front.



Formula of surface integral

- If surface S is given for z = g(x, y), surface integral of f(x, y, z) on S can be expressed as follows,
  - $\int_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{p^2 + q^2 + 1} dxdy$
  - $= \iint_D f(x, y, g(x, y)) \frac{dxdy}{|\mathbf{n} \cdot \mathbf{k}|}$
    - where, $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$, $\mathbf{n}$ is unit normal vector of S, D is projective of S to xy-coordinate



Proof

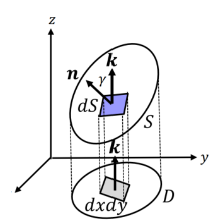
- Think small surface dS on S, its projective in xy-coordinate can express $dydx$
- Define angle of unit normal vectors $\boldsymbol{n}$, $\boldsymbol{k}$ as γ
  - dS|cos λ| = dxdy
- $\mathbf{n} = \frac{\pm 1}{\sqrt{p^2+q^2+1}}$ when $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$
- Thus, |cos γ| = |**n**·**k**| = $\frac{1}{\sqrt{p^2+q^2+1}}$
- $dS = \frac{dxdy}{|\cos\gamma|} = \frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$
- $\int_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$ (z = g(x, y))

Surface integral for vector

- For vector field $\boldsymbol{F}$ and unit vector $\boldsymbol{n}$ of surface $\boldsymbol{S}$integral of these inner products is called as surface integral of vector
  - $\int_S \mathbf{F}\cdot\mathbf{n}dS$
  - F_n is a **n** component of vector **F**($\mathbf{F}\cdot\mathbf{n} = F_n$)
  - Assume $\mathbf{n}dS = d\mathbf{S}$, $d\mathbf{S}$ is called area vector
    - $\int_S \mathbf{F}\cdot\mathbf{n}dS = \int_S F_n dS = \int_S \mathbf{F}\cdot d\mathbf{S} = \oint \mathbf{F}\cdot\mathbf{n}dS$ (If S is closed surface)
    - For $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$
      - $\int \mathbf{F}\cdot\mathbf{n}dS = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$
  - Several expressions for surface integral of vectors
    - $\int \mathbf{F}dS = \mathbf{i}\int_S F_1 dS + \mathbf{j}\int_S F_2 dS + \mathbf{k}\int_S F_3 dS$
    - $\int_S \mathbf{F}\times\mathbf{b}dS = \int_S \mathbf{F}\times d\mathbf{S}$

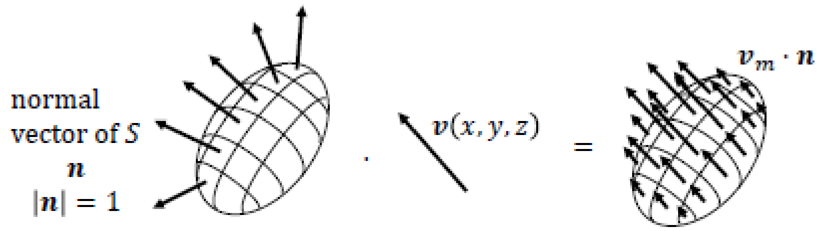
Formula of surface integral

- If surface S is given for $z = g(x, y)$, surface integral of $\boldsymbol{F}(x, y, z)$ on S can be expressed as follows,
  - $\int_S \mathbf{F}(x,y,z)dS = \iint_D \mathbf{F}(x,y,g(x,y))\sqrt{p^2+q^2+1}dxdy$
    - where, $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$, D is projective of S to xy-coordinate



Surface integral in physics

- In scalar: $\int_S \rho(x,y,z)dS$
  - In the case ρ is a function of mass density on surface S
    - Its integral: total mass of surface S
- In vector: $\int_S \mathbf{v}(x,y,z)\cdot dS$
  - In the case **v** is a function of liquid velocity on surface S
    - Its integral: total amount of liquid flow per unit time
    -

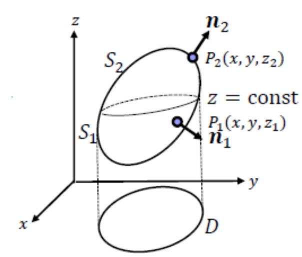


Volume integral

- Connect divergence on vector field and flow at the surface
  - Assume volume V surrounded by surface S
    - Volume integral of scalar f: $\int_V f(x,y,z)dV$
    - Volume integral of vector $\mathbf{F}$: $\int_V \mathbf{F}(x,y,z)dV$
      - $\int_V \mathbf{F}(x,y,z)dV = \mathbf{i}\int_V F_1 dV + \mathbf{j}\int_V F_2 dV + \mathbf{k}\int_V F_3 dV$
- Preliminary
  - For volume V surrounded by surface S, **n** = &cos; α**i** + &cos;**j** + &cos;**k**, folowing equation satisfies,
  - $\int_V \frac{\partial f}{\partial x} dV = \int_S f\cos\alpha dS$
    $\int_V \frac{\partial f}{\partial y} dV = \int_S f\cos\beta dS$
    $\int_V \frac{\partial f}{\partial z} dV = \int_S f\cos\gamma dS$

Proof

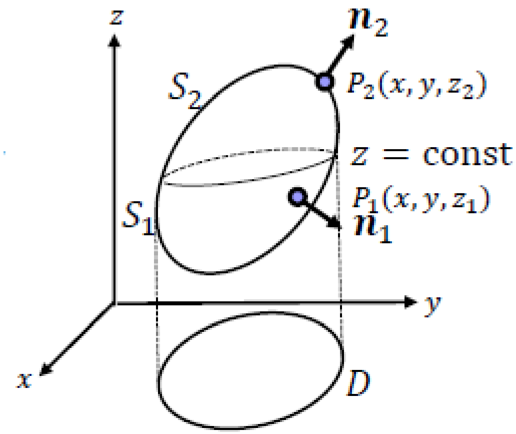
- Proof $\int_V \frac{\partial f}{\partial z} dV = \int_S f\cos\gamma dS$
- Assume two points P_1, P_2 on S
  - $z_2 \geq z_1$: z_2 coves upper side of S, z_1 coves lower side if S
  - $\int_V \frac{\partial f}{\partial z} dV$ means volume difference in z-axis, thus



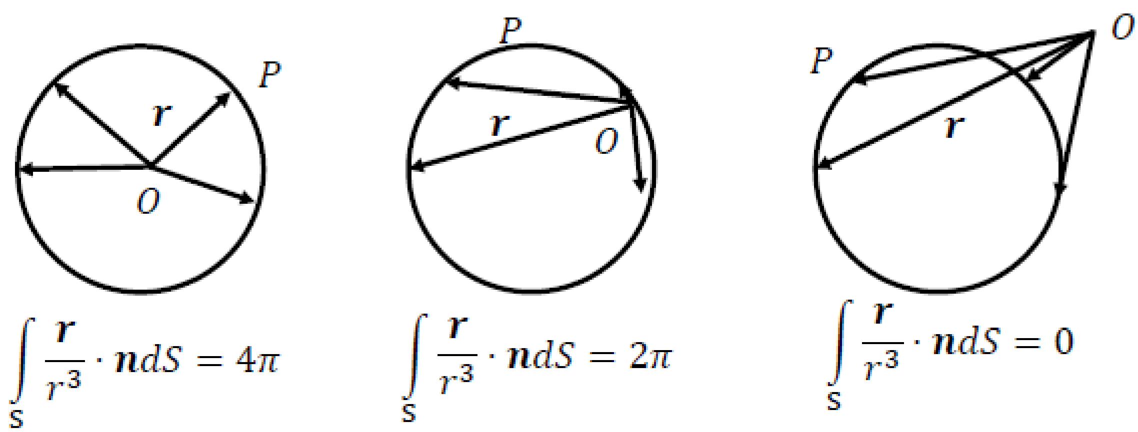
- $\int_V \frac{\partial f}{\partial z} dV = \iiint_V \frac{\partial f}{\partial z} dxdydz = \iint_D \int_{z_1}^{z_2} \frac{\partial f}{\partial z} dzdxdy$
  $= \iint_D [f]_{z_1}^{z_2} dxdy = \iint_D f(x,y,z_2) - f(x,y,z_1)dxdy$
- For z-axis, z_2 is upper ($dS\cos\gamma = dxdy$), z_1 is lower thus ($dS\cos\gamma = -dxdy$)
  - $\iint_D f(x,y,z_2)dxdy = \int_{S_2} f(x,y,z)dS$
  - $\iint_D f(x,y,z_1)dxdy = -\int_{S_1} f(x,y,z)dS$
  - $\int_V \frac{\partial f}{\partial z} dV = \int_{S_2} f\cos\gamma dS + \int_{S_1} f\cos\gamma dS$
    $= \int_S f\cos\gamma dS$

Divergence theorem(Dauss' theorem)

- Connect divergence on vector field and flow at the surface
  - Assume volume V surrounded bt surface S w/unit vector
    - $\int_V div\mathbf{F}dV = \int_V \nabla\cdot\mathbf{F}dV = \int_S \mathbf{F}\cdot\mathbf{n}dS$

- Physical meaning
  - $\int_S \mathbf{F} \cdot \mathbf{n} : amountofflowwhichpaththroughtheareaS$
  - $\int_V div\mathbf{F}dV : amountofflowout$
  - 

Extension of Gauss' theorem

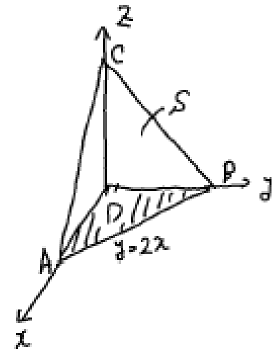
- Assume the point P on close surface S , express vector from origin $O(0, 0, 0)$ to P as $\vec{P} = \mathbf{r}, \mathbf{n}$ is unit normal vector of S
- Following equation satisfy the following
  - $\int_S \frac{\mathbf{r}}{r^3} \cdot \mathbf{n}dS = \begin{cases} 0(\text{when O is outside of S}) \\ 2\pi(\text{when O is on the surface S}) \\ 4\pi(\text{when O is inside of S}) \end{cases}$
  - 

Exercise

- For function $f(x, y, z) = x^2 - yz + z^2$, calculate its curvilinear integral $\int_C fds$
  - Case 1: C is a line from $P_1(1,2,0)$ to $P_2(1,2,3)$
    - $\int_C fds = \int_0^3 (1^2 - 2z + z^2)dz = [z - z^2 + \frac{z^3}{3}]_0^3 = 3$
  - Case 2: C is a line from $P_1(0,0,0)$ to $P_2(1,2,3)$
    - x = t, y = 2t, z = 3t ($0 \le t \le 1$)

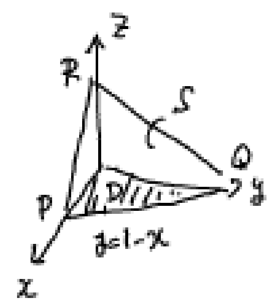
- $ds = \sqrt{1^2+2^2+3^2}dt = \sqrt{14}dt$
- $\int_C f(x(s)+y(s)+z(s))ds = \int_0^1 (t^2-6t^2+9t^2)\sqrt{14}dt$
  $= 4\sqrt{14}\int_0^1 t^2 dt = \frac{4\sqrt{14}}{3}$

- Assume the surface function 2x + 2y + z - 4 = 0, and its intercepts are points A, B, C and ABC create surface S
  - Calculate surface integral of $f(x,y,z) = 4x - y^2 + 2x - 12$



    - S: 2x + 2y + z - 4 = 0, f = 4x -y^2 + 2z -12
    - Surface dunction:$z = g(x,y) = 4-2x-2y$
      $p = \frac{\partial z}{\partial x} = -2, q = \frac{\partial z}{\partial y} = -2$
      $dS = \sqrt{p^2+q^2+1}dxdy = 3dxdy$
    - $f(x,y,g(x,y)) = -(y+2)^2$
    - $\int_S fdS = -\iint_D (y+2)^2 \cdot dxdy$
      $= -\int_0^2\int_0^{2-x} 3(y+2)^2 dydx$
      $= -\int_0^2 [y+2]^3{}_0^{2-x} dx$
      $= -\int_0^2((4-x)^3-8)dx = [\frac{(4-x)^3}{4}+8x]_0^2 = -44$

- Assume the surface function: x + y + z - 1 = 0, and its intercepts are points P, Q, R and PQR craete surface S
  - Calculate surface integral $\int_S \mathbf{F}\times\mathbf{n}dS$ for **F** = y**k**
    - Surface function $S = g(x,y) = 1-x-y$
      $p = \frac{\partial z}{\partial x} = -1, q = \frac{\partial z}{\partial y} = -1$
    - unit normal vector of S is : **n** = $\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$
    - **F** × **n** = $\mathbf{i}\int_C(F_2dz - F_3dy) + \mathbf{j}\int_C(F_3dz - F_1dy) + \mathbf{k}\int_C(F_1dz - F_2dy)$
      $= \mathbf{i}(0\cdot\frac{1}{\sqrt3} - \frac{1}{\sqrt3}\cdot y) + \mathbf{j}(y\cdot\frac{1}{\sqrt3} - \frac{1}{\sqrt3}\cdot 0) + \mathbf{k}(0\cdot\frac{1}{\sqrt3} - \frac{1}{\sqrt3}\cdot 0)$
      $-\frac{y}{\sqrt3}(\mathbf{i}-\mathbf{y})$
    - $dS = \sqrt{p^2+q^2+1}dxdy = \sqrt{3}dxdy$, thus
    - $\int_S \mathbf{F}\times\mathbf{n}dS = -\iint_D \frac{y}{\sqrt3}(\mathbf{i}-\mathbf{y})\sqrt{3}dxdy$
      $= -(\mathbf{i}-\mathbf{j})\int_0^1\int^{(} 1-x)_0 ydxdy$
      $-(\mathbf{i}-\mathbf{j})\int_0^1 \frac{1-x^2}{2}dx$
      $-(\mathbf{i}-\mathbf{j})[\frac{1-x^3}{6}]_0^1 = \frac{\mathbf{i}-\mathbf{j}}{6}$



- Assume the volume and surface of unit sphere as V , S , and **F** = ax**i** + by**j** + cz**k**.
  Calculate integral $\int_S \mathbf{F}\cdot\mathbf{n}dS$
  - from divergence law
  - $\mathbf{F}\cdot\mathbf{n}dS = \int_V \nabla\cdot\mathbf{F}dV$
    $\int_V(\frac{\partial F_1}{x} + \frac{\partial F_2}{y} + \frac{\partial F_3}{z})dV$
    $\int_V(a+b+c)dV$
    $= (a+b+c)\int_V dV = \frac{4\pi}{3}(a+b+c)$ `Volume of unit sphere r= 1` , V = $4\pi r^2/3$