Fundamental Mathematics (Engineering Mathematics)

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Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4,5)
- Vector analysis (#6,7)
- Complex function theory (#8,9)
- Fourier transform (#10,11)
- Laplace transform (#12,13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

Euler's formula

- The trigonometric functions (sin cos) and complex exponential function satisfy following relationship
  - $\underline{e^{ix} = \cos x + i \sin x}$
  - e: base of natural logarithm, $i(\mathrm{or}\ j)$: imaginary unit
- Euler's formula is useful for circuit analysis, cause…
  - Easy for integral, differential
    - $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}, \int e^{\lambda x}\,dx = \frac{1}{\lambda}e^{\lambda x} + c$ (c constant)
  - <u>Phasor:</u> expression of sine func. in complex exponent
    - $A\cos\omega x = Re\{A\cos\omega x + iA\sin\omega x\} = Re\{Ae^{i\omega x}\}$
    - Calculate circuit in complex exponent, then convert to original sine functions

2^{nd} order differential equation

- Introduce 2^{nd} order differential equation
  - $y'' + ay' + by = r(x)$ (a, b are constants) (eq.3.1)
    - If $r(x) = 0$, eq.3.1 is homogeneous
    - If $r(x) \neq 0$, eq.3.1 is inhomogeneous
- Inhomogeneous form is very tough for hand calculation
  - If $r(x)$ is constant, sine, or exponential we can use method of indeterminate coefficient
    - In physics, circuits, we can use this assumption

Characteristic equation

- If $r(x) = 0$ and $y(x) = ce^{\lambda x}$ (c, λ: constant), eq 3.1 is
  - $y'' + ay' + by = (\lambda^2 + a\lambda + b)ce^{\lambda x} = 0$, $ce^{\lambda x} \neq 0$ thus
    - $\underline{\lambda^2 + a\lambda + b = 0\text{: characteristic equation}}$

  - Solutoin and $\lambda = \frac{-\mathrm{a} \pm \sqrt{a^2 - 4b}}{2}$ changes depend on $\cdots$
    - $a^2 - 4b > 0$: λ_1, λ_2 in real. Solutions: $c_1 e^{\lambda_1 x}, c_2 e^{\lambda_2 x}$
    - $a^2 - 4b = 0$: $\lambda = -\frac{a}{2}$. Solutions: $c_1 e^{\lambda x}, c_2 x e^{\lambda x}$
    - $a^2 - 4b < 0$: λ_1, λ_2 in imaginary value.
      - $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$
      - Solutions: $c_1 e^{\lambda_1 x}, c_2 e^{\lambda_2 x}$

Linearity of solution

- Use linearity of solution
- Theorem: If $y(x)$and $w(x)$ are the solution of linear equation (eq.3.1), sum $c_1 y(x) + c_2 w(x)$ is also the solution
- Proof: since $y(x)$ and $w(x)$ are solution, it should satisfy
  - $y'' + ay' + by = 0,\ w'' + aw' + bw = 0,$
  - Multiply const c_1 and c_2 and get its sum
    - $c_1 y'' + c_1 ay' + c_1 by + c_2 w'' + c_2 aw' + c_2 bw = 0$
  - $(c_1 y + c_2 w)'' + a(c_1 y + c_2 w)' + b(c_1 y + c_2 w) = 0$
  - So, $c_1 y(x) + c_2 w(x)$ is also the solution

- Solution is the sum of exponents, comes from characteristic equation

General solution

- Theorem: General solution of 2nd order homogeneous differential equation is
  - $a^2 - 4b = 0$: $y(x) = c_1 e^{\lambda_1 x} + c_2 \boldsymbol{x} e^{\lambda_1 x}$ λ_1: multiple root of char. eq.
  - $a^2 - 4b \neq 0$: $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ λ_1 λ_2 : root of char. eq.
- Proof: if $y(x)$ is the solution of eq.3.1, multiply $e^{-\lambda x}$
  - $e^{-\lambda x} y'' + e^{-\lambda x} a y' + e^{-\lambda x} b y = 0$
  - $(e^{-\lambda x} y)'' + (a + 2\lambda)(e^{-\lambda x} y)' + (\lambda^2 + a\lambda + b) e^{-\lambda x} y = 0$
    - If we assume λ_1 is root of char. eq., $(\lambda_1^2 + a\lambda_1 + b) = 0$, thus
    - $(e^{-\lambda x} y)'' + (a + 2\lambda_1)\left(e^{-\lambda x} y\right)' = 0$
    - $u'' + (a + 2\lambda_1) u' = 0$, when $e^{-\lambda_1 x} y(x) = u(x)$

General solution (cont.)

- $u'' + (a + 2\lambda_1)u' = 0$, when $e^{-\lambda_1 x}y(x) = u(x)$
  - Case $(a^2 - 4b = 0)$: $\lambda = -\frac{a}{2}$, thus $u'' = 0$
    - $u(x) = c_1 + c_2 x$, thus $y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$
  - Case $(a^2 - 4b \neq 0)$:
    - $v' + (a + 2\lambda_1)v = 0$, when $v = u'$, solve this then
    - $v = Ce^{-(a+2\lambda_1)x}$, C is constant. Then integrate this
      - $u(x) = c_1 - \frac{C}{a+2\lambda_1} e^{-(a+2\lambda_1)x}$, thus
      - $y(x) = c_1 e^{\lambda_1 x} - \frac{C}{a+2\lambda_1} e^{-(a+\lambda_1)x}$, since $(a + 2\lambda_1)$ is the solution λ_2
    - $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ $\left(c_2 = -\frac{C}{a+2\lambda_1}, \lambda_2 = a + 2\lambda_1\right)$

Constants

- Now we get a general solution
  - For particular solution, we need to fix constants
    - Use initial value or boundary value

- Replace equation in sine function
  - Case $(a^2 - 4b < 0)$: $y(x) = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$
  - $y(x) = c_1 e^{ax}(\cos bx + i \sin bx) + c_2 e^{ax}(\cos bx - i \sin bx)$
  - $\quad\quad = (c_1 + c_2) e^{ax} \cos bx + i(c_1 - c_2) e^{ax} \sin bx$
  - $\quad\quad = d_1 e^{ax} \cos bx + i d_2 e^{ax} \sin bx$
  - We can use both sine or exponential
    - But exponential is useful to take differential

Exercise (1)

- Solve general solutions for following equations
  - by Variation of constants method
    - $y' - xy = x$
    - $y' + \frac{y}{x} = x^2 + 2x$
  - by Method of indeterminate coefficient
    - $2y' + 3y = 3x^2 + x$
    - $y' + 4y = 3e^{-x}$

Exercise (2)

- Solve characteristic equation and general solutions for following equations
  - by Method of indeterminate coefficient
    - $y'' + 2y' + y = 0$
    - $y'' + 2y' + 3y = 0$
    - $y'' - 4y' - 5y = 0$

Sample solutions

Ex1.

1 by variation of const

(1) $y' - xy = x$

① Think homogeneous eq.

$y' - xy = 0$

② Solve general solution of ①

$f(x) = -x, \quad F(x) = -\frac{x^2}{2}$

$y = c_1 e^{-F(x)} = c_1 e^{\frac{x^2}{2}} \quad (c_1 \text{ is const})$

③ Replace $c_1 \to u(x)$

$y = u(x) e^{\frac{x^2}{2}}, \quad u(x) = y e^{\frac{-x^2}{2}}$

④ substitute $u(x)$ to given inhomogeneous eq.

$y' - xy = x$

$(u(x) e^{x^2/2})' - x(u(x) e^{x^2/2}) = x$

$u'(x) e^{x^2/2} + x u(x) e^{x^2/2} - x u(x) e^{x^2/2} = x$

$u'(x) = x e^{-x^2/2}$

$u(x) = \int x e^{-x^2/2} dx \quad$ change param $-\frac{x^2}{2} = t, \ -x dx = dt$

$= \int e^t (-dt) = -e^t + C_2 \quad (C_2 : \text{const})$

⑤ Substitute $u(x)$ to the solution of homogeneous eq.

$y = (-e^{-\frac{x^2}{2}} + C_2) e^{\frac{x^2}{2}} = -1 + C_2 e^{\frac{x^2}{2}}$ //

(2) $y' + y/x = x^2 + 2x$

① homogeneous eq.: $y' + \frac{y}{x} = 0$

② general solution: $f(x) = 1/x, \quad F(x) = \log x$

$y = c_1 e^{-\log x} \quad (c_1 : \text{const})$

③ $c_1 \to u(x)$

$y = u(x) e^{-\log x} = \frac{u(x)}{x} \qquad (e^{\log x} = x)$

④ substitute $u(x)$ to given inhomogeneous eq.

$\left(\frac{u(x)}{x}\right)' + \frac{u(x)}{x^2} = x^2 + 2x$

$\frac{u'(x)}{x} - \frac{u(x)}{x^2} + \frac{u(x)}{x^2} = x^2 + 2x$

$u'(x) = x^3 + 2x^2$

$u(x) = \int (x^3 + 2x^2) dx = \frac{x^4}{4} + \frac{2}{3} x^3 + C_2 \quad (C_2 : \text{const})$

⑤ substitute to ③

$y = \frac{x^3}{4} + \frac{2}{3} x^2 + \frac{C_2}{x}$

2 indeterminate coeff.

(3) $2y' + 3y = 3x^2 + x.$

① Particular solution $y_p = \alpha x^2 + \beta x + \gamma$.

② Substitute

$2(2\alpha x + \beta) + 3(\alpha x^2 + \beta x + \gamma) = 3x^2 + x$

$3\alpha = 3, \quad 4\alpha + 3\beta = 1, \quad 2\beta + 3\gamma = 0$

$\alpha = 1, \quad \beta = -1, \quad \gamma = \frac{2}{3}$

④ calc $F(x)$

$y' + \frac{3}{2}y = \frac{3}{2}x^2 + \frac{x}{2}, \quad f(x) = \frac{3}{2}, \quad F(x) = \frac{3}{2}x$

⑤ calc solutions.

particular solution $y_p = x^2 - x + \frac{2}{3}$

general " $y = x^2 - x + 2/3 + ce^{-\frac{3}{2}x}$

(c: const)

(4) $y' + 4y = 3e^{-x}$

$y_p = \alpha e^{-x}$ substitute

$-\alpha e^{-x} + 4\alpha e^{-x} = 3e^{-x}, \quad \alpha = 1$

$f(x) = 4, \quad F(x) = 4x$

$y_p = e^{-x}, \quad y_g = e^{-x} + ce^{-4x}$ (c: const)

3 solve characteristic eq. and general solutions.

(4) $y'' + 2y' + y = 0$

assume $y = ce^{\lambda x}$ (c, λ : unknown), substitute

$\lambda^2 ce^{\lambda x} + 2\lambda ce^{\lambda x} + ce^{\lambda x} = 0$

$(\lambda^2 + 2\lambda + 1)ce^{\lambda x} = 0. \quad ce^{\lambda x} \neq 0$ thus

characteristic eq $\underline{\lambda^2 + 2\lambda + 1} = (\lambda + 1)^2 = 0 \rightarrow \lambda = -1.$

general solution $y = \underline{c_1 e^{-x} + c_2 x e^{-x}}$ (c_1, c_2 : const)

(5) $y'' + 2y' + 3y = 0$

assume $y = ce^{\lambda x}$

$(\lambda^2 + 2\lambda + 3)ce^{\lambda x} = 0$ characteristic eq $\underline{\lambda^2 + 2\lambda + 3 = 0}$

$\lambda = \frac{-2 \pm \sqrt{4 - 4 \times 3}}{2} = \frac{-2 \pm 2\sqrt{2}i}{2} = -1 \pm \sqrt{2}i$

general solution $\underline{y = c_1 e^{-(1+\sqrt{2}i)x} + c_2 e^{-(1-\sqrt{2}i)x}}$ (c_1, c_2 : const)

(6) $y'' - 4y' - 5y = 0$

assume $y = ce^{\lambda x}$

characteristic eq: $\underline{\lambda^2 - 4\lambda - 5 = 0} = (\lambda - 5)(\lambda + 1)$

general solution $\underline{y = c_1 e^{5x} + c_2^{-x}}$ $\quad \lambda = 5, -1$

(c_1, c_2 : const)

Fundamental Mathematics
- Differential equations and physics -

2023/10/24

2nd order differential equation

- Introduce 2nd order differential equation
  - $y'' + ay' + by = r(x)$ (a, b are constants) (eq.2.12)
    - If $r(x) = 0$, eq.2.12 is homogeneous (eq.2.2)
    - If $r(x) \neq 0$, eq.2.12 is inhomogeneous
- Inhomogeneous form is very tough for hand calculation
  - If $r(x)$ is constant, sine, or exponential we can use method of indeterminate coefficient
    - In physics, circuits, we can use this assumption
  - (Variation of constants)
  - Method of indeterminate coefficient

Structure of solution for inhomogeneous equation

- Theorem:
  - Assume solution $u(x)$ for $u'' + au' + bu = 0$ and particular solution $y_p(x)$ for $y'' + ay' + by = r(x)$.
  - General solution for $y'' + ay' + by = r(x)$ is $y(x) = y_p(x) + u(x)$.
- Proof:
  - Calculate differential for $y(x) + u(x)$
    - 1st order diff: $\left(y(x) + u(x)\right)' = y'(x) + u'(x)$
    - 2nd order diff: $\left(y(x) + u(x)\right)'' = y''(x) + u''(x)$
  - (continue)

$y(x)$ is general solution
$u(x)$ is solution for homogeneous

Structure of solution for inhomogeneous equation (cont.)

- $y(x) + u(x)$ is also the solution for eq.2.12
  - $(y+u)'' + a(y+u)' + b(y+u) = y'' + ay' + by + u'' + au' + bu = r(x)$
- Next, assume $y_1(x)$ and $y_2(x)$ are the solution for inhomogeneous equation (eq.2.1).
  - The difference $y_1(x) - y_2(x)$ is solution for homogeneous
    - $(y_1 - y_2)'' + a(y_1 - y_2)' + b(y_1 - y_2) = (y_1'' + ay'_1 + by_1) - \left(y_2'' + ay'_2 + by_2\right) = r(x) - r(x) = 0$
- $y(x) = y_p(x) + u(x)$
  - General solution for inhomogeneous equation $(y(x))$ is sum of one particular solution for inhomogeneous $(y_p(x))$ and general solution for homogenesis $(u(x))$

Structure of solution for inhomogeneous equation (cont.)

- General solution for inhomogeneous equation
  - $y(x) = y_p(x) + y_0(x)$
  - Need particular solution for inhomogeneous eq. ($y_p(x)$)

- We can calculate solution for inhomogeneous eq. with sum assumption
  - Method of indeterminate coefficient
  - (Variation of constants need to calculate array…)

Method of indeterminate coefficient
(recall)

- With some assumptions, we can easily solve differential equation
  - Guess the candidate of particular solution
  - If the right side of an equation is…
    - n-order polynormal: candidate should be n-polynormal
    - sine function: candidate should be in sine
    - exponential: candidate should be in exponential

Method of indeterminate coefficient(exponent)

- Solve general solution $y(x)$ of : $y'' + 3y' + 2y = e^{2x}$
  - Get general solution $y_0(x)$ for homogeneous equation
    - $y'' + 3y' + 2y = 0$
    - Its characteristic equation:
      - $(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2) = 0, \lambda = -1, -2$
    - $y_0(x) = c_1 e^{-x} + c_2 e^{-2x}$
  - Get particular solution $y_p(x)$ for inhomogeneous equation
    - Assume $y_p(x) = Ae^{2x}$, (A is const., e^{2x} is right side)
      - $4Ae^{2x} + 3 \cdot 2Ae^{2x} + 2 \cdot Ae^{2x} = e^{2x}$
      - $A = \frac{1}{12}$
  - $y(x) = y_0(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{12} e^{2x}$

Method of indeterminate coefficient(exponent, cont.)

- Solve general solution $y(x)$ of : $y'' + 3y' + 2y = e^{-x}$
  - Get general solution $y_0(x)$ for homogeneous equation
    - $y_0(x) = c_1 e^{-x} + c_2 e^{-2x}$
  - Get particular solution $y_p(x)$ for inhomogeneous equation
    - Assume $y_p(x) = Ae^{-x}$, (A is const., e^{-x} is right side)
      - $Ae^{-x} - 3Ae^{-x} + 2Ae^{-x} = 0$ *??*
    - Assume $y_p(x) = Axe^{-x}$, (A is const., e^{-x} is right side)
      - $Ax^2e^{-x} - 3xAe^{-x} + 2Ae^{-x} = e^{-x}$-> A=1
  - $y(x) = y_0(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x} + xe^{x}$

Method of indeterminate coefficient(sine)

- Solve particular solution of : $y'' + 3y' + 2y = \cos x$
  - Ex1: Assume particular solution is $y_p = \alpha \cos x \ + \beta \sin x$
    - $\alpha, \beta$are constant. Substitute y_p to equation
    - $\alpha = \frac{1}{10}, \beta = \frac{3}{10}$, thus $y_p = \frac{1}{10}\cos x + \frac{3}{10}\sin x$
  - Ex2: Solve it in imaginary space, then take real part
    - Assume target solution is $u'' + 3u' + 2u = e^{ix}$
    - Assume particular solution is $u_p = Ae^{ix}$ (A is const)
    - $A = \frac{1}{10} - \frac{3}{10}i$, $u_p = \left(\frac{1}{10} - \frac{3}{10}i\right)(\cos x + i \sin x)$
    - $y_p = Re\{u_p\} = \frac{1}{10}\cos x + \frac{3}{10}\sin x$

Method of indeterminate coefficient (polynormal)

- Solve particular solution of : $y'' + 3y' + 2y = x^2$
  - Assume particular solution is $y_p = \alpha x^2 + \beta x + \gamma$
    - α, β, γ are constant. Substitute y_p to equation
    - $2\alpha x^2 + (6\alpha + 2\beta)x + \ (2\alpha + 3\beta + 2\gamma) = x^2$
  - This equation should satisfy following conditions
    - x^2: $2\alpha = 1$, x^1: $6\alpha + 2\beta = 0$, x^0: $2\alpha + 3\beta + 2\gamma = 0$, thus
  - $y_p = \frac{1}{2}x^2 - \frac{3}{2}x - \frac{7}{4}$

Method of indeterminate coefficient (polynormal)

- Solve general solution of : $y'' + y' = x^2$
  - Get general solution $y_0(x)$ for homogeneous equation
    - Characteristic equation: $\lambda(\lambda + 1) = 0$
    - $y_0(x) = c_1 + c_2 e^{-x}$
  - Particular solution: cannot fix coefficient cx^0
  - Assume particular solution is $y_p = \alpha x^3 + \beta x^2 + \gamma x^1$
    - α, β, γ are constant. Substitute y_p to equation
    - $(3\alpha)x^2 + (6\alpha + 2\beta)x + \ (2\beta + \gamma) = x^2$
  - This equation should satisfy following conditions
    - x^2: $3\alpha = 1$, x^1:$6\alpha + 2\beta = 0$, x^0:$2\beta + \gamma = 0$, thus
  - $y_p = \frac{1}{3}x^2 - x + 2$, $y = \frac{1}{3}x^2 - x + 2 + c_1 + c_2 e^{-x}$

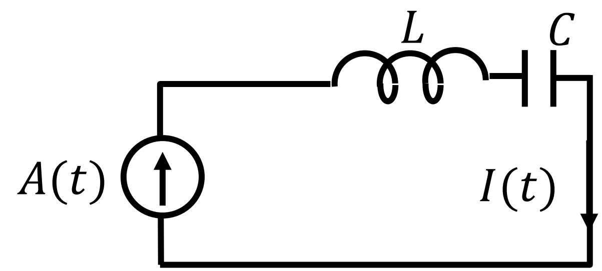
Initial condition

- Use initial condition to calculate particular solution
  - $y'' + ay' + by = r(x)$, use $y(0) = A$, $y'(0) = B$.(A, B:const)
- If one particular solution y_p is known, general solution $y(x)$:
  - $y(x) = c_1\varphi(x) + c_2\psi(x) + y_p$ ($\varphi(x)$ and $\psi(x)$: shape of basic functions)
  - Calculate c_1 and c_2 using initial conditions
- Generally, solve next simultaneous equation
  - $\begin{bmatrix} \varphi(0) & \psi(0) \\ \varphi'(0) & \psi'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} A - y_p(0) \\ B - y_p(0) \end{bmatrix}$

Method of indeterminate coefficient (w/ init. val)

- Solve particular solution $y(x)$
  - $y'' + 3y' + 2y = e^{2x}$, $y(0) = 0$, $y'(0) = 1$
  - Get general solution $y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{12} e^{2x}$
  - $y(0) = c_1 + c_2 + \frac{1}{12} = 0$
  - $y'^{(0)} = -c_1 - 2c_2 + \frac{1}{6} = 1$, thus $c_1 = \frac{2}{3}$, $c_1 = -\frac{3}{4}$
  - $y(x) = \frac{2}{3} e^{-x} - \frac{3}{4} e^{-2x} + \frac{1}{12} e^{2x}$

Example 1: LC circuit

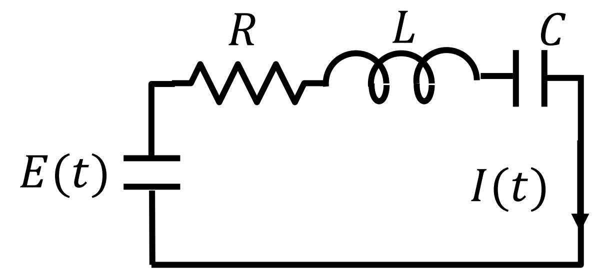


- Derive current $I(t)$ of LC circuit
  - Initial conditions:
    - For $t = 0$, $A(t) = I(t) = 2$,
    - Otherwise, $A(0) = 0$ (no impact on circuit)
    - $I'(0) = 0$
- Voltage of L (V_L) C (V_C) are: $(I(t) = Q'(t))$
  - $V_L = L\frac{dI(t)}{dt}$, $V_L = \frac{Q(t)}{C}$, $\mathrm{L}I'(t) + \frac{Q(t)}{C} = E(t)$
  - For the current $I(t)$, $I''(t) + \frac{I(t)}{LC} = \frac{E\prime(t)}{L}$
  - (You will learn this in electric circuit class)

LC circuit, $E(t) = V$

- $I''(t) + \frac{I(t)}{LC} = 0$ $\left(E'(t) = 0\right)$, assume $I(t) = ce^{\lambda t}$
  - Characteristic equation:$\lambda^2 + \frac{1}{LC} = 0$, $\lambda = \pm\sqrt{\frac{1}{LC}}i = \pm\omega i$,
- General solution $I_g(t)$:
  - $I_g(t) = \left(c_1 e^{\omega i t} + c_2 e^{-\omega i t}\right) = (d_1 \cos \omega t + d_2 \sin \omega t)$
- Select θ which satisfy $\cos\theta = \frac{d_2}{\sqrt{d_1^2+d_2^2}}$, $\sin\theta = \frac{d_1}{\sqrt{d_1^2+d_2^2}}$
  - $I_g(t) = \sqrt{d_1^2 + d_2^2}\sin(\omega t + \theta)$,
- $I_p(0) = \sqrt{d_1^2 + d_2^2}\sin(\theta) = 2$, $I_p{}'(0) = \sqrt{d_1^2 + d_2^2}\cos(\theta) = 0$
  - $I_p(t) = 2\sin(\omega t)$ (it will oscillate)

Example: RLC circuit



- Derive current $I(t)$ of RLC circuit
  - Initial condition: $I(0) = 0$
- Voltage of R (V_R) L (V_L) C (V_C) are:
  - $V_R = RI(t)$, $V_L = L\frac{dI(t)}{dt}$, $V_L = \frac{Q(t)}{C}$, $\mathrm{L}I'(t) + RI(t) + \frac{Q(t)}{C} = E(t)$
  - For the current $I(t)$, $I''(t) + \frac{R}{L}I'(t) + \frac{I(t)}{LC} = \frac{E\prime(t)}{L}$ $\quad (I(t) = Q'(t))$
  - (You will learn this in electric circuit class)
- Solve this equation
  - For $E(t) = V$ (V is constant)
  - For $E(t) = Vt$ (V is constant)

RLC circuit, $E(t) = V$

- $I''(t) + \frac{R}{L}I'(t) + \frac{I(t)}{LC} = 0$ $\left(E'(t) = 0\right)$, assume $I(t) = ce^{\lambda t}$
  - Characteristic equation:$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$
  - $\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$, three candidates of solution
  - $R^2 > \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{R^2 - 4L/C}/2L} + c_2 e^{-t\sqrt{R^2 - 4L/C}/2L}\right)$
  - $R^2 = \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}(c_1 + c_2 t)$
  - $R^2 < \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{4L/C - R^2}/2L} + c_2 e^{-t\sqrt{4L/C - R^2}/2L}\right)$

RLC circuit, $E(t) = Vt$

- $I''(t) + \frac{R}{L} I'(t) + \frac{I(t)}{LC} = V$ assume particular solutions are $I_{p1}(t)$, $I_{p2}(t)$, $I_{p3}(t)$
  - General solutions are
  - $R^2 > \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{R^2-4L/C}/2L} + c_2 e^{-t\sqrt{R^2-4L/C}/2L}\right) + I_{p1}(t)$
  - $R^2 = \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}(c_1 + c_2 t) + I_{p2}(t)$
  - $R^2 < \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{4L/C-R^2}/2L} + c_2 e^{-t\sqrt{4L/C-R^2}/2L}\right) + I_{p3}(t)$

Exercise

- Solve general solutions
  - $y'' + 3y' + 2y = \cos x$
  - $y'' - 2y' + 3y = x^2$
  - $y'' - 2y' - 3y = e^x$
  - $y'' - 2y' - 3y = e^{-x}$
- Solve particular solution
  - $y'' + 3y' + 2y = \cos x,\ y(\pi) = 0,\ y'(\pi) = 1$

example solutions

class 4

① $y''+3y'+2y=\cos x$. $\quad y_0=c_1e^{-x}+c_2e^{-2x}$

c_1, c_2: constants

(1-1) $y_p=\alpha\cos x+\beta\sin x$ $\quad \alpha,\beta$, constant

$$(-\alpha\cos x-\beta\sin x)+3(-\alpha\sin x+\beta\cos x)$$
$$+2(\alpha\cos x+\beta\sin x)=\cos x$$
$$(-\alpha+3\beta+2\alpha)\cos x+(-\beta-3\alpha+2\beta)\sin x=\cos x$$
$$3\beta+\alpha=1$$
$$\beta-3\alpha=0 \qquad \alpha=\frac{1}{10},\ \beta=\frac{3}{10}$$

$$y_p=\frac{1}{10}\cos x+\frac{3}{10}\sin x$$
$$y=\frac{1}{10}\cos x+\frac{3}{10}\sin x+c_1e^{-x}+c_2e^{-2x}\ /\!/$$

(1-2) $y_p=\mathrm{Re}\{u_p\}$, $u_p=Ae^{ix}$, $\cos x=\mathrm{Re}\{\cos x+i\sin x\}$

A: constant

$$(-Ae^{ix})+3(Aie^{ix})+2(Ae^{ix})=e^{ix}$$
$$A(1+3i)e^{ix}=e^{ix}$$
$$A=\frac{1}{1+3i}=\frac{1-3i}{(1+3i)(1-3i)}=\frac{1}{10}-\frac{3}{10}i$$

$$y_p=\mathrm{Re}\{u_p\}=\frac{1}{10}\cos x+\frac{3}{10}\sin x$$
$$y=y_0+y_p=c_1e^{-x}+c_2e^{-2x}+\frac{1}{10}\cos x+\frac{3}{10}\sin x\ /\!/$$

② $y''-2y'+3y=x^2$

Charactarisic eguation $\lambda^2-2\lambda+3=0$

$$\lambda=\frac{2\pm\sqrt{4-12}}{2}=1\pm\sqrt{2}i$$
$$y_0=c_1e^{1+\sqrt{2}i}+c_2e^{1-\sqrt{2}i}$$

c_1,c_2: const

Assume particular solution $y_p=\alpha x^2+\beta x+\delta$ $\quad \alpha,\beta,\delta$: const

$$(2\alpha)-2(2\alpha x+\beta)+3(\alpha x^2+\beta x+\delta)=x^2$$
$$3\alpha x^2+(-4\alpha+3\beta)x+(2\alpha-2\beta+3\delta)=x^2.$$
$$\alpha=\frac{1}{3},\ \beta=\frac{4}{9},\ \delta=\frac{-2\alpha+2\beta}{3}=\frac{-6+8}{27}=\frac{2}{27}$$

$$y=y_0+y_p=c_1e^{1+\sqrt{2}i}+c_2e^{1-\sqrt{2}i}+\frac{1}{3}x^2+\frac{4}{9}x+\frac{2}{27}\ /\!/$$

example solutions

③ $y'' - 2y' - 3y = e^x$

Characteristic equation $\lambda^2 - 2\lambda - 3 = 0$

$(\lambda-3)(\lambda+1) = 0 \quad \lambda = 3, -1$

$y_0 = C_1 e^{3x} + C_2 e^{-x}$

(C_1, C_2, const)

Assume particular solution $y_p = Ae^x$ (A. const)

$A(1-2-3) = 1 \ , \quad A = -\frac{1}{4}$

$y = y_0 + y_p = y_0 C_1 e^{3x} + C_2 e^{-x} - \frac{1}{4} e^x$ //

④ $y'' - 2y' - 3y = e^{-x}$

$y_0 = C_1 e^{3x} + C_2 e^{-x}$ $(C_1, C_2 : \text{constant})$

Assume particular solution $y_p = Axe^{-x}$ (A : const)

$y_p' = A(e^{-x} - xe^{-x})$

$= A(1-x)e^{-x}$

$y_p'' = A(-e^{-x} - e^{-x} + xe^{-x})$

$= A(-2+x)e^{-x}$ thus

$A(x-2) - 2A(1-x) - 3Ax = 1$

$A = -\frac{1}{4}$ thus

$y = y_0 + y_p$

$= C_1 e^{3x} + C_2 e^{-x} - \frac{1}{4} x e^{-x}$ //

⑤

General solution is $y(x) = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x$

$y'(x) = -C_1 e^{-x} - 2C_2 e^{-2x} - \frac{1}{10}\sin x + \frac{3}{10}\cos x$

$y(\pi) = C_1 e^{-\pi} + C_2 e^{-2\pi} - \frac{1}{10} = 0$

$y'(\pi) = -C_1 e^{-\pi} - 2C_2 e^{-2\pi} - \frac{3}{10} = 1$

$-C_2 e^{-2\pi} - \frac{2}{5} = 1 \ , \quad C_2 = -\frac{7}{5} e^{2\pi}$

$C_1 e^{-\pi} = \frac{7}{5} + \frac{1}{10} = \frac{3}{2} \ , \quad C_1 = \frac{3}{2} e^{\pi}$

$y(x) = \frac{3}{2} e^{-(x-\pi)} - \frac{7}{5} e^{-2(x-\pi)} + \frac{1}{10}\cos x + \frac{3}{10}\sin x$

Conclusion

- Introduction of solve 2nd order inhomogeneous differential equation for engineering mathematics
  - Use some assumptions, but useful enough for engineering
  - Characteristic equation is one key of 2nd order differential equation
    - Its indicate the function of solution
      - In real values, or in imaginary

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