Fundamental Mathematics (Engineering Mathematics)

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Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

Fundamental Mathematics

- Complex function theory -

Motivation

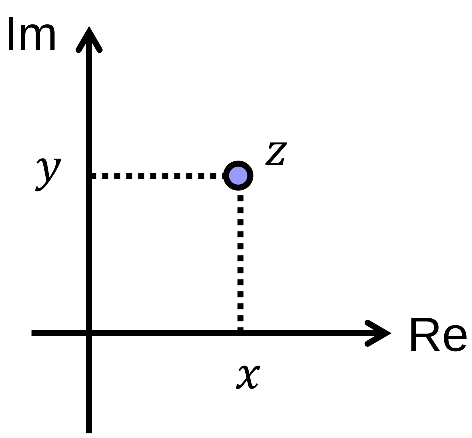
- Introduce Fourier transform and Laplace transform
  - Fourier transform: $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$
    - Conversion of time-domain function $f(t)$ to frequency-domain function $F(\omega)$
  - Laplace transform: $F(s) = \int_0^{\infty} f(x) e^{-st} dt$
    - Conversion (map) of differential equation in time-domain function $f(t)$ to s-domain function $F(s)$
    - s: complex number
    - Use for AC circuit analysis

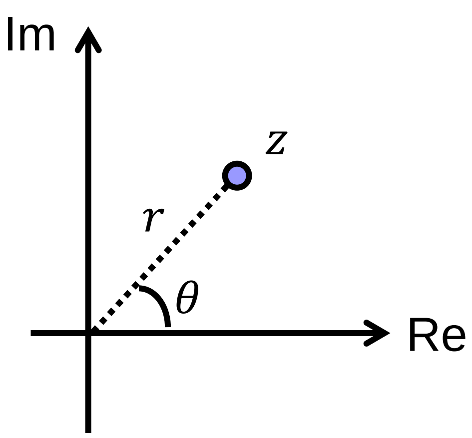
Complex number (複素数)

- $z = x + iy$ is called complex number ($x, y \in \mathbb{R}$: real number)
  - i: the imaginary unit (in electric circuit, use j instead)
    - $\mathrm{Re}\{z\} = x$, $\mathrm{Im}\{z\} = y$
  - Conjugate complex (共役複素数) of z: $\bar{z} = x - iy$
    - $\mathrm{Re}\{z\} = x = \frac{z+\bar{z}}{2}$, $\mathrm{Im}\{z\} = y = \frac{z-\bar{z}}{2i}$
  - Absolute value $|z|$ is real number
    - $z^2 = z\bar{z} = x^2 + y^2 \in \mathbb{R}$

The complex plane

- Complex plane: express points in rectangular coordinate system w/ complex value
- Polar coordinate system: express points in rectangular coordinate system w/ length of origin r and angle θ
  - $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ (Euler's law)
  - Conversion:
    - $r = \sqrt{x^2 + y^2},\ \theta = \arg z = \tan^{-1}\frac{y}{x}$





de Moivre's (ド・モアブル) theorem

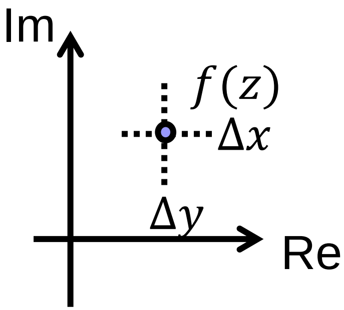
- Products, Quotients, de Moivre's theorem
  - Assume $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
  - $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$
  - $z_1/z_2 = r_1/r_2(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$
    - Length: multiple of two length
    - Angle: sum of two angle
- de Moivre's theorem
  - $z^n = r^n(\cos(n\theta) + i\sin(n\theta))\ (n \in \mathbb{Z})$
  - $\sqrt[n]{z} = r^{1/n}\left(\cos\left(\frac{\theta}{n} + \frac{2m\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2m\pi}{n}\right)\right)\ (n, m \in \mathbb{Z})$
  - n candidates of complex values satisfy above equation.

Differential for complex function

- Assume Complex function $w = f(z)$ $(w, z) \in \mathbb{C}$: complex
- Definition of differential
  - If following is satisfied, $f(z)$ is continuous at $z = z_0$
    - $\lim_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$
  - If following is available, $f(z)$ differentiable at $z = z_0$
    - $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{df}{dz}(z_0) = f'(z_0)$
    - $f(z)$ is called as regular analytic function
  - Similar to the definition in differential in real function, but this should take convergence from any angle of Δz in complex plane

"Differentiable" of complex func.

- $f(z)$ is differentiable at $z = z_0$ if following is available
  - $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{df}{dz}(z_0) = f'(z_0)$
  - For complex func. any Δz satisify its limits $\Delta z \to 0$
- Calculate limit in real/imaginary axis



  - Assume $f(z) = u(x,y) + iv(x,y)$ $\left(x, y, u, v \in \mathbb{R}\right)$
    - Take limit in real axis
      - $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta x \to 0} \left[\frac{u(x_0+\Delta x, y_0)}{\Delta x} + i\frac{v(x_0+\Delta x, y_0)}{\Delta x}\right] = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$
    - Take limit in imaginary axis
      - $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta y \to 0} \left[\frac{u(x_0, y_0+\Delta y)}{i\Delta y} + i\frac{v(x_0, y_0+\Delta y)}{i\Delta y}\right] = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)$
  - If this is differentiable, both limits should be the same

Cauchy–Riemann equations

- If $f(z)$ is differentiable, all of limits should be the same
  - $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (from the real part of the limits)
  - $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (from the imaginary part of the limits)
    - This is called <u>Cauchy–Riemann equations (コーシー・リーマン方程式)</u>
    - $f(z)$ is called as <u>regular analytic function (正則関数)</u>
      - $\frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}$
    - Real part and imaginary part of regular analytic function satisfy following Laplace equation
      - $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Regular analytic functions

- $f(z)$ is differentiable at $z = z_0$ if following is available
  - $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{df}{dz}(z_0) = f'(z_0)$
  - If above follows all points z in region D, $f(z)$ is regular analytic function in region D
- If both f and g are regular analytic functions, $f \pm g$, fg, f/g are also regular, and they satisfy
  - $(f \pm g)' = f' \pm g'$, $(fg)' = f'g + fg'$, $(f/g)' = (f'g - fg')/g^2$
- Next, check the regularity of several functions

Basic regular analytical functions 1

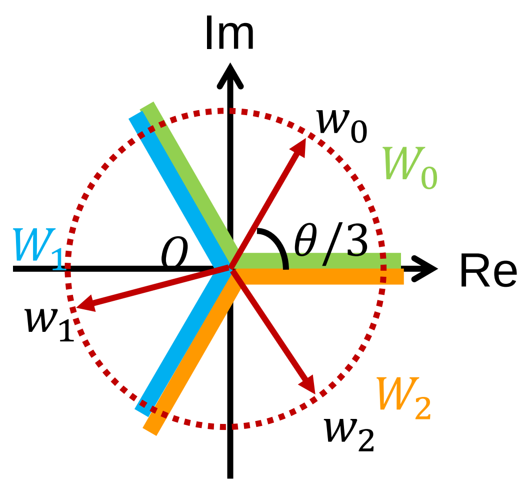
- Exponent function
  - $w = e^z = e^x e^{iy} = e^x(\cos y + i \sin y) = u + iv$
  - $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$
    - Cauchy–Riemann equations are satisfied
  - $(e^z)' = e^x(\cos y + i \sin y) = e^z$
    - $\frac{de^z}{dz} = e^z$
- Sine function
  - Exponent func. is regular analytical -> its sum also regular analytical
  - $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \cosh z = \frac{e^z+e^{-z}}{2}, \sinh z = \frac{e^z-e^{-z}}{2}$

Basic regular analytical functions 2

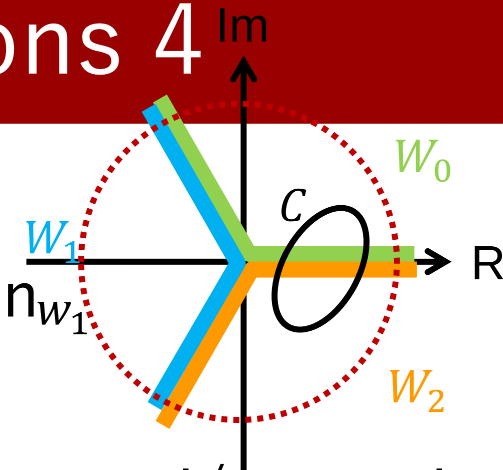
- Sine function
  - Exponent func. is regular analytical -> its sum also regular analytical
  - $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^{z}+e^{-z}}{2}$, $\sinh z = \frac{e^{z}-e^{-z}}{2}$
  - $\frac{d \cos z}{dz} = \frac{ie^{iz}-ie^{-iz}}{2} = -\sin z$
  - $\frac{d \sin z}{dz} = \frac{ie^{iz}+ie^{-iz}}{2i} = \cos z$
  - $\frac{d \cosh z}{dz} = \frac{e^{z}-e^{-z}}{2} = \sinh z$
  - $\frac{d \sinh z}{dz} = \frac{e^{z}+e^{-z}}{2} = \cosh z$

Basic regular analytical functions 3

- Inverse function
  - For $w = f(z)$, if we can swap w and z and the function $z = f(w)$ can be solved by $w = g(z)$, this is inverse func.
  - $w^3 = z$: inverse of $w = z^3$
    - Solutions for $z = re^{i\theta}$ $(0 \le \theta < 2\pi)$
      - $w_0 = \sqrt[3]{r}e^{\frac{\theta}{3}i}$, $w_1 = w_0 e^{\frac{2\pi}{3}i}$, $w_2 = w_0 e^{\frac{4\pi}{3}i}$ (multifunction, branch)
    - Three solutions are available in region W_0, W_1, W_2
      - w_0 in W_0 $\left(0 \le \arg w < \frac{2\pi}{3}\right)$
      - w_1 in W_1 $\left(\frac{2\pi}{3} \le \arg w < \frac{4\pi}{3}\right)$
      - w_2 in W_2 $\left(\frac{2\pi}{3} \le \arg w < 2\pi\right)$
    - Origin O is not differentiable
      - Branch point



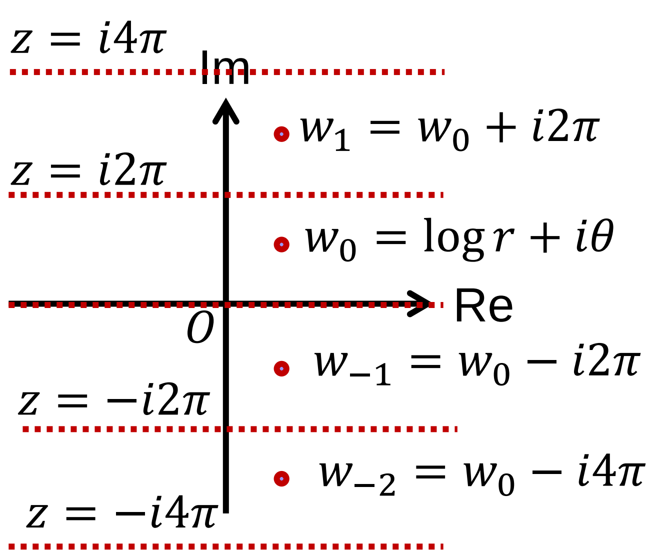
Basic regular analytical functions 4



- Think about close curve C (for integral)
  - If C is within the region, w is same function
  - If C covers the branch, w become change
  - We should assume proper branch for differential/integral
- For inverse function $w = z^n$ $(z = re^{i\theta}$ $(r \geq 0, 0 \leq \theta < 2\pi))$
  - nth branches (solutions): $w_0 = \sqrt[n]{r}e^{\frac{\theta}{n}i}$, $w_1 = w_0 e^{\frac{2\pi}{n}i}$, $\cdots$
  - $\frac{d}{dz}\sqrt[n]{z} = \frac{1}{n}\frac{1}{\left(\sqrt[n]{z}\right)^{n-1}}$ $(\mathrm{z} \neq 0)$
  - Both right and left eq. should within the same branch

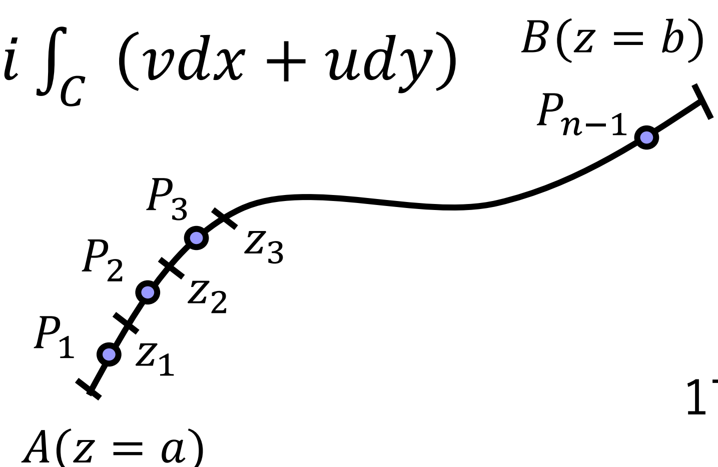
Basic regular analytical functions 5

- Logarithmic function $w = \log z$
  - For exponential $z = re^{i\theta}$ $(r \geq 0, 0 \leq \theta < 2\pi)$
    - Assume $w = \log z = u + iv$
      - $r = e^{u}$, $e^{i\theta} = e^{v}$-> $u = \log r$, $v = \theta + 2n\pi$ $(n \in \mathbb{N})$
      - w is multifunction, it has infinite branches
      - Point $z = 0$ is not differentiable
  - $\log z = \log r + i(\theta + 2n\pi)$ $(r \geq 0, 0 \leq \theta < 2\pi)$
    - $\frac{d}{dz}\log z = \frac{1}{z}$ $(\mathrm{z} \neq 0)$



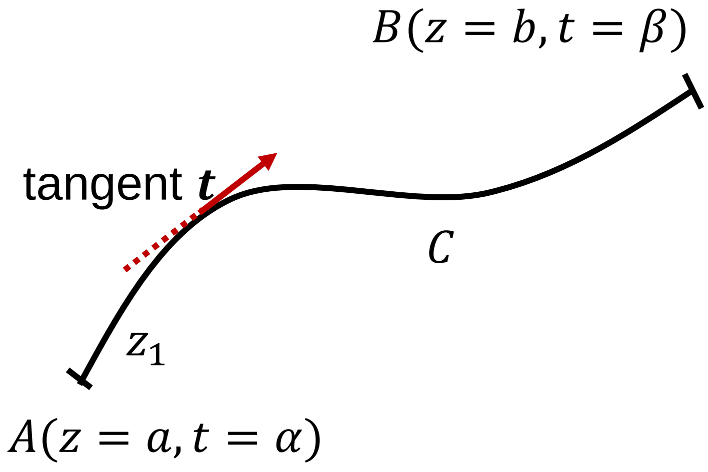
Integral of complex function

- Assume a smooth curve C from point $A(a = z)$ to $B(z = b)$, and scalar function $f(z)$ is continuous in curve C
  - Think curve C can divide into several arcs $\Delta z_1 \cdots \Delta z_n$
    - Points z_n divide a curve, these weight are points P_n
    - Limit of $n \to \infty$, $\Delta \mathrm{z_i} \to 0$; <u>complex integral (複素積分)</u>
    - $\lim_{\substack{n\to\infty \\ \Delta \mathrm{z_i}\to 0}} \sum_{i=1}^{n} f(P_i)\Delta \mathrm{z_i} = \int_C f(z)\,dz$
  - Assume $\Delta z_\mathrm{i} = \Delta x_\mathrm{i} + i\Delta y_\mathrm{i}$, $f(z) = u + iv$
    - $\int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$



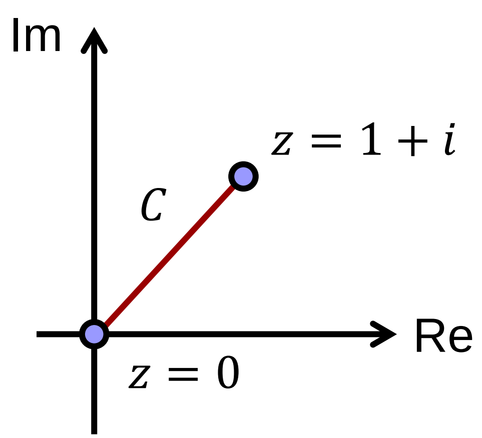
Integral of complex function

- Assume function of C is $z = z(t) = x(t) + iy(t)$ $(\alpha \leq t < \beta)$
  - Derivative: $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$
  - This derivative (vector) is a tangent of curve C
    - $\int_C f(z)\,dz = \int_\alpha^\beta f\big(z(t)\big)\frac{dz}{dt}dt$
  - Convert complex integral to definite integral



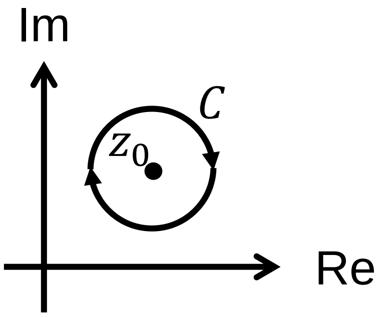
Integral of complex function

- Convert complex integral to definite integral
- Ex. integrate z^2 in line C from $z=0$ to $z=1+i$
  - Solution: re-write line C using parameter t (媒介変数)
    - $z(t) = t + it \ (0 \leq t < 1)$
  - Derivative is; $dz = \frac{dz}{dt}dt = \frac{d(t+it)}{dt}dt = (1+i)dt$
    - $\int_C z^2\, dz = \int_0^1 (t+it)(1+i)dt = (-2+2i)\left[\frac{1}{3}t^3\right]_0^1 = -\frac{2}{3} + \frac{2}{3}i$



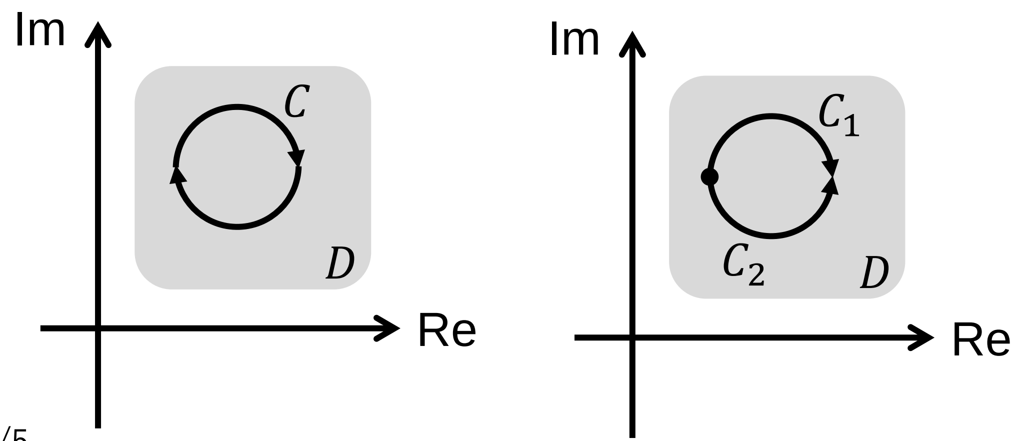
Integral of complex function

- Convert complex integral to definite integral
- Ex. $(z - z_0)^n$ $(n \in \mathbb{Z})$ in circle $|z - z_0| = \rho$
  - Solution: re-write circle using parameter θ
    - $z(\theta) = z_0 + \rho e^{i\theta}$ $(0 \le \theta < 2\pi)$, $d\theta = \frac{d(z_0+\rho e^{i\theta})}{d\theta} d\theta = i\rho e^{i\theta} d\theta$
    - $\oint_C (z - z_0)^n\, dz = i\rho e^{i\theta} \int_0^{2\pi} \rho e^{i(n+1)\theta}\, d\theta$ $(*)$
  - For $n \neq 1$:$(*) = i\rho^{(n+1)} \int_0^{2\pi} [\cos(n+1)\theta + i \sin(n+1)\theta]\, d\theta$
    - $= \frac{i\rho^{(n+1)}}{n+1} [\sin(n+1)\theta - i \cos(n+1)\theta]_0^{2\pi} = 0$
  - For $n = 1$:$(*)$ $\oint_C (z - z_0)^1\, dz = i \int_0^{2\pi} 1\, d\theta = 2\pi i$



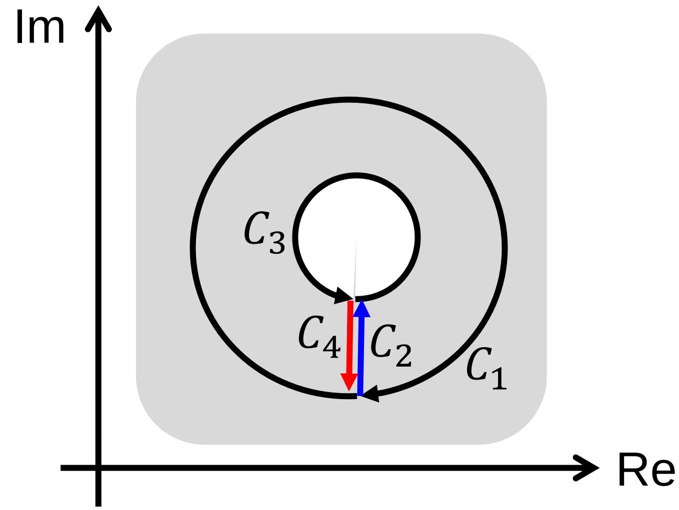
Cauchy's theorem (コーシーの定理)

- If $f(z)$ is regular analytical in region D, and curve C is a closed curve, its integral is:
  - $\oint_C f(z)\,dz = 0$: Cauchy's theorem
- If C is divided into two curves, C_1, C_2
  - $\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz$, thus $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$
  - Note: route must not cross the non-analytical points and branches



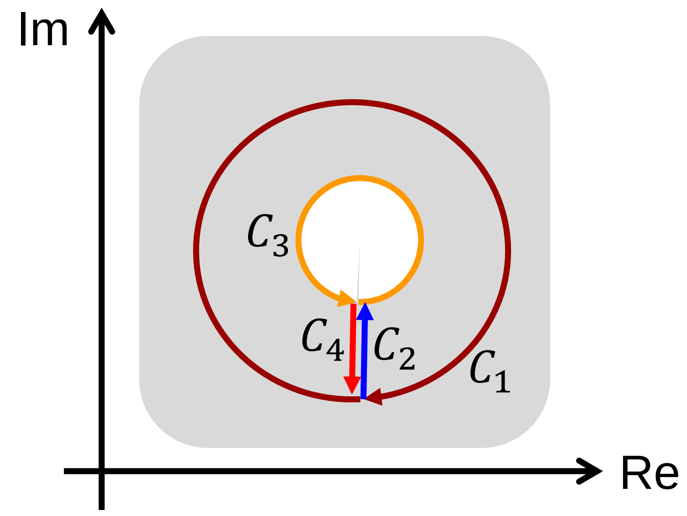
Cauchy's theorem for multiply connected domain

- For multiply connected domain (non-uniform domain, domain w/ hole), divide domain into several domains
  - Red part and blue part are cancel out
    - thus $\oint_{C_2} f(z)\, dz = -\oint_{C_4} f(z)\, dz$
  - Use for equation w/ non-analytical points
    - $(z = 0$ for $f(z) = 1/z)$



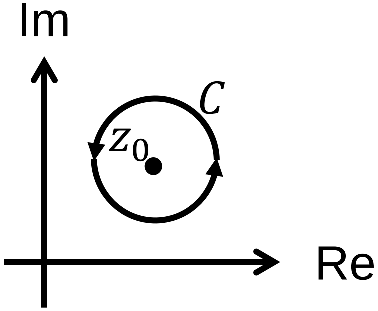
Cauchy's theorem for multiply connected domain

- For the path $C' = C_1 + C_2 + C_3 + C_4$, C' is close circle
  - $\oint_{C'} f(z)\,dz = 0$: Cauchy's theorem
  - $\oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \oint_{C_4} f(z)\,dz = 0$
    - $\oint_{C_1} f(z)\,dz + \oint_{C_3} f(z)\,dz = 0$
- If $f(z)$ is regular analytical for two closed circles C_1, C'_3(inverse of C_3), its integral becomes same
  - We can change the route



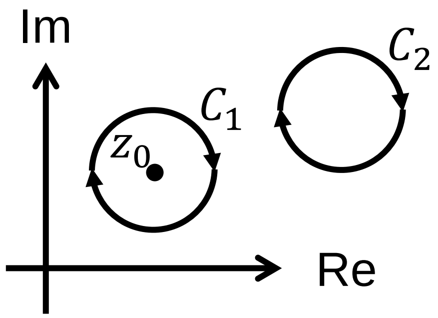
Cauchy's integral theorem

- Describe the value of complex function $f(z_0)$ at $z = z_0$ using circle integral
  - $2\pi i f(z_0) = \oint_C \frac{f(z)}{z-z_0} dz$, where z_0 and C are any point and circle in region D where $f(z)$ is a regular analytical



Usage of Cauchy's integral theorem

- Assume to take circle integral over C, and $f(z)$ is a regular analytical in region D
  - If point $z = z_0$ is inside of the circle C_1
    - $\oint_{C_1} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$
  - Else; (point $z = z_0$ is outside of the circle C_2)
    - $\oint_{C_2} \frac{f(z)}{z-z_0} dz = 0$



Theorem for regular analytical function

- Assume to take circle integral over C, and $f(z)$ is a regular analytical in region D
  - $f(z)$ can take n-th order differentiate $f^{(n)}(z)$
  - $f^{(n)}(z)$ can be expressed as
    - $f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad (n = 0,1,2,\ldots)$
- If $f(z)$ is a regular analytical in region D, $f^{(n)}(z)$ is available
  - Regular analytical means very limited case of function

Exercise

- Translate following equations in polar coordinate system
  - $z = \sqrt[3]{1+i}$
- Answer following for $w = z^4 = u + vi$, assume $z = x + yi$
  - Calculate u and v
  - Proof u and v satisfy Cauchy–Riemann equations
  - Calculate w'
- Integrate $f(z) = 1/z$ in unit circle C
- Integrate $f(z) = \cos z$ from $z = 0$ to $z = i$

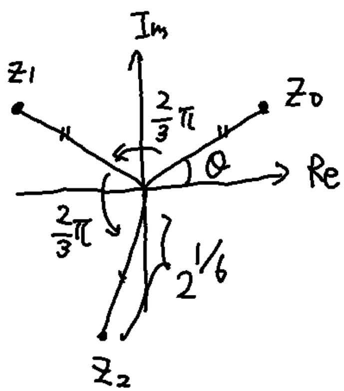
$$\sqrt[3]{1+i} = \sqrt[3]{\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)} = 2^{1/6}\cdot\sqrt[3]{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}}$$

$$= 2^{1/6}\cdot\left(e^{\left(\frac{\pi}{4}+2n\pi\right)i}\right)^{\frac{1}{3}}$$

$$= 2^{1/6}\cdot\left(e^{\frac{\pi}{12}i+\frac{2n\pi}{3}i}\right)$$

$$= 2^{1/6}\left(\cos\left(\frac{\pi}{12}i+\frac{2n\pi}{3}i\right)+i\sin\left(\frac{\pi}{12}i+\frac{2n\pi}{3}i\right)\right)$$

$n=0,1,2$



$\boxed{2}$ $w = z^4 = u + iv$

① Calc. u and v

$$z^4 = (x+iy)^4 = (x^2 + 2ixy - y^2)^2$$

$$= x^4 + 2ix^3y - x^2y^2 + 2ix^3y - 4x^2y^2 - 2ixy^3$$

$$- x^2y^2 - 2ixy^3 + y^4$$

$$= \underbrace{(x^4 - 6x^2y^2 + y^4)}_{u} + \underbrace{4xy(x^2 - y^2)}_{v}\, i$$

② Cauchy-Riemann

$$\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}_{(1)}, \quad \underbrace{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{(2)}$$

(1) $\dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2$

$\dfrac{\partial v}{\partial y} = 4x^3 - 12xy^2$

(2) $\dfrac{\partial u}{\partial y} = -12xy + 4y^3$

$-\dfrac{\partial v}{\partial x} = -12xy + 4y^3$ Satisfy.

③ $w' = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = 4\{(x^3 - 3xy^2) - i(3xy - y^3)\}$

$\boxed{3}$ integral

(1) $f(z) = 1/z$ for unit circle C.

use angle θ

$$z(\theta) = e^{i\theta} \quad (0 \leq \theta < 2\pi)$$

$$dz = de^{i\theta}\frac{d\theta}{d\theta} = ie^{i\theta}d\theta$$

$$\oint_C \frac{1}{z}dz = \int_0^{2\pi} e^{-i\theta} \cdot i e^{i\theta} d\theta = i[\theta]_0^{2\pi} = 2\pi i$$

(2) $f(z) = \cos z$, from $z = 0$ to $z = i$

$f(z)$ is regular analytical thus

$$\int_0^i \cos z \, dz = [\sin z]_0^i = \sin(i) - \sin 0. = \sin(i)$$

$$\sin i = \frac{e^{i \cdot i} - e^{-i \cdot i}}{2i} = \frac{e^{-1} - e^{1}}{2i} = \frac{-1}{i}\frac{e - e^{-1}}{2} = i\sinh(1)$$

Fundamental Mathematics

- Complex function theory 2-

Power series (数列) and convergence (収束)

- Power series equation $f(z)$: power-sum of coef. a and var. $(z-a)$ $(a, z, b_n \in \mathbb{C})$
  - $f(z) = \sum_{n=0}^{\infty} b_n(z-a)^n = b_0 + b_1(z-a) + \cdots + b_n(z-a)^n + \cdots$ (*1)
  - “Power series with centered on a”
- This equation has following characteristics
  - $f(z)$ has convergence range (収束半径) R $(\in \mathbb{R})$
  - If z satisfy $|z-a| < R$, $f(z)$ should converge (収束)
  - Else, $f(z)$ should diverge (発散)
    - Convergent circle: $|z-a| = R$
  - If $f(z)$ converge only at $z = a$ -> $R = 0$
  - If $f(z)$ converge all of complex values -> $R = \infty$

Power series and convergence (cont.)

- When power series equation $f(z)$ converge at $R > 0$,

1. $f(z)$ can take its differential inside the circle R
   - $f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$ is regular analytical
     - $f'^{(z)} = [\sum_{n=0}^{\infty} b_n (z-a)^n]' = b_1 + \cdots + nb_n (z-a)^{n-1} + \cdots$
2. $f(z)$ can calculate its integral at line C inside the circle R
   - $\int_C f(z)dz = \sum_{n=0}^{\infty} b_n \int_C (z-a)^n dz$
   - $= b_0 \int_C dz + b_1 \int_C (z-a)dz + \cdots + b_n \int_C (z-a)^n dz + \cdots$
3. Line integral from points b to z inside the circle R is
   - $\int_b^z f(z)dz = \sum_{n=0}^{\infty} b_n \int_b^z (z-a)^n dz$
   - $= k + b_0(z-a) + \frac{b_1}{2}(z-a)^2 + \cdots + \frac{b_n}{n+1}(z-a)^{n+1} + \cdots$

(k is constant)

Power series and convergence (cont.)

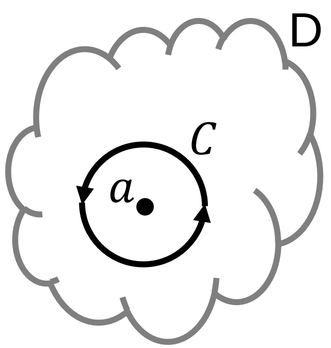
- Convergent circle R of eq. (*1) can calculate as follows (same as real)
  - $\frac{1}{R} = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right|, \frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{|b_n|}$
  - Note: if $\frac{1}{R} = 0$ then $R = \infty, \frac{1}{R} = \infty$ then $R = 0$

Power series and convergence (cont.)

- Similarly, negative power series is:
  - $g(z) = \sum_{n=0}^{\infty} c_n (z-a)^{-n} = c_0 + \frac{c_1}{(z-a)} + \cdots + \frac{c_n}{(z-a)^n} + \cdots$
  - If $g(z)$ has its convergent circle R', this negative power series $g(z)$..
    - has convergence within range $|z-a| > \frac{1}{R\prime}$
    - its differential, integral can be individually calculated within range $|z-a| > \frac{1}{R\prime}$
      - $g(z)$ is regular analytical in region $|z-a| > \frac{1}{R\prime}$

Taylor series in complex

- Taylor series (テイラー展開) in complex space
  - Assume $f(z)$ is regular analytical in region D, and it has circle C with center $z = a$, radius R. Taylor series of $f(z)$:
    - $f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$
  - If $a = 0$, this is called Maclaurin series (マクローリン展開)
    - $f(0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$



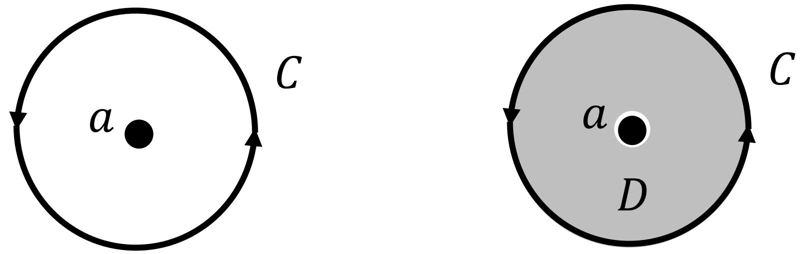
Maclaurin series in complex

- Same as real space, Maclaurin series can be calculated
  - $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$ (for all z)
  - $\sin z = 1 - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots$ (for all z)
  - $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$ (for all z)
  - $\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots$ (for all $|z| < 1$)
  - $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{4} + \cdots + (-1)^{n-1} \frac{z^n}{n} + \cdots$ (for all $|z| < 1$)
  - $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} + \cdots + (-1)^n \frac{z^{2n+1}}{2n+1} + \cdots$ (for all $|z| < 1$)
  - $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!} + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!} + \cdots$ (for all $|z| < 1$)

(z, p are complex value)

Singularity/singular point (特異点)

- If $f(z)$ is not regular analytical at point a, but regular analytical at circle C w/o point a, a is called singularity or singular point
- Theorem: assume a is singularity of $f(z)$. $f(z)$ can take Laurent series at region D which exclude a from circle C
  - $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n = \cdots + \frac{b_{-m}}{(z-a)^m} + \cdots + b_0 + b_1(z-a) + \cdots + b_n(z-a)^n + \cdots$
  - where, circle C is positive direction, $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}} dz$



Isolated singular point

- There are some important singular points
  - Isolated singular points
    - Pole (極)
      - Singular points when its numerator is zero
    - Removable singular points (除去可能な特異点)
      - Caused by the function is undefined at the point, but can define proper value to make regular analytical
    - Essential singular points (真性特異点)
      - Show different limit by different direction, or it has been divergence

Pole (極)

- Similar to real function, complex space support residue theorem
  - Assume a is singularity of $f(z)$. If its Laurent series is
    - $f(z) = \frac{b_{-k}}{(z-a)^k} + \cdots \frac{b_{-1}}{z-a} + +b_0 + b_1(z-a) + \cdots + b_n(z-a)^n + \cdots$
      - It means $b_{-k} \neq 0$ but $b_{-k-1} = b_{-k-2} = \cdots = 0$
    - a is called as (k-th) <u>pole</u> of $f(z)$
  - In this case, $g(z) = (z-a)^k f(z)$ is regular analytical at a

- In oppositely, if f(z) has infinite non-zero coeff. b_{-k}, a is called as <u>essential singularity</u> (真性特異点)

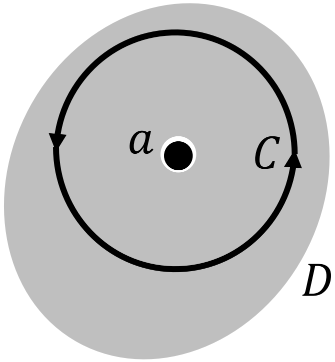
Removable singular point

- Some function has no negative series in its Laurent series
  - E.x. $f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots\right)\right] = \frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} \cdots$
  - In this case, singular point is called as <u>removable</u>
- If the singularity a is removable for $f(z)$, its Laurent series
    - $f(z) = b_0 + b_1(z-a) + \cdots + b_n(z-a)^n + \cdots$
    - has some limit: $\lim_{n\to a} f(z) = b_0$
  - Or, if $f(z)$ satisfy $\lim_{n\to a} f(z) = b_0$, singular point a is removable

Residue (留数)

- Residue: result of closed curve integral surrounds isolated singularities (removable singular, pole, essential singular)
- Assume regular analytical function $f(z)$ and its pole a, closed curve C, all in region D. Its closed curve integral is called <u>residue:</u> $\mathrm{Res}[f, a]$
  - $\mathrm{Res}[f, a] = \frac{1}{2\pi i}\int_C f(z)dz$
  - If a is not a pole ($= f(z)$ is regular analytical at a)
    - $\mathrm{Res}[f, a] = 0$
- If a is k-th pole of $f(z)$, its residue is
  - $\mathrm{Res}[f, a] = \frac{1}{(k-1)!}\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}[(z-a)^k f(z)]$

    (k is natural number (1,2,3…))

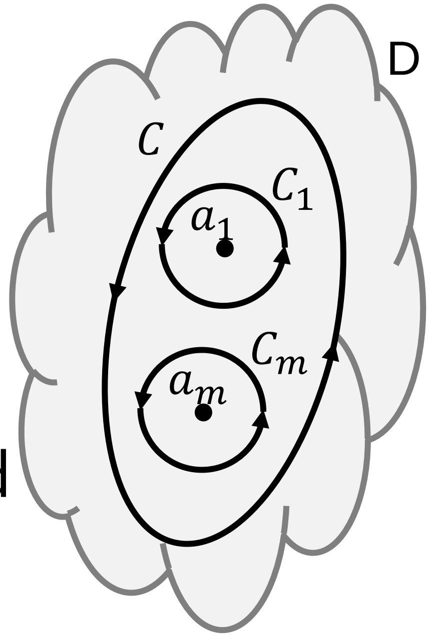


Residue: example

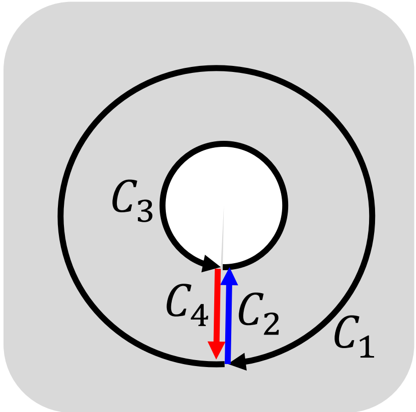
- Assume $f(z) = \frac{e^z}{(z-1)(z+3)^2}$. Calculate Residues
  - (1) $\mathrm{Res}[f,1]$, (2) $\mathrm{Res}[f,-3]$,
  - $z = 1$ is 1st pole, $z = -3$ is 2nd pole
  - $\mathrm{Res}[f,a] = \frac{1}{(k-1)!} \lim_{n \to a} \frac{d^{k-1}}{dz^{k-1}} [(z-a)^k f(z)]$
- Ans1: $\mathrm{Res}[f,1] = \lim_{z \to 1} [(z-1) f(z)] = \lim_{z \to 1} \left[\frac{e^z}{(z+3)^2}\right] = \frac{e}{16}$
- Ans2: $\mathrm{Res}[f,-3] = \lim_{z \to -3} \frac{d}{dz} [(z+3)^2 f(z)] = \lim_{z \to -3} \frac{d}{dz} \left[\frac{e^z}{(z-1)}\right] = -\frac{5e^{-3}}{16}$

Residue theorem (留数定理)

- Residue theorem: If circle C contain m poles $a_1, \cdots, a_m$, its circle integral is same as the sum of residues
  - $\frac{1}{2\pi i}\int_C f(z)dz = \mathrm{Res}[f, a_1] + \cdots + \mathrm{Res}[f, a_m]$

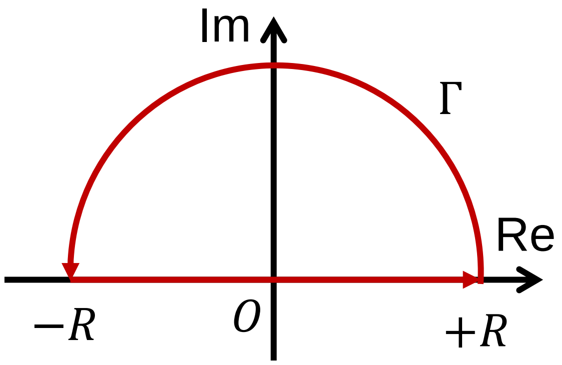


- Use Cauchy's theorem for multiply connected domain
  - For multiply connected domain (non-uniform domain, domain w/ hole), divide domain into several domains
    - Red part and blue part are cancel out
      - thus $\oint_{C_2} f(z)\,dz = -\oint_{C_4} f(z)\,dz$



Application of residue theorem

- Use residue theorem to calculate integral $\int_{-\infty}^{\infty} F(x)dx$
- Preliminary: assume $|f(z)| \leq \frac{M}{R^k}$ at $|z| = R(k > 1, M$: const.)*1
  - It satisfy $\lim_{R\to\infty} \int_\Gamma f(z)dz = 0$
    - Γ is half circle of route



- Proof: from eq *1,
  - $\left|\int_\Gamma f(z)dz\right| \leq \int_\Gamma |f(z)|ds \leq \frac{M}{R^k}\pi R = \frac{\pi M}{R^{k-1}}$ s: length of half circle
  - For all region of complex space $(R \to \infty)$, since $k > 1$,
    - $\lim_{R\to\infty} \left|\int_\Gamma f(z)dz\right| = 0$

Application of residue theorem

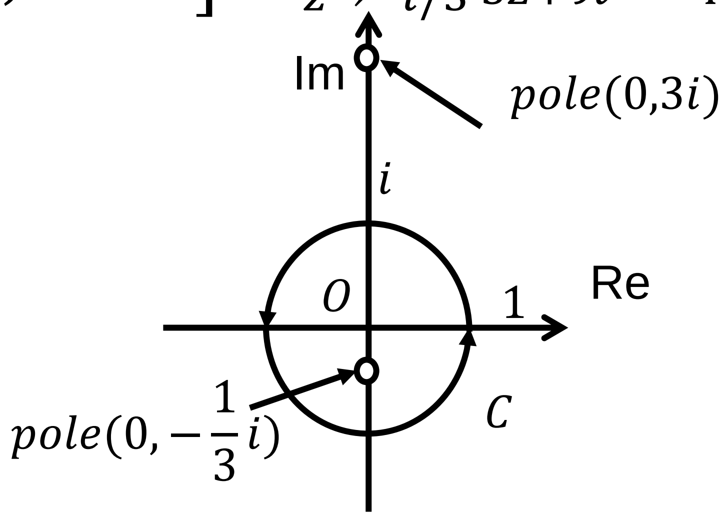
- Calculate integral $\int_0^{2\pi} F(\cos\theta\,)d\theta$
- Replace θ by $\mathrm{z} = e^{i\theta}$
  - $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, $d\theta = \frac{1}{iz}dz$
  - Integral of $(0 \leq \theta < 2\pi) \Leftrightarrow$ circle Integral of $|z| = 1$
    - $\int_0^{2\pi} F(\cos\theta\,)d\theta = \int_C F\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\frac{1}{iz}dz$

Example

- Calculate integral of $\int_0^{\infty} \frac{1}{z^4+1} dz$
- Solution: assume $|z| = R$
  - $\left|\frac{1}{z^4+1}\right| = \frac{1}{|z^4+1|} \leq \frac{1}{|z^4|-1} = \frac{1}{R^4-1} \leq \frac{2}{R^4}$,
  - If take $R \to \infty$, $\frac{2}{R^4} \to 0$
    - $\int_0^{\infty} \frac{1}{z^4+1} dz = 0$

Example

- Calculate integral of $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta$
  - Solution: replace θ by $z = e^{i\theta}$, C is positive unit circle
    - $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta = \int_C \frac{2}{3z^2+i10z-3} dz = \int_C \frac{2}{(3z+i)(z-3i)} dz$
    - C contain one residual $z = -i/3$ at 1st pole
    - $\mathrm{Res}\left[f, -\frac{i}{3}\right] = \lim_{z \to -i/3} \left[\left(z + \frac{i}{3}\right)^1 f(z)\right] = \lim_{z \to -i/3} \frac{2}{3z+9i} = \frac{1}{4i}$
    - $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$



Conclusion

- Introduce complex function theory
  - Power series and convergence
  - Laurent series
    - Singular points and isolated singular points
      - Pole, removable singular points, Essential singular points
  - Residue and residue theorem
  - Application of residue theorem for calculating integral
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