Fundamental Mathematics (Engineering Mathematics)

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Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

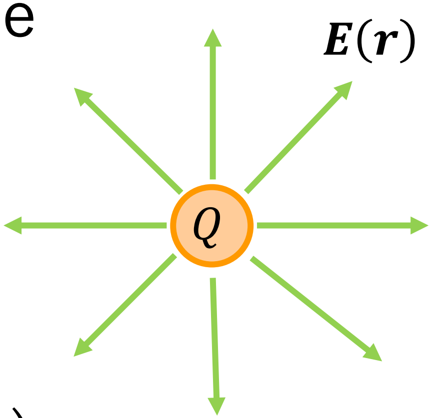
- Score: Exam (70%) + Report (20%) + Attendance (10%)

Motivation

- Many physics can be expressed by vectors
  - Good to explain in simple way (if we know vectors)
- Target: understand the meaning of Maxwell's equation
  - $\mathrm{div}\,\boldsymbol{D} = \rho$
    - $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho dV$ (Gauss's eq of electric-field)
  - $\mathrm{div}\,\boldsymbol{B} = 0$
    - $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \iiint \mathrm{div}\,\boldsymbol{B} dV$ (Gauss's eq of magnetic-field)
  - $\mathrm{rot}\,\boldsymbol{H} = i + \frac{\delta D}{\delta t}$: $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(i + \frac{\delta \boldsymbol{D}}{\delta t}\right) \cdot d\boldsymbol{S}$ (Ampele's law)
  - $\mathrm{rot}\,\boldsymbol{E} = -\frac{\delta B}{\delta t}$: $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\delta}{\delta t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$ (Faraday's law)

Electron and Electric field

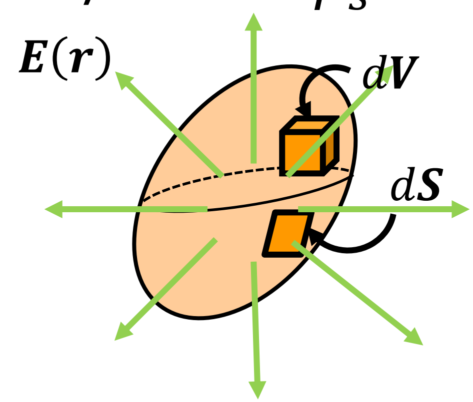
- Two electrons q_1 q_2 with distance r have attracting/repulsion force F
  - Coulomb's law $F = \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{r^2}$
    - $\epsilon = \epsilon_0 \epsilon_r$,
    - ϵ is dielectric constant (permittivity)
    - ϵ_0 is vacuum space permittivity $(=8.854 \times 10^{-12}\ \mathrm{C^2 N^{-1} m^{-2}})$
    - ϵ_r is relative permittivity

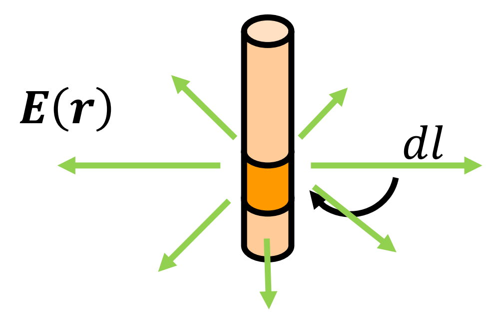


- Electron Q create vector field $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\frac{\boldsymbol{r}}{r}$
  - Coulomb's law in vector $\boldsymbol{F} = \frac{1}{4\pi\epsilon}\frac{qQ}{r^2}\frac{\boldsymbol{r}}{r} = q\boldsymbol{E}(\boldsymbol{r})$

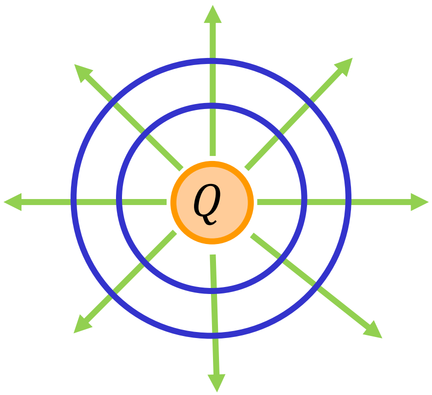
Electron and Electric field

- Electric field follows superposition law
  - For electron q_j for vector r_j $(j = 1, \ldots, N)$
    - $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon} \sum_{j=1}^{N} \frac{(\boldsymbol{r}-\boldsymbol{r}_j)}{|\boldsymbol{r}-\boldsymbol{r}_j|^3}$
  - For continuous distribution of electron $\rho_s d\boldsymbol{V}$, where ρ_s: electron density
    - For volume $d\boldsymbol{V}$: $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon} \iiint \frac{\rho_s(\boldsymbol{r}-\boldsymbol{s})}{|\boldsymbol{r}-\boldsymbol{s}|^3} d\boldsymbol{V}$
    - For surface $d\boldsymbol{S}$: $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon} \iint \frac{\rho_s(\boldsymbol{r}-\boldsymbol{s})}{|\boldsymbol{r}-\boldsymbol{s}|^3} d\boldsymbol{S}$
    - For line dl: $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon} \int \frac{\eta_s(\boldsymbol{r}-\boldsymbol{s})}{|\boldsymbol{r}-\boldsymbol{s}|^3} dl$





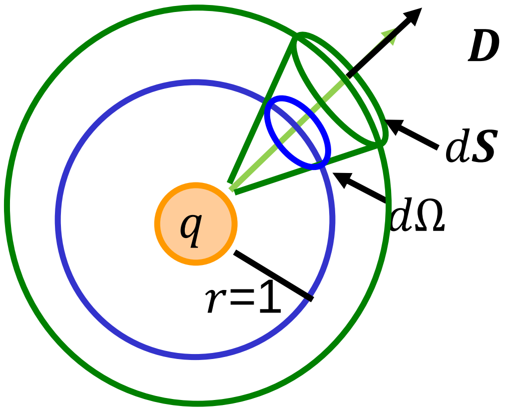
Electric flux



- Assume electron generate line of divergence
  - Electric flux (similar: electric line)
  - Electron Q generate Q-lines of electric flus
  - Density $\boldsymbol{D}$ should be change by the position
    - $\boldsymbol{D} = \epsilon \boldsymbol{E}$
  - Assume area vector $\boldsymbol{S}$ with its unit normal vector $\boldsymbol{n}$
    - $\boldsymbol{S} = \boldsymbol{n} S$
  - Amount of electric flux penetrate area S
    - $\phi = \boldsymbol{D} \cdot \boldsymbol{S}$
  - For small area $d\boldsymbol{S}$
    - $d\phi = \boldsymbol{D} \cdot d\boldsymbol{S}$, $\phi = \iint d\phi = \iint \boldsymbol{D} \cdot d\boldsymbol{S}$

Gauss's law for electric field

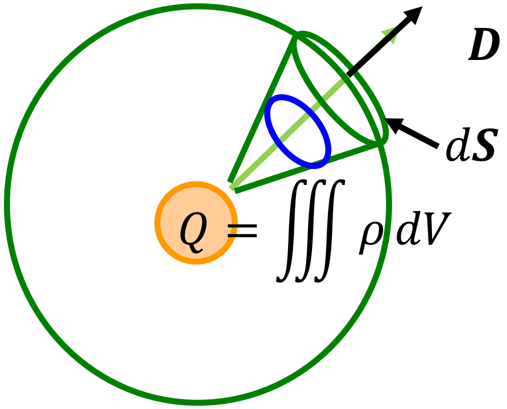
- Relationship between electric flux $d\phi$ generated by electron q, and its penetrating area $d\boldsymbol{S}$ on any shape
  - $d\phi = \boldsymbol{D} \cdot d\boldsymbol{S}$
- For the sphere w/ diameter of one, electric flux ratio should
  - $q\frac{d\Omega}{4\pi} = \begin{cases} \boldsymbol{D} \cdot d\boldsymbol{S} & (\boldsymbol{r} \cdot d\boldsymbol{S} > 0) \\ -\boldsymbol{D} \cdot d\boldsymbol{S} & (\boldsymbol{r} \cdot d\boldsymbol{S} < 0) \end{cases}$



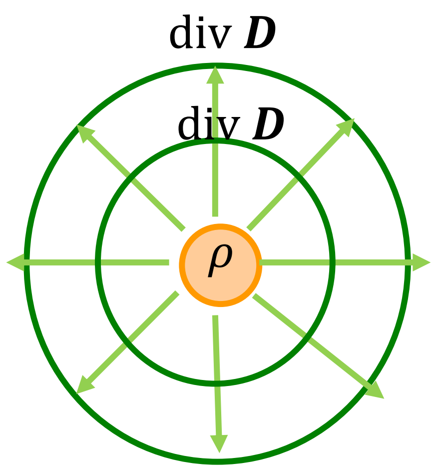
- If $\boldsymbol{r} \cdot d\boldsymbol{S} > 0$ (curve is convex)
  - $\phi = \iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iint q\frac{d\Omega}{4\pi} = q$
  - For volume w/ electron density ρ
    - $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho \, dV$

Physical meaning of Gauss's law for electric field

- $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho \, dV$ (integral from)
  - $\iiint \rho \, dV$: Total amount of electrons inside the volume
  - $\iint \boldsymbol{D} \cdot d\boldsymbol{S}$: Total amount of electric flux flow-outs from the surface



- $\operatorname{div} \boldsymbol{D} = \rho$ (differential form)
  - $\operatorname{div} \boldsymbol{D}$: divergence electric flux (density)
  - ρ : electron (density)

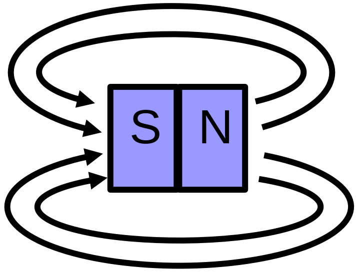


Magnetics and magnetic field

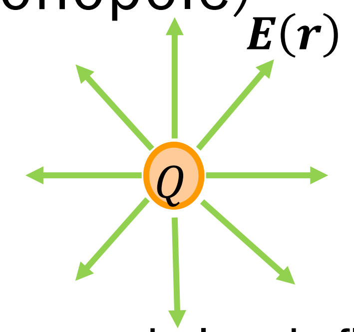
- Two amount of magnetics q_{m1} q_{m2} with distance r have attracting/repulsion force F
  - Coulomb's law for magnetics $F = \frac{1}{4\pi\mu}\frac{q_{m1}q_{m2}}{r^2}$
    - $\mu = \mu_0\mu_r$,
    - μ is magnetic permeability (permeability)
    - μ_0 is permeability in vacuum $(=4\pi \times 10^{-7}\ \mathrm{H/m})$
    - μ_r is relative permeability
- Monopole Q_m create magnetic field $\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\frac{Q_m}{r^2}\frac{\boldsymbol{r}}{r}$
  - Coulomb's law in vector $\boldsymbol{F} = \frac{1}{4\pi\epsilon}\frac{q_m Q_m}{r^2}\frac{\boldsymbol{r}}{r} = q_m\boldsymbol{H}(\boldsymbol{r})$

Gauss's law for magnetic field

- Magnetic pole of q_m generates q_m-lines of magnetic flux
  - Magnetic flux density $\boldsymbol{B}$ create magnetic field $\boldsymbol{H}$
    - $\boldsymbol{B} = \mu \boldsymbol{H}$
- Magnetic should in dipole (set of S and N, no monopole)
  - Same amount of flux from N to S
    - $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \iiint \mathrm{div}\, \boldsymbol{B} dV = 0$ (integral form)
    - $\mathrm{div}\, \boldsymbol{B} = 0$ (differential form)
- Gauss's law for magnetic field
  - No divergence in magnetic field (no monopole)



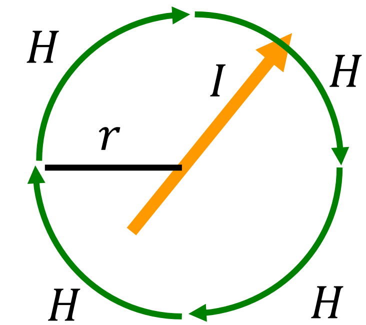
dipole of magnetics and magnetic flux



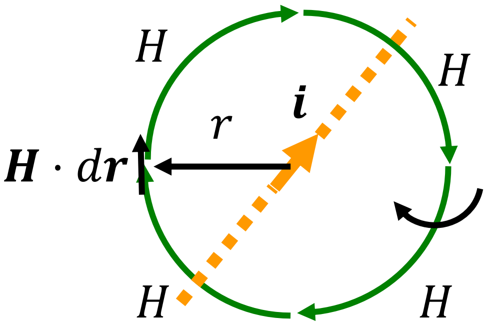
electron and electric flux

Magnetics and current flow

- Biot-Savart law: Constant current I create magnetic field H at the position of r
  - $H = \frac{I}{2\pi r}$
  - Right-hand turning (clockwise)

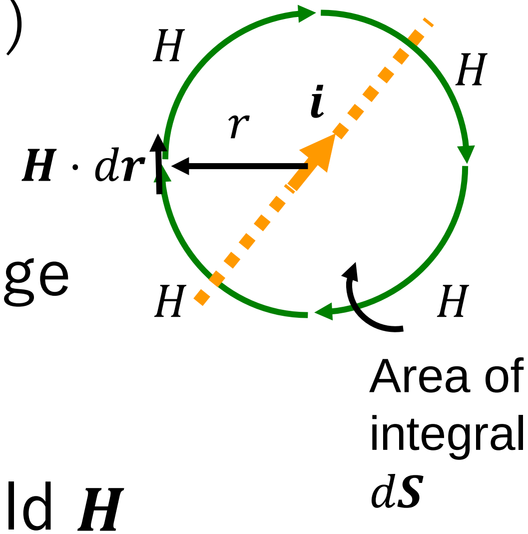


- Ampele's law: relationship of current I and magnetic field H
  - $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = I$ (take integral of Biot-Savart law)
  - For the continuous current, use current density $\boldsymbol{i}$ then
    - $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \boldsymbol{i} \cdot d\boldsymbol{S}$
  - Include current change term
    - $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}\right) \cdot d\boldsymbol{S}$

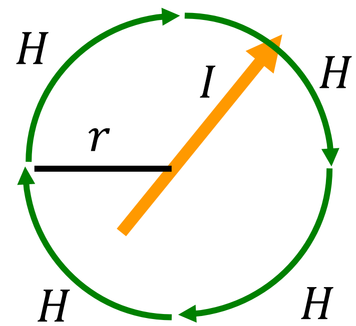


Physical meaning of Ampele's law

- $\oint \boldsymbol{H}\cdot d\boldsymbol{r} = \iint \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}\right)\cdot d\boldsymbol{S}$ (integral form)
  - $\iint \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}\right)\cdot d\boldsymbol{S}$: amount of constant current $\boldsymbol{i}$ and current change (=change in electric filed density $\frac{\partial \boldsymbol{D}}{\partial t}$) in area $\boldsymbol{S}$
  - $\oint \boldsymbol{H}\cdot d\boldsymbol{r}$: line integral of magnetic field $\boldsymbol{H}$

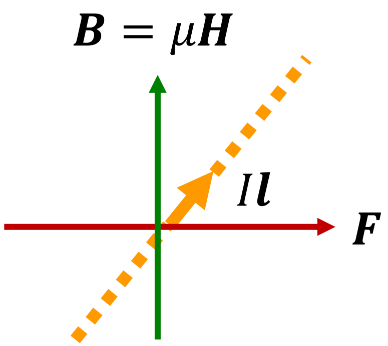


- rot $\boldsymbol{H} = i + \frac{\delta D}{\delta t}$
  - If rotating vector H exists, constant current $\boldsymbol{i}$ or current change $\frac{\partial \boldsymbol{D}}{\partial t}$ exists



Fleming's left-hand rule

- Current create magnetic flux
  - Behave like magnetic dipole
  - Constant current I in length $\boldsymbol{l}$ in uniform magnetic flux density $\boldsymbol{B}$ is force $\boldsymbol{F}$
    - $\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B}$ (outer product)

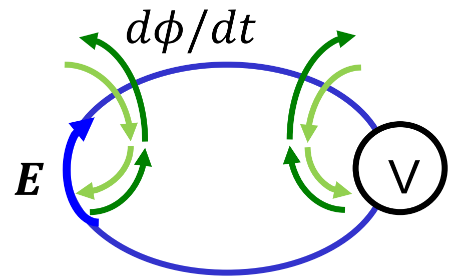


Lorentz force

- Fleming's left-hand rule: current I receive a force F from magnetic field density B
  - Moving electron receive a power by magnetic field
- Assume electrons qn with speed v, cross section of line S
  - $I = nqSv$ thus $\boldsymbol{F} = nqS\boldsymbol{v} \times \boldsymbol{B}l$
  - For one electron: $\boldsymbol{f} = q\boldsymbol{v} \times \boldsymbol{B}$: Lorentz force

Faraday's electromagnetic induction law

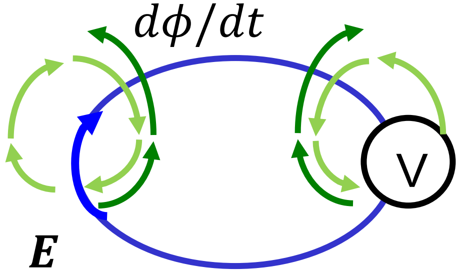
- Change of magnetic flux ϕ on inductor create electro motive force V
  - $V = -\frac{d\phi}{dt}$ (Faraday's electromagnetic induction law)
  - Change of magnetic flux:



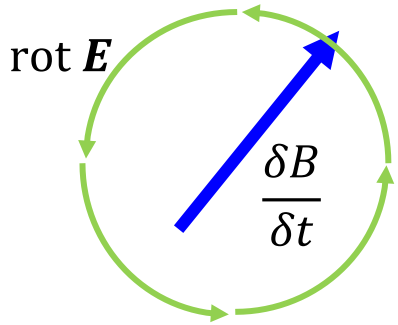
- Change of magnetic flux ϕ : try to create magnetic flux $-\phi$ to cancel out
  - Generate electro motive force $\boldsymbol{E}$ (Lenz's law)
- Total sum of electro motive force $V = \oint \boldsymbol{E} \cdot d\boldsymbol{r}$
- Total sum of magnetic flux $\phi = \iint \boldsymbol{B} \cdot d\boldsymbol{S}$
  - $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\partial}{\partial t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$

Physical meaning of Faraday's electromagnetic induction law

- $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\delta}{\delta t}\iint \boldsymbol{B} \cdot d\boldsymbol{S}$ (integral form)
  - $-\frac{\delta}{\delta t}\iint \boldsymbol{B} \cdot d\boldsymbol{S}$: amount of magnetic flux change in area S
  - $\oint \boldsymbol{E} \cdot d\boldsymbol{r}$: Total sum of electro motive force



- $\mathrm{rot}\,\boldsymbol{E} = -\frac{\delta B}{\delta t}$ (differential form)
  - $\mathrm{rot}\,\boldsymbol{E}$: electro motive force exists in rotation
  - $-\frac{\delta B}{\delta t}$: amount of magnetic flux changes



Fundamental Mathematics

- Complex function theory -

Motivation

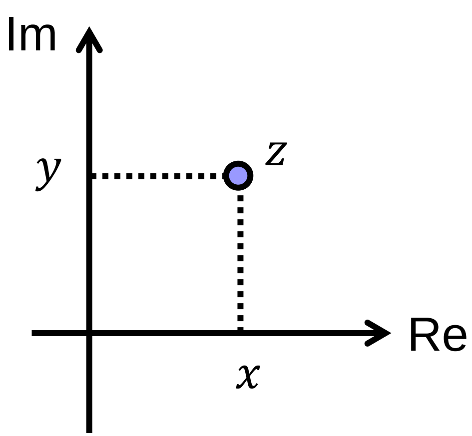
- Introduce Fourier transform and Laplace transform
  - Fourier transform: $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$
    - Conversion of time-domain function $f(t)$ to frequency-domain function $F(\omega)$
  - Laplace transform: $F(s) = \int_{0}^{\infty} f(x) e^{-st} dt$
    - Conversion (map) of differential equation in time-domain function $f(t)$ to s-domain function $F(s)$
    - s: complex number
    - Use for AC circuit analysis

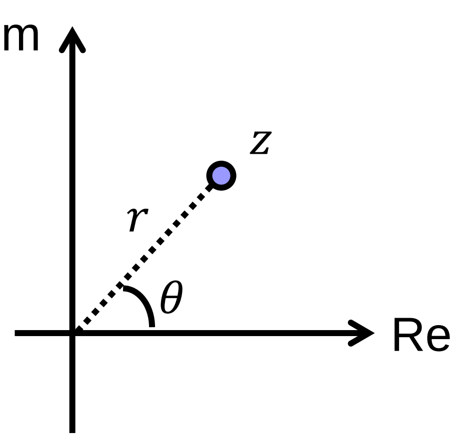
Complex number (複素数)

- $z = x + iy$ is called complex number ($x, y \in \mathbb{R}$: real number)
  - i: the imaginary unit (in electric circuit, use j instead)
    - $\mathrm{Re}\{z\} = x$, $\mathrm{Im}\{z\} = y$
  - Conjugate complex (共役複素数) of z: $\bar{z} = x - iy$
    - $\mathrm{Re}\{z\} = x = \frac{z+\bar{z}}{2}$, $\mathrm{Im}\{z\} = y = \frac{z-\bar{z}}{2i}$
  - Absolute value $|z|$ is real number
    - $z^2 = z\bar{z} = x^2 + y^2 \in \mathbb{R}$

The complex plane

- Complex plane: express points in rectangular coordinate system w/ complex value
- Polar coordinate system: express points in rectangular coordinate system w/ length of origin r and angle θ
  - $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ (Euler's law)
  - Conversion:
    - $r = \sqrt{x^2 + y^2}$, $\theta = \arg z = \tan^{-1}\frac{y}{x}$





de Moivre's (ド・モアブル) theorem

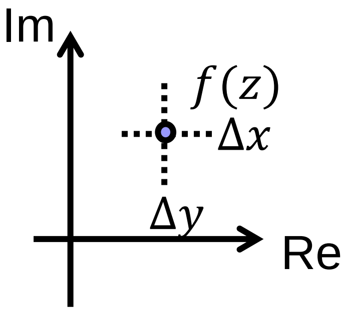
- Products, Quotients, de Moivre's theorem
  - Assume $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
  - $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$
  - $z_1/z_2 = r_1/r_2(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$
    - Length: multiple of two length
    - Angle: sum of two angle
- de Moivre's theorem
  - $z^n = r^n(\cos(n\theta) + i\sin(n\theta))\ (n \in \mathbb{Z})$
  - $\sqrt[n]{z} = r^{1/n}\left(\cos\left(\frac{\theta}{n} + \frac{2m\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2m\pi}{n}\right)\right)\ (n, m \in \mathbb{Z})$
  - n candidates of complex values satisfy above equation.

Differential for complex function

- Assume Complex function $w = f(z)$ $(w, z) \in \mathbb{C}$: complex
- Definition of differential
  - If following is satisfied, $f(z)$ is continuous at $z = z_0$
    - $\lim_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$
  - If following is available, $f(z)$ differentiable at $z = z_0$
    - $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{df}{dz}(z_0) = f'(z_0)$
    - $f(z)$ is called as regular analytic function
  - Similar to the definition in differential in real function, but this should take convergence from any angle of Δz in complex plane

“Differentiable” of complex func.

- $f(z)$ is differentiable at $z = z_0$ if following is available
  - $\lim\limits_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{df}{dz}(z_0) = f'(z_0)$
  - For complex func. any Δz satisify its limits $\Delta z \to 0$
- Calculate limit in real/imaginary axis



  - Assume $f(z) = u(x, y) + jv(x, y)$ $(x, y, u, v \in \mathbb{R})$
    - Take limit in real axis
      - $\lim\limits_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim\limits_{\Delta z \to 0} \left[\frac{u(x_0+\Delta x, y_0)}{\Delta x} + i\frac{v(x_0+\Delta x, y_0)}{\Delta x}\right] = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$
    - Take limit in imaginary axis
      - $\lim\limits_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim\limits_{\Delta z \to 0} \left[\frac{u(x_0, y_0+\Delta y)}{i\Delta y} + i\frac{v(x_0, y_0+\Delta y)}{i\Delta y}\right] = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)$
  - If this is differentiable, both limits should be the same

Cauchy–Riemann equations

- If $f(z)$ is differentiable, all of limits should be the same
  - $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (from the real part of the limits)
  - $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (from the imaginary part of the limits)
    - This is called Cauchy–Riemann equations (コーシー・リーマン方程式)
    - $f(z)$ is called as regular analytic function (正則関数)
      - $\frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}$
    - Real part and imaginary part of regular analytic function satisfy following Laplace equation
      - $\frac{\partial u^2}{\partial x^2} + \frac{\partial v^2}{\partial y^2} = 0, \frac{\partial v^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0$

Regular analytic functions

- $f(z)$ is differentiable at $z = z_0$ if following is available
  - $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{df}{dz}(z_0) = f'(z_0)$
  - If above follows all points z in region D, $f(z)$ is regular analytic function in region D
- If both f and g are regular analytic functions, $f \pm g$, fg, f/g are also regular, and they satisfy
  - $(f \pm g)' = f' \pm g'$, $(fg)' = f'g + fg'$, $(f/g)' = (f'g - fg')/g^2$

- Next, check the regularity of several functions

Basic regular analytical functions 1

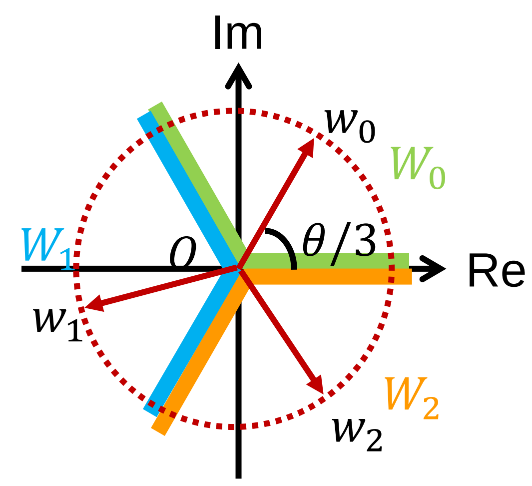
- Exponent function
  - $w = e^z = e^x e^{iy} = e^x(\cos y + i \sin y) = u + iv$
  - $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$
    - Cauchy–Riemann equations are satisfied
  - $(e^z)' = e^x(\cos y + i \sin y) = e^z$
    - $\frac{de^z}{dz} = e^z$
- Sine function
  - Exponent func. is regular analytical -> its sum also regular analytical
  - $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \cosh z = \frac{e^z+e^{-z}}{2}, \sinh z = \frac{e^z-e^{-z}}{2}$

Basic regular analytical functions 2

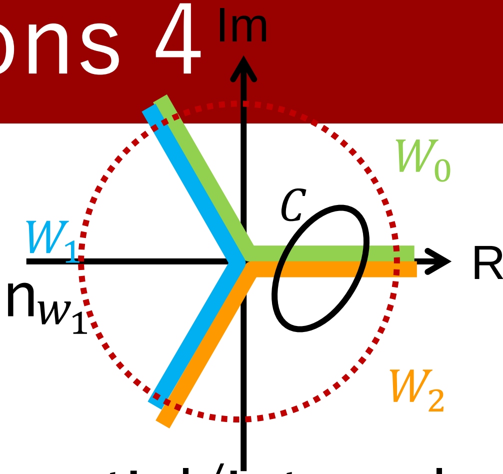
- Sine function
  - Exponent func. is regular analytical -> its sum also regular analytical
  - $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^{z}+e^{-z}}{2}$, $\sinh z = \frac{e^{z}-e^{-z}}{2}$
  - $\frac{d \cos z}{dz} = \frac{ie^{iz}-ie^{-iz}}{2} = -\sin z$
  - $\frac{d \sin z}{dz} = \frac{ie^{iz}+ie^{-iz}}{2i} = \cos z$
  - $\frac{d \cosh z}{dz} = \frac{e^{z}-e^{-z}}{2} = \sinh z$
  - $\frac{d \sinh z}{dz} = \frac{e^{z}+e^{-z}}{2} = \cosh z$

Basic regular analytical functions 3

- Inverse function
  - For $w = f(z)$, if we can swap w and z and the function $z = f(w)$ can be solved by $w = g(x)$, this is inverse func.
  - $w^3 = z$: inverse of $w = z^3$
    - Solutions for $z = re^{i\theta}$ $(0 \leq \theta < 2\pi)$
      - $w_0 = \sqrt[3]{r}e^{\frac{\theta}{3}i}$, $w_1 = w_0 e^{\frac{2\pi}{3}i}$, $w_2 = w_0 e^{\frac{4\pi}{3}i}$ (multifunction, branch)
    - Three solutions are available in region W_0, W_1, W_2
      - w_0 in W_0 $\left(0 \leq \arg w < \frac{2\pi}{3}\right)$
      - w_1 in W_1 $\left(\frac{2\pi}{3} \leq \arg w < \frac{4\pi}{3}\right)$
      - w_2 in W_2 $\left(\frac{2\pi}{3} \leq \arg w < 2\pi\right)$
    - Origin O is not differentiable
      - Branch point



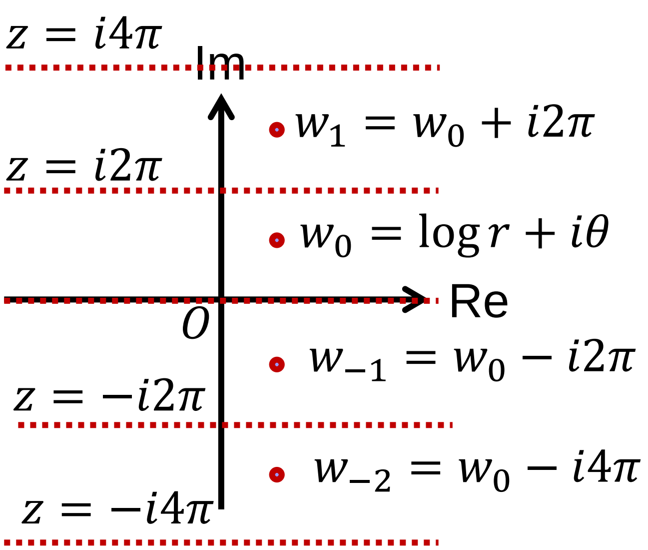
Basic regular analytical functions 4



- Think about close curve C (for integral)
  - If C is within the region, w is same function w_1
  - If C covers the branch, w become change
  - We should assume proper branch for differential/integral
- For inverse function $w = z^n$ $(z = re^{i\theta}$ $(r \geq 0, 0 \leq \theta < 2\pi))$
  - nth branches (solutions): $w_0 = \sqrt[n]{r}e^{\frac{\theta}{n}i}$, $w_1 = w_0 e^{\frac{2\pi}{n}i}$, $\cdots$
  - $\frac{d}{dz}\sqrt[n]{z} = \frac{1}{n}\frac{1}{\left(\sqrt[n]{z}\right)^{n-1}}$ $(z \neq 0)$
  - Both right and left eq. should within the same branch

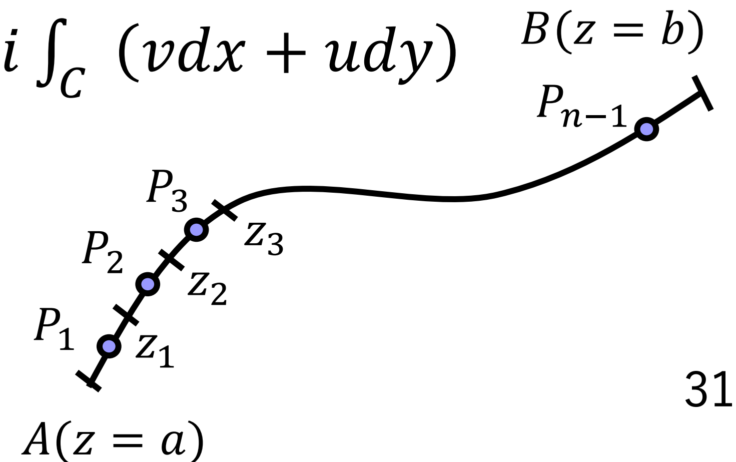
Basic regular analytical functions 5

- Logarithmic function $w = \log z$
  - For exponential $z = re^{i\theta}$ $(r \geq 0, 0 \leq \theta < 2\pi)$
    - Assume $w = \log z = u + iv$
      - $r = e^u$, $e^{i\theta} = e^v$-> $u = \log r$, $v = \theta + 2n\pi$ $(n \in \mathbb{N})$
      - w is multifunction, it has infinite branches
      - Point $z = 0$ is not differentiable
  - $\log z = \log r + i(\theta + 2n\pi)$ $(r \geq 0, 0 \leq \theta < 2\pi)$
    - $\frac{d}{dz}\log z = \frac{1}{z}$ $(\mathrm{z} \neq 0)$



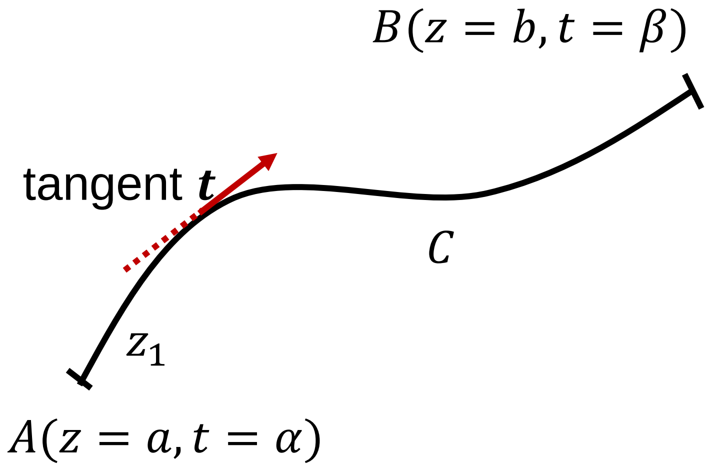
Integral of complex function

- Assume a smooth curve C from point $A(a = z)$ to $B(z = b)$, and scalar function $f(z)$ is continuous in curve C
  - Think curve C can divide into several arcs $\Delta z_1 \cdots \Delta z_2$
    - Points z_n divide a curve, these weight are points P_n
    - Limit of $n \to \infty$, $\Delta \mathrm{z_i} \to 0$; <u>complex integral (複素積分)</u>
    - $\lim_{\substack{n\to\infty \\ \Delta \mathrm{z_i}\to 0}} \sum_{i=1}^{n} f(P_i)\Delta \mathrm{z_i} = \int_C f(z)\, dz$
  - Assume $\Delta z_{\mathrm{i}} = \Delta x_{\mathrm{i}} + i\Delta y_{\mathrm{i}}$, $f(z) = u + iv$
    - $\int_C f(z)\, dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$



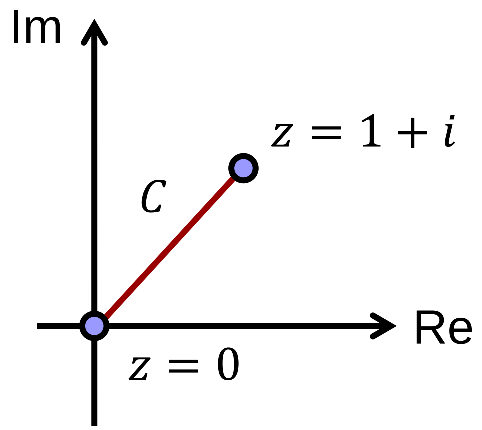
Integral of complex function

- Assume function of C is $z = z(t) = x(t) + iy(t)\ (\alpha \le t < \beta)$
  - Derivative: $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$
  - This derivative (vector) is a tangent of curve C
    - $\int_C f(z)\,dz = \int_\alpha^\beta f\big(z(t)\big)\frac{dz}{dt}dt$
  - Convert complex integral to definite integral



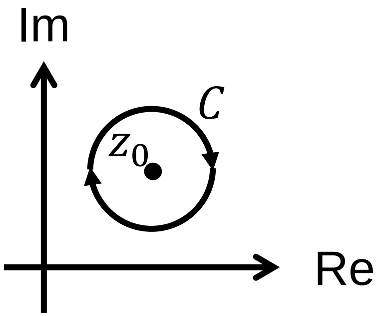
Integral of complex function

- Convert complex integral to definite integral
- Ex. integrate z^2 in line C from $z = 0$ to $z = 1 + i$
  - Solution: re-write line C using parameter t (媒介変数)
    - $z(t) = t + it \ (0 \leq t < 1)$
  - Derivative is; $dz = \frac{dz}{dt}dt = \frac{d(t+it)}{dt}dt = (1 + i)dt$
    - $\int_C z^2 \, dz = \int_0^1 (t + it)(1 + i)dt = (-2 + 2i)\left[\frac{1}{3}t^3\right]_0^1 = -\frac{2}{3} + \frac{2}{3}i$



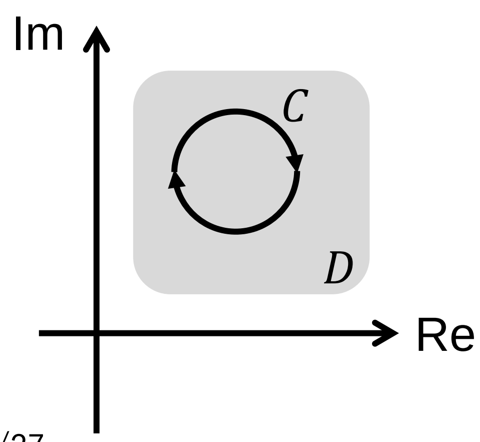
Integral of complex function

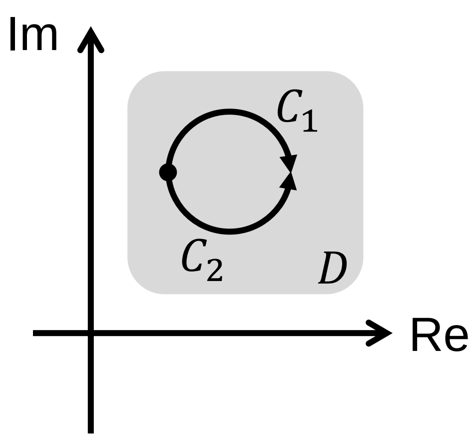
- Convert complex integral to definite integral
- Ex. $(z - z_0)^n$ $(n \in \mathbb{Z})$ in circle $|z - z_0| = \rho$
  - Solution: re-write circle using parameter θ
    - $z(\theta) = z_0 + \rho e^{i\theta}$ $(0 \leq \theta < 2\pi)$, $d\theta = \frac{d(z_0 + \rho e^{i\theta})}{d\theta} d\theta = i\rho e^{i\theta} d\theta$
    - $\oint_C (z - z_0)^n dz = i\rho e^{i\theta} \int_0^{2\pi} \rho e^{i(n+1)\theta} d\theta$ (*)
  - For $n \neq 1$:(*) $= i\rho^{(n+1)} \int_0^{2\pi} [\cos(n+1)\theta + i \sin(n+1)\theta] \, d\theta$
    - $= \frac{i\rho^{(n+1)}}{n+1} [\sin(n+1)\theta - i \cos(n+1)\theta]_0^{2\pi} = 0$
  - For $n = 1$:(*) $\oint_C (z - z_0)^1 dz = i \int_0^{2\pi} 1 \, d\theta = 2\pi i$



Cauchy's theorem (コーシーの定理)

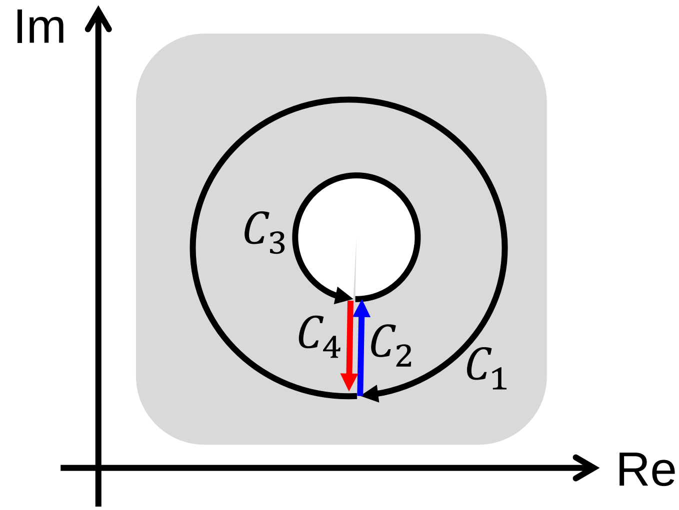
- If $f(z)$ is regular analytical in region D, and curve C is a closed curve, its integral is:
  - $\oint_C f(z)\,dz = 0$: Cauchy's theorem
- If C is divided into two curves, C_1, C_2
  - $\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz$, thus $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$
  - Note: route must not cross the non-analytical points and branches





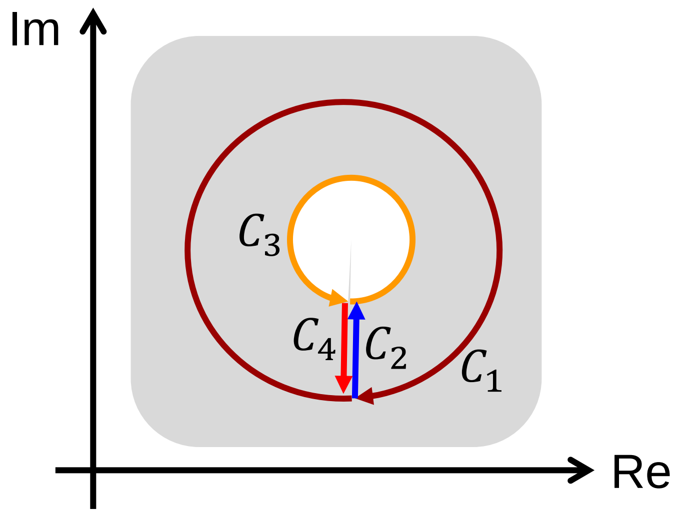
Cauchy's theorem for multiply connected domain

- For multiply connected domain (non-uniform domain, domain w/ hole), divide domain into several domains
  - Red part and blue part are cancel out
    - thus $\oint_{C_2} f(z)\,dz = -\oint_{C_4} f(z)\,dz$
  - Use for equation w/ non-analytical points
    - $(z = 0$ for $f(z) = 1/z)$



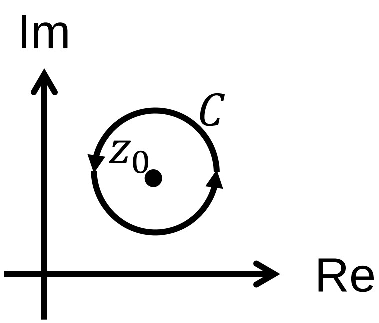
Cauchy's theorem for multiply connected domain

- For the path $C' = C_1 + C_2 + C_3 + C_4$, C' is close circle
  - $\oint_{C'} f(z)\,dz = 0$: Cauchy's theorem
  - $\oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \oint_{C_4} f(z)\,dz = 0$
    - $\oint_{C_1} f(z)\,dz + \oint_{C_3} f(z)\,dz = 0$
- If $f(z)$ is regular analytical for two closed circles C_1, C'_3(inverse of C_3), its integral becomes same
  - We can change the route



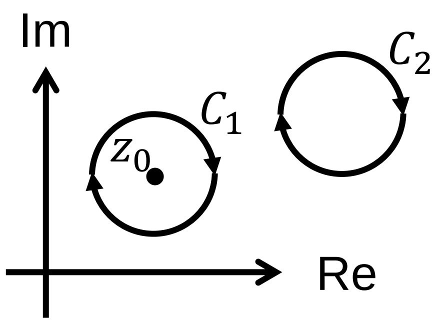
Cauchy's integral theorem

- Describe the value of complex function $f(z_0)$ at $z = z_0$ using circle integral
  - $2\pi i f(z_0) = \oint_C \frac{f(z)}{z-z_0} dz$, where z_0 and C are any point and circle in region D where $f(z)$ is a regular analytical



Usage of Cauchy's integral theorem

- Assume to take circle integral over C, and $f(z)$ is a regular analytical in region D
  - If point $z = z_0$ is inside of the circle C_1
    - $\oint_{C_1} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$
  - Else; (point $z = z_0$ is outside of the circle C_2)
    - $\oint_{C_2} \frac{f(z)}{z-z_0} dz = 0$



Theorem for regular analytical function

- Assume to take circle integral over C, and $f(z)$ is a regular analytical in region D
  - $f(z)$ can take n-th order differentiate $f^{(n)}(z)$
  - $f^{(n)}(z)$ can be expressed as
    - $f^{(n)}(z)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \quad (n=0,1,2,\ldots)$
- If $f(z)$ is a regular analytical in region D, $f^{(n)}(z)$ is available
  - Regular analytical means very limited case of function

Exercise

- Translate following equations in polar coordinate system
  - $z = \sqrt[3]{1+i}$
- Answer following for $w = z^4 = u + vi$, assume $z = x + yi$
  - Calculate u and v
  - Proof u and v satisfy Cauchy–Riemann equations
  - Calculate w'
- Integrate $f(z) = 1/z$ in unit circle C
- Integrate $f(z) = \cos z$ from $z = 0$ to $z = i$

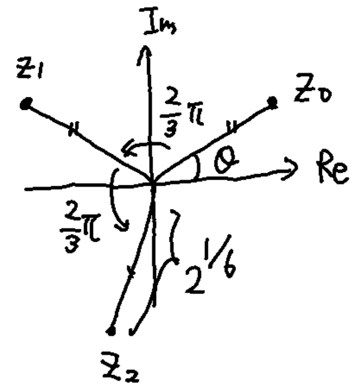
$$\sqrt[3]{1+i} = \sqrt[3]{\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)} = 2^{1/6}\cdot\sqrt[3]{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}}$$

$$= 2^{1/6}\cdot\left(e^{\left(\frac{\pi}{4}+2n\pi\right)i}\right)^{\frac{1}{3}}$$

$$= 2^{1/6}\cdot\left(e^{\frac{\pi}{12}i+\frac{2n\pi}{3}i}\right)$$

$$= 2^{1/6}\left(\cos\left(\frac{\pi}{12}i+\frac{2n\pi}{3}i\right)+i\sin\left(\frac{\pi}{12}i+\frac{2n\pi}{3}i\right)\right)$$

$n=0,1,2$



[2] $w = z^4 = u + iv$

① Calc. u and v

$$z^4 = (x+iy)^4 = (x^2+2ixy-y^2)^2$$

$$= x^4+2ix^3y-x^2y^2+2ix^3y-4x^2y^2-2ixy^3$$

$$-x^2y^2-2ixy^3+y^4$$

$$= \underbrace{(x^4-6x^2y^2+y^4)}_{u} + \underbrace{4xy(x^2-y^2)}_{v}\, i$$

② Cauchy-Riemann

$$\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}_{(1)}, \quad \underbrace{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{(2)}$$

(1) $\dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2$

$\dfrac{\partial v}{\partial y} = 4x^3 - 12xy^2$

(2) $\dfrac{\partial u}{\partial y} = -12xy + 4y^3$

$-\dfrac{\partial v}{\partial x} = -12xy + 4y^3$ Satisfy.

③ $$w' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 4\{(x^3-3xy^2) - i(3xy - y^3)\}$$

$\boxed{3}$ integral

(1) $f(z) = 1/z$ for unit circle C.

use angle θ

$$z(\theta) = e^{i\theta} \quad (0 \le \theta < 2\pi)$$

$$dz = de^{i\theta}\frac{d\theta}{d\theta} = ie^{i\theta}d\theta$$

$$\oint_C \frac{1}{z}dz = \int_0^{2\pi} e^{-i\theta} \cdot i e^{i\theta} d\theta = i[\theta]_0^{2\pi} = 2\pi i$$

(2) $f(z) = \cos z$, from $z = 0$ to $z = i$

$f(z)$ is regular analytical thus

$$\int_0^i \cos z \, dz = [\sin z]_0^i = \sin(i) - \sin 0. = \sin(i)$$

$$\sin i = \frac{e^{i \cdot i} - e^{-i \cdot i}}{2i} = \frac{e^{-1} - e^{1}}{2i} = \frac{-1}{i}\frac{e - e^{-1}}{2} = i\sinh(1)$$

Conclusion

- Introduce complex function theory
  - Complex plane ~ similar to the vector (in 2D space)
  - de Moivre's theorem
  - Complex differential
    - If available, $f(z)$ is regular analytic function
    - Some functions have several solutions
      - Multifunction or it has branches
      - Do not take differential cross over the branches
  - Complex integral
    - Cauchy's theorem: $\oint_C f(z)\,dz = 0$
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