北京大學

习题6.4 HW

1(5) $\frac{\partial z}{\partial x} = \sqrt{1-yx^2}\cdot\sqrt{y}$ $\frac{\partial z}{\partial y} = \sqrt{1-yx^2}\cdot\frac{x}{2\sqrt{y}}$ (8) $\frac{\partial u}{\partial x} = z x^{z-1}\cdot(y)^z$

$\frac{\partial u}{\partial y} = z\cdot y^{z-1}\cdot x^z$ $\frac{\partial u}{\partial z} = (xy)^z\cdot\ln(xy)$

2.(1) $\frac{\partial z}{\partial x}\Big|_{(0,1)} = \frac{d\frac{x}{1+\sin x}}{dx}\Big|_{x=0} = \frac{1+\sin x - x\cos x}{(1+\sin x)^2}\Big|_{x=0} = 1$

$\frac{\partial z}{\partial y}\Big|_{(0,1)} = \frac{d\frac{1-y}{1+\sin(y-1)}}{dy}\Big|_{y=1} = \frac{-1-\sin(y-1)-\cos(y-1)(1-y)}{(1+\sin(y-1))^2}\Big|_{y=1} = -1$

3. $0 \le \lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{|x|+|y|} \le \lim\limits_{(x,y)\to(0,0)} \frac{|x|\cdot(|x|+|y|)+|y|\cdot(|x|+|y|)}{|x|+|y|} = \lim\limits_{(x,y)\to(0,0)} |x|+|y| = 0 = f(0,0)$

$f_x(0,0)$ $\lim\limits_{x\to 0}\frac{df(x,0)}{dx} = \lim\limits_{x\to 0}\frac{d|x|}{dx}$ ~~不存在~~ $f_x(x,0)$不存在

5.(3). $f_x(x,y) = 1+y^2+12x^2-\frac{2x}{x^2+1}$ $f_{xy}(x,y) = 2y$

8. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y\partial x}$ $\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial x\partial y}$ v有连续二阶偏导数

故 $\frac{\partial^2 v}{\partial y\partial x} = \frac{\partial^2 v}{\partial x\partial y} \Rightarrow \therefore \Delta u = 0$

$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x\partial y}$. $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y\partial x} \Rightarrow \Delta v = 0$

10.(1) $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = -\frac{y}{x^2}e^{\frac{y}{x}}dx + \frac{1}{x}e^{\frac{y}{x}}dy$

13. 若 $f(x_1,y_1)=c_1$, $f(x_2,y_2)=c_2$ $c_1\ne c_2$

$c_1-c_2 = f(x_1,y_1)-f(x_2,y_2) = f(x_1,y_1)-f(x_2,y_1)+f(x_2,y_1)-f(x_2,y_2)$

$= (x_1-x_2)f_x(x_1+\theta_1(x_1-x_2),y_1) + (y_1-y_2)f_y(x_2,y_1+\theta_2(y_1-y_2)) = 0$

矛盾. 故 $f(x,y)\equiv c$

17.(1) $f_x(x,y) = \frac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-y^3x)}{(x^2+y^2)^2}$

$f_y(x,y) = \frac{(x^3-3xy^2)(x^2+y^2)-2y(x^3y-y^3x)}{(x^2+y^2)^2}$

$f_x(0,y) = \frac{-y^5}{y^4} = -y$

(2) $f_x(0,0) = \lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0}\frac{0-0}{x} = 0$

$f_y(x,0) = \frac{x^5}{x^4} = x$

$f_y(0,0) = \lim\limits_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = 0$

(3) $f_{xy}(0,0) = \lim\limits_{y\to 0}\frac{f_x(0,y)-f_x(0,0)}{y-0} = \lim\limits_{y\to 0}\frac{-y-0}{y} = -1$

$f_{yx}(0,0) = \lim\limits_{x\to 0}\frac{f_y(x,0)-f_y(0,0)}{x-0} = 1$

北京大学

习题 6.5 HW

4. (1) $dz = \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy = \frac{y^2 e^{2x}(2x-1)}{x^2 \ln y}dx + \frac{e^{2x}(2y\ln y - y)}{x(\ln y)^2}\cdot dy$

$z_x = \frac{y^2 e^{2x}(2x-1)}{x^2 \ln y}$ $\quad z_y = \frac{e^{2x}(2y\ln y - y)}{x(\ln y)^2}$

(2) $\frac{\partial z}{\partial x} = f_1'\cdot y + f_2'\cdot\frac{1}{y}$ $\quad \frac{\partial z}{\partial y} = f_1' x - \frac{x}{y^2} f_2'$

6. $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} = e^s\cos t\cdot\frac{\partial u}{\partial x} + e^s\sin t\cdot\frac{\partial u}{\partial y}$

$\frac{\partial^2 u}{\partial s^2} = \cos t\cdot e^s\cdot\frac{\partial u}{\partial x} + (e^s\cos t)^2\cdot\frac{\partial^2 u}{\partial x^2} + e^s\sin t\cdot\frac{\partial u}{\partial y} + (e^s\sin t)^2\cdot\frac{\partial^2 u}{\partial y^2}$

$\frac{\partial u}{\partial t} = \cancel{\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}(=-e^s\sin t\cdot\frac{\partial u}{\partial x})+} - e^s\sin t\cdot\frac{\partial u}{\partial x} + e^s\cos t\frac{\partial u}{\partial y}$

$\frac{\partial^2 u}{\partial t^2} = -e^s\cos t\cdot\frac{\partial u}{\partial x} + (-e^s\sin t)^2\frac{\partial^2 u}{\partial x^2} + (e^s\cos t)^2\cdot\frac{\partial^2 u}{\partial y^2}$

$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = e^{2s}\cdot\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$

8. $f(tx, ty, tz) = t^n f(x,y,z)$ 看成关于 t 的函数 对 t 求导

$\cancel{t f_1'}$ $x f_1' + y f_2' + z f_3' = n\cdot t^{n-1} f(x,y,z) = n\cdot\frac{1}{t} f(tx, ty, tz)$

$\Rightarrow (tx)\cdot f_1' + (ty) f_2' + (tz) f_3' = n f(tx, ty, tz)$

令 $x_1 = tx,\ y_1 = ty,\ z_1 = tz \Rightarrow x_1 f_{x_1}(x_1, y_1, z_1) + y_1 f_{y_1}(x_1, y_1, z_1) + z_1 f_{z_1}(x_1, y_1, z_1) = n f(x_1, y_1, z_1)$

即 $x f_x(x,y,z) + y f_y(x,y,z) + z f_z(x,y,z) = n f(x,y,z)$

北京大学

习题 6.6 HW

1. $\mathrm{grad}\, f|_{(2+\sqrt{3},1+2\sqrt{3})} = ((2x-y)|_{(2+\sqrt{3},1+2\sqrt{3})}, (2y-x)|_{(2+\sqrt{3},1+2\sqrt{3})}) = (3, 3\sqrt{3})$

$(\cos\theta, \sin\theta) \parallel (3, \cancel{3+}\ 3\sqrt{3}) \Rightarrow \theta = \frac{\pi}{3} / \cancel{\frac{4}{3}\pi}\ \frac{4\pi}{3}$

$(\cos\theta, \sin\theta) \cdot (3, 3\sqrt{3}) = 0 \Rightarrow \theta = \frac{5\pi}{6}, \frac{11}{6}\pi$

当 $\theta=\frac{\pi}{3}$ 时，方向导数最大为6；$\theta=\frac{4\pi}{3}$ 时，方向导数最小，为-6

当 $\theta=\frac{5\pi}{6}$ 或 $\frac{11}{6}\pi$ 时，方向导数为0

7. $\cancel{\mathrm{grad}\, z|_A}$ $\frac{\partial z}{\partial x} = -\frac{1}{x}, \quad \frac{\partial z}{\partial y} = \frac{1}{y}$

$\vec{a} = \mathrm{grad}\, z|_A = (-3, 10) \quad \vec{b} = \mathrm{grad}\, z|_B = (-1, 6)$

$\cos\langle\vec{a}, \vec{b}\rangle = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{63}{\sqrt{109}\cdot\sqrt{37}}$

9. $d(x^2+2y^2) = 0 \Rightarrow x\,dx + 2y\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y}$

椭圆周上 (x_0, y_0) 处法方向方向导数 $\vec{u} = (\cos\alpha, \sin\alpha)$

则 $\tan\alpha = -\frac{dx}{dy} = \frac{2y_0}{x_0}$ 取 $\vec{u} = (\frac{x_0}{\sqrt{4y_0^2+x_0^2}}, \frac{2y_0}{\sqrt{4y_0^2+x_0^2}})$

$\mathrm{grad}\, f|_{(x_0,y_0)} = (-\frac{2y_0}{x_0^2}, \frac{1}{x_0})$

$\mathrm{grad}\, f|_{(x_0,y_0)} \cdot \vec{u} = 0$ 即 f 沿该处椭圆周法方向的方向导数为0

北京大学

习题 6.7

2(1) $\cos x = 1 - \frac{1}{2}x^2 + o(x^2) = 1 - \frac{1}{2}x^2 + o(\rho^2)$

$$\frac{1}{\cos y} = \frac{1}{1 - \frac{1}{2}y^2 + o(y^2)} = 1 + \frac{1}{2}y^2 + o(y^2) + O\left(\left(-\frac{1}{2}y^2 + o(y^2)\right)\right)^2 = 1 + \frac{1}{2}y^2 + o(\rho^2)$$

$$\frac{\cos x}{\cos y} = \left(1 - \frac{1}{2}x^2 + o(\rho^2)\right)\left(1 + \frac{1}{2}y^2 + o(\rho^2)\right) = 1 - \frac{1}{2}x^2 + \frac{1}{2}y^2 + o(\rho^2)$$

3. $\cancel{\ln(1+u) = u - \frac{1}{2}u^2}$ $\ln(1+u) = u - \frac{u^2}{2(1+\xi u)^2}$, $\xi \in (0,1)$

$$f(x,y) = x + y - \frac{(x+y)^2}{2(1+\xi x + \xi y)^2}$$

5. $f(x,y) = f(0,0) + f_x(\theta x, \theta y)\cdot x + f_y(\theta x, \theta y)\, y \quad \theta \in (0,1)$

当 $(x,y) \in D$，有 ~~(0,0)与(x,y)~~ 有 $u^2x^2 + u^2y^2 < 1$ (当 $u \in (0,1)$) 时 的即 $(0,0)$ 与 (x,y) 连线上任意一点在 D 中

$$= f(0,0) + \frac{1}{\theta}\left(\theta x f_x(\theta x, \theta y) + \theta y f_y(\theta x, \theta y)\right)$$

$= f(0,0)$ 对 $\forall (x,y) \in D$ 成立

故 $f(x,y)$ 为常数

习题 6.10

1.(4) $F_x(2,1,0)(x-2) + F_y(2,1,0)(y-1) + F_z(2,1,0)\, z = 0$

即 $x + 2y + z = 4$

2. $F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

即 $\frac{x}{2\sqrt{x_0}} - \frac{\sqrt{x_0}}{2} + \frac{y}{2\sqrt{y_0}} - \frac{\sqrt{y_0}}{2} + \frac{z}{2\sqrt{z_0}} - \frac{\sqrt{z_0}}{2} = 0$

即 $\frac{x}{\sqrt{a}\sqrt{x_0}} + \frac{y}{\sqrt{a}\sqrt{y_0}} + \frac{z}{\sqrt{a}\sqrt{z_0}} = \cancel{a}\ 1$

截距之和为 $\sqrt{a}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = a$