Fundamental Mathematics (Engineering Mathematics)

Shinichi Nishizawa

Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

2nd order differential equation

- Introduce 2nd order differential equation
  - $y'' + ay' + by = r(x)$ (a, b are constants) (eq.4.1)
    - If $r(x) = 0$, eq.4.1 is homogeneous (eq.4.2)
    - If $r(x) \neq 0$, eq.4.1 is inhomogeneous
- Inhomogeneous form is very tough for hand calculation
  - If $r(x)$ is constant, sine, or exponential we can use method of indeterminate coefficient
    - In physics, circuits, we can use this assumption
  - (Variation of constants)
  - Method of indeterminate coefficient

Structure of solution for inhomogeneous equation

- Theorem:
  - Assume general solution $u(x)$ for $y'' + ay' + by = 0$ and particular solution $y_p(x)$ for $y'' + ay' + by = r(x)$.
  - General solution for $y'' + ay' + by = r(x)$ is $y(x) = y_p(x) + u(x)$.
- Proof:
  - Calculate differential for $y(x) + u(x)$
    - 1st order diff: $\big(y(x) + u(x)\big)' = y'(x) + u'(x)$
    - 2nd order diff: $\big(y(x) + u(x)\big)'' = y''(x) + u''(x)$
  - (continue)

$y(x)$ is any solution
$u(x)$ is solution for homogeneous

Structure of solution for inhomogeneous equation (cont.)

- $y(x) + u(x)$ is also the solution for eq.4.1
  - $(y+u)'' + a(y+u)' + b(y+u) = y'' + ay' + by + u'' + au' + bu = r(x)$
- Next, assume $y_1(x)$ and $y_2(x)$ are the solution for inhomogeneous equation (eq.4.1).
  - The difference $y_1(x) - y_2(x)$ is solution for homogeneous
    - $(y_1 - y_2)'' + a(y_1 - y_2)' + b(y_1 - y_2) = (y_1'' + a{y'}_1 + by_1) - \left(y_2'' + a{y'}_2 + by_2\right) = r(x) - r(x) = 0$
- General solution for inhomogeneous equation $(y(x))$:
  - Particular solution for inhomogeneous $(y_p(x))$ + general solution for homogenesis $(u(x))$

Structure of solution for inhomogeneous equation (cont.)

- General solution for 2nd order homogeneous equation:
  - $y_0(x) = c_1\varphi(x) + c_2\psi(x)$

    ($y_0(x)$ is $u(x)$ in previous slide)
    ($\varphi(x)$ and $\psi(x)$: shape of basic functions)

- General solution for inhomogeneous equation
  - $y(x) = y_p(x) + y_0(x)$
  - Need particular solution for inhomogeneous eq. ($y_p(x)$)

- Use method of indeterminate coefficient
- (Variation of constants need to calculate array…)

Method of indeterminate coefficient

- With some assumptions, we can easily solve differential equation
  - Guess the candidate of particular solution
  - If the right side of an equation is…
    - n-order polynormal: candidate should be n-polynormal
    - sine function: candidate should be in sine
    - exponential: candidate should be in exponential

Method of indeterminate coefficient(exponent)

- Solve general solution $y(x)$ of : $y'' + 3y' + 2y = e^{2x}$
  - Get general solution $y_0(x)$ for homogeneous equation
    - $y'' + 3y' + 2y = 0$
    - Its characteristic equation:
      - $(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2) = 0, \lambda = -1, -2$
    - $y_0(x) = c_1 e^{-x} + c_2 e^{-2x}$
  - Get particular solution $y_p(x)$ for inhomogeneous equation
    - Assume $y_p(x) = Ae^{2x}$, (A is const., e^{2x} is right side)
      - $4Ae^{2x} + 3 \cdot 2Ae^{2x} + 2 \cdot Ae^{2x} = e^{2x}$
      - $A = \frac{1}{12}$
  - $y(x) = y_0(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{12} e^{2x}$

Method of indeterminate coefficient(exponent, cont.)

- Solve general solution $y(x)$ of : $y'' + 3y' + 2y = e^{-x}$
  - Get general solution $y_0(x)$ for homogeneous equation
    - $y_0(x) = c_1e^{-x} + c_2e^{-2x}$
  - Get particular solution $y_p(x)$ for inhomogeneous equation
    - Assume $y_p(x) = Ae^{-x}$, (A is const., e^{-x} is right side)
      - $Ae^{-x} - 3Ae^{-x} + 2Ae^{-x} = 0$??
    - Assume $y_p(x) = Axe^{-x}$, (A is const., e^{-x} is right side)
      - $Ax^2e^{-x} - 3xAe^{-x} + 2Ae^{-x} = e^{-x}$-> A=1
- $y(x) = y_0(x) + y_p(x) = c_1e^{-x} + c_2e^{-2x} + xe^{-x}$

Method of indeterminate coefficient(sine)

- Solve particular solution of : $y'' + 3y' + 2y = \cos x$
  - Ex1: Assume particular solution is $y_p = \alpha \cos x \ + \beta \sin x$
    - α, β are constant. Substitute y_p to equation
    - $\alpha = \frac{1}{10}, \beta = \frac{3}{10}$, thus $y_p = \frac{1}{10}\cos x + \frac{3}{10}\sin x$
  - Ex2: Solve in imaginary space, then take real part
    - Assume target solution is $u'' + 3u' + 2u = e^{ix}$
    - Assume particular solution is $u_p = Ae^{ix}$ (A is const)
    - $A = \frac{1}{10} - \frac{3}{10}i$, $u_p = \left(\frac{1}{10} - \frac{3}{10}i\right)(\cos x + i \sin x)$
    - $y_p = Re\{u_p\} = \frac{1}{10}\cos x + \frac{3}{10}\sin x$

Method of indeterminate coefficient (polynormal)

- Solve particular solution of : $y'' + 3y' + 2y = x^2$
  - Assume particular solution is $y_p = \alpha x^2 + \beta x + \gamma$
    - α, β, γ are constant. Substitute y_p to equation
    - $2\alpha x^2 + (6\alpha + 2\beta)x + \ (2\alpha + 3\beta + 2\gamma) = x^2$
  - This equation should satisfy following conditions
    - x^2: $2\alpha = 1$, x^1: $6\alpha + 2\beta = 0$, x^0: $2\alpha + 3\beta + 2\gamma = 0$, thus
  - $y_p = \frac{1}{2}x^2 - \frac{3}{2}x - \frac{7}{4}$

Method of indeterminate coefficient (polynormal)

- Solve general solution of : $y'' + y' = x^2$
  - Get general solution $y_0(x)$ for homogeneous equation
    - Characteristic equation: $\lambda(\lambda + 1) = 0$
    - $y_0(x) = c_1 + c_2 e^{-x}$
  - Particular solution: cannot fix coefficient cx^0
  - Assume particular solution is $y_p = \alpha x^3 + \beta x^2 + \gamma x^1$
    - α, β, γ are constant. Substitute y_p to equation
    - $(3\alpha)x^2 + (6\alpha + 2\beta)x + \ (2\beta + \gamma) = x^2$
  - This equation should satisfy following conditions
    - x^2: $3\alpha = 1$, x^1:$6\alpha + 2\beta = 0$, x^0:$2\beta + \gamma = 0$, thus
  - $y_p = \frac{1}{3}x^2 - x + 2$, $y = \frac{1}{3}x^2 - x + 2 + c_1 + c_2 e^{-x}$

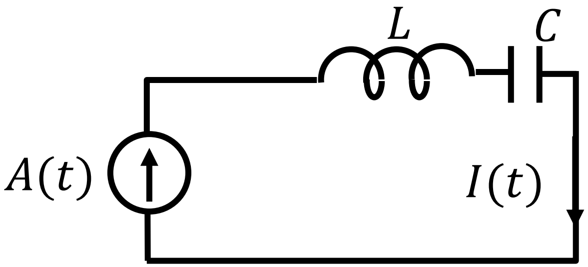
Initial condition

- Use initial condition to calculate particular solution
  - $y'' + ay' + by = r(x)$, use $y(0) = A$, $y'(0) = B$.$(A, B$:const$)$
- If one particular solution y_p is known, general solution $y(x)$:
  - $y(x) = c_1\varphi(x) + c_2\psi(x) + y_p$ ($\varphi(x)$ and $\psi(x)$: shape of basic functions)
  - Calculate c_1 and c_2 using initial conditions
- Generally, solve next simultaneous equation
  - $\begin{bmatrix} \varphi(0) & \psi(0) \\ \varphi'(0) & \psi'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} A - y_p(0) \\ B - y_p(0) \end{bmatrix}$

Method of indeterminate coefficient(exponent)

- Solve particular solution $y(x)$
  - $y'' + 3y' + 2y = e^{2x}, y(0) = 0, y'(0) = 1$
  - Get general solution $y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{12} e^{2x}$
  - $y(0) = c_1 + c_2 + \frac{1}{12} = 0$
  - $y'^{(0)} = -c_1 - 2c_2 + \frac{1}{6} = 1$, thus $c_1 = \frac{2}{3}, c_1 = -\frac{3}{4}$
  - $y(x) = \frac{2}{3} e^{-x} - \frac{3}{4} e^{-2x} + \frac{1}{12} e^{2x}$

Example 1: LC circuit



- Derive current $I(t)$ of LC circuit
  - Initial conditions:
    - For $t = 0$, $A(0) = I(0) = 2$,
    - Otherwise, $A(t) = 0$ (no impact on circuit)
    - $I'(0) = 0$
- Voltage of L (V_L) C (V_C) are:

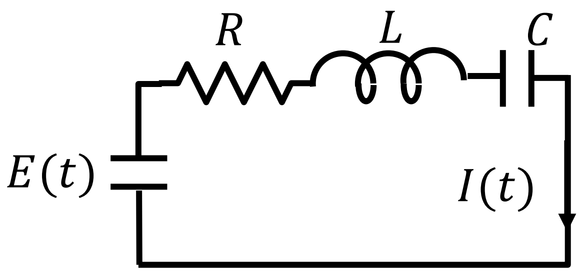
  $(I(t) = Q'(t))$

  - $V_L = L\frac{dI(t)}{dt}$, $V_C = \frac{Q(t)}{C}$, $\mathrm{L}I'(t) + \frac{Q(t)}{C} = 0$
  - For the current $I(t)$, $I''(t) + \frac{I(t)}{LC} = 0$
  - (You will learn this in electric circuit class)

LC circuit, $E(t) = V$

- $I''(t) + \frac{I(t)}{LC} = 0$ $(E'(t) = 0)$, assume $I(t) = ce^{\lambda t}$
  - Characteristic equation:$\lambda^2 + \frac{1}{LC} = 0,\ \lambda = \pm\sqrt{\frac{1}{LC}}i = \pm\omega i,$
- General solution $I_g(t)$:
  - $I_g(t) = \left(c_1 e^{\omega i t} + c_2 e^{-\omega i t}\right) = (d_1 \cos \omega t + d_2 \sin \omega t)$
- Select θ which satisfy $\cos\theta = \frac{d_2}{\sqrt{d_1^2+d_2^2}},\ \sin\theta = \frac{d_1}{\sqrt{d_1^2+d_2^2}}$
  - $I_g(t) = \sqrt{d_1^2 + d_2^2}\sin(\omega t + \theta),$
- $I_p(0) = \sqrt{d_1^2 + d_2^2}\sin(\theta) = 2,\ I_p{}'(0) = \sqrt{d_1^2 + d_2^2}\cos(\theta) = 0$
  - $I_p(t) = 2\sin(\omega t + \pi/2)$ (it will oscillate, "resonance")
  - Frequency $f = 1/(2\pi\sqrt{LC})$

Example 2: RLC circuit



- Derive current $I(t)$ of RLC circuit
  - Initial condition: $I(0) = 0$
- Voltage of R (V_R) L (V_L) C (V_C) are:
  - $V_R = RI(t)$, $V_L = L\frac{dI(t)}{dt}$, $V_C = \frac{Q(t)}{C}$, $LI'(t) + RI(t) + \frac{Q(t)}{C} = E(t)$
  - For the current $I(t)$, $I''(t) + \frac{R}{L}I'(t) + \frac{I(t)}{LC} = \frac{E'(t)}{L}$ $\quad (I(t) = Q'(t))$
  - (You will learn this in electric circuit class)
- Solve this equation
  - For $E(t) = V$ (V is constant)
  - For $E(t) = Vt$ (V is constant)

RLC circuit, $E(t) = V$

- $I''(t) + \frac{R}{L} I'(t) + \frac{I(t)}{LC} = 0$ $(E'(t) = 0)$, assume $I(t) = ce^{\lambda t}$
  - Characteristic equation:$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$
  - $\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$, three candidates of solution
  - $R^2 > \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{R^2-4L/C}/2L} + c_2 e^{-t\sqrt{R^2-4L/C}/2L}\right)$
  - $R^2 = \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}(c_1 + c_2 t)$
  - $R^2 < \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{4L/C-R^2}/2L} + c_2 e^{-t\sqrt{4L/C-R^2}/2L}\right)$

RLC circuit, $E(t) = Vt$

- $I''(t) + \frac{R}{L} I'(t) + \frac{I(t)}{LC} = V$ assume particular solutions are $I_{p1}(t)$, $I_{p2}(t)$, $I_{p3}(t)$
  - General solutions are
  - $R^2 > \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{R^2-4L/C}/2L} + c_2 e^{-t\sqrt{R^2-4L/C}/2L}\right) + I_{p1}(t)$
  - $R^2 = \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}(c_1 + c_2 t) + I_{p2}(t)$
  - $R^2 < \frac{4L}{C}$: $I(t) = e^{-\frac{Rt}{2L}}\left(c_1 e^{t\sqrt{4L/C-R^2}/2L} + c_2 e^{-t\sqrt{4L/C-R^2}/2L}\right) + I_{p3}(t)$

Fundamental Mathematics
- Array and vector-

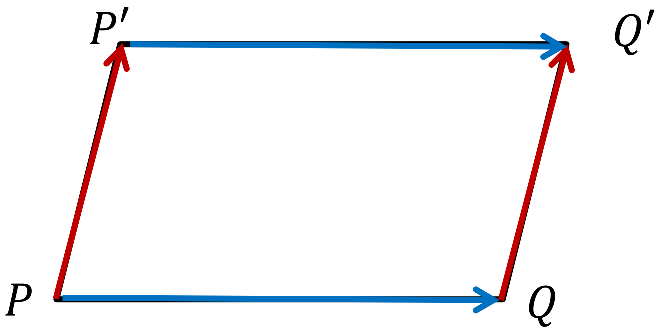
2023/10/31

Motivation

- Many physics can be expressed by vectors
  - Good to explain in simple way (if we know vectors)
- Target: understand the meaning of Maxwell's equation
  - $\mathrm{div}\,\boldsymbol{D} = \rho$
    - $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho dV$ (Gauss's eq of electricfield)
  - $\mathrm{div}\,\boldsymbol{B} = 0$
    - $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \iiint \mathrm{div}\,\boldsymbol{B} dV$ (Gauss's eq of magneticfield)
  - $\mathrm{rot}\,\boldsymbol{H} = i + \frac{\delta D}{\delta t}$: $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(i + \frac{\delta \boldsymbol{D}}{\delta t}\right) \cdot d\boldsymbol{S}$ (Ampele's law)
  - $\mathrm{div}\,\boldsymbol{E} = -\frac{\delta B}{\delta t}$: $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\delta}{\delta t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$ Faraday's law)

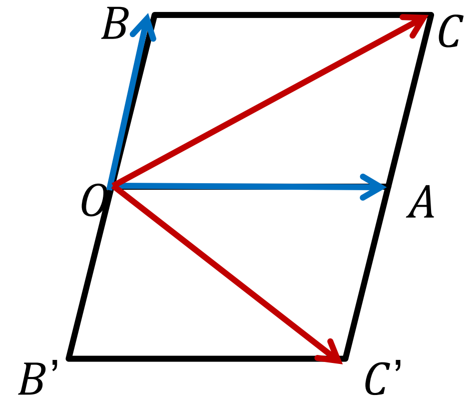
Scalar and Vector

- Scalar: Value (only)
- Vector: Value (length) and its direction
  - Vector from point P to Q is: $\overrightarrow{PQ}$
    - P: start point, Q: end point
    - If $\overrightarrow{P'Q'}$ is equal to $\overrightarrow{PQ}$, $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ is in the same class
    - If two points are the same, it is zero vector $\overrightarrow{PP}$, $\overrightarrow{QQ}$
- To show the vector, we use **bold**
  - Vector: $\boldsymbol{a} = \overrightarrow{PQ}$
  - Zero vector $\mathbf{0} = \overrightarrow{PP}$



Add, sub, extension

- Assume $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$,$\boldsymbol{c} = \overrightarrow{OC}$, where O, A, B, C composes parallelogram
  - Define:$-\boldsymbol{a} = -\overrightarrow{OA} = \overrightarrow{AO}$
  - Define: $\boldsymbol{a} + \boldsymbol{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$
  - Define: $\boldsymbol{a} - \boldsymbol{b} = \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OC'}$



- For real value λ, its product to the vector $\boldsymbol{a}$ is
  - $\boldsymbol{a}\lambda = \lambda\boldsymbol{a}$
- If the three points P, Q, R are on the same line: $\overrightarrow{PQ} = \lambda\overrightarrow{PR}$
- If the two vectors are in parallel: $\boldsymbol{a}\lambda = \boldsymbol{b}$
  - Geometric vector space
    - Vector space: more general and abstract

Vector space

- L is called vector space if element of L satisfy following definition and notation
  - Addition: result of $\boldsymbol{a}+\boldsymbol{b}$ is unique $(\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{L})$
  - Scalar multiply: result of $\boldsymbol{a}\lambda$ is unique $(\boldsymbol{a} \in \boldsymbol{L}, \lambda \in R)$
- Both satisfy following:
  - Association law: $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$
  - Exchange low: $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$
  - Identity element: $\boldsymbol{a}+\boldsymbol{o}=\boldsymbol{a}$
  - inverse element: $\boldsymbol{a}+(-\boldsymbol{a})=\boldsymbol{0}$

Component

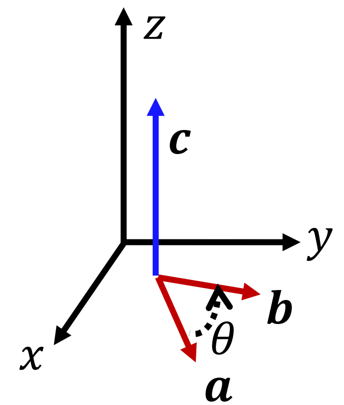
- Vector $\boldsymbol{a}$ is also defined by its components $[a_1, \cdots, a_n]$
  - n: its #dimension
- For the xyz-coordinate system, $\boldsymbol{a} = [a_x, a_y, a_z]$
  - This also satisfy the rules of vector space
- Or, using unit vector (基本ベクトル) $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, for xyz-coord. system,
  - $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, where $a_1 = |a_x|$, $a_2 = |a_y|$, $a_3 = |a_z|$
- Length: $|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$, unit vector $\boldsymbol{u} = \boldsymbol{a}/|\boldsymbol{a}|$
- These definitions can be easily implemented as array of computer program
  - In C-language: $a[3] = [a_x, a_y, a_z]$

Inner product (内積)

- For two vectors $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{a} \cdot \boldsymbol{b} = \mathrm{c} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ is called as inner product in scaler value $(\theta = \angle AOB)$
- Inner products has following characteristics
  - $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
  - $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
  - $\lambda\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \lambda\boldsymbol{b} = \lambda(\boldsymbol{a} \cdot \boldsymbol{b})$
- For unit vector $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$,
  - $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = \boldsymbol{1}$
  - $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{0}$

Outer product (外積)

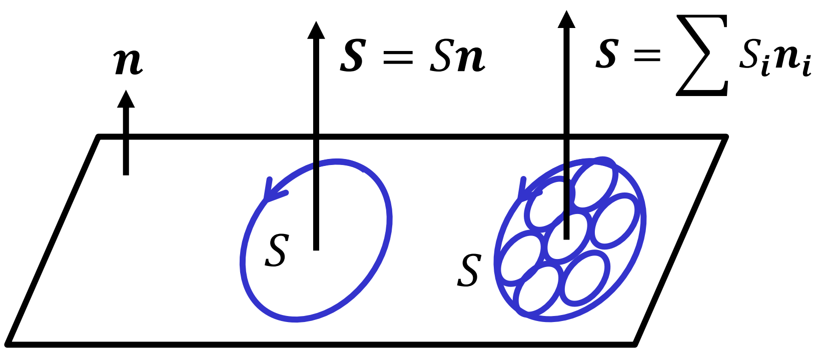
- (Assume right-hand side coordinate system)
- For $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$: outer product
  - $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$
  - Angle of $\boldsymbol{c}$: perpendicular to the surface of $\boldsymbol{a}, \boldsymbol{b}$
  - If $\boldsymbol{a}$ and $\boldsymbol{b}$ are in parallel $(\sin\theta = 0)$, $\boldsymbol{a}$ or $\boldsymbol{b} = \boldsymbol{o}$, c $= \boldsymbol{o}$



- Theorem:
  - $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{o}$
  - $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
  - $\lambda\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \lambda\boldsymbol{b} = \lambda(\boldsymbol{a} \times \boldsymbol{b})$
  - $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = \boldsymbol{0}$
  - $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{i},\ \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{j}$

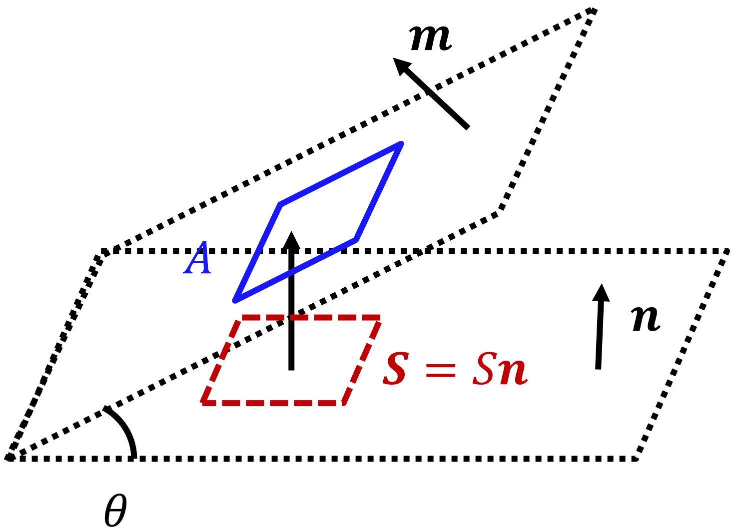
Vector area(面積ベクトル)

- Vector area: vector combining an area quality w/ dimension
- Assume surface S on signed area in two dimension system
  - Vector area $\boldsymbol{S}$ can be expressed with its unit vector $\boldsymbol{n}$
    - $\boldsymbol{S} = S\boldsymbol{n}$
  - Rotation of vector $\boldsymbol{n}$ express the sign
    - anticlockwise (right-hand screw) : plus
    - clockwsise (left-hand screw): minus
  - If S is subset of Si, the vector area $\boldsymbol{S}$ can be
    - $\boldsymbol{S} = \sum S_i \boldsymbol{n}_i$



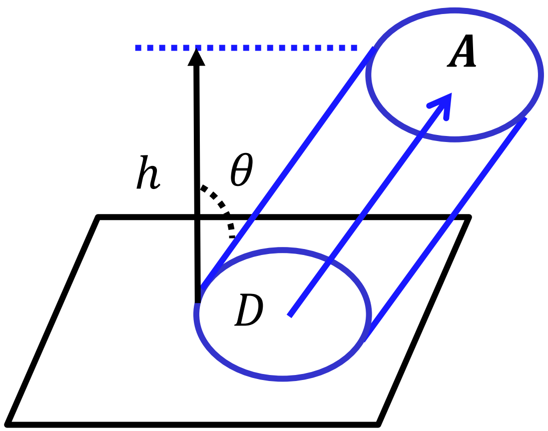
Projection (射影)

- Area vector is used to calculate surface integral
  - Treat flus of a vector filed through a surface
- Projection area A on plane $\boldsymbol{S}$ can be calculated by dot product with target plane unit normal $\boldsymbol{m}$
  - $A = \boldsymbol{S} \cdot \boldsymbol{m}$
- If the two surface has same xy and angle θ for z-coordinate,
  - $A = |\boldsymbol{S}| \cos \theta$



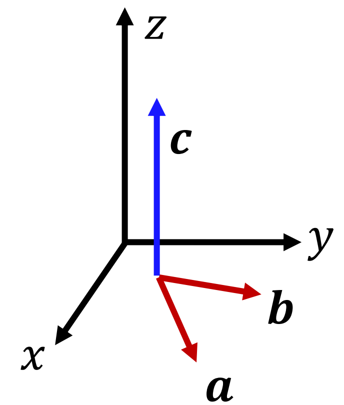
Volume (体積)

- Volume V can be calculated by area vector
  - Calculate volume V of tilted cylinder
    - Bottom plane: area vector $\boldsymbol{D}$
    - Direction: $\boldsymbol{A}$
    - Assume its angle: θ
  - Hight $h = |\boldsymbol{A}| \cos\theta$
  - Volume $V = h|\boldsymbol{S}| = |\boldsymbol{A}||\boldsymbol{S}| \cos\theta = \boldsymbol{A} \cdot \boldsymbol{S}$
- Volume $V = \boldsymbol{A} \cdot \boldsymbol{S}$ express the amount of flow $\boldsymbol{A}$ which punctulate the plane $\boldsymbol{D}$



Conclusion

- Start to learn for vector and array
- Scalar: Value (only)
- Vector: Value (length) and its direction
  - Vector from point P to Q is: $\overrightarrow{PQ} = \boldsymbol{a}$
- Inner product: $\boldsymbol{a} \cdot \boldsymbol{b} = \mathrm{c} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ (in scalar)
- Outer product: $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$
  - Orthogonal to the parallelogram of $\boldsymbol{a}$ and $\boldsymbol{b}$
  - Lentgh : $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$



- nishizawa@aoni.waseda.jp