Fundamental Mathematics (Engineering Mathematics)

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Differential equation

- Differential equation for engineering
  - Express natural behaviors (physics, electrics) as equation
    - Handle simple models
    - Easy to understand
    - Show very primitive “solution”

Linear differential equation

- Differential equation defined by linear polynomial in the unknown function and its derivatives
  - $y' + ay = 0$ (a is constant) (eq.1.1)
    - If $y(x)$ satisfy above, $y(x)$ is called as solution
    - Solve above equation to obtain y(x)
    - Before that, y is called as unknown function

  - How to solve the equation?
    - Algebraically
      - Formula
      - Use assumption (Method of undetermined multiplier)
    - Program solver (Mathematica, Maxima)

Linear differential equation (cont.)

- Mission: solve function $y' + ay = 0$ (a is constant) (eq.1.1)
  - Use nature of exponential function
    - $(e^{ax})' = ae^{ax}$
  - Multiple e^{ax} to (eq.1.1)
    - $e^{ax}y' + ae^{ax}y = 0$
  - Recall the differential for products
    - $\left(g(x)h(x)\right)' = g(x)h(x)' + g(x)'h(x)$
  - (e.q.1.1) should be
    - $(e^{ax}y)' = 0.$ -> $y = ce^{-ax}$(c is <u>arbitrary constant</u>)
    - Solution w/o constant: <u>a general solution</u>
    - If c has some specific value -> <u>a particular solution</u>

Initial value problem

- Shape of function depends arbitrary constant
  - We may don't know the arbitrary constant itself
  - We may know the value(y_0) on specific point(x_0)
    - y_0: <u>Initial value</u> or <u>initial condition</u>
  - e.x. $y' + ay = 0,\ y_0 = y(x_0) = ce^{-ax_0}$
    - $c = y_0 e^{ax_0}$

  - General form of (eq.1.1) should be
    - $y = y_0 e^{-a(x-x_0)}$

Homogeneous differential eq.

- Think about following const. coeff. diff. equation
  - $y' + ay = r(x)$ (eq.1.7)
- A differential equation is homogeneous when
  - $f(x, y)dy = -g(x, y)dx$ -> $f(x, y) + g(x, y)\frac{dx}{dy} = 0$
  - If $r(x) = 0$, eq.1.7 is homogeneous
- If not, a differential equation is inhomogeneous
  - If $r(x) \neq 0$, eq.1.7 is inhomogeneous
- A general solution of inhomogeneous function eq.1.7 is
  - $y = \left(\int r(x)e^{ax}dx + c\right)e^{-ax}$ (c is constant)
  - Somewhat difficult (we'll introduce more easy way)

Homogeneous differential eq.

- Multiple e^{ax} to (eq.1.7)
  - $(y' + ay)e^{ax} = r(x)e^{ax}$ -> $(ye^{ax})' = r(x)e^{ax}$
- Take integral
  - $ye^{ax} = \int r(x)e^{ax}dx + c$ (c is constant)
  - $y = \left(\int r(x)e^{ax}dx + c\right)e^{-ax}$

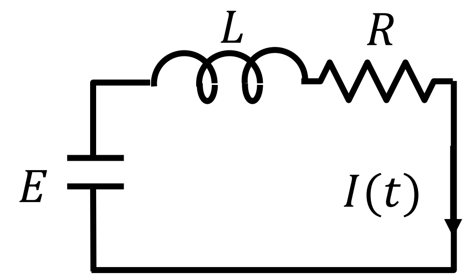
Homogeneous differential eq.

- Similarly, variable coeff. diff. equation
  - $y' + f(x)y = r(x)$ (eq.1.14)
- Homogeneous case
  - $y' + f(x)y = 0$(eq.1.15)
  - Assume $F(x)$ as primitive function of $f(x)$
    - $(e^{F(x)})' = e^{F(x)}F'(x) = f(x)e^{F(x)}$
  - Multiply $e^{F(x)}$ to eq.1.15
    - $e^{F(x)}y' + e^{F(x)}f(x)y = \underbrace{e^{F(x)}y' + (e^{F(x)})'y = (e^{F(x)}y)'}_{\textbf{Leibniz product rule}} = 0$
  - Thus: $e^{F(x)}y = c$ (c is constant)
- General solution for homogeneous eq. $y' + f(x)y = 0$
  - $y = ce^{-F(x)}$

Homogeneous differential eq.

- Inhomogeneous case
  - $y' + f(x)y = r(x)$ (eq.1.14)
  - Multiply $e^{F(x)}$ to eq.1.14
    - $e^{F(x)}y' + e^{F(x)}f(x)y = (e^{F(x)}y)' = e^{F(x)}r(x)$
  - Take integral
    - $e^{F(x)}y = \left(\int r(x)e^{F(x)}dx + c\right)$
- General solution for inhomogeneous eq. $y' + f(x)y = r(x)$
  - $y = \left(\int r(x)e^{F(x)}dx + c\right)e^{-F(x)}$

Example: RL circuit



- Derive current $I(t)$ of RL circuit
  - Initial condition: $I(0) = 0$
- Voltage of R (V_R) and L (V_L) are:
  - $V_R = RI(t)$, $V_L = L\frac{dI(t)}{dt}$, $E = V_R + V_L$, thus $\frac{dI(t)}{dt} + \frac{R}{L}I(t) = \frac{E}{L}$
  - Equation is same as eq.1.7 ->
  - From general solution: $I(t) = \left(\int \frac{E(t)}{L} e^{\left(\frac{R}{L}\right)t} dt + c\right) e^{-a\left(\frac{R}{L}\right)t}$,
  - E is constant: $I(t) = \frac{E}{R} + ce^{-a\left(\frac{R}{L}\right)t}$
  - Apply initial condition, and final result should be
    - $I(t) = \frac{E}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$

Fundamental Mathematics
- Differential equations 2 -

2023/10/17

Solution of differential equation

- Constant coefficient 1st order differential equation
  - Homogeneous: $y' + ay = 0$ <-> $y = ce^{-ax}$ (c is constant)
  - Inhomogeneous:
    - $y' + ay = r(x)$ <->$y = \left(\int r(x)e^{ax}dx + c\right)e^{-ax}$
- Variable coefficient 1st order differential equation
  - Homogeneous: $y' + f(x)y = 0$ <-> $y = ce^{-Fx}$
  - Inhomogeneous:
    - $y' + f(x)y = r(x)$ <->$y = \left(\int r(x)e^{F(x)}dx + c\right)e^{-F(x)}$
- We can calculate, but time consuming…
  - Too tough to solve 2nd order differential equation

Solution of differential equation

- There are many way to solve inhomogeneous differential equation for engineering mathematics
  - Note: These solutions cannot solve all of the differential equations
  - Use some assumptions, but useful enough for engineering

- Variation of constants (定数変化法)
- Method of indeterminate coefficient (未定係数法)

Variation of constants method

- Can solve linear (inhomogeneous) differential equation
  - Difficulty to solve high order equation
  - Equation becomes complex for high order equation

- Strategy

1. Change given inhomogeneous equation to homogeneous
2. Solve general solution for the homogeneous equation
3. Replace constant c to function $u(x)$
4. Substitute $u(x)$ to given inhomogeneous equation
   - Calculate general solution of $u(x)$
5. Substitute $u(x)$ to solution of homogeneous equation

Variation of constants method (cont.)

- Example: get general solution of : $y' + f(x)y = r(x)$ (eq.1.7)

1. Change given inhomogeneous equation to homogeneous
   - $y' + f(x)y = 0$
2. Solve general solution for the homogeneous equation
   - Use this relationship: $\left(e^{F(x)}\right)' = e^{F(x)}F'(x) = e^{F(x)}f(x)$
   - $e^{F(x)}y' + e^{F(x)}f(x)y = e^{F(x)}y' + e^{F(x)}\left(e^{F(x)}\right)'y = 0$
   - thus, $\left(e^{F(x)}y\right)' = 0,$ ➡ $e^{F(x)}y = c$ (c is constant)
3. Replace constant c to function $u(x)$
   - $y = ce^{-F(x)}$ ➡ $y = u(x)e^{-F(x)}$ ➡ $u(x) = ye^{+F(x)}$

Variation of constants method (cont.)

- Example: get general solution of : $y' + f(x)y = r(x)$ (eq.1.7)

4. Substitute $u(x)$ to given inhomogeneous equation
   - $\left(u(x)e^{-F(x)}\right)' + f(x)u(x)e^{-F(x)} = r(x)$
   - $u'(x)e^{-F(x)} + u(x)e^{-F(x)}(-f(x))' + f(x)u(x)e^{-F(x)} = r(x)$
   - $u'(x) = r(x)e^{+F(x)}$ ➡ $u(x) = \int r(x)e^{+F(x)}dx + C$ (C is const.)

5. Substitute $u(x)$ to solution of homogeneous equation
   - $y = u(x)e^{-F(x)} = \left(\int r(x)e^{+F(x)}dx + c\right)e^{-F(x)}$

Method of indeterminate coefficient

- With some assumptions, we can easily solve differential equation
  - Guess the candidate of particular solution
  - If the right side of an equation is…
    - n-order polynormal: candidate should be n-polynormal
    - sine function: candidate should be in sine
    - exponential: candidate should be in exponential

Method of indeterminate coefficient (polynormal)

- Example: get general solution of : $y' + 3y = x^2 - 1$(eq.1.23)
  - Assume particular solution is $y_p = \alpha x^2 + \beta x + \gamma$
    - α, β, γ are constant. Substitute y_p to eq.1.23
    - $y_p' + 3y_p = (2\alpha x + \beta) + 3(\alpha x^2 + \beta x + \gamma) = x^2 - 1$
  - This equation should satisfy following conditions
    - x^2: $3\alpha = 1$, x^1: $2\alpha + 3\beta = 0$, x^0: $\beta + 3\gamma = -1$, thus
  - $y_p = \frac{1}{3}x^2 - \frac{2}{9}x - \frac{7}{27}$

Structure of solution

- If one particular solution is clear, general solution can be easily solved.
- Example: get general solution of : $y' + f(x)y = r(x)$ (eq.1.7)
  - Assume particular solution y_p, general solution y, and its difference $y_h = y - y_p$. eq.1.7 is
    - $y_h' + f(x)y_h = \left(y - y_p\right)' + f(x)\left(y - y_p\right)$
    - $= y' + f(x)y - \left(y_p{}' + f(x)y_p\right) = r(x) - r(x) = 0$
    - This is homogeneous: $y_h = ce^{-F(x)}$
  - $y = y_p + y_h = y_p + ce^{-F(x)}$ (c is constant)
    - We can use this as theorem
  - general solution of eq.1.23: $y = \frac{1}{3}x^2 - \frac{2}{9}x - \frac{7}{27} + ce^{-3x}$

Method of indeterminate coefficient(sine)

- Example: get general solution of : $y' + 2y = \cos x$(eq.1.25)
  - Assume particular solution is $y_p = \alpha \cos x + \beta \sin x$
    - $\alpha, \beta$are constant. Substitute y_p to eq.1.25
    - $y_p' + 2y_p = -\alpha \sin x + \beta \cos x + 2(\alpha \cos x + \beta \sin x) = \cos x$
    - This equation should satisfy following conditions
    - $\cos x$: $2\alpha + \beta = 1$, $\sin x$: $-\alpha + 2\beta = 0$, thus
  - $y_p = \frac{2}{5} \cos x + \frac{1}{5} \sin x$
  - $y = \frac{2}{5} \cos x + \frac{1}{5} \sin x + ce^{-2x}$

Method of indeterminate coefficient(exponent)

- Example: get general solution of : $y' - y = 2e^{2x}$(eq.1.28)
  - Assume particular solution is $y_p = \alpha e^{2x}$
    - α is constant. Substitute y_p to eq.1.28
    - $y_p' - y_p = 2\alpha e^{2x} - \alpha e^{2x} = \alpha e^{2x} = 2e^{2x}$, thus
  - $y_p = 2e^{2x}$
  - $y = 2e^{2x} + ce^{x}$
  - However, this is not true for all of solution

Method of indeterminate coefficient (exponent) (cont.)

- Example: get general solution of : $y' - 2y = 2e^{2x}$(eq.1.29)
  - Assume particular solution is $y_p = \alpha e^{2x}$
    - α is constant. Substitute y_p to eq.1.29
    - $y_p' - y_p = 2\alpha e^{2x} - 2\alpha e^{2x} = 0$, ?? ➡ wrong assumption
  - Assume particular solution is $y_p = \alpha x e^{2x}$
    - α is constant. Substitute y_p to eq.1.29
    - $y_p' - y_p = (\alpha e^{2x} + 2\alpha x e^{2x}) - 2\alpha x e^{2x} = 2e^{2x}$
    - $y_p = 2xe^{2x}$
    - $y = (2x + c)e^{2x}$
  - If general solution is $y' + ay = ke^{-ax}$, particular solution should be $y_p = kxe^{-ax}$

Exercise (1)

- Solve general solutions for following equations
  - by Variation of constants method
    - $y' - xy = x$
    - $y' + \frac{y}{x} = x^2 + 2x$
  - by Method of indeterminate coefficient
    - $2y' + 3y = 3x^2 + x$
    - $y' + 4y = 3e^{-x}$

Euler's formula

- The trigonometric functions (sin cos) and complex exponential function satisfy following relationship
  - $\underline{e^{ix} = \cos x + i \sin x}$
  - e: base of natural logarithm, i(or j): imaginary unit
- Euler's formula is useful for circuit analysis, cause…
  - Easy to take integral, differential
    - $\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}, \int e^{\lambda x}\, dx = \frac{1}{\lambda} e^{\lambda x} + c$ (c constant)
  - <u>Phasor: </u>expression of sine func. in complex exponent
    - $A \cos \omega x = Re\{A \cos \omega x + iA \sin \omega x\} = Re\{Ae^{i\omega}\}$
    - Calculate circuit in complex exponent, then convert to original sine functions

2^{nd} order differential equation

- Introduce 2^{nd} order differential equation
  - $y'' + ay' + by = r(x)$ (a, b are constants) (eq.3.1)
    - If $r(x) = 0$, eq.3.1 is homogeneous
    - If $r(x) \neq 0$, eq.3.1 is inhomogeneous
- Inhomogeneous form is very tough for hand calculation
  - If $r(x)$ is constant, sine, or exponential we can use method of indeterminate coefficient
    - In physics, circuits, we can use this assumption

Characteristic equation

- If $r(x) = 0$ and $y(x) = ce^{\lambda x}$ (c, λ: constant), eq 3.1 is
  - $y'' + ay' + by = (\lambda^2 + a\lambda + b)ce^{\lambda x} = 0$, $ce^{\lambda x} \neq 0$ thus
    - $\underline{\lambda^2 + a\lambda + b = 0\text{: characteristic equation}}$
  - Solutoin and $\lambda = \frac{-\mathrm{a} \pm \sqrt{a^2 - 4b}}{2}$ changes depend on discriminant function $(a^2 - 4b)$
    - $a^2 - 4b > 0$: λ_1, λ_2 in real. Solutions: $c_1 e^{\lambda_1 x}, c_2 e^{\lambda_2 x}$
    - $a^2 - 4b = 0$: $\lambda = -\frac{a}{2}$. Solutions: $c_1 e^{\lambda x}, c_2 x e^{\lambda x}$
    - $a^2 - 4b < 0$: λ_1, λ_2 in imaginary value.
      - $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$
      - Solutions: $c_1 e^{\lambda_1 x}, c_2 e^{\lambda_2 x}$

Linearity of solution

- Use linearity of solution
- Theorem: If $y(x)$and $w(x)$ are the solution of linear equation (eq.3.1), sum $c_1y(x) + c_2w(x)$ is also the solution
- Proof: since $y(x)$ and $w(x)$ are solution, it should satisfy
  - $y'' + ay' + by = 0,\ w'' + aw' + bw = 0,$
  - Multiply const c_1 and c_2 and get its sum
    - $c_1y'' + c_1ay' + c_1by + c_2w'' + c_2aw' + c_2bw = 0$
  - $(c_1y + c_2w)'' + a(c_1y + c_2w)' + b(c_1y + c_2w) = 0$
  - So, $c_1y(x) + c_2w(x)$ is also the solution

- Solution is the sum of exponents, comes from characteristic equation

General solution

- Theorem: General solution of 2nd order homogeneous differential equation is
  - $a^2 - 4b = 0$: $y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$ λ_1: multiple root of char. eq.
  - $a^2 - 4b \neq 0$: $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ λ_1 λ_2 : roots of char. eq.
- Proof: if $y(x)$ is the solution of eq.3.1, multiply $e^{-\lambda x}$
  - $e^{-\lambda x} y'' + e^{-\lambda x} a y' + e^{-\lambda x} b y = 0$
  - $(e^{-\lambda x} y)'' + (a + 2\lambda)(e^{-\lambda x} y)' + (\lambda^2 + a\lambda + b) e^{-\lambda x} y = 0$
    - If we assume λ_1 is root of char. eq., $(\lambda_1^2 + a\lambda_1 + b) = 0$, thus
    - $(e^{-\lambda x} y)'' + (a + 2\lambda_1)\left(e^{-\lambda x} y\right)' = 0$
    - $u'' + (a + 2\lambda_1) u' = 0$, when $e^{-\lambda_1 x} y(x) = u(x)$

General solution (cont.)

- $u'' + (a + 2\lambda_1)u' = 0$, when $e^{-\lambda_1 x} y(x) = u(x)$
  - Case $(a^2 - 4b = 0)$: $\lambda = -\frac{a}{2}$, thus $u'' = 0$
    - $u(x) = c_1 + c_2 x$, thus $y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$
  - Case $(a^2 - 4b \neq 0)$:
    - $v' + (a + 2\lambda_1)v = 0$, when $v = u'$, solve this then
    - $v = Ce^{-(a+2\lambda_1)x}$, C is constant. Then integrate this
      - $u(x) = c_1 - \frac{C}{a+2\lambda_1} e^{-(a+2\lambda_1)x}$, thus
      - $y(x) = c_1 e^{\lambda_1 x} - \frac{C}{a+2\lambda_1} e^{-(a+\lambda_1)x}$, since $(a + 2\lambda_1)$ is the solution λ_2
    - $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ $\left(c_2 = -\frac{C}{a+2\lambda_1}, \lambda_2 = a + 2\lambda_1\right)$

Constants and sine/exp. transformation

- Now we get a general solution
  - For particular solution, we need to fix constants
    - Use initial value or boundary value
- Transform from/to sine to/from exponent
  - Case $(a^2 - 4b < 0)$:
  - $y(x) = c_1 e^{(A+iB)x} + c_2 e^{(A-iB)x}$
  - $y(x) = c_1 e^{Ax}(\cos Bx + i \sin Bx) + c_2 e^{Ax}(\cos Bx - i \sin Bx)$
  - $\qquad = (c_1 + c_2) e^{Ax} \cos Bx + i(c_1 - c_2) e^{Ax} \sin Bx$
  - $\qquad = d_1 e^{Ax} \cos Bx + d_2 e^{Ax} \sin Bx$
- We can use both sine or exponential
  - Exponential is useful to take differential

Exercise (2)

- Solve characteristic equation and general solutions for following equations
  - by Method of indeterminate coefficient
    - $y'' + 2y' + y = 0$
    - $y'' + 2y' + 3y = 0$
    - $y'' - 4y' - 5y = 0$

Sample solutions

Ex1.

① by variation of const

(1) $y' - xy = x$

① Think homogeneous eq.

$y' - xy = 0$

② Solve general solution of ①

$f(x) = -x, \quad F(x) = -\frac{x^2}{2}$

$y = c_1 e^{-F(x)} = c_1 e^{\frac{x^2}{2}}$ (c_1 is const)

③ Replace $c_1 \to u(x)$

$y = u(x) e^{\frac{x^2}{2}}, \quad u(x) = y e^{\frac{-x^2}{2}}$

④ substitute $u(x)$ to given inhomogeneous eq.

$y' - xy = x$

$(u(x) e^{x^2/2})' - x(u(x) e^{x^2/2}) = x$

$u'(x) e^{x^2/2} + x u(x) e^{x^2/2} - x u(x) e^{x^2/2} = x$

$u'(x) = x e^{-x^2/2}$

$u(x) = \int x e^{-x^2/2} dx$ change param $-\frac{x^2}{2} = t, \ -x dx = dt$

$= \int e^{t} \cdot (-dt) = -e^{t} + C_2$ (C_2: const)

⑤ Substitute $u(x)$ to the solution of homogeneous eq.

$y = (-e^{-\frac{x^2}{2}} + C_2) e^{\frac{x^2}{2}} = -1 + C_2 e^{\frac{x^2}{2}}$ #

(2) $y' + y/x = x^2 + 2x$

① homogeneous eq.: $y' + \frac{y}{x} = 0$

② general solution: $f(x) = 1/x, \quad F(x) = \log x$

$y = C_1 e^{-\log x}$ (C_1: const)

③ $C_1 \to u(x)$

$y = u(x) e^{-\log x} = \frac{u(x)}{x}$ $\quad (e^{\log x} = x)$

④ substitute $u(x)$ to given inhomogeneous eq.

$\left(\frac{u(x)}{x}\right)' + \frac{u(x)}{x^2} = x^2 + 2x$

$\frac{u'(x)}{x} - \frac{u(x)}{x^2} + \frac{u(x)}{x^2} = x^2 + 2x$

$u'(x) = x^3 + 2x^2$

$u(x) = \int (x^3 + 2x^2) dx = \frac{x^4}{4} + \frac{2}{3}x^2 + C_2$ (C_2: const)

⑤ substitute to ③

$y = \frac{x^3}{4} + \frac{2}{3}x^2 + \frac{C_2}{x}$

[2] indeterminate coeff.

(3) $2y' + 3y = 3x^2 + x$.

① Particular solution $y_p = \alpha x^2 + \beta x + \gamma$

② Substitute

$$2(2\alpha x + \beta) + 3(\alpha x^2 + \beta x + \gamma) = 3x^2 + x$$

$$3\alpha = 3,\quad 4\alpha + 3\beta = 1,\quad 2\beta + 3\gamma = 0$$

$$\alpha = 1,\quad \beta = -1,\quad \gamma = \frac{2}{3}$$

④ calc $F(x)$

$$y' + \frac{3}{2}y = \frac{3}{2}x^2 + \frac{x}{2},\quad f(x) = \frac{3}{2},\quad F(x) = \frac{3}{2}x$$

⑤ calc solutions.

particular solution $y_p = x^2 - x + \frac{2}{3}$

general 〃 $y = x^2 - x + 2/3 + Ce^{-\frac{3}{2}x}$

(C: const)

(4) $y' + 4y = 3e^{-x}$

$y_p = \alpha e^{-x}$ substitute

$-\alpha e^{-x} + 4\alpha e^{-x} = 3e^{-x}$, $\alpha = 1$

$f(x) = 4$, $F(x) = 4x$

$y_p = e^{-x}$, $y_g = e^{-x} + Ce^{-4x}$ (C: const)

[3] solve characteristic eq. and general solutions.

(4) $y'' + 2y' + y = 0$

assume $y = Ce^{\lambda x}$ (C, λ : unknown), substitute

$$\lambda^2 Ce^{\lambda x} + 2\lambda Ce^{\lambda x} + Ce^{\lambda x} = 0$$

$(\lambda^2 + 2\lambda + 1)Ce^{\lambda x} = 0$. $Ce^{\lambda x} \neq 0$ thus

characteristic eq $\underline{\lambda^2 + 2\lambda + 1} = (\lambda + 1)^2 = 0 \to \lambda = -1$.

general solution $y = \underline{C_1 e^{-x} + C_2 x e^{-x}}$ (C_1, C_2 : const)

(5) $y'' + 2y' + 3y = 0$

assume $y = Ce^{\lambda x}$

$(\lambda^2 + 2\lambda + 3)Ce^{\lambda x} = 0$ characteristic eq $\underline{\lambda^2 + 2\lambda + 3 = 0}$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \times 3}}{2} = \frac{-2 \pm 2\sqrt{2}i}{2} = -1 \pm \sqrt{2}i$$

general solution $\underline{y = C_1 e^{-(1+\sqrt{2}i)x} + C_2 e^{-(1-\sqrt{2}i)x}}$ (C_1, C_2 : const)

(6) $y'' - 4y' - 5y = 0$

assume $y = Ce^{\lambda x}$

characteristic eq: $\underline{\lambda^2 - 4\lambda - 5 = 0} = (\lambda - 5)(\lambda + 1)$

general solution $\underline{y = C_1 e^{5x} + C_2^{-x}}$ $\lambda = 5, -1$

(C_1, C_2 : const)

Conclusion

- Introduction of solve inhomogeneous differential equation for engineering mathematics
  - These solutions cannot solve all of the differential equations
  - Use some assumptions, but useful enough for engineering

- Variation of constants
- Method of indeterminate coefficient