Fundamental Mathematics (Engineering Mathematics)

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Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

Motivation

- Many physics can be expressed by vectors
  - Good to explain in simple way (if we know vectors)
- Target: understand the meaning of Maxwell's equation
  - $\mathrm{div}\,\boldsymbol{D} = \rho$
    - $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho dV$ (Gauss's eq of electricfield)
  - $\mathrm{div}\,\boldsymbol{B} = 0$
    - $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \iiint \mathrm{div}\,\boldsymbol{B} dV$ (Gauss's eq of magneticfield)
  - $\mathrm{rot}\,\boldsymbol{H} = i + \frac{\delta D}{\delta t}$: $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(i + \frac{\delta \boldsymbol{D}}{\delta t}\right) \cdot d\boldsymbol{S}$ (Ampele's law)
  - $\mathrm{div}\,\boldsymbol{E} = -\frac{\delta B}{\delta t}$: $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\delta}{\delta t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$ Faraday's law)

Motivation (1)

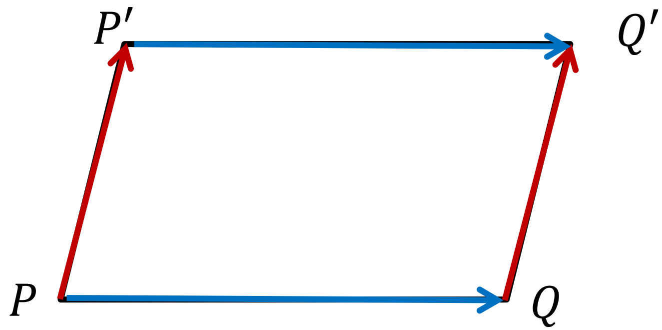
- Many physics can be expressed by vectors
  - Good to explain in simple way (if we know vectors)
- Target: understand the meaning of Maxwell's equation
  - $\operatorname{div} \boldsymbol{D} = \rho$
    - $\iint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho dV$ (Gauss's eq of electricfield)
    - $\boldsymbol{D}$: electric flux density, S: area, ρ: charge density, V: volume
  - $\operatorname{div} \boldsymbol{B} = 0$
    - $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \iiint \operatorname{div} \boldsymbol{B} dV$ (Gauss's eq of magneticfield)
    - $\boldsymbol{B}$: Magnetic flux density,

Motivation (2)

- rot $\boldsymbol{H} = i + \frac{\delta D}{\delta t}$: $\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \iint \left(i + \frac{\delta \boldsymbol{D}}{\delta t}\right) \cdot d\boldsymbol{S}$ (Ampele's law)
  - $\boldsymbol{H}$: magnetic field, i: current, $\boldsymbol{D}$: electric flux density
- div $\boldsymbol{E} = -\frac{\delta B}{\delta t}$: $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\delta}{\delta t} \iint \boldsymbol{B} \cdot d\boldsymbol{S}$ Faraday's law)
  - $\boldsymbol{E}$: electromotive force, $\boldsymbol{B}$: Magnetic flux density

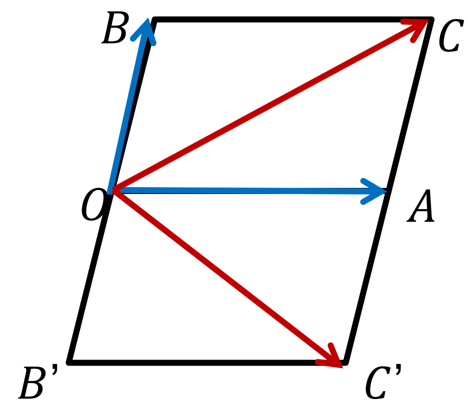
Scalar and Vector

- Scalar: Value (only)
- Vector: Value (length) and its direction
  - Vector from point P to Q is: $\overrightarrow{PQ}$
    - P: start point, Q: end point
    - If $\overrightarrow{P'Q'}$ is equal to $\overrightarrow{PQ}$, $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ is in the same class
    - If two points are the same, it is zero vector $\overrightarrow{PP}$, $\overrightarrow{QQ}$
- To show the vector, we use **bold**
  - Vector: $\boldsymbol{a} = \overrightarrow{PQ}$
  - Zero vector $\mathbf{0} = \overrightarrow{PP}$



Add, sub, extension

- Assume $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$,$\boldsymbol{c} = \overrightarrow{OC}$, where O, A, B, C composes parallelogram
  - Define:$-\boldsymbol{a} = -\overrightarrow{OA} = \overrightarrow{AO}$
  - Define: $\boldsymbol{a} + \boldsymbol{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$
  - Define: $\boldsymbol{a} - \boldsymbol{b} = \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OC'}$



- For real value λ, its product to the vector $\boldsymbol{a}$ is
  - $\boldsymbol{a}\lambda = \lambda\boldsymbol{a}$
- If the three points P, Q, R are on the same line: $\overrightarrow{PQ} = \lambda\overrightarrow{PR}$
- If the two vectors are in parallel: $\boldsymbol{a}\lambda = \boldsymbol{b}$
  - Geometric vector space
    - Vector space: more general and abstract

Vector space

- L is called vector space if element of L satisfy following definition and notation
  - Addition: result of $\boldsymbol{a}+\boldsymbol{b}$ is unique $(\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{L})$
  - Scalar multiply: result of $\boldsymbol{a}\lambda$ is unique $(\boldsymbol{a} \in \boldsymbol{L}, \lambda \in R)$
- Both satisfy following:
  - Association law: $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$
  - Exchange low: $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$
  - Identity element: $\boldsymbol{a}+\boldsymbol{o}=\boldsymbol{a}$
  - inverse element: $\boldsymbol{a}+(-\boldsymbol{a})=\boldsymbol{0}$

Component

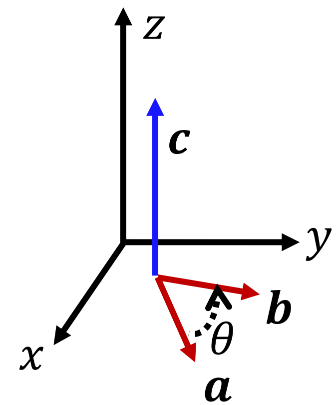
- Vector $\boldsymbol{a}$ is also defined by its components $[a_1, \cdots, a_n]$
  - n: its #dimension
- For the xyz-coordinate system, $\boldsymbol{a} = [a_x, a_y, a_z]$
  - This also satisfy the rules of vector space
- Or, using unit vector (基本ベクトル) $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, for xyz-coord. system,
  - $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, where $a_1 = |a_x|$, $a_2 = |a_y|$, $a_3 = |a_z|$
- Length: $|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$, unit vector $\boldsymbol{u} = \boldsymbol{a}/|\boldsymbol{a}|$
- These definitions can be easily implemented as array of computer program
  - In C-language: $a[3] = [a_x, a_y, a_z]$

Inner product (内積)

- For two vectors $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{a} \cdot \boldsymbol{b} = \mathrm{c} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ is called as inner product in scaler value $(\theta = \angle AOB)$
- Inner products has following characteristics
  - $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
  - $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
  - $\lambda\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \lambda\boldsymbol{b} = \lambda(\boldsymbol{a} \cdot \boldsymbol{b})$
- For unit vector $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$,
  - $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = \boldsymbol{1}$
  - $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{0}$

Outer product (外積)

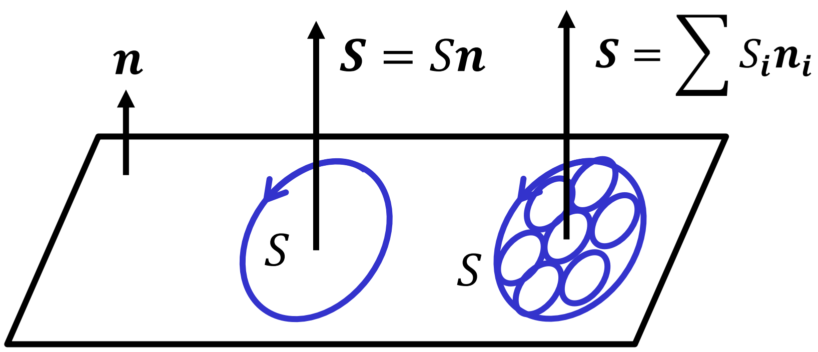
- (Assume right-hand side coordinate system)
- For $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$: outer product
  - $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$
  - Angle of $\boldsymbol{c}$: perpendicular to the surface of $\boldsymbol{a}, \boldsymbol{b}$
  - If $\boldsymbol{a}$ and $\boldsymbol{b}$ are in parallel $(\sin\theta = 0)$, $\boldsymbol{a}$ or $\boldsymbol{b} = \boldsymbol{o}$, c $= \boldsymbol{o}$



- Theorem:
  - $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{o}$
  - $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
  - $\lambda\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \lambda\boldsymbol{b} = \lambda(\boldsymbol{a} \times \boldsymbol{b})$
  - $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = \boldsymbol{0}$
  - $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{i},\ \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{j}$

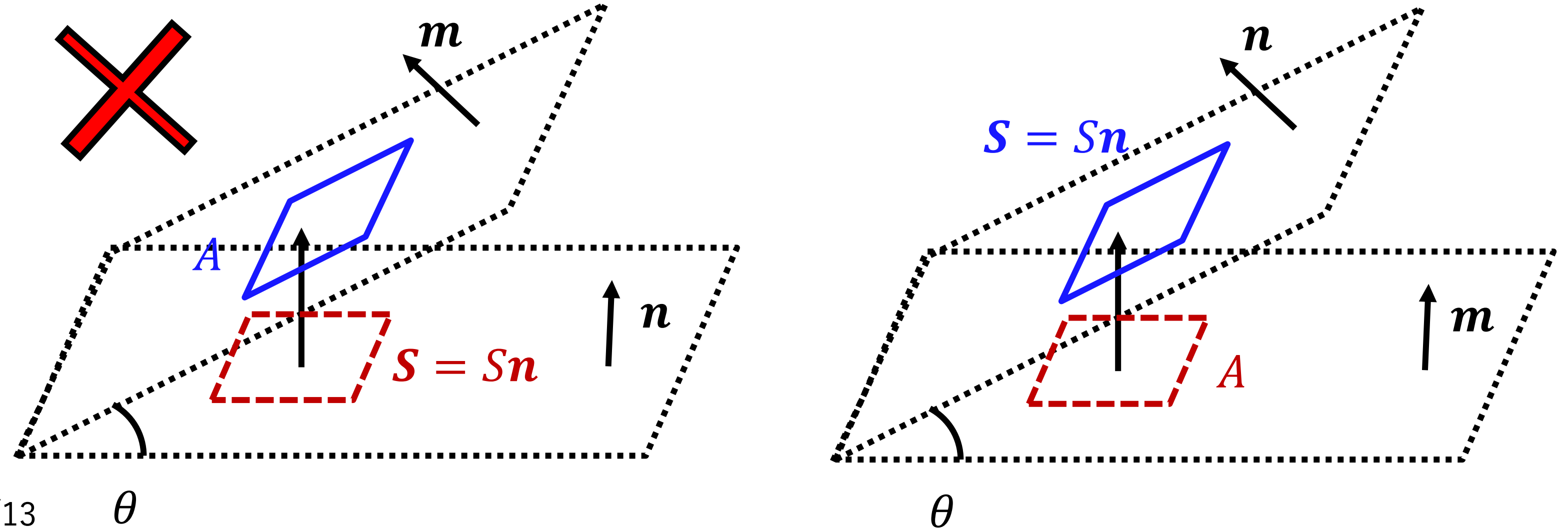
Vector area(面積ベクトル)

- Vector area: vector combining an area quality w/ dimension
- Assume surface S on signed area in two dimension system
  - Vector area $\boldsymbol{S}$ can be expressed with its unit vector $\boldsymbol{n}$
    - $\boldsymbol{S} = S\boldsymbol{n}$
  - Rotation of vector $\boldsymbol{n}$ express the sign
    - anticlockwise (right-hand screw) : plus
    - clockwsise (left-hand screw): minus
  - If S is subset of S_i, the vector area $\boldsymbol{S}$ can be
    - $\boldsymbol{S} = \sum S_i \boldsymbol{n}_i$



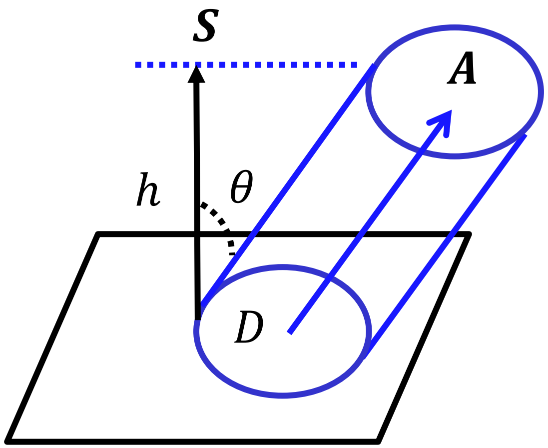
Projection (射影)

- Area vector is used to calculate surface integral
  - Treat fluxs of a vector filed through a surface
- Projection area A on plane $\boldsymbol{S}$ can be calculated by dot product with target plane unit normal $\boldsymbol{m}$
  - $A = \boldsymbol{S} \cdot \boldsymbol{m}$
- If the two surface has same xy and angle θ for z-coordinate,
  - $A = |\boldsymbol{S}| \cos \theta$



Volume (体積)

- Volume V can be calculated by area vector
  - Calculate volume V of tilted cylinder
    - Bottom plane: D
    - Area vector: $\boldsymbol{S}$
    - Direction: $\boldsymbol{A}$
    - Assume its angle: θ
  - Hight $h = |\boldsymbol{A}| \cos\theta$
  - Volume $V = h|\boldsymbol{S}| = |\boldsymbol{A}||\boldsymbol{S}| \cos\theta = \boldsymbol{A} \cdot \boldsymbol{S}$
- Volume $V = \boldsymbol{A} \cdot \boldsymbol{S}$ express the amount of flow $\boldsymbol{A}$ which punctulate the plane D

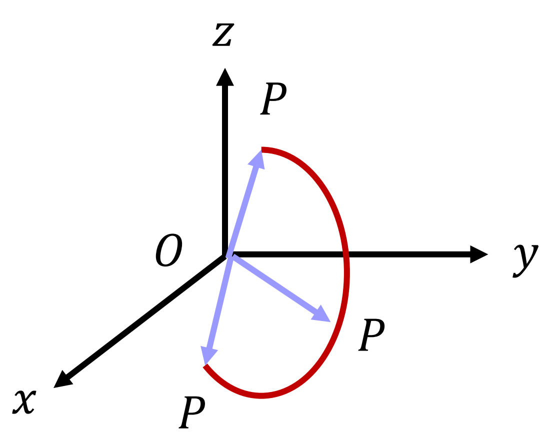


Fundamental Mathematics

- Derivation of vector-

Derivation for vector func.

- Vector function $\boldsymbol{F}(t)$: vector $\boldsymbol{F}$ is a function of scalar t
  - If vector $\boldsymbol{F}$ is continuous to the t: $\boldsymbol{F}$ is continuous
- Assume vector $\boldsymbol{F}(t) = \overrightarrow{OP}$, where O is origin (fixed point)
  - Point P draw a curved line

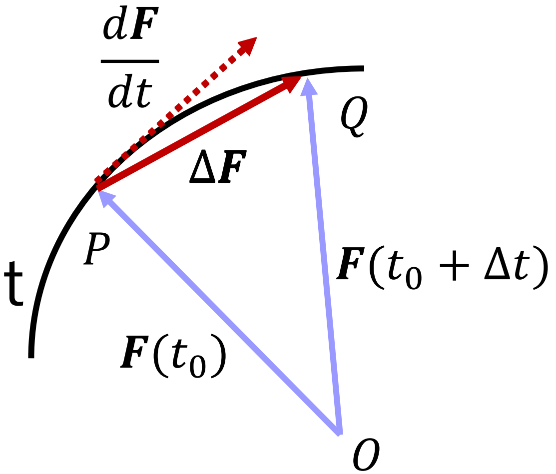


Characteristics

- A limit: if vector $\boldsymbol{A}$ satisfy $\lim_{n\to\infty}|\boldsymbol{A}_n - \boldsymbol{A}| = 0$ for $\boldsymbol{A}_0 \cdots \boldsymbol{A}_n$
  - $\lim_{n\to\infty} \boldsymbol{A}_n = \boldsymbol{A}$, and $\boldsymbol{A}$ is a limit of $\boldsymbol{A}_0 \cdots \boldsymbol{A}_n$
- A limit: if vector func. $\boldsymbol{F}(t)$ has const. vector $\boldsymbol{A}$, and it satisfy $\lim_{t\to t_0}|\boldsymbol{F}(t) - \boldsymbol{A}| = 0$ for $t \to t_0$
  - $\lim_{t\to t_0} \boldsymbol{F}(t) = \boldsymbol{A}$, and $\boldsymbol{A}$ is a limit of $\boldsymbol{F}(t)$ for $t \to t_0$
  - For $\boldsymbol{F}(t) = F_1(t)\boldsymbol{i} + F_2(t)\boldsymbol{j} + F_3(t)\boldsymbol{k}$, $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$
    - $\lim_{t\to t_0} F_1(t) = A_1$, $\lim_{t\to t_0} F_2(t) = A_2$, $\lim_{t\to t_0} F_3(t) = A_3$
- Continuity: if vector func. $\boldsymbol{F}(t)$ satisfy $\lim_{t\to t_0} \boldsymbol{F}(t) = \boldsymbol{F}(t_0)$ for $t \to t_0$, $\boldsymbol{F}(t)$ is continuous

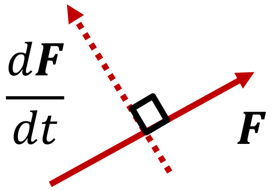
Characteristics

- Derivative(導関数): if $\lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{F}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{F}(t_0+\Delta t)-\boldsymbol{F}(t_0)}{\Delta t}$ is available, this is called as differential coefficient $\boldsymbol{F}'(t_0)$
  - For each t, the vector function $\boldsymbol{F}'(t_0)$ or $\frac{d\boldsymbol{F}}{dt}$ is called as derivative or derivative vector
  - Similarly, derivative can be taken as $\boldsymbol{F}'(t_0)$ and $\boldsymbol{F}^{(n)}(t_0)$
- Geometric meaning
  - Assume $\overrightarrow{OP} = \boldsymbol{F}\,(t)$, $\overrightarrow{OQ} = \boldsymbol{F}\,(t + \Delta t)$,
    - $\Delta \boldsymbol{F} = \boldsymbol{F}\,(t + \Delta t) - \boldsymbol{F}\,(t) = \overrightarrow{PQ}$
    - Take $\Delta t \to 0$ then $\Delta \boldsymbol{F}$ becomes tangent



Theorems for derivation

- Vector func. $\boldsymbol{F}(t)$ and $\boldsymbol{G}(t)$, scalar func $f(t)$, satisfy followings
  - (sum) : $\frac{d}{dt}(\boldsymbol{F}+\boldsymbol{G}) = \frac{d}{dt}\boldsymbol{F} + \frac{d}{dt}\boldsymbol{G}$
  - (scalar prod.) : $\frac{d}{dt}(f\boldsymbol{F}) = \frac{df}{dt}\boldsymbol{F} + f\frac{d}{dt}\boldsymbol{F}$
  - (inner prod.) : $\frac{d}{dt}(\boldsymbol{F}\cdot\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt}\cdot\boldsymbol{G} + \boldsymbol{F}\cdot\frac{d\boldsymbol{G}}{dt}$
  - (outer prod.) : $\frac{d}{dt}(\boldsymbol{F}\times\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt}\times\boldsymbol{G} + \boldsymbol{F}\times\frac{d\boldsymbol{G}}{dt}$
  - For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$, $\frac{d\boldsymbol{F}}{dt} = \frac{dF_1}{dt}\boldsymbol{i} + \frac{dF_2}{dt}\boldsymbol{j} + \frac{dF_3}{dt}\boldsymbol{k}$
  - If $\boldsymbol{F}$ is constant, $\frac{d\boldsymbol{F}}{dt}$ is $\boldsymbol{o}$, or perpendicular s.t. $\boldsymbol{F}\cdot\frac{d\boldsymbol{F}}{dt} = 0$



High order derivatives, partial difference

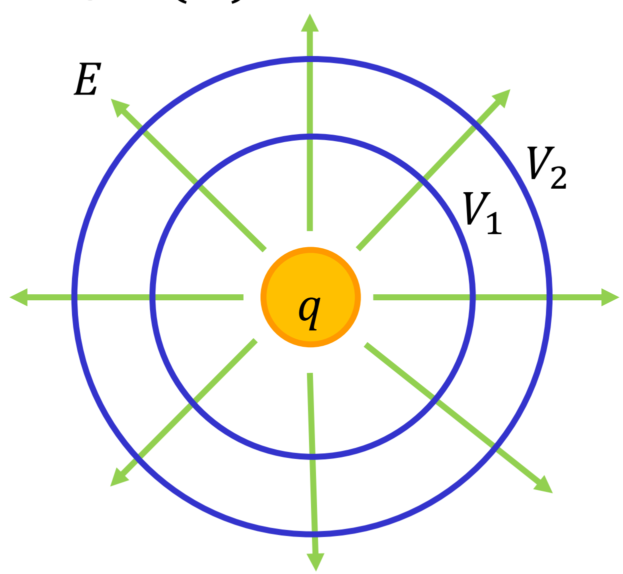
- High order derivatives can defined as similar to 1st order
  - $\frac{d^2\boldsymbol{F}}{dt^2}, \frac{d^3\boldsymbol{F}}{dt^3}, \cdots, \frac{d^n\boldsymbol{F}}{dt^n}$
  - For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k},\ \frac{d^n\boldsymbol{F}}{dt^n} = \frac{d^nF_1}{dt^n}\boldsymbol{i} + \frac{d^nF_2}{dt^n}\boldsymbol{j} + \frac{d^nF_3}{dt^n}\boldsymbol{k}$
- Partial difference also defined like derivation
  - $\boldsymbol{A} = \boldsymbol{A}(u, v),\ \frac{\delta\boldsymbol{A}}{\delta u}, \frac{\delta\boldsymbol{A}}{\delta v}, \frac{\delta^2\boldsymbol{A}}{\delta v^2}, \frac{\delta^2\boldsymbol{A}}{\delta v\delta u}, \frac{\delta^2\boldsymbol{A}}{\delta u\delta v}, \frac{\delta^2\boldsymbol{A}}{\delta u^2}$
- Total difference of $\boldsymbol{A}(u, v)$ can be defined as
  - $\delta\boldsymbol{A}(u, v) = \frac{\delta\boldsymbol{A}}{\delta v}du + \frac{\delta\boldsymbol{A}}{\delta u}dv$
  - It approx. small delta of $\delta\boldsymbol{A}$ by small delta of du, $d\cancel{u}$ v
    - For $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}, \delta\boldsymbol{A} = \delta A_1\boldsymbol{i} + \delta A_2\boldsymbol{j} + \delta A_3\boldsymbol{k},$

Gradient of scalar

- Scalar function: $f(x, y, z)$ can be defined in unique
  - This field is called scalar field f
    - Distribution of temperature, mass, voltage
- Vector function: $\boldsymbol{F}(x, y, z)$ can be defined in unique
  - This field is called vector field $\boldsymbol{F}$
    - Electric field, magnetic field, gravity field
- <u>Gradient</u> of scalar: $\mathrm{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$
  - ∇: Hamilton operator
    - $\nabla(f+g) = \nabla f + \nabla g$, $\nabla \lambda f = \lambda \nabla f$, $\nabla(fg) = g\nabla f + f\nabla g$
    - $\nabla \phi(f) = \frac{d\phi}{df}\nabla f$, where $\phi(f)$ is a function of f

Equipotential surface

- If group of points $P(x, y, z)$ satisfy $f(x, y, z) = c$ (c: const), P is called equipotential surface
  - In the case of $f(x, y, z) = x^2 + y^2 + z^2$
    - Surface of sphere
- In electro-magnetics, electron (q) create divergence of electric lines (electric field: E), and electric line create equipotential voltage (V)

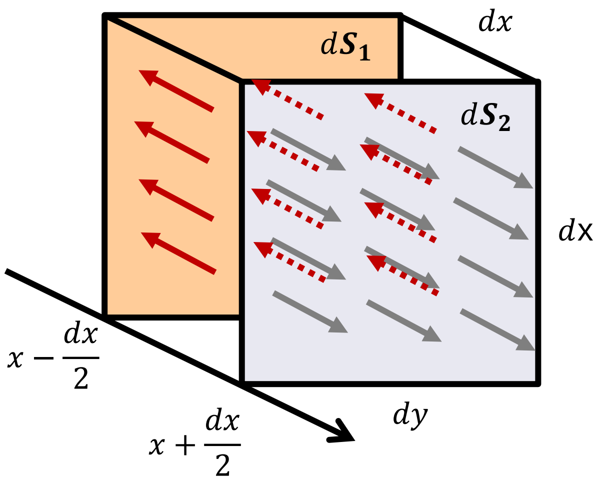


Divergence of vector

- For vector $\boldsymbol{F}(x, y, z) = F_1(x, y, z)\boldsymbol{i} + F_2(x, y, z)\boldsymbol{j} + F_3(x, y, z)\boldsymbol{k}$, $\mathrm{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \boldsymbol{F}$ is called as divergence
- Vector $\boldsymbol{F}, \boldsymbol{G}$, scalar f satisfy following conditions
  - $\mathrm{div}\,(\boldsymbol{F} + \boldsymbol{G}) = \mathrm{div}\,(\boldsymbol{F}) + \mathrm{div}\,(\boldsymbol{G})$
  - $\mathrm{div}\,(f\boldsymbol{G}) = \mathrm{grad}(f) \cdot \boldsymbol{G} + f\,\mathrm{div}\boldsymbol{G}$
  - $\mathrm{div}\,\mathrm{grad}(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
- Physical meaning
  - $\mathrm{div}\boldsymbol{F} > 0$: something spout (flow out)
  - $\mathrm{div}\boldsymbol{F} < 0$: something swallowed (flow in)

Divergence of vector

- $\mathrm{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \boldsymbol{F}$
- Assume flow $\boldsymbol{F}$ of small box $dxdydz$
  - Assume flow $\boldsymbol{F}$ of area $d\boldsymbol{S}_1 = (-dydz, 0,0)$ at $x - \frac{dx}{2}$
  - Assume flow $\boldsymbol{F}$ of area $d\boldsymbol{S}_2 = (+dydz, 0,0)$ at $x + \frac{dx}{2}$
    - $\boldsymbol{F} \cdot d\boldsymbol{S} = \boldsymbol{F} \cdot d\boldsymbol{S}_2 - \boldsymbol{F} \cdot d\boldsymbol{S}_1$
    - $= F_1\left(x + \frac{dx}{2}, y, z\right) dydz$ $+$
      $F_1\left(x - \frac{dx}{2}, y, z\right)(-dydz)$
    - $= \frac{\partial F_1}{\partial x} dxdydz$
  - Diff. flow in ( ) and out ( )

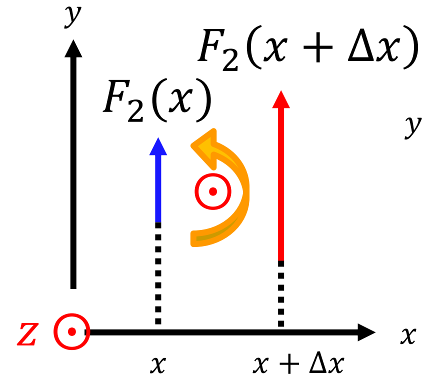


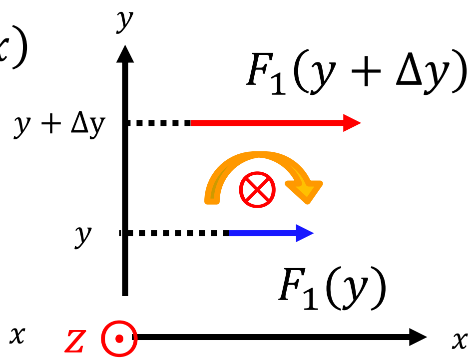
Rotation of vector

- For vector $\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}$, $\mathrm{rot}\,\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} = \nabla \times \boldsymbol{F}$ is called as <u>rotation</u>
- $\mathrm{rot}\,\boldsymbol{F} = (\mathrm{rot}_1\,\boldsymbol{F})\boldsymbol{i} + (\mathrm{rot}_2\,\boldsymbol{F})\boldsymbol{j} + (\mathrm{rot}_3\,\boldsymbol{F})\boldsymbol{k}$
- Vector $\boldsymbol{F}, \boldsymbol{G}$, scalar f satisfy following conditions
  - $\mathrm{rot}\,(\boldsymbol{F} + \boldsymbol{G}) = \mathrm{rot}\,(\boldsymbol{F}) + \mathrm{rot}\,(\boldsymbol{G})$
  - $\mathrm{rot}\,(f\boldsymbol{G}) = \mathrm{grad}(f) \times G + f\nabla \times \boldsymbol{G}$
- Physical meaning
  - $\mathrm{rot}\,\boldsymbol{F} > 0$: right-hand side (screw) rotation ($\otimes$)
  - $\mathrm{rot}\boldsymbol{F} < 0$: left-hand side (screw) rotation ($\odot$)

Physical meaning of rotation

- Link physical notation to the rotation of vector
- Focus 3rd term ($\boldsymbol{k}$) of rotation
  - $\mathrm{rot}\,\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}$
  - $\frac{\partial F_2}{\partial x} = \lim_{\Delta x \to 0} \frac{F_2(x+\Delta x) - F_2(x)}{\Delta x}$
  - $\frac{\partial F_1}{\partial y} = \lim_{\Delta y \to 0} \frac{F_1(y+\Delta y) - F_1(y)}{\Delta y}$

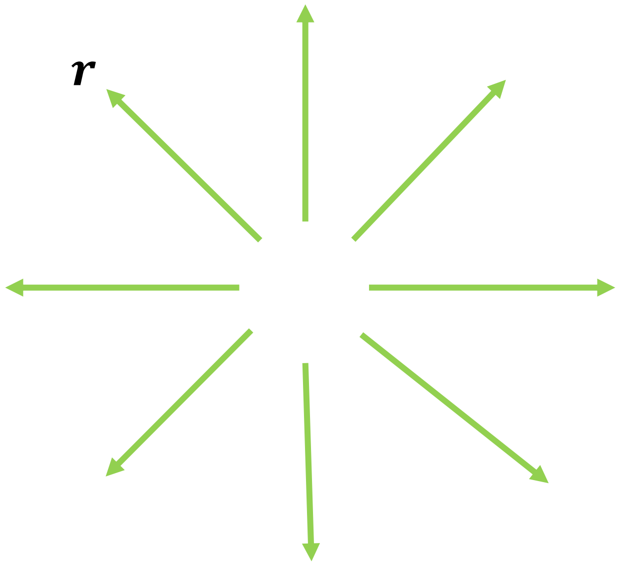




  - If $\frac{\partial F_2}{\partial x} > 0$, it generates right-hand side rotation
  - If $-\frac{\partial F_1}{\partial y} > 0$, it generates right-hand side rotation
  - $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} > 0$ means right-hand side rotation is available

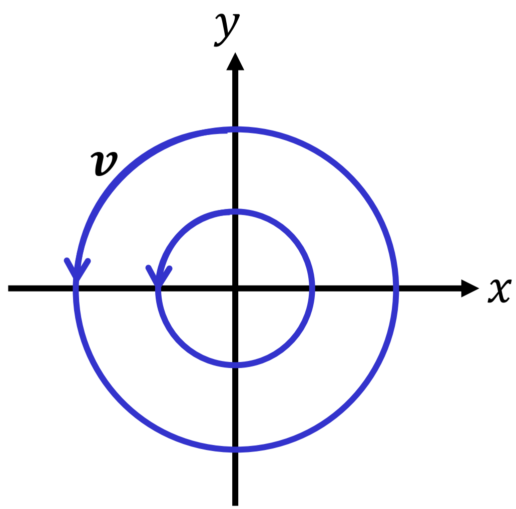
Examples

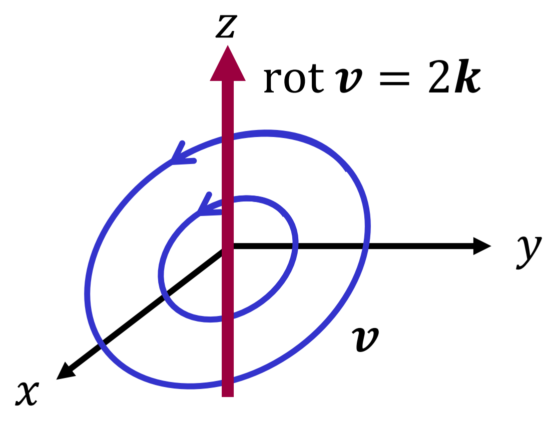
- For $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$,
  - (Q1) Calculate $\mathrm{div}\,\boldsymbol{r}$
  - (A1) $\mathrm{div}\,\boldsymbol{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$
    - (Volume is positive for all xyz)
  - (Q2) Calculate $\mathrm{rot}\,\boldsymbol{r}$
  - (A2) $\mathrm{rot}\,\boldsymbol{r} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\boldsymbol{k} = 0$
    - (No rotating vector here)



Examples

- For $\boldsymbol{v} = -y\boldsymbol{i} + x\boldsymbol{j}$,
  - (Q1) Calculate $\mathrm{div}\,\boldsymbol{v}$
  - (A1) $\mathrm{div}\,\boldsymbol{v} = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0$
  - This equation is $x^2 + y^2 = c$ (c: const.)
    - No flow in/out, rotation
  - (Q2) Calculate $\mathrm{rot}\,\boldsymbol{v}$
  - (A2) $\mathrm{rot}\,\boldsymbol{v} = \left(\frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial(-y)}{\partial z} - \frac{\partial 0}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\boldsymbol{k} = 2\boldsymbol{k}$





Exercise

- Assume $\boldsymbol{a}, \boldsymbol{b}$ is constant vector, $|\boldsymbol{r}(t)| = r(t)$, calculate its derivation
  - $r\boldsymbol{r} + (\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{b}$
  - $\frac{\boldsymbol{r}}{r^2}$
- Calculate gradient for following functions
  - $f = xz^3 - x^2y$, calculate ∇f at point $P(1, -2, 2)$
  - $f = x^2y^2 - 2xz^3$, calculate ∇f at point $P(1, -2, 1)$
- Calculate divergence of following functions
  - $x^2y\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + 3z^3x^3\boldsymbol{k}$
- Calculate rotation of following functions
  - $x^2\boldsymbol{i} - 2xz\boldsymbol{j} + y^2z\boldsymbol{k}$

sample solution

Math ⑥

(1)

$$(r\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\mathbf{b})' = r'\mathbf{r}+r\mathbf{r}'+(\mathbf{a}\cdot\mathbf{r}')\mathbf{b}$$

$$\left(\frac{\mathbf{r}}{r^2}\right)' = \frac{\mathbf{r}'}{r^2} - \frac{2r'}{r^3}\mathbf{r}$$

(2) $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

$$= (z^3-2xy)\mathbf{i} + (-x^2)\mathbf{j} + (3xz^2)\mathbf{k}$$

$$= (+8+4)\mathbf{i} + (-1)\mathbf{j} + 12\mathbf{k}$$

$$= 12\mathbf{i} - \mathbf{j} + 12\mathbf{k}$$

$$\Delta f = (2xy^2-2z^3)\mathbf{i} + (2x^2y)\mathbf{j} + (-6xz^2)\mathbf{k}$$

$$= (8-2)\mathbf{i} + (-4)\mathbf{j} + (-6)\mathbf{k}$$

$$= +6\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}$$

(3) $\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\operatorname{div}(x^2y\mathbf{i} - 2y^2z^2\mathbf{j} + 3z^3x^3\mathbf{k})$$

$$= 2xy - 4yz^2 + 9z^2x^3$$

(5) $\mathbf{f} = x^2\mathbf{i} - 2xz\mathbf{j} + y^2z\mathbf{k}$

$$\operatorname{rot}\mathbf{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}$$

$$= \left(2yz - (-2x)\right)\mathbf{i} + \left(0-0\right)\mathbf{j} + \left((-2z) - 0\right)\mathbf{k}$$

$$= 2(yz+x)\mathbf{i} - 2z\mathbf{k}$$

Conclusion

- Learn derivation of vectors
  - Definition of derivation
  - Gradient
  - Divergence
  - Rotation

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