Fundamental Mathematics (Engineering Mathematics)

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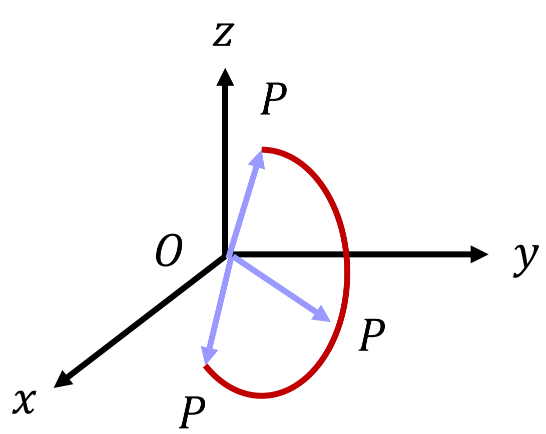
Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

Derivation for vector func.

- Vector function $\boldsymbol{F}(t)$: vector $\boldsymbol{F}$ is a function of scalar t
  - If vector $\boldsymbol{F}$ is continuous to the t: $\boldsymbol{F}$ is continuous
- Assume vector $\boldsymbol{F}(t) = \overrightarrow{OP}$, where O is origin (fixed point)
  - Point P draw a curved line

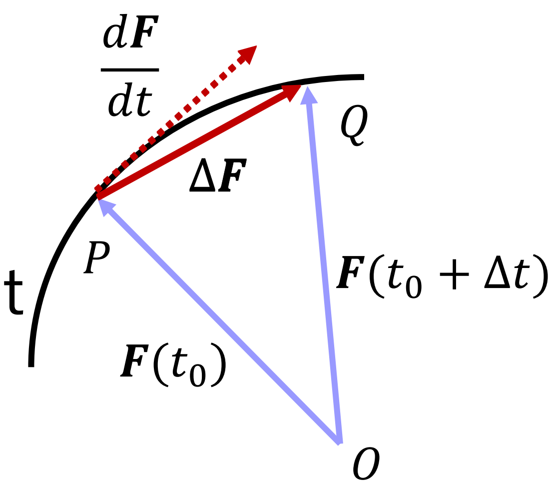


Characteristics

- A limit: if vector $\boldsymbol{A}$ satisfy $\lim_{n\to\infty}|\boldsymbol{A}_n - \boldsymbol{A}| = 0$ for $\boldsymbol{A}_0 \cdots \boldsymbol{A}_n$
  - $\lim_{n\to\infty} \boldsymbol{A}_n = \boldsymbol{A}$, and $\boldsymbol{A}$ is a limit of $\boldsymbol{A}_0 \cdots \boldsymbol{A}_n$
- A limit: if vector func. $\boldsymbol{F}(t)$ has const. vector $\boldsymbol{A}$, and it satisfy $\lim_{t\to t_0}|\boldsymbol{F}(t) - \boldsymbol{A}| = 0$ for $t \to t_0$
  - $\lim_{t\to t_0} \boldsymbol{F}(t) = \boldsymbol{A}$, and $\boldsymbol{A}$ is a limit of $\boldsymbol{F}(t)$ for $t \to t_0$
  - For $\boldsymbol{F}(t) = F_1(t)\boldsymbol{i} + F_2(t)\boldsymbol{j} + F_3(t)\boldsymbol{k}$, $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$
    - $\lim_{t\to t_0} F_1(t) = A_1$, $\lim_{t\to t_0} F_2(t) = A_2$, $\lim_{t\to t_0} F_3(t) = A_3$
- Continuity: if vector func. $\boldsymbol{F}(t)$ satisfy $\lim_{t\to t_0} \boldsymbol{F}(t) = \boldsymbol{F}(t_0)$ for $t \to t_0$, $\boldsymbol{F}(t)$ is continuous

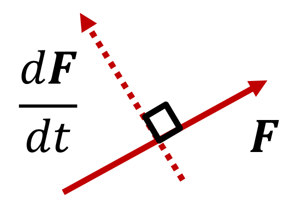
Characteristics

- Derivative(導関数): if $\lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{F}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{F}(t_0+\Delta t)-\boldsymbol{F}(t_0)}{\Delta t}$ is available, this is called as differential coefficient $\boldsymbol{F}'(t_0)$
  - For each t, the vector function $\boldsymbol{F}'(t_0)$ or $\frac{d\boldsymbol{F}}{dt}$ is called as derivative or derivative vector
  - Similarly, derivative can be taken as $\boldsymbol{F}'(t_0)$ and $\boldsymbol{F}^{(n)}(t_0)$
- Geometric meaning
  - Assume $\overrightarrow{OP} = \boldsymbol{F}\,(t)$, $\overrightarrow{OQ} = \boldsymbol{F}\,(t + \Delta t)$,
    - $\Delta \boldsymbol{F} = \boldsymbol{F}\,(t + \Delta t) - \boldsymbol{F}\,(t) = \overrightarrow{PQ}$
    - Take $\Delta t \to 0$ then $\Delta \boldsymbol{F}$ becomes tangent



Theorems for derivation

- Vector func. $\boldsymbol{F}(t)$ and $\boldsymbol{G}(t)$, scalar func $f(t)$, satisfy followings
  - (sum) : $\frac{d}{dt}(\boldsymbol{F}+\boldsymbol{G}) = \frac{d}{dt}\boldsymbol{F} + \frac{d}{dt}\boldsymbol{G}$
  - (scalar prod.) : $\frac{d}{dt}(f\boldsymbol{F}) = \frac{df}{dt}\boldsymbol{F} + f\frac{d}{dt}\boldsymbol{F}$
  - (inner prod.) : $\frac{d}{dt}(\boldsymbol{F}\cdot\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt}\cdot\boldsymbol{G} + \boldsymbol{F}\cdot\frac{d\boldsymbol{G}}{dt}$
  - (outer prod.) : $\frac{d}{dt}(\boldsymbol{F}\times\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt}\times\boldsymbol{G} + \boldsymbol{F}\times\frac{d\boldsymbol{G}}{dt}$
  - For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$, $\frac{d\boldsymbol{F}}{dt} = \frac{dF_1}{dt}\boldsymbol{i} + \frac{dF_2}{dt}\boldsymbol{j} + \frac{dF_3}{dt}\boldsymbol{k}$
  - If $\boldsymbol{F}$ is constant, $\frac{d\boldsymbol{F}}{dt}$ is $\boldsymbol{o}$, or perpendicular s.t. $\boldsymbol{F}\cdot\frac{d\boldsymbol{F}}{dt} = 0$



High order derivatives, partial difference

- High order derivatives can defined as similar to 1st order
  - $\frac{d^2\boldsymbol{F}}{dt^2}, \frac{d^3\boldsymbol{F}}{dt^3}, \cdots, \frac{d^n\boldsymbol{F}}{dt^n}$
  - For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k},\ \frac{d^n\boldsymbol{F}}{dt^n} = \frac{d^nF_1}{dt^n}\boldsymbol{i} + \frac{d^nF_2}{dt^n}\boldsymbol{j} + \frac{d^nF_3}{dt^n}\boldsymbol{k}$
- Partial difference also defined like derivation
  - $\boldsymbol{A} = \boldsymbol{A}(u,v),\ \frac{\delta\boldsymbol{A}}{\delta u}, \frac{\delta\boldsymbol{A}}{\delta v}, \frac{\delta^2\boldsymbol{A}}{\delta v^2}, \frac{\delta^2\boldsymbol{A}}{\delta v\delta u}, \frac{\delta^2\boldsymbol{A}}{\delta u\delta v}, \frac{\delta^2\boldsymbol{A}}{\delta u^2}$
- Total difference of $\boldsymbol{A}(u,v)$ can be defined as
  - $\delta\boldsymbol{A}(u,v) = \frac{\delta\boldsymbol{A}}{\delta v}du + \frac{\delta\boldsymbol{A}}{\delta u}dv$
  - It approx. small delta of $\delta\boldsymbol{A}$ by small delta of du, dv
    - For $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}, \delta\boldsymbol{A} = \delta A_1\boldsymbol{i} + \delta A_2\boldsymbol{j} + \delta A_3\boldsymbol{k},$

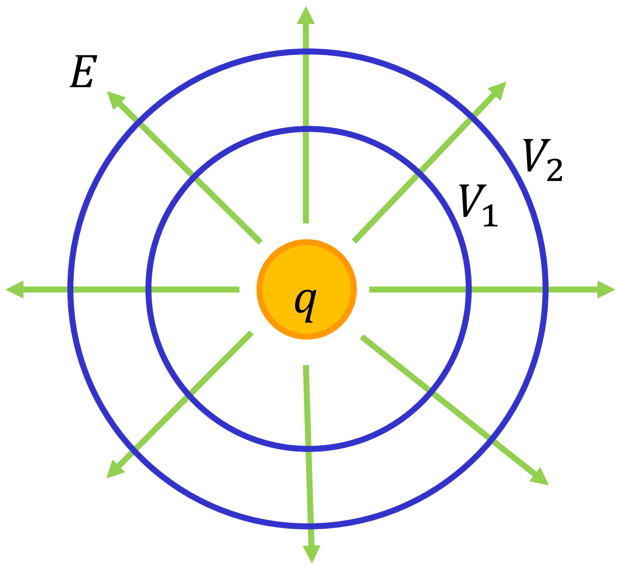
Gradient of scalar

- Scalar function: $f(x, y, z)$ can be defined in unique
  - This field is called scalar field f
    - Distribution of temperature, mass, voltage
- Vector function: $\boldsymbol{F}(x, y, z)$ can be defined in unique
  - This field is called vector field $\boldsymbol{F}$
    - Electric field, magnetic field, gravity field

- Gradient of scalar: $\mathrm{grad}\, f \;=\; \nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$
  - ∇: Hamilton operator
    - $\nabla(f + g) = \nabla f + \nabla g$, $\nabla \lambda f = \lambda \nabla f$, $\nabla(fg) = g\nabla f + f\nabla g$
    - $\nabla \phi(f) = \frac{d\phi}{df}\nabla f$, where $\phi(f)$ is a function of f

Equipotential surface

- If group of points $P(x, y, z)$ satisfy $f(x, y, z) = c$ $(c$: const$)$, P is called equipotential surface
  - In the case of $f(x, y, z) = x^2 + y^2 + z^2$
    - Surface of sphere
- In electro-magnetics, electron (q) create divergence of electric lines $($electric field: $E)$, and electric line create equipotential voltage (V)

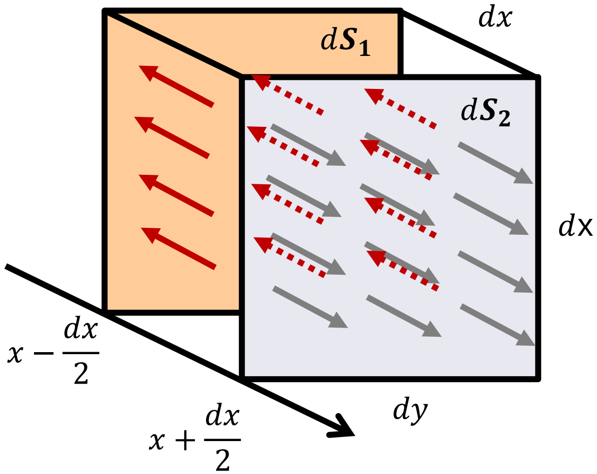


Divergence of vector

- For vector $\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}$,
  $\mathrm{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \boldsymbol{F}$ is called as <u>divergence</u>
- Vector $\boldsymbol{F}, \boldsymbol{G}$, scalar f satisfy following conditions
  - $\mathrm{div}\,(\boldsymbol{F} + \boldsymbol{G}) = \mathrm{div}\,(\boldsymbol{F}) + \mathrm{div}\,(\boldsymbol{G})$
  - $\mathrm{div}\,(f\boldsymbol{G}) = \mathrm{grad}(f) \cdot \boldsymbol{G} + f\,\mathrm{div}\boldsymbol{G}$
  - $\mathrm{div\,grad}(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
- Physical meaning
  - $\mathrm{div}\boldsymbol{F} > 0$: <u>something spout (flow out)</u>
  - $\mathrm{div}\boldsymbol{F} < 0$: <u>something swallowed (flow in)</u>

Divergence of vector

- $\mathrm{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \boldsymbol{F}$
- Assume flow $\boldsymbol{F}$ of small box $dxdydz$
  - Assume flow $\boldsymbol{F}$ of area $d\boldsymbol{S}_1 = (-dydz, 0,0)$ at $x - \frac{dx}{2}$
  - Assume flow $\boldsymbol{F}$ of area $d\boldsymbol{S}_2 = (+dydz, 0,0)$ at $x + \frac{dx}{2}$
    - $\boldsymbol{F} \cdot d\boldsymbol{S} = \boldsymbol{F} \cdot d\boldsymbol{S}_2 - \boldsymbol{F} \cdot d\boldsymbol{S}_1$
    - $= F_1\left(x + \frac{dx}{2}, y, z\right) dydz + F_1\left(x - \frac{dx}{2}, y, z\right)(-dydz)$
    - $= \frac{\partial F_1}{\partial x} dxdydz$
  - Diff. flow in ( )and out ( )

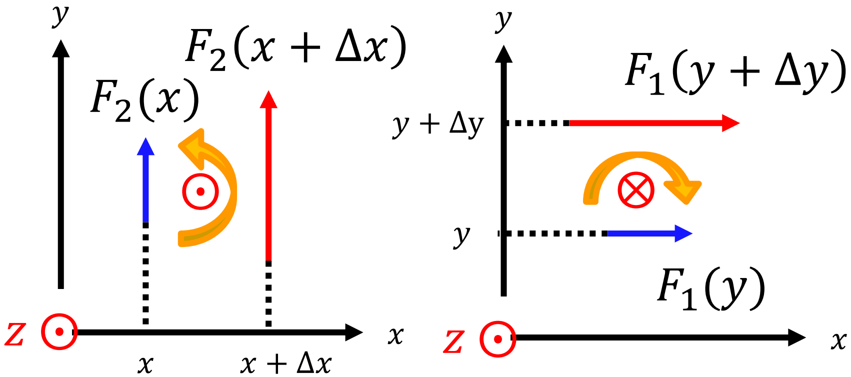


Rotation of vector

- For vector $\boldsymbol{F}(x, y, z) = F_1(x, y, z)\boldsymbol{i} + F_2(x, y, z)\boldsymbol{j} + F_3(x, y, z)\boldsymbol{k}$, $\mathrm{rot}\,\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} = \nabla \times \boldsymbol{F}$ is called as <u>rotation</u>
- $\mathrm{rot}\,\boldsymbol{F} = (\mathrm{rot}_1\,\boldsymbol{F})\boldsymbol{i} + (\mathrm{rot}_2\,\boldsymbol{F})\boldsymbol{j} + (\mathrm{rot}_3\,\boldsymbol{F})\boldsymbol{k}$
- Vector $\boldsymbol{F}, \boldsymbol{G}$, scalar f satisfy following conditions
  - $\mathrm{rot}\,(\boldsymbol{F} + \boldsymbol{G}) = \mathrm{rot}\,(\boldsymbol{F}) + \mathrm{rot}\,(\boldsymbol{G})$
  - $\mathrm{rot}\,(f\boldsymbol{G}) = \mathrm{grad}(f) \times G + f\nabla \times \boldsymbol{G}$
- Physical meaning
  - $\mathrm{rot}\,\boldsymbol{F} > 0$: right-hand side (screw) rotation ($\otimes$)
  - $\mathrm{rot}\boldsymbol{F} < 0$: left-hand side (screw) rotation ($\odot$)

Physical meaning of rotation

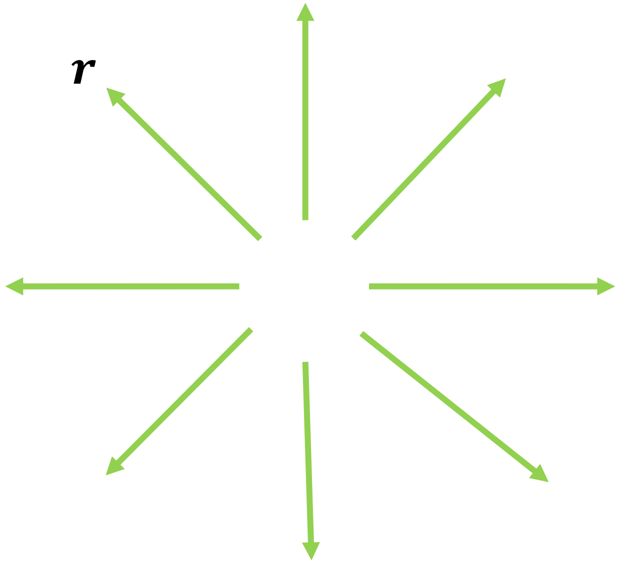
- Link physical notation to the rotation of vector
- Focus 3rd term ($\boldsymbol{k}$) of rotation
  - $\mathrm{rot}\,\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}$
  - $\frac{\partial F_2}{\partial x} = \lim_{\Delta x \to 0} \frac{F_2(x+\Delta x) - F_2(x)}{\Delta x}$
  - $\frac{\partial F_1}{\partial y} = \lim_{\Delta y \to 0} \frac{F_1(y+\Delta y) - F_1(y)}{\Delta y}$



  - If $\frac{\partial F_2}{\partial x} > 0$, it generates right-hand side rotation
  - If $-\frac{\partial F_1}{\partial y} > 0$, it generates right-hand side rotation
  - $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} > 0$ means right-hand side rotation is available

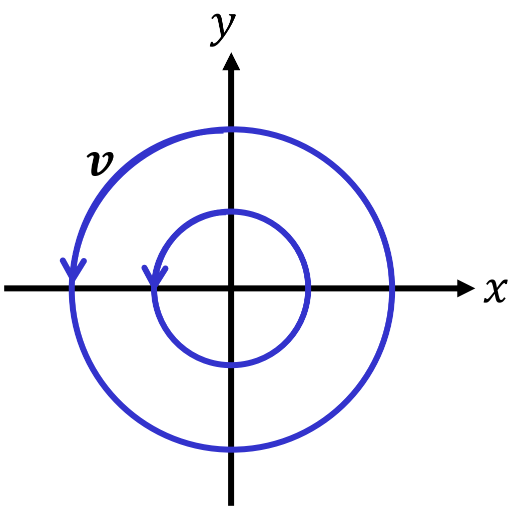
Examples

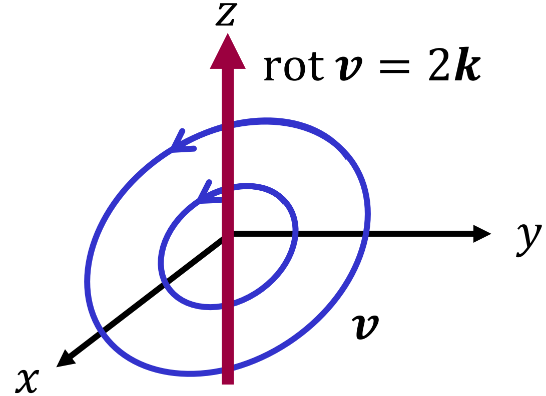
- For $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$,
  - (Q1) Calculate $\mathrm{div}\,\boldsymbol{r}$
  - (A1) $\mathrm{div}\,\boldsymbol{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$
    - (Volume is positive for all xyz)
  - (Q2) Calculate $\mathrm{rot}\,\boldsymbol{r}$
  - (A2) $\mathrm{rot}\,\boldsymbol{r} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\boldsymbol{k} = 0$
    - (No rotating vector here)



Examples

- For $\boldsymbol{v} = -y\boldsymbol{i} + x\boldsymbol{j}$,
  - (Q1) Calculate $\operatorname{div} \boldsymbol{v}$
  - (A1) $\operatorname{div} \boldsymbol{v} = \dfrac{\partial(-y)}{\partial x} + \dfrac{\partial x}{\partial y} = 0$
  - This equation is $x^2 + y^2 = c$ (c: const.)
    - No flow in/out, rotation
  - (Q2) Calculate $\operatorname{rot} \boldsymbol{v}$
  - (A2) $\operatorname{rot} \boldsymbol{v} = \left(\dfrac{\partial 0}{\partial y} - \dfrac{\partial x}{\partial z}\right)\boldsymbol{i} + \left(\dfrac{\partial(-y)}{\partial z} - \dfrac{\partial 0}{\partial x}\right)\boldsymbol{j} + \left(\dfrac{\partial x}{\partial x} - \dfrac{\partial(-y)}{\partial y}\right)\boldsymbol{k} = 2\boldsymbol{k}$





Exercise

- Assume $\boldsymbol{a}, \boldsymbol{b}$ is constant vector, $|\boldsymbol{r}(t)| = r(t)$, calculate its derivation
  - $r\boldsymbol{r} + (\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{b}$
  - $\frac{\boldsymbol{r}}{r^2}$
- Calculate gradient for following functions
  - $f = xz^3 - x^2y$, calculate ∇f at point $P(1, -2, 2)$
  - $f = x^2y^2 - 2xz^3$, calculate ∇f at point $P(1, -2,1)$
- Calculate divergence of following functions
  - $x^2y\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + 3z^3x^3\boldsymbol{k}$
- Calculate rotation of following functions
  - $x^2\boldsymbol{i} - 2xz\boldsymbol{j} + y^2z\boldsymbol{k}$

sample solution

Math ⑥

(1)

$$(r\mathbb{r} + (\mathbb{a}\cdot\mathbb{r})\mathbb{b})' = r'\mathbb{r} + r\mathbb{r}' + (\mathbb{a}\cdot\mathbb{r}')\mathbb{b}$$

$$\left(\frac{\mathbb{r}}{r^2}\right)' = \frac{\mathbb{r}'}{r^2} - \frac{2r'}{r^3}\mathbb{r}$$

(2) $\nabla f = \frac{\partial f}{\partial x}\mathbb{i} + \frac{\partial f}{\partial y}\mathbb{j} + \frac{\partial f}{\partial z}\mathbb{k}$

$$= (z^3 - 2xy)\mathbb{i} + (-x^2)\mathbb{j} + (3xz^2)\mathbb{k}$$

$$= (+8+4)\mathbb{i} + (-1)\mathbb{j} + 12\mathbb{k}$$

$$= 12\mathbb{i} - \mathbb{j} + 12\mathbb{k}$$

$$\Delta f = (2xy^2 - 2z^3)\mathbb{i} + (2x^2y)\mathbb{j} + (-6xz^2)\mathbb{k}$$

$$= (8-2)\mathbb{i} + (-4)\mathbb{j} + (-6)\mathbb{k}$$

$$= +6\mathbb{i} - 4\mathbb{j} - 6\mathbb{k}$$

(3) $\operatorname{div}\mathbb{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\operatorname{div}(x^2y\mathbb{i} - 2y^2z^2\mathbb{j} + 3z^3x^3\mathbb{k})$$

$$= 2xy - 4yz^2 + 9z^2x^3$$

(5) $\mathbb{f} = x^2\mathbb{i} - 2xz\mathbb{j} + y^2z\mathbb{k}$

$$\operatorname{rot}\mathbb{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbb{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbb{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbb{k}$$

$$= \big(2yz - (-2x)\big)\mathbb{i} + (0-0)\mathbb{j} + \big((-2z) - 0\big)\mathbb{k}$$

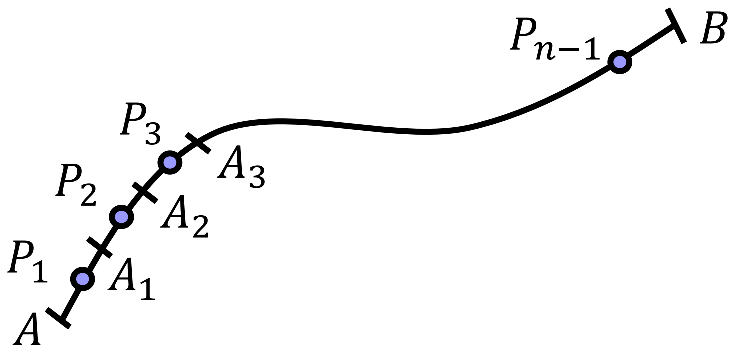
$$= 2(yz + x)\mathbb{i} - 2z\mathbb{k}$$

Fundamental Mathematics

- Integral of vector-

Curvilinear integral (線積分)

- Assume a smooth curve C from point A to B, and scalar function $f(P) = f(x, y, z)$ is continuous in curve C
  - Think curve C can divide into several arcs $\Delta s_1 \cdots \Delta s_n$
    - Points A_n divide a curve, these weight are points P_n
    - Assume limit of $n \to \infty$, $\Delta \mathrm{s_i} \to 0$; <u>curvilinear integral</u>
    - $\lim\limits_{\substack{n\to\infty \\ \Delta \mathrm{s_i}\to 0}} \sum_{i=1}^{n} f(P_i)\Delta \mathrm{s_i} = \int_C \ f(P)\, ds = \int_C \ f(x, y, z)\, ds$
  - Point D on curve C is function of the length (s) of arc $\widehat{AB}$



Curvilinear integral

- Point D on curve C is function of the length (s) of arc $\widehat{AB}$
  - (Any) point D can be expressed as function of length s
    - $\boldsymbol{r} = \boldsymbol{r}(s) = x(s)\boldsymbol{i} + y(s)\boldsymbol{j} + z(s)\boldsymbol{k}$
    - $\int_C f(x, y, z)\, ds = \int_a^d f(x(s), y(s), z(s)\ )\, ds$
    - d and a in s are correspond D and A on curve C
- If we use general parameter t to express the curve C;
  - $\boldsymbol{r} = \boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$
  - $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$
  - $\int_C f(x, y, z)\, ds = \int_\alpha^\beta f(x(s), y(s), z(s)\ ) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$
    - where A, B of curve C are point $\alpha, = t\ \beta = t$

Expressions of curvilinear integral

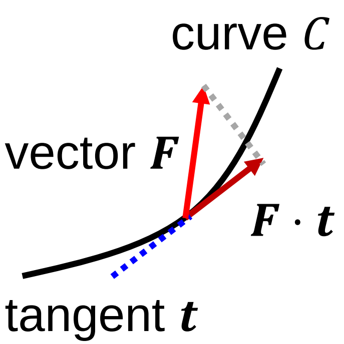
- Several expressions are available for curvilinear integral
  - $\int_C f\,ds = \int_A^B f\,ds = \int_{AB} f\,ds$
  - $\int_{AB} f\,ds = -\int_{BA} f\,ds$
- If point P is on the curve C, $\int_{AB} f\,ds = \int_{AP} f\,ds + \int_{PB} f\,ds$
- If the curve C is a closed curve, $\oint_C f\,ds = \oint_{AB} f\,ds$

Example of curvilinear integral

- Calculate curvilinear integral of $f(x,y,z) = y^2z + z^2x + x^2y$
  - Route 1: $O(0,0,0) \to Q(3,0,0) \to R(3,1,0) \to P(3,1,2)$
    - $\int_{R1} f\,ds = \int_O^Q f\,ds + \int_Q^R f\,ds + \int_R^P f\,ds$
    - $= \int_O^3 f(x,0,0)\,dx + \int_0^1 f(3,y,0)\,dy + \int_0^2 f(3,1,z)\,dz = \frac{65}{2}$
  - Route 2: $\overrightarrow{OP}$
    - $\overrightarrow{OP} = \boldsymbol{r} = 3t\boldsymbol{i} + t\boldsymbol{j} + 2t\boldsymbol{k}\ (0 \le t \le 1)$
    - $ds = \sqrt{(3dt)^2 + (1dt)^2 + (2dt)^2} = \sqrt{14}dt$
    - $\int_{R2} f\,ds = \int_O^{\sqrt{14}} (y^2z + z^2x + x^2y\ )\,ds$
    - $= \int_O^1 (2t^3 + 12t^3 + 9t^3)\sqrt{(3)^2 + (1)^2 + (2)^2}\,dt = \frac{23\sqrt{14}}{4}$

Curvilinear integral for vector

- Assume a smooth curve C from point A to B, and vector function $\boldsymbol{F}(P) = \boldsymbol{F}(x, y, z)$ is continuous in curve C
  - $\boldsymbol{r}(s)$ is a position vector from origin O to the point P on C
  - Assume $\boldsymbol{t} = \frac{d\boldsymbol{r}}{ds}$ is a tangent of curve C at point P
    - <u>Curvilinear integral for the vector</u> $\boldsymbol{F}$: $\int_C \boldsymbol{F} \cdot \boldsymbol{t}\, ds$
- Assume func. of C: $\boldsymbol{r}(s) = x(s)\boldsymbol{i} + y(s)\boldsymbol{j} + z(s)\boldsymbol{k}$, $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$
  - $\int_C \boldsymbol{F} \cdot \boldsymbol{t}\, ds = \int_C \left(\frac{F_1 dx}{ds} + \frac{F_2 dy}{ds} + \frac{F_3 dz}{ds}\right)$
- Scalar $\boldsymbol{F} \cdot \boldsymbol{t}$ is a tangent component of vector $\boldsymbol{F}$



Characteristics of curvilinear integral for vector

- Curvilinear integral for vector has following characteristics
  - For scalar field $f(x, y, z)$ and vector field $\boldsymbol{F}(x, y, z)$
    - $\int_C f(x, y, z)\, d\boldsymbol{r} = \boldsymbol{i}\int_C fdx + \boldsymbol{j}\int_C fdy + \boldsymbol{k}\int_C fdz$
    - $\int_C \boldsymbol{F}(x, y, z)\, ds = \boldsymbol{i}\int_C F_1 ds + \boldsymbol{j}\int_C F_2 ds + \boldsymbol{k}\int_C F_3 ds$
    - $\int_C \boldsymbol{F} \times d\boldsymbol{r} = \int_C \boldsymbol{F} \times \boldsymbol{r} ds = \boldsymbol{i}\int_C (F_2 dz - F_3 dy) +$
      $\boldsymbol{j}\int_C (F_3 dx - F_1 dz) + \boldsymbol{k}\int_C (F_x dy - F_2 dx)$

Exercise

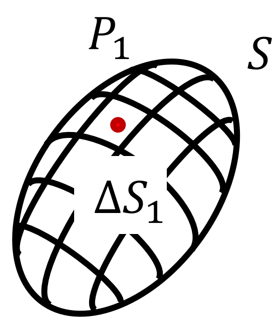
- Calculate curvilinear integral $\int_C \; y \, d\boldsymbol{r}$
  - C: $x = a\cos t$, $y = a\sin t$, $z = ht$, $(0 \leq t \leq 2\pi)$
- Solution
  - $\int_C \; y \, d\boldsymbol{r} = \int_C \; a\sin t \, (\boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz)$
  - $= -\boldsymbol{i}\int_0^{2\pi} \; a^2\sin^2 t \, dt + \boldsymbol{j}\int_0^{2\pi} \; a^2 \sin t \cos t \, dt + \boldsymbol{k}\int_0^{2\pi} \; ah \sin t \, dt$
  - $= -\pi a^2 \boldsymbol{i}$

Potential

- If scalar func. $\varphi(x,y,z)$ is available for $\boldsymbol{F}(x,y,z) = -\mathrm{grad}\varphi$; φ is called as <u>potential</u> or <u>scalar potential</u> of $\boldsymbol{F}$
- Potential has following characteristics;
  - Assume vector field $\boldsymbol{F}(x,y,z)$ has potential φ
    - $\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_A^B \nabla\varphi \cdot d\boldsymbol{r} = \varphi(A) - \varphi(B)$
  - If curve C is a closed curve
    - $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = -\oint_C \nabla\varphi \cdot d\boldsymbol{r} = 0$

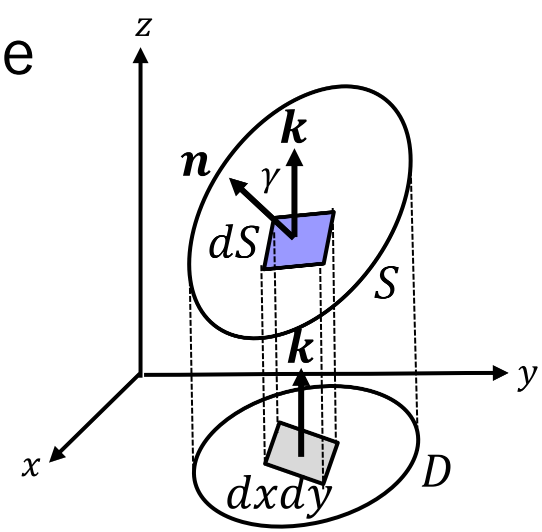
Surface integral (面積分) for scalar

- Assume smooth curved surface S
  - Scalar function $f(P) = f(x, y, z)$ is continuous in S
    - Assume S can be divided into small area $\Delta S_1 \cdots \Delta S_n$, and any point of $P_1 \cdots P_n$
    - If $\lim\limits_{\substack{n\to\infty \\ \Delta S_i \to 0}} \sum_{i=1}^{n} f(P_i)\Delta S_i$ is available, this is called <u>surface integral for scalar</u> $\int_S f(x, y, z)dS$
  - If $f(P) = 1$, $\int_S f(x, y, z)dS$ is area of S
- For the curved surface, outside is the front



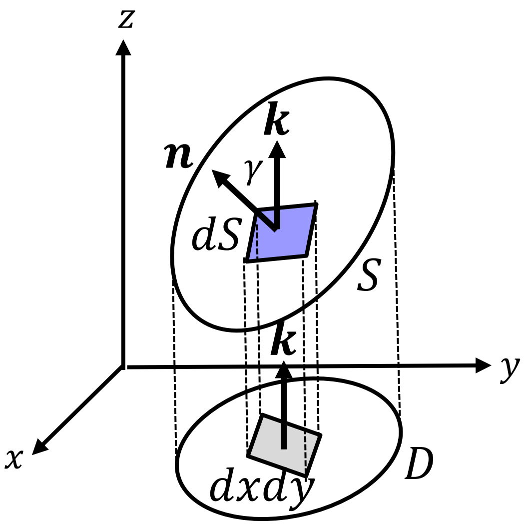
Formula of surface integral

- If surface S is given for $z = g(x, y)$, surface integral of $f(x, y, z)$ on S can be expressed as follows,
  - $\int_S f(x, y, z) dS = \iint_D f(x, y, g(x, y))\sqrt{p^2 + q^2 + 1}\, dxdy$
  - $= \iint_D f(x, y, g(x, y)) \frac{dxdy}{|\boldsymbol{n}\cdot\boldsymbol{k}|}$,
    - where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $\boldsymbol{n}$ is unit normal vector of S, D is projective of S to xy-coordinate



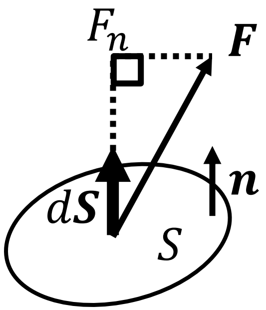
Formula of surface integral (proof)

- Think small surface dS on S, its projective in xy-coordinate can express $dydx$
- Define angle of unit normal vectors $\boldsymbol{n}, \boldsymbol{k}$ as γ
  - $dS|\cos\gamma| = dxdy$
- $\boldsymbol{n} = \frac{\pm 1}{\sqrt{p^2+q^2+1}}$ when $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$
- Thus, $|\cos\gamma| = |\boldsymbol{n}\cdot\boldsymbol{k}| = \frac{1}{\sqrt{p^2+q^2+1}}$
- $dS = \frac{dxdy}{|\cos\gamma|} = \frac{dxdy}{|\boldsymbol{n}\cdot\boldsymbol{k}|}$
- $\int_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\frac{dxdy}{|\boldsymbol{n}\cdot\boldsymbol{k}|}$
  - $(z = g(x,y))$



Surface integral for vector

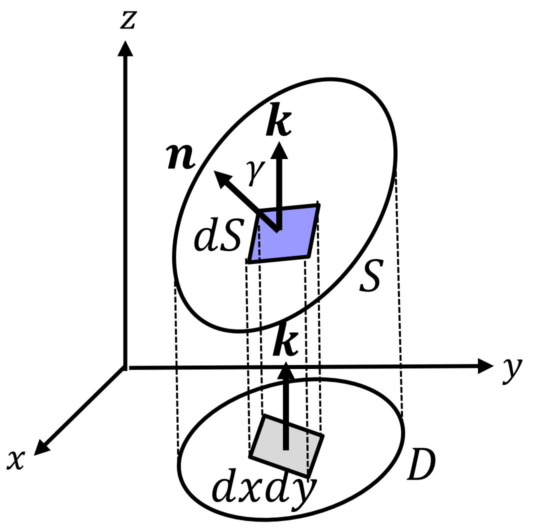
- For vector field $\boldsymbol{F}$ and unit vector $\boldsymbol{n}$ of surface S, integral of these inner products is called as <u>surface integral of vector</u>
  - $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$
  - F_n is a $\boldsymbol{n}$ component of vector $\boldsymbol{F}$ $(\boldsymbol{F} \cdot \boldsymbol{n} = F_n)$
  - Assume $\boldsymbol{n}\, dS = d\boldsymbol{S}$, $d\boldsymbol{S}$ is called area vector



    - $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \int_S F_n\, dS = \int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \oint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ (If S is closed surface)
  - For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$,
    - $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$
- Several expressions for surface integral of vectors
    - $\int_S \boldsymbol{F} dS = \boldsymbol{i} \int_S F_1 dS + \boldsymbol{j} \int_S F_2 dS + \boldsymbol{k} \int_S F_3 dS$
    - $\int_S \boldsymbol{F} \times \boldsymbol{n} dS = \int_S \boldsymbol{F} \times d\boldsymbol{S}$

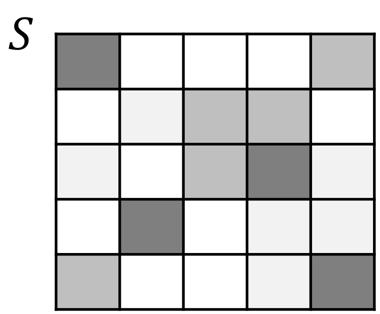
Formula of surface integral

- If surface S is given for $z = g(x, y)$, surface integral of $\boldsymbol{F}(x, y, z)$ on S can be expressed as follows,
  - $\int_S \boldsymbol{F}(x, y, z) dS = \iint_D \boldsymbol{F}(x, y, g(x, y))\sqrt{p^2 + q^2 + 1}\, dxdy$
    - where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, D is projective of S to xy-coordinate

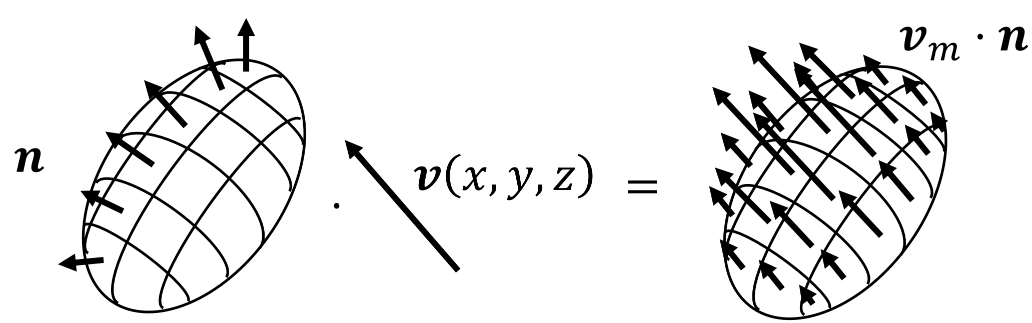


Surface integral in physics

- In scalar: $\int_S \rho(x,y,z)dS$
  - In the case ρ is a function of mass density on surface S
    - Its integral: total mass of surface S
- In vector: $\int_S \boldsymbol{v}(x,y,z)\cdot\boldsymbol{n}dS$
  - In the case $\boldsymbol{v}$ is a function of liquid velocity on surface S
    - Its integral: total amount of liquid flow per unit time



Mass density $\rho(x,y,z)$ on surface S



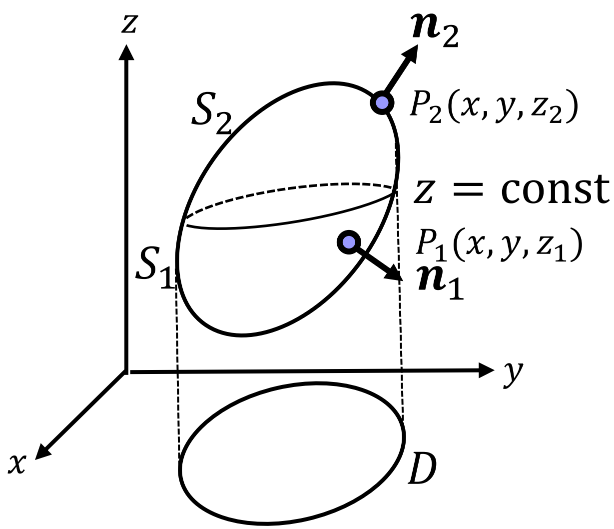
Liquid velocity $\boldsymbol{v}$ penetrate surface S

Volume integral (体積分)

- Connect divergence on vector field and flow at the surface
  - Assume volume V surrounded by surface S
    - Volume integral of scalar f: $\int_V f(x,y,z)dV$
    - Volume integral of vector $\boldsymbol{F}$: $\int_V \boldsymbol{F}(x,y,z)dV$
      - $\int_V \boldsymbol{F}(x,y,z)dV = \boldsymbol{i}\int_V F_1 dV + \boldsymbol{j}\int_V F_2 dV + \boldsymbol{k}\int_V F_3 dV$
- Preliminary
  - For volume V surrounded by surface S, $\boldsymbol{n} = \cos\alpha\,\boldsymbol{i} + \cos\beta\,\boldsymbol{j} + \cos\gamma\,\boldsymbol{k}$, following equation satisfies,
  - $\int_V \frac{\partial f}{\partial x} dV = \int_S f \cos\alpha\, dS$, $\int_V \frac{\partial f}{\partial y} dV = \int_S f \cos\beta\, dS$, $\int_V \frac{\partial f}{\partial z} dV = \int_S f \cos\gamma\, dS$

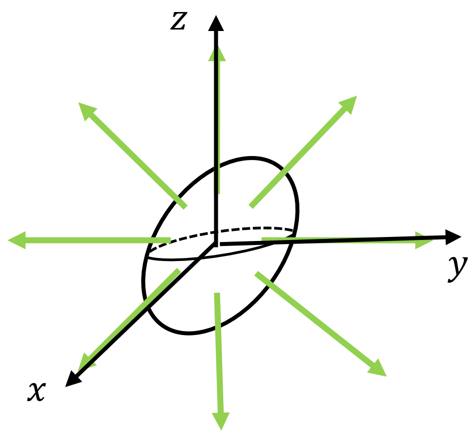
Volume integral (proof)

- Proof $\int_V \frac{\partial f}{\partial z} dV = \int_S f \cos\gamma \, dS$;
- Assume two points P_1 , P_2 on S
  - $z_2 \geq z_1$: z_2 coves upper side of S, z_1 coves lower side of S
  - $\int_V \frac{\partial f}{\partial z} dV$ means volume difference in z-axis, thus
- $\int_V \frac{\partial f}{\partial z} dV = \iiint_V \frac{\partial f}{\partial z} dxdydz = \iint_D \left\{ \int_{z_1}^{z_2} \frac{\partial f}{\partial z} dz \right\} dxdy =$
  $\iint_D [f]_{z_1}^{z_2} dxdy = \iint_D \{f(x,y,z_2) - f(x,y,z_1)\} dxdy$
- For z-axis, z_2 is upper $(dS \cos\gamma = dxdy)$, z_1 is lower thus $(dS \cos\gamma = -dxdy)$
  - $\iint_D f(x,y,z_2) dxdy = \int_{S_2} f(x,y,z) dS$
  - $\iint_D f(x,y,z_1) dxdy = -\int_{S_1} f(x,y,z) dS$
  - $\int_V \frac{\partial f}{\partial z} dV = \int_{S_2} f \cos\gamma \, dS + \int_{S_1} f \cos\gamma \, dS = \int_S f \cos\gamma \, dS$

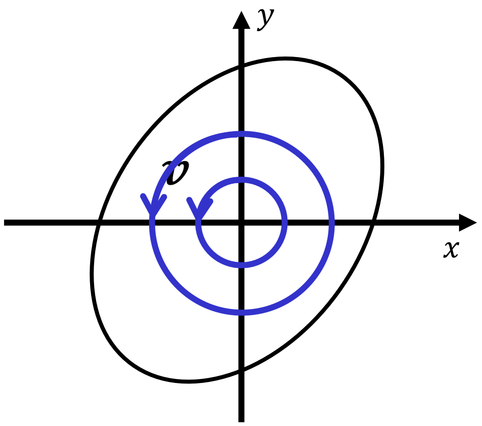


Divergence theorem (Gauss' theorem)

- Connect divergence on vector field and flow at the surface
  - Assume volume V surrounded by surface S w/ unit vec. $\boldsymbol{n}$
    - $\int_V \mathrm{div}\,\boldsymbol{F} dV = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{F} dV = \int_S \boldsymbol{F} \cdot \boldsymbol{n} dS$
- Physical meaning
  - $\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS$: amount of flow which path through the area S
  - $\int_V \mathrm{div}\,\boldsymbol{F} dV$: amount of flow out



For $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$,
$\mathrm{div}\,\boldsymbol{r} = 3$

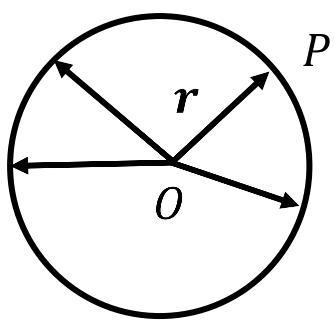


For $\boldsymbol{v} = -y\boldsymbol{i} + x\boldsymbol{j}$, and if volume V is outside of $\boldsymbol{v}$, $\mathrm{div}\,\boldsymbol{r} = 0$

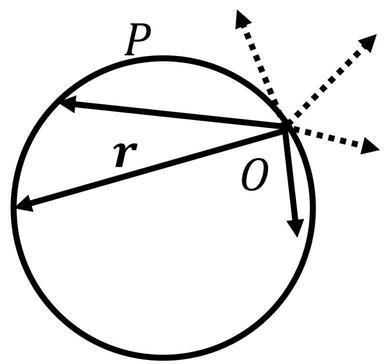
Extension of Gauss' theorem

- Assume the point P on close surface S, express vector from origin $O(0,0,0)$ to P as $\overrightarrow{OP} = \boldsymbol{r}$, $\boldsymbol{n}$ is unit normal vector of S
- Following equation satisfy the following conditions

- $$\int_S \frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{n} dS = \begin{cases} 0 & \text{(when } O \text{ is outside of } S\text{)} \\ 2\pi & \text{(when } O \text{ is on the surface } S\text{)} \\ 4\pi & \text{(when } O \text{ is inside of } S\text{)} \end{cases}$$

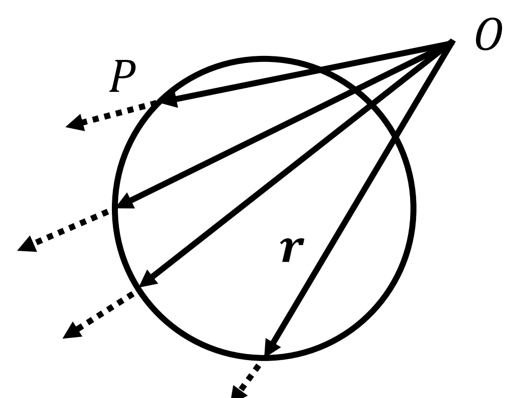


$$\int_S \frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{n} dS = 4\pi$$



$$\int_S \frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{n} dS = 2\pi$$

Only the half (solid) penetrate S



$$\int_S \frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{n} dS = 0$$

Same amount for flow in and flow out

Exercise

- For function $f(x, y, z) = x^2 - yz + z^2$, calculate its curvilinear integral $\int_C f\,ds$
  - Case 1: C is a line from $P_1(1,2,0)$ to $P_2(1,2,3)$
  - Case 2: C is a line from $P_1(0,0,0)$ to $P_2(1,2,3)$
- Assume the surface func. $2x + 2y + z - 4 = 0$, and its intercepts are points A, B, C, and ABC create surface S
  - Calculate surface integral of $f(x, y, z) = 4x - y^2 + 2z - 12$

Exercise

- Assume the surface func. $x + y + z - 1 = 0$, and its intercepts are points P, Q, R and PQR create surface S
  - Calculate surface integral $\int_S \boldsymbol{F} \times \boldsymbol{n}\, dS$ for $\boldsymbol{F} = y\boldsymbol{k}$
- Assume the volume and surface of unit sphere as V, S, and $\boldsymbol{F} = ax\boldsymbol{i} + by\boldsymbol{j} + cz\boldsymbol{k}$. Calculate integral $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$

Sample solution

Math 7

① $f = x^2 - yz + z^2$, take $\int_C f\,ds$

(1-1) C is $P_1(1,2,0)$ to $P_2(1,2,3)$

$$\int_C f\,ds = \int_0^3 (1^2 - 2z + z^2)\,dz = \left[z - z^2 + \frac{z^3}{3}\right]_0^3 = 3$$

(1-2) C is $P_1(0,0,0)$ to $P_2(1,2,3)$

$\Rightarrow x = t,\ y = 2t,\ z = 3t\ (0 \le t \le 1)$

$$ds = \sqrt{1^2 + 2^2 + 3^2}\,dt = \sqrt{14}\,dt$$

$$\int_C f(x(s) + y(s) + z(s))\,ds = \int_0^1 (t^2 - 6t^2 + 9t^2)\sqrt{14}\,dt$$

$$= 4\sqrt{14}\int_0^1 t^2\,dt = \frac{4\sqrt{14}}{3}$$

② $S: 2x + 2y + z - 4 = 0$, $f = 4x - y^2 + 2z - 12$

Calculate $\int_S f\,dS$

Surface func $z = g(x,y) = 4 - 2x - 2y$

$$p = \frac{\partial z}{\partial x} = -2,\quad q = \frac{\partial z}{\delta y} = -2$$

$$dS = \sqrt{p^2 + q^2 + 1}\,dx\,dy = 3\,dx\,dy$$

$$f(x, y, g(x,y)) = -(y+2)^2$$

$$\int_S f\,dS = -\iint_D (y+2)^2 \cdot 3\,dx\,dy$$

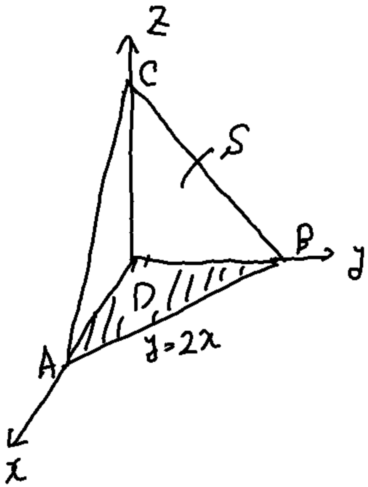
$$= -\underbrace{\int_0^2 \underbrace{\int_0^{2-x} 3(y+2)^2\,dy}_{y}\,dx}_{x}$$

$$= -\int_0^2 \left[(y+2)^3\right]_0^{2-x} dx$$

$$= -\int_0^2 \left((4-x)^3 - 8\right) dx$$

$$= \left[\frac{1}{4}(4-x)^3 + 8x\right]_0^2$$

$$= -44$$



Sample solution

③ $S: x+y+z-1=0,\ \mathbf{F}=y\mathbf{k}$, calc. $\int_S \mathbf{F}\times\mathbf{n}\,dS$

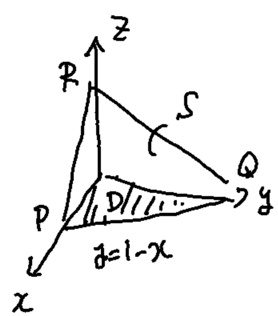
Surface func $S = g(x,y) = 1-x-y$.

$$p=\frac{\partial z}{\partial x}=-1,\quad q=\frac{\partial z}{\partial y}=-1$$

unit normal vector of S is : $\mathbf{n}=\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

$$\mathbf{F}\times\mathbf{n} = \mathbf{i}\left(0\cdot\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}y\right)+\mathbf{j}\left(y\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}0\right)+\mathbf{k}\left(0\cdot\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}0\right)$$
$$=-\frac{y}{\sqrt{3}}(\mathbf{i}-\mathbf{j})$$

$dS=\sqrt{p^2+q^2+1}\,dxdy=\sqrt{3}\,dxdy$ thus



$$\int_S \mathbf{F}\times\mathbf{n}\,dS = -\iint_D \frac{y}{\sqrt{3}}(\mathbf{i}-\mathbf{j})\sqrt{3}\,dxdy$$
$$=-(\mathbf{i}-\mathbf{j})\int_0^1\int_0^{1-x} y\,dxdy$$
$$=-(\mathbf{i}-\mathbf{j})\int_0^1 \frac{(1-x)^2}{2}dx$$
$$=(\mathbf{i}-\mathbf{j})\left[\frac{(1-x^3)}{6}\right]_0^1 = -\frac{1}{6}(\mathbf{i}-\mathbf{j})$$

④

Calculate $\int_S \mathbf{F}\cdot\mathbf{n}\,dS$ for $\mathbf{F}=ax\mathbf{i}+by\mathbf{j}+cz\mathbf{k}$

from divergence law

$$\int_S \mathbf{F}_S\cdot\mathbf{n}\,dS = \int_V \nabla\cdot\mathbf{F}\,dV$$
$$=\int_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dV$$
$$=\int_V (a+b+c)\,dV$$
$$=(a+b+c)\int_V dV$$
$$=\frac{4}{3}\pi(a+b+c)$$

(Volume of unit sphere $V=\frac{4}{3}\pi r^3$, $r=1$.)

Conclusion

- Learn integral of vectors
  - Curvilinear integral
  - Surface integral
  - Volume integral
- Next
  - Vector analysis and Maxwell’s equation
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