Fundamental Mathematics (Engineering Mathematics)

Shinichi Nishizawa

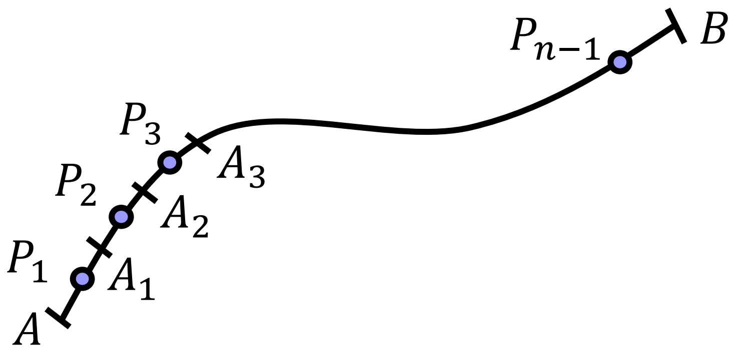
Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

Curvilinear integral

- Assume a smooth curve C from point A to B, and scalar function $f(P) = f(x, y, z)$ is continuous in curve C
  - Think curve C can divide into several arcs $\Delta s_1 \cdots \Delta s_2$
    - Points A_n divide a curve, these weight are points P_n
    - Assume limit of $n \to \infty$, $\Delta \mathrm{s_i} \to 0$; <u>curvilinear integral</u>
    - $\lim_{\substack{n\to\infty \\ \Delta \mathrm{s_i}\to 0}} \sum_{i=1}^{n} f(P_i)\Delta \mathrm{s_i} = \int_C f(P)\, ds = \int_C f(x, y, z)\, ds$
  - Point D on curve C is function of the length (s) of arc $\widehat{AD}$



Curvilinear integral

- Point D on curve C is function of the length (s) of arc $\widehat{AD}$
  - (Any) point D can be expressed as function of length s
    - $\boldsymbol{r} = \boldsymbol{r}(s) = x(s)\boldsymbol{i} + y(s)\boldsymbol{j} + z(s)\boldsymbol{k}$
    - $\int_C f(x, y, z)\, ds = \int_A^B f(x(s), y(s), z(s)\ )\, ds$
- If we use general parameter t to express the curve C;
  - $\boldsymbol{r} = \boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$
  - $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$
  - $\int_C f(x, y, z)\, ds = \int_\alpha^\beta f(x(s), y(s), z(s)\ ) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$
    - where A, B of curve C are point $\alpha, = t\ \beta = t$

Expressions of curvilinear integral

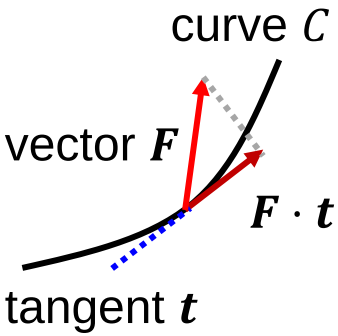
- Several expressions are available for curvilinear integral
  - $\int_C \ f \, ds = \int_A^B f \, ds = \int_{AB} f \, ds$
  - $\int_{AB} f \, ds = -\int_{BA} f \, ds$
- If point P is on the curve C, $\int_{AB} f \, ds = \int_{AP} f \, ds + \int_{PB} f \, ds$
- If the curve C is a closed curve,$\oint_C \ f \, ds = \oint_{AB} f \, ds$

Example of curvilinear integral

- Calculate curvilinear integral of $f(x,y,z) = y^2z + z^2x + x^2y$
  - Route 1: $O(0,0,0) \rightarrow Q(3,0,0) \rightarrow R(3,1,0) \rightarrow P(3,1,2)$
    - $\int_{R1} f\,ds = \int_O^Q f\,ds + \int_Q^R f\,ds + \int_R^P f\,ds$
    - $= \int_O^3 f(x,0,0)\,dx + \int_0^1 f(3,y,0)\,dy + \int_0^2 f(3,1,z)\,dz = \frac{65}{2}$
  - Route 2: $\overrightarrow{OP}$
    - $\overrightarrow{OP} = \boldsymbol{r} = 3t\boldsymbol{i} + t\boldsymbol{j} + 2t\boldsymbol{k}\ (0 \leq t \leq 1)$
    - $ds = \sqrt{(3dt)^2 + (1dt)^2 + (2dt)^2} = \sqrt{14}dt$
    - $\int_{R2} f\,ds = \int_O^{\sqrt{14}} (y^2z + z^2x + x^2y\ )\,ds$
    - $= \int_O^1 (2t^3 + 12t^3 + 9t^3)\sqrt{(3)^2 + (1)^2 + (2)^2}\,dt = \frac{23\sqrt{14}}{4}$

Curvilinear integral for vector

- Assume a smooth curve C from point A to B, and vector function $\boldsymbol{F}(P) = \boldsymbol{F}(x, y, z)$ is continuous in curve C
  - $\boldsymbol{r}(s)$ is a position vector from origin O to the point P on C
  - Assume $\boldsymbol{t} = \frac{d\boldsymbol{r}}{ds}$ is a tangent of curve C at point P
    - <u>Curvilinear integral for the vector </u>$\boldsymbol{F}$: $\int_C \boldsymbol{F} \cdot \boldsymbol{t}\, ds$
- Assume func. of C: $\boldsymbol{r}(s) = x(s)\boldsymbol{i} + y(s)\boldsymbol{j} + z(s)\boldsymbol{k}$, $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$
  - $\int_C \boldsymbol{F} \cdot \boldsymbol{t}\, ds = \int_C \left(\frac{F_1 dx}{ds} + \frac{F_2 dy}{ds} + \frac{F_3 dz}{ds}\right)$
- Scalar $\boldsymbol{F} \cdot \boldsymbol{t}$ is a tangent component of vector $\boldsymbol{F}$



Characteristics of curvilinear integral for vector

- Curvilinear integral for vector has following characteristics
  - For scalar field $f(x, y, z)$ and vector field $\boldsymbol{F}(x, y, z)$
    - $\int_C \; f(x, y, z)\, d\boldsymbol{r} = \boldsymbol{i} \int_C \; f dx + \boldsymbol{j} \int_C \; f dy + \boldsymbol{k} \int_C \; f dz$
    - $\int_C \; \boldsymbol{F}(x, y, z)\, ds = \boldsymbol{i} \int_C \; F_1 ds + \boldsymbol{j} \int_C \; F_2 ds + \boldsymbol{k} \int_C \; F_3 ds$
    - $\int_C \; \boldsymbol{F} \times d\boldsymbol{r} = \int_C \; \boldsymbol{F} \times \boldsymbol{r} ds = \boldsymbol{i} \int_C \; (F_2 dz - F_3 dy) +$
    $\boldsymbol{j} \int_C \; (F_3 dx - F_1 dz) + \boldsymbol{k} \int_C \; (F_x dy - F_2 dx)$

Exercise

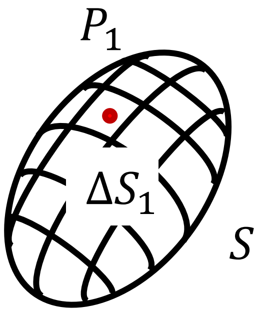
- Calculate curvilinear integral $\int_C \; y\, d\boldsymbol{r}$
  - C: $x = a\cos t,\ y = a\sin t,\ z = ht,\ (0 \le t \le 2\pi)$
- Solution
  - $\int_C \; \underline{y}\, \underline{d\boldsymbol{r}} = \int_C \; \underline{a\sin t}\,(\underline{\boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz})$
  - $= -\boldsymbol{i}\int_0^{2\pi} \; a^2 \sin^2 t\, dt + \boldsymbol{j}\int_0^{2\pi} \; a^2 \sin t \cos t\, dt + \boldsymbol{k}\int_0^{2\pi} \; ah \sin t\, dt$
  - $= -\pi a^2 \boldsymbol{i}$

Potential

- If scalar function $\varphi(x, y, z)$ is available for $\boldsymbol{F}(x, y, z) = \mathrm{grad}\varphi$; φ is called as potential or scalar potential of $\boldsymbol{F}$
- Potential has following characteristics;
  - Assume vector field $\boldsymbol{F}(x, y, z)$ has potential φ
    - $\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_A^B \nabla\varphi \cdot d\boldsymbol{r} = \varphi(A) - \varphi(B)$
  - If curve C is a closed curve
    - $\oint_C \ F \cdot d\boldsymbol{r} = -\oint_C \ \nabla\varphi \cdot d\boldsymbol{r} = 0$

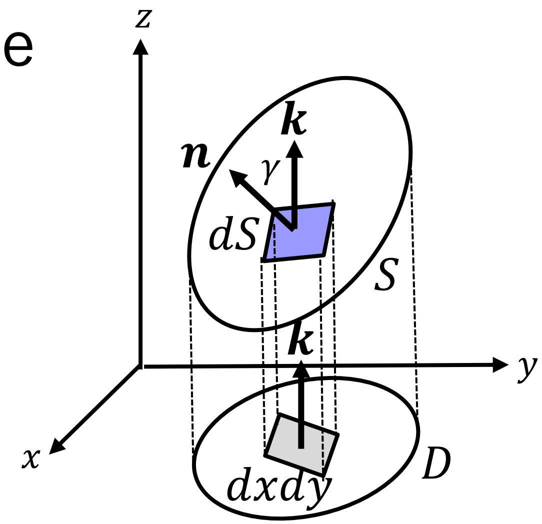
Surface integral for scalar

- Assume smooth curved surface S
  - Scalar function $f(P) = f(x, y, z)$ is continuous in S
    - Assume S can be divided into small area $\Delta S_1 \cdots \Delta S_n$, and any point of $P_1 \cdots P_n$
    - If $\lim\limits_{\substack{n \to \infty \\ \Delta S_i \to 0}} \sum_{i=1}^{n} f(P_i)\Delta S_i$ is available, this is called <u>surface integral for scalar</u> $\int_S f(x, y, z)dS$
  - If $f(P) = 1$, $\int_S f(x, y, z)dS$ is area of S
- For the curved surface, outside is the front



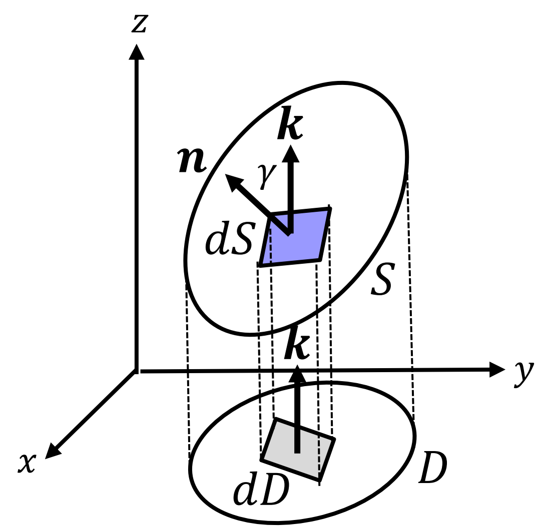
Formula of surface integral

- If surface S is given for $z = g(x, y)$, surface integral of $f(x, y, z)$ on S can be expressed as follows,
  - $\int_S f(x, y, z)dS = \iint_D f(x, y, g(x, y))\sqrt{p^2 + q^2 + 1}\,dxdy$
  - $= \iint_D f(x, y, g(x, y))\frac{dxdy}{|\boldsymbol{n}\cdot\boldsymbol{k}|}$,
    - where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $\boldsymbol{n}$ is unit normal vector of S, D is projective of S to xy-coordinate



Formula of surface integral (proof)

- Think small surface dS on S, its projective in xy-coordinate can express $dydx$
- Define angle of unit normal vectors $\boldsymbol{n}, \boldsymbol{k}$ as γ
  - $dS|\cos\gamma| = dxdy$



- $\boldsymbol{n} = \frac{\pm 1}{\sqrt{p^2+q^2+1}}$ when $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$
- Thus, $|\cos\gamma| = |\boldsymbol{n}\cdot\boldsymbol{k}| = \frac{1}{\sqrt{p^2+q^2+1}}$
- $dS = \frac{dxdy}{|\cos\gamma|} = \frac{dxdy}{|\boldsymbol{n}\cdot\boldsymbol{k}|}$
- $\int_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\frac{dxdy}{|\boldsymbol{n}\cdot\boldsymbol{k}|}$
  - $(z = g(x,y))$

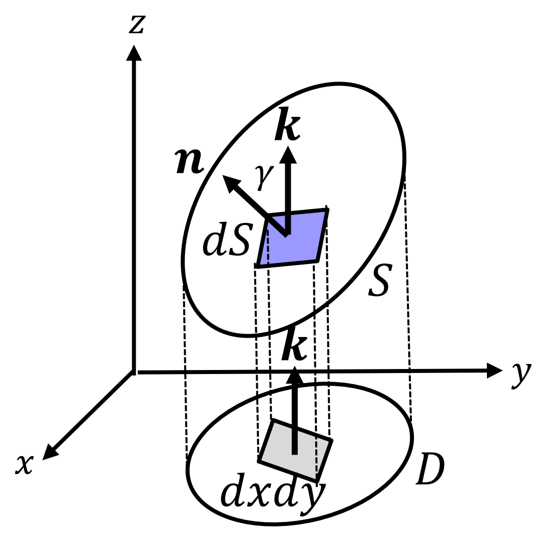
Surface integral for vector

- For vector field $\boldsymbol{F}$ and unit vector $\boldsymbol{n}$ of surface S, integral of these inner products is called as surface integral of vector
  - $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$
  - F_n is a $\boldsymbol{n}$ component of vector $\boldsymbol{F}$ $(\boldsymbol{F} \cdot \boldsymbol{n} = F_n)$
  - Assume $\boldsymbol{n}\, dS = d\boldsymbol{S}$, $d\boldsymbol{S}$ is called area vector
    - $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \int_S F_n\, dS = \int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \oint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$

      (If S is closed surface)
  - For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$,
    - $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$
- Several expressions for surface integral of vectors
  - $\int_S \boldsymbol{F} dS = \boldsymbol{i} \int_S F_1 dS + \boldsymbol{j} \int_S F_2 dS + \boldsymbol{k} \int_S F_3 dS$
  - $\int_S \boldsymbol{F} \times \boldsymbol{n} dS = \int_S \boldsymbol{F} \times d\boldsymbol{S}$

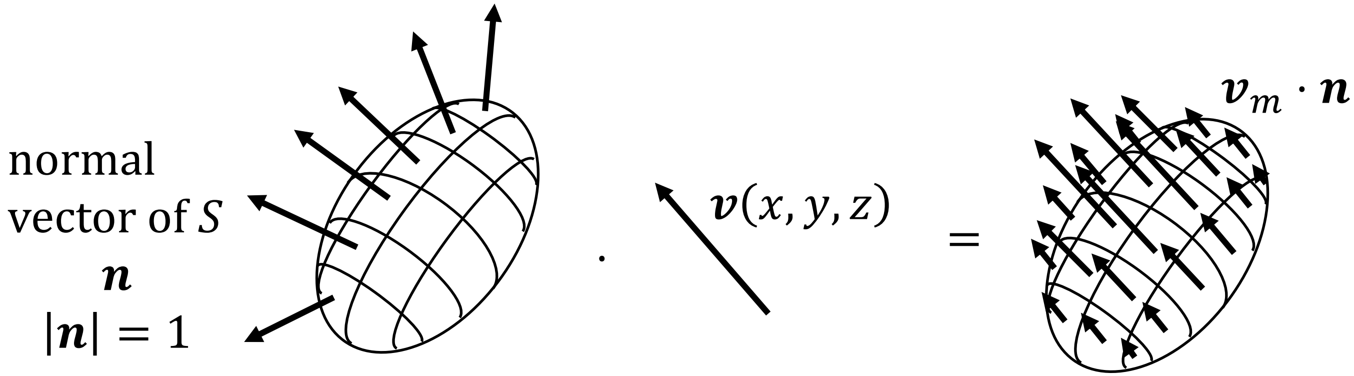
Formula of surface integral

- If surface S is given for $z = g(x, y)$, surface integral of $\boldsymbol{F}(x, y, z)$ on S can be expressed as follows,
  - $\int_S \boldsymbol{F}(x, y, z) dS = \iint_D \boldsymbol{F}(x, y, g(x, y))\sqrt{p^2 + q^2 + 1}\, dxdy$
    - where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, D is projective of S to xy-coordinate



Surface integral in physics

- In scalar: $\int_S \rho(x,y,z)dS$
  - In the case ρ is a function of mass density on surface S
    - Its integral: total mass of surface S
- In vector: $\int_S \boldsymbol{v}(x,y,z) \cdot \boldsymbol{n} dS$
  - In the case $\boldsymbol{v}$ is a function of liquid velocity on surface S
    - Its integral: total amount of liquid flow per unit time

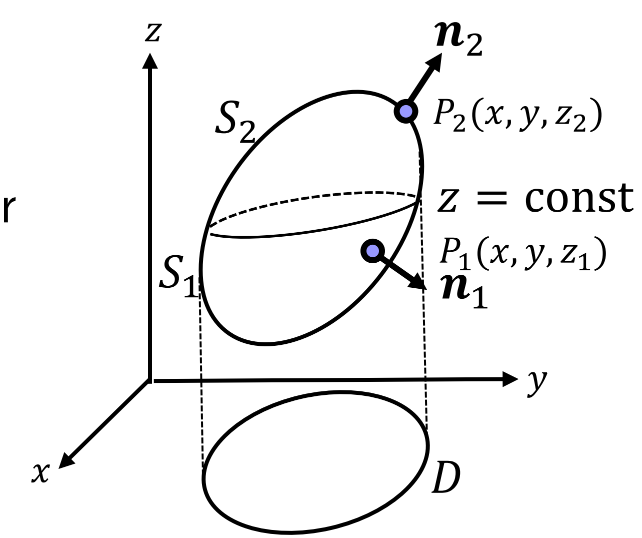


Volume integral

- Connect divergence on vector field and flow at the surface
  - Assume volume V surrounded by surface S
    - Volume integral of scalar f: $\int_V f(x,y,z)dV$
    - Volume integral of vector $\boldsymbol{F}$: $\int_V \boldsymbol{F}(x,y,z)dV$
      - $\int_V \boldsymbol{F}(x,y,z)dV = \boldsymbol{i}\int_V F_1 dV + \boldsymbol{j}\int_V F_2 dV + \boldsymbol{k}\int_V F_3 dV$
- Preliminary
  - For volume V surrounded by surface S, $\boldsymbol{n} = \cos\alpha\, \boldsymbol{i} + \cos\beta\, \boldsymbol{j} + \cos\gamma\, \boldsymbol{k}$, following equation satisfies,
  - $\int_V \frac{\partial f}{\partial x} dV = \int_S f \cos\alpha\, dS$, $\int_V \frac{\partial f}{\partial y} dV = \int_S f \cos\beta\, dS$, $\int_V \frac{\partial f}{\partial z} dV = \int_S f \cos\gamma\, dS$

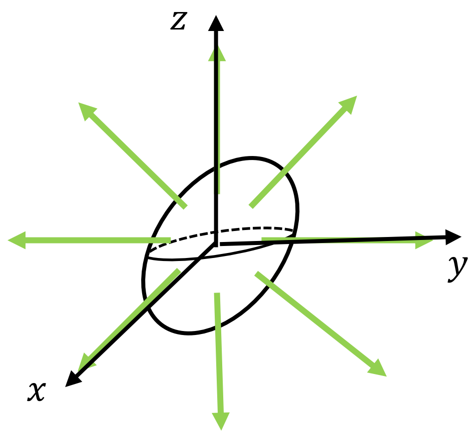
Volume integral (proof)

- Proof $\int_V \frac{\partial f}{\partial z} dV = \int_S f \cos\gamma \, dS$;
- Assume two points P_1 , P_2 on S
  - $z_2 \geq z_1$: z_2 coves upper side of S, z_1 coves lower side of S
  - $\int_V \frac{\partial f}{\partial z} dV$ means volume difference in z-axis, thus
- $\int_V \frac{\partial f}{\partial z} dV = \iiint_V \frac{\partial f}{\partial z} dxdydz = \iint_D \left\{\int_{z_1}^{z_2} \frac{\partial f}{\partial z} dz\right\} dxdy =$
  $\iint_D [f]_{z_1}^{z_2} dxdy = \iint_D \{f(x,y,z_2) - f(x,y,z_1)\} dxdy$
- For z-axis, z_2 is upper $(dS \cos\gamma = dxdy)$, z_1 is lower thus $(dS \cos\gamma = -dxdy)$
  - $\iint_D f(x,y,z_2) dxdy = \int_{S_2} f(x,y,z) dS$
  - $\iint_D f(x,y,z_1) dxdy = -\int_{S_1} f(x,y,z) dS$
  - $\int_V \frac{\partial f}{\partial z} dV = \int_{S_2} f \cos\gamma \, dS + \int_{S_1} f \cos\gamma \, dS = \int_S f \cos\gamma \, dS$

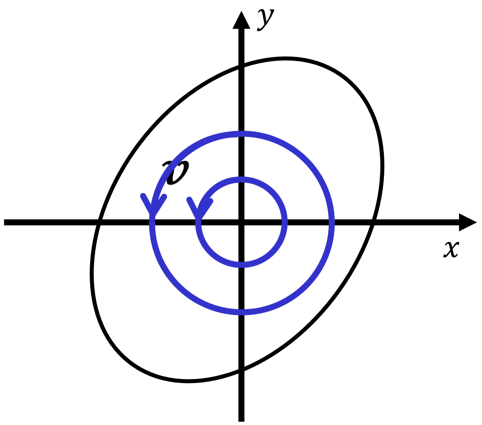


Divergence theorem (Gauss' theorem)

- Connect divergence on vector field and flow at the surface
  - Assume volume V surrounded by surface S w/ unit vec. n
    - $\int_V \operatorname{div} \boldsymbol{F} dV = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{F} dV = \int_S \boldsymbol{F} \cdot \boldsymbol{n} dS$
- Physical meaning
  - $\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS$: amount of flow which path through the area S
  - $\int_V \operatorname{div} \boldsymbol{F} dV$: amount of flow out



For $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$,
$\operatorname{div} \boldsymbol{r} = 3$

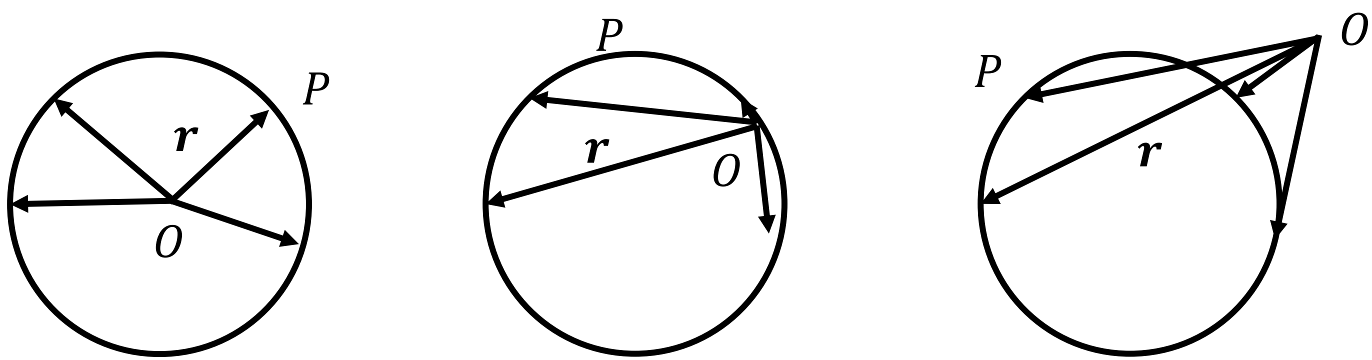


For $\boldsymbol{v} = -y\boldsymbol{i} + x\boldsymbol{j}$, and if volume V is outside of $\boldsymbol{v}$, $\operatorname{div} \boldsymbol{r} = 0$

Extension of Gauss' theorem

- Assume the point P on close surface S, express vector from origin $O(0,0,0)$ to P as $\overrightarrow{OP} = \boldsymbol{r}$, $\boldsymbol{n}$ is unit normal vector of S
- Following equation satisfy the following

- $$\int_S \frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{n} dS = \begin{cases} 0 & \text{(when } O \text{ is outside of } S\text{)} \\ 2\pi & \text{(when } O \text{ is on the surface } S\text{)} \\ 4\pi & \text{(when } O \text{ is inside of } S\text{)} \end{cases}$$



$$\int_S \frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{n} dS = 4\pi \qquad \int_S \frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{n} dS = 2\pi \qquad \int_S \frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{n} dS = 0$$

Exercise

- For function $f(x, y, z) = x^2 - yz + z^2$, calculate its curvilinear integral $\int_C f\, ds$
  - Case 1: C is a line from $P_1(1,2,0)$ to $P_2(1,2,3)$
  - Case 2: C is a line from $P_1(0,0,0)$ to $P_2(1,2,3)$
- Assume the surface func. $2x + 2y + z - 4 = 0$, and its intercepts are points A, B, C, and ABC create surface S
  - Calculate surface integral of $f(x, y, z) = 4x - y^2 + 2z - 12$

Exercise

- Assume the surface func. $x + y + z - 1 = 0$, and its intercepts are points P, Q, R and PQR create surface S
  - Calculate surface integral $\int_S \boldsymbol{F} \times \boldsymbol{n}\, dS$ for $\boldsymbol{F} = y\boldsymbol{k}$
- Assume the volume and surface of unit sphere as V, S, and $\boldsymbol{F} = ax\boldsymbol{i} + by\boldsymbol{j} + cz\boldsymbol{k}$. Calculate integral $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$

Sample solution

Math 7

① $f = x^2 - yz + z^2$, take $\int_C f\,ds$

(1-1) C is $P_1(1,2,0)$ to $P_2(1,2,3)$

$$\int_C f\,ds = \int_0^3 (1^2 - 2z + z^2)\,dz = \left[z - z^2 + \frac{z^3}{3}\right]_0^3 = 3_{/\!/}$$

(1-2) C is $P_1(0,0,0)$ to $P_2(1,2,3)$

$\Rightarrow x = t,\ y = 2t,\ z = 3t\ (0 \le t \le 1)$

$$ds = \sqrt{1^2 + 2^2 + 3^2}\,dt = \sqrt{14}\,dt$$

$$\int_C f(x(s) + y(s) + z(s))\,ds = \int_0^1 (t^2 - 6t^2 + 9t^2)\sqrt{14}\,dt$$

$$= 4\sqrt{14}\int_0^1 t^2\,dt = \frac{4\sqrt{14}}{3}_{/\!/}$$

② $S: 2x + 2y + z - 4 = 0$, $f = 4x - y^2 + 2z - 12$

Calculate $\int_S f\,dS$

Surface func $z = g(x,y) = 4 - 2x - 2y$

$$p = \frac{\partial z}{\partial x} = -2,\quad q = \frac{\partial z}{\partial x} = -2x$$

$$dS = \sqrt{p^2 + q^2 + 1}\,dx\,dy = 3\,dx\,dy$$

$$f(x, y, g(x,y)) = -(y+2)^2$$

$$\int_S f\,dS = -\iint_D (y+2)^2 \cdot 3\,dx\,dy$$

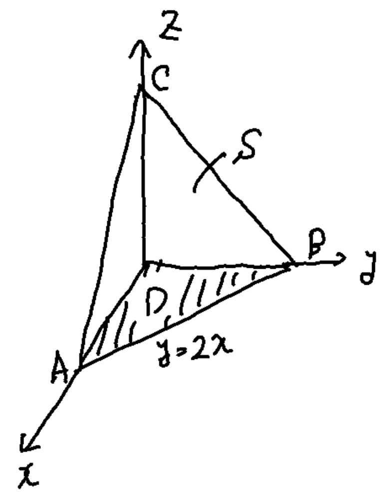
$$= -\underbrace{\int_0^2 \underbrace{\int_0^{2-x} 3(y+2)^2\,dy}_{y}\,dx}_{x}$$

$$= -\int_0^2 \left[(y+2)^3\right]_0^{2-x} dx$$

$$= -\int_0^2 \left((4-x)^3 - 8\right) dx$$

$$= \left[\frac{1}{4}(4-x)^3 + 8x\right]_0^2$$

$$= -44$$



Sample solution

③ $S: x+y+z-1=0,\ \mathbb{F}=y\mathbb{k},$ calc. $\int_S \mathbb{F}\times\mathbb{n}\,dS$

Surface func $S = g(x,y) = 1-x-y.$

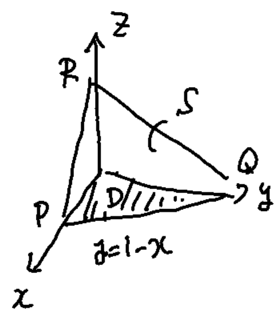
$$p=\frac{\partial z}{\partial x}=-1,\quad q=\frac{\partial z}{\partial y}=-1$$

unit normal vector of S is : $\mathbb{n}=\dfrac{\mathbb{i}+\mathbb{j}+\mathbb{k}}{\sqrt{3}}$

$$\mathbb{F}\times\mathbb{n}=\mathbb{i}\left(0\cdot\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}y\right)+\mathbb{j}\left(y\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}0\right)+\mathbb{k}\left(0\cdot\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}0\right)$$

$$=-\frac{y}{\sqrt{3}}(\mathbb{i}-\mathbb{j})$$

$dS=\sqrt{p^2+q^2+1}\,dxdy=\sqrt{3}\,dxdy$ thus



$$\int_S \mathbb{F}\times\mathbb{n}\,dS=-\iint_D \frac{y}{\sqrt{3}}(\mathbb{i}-\mathbb{j})\sqrt{3}\,dxdy$$

$$=-(\mathbb{i}-\mathbb{j})\int_0^1\int_0^{1-x} y\,dxdy$$

$$=-(\mathbb{i}-\mathbb{j})\int_0^1 \frac{(1-x^2)}{2}dx$$

$$=(\mathbb{i}-\mathbb{j})\left[\frac{(1-x^3)}{6}\right]_0^1=-\frac{1}{6}(\mathbb{i}-\mathbb{j})$$

④

Calculate $\int_S \mathbb{F}\cdot\mathbb{n}\,dS$ for $\mathbb{F}=ax\mathbb{i}+by\mathbb{j}+cz\mathbb{k}$

from divergence law

$$\int_S \mathbb{F}_S\cdot\mathbb{n}\,dS=\int_V \nabla\cdot\mathbb{F}\,dV$$

$$=\int_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dV$$

$$=\int_V (a+b+c)\,dV$$

$$=(a+b+c)\int_V dV$$

$$=\frac{4}{3}\pi(a+b+c)$$

$\left(\begin{array}{l}\text{Volume of unit sphere}\\ V=\frac{4}{3}\pi r^2,\ r=1.\end{array}\right)$

Fundamental Mathematics

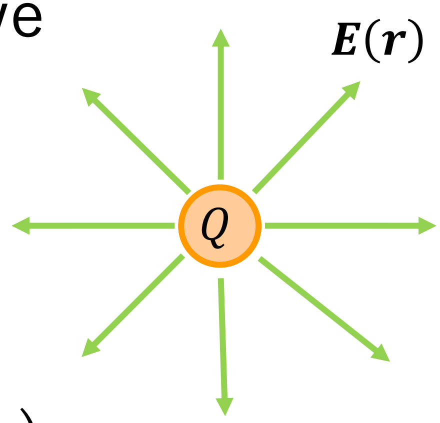
- Vector and Maxwell’s equation-

Motivation

- Many physics can be expressed by vectors
  - Good to explain in simple way (if we know vectors)
- Target: understand the meaning of Maxwell's equation
  - $\operatorname{div} \boldsymbol{D} = \rho$
    - $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho dV$ (Gauss's eq of electricfield)
  - $\operatorname{div} \boldsymbol{B} = 0$
    - $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \iiint \operatorname{div} \boldsymbol{B} dV$ (Gauss's eq of magneticfield)
  - $\operatorname{rot} \boldsymbol{H} = i + \frac{\delta D}{\delta t}$: $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(i + \frac{\delta \boldsymbol{D}}{\delta t}\right) \cdot d\boldsymbol{S}$ (Ampele's law)
  - $\operatorname{rot} \boldsymbol{E} = -\frac{\delta B}{\delta t}$: $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\delta}{\delta t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$ (Faraday's law)

Electron and Electric field

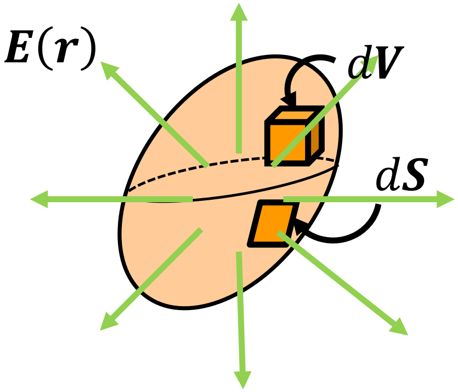
- Two electrons q_1 q_2 with distance r have attracting/repulsion force F
  - Coulomb's law $F = \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{r^2}$
    - $\epsilon = \epsilon_0 \epsilon_r$,
    - ϵ is dielectric constant (permittivity)
    - ϵ_0 is vacuum space permittivity $(=8.854 \times 10^{-12}\ \mathrm{C^2N^{-1}m^{-2}})$
    - ϵ_r is relative permittivity

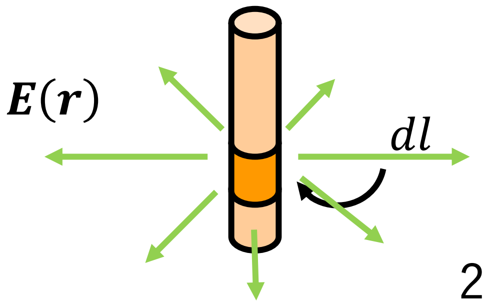


- Electron Q create vector field $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\frac{\boldsymbol{r}}{r}$
  - Coulomb's law in vector $\boldsymbol{F} = \frac{1}{4\pi\epsilon}\frac{qQ}{r^2}\frac{\boldsymbol{r}}{r} = q\boldsymbol{E}(\boldsymbol{r})$

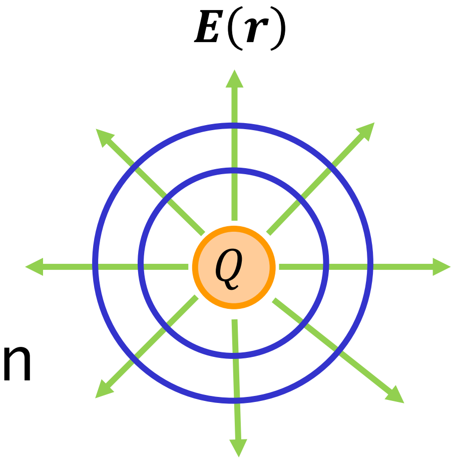
Electron and Electric field

- Electric field follows superposition law
  - For electron q_j for vector r_j $(j = 1, \dots, N)$
    - $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\sum_{j=1}^{N}\frac{(\boldsymbol{r}-\boldsymbol{r}_j)}{|\boldsymbol{r}-\boldsymbol{r}_j|^3}$
  - For continuous distribution of electron $\rho_s d\boldsymbol{V}$, where ρ_s: electron density
    - For volume $d\boldsymbol{V}$: $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\iiint\frac{\rho_s(\boldsymbol{r}-\boldsymbol{s})}{|\boldsymbol{r}-\boldsymbol{s}|^3}d\boldsymbol{V}$
    - For surface $d\boldsymbol{S}$: $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\iint\frac{\rho_s(\boldsymbol{r}-\boldsymbol{s})}{|\boldsymbol{r}-\boldsymbol{s}|^3}d\boldsymbol{S}$
    - For line dl: $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\int\frac{\eta_s(\boldsymbol{r}-\boldsymbol{s})}{|\boldsymbol{r}-\boldsymbol{s}|^3}dl$





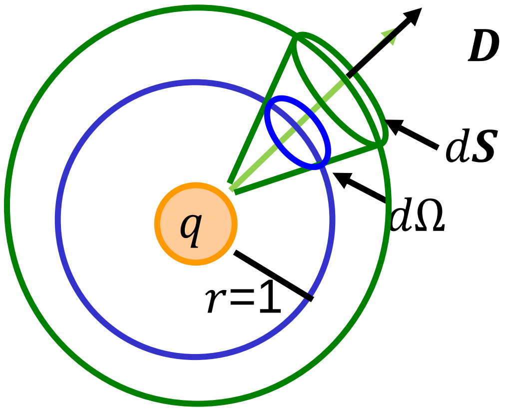
Electric flux



- Assume electron generate line of divergence
  - Electric flux (similar: electric line)
  - Electron Q generate Q-lines of electric flus
  - Density $\boldsymbol{D}$ should be changed by the position
    - $\boldsymbol{D} = \epsilon \boldsymbol{E}$
  - Assume area vector $\boldsymbol{S}$ with its unit normal vector $\boldsymbol{n}$
    - $\boldsymbol{S} = \boldsymbol{n}S$
  - Amount of electric flux penetrate area S
    - $\phi = \boldsymbol{D} \cdot \boldsymbol{S}$
  - For small area $d\boldsymbol{S}$
    - $d\phi = \boldsymbol{D} \cdot d\boldsymbol{S}$, $\phi = \iint d\phi = \iint \boldsymbol{D} \cdot d\boldsymbol{S}$

Gauss's law for electric field

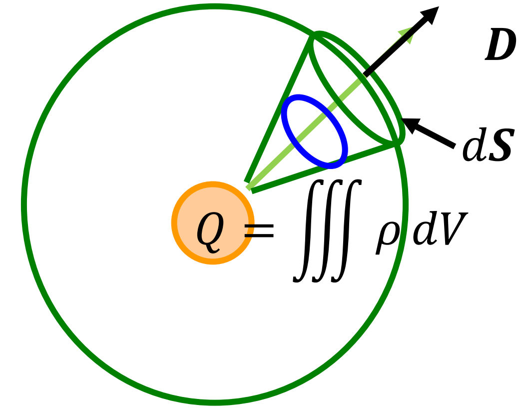
- Relationship between electric flux $d\phi$ generated by electron q, and its penetrating area $d\boldsymbol{S}$ on any shape
  - $d\phi = \boldsymbol{D} \cdot d\boldsymbol{S}$
- For the sphere w/ diameter of one, electric flux ratio should



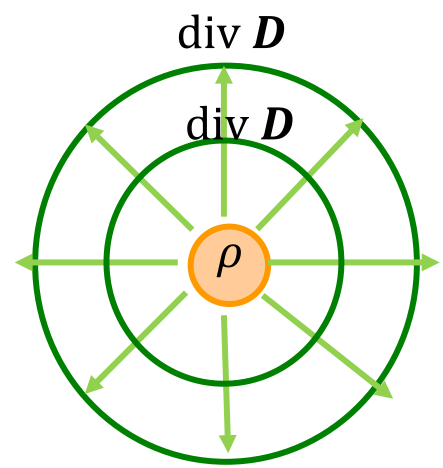
  - $q\frac{d\Omega}{4\pi} = \begin{cases} \boldsymbol{D} \cdot d\boldsymbol{S} & (\boldsymbol{r} \cdot d\boldsymbol{S} > 0) \\ -\boldsymbol{D} \cdot d\boldsymbol{S} & (\boldsymbol{r} \cdot d\boldsymbol{S} < 0) \end{cases}$
- If $\boldsymbol{r} \cdot d\boldsymbol{S} > 0$ (curve is convex)
  - $\phi = \iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iint q\frac{d\Omega}{4\pi} = q$
  - For volume w/ electron density ρ
    - $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho\, dV$

Physical meaning of Gauss's law for electric field

- $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho \, dV$ (integral from)
  - $\iint \boldsymbol{D} \cdot d\boldsymbol{S}$: Total amount of electric flux flow-outs from the surface
  - $\iiint \rho \, dV$: Total amount of electrons inside the volume



- $\text{div}\, \boldsymbol{D} = \rho$ (differential form)
  - $\text{div}\, \boldsymbol{D}$: divergence electric flux (density)
  - ρ : electron (density)

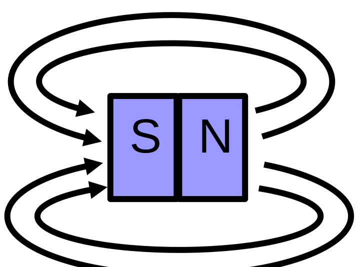


Magnetics and magnetic field

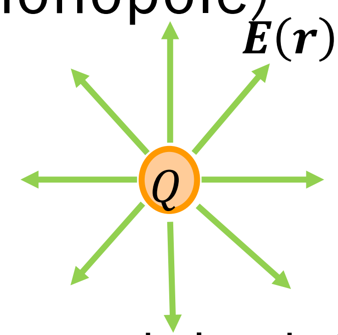
- Two amount of magnetics q_{m1} q_{m2} with distance r have attracting/repulsion force F
  - Coulomb's law for magnetics $F = \frac{1}{4\pi\mu}\frac{q_{m1}q_{m2}}{r^2}$
    - $\mu = \mu_0\mu_r$,
    - μ is magnetic permeability (permeability)
    - μ_0 is permeability in vacuum $(=4\pi \times 10^{-7}\ \mathrm{H/m})$
    - μ_r is relative permeability
- Monopole Q_m create magnetic field $\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\frac{Q_m}{r^2}\frac{\boldsymbol{r}}{r}$
  - Coulomb's law in vector $\boldsymbol{F} = \frac{1}{4\pi\epsilon}\frac{q_m Q_m}{r^2}\frac{\boldsymbol{r}}{r} = q_m\boldsymbol{H}(\boldsymbol{r})$

Gauss's law for magnetic field

- Magnetic pole of q_m generates q_m-lines of magnetic flux
  - Magnetic flux density $\boldsymbol{B}$ create magnetic field $\boldsymbol{H}$
    - $\boldsymbol{B} = \mu \boldsymbol{H}$
- Magnetic should in dipole (set of S and N, no monopole)
  - Same amount of flux from N to S
    - $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \iiint \mathrm{div}\, \boldsymbol{B} dV = 0$ (integral form)
    - $\mathrm{div}\, \boldsymbol{B} = 0$ (differential form)
- Gauss's law for magnetic field
  - No divergence in magnetic field (not monopole)



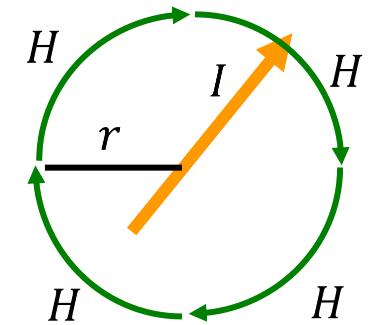
dipole of magnetics and magnetic flux



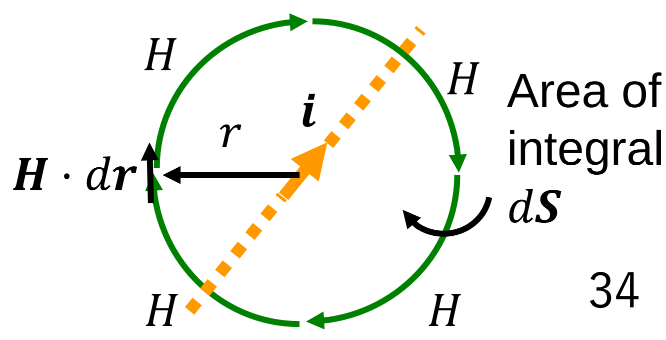
electron and electric flux

Magnetics and current flow

- Biot-Savart law: Constant current I create magnetic field H at the position of r
  - $H = \frac{I}{2\pi r}$
  - Right-hand turning (clockwise)

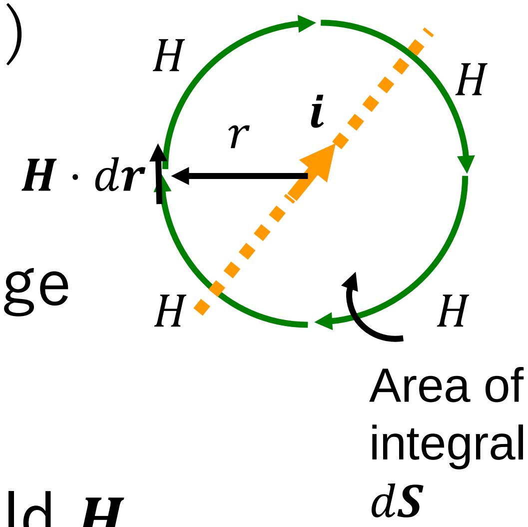


- Ampele’s law: relationship of current I and magnetic field H
    - $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = I$ (take integral of Biot-Savart law)
  - For the continuous current, use current density $\boldsymbol{i}$ then
    - $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \boldsymbol{i} \cdot d\boldsymbol{S}$
  - Include current change term
    - $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}\right) \cdot d\boldsymbol{S}$

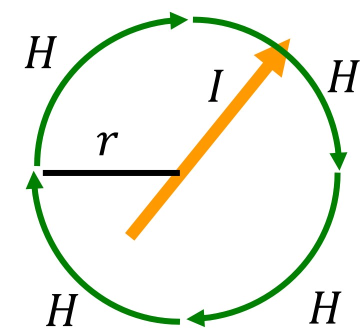


Physical meaning of Ampele’s law

- $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}\right) \cdot d\boldsymbol{S}$ (integral form)
  - $\iint \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}\right) \cdot d\boldsymbol{S}$: amount of constant current $\boldsymbol{i}$ and current change (=change in electric filed density $\frac{\partial \boldsymbol{D}}{\partial t}$) in area $\boldsymbol{S}$
  - $\oint \boldsymbol{H} \cdot d\boldsymbol{r}$: line integral of magnetic field $\boldsymbol{H}$

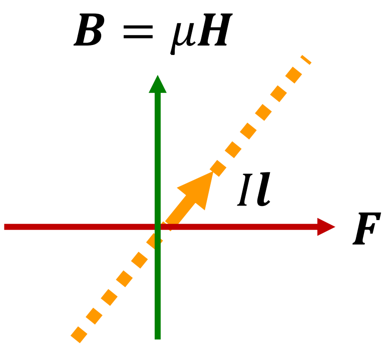


- rot $\boldsymbol{H} = i + \frac{\delta D}{\delta t}$
  - If rotating vector H exists, constant current $\boldsymbol{i}$ or current change $\frac{\partial \boldsymbol{D}}{\partial t}$ exists



Fleming's left-hand rule

- Current create magnetic flux
  - Behave like magnetic dipole
  - Constant current I in length $\boldsymbol{l}$ in uniform magnetic flux density $\boldsymbol{B}$ is force $\boldsymbol{F}$
    - $\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B}$ (outer product)

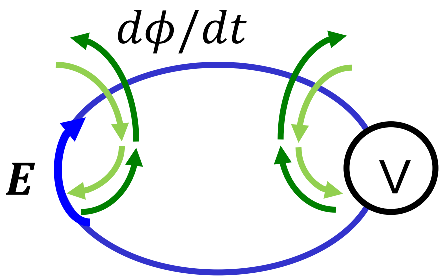


Lorentz force

- Fleming's left-hand rule: current I receive a force F from magnetic field density B
  - Moving electron receive a power by magnetic field
- Assume electrons qn with speed v, cross section of line S
  - $I = nqSv$ thus $\boldsymbol{F} = nqS\boldsymbol{v} \times \boldsymbol{B}l$
  - For one electron: $\boldsymbol{f} = q\boldsymbol{v} \times \boldsymbol{B}$: Lorentz force

Faraday's electromagnetic induction law

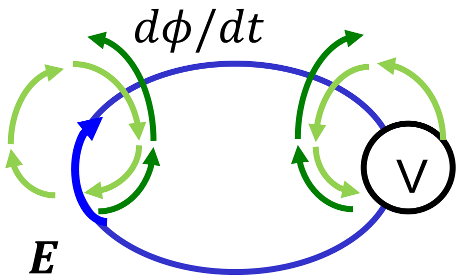
- Change of magnetic flux ϕ on inductor create electro motive force V
  - $V = -\frac{d\phi}{dt}$ (Faraday's electromagnetic induction law)
  - Change of magnetic flux:



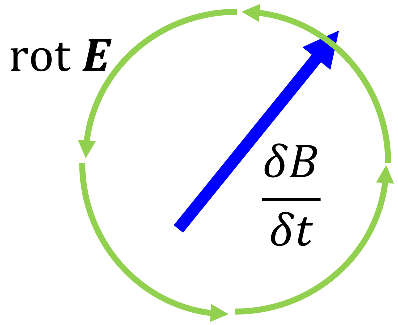
- Change of magnetic flux ϕ : try to create magnetic flux $-\phi$ to cancel out
  - Generate electromotive force $\boldsymbol{E}$ (Lenz's law)
- Total sum of electro motive force $V = \oint \boldsymbol{E} \cdot d\boldsymbol{r}$
- Total sum of magnetic flux $\phi = \iint \boldsymbol{B} \cdot d\boldsymbol{S}$
  - $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\partial}{\partial t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$

Physical meaning of Faraday's electromagnetic induction law

- $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\delta}{\delta t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$ (integral form)
  - $-\frac{\delta}{\delta t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$: amount of magnetic flux change in area S
  - $\oint \boldsymbol{E} \cdot d\boldsymbol{r}$: Total sum of electro motive force

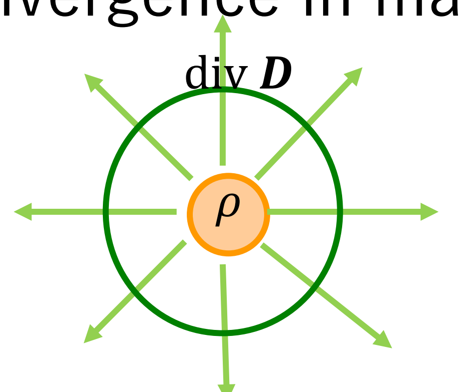


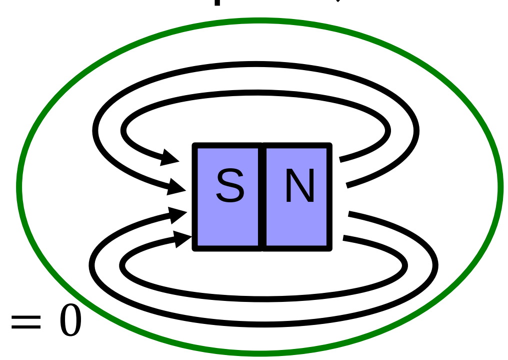
- $\mathrm{rot}\,\boldsymbol{E} = -\frac{\delta B}{\delta t}$ (differential form)
  - $\mathrm{rot}\,\boldsymbol{E}$: electro motive force exists in rotation
  - $-\frac{\delta B}{\delta t}$: amount of magnetic flux changes



Conclusion

- Understand the meaning of Maxwell's equation using vector form
  - Gauss's eq of electric field
    - $\mathrm{div}\,\boldsymbol{D} = \rho$, $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho dV$
    - If charge (density) ρ is available, it generate and diverge electric flux density $\boldsymbol{D}$
  - Gauss's eq of magnetic field
    - $\mathrm{div}\,\boldsymbol{B} = 0$, $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \iiint \mathrm{div}\,\boldsymbol{B} dV$
  - No divergence in magnetic field (not monopole)





$\mathrm{div}\,\boldsymbol{E} = 0$

Conclusion

- Ampele's law
  - rot $\boldsymbol{H} = i + \frac{\delta D}{\delta t}$: $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(i + \frac{\delta \boldsymbol{D}}{\delta t}\right) \cdot d\boldsymbol{S}$
  - If constant current $\boldsymbol{i}$ or current change $\frac{\partial \boldsymbol{D}}{\partial t}$ exists, it generate magnetic filed $\boldsymbol{H}$ as rotating vector
- Faraday's law
  - rot $\boldsymbol{E} = -\frac{\delta B}{\delta t}$: $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\delta}{\delta t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$
  - If $-\frac{\delta B}{\delta t}$ amount of magnetic flux changes, it generate electromotive force $\boldsymbol{E}$ as rotating vector
- nishizawa@aoni.waseda.jp