Fundamental Mathematics (Engineering Mathematics)

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Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

Fundamental Mathematics

- Complex function theory 2-

Power series (数列) and convergence (収束)

- Power series equation $f(z)$: power-sum of coef. a and var. $(z-a)$ $(a, z, b_n \in \mathbb{C})$
  - $f(z) = \sum_{n=0}^{\infty} b_n(z-a)^n = b_0 + b_1(z-a) + \cdots + b_n(z-a)^n + \cdots$ (*1)
  - “Power series with centered on a”
- This equation has following characteristics
  - $f(z)$ has convergence range (収束半径) R $(\in \mathbb{R})$
  - If z satisfy $|z-a| < R$, $f(z)$ should converge (収束)
  - Else, $f(z)$ should diverge (発散)
    - Convergent circle: $|z-a| = R$
  - If $f(z)$ converge only at $z = a$ -> $R = 0$
  - If $f(z)$ converge all of complex values -> $R = \infty$

Power series and convergence (cont.)

- When power series equation $f(z)$ converge at $R > 0$,

1. $f(z)$ can take its differential inside the circle R
   - $f(z) = \sum_{n=0}^{\infty} b_n(z-a)^n$ is regular analytical
     - $f'^{(z)} = [\sum_{n=0}^{\infty} b_n(z-a)^n]' = b_1 + \cdots + nb_n(z-a)^{n-1} + \cdots$
2. $f(z)$ can calculate its integral at line C inside the circle R
   - $\int_C f(z)dz = \sum_{n=0}^{\infty} b_n \int_C (z-a)^n dz$
   - $= b_0 \int_C dz + b_1 \int_C (z-a)dz + \cdots + b_n \int_C (z-a)^n dz + \cdots$
3. Line integral from points b to z inside the circle R is
   - $\int_b^z f(z)dz = \sum_{n=0}^{\infty} b_n \int_b^z (z-a)^n dz$
   - $= k + b_0(z-a) + \frac{b_1}{2}(z-a)^2 + \cdots + \frac{b_n}{n+1}(z-a)^{n+1} + \cdots$

(k is constant)

Power series and convergence (cont.)

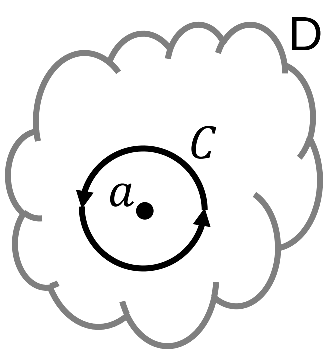
- Convergent circle R of eq. (*1) can calculate as follows (same as real)
  - $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right|, \frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|b_n|}$
  - Note: if $\frac{1}{R} = 0$ then $R = \infty$, $\frac{1}{R} = \infty$ then $R = 0$

Power series and convergence (cont.)

- Similarly, negative power series is:
  - $g(z) = \sum_{n=0}^{\infty} c_n (z-a)^{-n} = c_0 + \frac{c_1}{(z-a)} + \cdots + \frac{c_n}{(z-a)^n} + \cdots$
  - If $g(z)$ has its convergent circle R', this negative power series $g(z)$..
    - has convergence within range $|z-a| > \frac{1}{R\prime}$
    - its differential, integral can be individually calculated within range $|z-a| > \frac{1}{R\prime}$
      - $g(z)$ is regular analytical in region $|z-a| > \frac{1}{R\prime}$

Taylor series in complex

- Taylor series (テイラー展開) in complex space
  - Assume $f(z)$ is regular analytical in region D, and it has circle C with center $z = a$, radius R. Taylor series of $f(z)$:
    - $f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$
  - If $a = 0$, this is called Maclaurin series (マクローリン展開)
    - $f(0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$



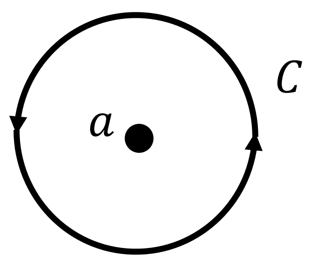
Maclaurin series in complex

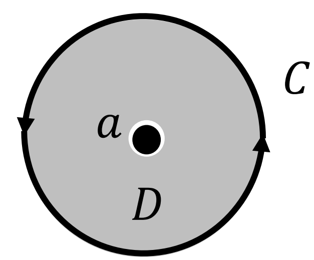
- Same as real space, Maclaurin series can be calculated
  - $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$ (for all z)
  - $\sin z = 1 - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots$ (for all z)
  - $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$ (for all z)
  - $\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots$ (for all $|z| < 1$)
  - $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{4} + \cdots + (-1)^{n-1} \frac{z^n}{n} + \cdots$ (for all $|z| < 1$)
  - $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} + \cdots + (-1)^n \frac{z^{2n+1}}{2n+1} + \cdots$ (for all $|z| < 1$)
  - $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!} + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!} + \cdots$ (for all $|z| < 1$)

(z, p are complex value)

Singularity/singular point (特異点)

- If $f(z)$ is not regular analytical at point a, but regular analytical at circle C w/o point a, a is called <u>singularity</u> or <u>singular point</u>
- Theorem: assume a is singularity of $f(z)$. $f(z)$ can take <u>Laurent series</u> at region D which exclude a from circle C
    - $f(z) = \sum_{n=-\infty}^{\infty} b_n(z-a)^n = \cdots + \frac{b_{-m}}{(z-a)^m} + \cdots + b_0 + b_1(z-a) + \cdots + b_n(z-a)^n + \cdots$
    - where, circle C is positive direction, $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}} dz$





Isolated singular point

- There are some important singular points
  - Isolated singular points
    - Pole (極)
      - Singular points when its numerator is zero
    - Removable singular points (除去可能な特異点)
      - Caused by the function is undefined at the point, but can define proper value to make regular analytical
    - Essential singular points (真性特異点)
      - Show different limit by different direction, or it has been divergence

Pole (極)

- Similar to real function, complex space support residue theorem
  - Assume a is singularity of $f(z)$. If its Laurent series is
    - $f(z) = \frac{b_{-k}}{(z-a)^k} + \cdots \frac{b_{-1}}{z-a} + +b_0 + b_1(z-a) + \cdots + b_n(z-a)^n + \cdots$
      - It means $b_{-k} \neq 0$ but $b_{-k-1} = b_{-k-2} = \cdots = 0$
    - a is called as (k-th) <u>pole</u> of $f(z)$
  - In this case, $g(z) = (z-a)^k f(z)$ is regular analytical at a

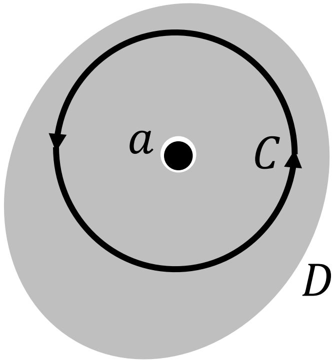
- In oppositely, if f(z) has infinite non-zero coeff. b_{-k}, a is called as <u>essential singularity</u> (真性特異点)

Removable singular point

- Some function has no negative series in its Laurent series
  - E.x. $f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!}\cdots\right)\right] = \frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!}\cdots$
  - In this case, singular point is called as <u>removable</u>
- If the singularity a is removable for $f(z)$, its Laurent series
    - $f(z) = b_0 + b_1(z-a) + \cdots + b_n(z-a)^n + \cdots$
    - has some limit: $\lim_{n\to a} f(z) = b_0$
  - Or, if $f(z)$ satisfy $\lim_{n\to a} f(z) = b_0$, singular point a is removable

Residue (留数)

- Residue: result of closed curve integral surrounds isolated singularities (removable singular, pole, essential singular)
- Assume regular analytical function $f(z)$ and its pole a, closed curve C, all in region D. Its closed curve integral is called <u>residue:</u> $\mathrm{Res}[f,a]$
    - $\mathrm{Res}[f,a] = \frac{1}{2\pi i}\int_C f(z)dz$
  - If a is not a pole $(= f(z)$ is regular analytical at $a)$
    - $\mathrm{Res}[f,a] = 0$
- If a is k-th pole of $f(z)$, its residue is



  - $\mathrm{Res}[f,a] = \frac{1}{(k-1)!}\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}[(z-a)^k f(z)]$

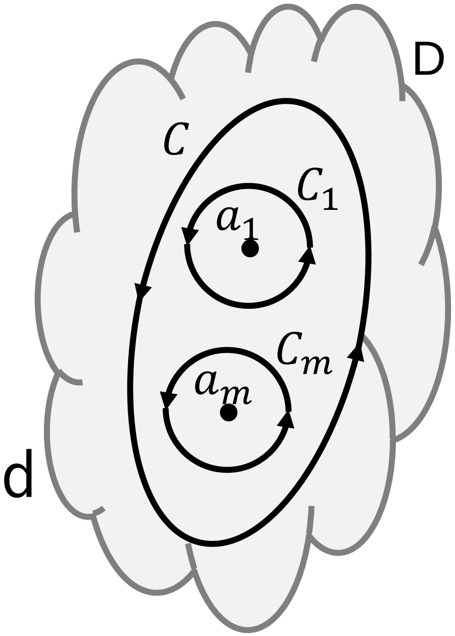
  (k is natural number (1,2,3…))

Residue: example

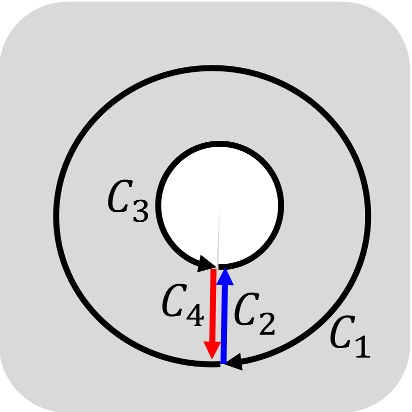
- Assume $f(z) = \frac{e^z}{(z-1)(z+3)^2}$. Calculate Residues
  - (1) $\mathrm{Res}[f, 1]$, (2) $\mathrm{Res}[f, -3]$,
  - $z = 1$ is 1st pole, $z = -3$ is 2nd pole
  - $\mathrm{Res}[f, a] = \frac{1}{(k-1)!} \lim_{n \to a} \frac{d^{k-1}}{dz^{k-1}} [(z-a)^k f(z)]$
- Ans1: $\mathrm{Res}[f, 1] = \lim_{z \to 1} [(z-1) f(z)] = \lim_{z \to 1} \left[\frac{e^z}{(z+3)^2} \right] = \frac{e}{16}$
- Ans2: $\mathrm{Res}[f, -3] = \lim_{z \to -3} \frac{d}{dz} [(z+3)^2 f(z)] = \lim_{z \to -3} \frac{d}{dz} \left[\frac{e^z}{(z-1)} \right] = -\frac{5e^{-3}}{16}$

Residue theorem (留数定理)

- Residue theorem: If circle C contain m poles $a_1, \cdots, a_m$, its circle integral is same as the sum of residues
  - $\frac{1}{2\pi i} \int_C \ f(z)dz = \mathrm{Res}[f, a_1] + \cdots + \mathrm{Res}[f, a_m]$

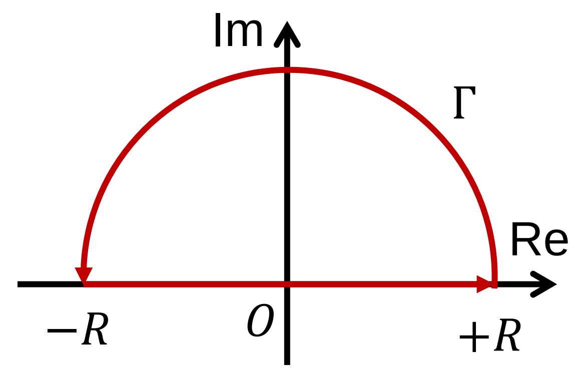


- Use Cauchy's theorem for multiply connected domain
  - For multiply connected domain (non-uniform domain, domain w/ hole), divide domain into several domains
    - Red part and blue part are cancel out
      - thus $\oint_{C_2} f(z)\, dz = -\oint_{C_4} f(z)\, dz$



Application of residue theorem

- Use residue theorem to calculate integral $\int_{-\infty}^{\infty} F(x)dx$
- Preliminary: assume $|f(z)| \leq \frac{M}{R^k}$ at $|z| = R(k > 1, M$: const.$)$*1
  - It satisfy $\lim_{R\to\infty} \int_\Gamma f(z)dz = 0$
    - Γ is half circle of route



- Proof: from eq *1,
  - $\left|\int_\Gamma f(z)dz\right| \leq \int_\Gamma |f(z)|ds \leq \frac{M}{R^k}\pi R = \frac{\pi M}{R^{k-1}}$ s: length of half circle
  - For all region of complex space $(R \to \infty)$, since $k > 1$,
    - $\lim_{R\to\infty} \left|\int_\Gamma f(z)dz\right| = 0$

Application of residue theorem

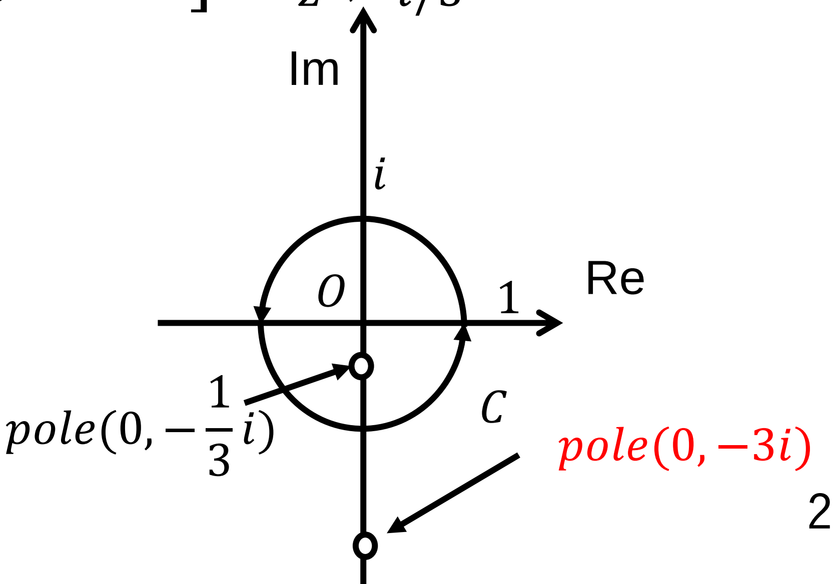
- Calculate integral $\int_0^{2\pi} F(\cos\theta\ )d\theta$
- Replace θ by $\mathrm{z} = e^{i\theta}$
  - $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, $d\theta = \frac{1}{iz}dz$
  - Integral of $(0 \leq \theta < 2\pi)$ ⇔ circle Integral of $|z| = 1$
    - $\int_0^{2\pi} F(\cos\theta\ )d\theta = \int_C F\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\frac{1}{iz}dz$

Example

- Calculate integral of $\int_0^\infty \frac{1}{z^4+1} dz$
- Solution: assume $|z| = R$
  - $\left|\frac{1}{z^4+1}\right| = \frac{1}{|z^4+1|} \leq \frac{1}{|z^4|-1} = \frac{1}{R^4-1} \leq \frac{2}{R^4}$,
  - If take $R \to \infty$, $\frac{2}{R^4} \to 0$
    - $\int_0^\infty \frac{1}{z^4+1} dz = 0$

Example

- Calculate integral of $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta$
  - Solution: replace θ by $z = e^{i\theta}$, C is positive unit circle
    - $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta = \int_C \frac{2}{3z^2+i10z-3} dz = \int_C \frac{2}{(3z+i)(z+3i)} dz$
    - C contain one residual $z = -i/3$ at 1st pole
    - $\mathrm{Res}\left[f, -\frac{i}{3}\right] = \lim_{z\to -i/3} \left[\left(z+\frac{i}{3}\right)^1 f(z)\right] = \lim_{z\to -i/3} \frac{2}{3z+9i} = \frac{1}{4i}$
    - $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$

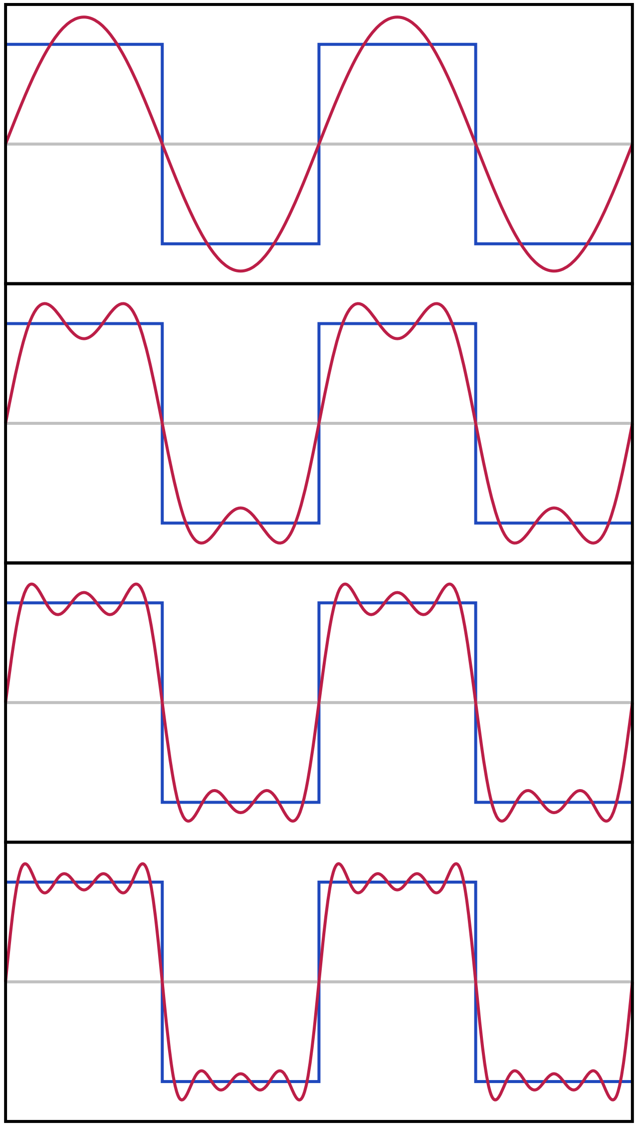


Fundamental Mathematics

- Complex function theory 2 Fourier series-

Fourier series (フーリエ級数)

- Fourier series: summation of harmonically related sine function
  - Summation is a periodic function, determined by
    - the choices of cycle length (period)
    - the number of components, amplitude and phase
  - Originally, developed to solve thermal conduction (differential equation)



Fourier series (from 1st order to 4th order)
Source: https://en.wikipedia.org/wiki/Fourier_series

Fourier series: definition

- Assume $f(x)$ is defined in $x \in \mathbb{R}$, and it has period 2π
- Fourie series in sine functions can be expressed as
  - $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (1)
- Here, we want to know the value of a_n and b_n
  - Take integral $-\pi \sim \pi$
    - $\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\, dx$
  - Since $\int_{-\pi}^{\pi} \cos nx\, dx = \int_{-\pi}^{\pi} \sin nx\, dx = 0$
    - $a_0 = 1/\pi \int_{-\pi}^{\pi} f(x)dx$
- Note: $\sim$ means the Fourier series need some conditions to be equal

Fourier series: definition (cont.)

- Multiply $\cos mx$ to (1) of both side, then take integral
  - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (1)
  - $\int_{-\pi}^{\pi} f(x) \cos mx\, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx\, dx +$
    $\sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} \cos mx \cos nx\, dx + b_n \int_{-\pi}^{\pi} \cos mx \sin nx\, dx)$
  - Use following relationships (w/o proof)
    - $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} \pi & (n = m) \\ 0 & (n \neq m) \end{cases}$
    - $\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0$
    - $a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx,\ b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx\, dx,$

Fourier series: definition (cont.)

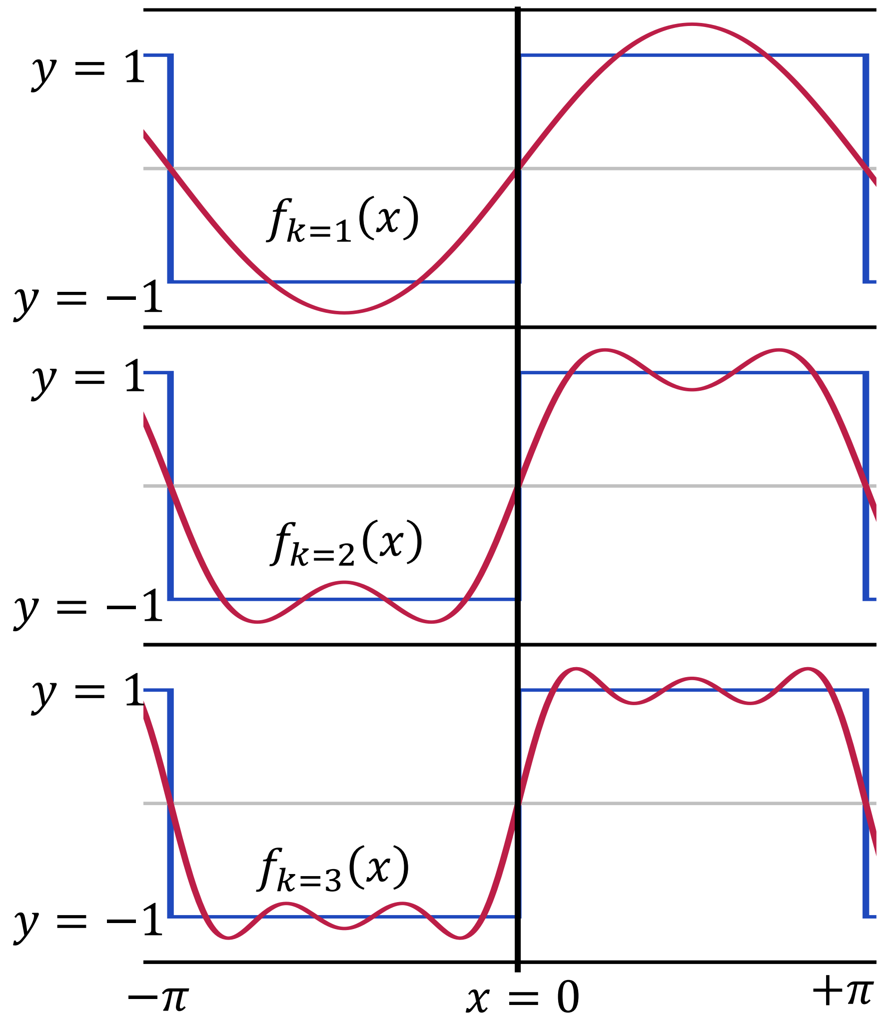
- Fourier series can be defined as
  - $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (1)
  - where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$
- Note: ~ means the Fourier series need some conditions

Fourier series: example 1

- Calculate Fourier series for periodic function (period 2π)
  - $f(x) = \begin{cases} -1 & (-\pi \le x < 0, x = \pi) \\ +1 & (0 \le x < \pi) \end{cases}$
- Solution:
  - $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = -\frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = 0$
  - $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -\frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx =$
    $\frac{2}{\pi}\int_{0}^{\pi}\sin nx\,dx = \begin{cases} 0 & n \text{ is even} \\ 4/n\pi & n \text{ is odd} \end{cases}$
  - $f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1}$
  - Note: $f(0) = 1$ in definition, but Fourier series converge to 0 at $x = 0$: convergence problem

Fourier series: example 1

- $f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1}$ can be decomposed to
  - $f_{k=1}(x) = \frac{4}{k}\sin x$
  - $f_{k=2}(x) = f_{k=1}(x) + \frac{4}{3k}\sin 3x$
  - $f_{k=3}(x) = f_{k=2}(x) + \frac{4}{5k}\sin 5x$
- NOTE: Conversion
  - Original func. $f(0) = +1$
  - Fourier series converse to 0 (not +1)



Fourier series: example 2

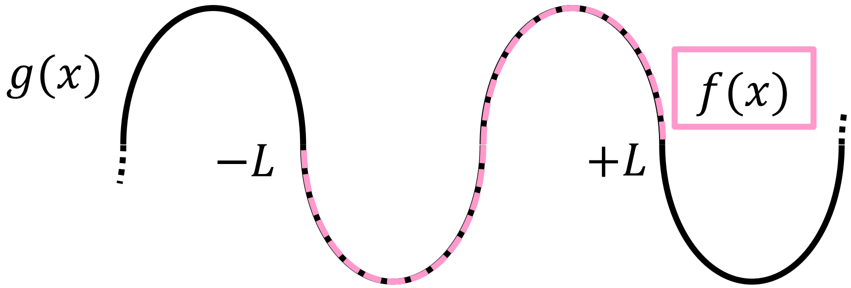
- Calculate Fourier series for periodic function (period 2π)
  - $f(x) = \begin{cases} \frac{\pi}{2} + x & (-\pi \leq x \leq 0,) \\ \frac{\pi}{2} - x & (0 \leq x \leq \pi) \end{cases}$
- Solution: $(n \neq 0)$
  - $a_n = \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{\pi}{2} + x\right)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx\, dx$
  - $= \frac{1}{\pi}\left[\left(\frac{\pi}{2} + x\right)\frac{\sin nx}{n}\right]_{-\pi}^{0} - \frac{1}{n\pi}\int_{-\pi}^{0}\left(\frac{\pi}{2} + x\right)\sin nx dx + \frac{1}{\pi}\left[\left(\frac{\pi}{2} - x\right)\frac{\sin nx}{n}\right]_{0}^{\pi} - \frac{1}{n\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - x\right)\sin nx\, dx$
  - $=\frac{2(1-\cos n\pi)}{n^2 x} = \frac{2(1-(-1)^n)}{n^2 x}$
  - Similarly, a_0 and b_n can be calculated
- $f(x) \sim \frac{4}{\pi}\cos x + \cdots \frac{4}{3\pi}\cos 3x + \cdots = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}$

Fourier series: even/odd func

- Introduce variable $t = x\pi/L$ w/ period $0 \sim 2L$
  - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$
  - where $a_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$, $b_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$
- If $f(x)$ is even function (偶関数):
  - $f(x)\sin\frac{n\pi x}{L}$ is odd function (奇関数) -> $b_n = 0$
- If $f(x)$ is odd function:
  - $f(x)\cos\frac{n\pi x}{L}$ is even function -> $a_n = 0$

Fourier series in finite interval

- Fourier series of $f(x)$ is available for finite interval $[L, -L]$
  - Assume infinite func. $g(x)$ is available which matched with $f(x)$ in finite interval $[L, -L]$
  - Fourier series of $f(x)$ (finite) is same as $g(x)$ (infinite)
- Case if $f(-L) \neq f(L)$?
  - Re-define $f(x = L) = \frac{1}{2}\{f(-L+0) + f(L-0)\}$ to $f(-L) = f(L)$
    - (this re-definition also show same as original result of $f(x)$)

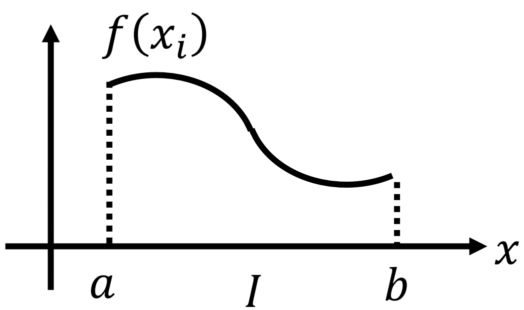


Fourier series in complex space

- Use Euler's formula to extend Fourier series in complex
  - $\cos nx = \frac{e^{inx}+e^{-inx}}{2}, \sin nx = \frac{e^{inx}-e^{-inx}}{2i}$
  - $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx} dx, (x \in \mathbb{R})$
- Also, Fourier series is available for finite interval $[-L, L]$
  - $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{in\pi x}{L}}, a_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{\frac{in\pi x}{L}} dx, (x \in \mathbb{R})$

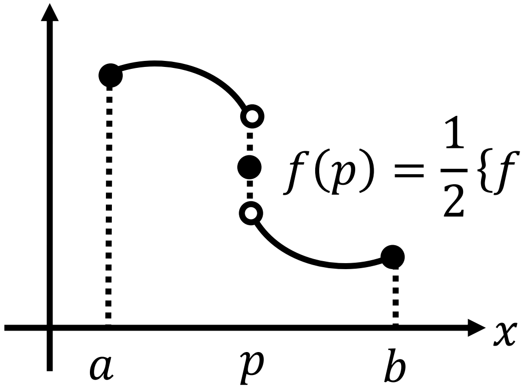
Piecewise smooth (区分的連続)

- Function $f(x)$ should be piecewise smooth at range I $[a, b]$ to converse its Fourier series
- Piecewise smooth
  - Derivative of $f(x)$ should continuous (連続) (exclude finite non-continuous points)
  - At the non-continuous points, $f(x)$ and $f'(x)$ of both right-side and left-side limits are available and not infinite
    - $f(x_i - 0) = \lim_{h \to 0} f(x_i - h)$, $f(x_i + 0) = \lim_{h \to 0} f(x_i + h)$
    - $f'(a + 0) = \lim_{h \to 0} f'(a + h)$, $f'(b - 0) = \lim_{h \to 0} f'(b - h)$



Piecewise smooth

- Redefine non-continuous func. to piecewise smooth
  - Assume point p is non-continuous in range $I\ [a, b]$
  - Redefine $f(p)$ by average of left limit and right limit
    - $f(p) = \frac{1}{2}\{f(p+0) + f(p-0)\}$
  - This operation make function to piecewise smooth
- Following contents assume this operation for all of non-continuous points in Fourier series



Fourier series and conversion

- Theorem1: If $f(x)$ is periodic function w/ period 2L and piecewise smooth, and its derivative $f'(x)$ is also piecewise smooth. Its Fourier series
  - converge to $f(x)$ when x is continuous
  - converge to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ when x is non-continuous
- If above is satisfied, its Fourier series is
  - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$
  - ($\sim$ -> $=$)

Termwise integral (項別積分)

- Theorem2: If $f(x)$ is periodic function w/ period $2L$ and piecewise smooth, and its Fourier series as
  - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$
- Its termwise integral for $(-L \leq a, x \leq L)$ can be calculated as
  - $\int_a^x f(x)dx = \frac{a_0}{2}\int_a^x dx + \sum_{n=1}^{\infty}(a_n \int_a^x \cos\frac{n\pi x}{L}dx + b_n \int_a^x \sin\frac{n\pi x}{L}dx)$
  - $= \frac{a_0}{2}(x-a) - \sum_{n=1}^{\infty}\frac{L}{n\pi}\left(a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\frac{L}{n\pi}(a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L})$

Termwise differential (項別微分)

- Theorem3: If $f(x)$ and $f'(x)$ continuous, and $f''(x)$ piecewise smooth, Fourier series of $f(x)$ and $f'(x)$ will converge to $f(x)$ and $f'(x)$, and Fourier series of $f'(x)$ can be calculated by ternmwise differential of $f(x)$
  - For $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$
  - $f'(x) = \left(\frac{a_0}{2}\right)' + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)'$
  - $f'(x) = \frac{n\pi}{L}\sum_{n=1}^{\infty}\left(-a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right)$

Conclusion

- Introduce complex function theory
  - Fourier series in complex space
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