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Spherical Spline Implementation

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1 Introduction

Spherical spline interpolation has emerged as a critical tool in addressing approximation problems on the sphere, particularly in fields such as geophysics, physical geodesy, and environmental sciences. The increasing availability of high-resolution, spatially distributed data, such as those generated by satellite-based techniques, has necessitated the development of robust mathematical methods to approximate functions defined on spherical domains. Such functions often represent physical quantities like temperature, pressure, ozone concentration, gravitational and magnetic forces, or elastic deformation. These quantities are typically sampled at discrete, irregularly spaced points on the Earth's surface, requiring interpolation methods that can accurately reconstruct them over the entire spherical domain.

Traditional Euclidean methods for localized approximation are often inadequate for global problems on the sphere, such as gravity field determination. The fundamental challenge lies in the lack of a differential mapping that can transform the entire sphere into a bounded planar region without distortion. This limitation has driven the development of approximation methods specifically tailored to spherical domains. Among these, spherical spline interpolation stands out for its ability to model global phenomena effectively while maintaining desirable mathematical properties, such as smoothness and stability.

This report is focused on spherical spline interpolation for gravity potential data, where three distinct kernels (Abel-Poisson, Singularity, and Logarithmic) are employed for interpolating data from a global 6-minute regular grid to a finer 3-minute grid.

2 Implementation Steps

The answer for spherical spline is linear combination of the kernel function, as shown in the equation 1. Where y is a desired point, x_i are the input points, a_i are the coefficients for spherical spline interpolation, and N is the number of input points.

$$S(y) = \sum_{i=1}^{N} a_i K(x_i, y) \tag{1}$$

The coefficients can be calculated from the linear system of equations shown in equation 2. The matrix equation is shown in equation 3. Where both x_i and y_i represent input points and $F(y_j)$ are the values at input points. Thus the coefficients can be calculated simply by inverting the design matrix (A).

$$F(y_j) = \sum_{i=1}^{N} a_i K(x_i, y_j) \quad , \quad j = 1, 2, \ldots, N \tag{2}$$

$$\begin{bmatrix} F(y_1) \\ F(y_2) \\ \vdots \\ F(y_N) \end{bmatrix} = \begin{bmatrix} K(x_1, y_1) & K(x_2, y_1) & \ldots & K(x_N, y_1) \\ K(x_1, y_2) & K(x_2, y_2) & \ldots & K(x_N, y_2) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_1, y_N) & K(x_2, y_N) & \ldots & K(x_N, y_N) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \equiv l = Ax \tag{3}$$

$$\hat{x} = A^{-1} l \tag{4}$$

After the coefficients are calculated, the value at any desired point can be calculated from the equation 1, or the matrix equation of 5 for U Points.

$$\begin{bmatrix} S(y_1) \\ S(y_2) \\ \vdots \\ S(y_U) \end{bmatrix} = \begin{bmatrix} K(x_1, y_1) & K(x_2, y_1) & \dots & K(x_N, y_1) \\ K(x_1, y_2) & K(x_2, y_2) & \dots & K(x_N, y_2) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_2, y_U) & K(x_2, y_U) & \dots & K(x_N, y_U) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \tag{5}$$

As mentioned, three kernel functions of Abel-Poisson, Singularity, and Logarithmic are used in this report. These three functions are defined in the equations 6 to 8. Where x and y Cartesian coordinate vectors of a point on the sphere, and h is the smoothing parameter of the kernel which it's value is in range of $(0, 1)$.

$$K_{Abel-Poisson}(x, y) = \frac{1}{4\pi} \frac{1 - h^2}{(L_h(x, y))^{\frac{3}{2}}} \tag{6}$$

$$K_{Singularity}(x, y) = \frac{1}{2\pi} \frac{1}{(L_h(x, y))^{\frac{1}{2}}} \tag{7}$$

$$K_{Logarithmic}(x, y) = \frac{1}{2\pi h} \ln \left(1 + \frac{2h}{(L_h(x, y))^{\frac{1}{2}} + 1 - h} \right) \tag{8}$$

$$L_h(x, y) = 1 + h^2 - 2h \langle x, y \rangle \tag{9}$$

As mentioned before, data used in this project is gravity potential. This data was acquired from ICGEM [1]. This data is shown in the figure 1.

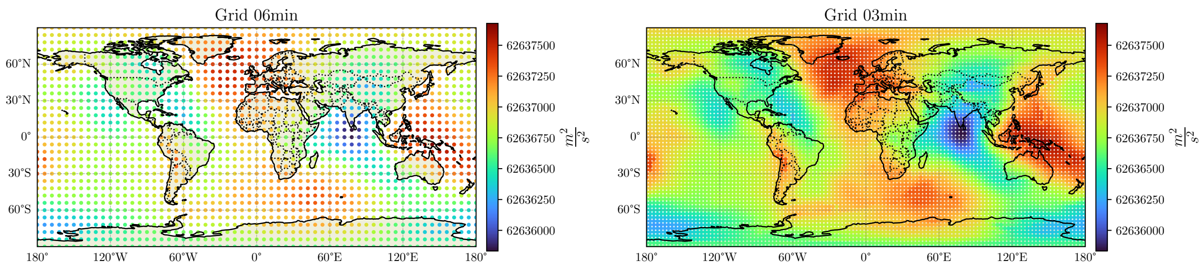

Figure 1: Gravity potential data used for interpolation and validation.

The design matrix A for solving coefficients is ill-conditioned, with the condition number of $7.15 * 10^{49}$. The Picard plot for this matrix is shown in figure 2.

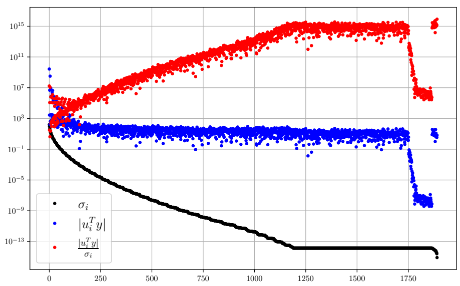

Figure 2: Picard plot of the design matrix.

[1]International Center for Global Earth Models: `https://icgem.gfz-potsdam.de`

Therefore methods of regularization is required for calculating the coefficients, and the solution by simply inverting the design matrix is not a reasonable solution, as shown in figure 3.

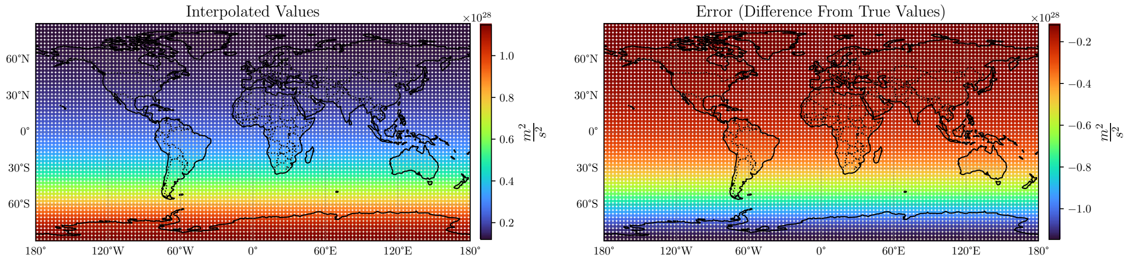

Figure 3: Results without using regularization methods.

Regularization methods used are TSVD and Tikhonov (With regularization parameter calculated from VCE). Cholesky decomposition is also used, which is a method for cases with numerical problems.

Error of interpolation is defined as shown in equation 10, Where l is the true value acquired from XGM2019 model, and $\hat{l}$ is the interpolated value.

$$e = l - \hat{l} \tag{10}$$

3 Results

3.1 Abel-Poisson

Results for Abel-Poisson kernel is shown in the figures 4 to 6. The value of parameter h for this kernel was considered 0.360. Mean and norm (L_2) of these three methods are shown in table 1.

Table 1: Error for interpolation using Abel-Poisson kernel (unit of values are $\frac{m^2}{s^2}$).

Method	Cholesky	TSVD	Tikhonov (VCE)
Mean Error	0.1612	0.1766	-2.4909
Norm of Errors	2014.1234	4175.8890	3931.1049

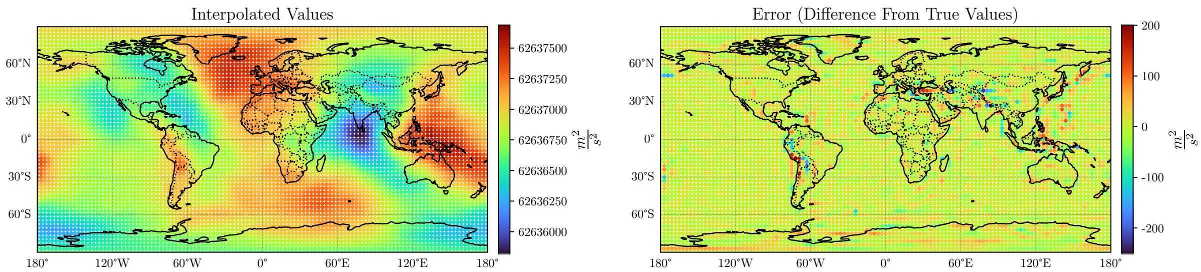

Figure 4: Results of Abel-Poisson kernel with Cholesky decomposition.

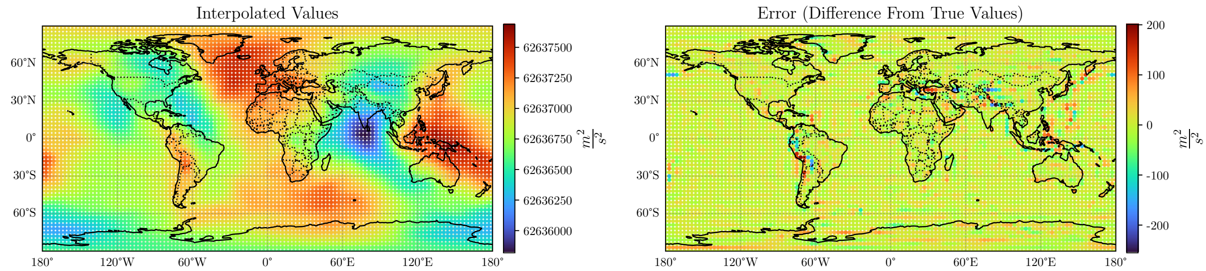

Figure 5: Results of Abel-Poisson kernel with TSVD method.

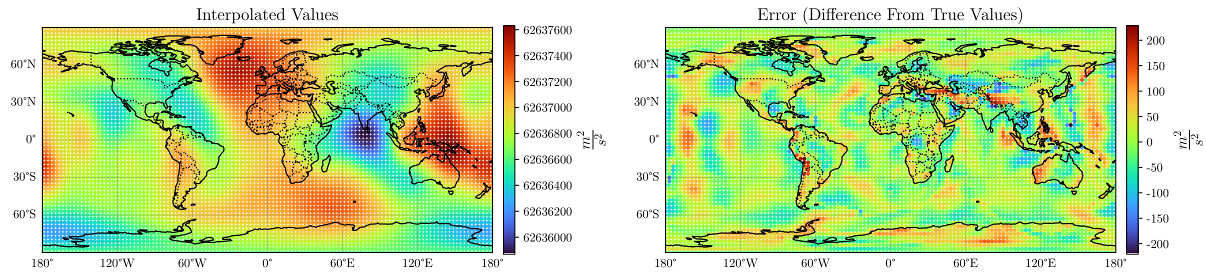

Figure 6: Results of Abel-Poisson kernel with Tikhonov (VCE) method.

3.2 Singularity

Results for Singularity kernel is shown in the figures 7 to 9. The value of parameter h for this kernel was considered 0.405. Mean and norm (L_2) of these three methods are shown in table 2.

Table 2: Error for interpolation using Singularity kernel (unit of values are $\frac{m^2}{s^2}$).

Method	Cholesky	TSVD	Tikhonov (VCE)
Mean Error	0.1814	0.1696	2.0778
Norm of Errors	2014.3146	2016.4105	3931.1049

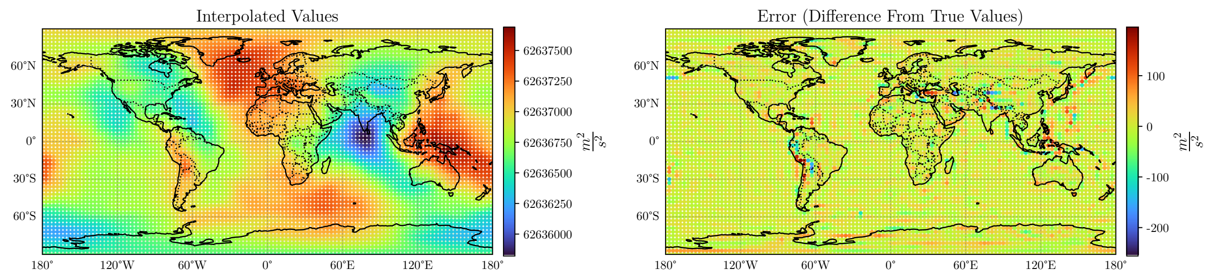

Figure 7: Results of Singularity kernel with Cholesky decomposition.

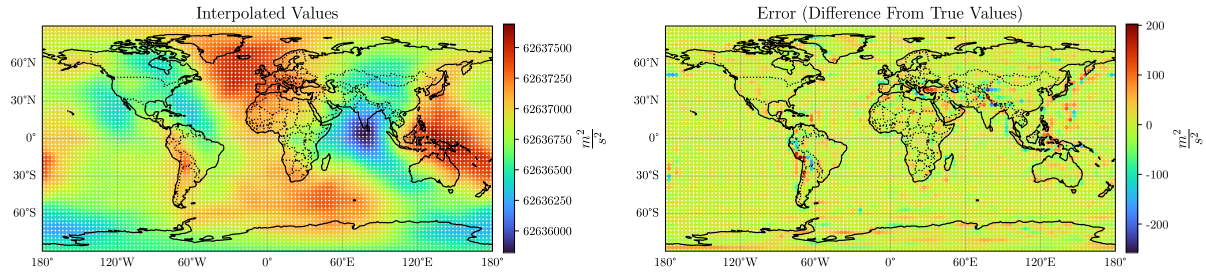

Figure 8: Results of Singularity kernel with TSVD method.

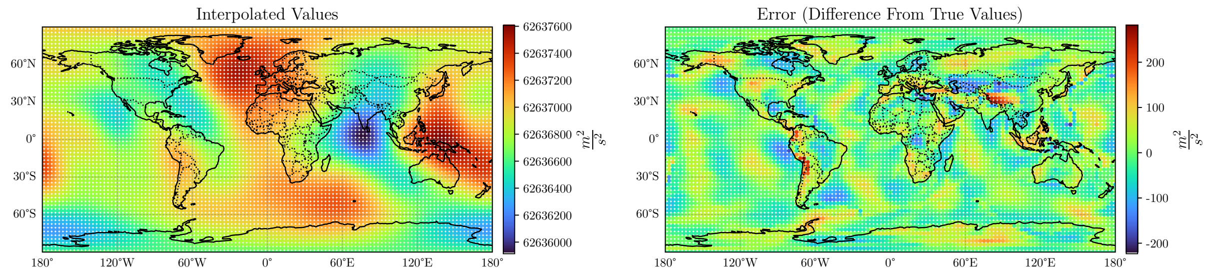

Figure 9: Results of Singularity kernel with Tikhonov (VCE) method.

3.3 Logarithmic

Results for Singularity kernel is shown in the figures 10 to 12. The value of parameter h for this kernel was considered 0.457. Mean and norm (L_2) of these three methods are shown in table 3.

Table 3: Error for interpolation using Logarithmic kernel (unit of values are $\frac{m^2}{s^2}$).

Method	Cholesky	TSVD	Tikhonov (VCE)
Mean Error	0.1503	0.1746	0.4764
Norm of Errors	2005.1889	2013.2417	5436.0035

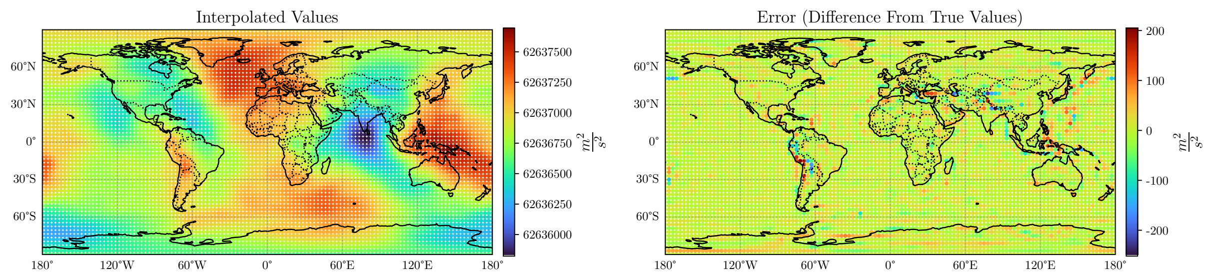

Figure 10: Results of Logarithmic kernel with Cholesky decomposition.

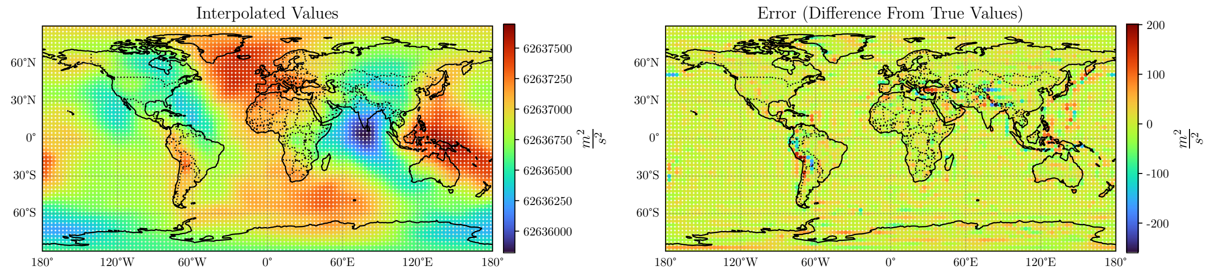

Figure 11: Results of Logarithmic kernel with TSVD method.

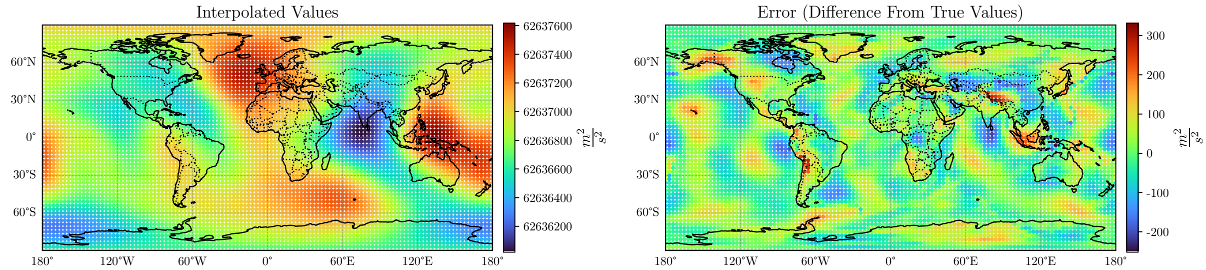

Figure 12: Results of Logarithmic kernel with Tikhonov (VCE) method.

3.4 Abel-Poisson With Noise

In this case, white-noise with standard deviation of $200\frac{m^2}{s^2}$ was added to the input values. Results are shown in the figures 13 to 15.Also, mean and norm (L_2) of the three methods are shown in table 4. As it can be understood from results, by adding white-noise to the data, Cholesky and TSVD can not achieve satisfactory results. But Tikhonov method using VCE can have much realistic and smoother results that other two methods.

Table 4: Error for interpolation using Abel-Poisson kernel and added white-noise (unit of values are $\frac{m^2}{s^2}$).

Method	Cholesky	TSVD	Tikhonov (VCE)
Mean Error	-3.1558	-3.1328	71.1711
Norm of Errors	5548.1711	12137.6909	8200.7749

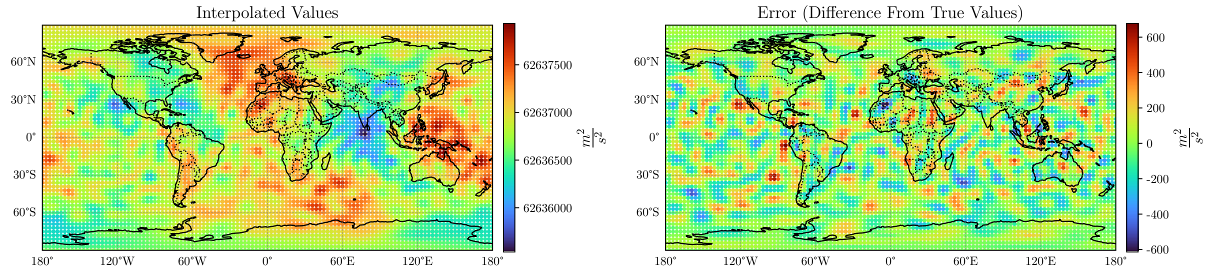

Figure 13: Results of Abel-Poisson kernel with Cholesky decomposition and added white-noise.

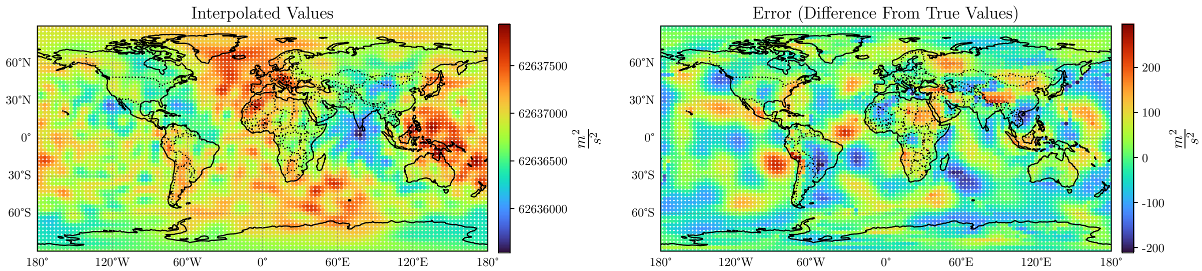

Figure 14: Results of Abel-Poisson kernel with TSVD method and added white-noise.

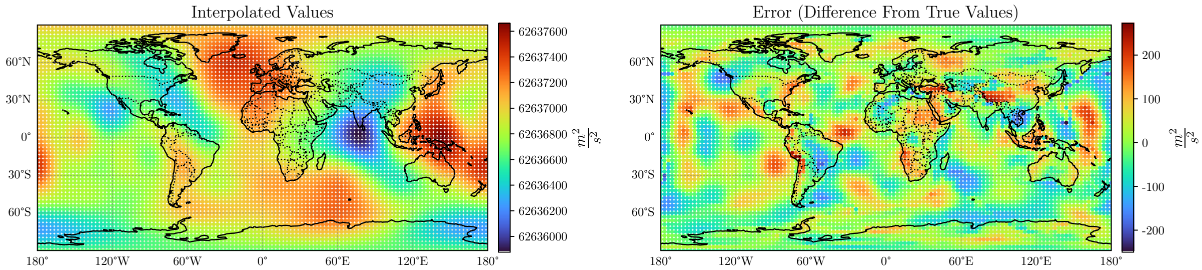

Figure 15: Results of Abel-Poisson kernel with Tikhonov (VCE) method and added white-noise.

3.5 Singularity With Noise

The same amount of white-noise was added to the input values. Results are shown in the figures 16 to 18.Also, mean and norm (L_2) of the three methods are shown in table 5.

Table 5: Error for interpolation using Singularity kernel and added white-noise (unit of values are $\frac{m^2}{s^2}$).

Method	Cholesky	TSVD	Tikhonov (VCE)
Mean Error	-8.9409	-8.1949	-7.2813
Norm of Errors	12540.6580	5736.3606	5770.8221

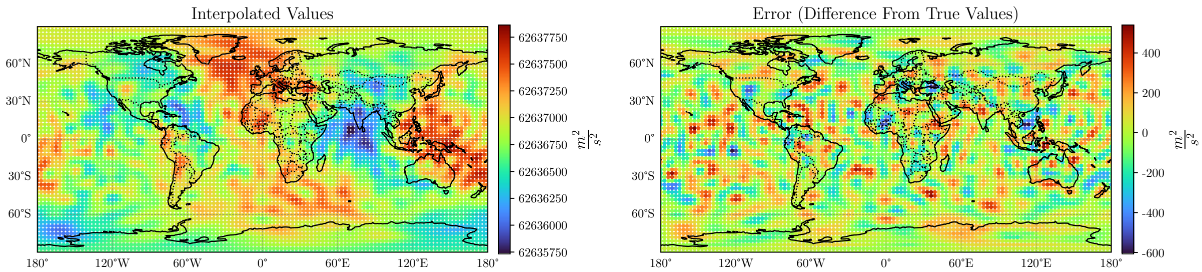

Figure 16: Results of Singularity kernel with Cholesky decomposition and added white-noise.

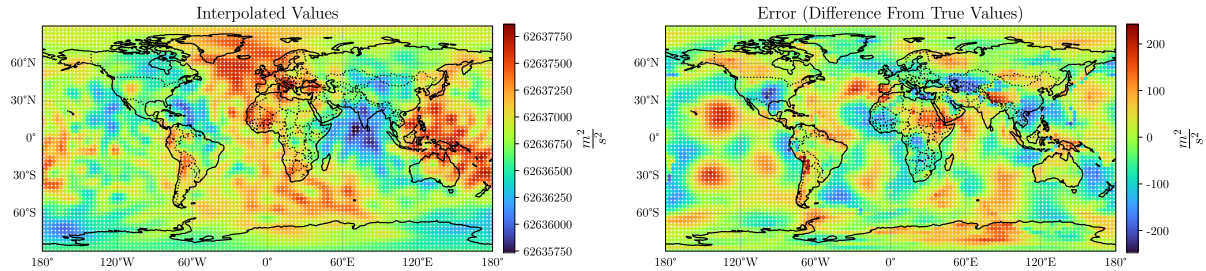

Figure 17: Results of Singularity kernel with TSVD method and added white-noise.

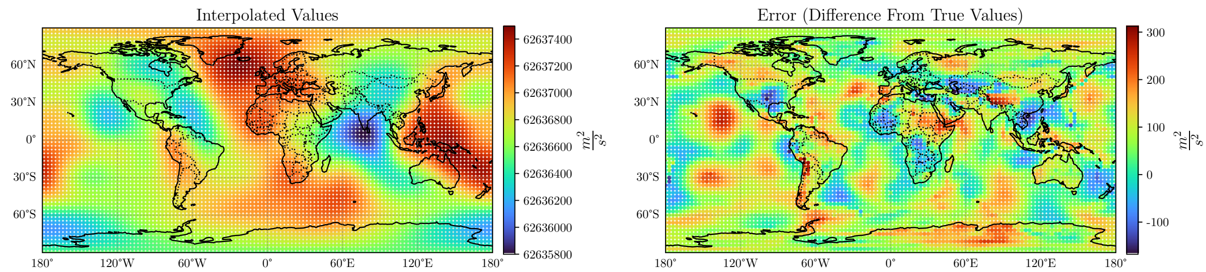

Figure 18: Results of Singularity kernel with Tikhonov (VCE) method and added white-noise.

3.6 Logarithmic With Noise

Results are shown in the figures 19 to 21.Also, mean and norm (L_2) of the three methods are shown in table 6.

Table 6: Error for interpolation using Logarithmic kernel and added white-noise (unit of values are $\frac{m^2}{s^2}$).

Method	Cholesky	TSVD	Tikhonov (VCE)
Mean Error	7.5111	7.3757	74.2444
Norm of Errors	5155.9012	12404.6648	8167.3250

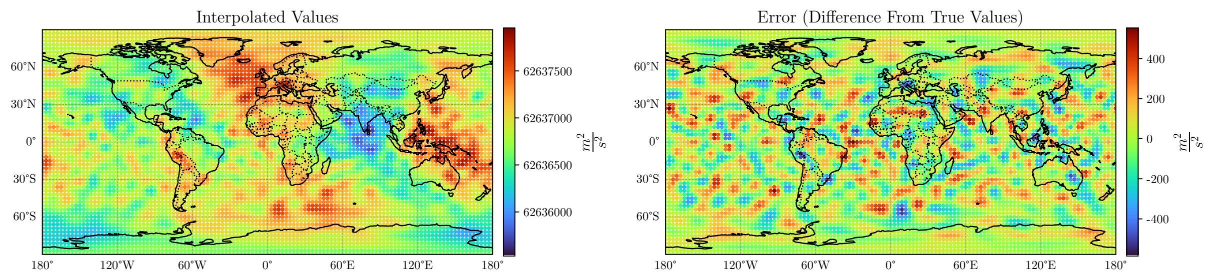

Figure 19: Results of Logarithmic kernel with Cholesky decomposition and added white-noise.

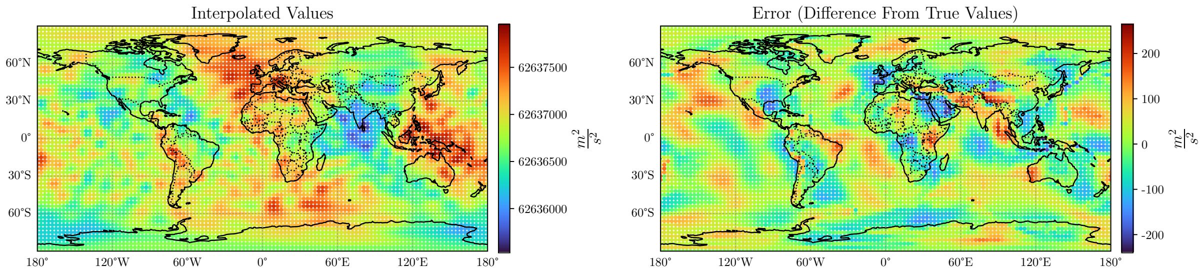

Figure 20: Results of Logarithmic kernel with TSVD method and added white-noise.

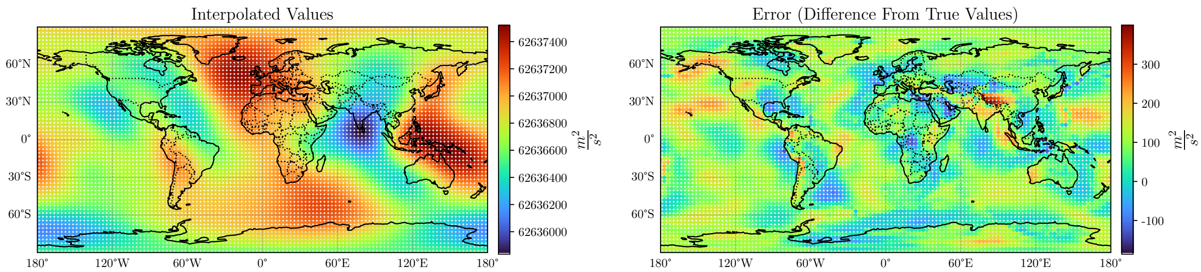

Figure 21: Results of Logarithmic kernel with Tikhonov (VCE) method and added white-noise.