Fundamental Mathematics (Engineering Mathematics)

Shinichi Nishizawa

Course schedule

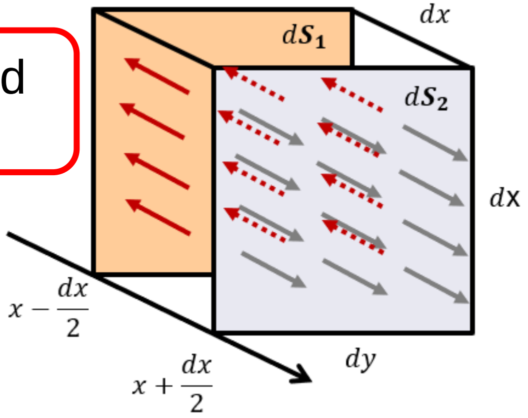
- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

Issue in lecture #5

Divergence of vector

- $\mathrm{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \boldsymbol{F}$
- Assume flow $\boldsymbol{F}$ of small box $dxdydz$
  - Assume flow $\boldsymbol{F}$ of area $d\boldsymbol{S}_1 = (-dydz, 0{,}0)$ at $x - \frac{dx}{2}$
  - Assume flow $\boldsymbol{F}$ of area $d\boldsymbol{S}_2 = (+dydz, 0{,}0)$ at $x + \frac{dx}{2}$
    - $\boldsymbol{F} \cdot d\boldsymbol{S} = \boldsymbol{F} \cdot d\boldsymbol{S}_2 + \boldsymbol{F} \cdot d\boldsymbol{S}_1$
    - $= F_1\left(x + \frac{dx}{2}, y, z\right) dydz +$ This should be plus
      $F_1\left(x - \frac{dx}{2}, y, z\right)(-dydz)$
    - $= \frac{\partial F_1}{\partial x} dxdydz$
  - Diff. flow in ( ) and out ( )

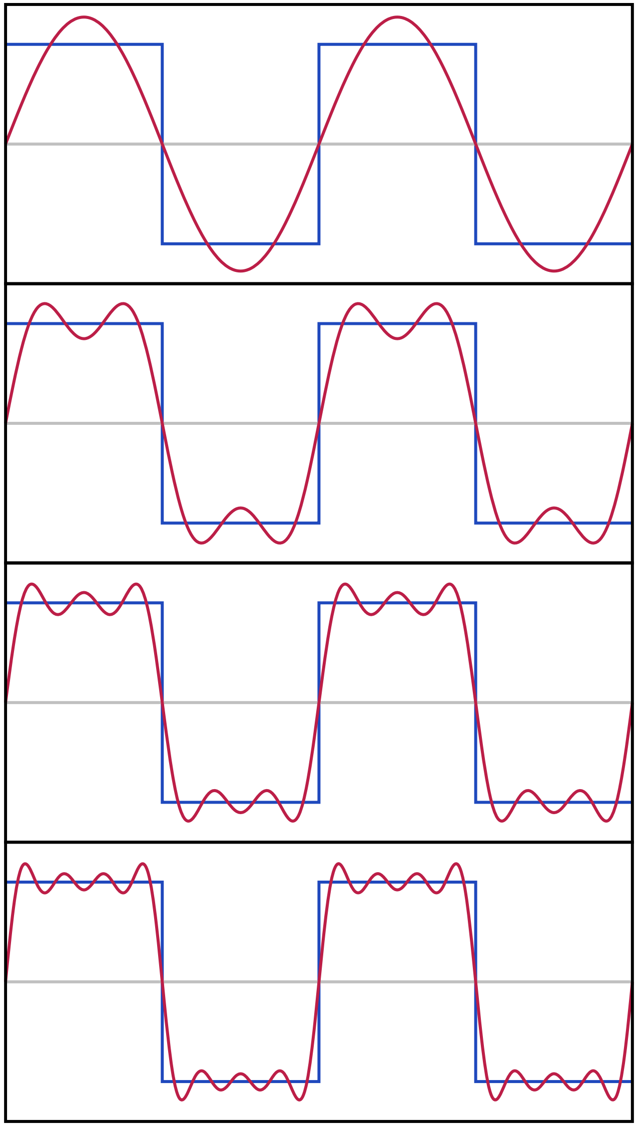


Fundamental Mathematics

- Fourier series/transform 1-

Fourier series (フーリエ級数)

- Fourier series: summation of harmonically related sine function
  - Summation is a periodic function, determined by
    - the choices of cycle length (period)
    - the number of components, amplitude and phase
  - Originally, developed to solve thermal conduction (differential equation)



Fourier series (from 1st order to 4th order)
Source: https://en.wikipedia.org/wiki/Fourier_series

Fourier series: definition

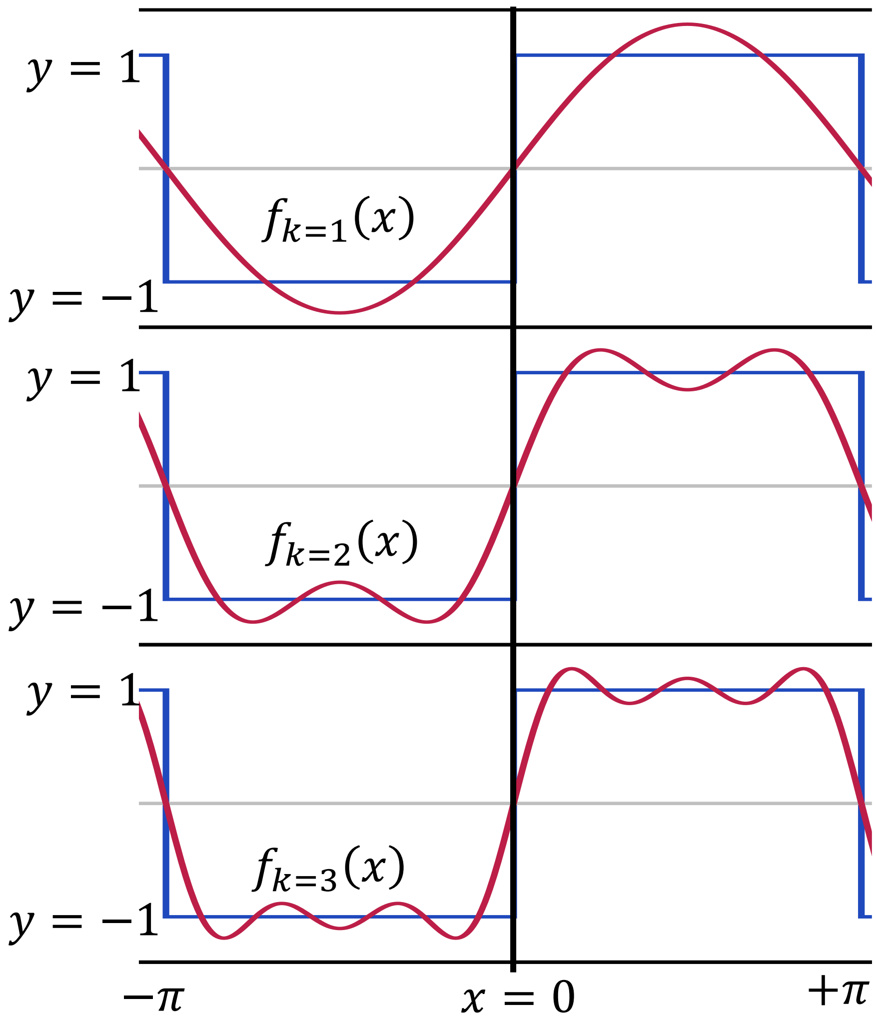
- Assume $f(x)$ is defined in $x \in \mathbb{R}$, and it has period 2π
- Fourie series in sine functions can be expressed as
  - $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (1)
- Here, we want to know the value of a_n and b_n
  - Take integral $-\pi \sim \pi$
    - $\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\, dx$
  - Since $\int_{-\pi}^{\pi} \cos nx\, dx = \int_{-\pi}^{\pi} \sin nx\, dx = 0$
    - $a_0 = 1/\pi \int_{-\pi}^{\pi} f(x)dx$
- Note: $\sim$ means the Fourier series need some conditions to be equal

Fourier series: example 1

- Calculate Fourier series for periodic function (period 2π)
  - $f(x) = \begin{cases} -1 & (-\pi \le x < 0, x = \pi) \\ +1 & (0 \le x < \pi) \end{cases}$
- Solution:
  - $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = -\frac{1}{\pi}\int_{-\pi}^{0}\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\, dx = 0$
  - $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = -\frac{1}{\pi}\int_{-\pi}^{0}\sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\, dx =$
    $\frac{2}{\pi}\int_{0}^{\pi}\sin nx\, dx = \begin{cases} 0 & n \text{ is even} \\ 4/n\pi & n \text{ is odd} \end{cases}$
  - $f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1}$
  - Note: $f(0) = 1$ in definition, but Fourier series converge to 0 at $x = 0$: convergence problem

Fourier series: example 1

- $f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1}$ can be decomposed to
  - $f_{k=1}(x) = \frac{4}{k}\sin x$
  - $f_{k=2}(x) = f_{k=1}(x) + \frac{4}{3k}\sin 3x$
  - $f_{k=3}(x) = f_{k=2}(x) + \frac{4}{5k}\sin 5x$
- NOTE: Conversion
  - Original func. $f(0) = +1$
  - Fourier series converse to 0 (not +1)

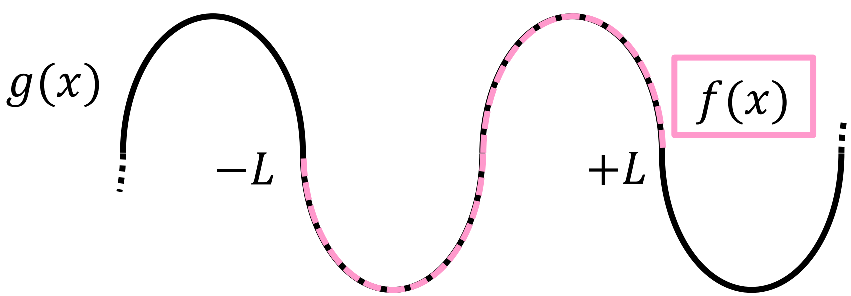


Fourier series: even/odd func

- Introduce variable $t = x\pi/L$ w/ period $0 \sim 2L$
  - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$
  - where $a_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$, $b_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$
- If $f(x)$ is even function (偶関数):
  - $f(x)\sin\frac{n\pi x}{L}$ is odd function (奇関数) -> $b_n = 0$
- If $f(x)$ is odd function:
  - $f(x)\cos\frac{n\pi x}{L}$ is even function -> $a_n = 0$

Fourier series in finite interval

- Fourier series of $f(x)$ is available for finite interval $[L, -L]$
  - Assume infinite func. $g(x)$ is available which matched with $f(x)$ in finite interval $[L, -L]$
  - Fourier series of $f(x)$ (finite) is same as $g(x)$ (infinite)
- Case if $f(-L) \neq f(L)$?
  - Re-define $f(x = L) = \frac{1}{2}\{f(-L+0) + f(L-0)\}$ to $f(-L) = f(L)$
    - (this re-definition also show same as original result of $f(x)$)

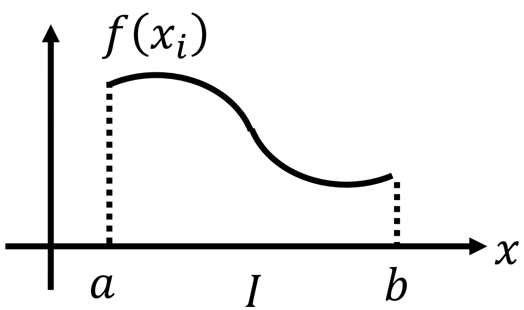


Fourier series in complex space

- Use Euler's formula to extend Fourier series in complex
  - $\cos nx = \frac{e^{inx}+e^{-inx}}{2}$, $\sin nx = \frac{e^{inx}-e^{-inx}}{2i}$
  - $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$, $(x \in \mathbb{R})$
- Also, Fourier series is available for finite interval $[-L, L]$
  - $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{in\pi x}{L}}$, $a_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{in\pi x}{L}}dx$, $(x \in \mathbb{R})$

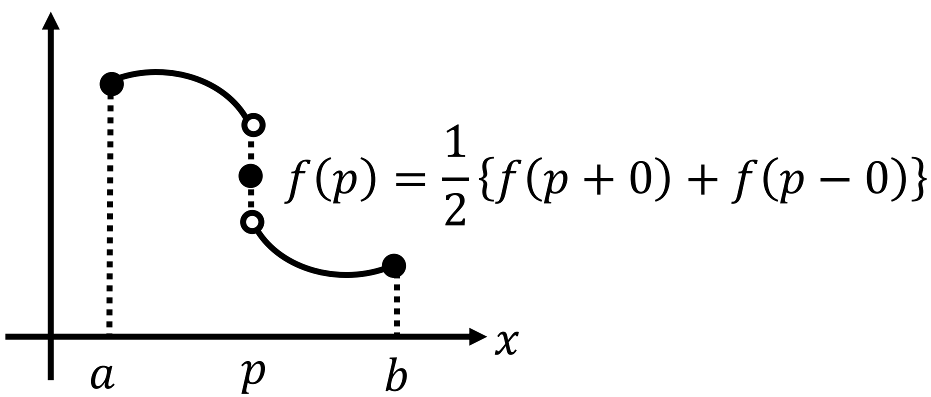
Piecewise smooth (区分的連続)

- Function $f(x)$ should be piecewise smooth at range I $[a, b]$ to converse its Fourier series
- Piecewise smooth
  - Derivative of $f(x)$ should continuous (連続) (exclude finite non-continuous points)
  - At the non-continuous points, $f(x)$ and $f'(x)$ of both right-side and left-side limits are available and not infinite
    - $f(x_i - 0) = \lim_{h \to 0} f(x_i - h)$, $f(x_i + 0) = \lim_{h \to 0} f(x_i + h)$
    - $f'(a + 0) = \lim_{h \to 0} f'(a + h)$, $f'(b - 0) = \lim_{h \to 0} f'(b - h)$



Piecewise smooth

- Redefine non-continuous func. to piecewise smooth
  - Assume point p is non-continuous in range I $[a, b]$
  - Redefine $f(p)$ by average of left limit and right limit
    - $f(p) = \frac{1}{2}\{f(p+0) + f(p-0)\}$
  - This operation make function to piecewise smooth
- Following contents assume this operation for all of non-continuous points in Fourier series



Fourier series and conversion

- Theorem1: If $f(x)$ is periodic function w/ period 2L and piecewise smooth, and its derivative $f'(x)$ is also piecewise smooth. Its Fourier series
  - converge to $f(x)$ when x is continuous
  - converge to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ when x is non-continuous
- If above is satisfied, its Fourier series is
  - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$
  - ($\sim$ -> $=$)

Termwise integral (項別積分)

- Theorem2: If $f(x)$ is periodic function w/ period $2L$ and piecewise smooth, and its Fourier series as
  - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$
- Its termwise integral for $(-L \leq a, x \leq L)$ can be calculated as
  - $\int_a^x f(x)dx = \frac{a_0}{2}\int_a^x dx + \sum_{n=1}^{\infty}(a_n \int_a^x \cos\frac{n\pi x}{L}dx + b_n \int_a^x \sin\frac{n\pi x}{L}dx)$
  - $= \frac{a_0}{2}(x-a) - \sum_{n=1}^{\infty}\frac{L}{n\pi}\left(a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\frac{L}{n\pi}(a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L})$

Termwise differential (項別微分)

- Theorem3: If $f(x)$ and $f'(x)$ continuous, and $f''(x)$ piecewise smooth, Fourier series of $f(x)$ and $f'(x)$ will converge to $f(x)$ and $f'(x)$, and Fourier series of $f'(x)$ can be calculated by ternmwise differential of $f(x)$
  - For $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$
  - $f'(x) = \left(\frac{a_0}{2}\right)' + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)'$
  - $f'(x) = \frac{n\pi}{L}\sum_{n=1}^{\infty}\left(-a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right)$

Fundamental Mathematics

- Fourier series/transform 2-

Fourie integral (フーリエ積分)

- Fourie series: express (1) periodic function or (2) function defined within finite range $[-L, L]$, w/ sum of sine functions
  - For periodic function $f(x)$ w/ period $0 \sim 2L$
    - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$ (a)
    - where $a_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$, $b_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$ (b)
- Fourie integral: extension of (2) to infinite range $(-\infty, \infty)$
  - $f(x) = \int_0^{\infty}\{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\}\,d\alpha$
    - where $A_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,du$, $B_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,du$

Introduction of Fourie integral

- Introduce Fourie integral from Fourie series
  - Substitute (b) to (a)
    - $f(x) = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}(\cos\frac{n\pi x}{L}\frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx + \sin\frac{n\pi x}{L}\frac{1}{\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx)$
  - If integral of $f(x)$ has finite value C: $\int_{-\infty}^{\infty} f(x)\,dx = C$
    - $\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} f(x)\,dx \quad = 0$
  - Replace variables
    - $\alpha_n = \frac{n\pi}{L}$, $\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{L}$, $\lim_{L\to\infty}\Delta\alpha = 0$
    - $f(x) = \lim_{\Delta\alpha\to\infty}\frac{\Delta\alpha}{\pi}\sum_{n=1}^{\infty}\left[\cos a_n x\int_{-\infty}^{\infty} f(u)\cos\alpha u\,du + \sin a_n x\int_{-\infty}^{\infty} f(u)\sin\alpha u\,du\right]$
  - Replaced by $A_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,du$, $B_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,du$

Introduction of Fourie integral (cont.)

- $f(x) = \lim_{\Delta\alpha \to \infty} \Delta\alpha \sum_{n=1}^{\infty} [\cos a_n x \, A(a_n) + \sin a_n x \, B(a_n)]$
  - $= \lim_{\Delta\alpha \to \infty} [\sum_{n=1}^{\infty} [\Delta\alpha \cos a_n x \, A(a_n)\,] + \sum_{n=1}^{\infty} [\Delta\alpha \sin a_n x \, B(a_n)]]$
  - This is a Riemann sum, thus re-write and obtain the Fourie integral is as follows
- $f(x) = \int_0^{\infty} A(\alpha) \cos \alpha x \, d\alpha + \int_0^{\infty} B(\alpha) \sin \alpha x \, d\alpha$ (c)
  - where $A_n = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \alpha u \, du$, $B_n = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin \alpha u \, du$ (d)
- $f(x)$ can be simplified substituting (d) to (c)
  - $f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(u) [\cos \alpha u \cos \alpha x + \sin \alpha u \sin \alpha x] \, du d\alpha$
    - $= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(u) \cos \alpha (x - u) \, du d\alpha$ (e)
    - This is also Fourie integral

Fourie integral in exponent

- Use Euler's theorem ($\cos\theta = (e^{i\theta} + e^{-i\theta})/2$) to introduce Fourie integral in exponent function. Recall eq. (e)
  - $f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(u)\cos\alpha(x-u)\,du d\alpha$
    - $= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \frac{f(u)\left(e^{i\alpha(x-u)}+e^{-i\alpha(x-u)}\right)}{2}\,du d\alpha$
    - $= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,du d\alpha + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{-i\alpha(x-u)}\,du d\alpha$
- Replace α to $-\alpha$ for second term; $d\alpha \to -da$, $\infty \to -\infty$
  - $f(x) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,du d\alpha - \frac{1}{2\pi}\int_0^{-\infty} \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,du d\alpha$
    - $= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,du d\alpha + \frac{1}{2\pi}\int_{-\infty}^0 \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,du d\alpha$
    - $= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,du d\alpha$ (f) ⇦ Fourie integral (exponent)

Fourie integral in exponent (cont.)

- Fourie integral has another form (this form is widely recognized). From (f)
  - $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\alpha(x-u)}\,du d\alpha$
    - $= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\alpha x}e^{-i\alpha u}\,du d\alpha$
    - $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\,du\right]e^{i\alpha x}d\alpha$

Fourie transform (フーリエ変換)

- Fourie transform can be obtained by replacing variables $u \to t$, $x \to t$, $\alpha \to \omega$,
  - $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$
  - $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \quad \text{(g)}$
- Equation shows conversion of time-domain function $f(t)$ to frequency-domain ($F(\omega)$)

Fourie integral for odd/even function

- If the function $f(x)$ is odd, (c) only has $\cos \alpha x$ component
- If the function $f(x)$ is even, (c) only has $\sin \alpha x$ component
- Fourie integral is simplified as follows
  - $f(x)$ is even: called cosine-transform
    - $f(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} C(\alpha) \cos \alpha x \, d\alpha, \quad C(\alpha) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(u) \cos \alpha u \, du$
  - $f(x)$ is odd: called sine-transform
    - $f(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} S(\alpha) \sin \alpha x \, d\alpha, \quad S(\alpha) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(u) \sin \alpha u \, du$

Characteristics of Fourie integral

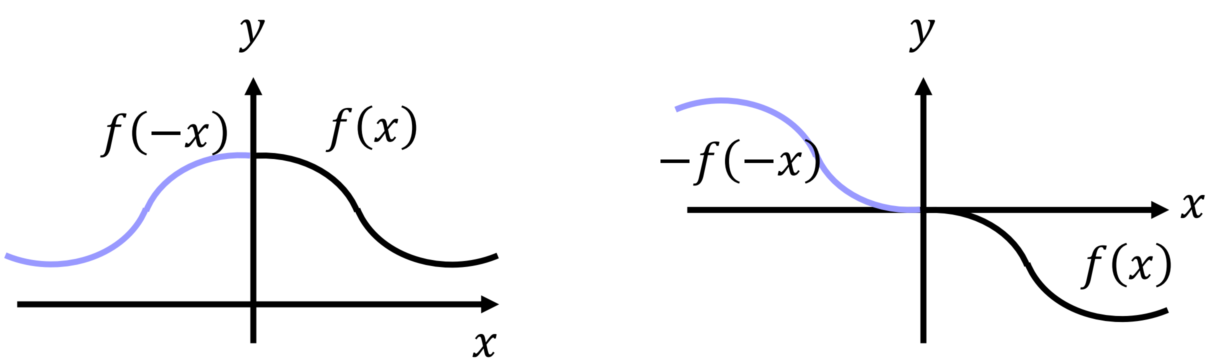
- Fourie integral need some condition to have a limit

  If integral of $f(x)$ has finite value C: $\int_{-\infty}^{\infty} f(x)\,dx = C$

  - Same conditions are required as Fourie series
- Theorem 4: Assume $f(x)$ is defined in $(-\infty, \infty)$, $f(x)$ and $f'(x)$ are piecewise smooth, $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite.
  - If $f(x)$ is continuous on x,
    - $f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,dud\alpha$
  - If $f(x)$ is not continuous on x,
    - $f(x) = \frac{f(x+0)+f(x-0)}{2} = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,dud\alpha$

Characteristics of Fourie integral (cont)

- Assume $f(x)$ is defined in $[0, \infty)$.
  - Use $f(x) = f(-x)$ to expand its range to $(-\infty, \infty)$.
    - New $f(x)$ is even-function
    - “Cosine translation of original $f(x)$ ”
  - Use $f(x) = -f(-x)$ to expand its range to $(-\infty, \infty)$.
    - New $f(x)$ is odd-function
    - “Sine translation of original $f(x)$ ”



Application of Fourie transform

- Fourie transform has wide applications
  - Try to apply electric circuit analysis

- Introduce Fourie transform for derivatives
  - Take partial difference for Fourie transform (g)
    - $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df(t)}{dt}e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\left[f(t)e^{-i\omega t}\right]_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\,dt$
  - Our target is nature, thus $\lim_{t\to-\infty} f(t) = \lim_{t\to\infty} f(t) = 0$ (dump)
    - $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df(t)}{dt}e^{-i\omega t}\,dt = \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\,dt = i\omega F(\omega)$
  - Fourie transform for derivatives : multiply $i\omega$ to its original Fourie transform
    - Also this is true for higher order of derivatives

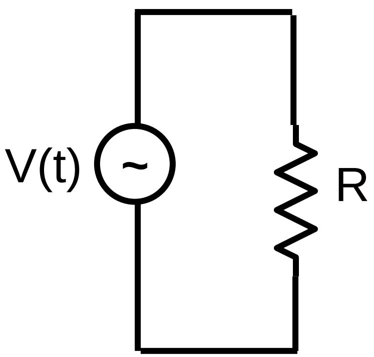
Application for circuit analysis

- Analyze frequency dependency of resistance (R), inductance (L), capacitance (C)

- Extract impedance $Z(\omega)$ on frequency domain
  - $Z(\omega) = V(\omega)/I(\omega)$
  - $V(\omega)$: voltage on frequency domain
    - $V(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(t) e^{-i\omega t}\, dt$
  - $I(\omega)$: current on frequency domain
    - $I(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t) e^{-i\omega t}\, dt$
  - ω: angular frequency

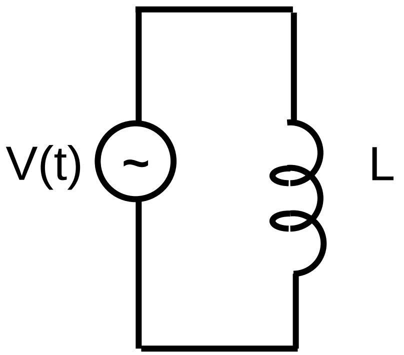
Resistance analysis

- From the Kirchhoff's voltage Law
  - $-V(t) + RI(t) = 0$
- Multiply $\frac{1}{\sqrt{2\pi}} e^{-i\omega t}$ and take integral at $[-\infty, \infty]$
  - $-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(t) e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} RI(t) e^{-i\omega t}\, dt$
  - $V(\omega) = RI(\omega)$
  - $Z(\omega) = \frac{V(\omega)}{I(\omega)} = R$
  - No frequency dependence



Inductance analysis

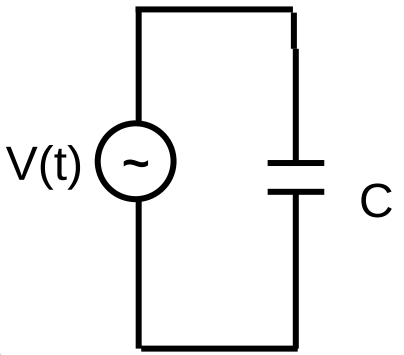
- Inductance create electromotive force $V_L(t)$ when the current flow changes $(I(t+\Delta t) - I(t))$
  - Its amplitude is called inductance L
    - $V_L(t) = L \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t} = L \frac{dI(t)}{dt}$
  - From the Kirchhoff's voltage Law
    - $-V(t) + V_L(t) = 0$
    - $-V(t) + L \frac{dI(t)}{dt} = 0$



- Multiply $\frac{1}{\sqrt{2\pi}} e^{-i\omega t}$ and take integral at $[-\infty, \infty]$
  - $V(\omega) = i\omega L I(\omega) => Z(\omega) = \frac{V(\omega)}{I(\omega)} = i\omega L$
  - Frequency dependence
    - Impedance increases as freq. (ω) increase

Capacitance analysis

- Capacitance create voltage $V_C(t)$ as the electrons $Q(t)$ are charged. Its amplitude is called Capacitance C
  - $V_C(t) = \frac{Q(t)}{C}$
- From the Kirchhoff's voltage Law
  - $-V(t) + V_C(t) = 0$ -> $-V(t) + \frac{Q(t)}{C} = 0$
- Take differential
  - $-\frac{dV(t)}{dt} + \frac{1}{C}\frac{dQ(t)}{dt} = 0$ -> $-\frac{dV(t)}{dt} + \frac{1}{C}I(t) = 0$
  - $\frac{dQ(t)}{dt} = I(t)$ and/or $\int I(t)dt = Q(t)$



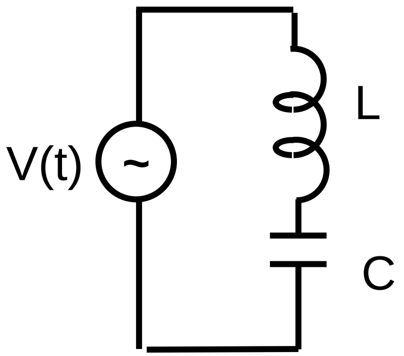
Capacitance analysis (cont.)

- Multiply $\frac{1}{\sqrt{2\pi}}e^{-i\omega t}$ and take integral at $[-\infty, \infty]$
  - $i\omega V(\omega) = \frac{1}{C}I(\omega) => Z(\omega) = \frac{V(\omega)}{I(\omega)} = \frac{1}{i\omega C}$
  - Frequency dependence
    - Impedance decreases as freq. (ω) increase

- Similar to Fourie transform, we introduce Laplace transform to solve differential equations
  - Fourie transform: transform to time- to freq- domain
  - Laplace transform: transform to time- to s- domain

Exercise

- Calculate Fourie transform
  - $f(x) = \begin{cases} 1 & (-a \leq x \leq a) \\ 0 & (x < -a, a < x) \end{cases}$
  - $f(x) = \begin{cases} 1 - x^2 & (|x| \leq 1) \\ 0 & (|x| > 1) \end{cases}$
  - $f(x) = e^{-a|x|}, (a > 0)$
- Calculate impedance of LC series circuit at AC supply $V(t)$
  - Capacitance: C
  - Inductance: L
  - Charge: $Q(t)$
  - Current: $I(t)$



Sample solution

Math 12

Calculate Fourie transform $F(x)$

① $f(x) = \begin{cases} 1 & (-a \le x \le a) \\ 0 & (x < -a,\ a < x) \end{cases}$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\alpha u} du$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha} \left[e^{-i\alpha u} \right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{2}{\alpha} \frac{e^{i\alpha a} - e^{-i\alpha a}}{2i} = \sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha}$$

$$f(x) = e^{-a|x|} \quad (a > 0)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$\sqrt{\frac{2}{\pi}}$

$$= \int_{-\infty}^{0} e^{-a(-x)} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-ax} e^{-i\omega x} dx$$

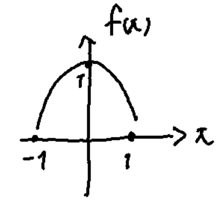
$$= \int_{-\infty}^{0} e^{(a-i\omega)x} dx + \int_{0}^{\infty} e^{-(a+i\omega)} dx$$

$$= \left[\frac{e^{(a-i\omega)x}}{a-i\omega} \right]_{-\infty}^{0} + \left[\frac{-e^{-(a+i\omega)x}}{a+i\omega} \right]_{0}^{\infty}$$

$$= \left(\frac{1}{a-i\omega} - 0 - 0 + \frac{1}{a+i\omega} \right) = \frac{2a}{a^2 - \omega^2} \sqrt{\frac{2}{\pi}}$$

$2 \quad \sqrt{\frac{2}{\pi}} \times$

$$f(x) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$



even func

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(u) \cos \alpha u \, du$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\sin \alpha u}{\alpha u} \right]_{-1}^{1} - \int_{-1}^{1} u^2 \cos \alpha u \, du \right\}$$

$$\int x^2 \cos \alpha x \, dx = x^2 \frac{\sin \alpha x}{\alpha} - \int 2x \frac{\sin \alpha x}{\alpha} dx$$

$$= \text{〃} - \left[2x \frac{-\cos \alpha x}{\alpha^2} - \int 2 \frac{-\cos \alpha x}{\alpha^2} dx \right]$$

$$= \frac{x^2 \sin \alpha x}{\alpha} + \frac{2x \cos \alpha x}{\alpha^2} - \frac{2 \sin \alpha x}{\alpha^3} + C$$

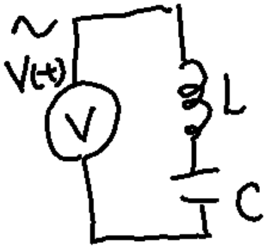
$$F(\alpha) = \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin \alpha}{\alpha} - \frac{\sin(-\alpha)}{-\alpha} - \left\{ \frac{\sin \alpha}{\alpha} - \frac{\sin(-\alpha)}{-\alpha} \right. \right.$$

$$\left. \left. + \frac{2\cos \alpha}{\alpha^2} - \frac{-2\cos(-\alpha)}{\alpha^2} - \left(\frac{2 \sin \alpha}{\alpha^3} - \frac{2 \sin(-\alpha)}{\alpha^3} \right) \right\} \right.$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{4 \cos \alpha}{\alpha^2} + \frac{4 \sin \alpha}{\alpha^3} \right)$$

Sample solution

LC series



$$-V(t)+\frac{Q(t)}{C}+L\frac{dI(t)}{dt}=0$$

differentiate

$$-\frac{dV(t)}{dt}+\frac{I(t)}{C}+L\frac{d^2I(t)}{dt^2}=0$$

Fourie transform

$$-i\omega V(\omega)+\frac{I(\omega)}{C}-\omega^2 L I(\omega)=0$$

$$Z(\omega)=\frac{V(\omega)}{I(\omega)}=\frac{1}{i\omega C}+i\omega L$$

+

Report

- In engineering, some mathematic methods are used to analyze and model the natural behavior and/or systems.
  - Find one example of application which uses mathematic methods, and explain how these mathematic methods are used for the application.
  - Length: no limit
  - Due: 2024/02/02 (Fri.)

Exam:
60min. You can use your note (printed materials) and calculator. Smartphone, Tablet, PC is not allowed.

Conclusion

- Introduce complex function theory
  - Fourier series in complex space
  - nishizawa@aoni.waseda.jp