Fundamental Mathematics (Engineering Mathematics)

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Course schedule

- Guidance + Differential equations (#1,2)
- Differential equations and physics (#3)
- Array and vector (#4, 5)
- Vector analysis (#6, 7)
- Complex function theory (#8, 9)
- Fourier transform (#10, 11)
- Laplace transform (#12, 13)
- Final examination and explanation(#14)

- Score: Exam (70%) + Report (20%) + Attendance (10%)

Fundamental Mathematics

- Fourier series/transform 2-

Fourie integral (フーリエ積分)

- <u>Fourie series</u>: express (1) periodic function or (2) function defined within finite range $[-L, L]$, w/ sum of sine functions
  - For periodic function $f(x)$ w/ period $0 \sim 2L$
    - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$ (a)
    - where $a_n = \frac{1}{\pi}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} dx$, $b_n = \frac{1}{\pi}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L} dx$ (b)
- <u>Fourie integral</u>: extension of (2) to infinite range $(-\infty, \infty)$
  - $f(x) = \int_0^{\infty}\{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\}\, d\alpha$
    - where $A_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\, du$, $B_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\, du$

Introduction of Fourie integral

- Introduce Fourie integral from Fourie series
  - Substitute (b) to (a)
    - $f(x) = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}(\cos\frac{n\pi x}{L}\frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx + \sin\frac{n\pi x}{L}\frac{1}{\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx)$
  - If integral of $f(x)$ has finite value C: $\int_{-\infty}^{\infty} f(x)\,dx = C$
    - $\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} f(x)\,dx \;\; = 0$
  - Replace variables
    - $\alpha_n = \frac{n\pi}{L}$, $\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{L}$, $\lim_{L\to\infty}\Delta\alpha = 0$
    - $f(x) = \lim_{\Delta\alpha\to\infty}\frac{\Delta\alpha}{\pi}\sum_{n=1}^{\infty}\left[\cos a_n x\int_{-\infty}^{\infty} f(u)\cos\alpha u\,du + \sin a_n x\int_{-\infty}^{\infty} f(u)\sin\alpha u\,du\right]$
  - Replaced by $A_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,du$, $B_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,du$

Introduction of Fourie integral (cont.)

- $f(x) = \lim_{\Delta\alpha\to\infty} \Delta\alpha \sum_{n=1}^{\infty}[\cos a_n x\, A(a_n) + \sin a_n x\, B(a_n)]$
  - $= \lim_{\Delta\alpha\to\infty} [\sum_{n=1}^{\infty}[\Delta\alpha \cos a_n x\, A(a_n)\,] + \sum_{n=1}^{\infty}[\Delta\alpha \sin a_n x\, B(a_n)]]$
  - This is a Riemann sum, thus re-write and obtain the Fourie integral is as follows
- $f(x) = \int_0^{\infty} A(\alpha) \cos \alpha x\, d\alpha + \int_0^{\infty} B(\alpha) \sin \alpha x\, d\alpha$ (c)
  - where $A_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \cos \alpha u\, du\,,\, B_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \sin \alpha u\, du$ (d)
- $f(x)$ can be simplified substituting (d) to (c)
  - $f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)[\cos \alpha u \cos \alpha x + \sin \alpha u \sin \alpha x]\, du d\alpha$
    - $= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u) \cos \alpha(x-u)\, du d\alpha$ (e)
    - This is also Fourie integral

Fourie integral in exponent

- Use Euler's theorem ($\cos\theta = (e^{i\theta} + e^{-i\theta})/2$) to introduce Fourie integral in exponent function. Recall eq. (e)
  - $f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(u)\cos\alpha(x-u)\,dud\alpha$
    - $= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \frac{f(u)\left(e^{i\alpha(x-u)}+e^{-i\alpha(x-u)}\right)}{2}\,dud\alpha$
    - $= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,dud\alpha + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{-i\alpha(x-u)}\,dud\alpha$
- Replace α to $-\alpha$ for second term; $d\alpha \to -da$, $\infty \to -\infty$
  - $f(x) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,dud\alpha - \frac{1}{2\pi}\int_0^{-\infty} \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,dud\alpha$
    - $= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,dud\alpha + \frac{1}{2\pi}\int_{-\infty}^0 \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,dud\alpha$
    - $= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(x-u)}\,dud\alpha$ (f) ⇦ Fourie integral (exponent)

Fourie integral in exponent (cont.)

- Fourie integral has another form (this form is widely recognized). From (f)
  - $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i\alpha(x-u)}\, du d\alpha$
    - $= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i\alpha x} e^{-i\alpha u}\, du d\alpha$
    - $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\alpha u}\, du \right] e^{i\alpha x} d\alpha$

Fourie transform (フーリエ変換)

- Fourie transform can be obtained by replacing variables $u \to t,\ x \to t,\ \alpha \to \omega,$
  - $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$
  - $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$ (g)
- Equation shows conversion of time-domain function $f(t)$ to frequency-domain $(F(\omega))$

Fourie integral for odd/even function

- If the function $f(x)$ is odd, (c) only has $\cos \alpha x$ component
- If the function $f(x)$ is even, (c) only has $\sin \alpha x$ component
- Fourie integral is simplified as follows
  - $f(x)$ is even: called cosine-transform
    - $f(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} C(\alpha) \cos \alpha x \, d\alpha, \quad C(\alpha) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(u) \cos \alpha u \, du$
  - $f(x)$ is odd: called sine-transform
    - $f(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} S(\alpha) \sin \alpha x \, d\alpha, \quad S(\alpha) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(u) \sin \alpha u \, du$

Characteristics of Fourie integral

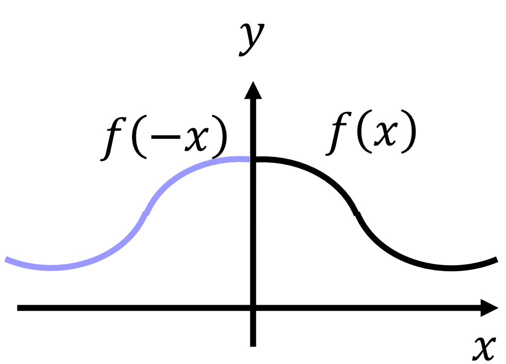
- Fourie integral need some condition to have a limit

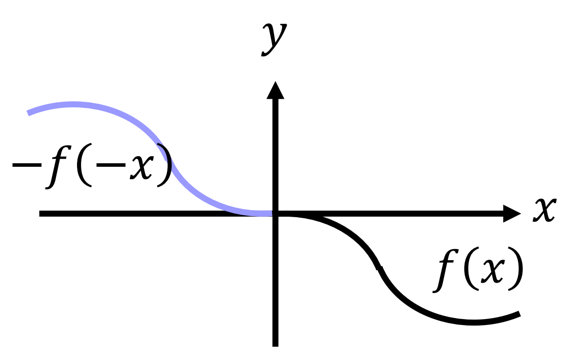
  If integral of $f(x)$ has finite value C: $\int_{-\infty}^{\infty} f(x)\,dx = C$

  - Same conditions are required as Fourie series
- Theorem 4: Assume $f(x)$ is defined in $(-\infty, \infty)$, $f(x)$ and $f'(x)$ are piecewise smooth, $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite.
  - If $f(x)$ is continuous on x,
    - $f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du d\alpha$
  - If $f(x)$ is not continuous on x,
    - $f(x) = \frac{f(x+0)+f(x-0)}{2} = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du d\alpha$

Characteristics of Fourie integral (cont)

- Assume $f(x)$ is defined in $[0, \infty)$.
  - Use $f(x) = f(-x)$ to expand its range to $(-\infty, \infty)$.
    - New $f(x)$ is even-function
    - "Cosine translation of original $f(x)$ "
  - Use $f(x) = -f(-x)$ to expand its range to $(-\infty, \infty)$.
    - New $f(x)$ is odd-function
    - "Sine translation of original $f(x)$ "





Application of Fourie transform

- Fourie transform has wide applications
  - Try to apply electric circuit analysis

- Introduce Fourie transform for derivatives
  - Take partial difference for Fourie transform (g)
    - $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df(t)}{dt}e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\left[f(t)e^{-i\omega t}\right]_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\,dt$
  - Our target is nature, thus $\lim_{\mathrm{t}\to-\infty} f(t) = \lim_{\mathrm{t}\to\infty} f(t) = 0$ (dump)
    - $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df(t)}{dt}e^{-i\omega t}\,dt = \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\,dt = i\omega F(\omega)$
  - Fourie transform for derivatives : multiply $i\omega$ to its original Fourie transform
    - Also this is true for higher order of derivatives

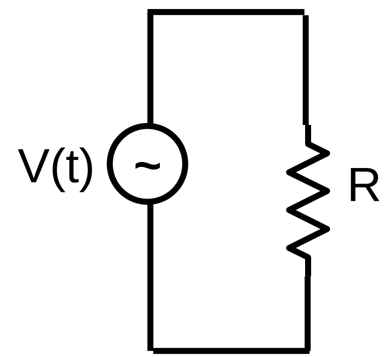
Application for circuit analysis

- Analyze frequency dependency of resistance (R), inductance (L), capacitance (C)

- Extract impedance $Z(\omega)$ on frequency domain
  - $Z(\omega) = V(\omega)/I(\omega)$
  - $V(\omega)$: voltage on frequency domain
    - $V(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(t) e^{-i\omega t}\, dt$
  - $I(\omega)$: current on frequency domain
    - $I(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t) e^{-i\omega t}\, dt$
  - ω: angular frequency

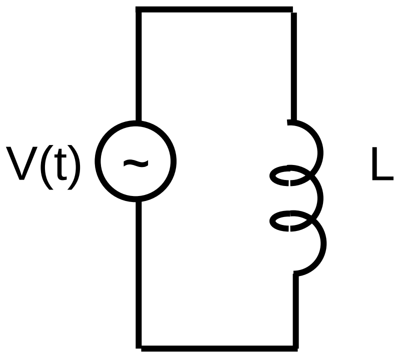
Resistance analysis

- From the Kirchhoff's voltage Law
  - $-V(t) + RI(t) = 0$
- Multiply $\frac{1}{\sqrt{2\pi}}e^{-i\omega t}$ and take integral at $[-\infty, \infty]$
  - $-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V(t)e^{-i\omega t}\,dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} RI(t)e^{-i\omega t}\,dt$
  - $V(\omega) = RI(\omega)$
  - $Z(\omega) = \frac{V(\omega)}{I(\omega)} = R$
  - No frequency dependence



Inductance analysis

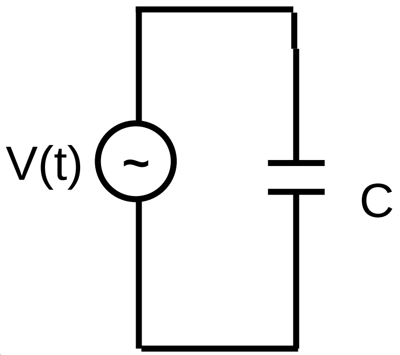
- Inductance create electromotive force $V_L(t)$ when the current flow changes $(I(t+\Delta t) - I(t))$
  - Its amplitude is called inductance L
    - $V_L(t) = L \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t} = L \frac{dI(t)}{dt}$
  - From the Kirchhoff's voltage Law
    - $-V(t) + V_L(t) = 0$
    - $-V(t) + L \frac{dI(t)}{dt} = 0$



- Multiply $\frac{1}{\sqrt{2\pi}} e^{-i\omega t}$ and take integral at $[-\infty, \infty]$
  - $V(\omega) = i\omega L I(\omega) => Z(\omega) = \frac{V(\omega)}{I(\omega)} = i\omega L$
  - Frequency dependence
    - Impedance increases as freq. (ω) increase

Capacitance analysis

- Capacitance create voltage $V_C(t)$ as the electrons $Q(t)$ are charged. Its amplitude is called Capacitance C
  - $V_C(t) = \frac{Q(t)}{C}$
- From the Kirchhoff's voltage Law
  - $-V(t) + V_C(t) = 0$ -> $-V(t) + \frac{Q(t)}{C} = 0$
- Take differential
  - $-\frac{dV(t)}{dt} + \frac{1}{C}\frac{dQ(t)}{dt} = 0$ -> $-\frac{dV(t)}{dt} + \frac{1}{C}I(t) = 0$
  - $\frac{dQ(t)}{dt} = I(t)$ and/or $\int I(t)dt = Q(t)$



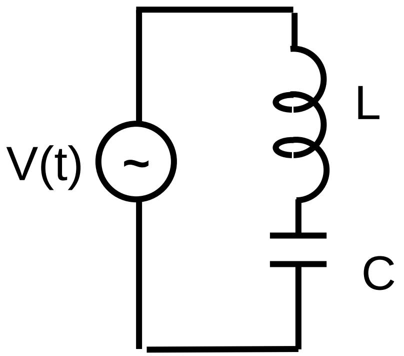
Capacitance analysis (cont.)

- Multiply $\frac{1}{\sqrt{2\pi}}e^{-i\omega t}$ and take integral at $[-\infty, \infty]$
  - $i\omega V(\omega) = \frac{1}{C}I(\omega) => Z(\omega) = \frac{V(\omega)}{I(\omega)} = \frac{1}{i\omega C}$
  - Frequency dependence
    - Impedance decreases as freq. (ω) increase

- Similar to Fourie transform, we introduce Laplace transform to solve differential equations
  - Fourie transform: transform to time- to freq-domain
  - Laplace transform: transform to time- to s-domain

Exercise

- Calculate Fourie transform
  - $f(x) = \begin{cases} 1 & (-a \leq x \leq a) \\ 0 & (x < -a, a < x) \end{cases}$
  - $f(x) = \begin{cases} 1 - x^2 & (|x| \leq 1) \\ 0 & (|x| > 1) \end{cases}$
  - $f(x) = e^{-a|x|}, (a > 0)$
- Calculate impedance of LC series circuit at AC supply $V(t)$
  - Capacitance: C
  - Inductance: L
  - Charge: $Q(t)$
  - Current: $I(t)$



Sample solution

Math 12

Calculate Fourie transform $F(x)$

① $f(x) = \begin{cases} 1 & (-a \le x \le a) \\ 0 & (x < -a,\ a < x) \end{cases}$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\alpha u} du$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha} \left[e^{-i\alpha u} \right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{2}{\alpha} \frac{e^{i\alpha a} - e^{-i\alpha a}}{2i} = \sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha}$$

$$f(x) = e^{-a|x|} \quad (a > 0)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$\sqrt{\frac{2}{\pi}}$

$$= \int_{-\infty}^{0} e^{-a(-x)} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-ax} e^{-i\omega x} dx$$

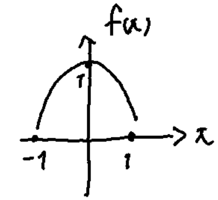
$$= \int_{-\infty}^{0} e^{(a-i\omega)x} dx + \int_{0}^{\infty} e^{-(a+i\omega)} dx$$

$$= \left[\frac{e^{(a-i\omega)x}}{a-i\omega} \right]_{-\infty}^{0} + \left[\frac{-e^{-(a+i\omega)x}}{a+i\omega} \right]_{0}^{\infty}$$

$$= \left(\frac{1}{a-i\omega} - 0 - 0 + \frac{1}{a+i\omega} \right) = \frac{2a}{a^2 - \omega^2} \sqrt{\frac{2}{\pi}}$$

2 $\quad \sqrt{\frac{2}{\pi}} \times$

$$f(x) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$



even func

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(u) \cos \alpha u \, du$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\sin \alpha u}{\alpha u} \right]_{-1}^{1} - \int_{-1}^{1} u^2 \cos \alpha u \, du \right\}$$

$$\int x^2 \cos \alpha x \, dx = x^2 \frac{\sin \alpha x}{\alpha} - \int 2x \frac{\sin \alpha x}{\alpha} dx$$

$$= \text{〃} - \left[2x \frac{-\cos \alpha x}{\alpha^2} - \int 2 \frac{-\cos \alpha x}{\alpha^2} dx \right]$$

$$= \frac{x^2 \sin \alpha x}{\alpha} + \frac{2x \cos \alpha x}{\alpha^2} - \frac{2 \sin \alpha x}{\alpha^3} + C$$

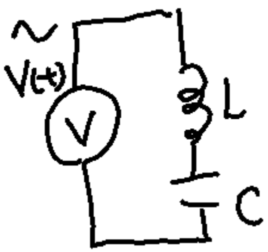
$$F(\alpha) = \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin a}{\alpha} - \frac{\sin(-a)}{-\alpha} - \left\{ \frac{\sin \alpha}{\alpha} - \frac{\sin(-\alpha)}{-\alpha} \right.\right.$$

$$\left.\left. + \frac{2\cos \alpha}{\alpha^2} - \frac{-2\cos(-\alpha)}{\alpha^2} - \left(\frac{2 \sin \alpha}{\alpha^3} - \frac{2 \sin(-\alpha)}{\alpha^3} \right) \right\} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{4 \cos \alpha}{\alpha^2} + \frac{4 \sin \alpha}{\alpha^3} \right)$$

Sample solution

LC series



$$-V(t)+\frac{Q(t)}{C}+L\frac{dI(t)}{dt}=0$$

differentiate

$$-\frac{dV(t)}{dt}+\frac{I(t)}{C}+L\frac{d^2I(t)}{dt^2}=0$$

Fourie transform

$$-i\omega V(\omega)+\frac{I(\omega)}{C}-\omega^2 L I(\omega)=0$$

$$Z(\omega)=\frac{V(\omega)}{I(\omega)}=\frac{1}{i\omega C}+i\omega L$$

$+$

Report

- In engineering, some mathematic methods are used to analyze and model the natural behavior and/or systems.
  - Find one example of application which uses mathematic methods, and explain how these mathematic methods are used for the application.
  - Length: no limit
  - Due: 2024/02/02 (Fri.)

Exam:
60min. You can use your note (printed materials) and calculator. Smartphone, Tablet, PC is not allowed.

Fundamental Mathematics

- Laplace transform 1-

Laplace transform (ラプラス変換)

- Assume function $f(x)$ within infinite range $(0, \infty)$. If its integral $F(s)$ is finite determinate for complex value s, this is called Laplace transform

- $F(s) = \int_0^\infty f(x) e^{-st} dt = \lim_{\substack{T \to \infty \\ \epsilon \to +0}} \int_\epsilon^T f(x) e^{-st} dt = \mathcal{L} f(t)$
  - $f(t)$: time domain function
  - $F(s)$: complex domain (s-domain) function
  - $\mathcal{L}$: Laplace-operator(ラプラス演算子), or Laplacian
  - Useful to calculate differential equation

Laplace transform definition

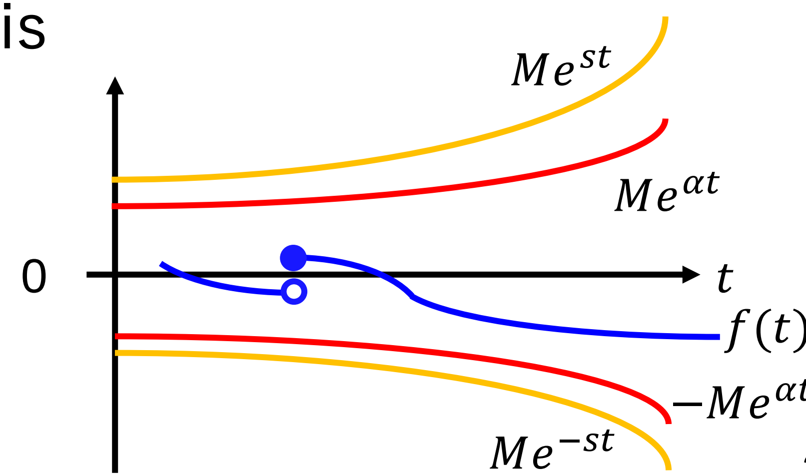
- Preliminaries:
  - If $\mathrm{Re}(s) > 0$
    - $\lim_{t \to \infty} t^n e^{-st} = 0 \ (n \in \mathbb{N})$
  - Proof:
    - Assume $s = a + bi$ and $t > 0$, since $\mathrm{Re}(s) = a > 0$
    - $|t^n e^{-st}| = t^n e^{-at}$
    - $\lim_{t \to \infty} |t^n e^{-st}| = \lim_{t \to \infty} \frac{t^n}{e^{at}} = \lim_{t \to \infty} \frac{nt^{n-1}}{ae^{at}} = \cdots = \lim_{t \to \infty} \frac{n!}{a^n e^{at}} = 0$

Examples (condition: $\mathrm{Re}[s] > 0$)

- $\mathcal{L}1 = \frac{1}{s}$
  - $\mathcal{L}1 = \int_0^\infty 1 \cdot e^{-st} dt = \left[-\frac{1}{s} e^{-st}\right]_0^\infty = -\frac{1}{s} \lim_{t\to\infty} e^{-st} + \frac{1}{s} = \frac{1}{s}$
- $\mathcal{L}t = \frac{1}{s^2}$
  - $\mathcal{L}t = \int_0^\infty t \cdot e^{-st} dt = \left[-\frac{t}{s} e^{-st}\right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt$
  - $= -\frac{1}{s} \lim_{t\to\infty} te^{-st} + \frac{1}{s} \mathcal{L}1 = \frac{1}{s^2}$
- $\mathcal{L}t^n = \frac{n!}{s^{n+1}}$: use inductive method, $n = 1$ satisfy, assume $n = k$ satisfy $\mathcal{L}t^k = \frac{k!}{s^{k+1}}$
  - $\mathcal{L}t^{k+1} = \int_0^\infty t^{k+1} \cdot e^{-st} dt = \left[-\frac{t^{k+1}}{s} e^{-st}\right]_0^\infty + \frac{k+1}{s} \int_0^\infty t^k e^{-st} dt$
  - $= -\frac{1}{s} \lim_{t\to\infty} t^{k+1} e^{-st} + \frac{k+1}{s} \mathcal{L}t^k = \frac{k+1}{s} \cdot \frac{k!}{s^{k+1}} = \frac{(k+1)!}{s^{k+2}}$

Convergence of Laplace transform

- Discuss the requirement of $f(x)$ to achieve $\int_0^\infty f(x)e^{-st}dt = F(s)$ ($\cancel{F(s)}$ ~~is const~~., not diverge)
- Assume function $f(x)$ is piecewise smooth within infinite range $(0, \infty)$. If $f(x)$ satisfy following conditions, $f(x)$ is called "exponential w/ index number α"
  - $|f(t)| < Me^{\alpha t}$ $(M > 0) \leftrightarrow -Me^{\alpha t} < f(t) < Me^{\alpha t}$
  - Speed of diverge of $|f(x)|$ is slower than $Me^{\alpha t}$
- $\int_0^\infty f(x)e^{-st}dt$ $\left(\mathrm{Re}(s) > \mathrm{Re}(\alpha)\right)$ is the condition to converge



Convergence theorem 1

- Theorem 1: Assume function $f(x)$ is piecewise smooth within infinite range $(0,\infty)$, and "exponential w/ index number a". Laplace transform $F(s)$ is available for any complex number s which satisfy $\mathrm{Re}(s) > \gamma$
- Proof: Assume $0 < T < T'$, $\mathrm{Re}(s) = \alpha$.
  - $\int_0^{T'} f(x)e^{-st}dt = \int_T^{T'} f(x)e^{-st}dt + \int_0^{T} f(x)e^{-st}dt$
    - $\int_0^{T} f(x)e^{-st}dt < Me^{\alpha t}$
    - $\left|\int_T^{T'} f(x)e^{-st}dt\right| \leq M\int_T^{T'} e^{-(a-\gamma)t}dt = \frac{M(e^{-(a-\gamma)T} - e^{-(a-\gamma)T'})}{a-\gamma}$
      - $T' \to \infty$, $\mathrm{Re}(s) = \alpha > \gamma$ means $\alpha - \gamma > 0$, then
    - $\left|\int_T^{T'} f(x)e^{-st}dt\right| \leq \frac{Me^{-(a-\gamma)T}}{a-\gamma}$
- Both term are less than exponential (Bounded)

Convergence theorem 2

- Theorem 2: Assume function $f(x)$ is piecewise smooth within infinite range $(0,\infty)$, and "exponential w/ index number γ". If Laplace transform $F(s_0)$ is available, Laplace transform $F(s)$ is also available for $\mathrm{Re}[s] > \mathrm{Re}[s_0]$.
- Proof: Assume $g(t) = \int_0^t f(u)e^{-s_0 u}du$, this is Laplace transform and bounded. $\mathcal{L}g(s)$ is available for any complex s
  - $\int_0^T f(x)e^{-st}dt = \int_0^T e^{-(s-s_0)t}e^{-s_0 t}f(t)dt$
    - $= e^{-(s-s_0)T}g(T) + (s-s_0)\int_0^T e^{-(s-s_0)t}g(x)\,dt$
  - Assume $T \to \infty$, $e^{-(s-s_0)T}g(T) \to 0$, $(s-s_0)\int_0^T e^{-(s-s_0)t}g(x)\,dt \to (s-s_0)\mathcal{L}g(s)$
- Thus, $\int_0^{\infty} f(x)e^{-st}dt$ has some limit value $F(s)$

Convergence coordinate, range

- Theorem 1,2, if $f(x)$ satisfy condition in theorem 1, and its Laplace transform $F(s_0)$ is available, Laplace transform $F(s)$ is available for $\mathrm{Re}[s] > \mathrm{Re}[s_0]$.
  - Guess boundary α which (1) satisfy $F(s) = \mathcal{L}f(t)$ for $\mathrm{Re}[s] > \alpha$ and (2) α is the lowest boundary of α (α is real value)
    - Convergence range: $\mathrm{Re}[s] > \alpha$
    - Convergence coordinate: α (Outside of convergence range or $\mathcal{L}f(t)$ is not available)

Characteristics of Laplace transform

- Laplace transform $F(s) = \int_0^{\infty} f(x)e^{-st}dt = \mathcal{L}f(t)$ has following characteristics
  1. $\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}f(t) + b\mathcal{L}g(t)$
  2. $\mathcal{L}f(at) = \frac{1}{a}F\left(\frac{s}{a}\right)$
  3. $\mathcal{L}\{e^{at}f(t)\} = F(s - a)$

  where $a > 0$

Characteristics of Laplace transform (cont)

- Characteristics of integrals, differentials
  - If $f(t)$ is continuous,
    1. $\mathcal{L}f'(t) = sF(s) - f(+0)$
  - If $f(t)$, $f'(t)$, $\cdots f^{(n-1)}(t)$ are continuous,
    2. $\mathcal{L}f^{(n)}(t) = s^n F(s) - f(+0)s^{n-1} - f'(+0)s^{n-2} \ldots - f^{(n)}(+0)$
  3. $\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$
  4. $\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
  5. $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_0^\infty F(s)ds$

Composite product (合成積)

- Assume $f(x)$ and $g(x)$ are defined within infinite range $(0, \infty)$, if $h(t)$ is available, $h(t) = f(t) * g(t)$ is composite product
  - $h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$
- Composite product support commutation relations (交換則)
  - $f(t) * g(t) = g(t) * f(t)$
- Can calculate Laplace transform by multiply of individual Laplace transform
  - $\mathcal{L}h(t) = \mathcal{L}(f(t) * g(t)) = \mathcal{L}\big(f(t)\big)\mathcal{L}\big(g(t)\big)$

Inverse Laplace transform

- Theorem 3: Assume function $f(x)$, $g(x)$ are piecewise smooth within infinite range $(0, \infty)$, and exponential. If $\mathcal{L}f(t) = \mathcal{L}g(t)$, $f(t) = g(t)$
  - If Laplace transform of $\mathcal{L}f(t)$ is known, $f(t)$ can be calculated
  - Inverse Laplace transform

Table: cheat sheet of Laplace transform

$F(s) = \mathcal{L}f(t)$	$f(t) = \mathcal{L}^{-1}F(\boldsymbol{s})$
$1/s$	1
$1/s^n$	$t^{n-1}/(n-1)!$
$1/(s-a)$	e^{at}
$\omega/(s^2+\omega^2)$	$\sin \omega t$
$s/(s^2+\omega^2)$	$\cos \omega t$
$\omega/((s-a)^2+\omega^2)$	$e^{at} \sin \omega t$
$(s-a)/((s-a)^2+\omega^2)$	$e^{at} \cos \omega t$

Exercise

- Proof following relationships
  - $\mathcal{L}e^{at} = 1/(s-a)$ (where $\mathrm{Re}[s] > a$)
  - $\mathcal{L}\sin\omega t = \omega/(s^2+\omega^2)$ (where $\mathrm{Re}[s] > 0$)
  - $\mathcal{L}\cos\omega t = s/(s^2+\omega^2)$ (where $\mathrm{Re}[s] > 0$)
- Calculate following Laplace transform
  - $*$: composite product
  - $f(t) = (t^2) * (te^{-t})$
  - $f(t) = (e^{at}\sin\omega t) * (e^{at}\cos\omega t)$

Solutions

Proof followings

(1) $\mathcal{L} e^{at} = \frac{1}{s-a}$ $(Re[s]>a)$

$$\mathcal{L} e^{at} = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt = \left[-\frac{e^{-(s-a)t}}{s-a} \right]_0^\infty$$

$$= \underbrace{-\frac{1}{s-a} \lim_{t\to\infty} e^{-(s-a)t}}_{\to 0} + \frac{1}{s-a} = \frac{1}{s-a}$$

(2) $\mathcal{L} \sin\omega t = \frac{\omega}{s^2+\omega^2}$, $\mathcal{L}\cos\omega t = \frac{s}{s^2+\omega^2}$ $(Re[s]>0)$

$$\mathcal{L}\sin\omega t = \int_0^\infty \sin\omega t\, e^{-st} dt = \left[-\frac{e^{-st}}{s} \sin\omega t \right]_0^\infty + \frac{\omega}{s}\int_0^\infty \cos\omega t\, e^{-st} dt$$

$$= \underbrace{-\frac{1}{s} \lim_{t\to\infty} e^{-st} \sin\omega t}_{\to 0} + \frac{\omega}{s} \mathcal{L}\cos\omega t$$

$$\therefore \mathcal{L}\sin\omega t = \frac{\omega}{s}\mathcal{L}\cos\omega t$$

$$\mathcal{L}\cos\omega t = \left[-\frac{e^{-st}}{s}\cos\omega t \right]_0^\infty - \frac{\omega}{s}\int_0^\infty \sin\omega t \cdot e^{-st} dt$$

$$= \underbrace{-\frac{1}{s}\lim_{t\to\infty} e^{-st}\cos\omega t}_{\to 0} + \frac{1}{s} - \frac{\omega}{s}\mathcal{L}\sin\omega t$$

$$\therefore \mathcal{L}\cos\omega t = \frac{1}{s} - \frac{\omega}{s}\mathcal{L}\sin\omega t$$

$$\frac{s}{\omega}\mathcal{L}\sin\omega t = \frac{1}{s} - \frac{\omega}{s}\mathcal{L}\sin\omega t$$

$$\left(\frac{s}{\omega} + \frac{\omega}{s}\right)\mathcal{L}\sin\omega t = \frac{1}{s} \qquad \mathcal{L}\sin\omega t = \frac{\omega}{s^2+\omega^2}, \ \mathcal{L}\cos\omega t = \frac{s}{s^2+\omega^2}$$

Calculate following Laplace transform

(1) $f(t) = (t^2) * (te^{-t})$

$$\mathcal{L} t^2 = \frac{2}{s^3}, \quad \mathcal{L}(te^{-t}) = -\left[\frac{1}{s+1}\right]' = \frac{1}{(s+1)^2}$$

$$\mathcal{L} f(t) = \frac{2}{s^3(s+1)}$$

(2) $f(t) = (e^{at}\sin\omega t) * (e^{at}\cos\omega t)$

$$\mathcal{L}(e^{at}\sin\omega t) = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}(e^{at}\cos\omega t) = \frac{s-a}{(s-a)^2+\omega^2}$$

from cheat sheet

$$\mathcal{L} f(t) = \frac{\omega(s-a)}{\{(s-a)^2+\omega^2\}}$$

Conclusion

- Use Laplace transform to solve practical differential equation
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