2020秋 高等数学A 期中考试(统考)

1 (1) $\lim\limits_{x\to+\infty}(\sqrt{x^2+x+1}-x)=\lim\limits_{x\to+\infty}\frac{x+1}{\sqrt{x^2+x+1}+x}=\lim\limits_{x\to+\infty}\frac{1+\frac{1}{x}}{1+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}}=\frac{1}{2}$.

(2) $\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}\frac{n}{n^2+k^2}=\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}\frac{1}{n}\frac{1}{1+\frac{k^2}{n^2}}=\lim\limits_{n\to+\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\frac{1}{1+\frac{k^2}{n^2}}$

$=\int_0^1 \arctan x\,dx=\frac{\pi}{4}$.

(3) $\lim\limits_{n\to\infty} n(n^{\frac{1}{n^2}}-1)$

先证 $\lim\limits_{n\to\infty} n^{\frac{1}{n^2}}=1$. $1\le n^{\frac{1}{n^2}}=\sqrt[n^2]{\sqrt{n}\cdot\sqrt{n}\cdot\underbrace{1\cdot1\cdots1}_{n^2-2个1}}\le\frac{2\sqrt{n}+n^2-2}{n^2}=1+\frac{2\sqrt{n}-2}{n^2}$

$\lim\limits_{n\to\infty}(1+\frac{2\sqrt{n}-2}{n^2})=1$ $\therefore \lim\limits_{n\to\infty} n^{\frac{1}{n^2}}=1$

$\therefore \lim\limits_{n\to\infty} n(n^{\frac{1}{n^2}}-1)=\lim\limits_{n\to\infty} n\ln n^{\frac{1}{n^2}}=\lim\limits_{n\to\infty}\frac{\ln n}{n}=\lim\limits_{n\to\infty}\ln\sqrt[n]{n}$

$1<\sqrt[n]{n}=\sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot\underbrace{1\cdot1\cdots1}_{n-2个1}}\le\frac{2\sqrt{n}+n-2}{n}=1+\frac{2\sqrt{n}-2}{n}$

$\lim\limits_{n\to\infty}(1+\frac{2\sqrt{n}-2}{n})=1$ $\therefore\lim\limits_{n\to\infty}\sqrt[n]{n}=1$

$\therefore$ 原式 $=0$.

(4) $\lim\limits_{x\to0}(1+2x)^{\frac{(x+1)^2}{x}}=\lim\limits_{x\to0}(1+2x)^{\frac{1}{2x}\cdot2x\cdot\frac{(x+1)^2}{x}}=\lim\limits_{x\to0}e^{2(x+1)^2}=e^2$

2 (1) $\int_0^1\ln(x+\sqrt{x^2+1})dx=x\ln(x+\sqrt{x^2+1})\Big|_0^1-\int_0^1 x\cdot\frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}dx$

$=\ln(1+\sqrt{2})-\int_0^1\frac{x}{\sqrt{x^2+1}}dx$

$=\ln(1+\sqrt{2})-\sqrt{x^2+1}\Big|_0^1=\ln(1+\sqrt{2})-\sqrt{2}+1$

(2) 设 $\frac{4x^3+2x^2+3x+1}{x(x+1)(x^2+1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{Dx+E}{x^2+1}$

$=\frac{A(x^3+x+x^2+1)+B(x^3+x)+Dx^3+(D+E)x^2+Ex}{x(x+1)(x^2+1)}$

$\therefore\begin{cases}A+B+D=4\\A+D+E=2\\A+B+E=3\\A=1\end{cases}$ $\therefore\begin{cases}A=1\\E=0\\B=2\\D=1\end{cases}$

$\therefore\int\frac{4x^3+2x^2+3x+1}{x(x+1)(x^2+1)}dx=\int(\frac{1}{x}+\frac{2}{x+1}+\frac{x}{x^2+1})dx$

$=\ln|x|+2\ln|x+1|+\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}$

$=\ln|x|+2\ln|x+1|+\frac{1}{2}\ln(x^2+1)+C$

(3) $\int_0^1x^4\sqrt{1-x^2}dx=-\frac{1}{3}\int_0^1x^3\cdot(-3x\sqrt{1-x^2})dx=-\frac{1}{3}\int_0^1x^3d(1-x^2)^{\frac{3}{2}}$

$=-\frac{x^3}{3}(1-x^2)^{\frac{3}{2}}\Big|_0^1+\int_0^1(1-x^2)^{\frac{3}{2}}x^2dx=0-\int_0^1x^4\sqrt{1-x^2}dx+\int_0^1x^2\sqrt{1-x^2}dx$

$\therefore\int_0^1x^4\sqrt{1-x^2}dx=\frac{1}{2}\int_0^1x^2\sqrt{1-x^2}dx$

$=-\frac{1}{6}\int_0^1x\cdot(-3x\sqrt{1-x^2})dx=-\frac{1}{6}\int_0^1x\,d(1-x^2)^{\frac{3}{2}}$

$=-\frac{1}{6}x(1-x^2)^{\frac{3}{2}}\Big|_0^1+\frac{1}{6}\int_0^1(1-x^2)\sqrt{1-x^2}dx$

$=\frac{1}{6}\int_0^1\sqrt{1-x^2}dx-\frac{1}{6}\int_0^1x^2\sqrt{1-x^2}dx$

$\therefore\int_0^1x^2\sqrt{1-x^2}dx=\frac{\pi}{16}$

$\therefore\int_0^1x^4\sqrt{1-x^2}dx=\frac{\pi}{32}$ (或设 $x=\sin a$ 进行换元)

(4) 设 $f(x)=(x^4+2x^2+1)\sin^3x$ 注意到 $f(x)=-f(-x)$

$\therefore$ 设 $I=\int_{-1}^1f(x)dx=\int_1^{-1}f(-x)d(-x)=-\int_{-1}^1f(x)dx$

$\therefore I=0$

3 $x_{n+1}=\frac{x_n+f(x_n)}{2}$ $x_{n+2}=\frac{x_{n+1}+f(x_{n+1})}{2}$

$x_{n+2}-x_{n+1}=\frac{x_{n+1}-x_n}{2}+\frac{f(x_{n+1})-f(x_n)}{2}$ 由题. $|f(x_{n+1})-f(x_n)|\le|x_{n+1}-x_n|$

若 $x_{n+1}\le x_n$:

$x_{n+1}-x_n\le f(x_{n+1})-f(x_n)\le x_n-x_{n+1}$

$x_{n+2}-x_{n+1}\le\frac{x_{n+1}-x_n}{2}+\frac{x_n-x_{n+1}}{2}=0$

$\therefore x_{n+2}\le x_{n+1}$

① 若 $\{x_n\}$ 单调增,则 $x_{n+1}=\frac{x_n+f(x_n)}{2}\le\frac{b+b}{2}=b$

$\therefore\{x_n\}$ 有上界

② 若 $\{x_n\}$ 非单调增,则 $\exists N_0$, $x_{N_0+1}<x_{N_0}$.

则 x_{N_0}, x_{N_0+1} --- 单调减. 又 $x_n\ge a$. $\therefore\{x_n\}$ 有下界

$\therefore\{x_n\}$ 有界,极限存在

$x_{n+1}>x_n$ 时同理

综上,$\{x_n\}$ 有极限

4 (1) $y'=\frac{2\arcsin x}{\sqrt{1-x^2}}$ $y'(0)=0$

$\therefore y''=2\frac{1+\frac{x\arcsin x}{\sqrt{1-x^2}}}{1-x^2}=\frac{2}{1-x^2}+\frac{2x\arcsin x}{(1-x^2)^{\frac{3}{2}}}$ $y''(0)=2$

$\therefore(1-x^2)y''=2+xy'$

求 $n-1$ 阶导数得

$y^{(n+1)}(1-x^2)+(n-1)\cdot(-2x)y^{(n)}+\frac{(n-1)(n-2)}{2}\cdot(-2)y^{(n-1)}=xy^{(n)}+(n-1)y^{(n-1)}$

$\therefore(1-x^2)y^{(n+1)}+x(1-2n)y^{(n)}-(n-1)^2y^{(n-1)}=0$

$\therefore y^{(n+1)}(0)=(n-1)^2y^{(n-1)}(0)$

$\therefore y^{(2k+1)}(0)=(2k-1)^2y^{(2k-1)}(0)$

$y^{(2k+2)}(0)=(2k)^2y^{(2k)}(0)$

$\therefore n$ 为奇数时, $y^{(n)}(0)=0$

n 为偶数时. $y^{(n)}(0)=(n!!)^2y^{(2)}(0)=2\cdot(n!!)^2$. $(n\ne0)$

(2) $\frac{d}{dx}\int_{x^3+1}^{2^x}\frac{\sin t}{t^4+2}dt=\frac{\sin2^x}{2^{4x}+2}\cdot2^x\ln2-\frac{3x^2\sin(x^3+1)}{(x^3+1)^4+2}$.

5 设 $g(x)=f(x)-f(x+\frac{1}{3})$

$g(0)=f(0)-f(\frac{1}{3})$ $g(\frac{1}{3})=f(\frac{1}{3})-f(\frac{2}{3})$

$g(\frac{2}{3})=f(\frac{2}{3})-f(1)$

$\therefore g(0)+g(\frac{1}{3})+g(\frac{2}{3})=f(0)-f(1)=0$

设 $m=g_{min}(x)$. $M=g_{max}(x)$

$\therefore 3m\le0\le3M$ $\therefore m\le0\le M$ 由介值定理

$\therefore\exists c\in[0,1]$ 使 $f(c)=f(c+\frac{1}{3})$

6 (法1)

设 $[|f(x)|]_{min}=|f(m)|$

对于 $\forall x\in[m,1]$, 由微积分基本定理有:

$f(x)=\int_m^xf'(t)dt+f(m)$

$\therefore|f(x)|=\left|\int_m^xf'(t)dt+f(m)\right|$

$\le\left|\int_m^xf'(t)dt\right|+|f(m)|$

$=\left|\int_m^xf'(t)dt\right|+\int_0^1|f(m)|dt$

$\le\int_m^x|f'(t)|dt+\int_0^1|f(m)|dt$

$\le\int_0^1|f'(t)|dt+\int_0^1|f(t)|dt$

类似地, $\forall x\in[0,m]$, 有:

$|f(x)|\le\int_0^1|f'(t)|dt+\int_0^1|f(t)|dt$

故 $\forall x\in[0,1]$, 原命题成立.

若等号成立, 注意到

$\int_0^1|f'(t)|dt+\int_0^1|f(t)|dt$ 为常数

故 $f(x)=C$ (C为常数)

验证: $f(x)=C$ 时

$\int_0^1|f'(t)|dt+\int_0^1|f(t)|dt$

$=\int_0^10dt+\int_0^1|C|dt=|C|=|f(x)|$ 成立

故取等条件是 $f(x)$ 为常函数.

(法2)

由分部积分公式有:

$\int_0^xtf'(t)dt=\int_0^xt\,df(t)=tf(t)\Big|_0^x-\int_0^xf(t)dt$

$=xf(x)-\int_0^xf(t)dt$ ①

$\int_x^1(t-1)f'(t)dt=\int_x^1(t-1)df(t)=(t-1)f(t)\Big|_x^1-\int_x^1f(t)dt$

$=-(x-1)f(x)-\int_x^1f(t)dt$ ②

①②相加得

$\int_0^xtf'(t)dt+\int_x^1tf'(t)dt-\int_x^1f'(t)dt=f(x)-\int_0^1f(x)dx$

$f(x)=\int_0^xtf'(t)dt+\int_x^1[tf'(t)-f'(t)]dt+\int_0^1f(x)dt$

$\therefore|f(x)|\le\int_0^x|f'(t)|dt+\int_x^1|f'(t)|dt+\int_0^1|f(t)|dt$

$=\int_0^1|f'(t)|dt+\int_0^1|f(t)|dt$

(法3) 要证 $|f(x)|\le\int_0^1|f(t)|dt+\int_0^1|f'(t)|dt$

要证 $|f(x)|\le\int_0^1|f(t)|dt+\int_0^1|f'(t)|dt$

即证 $|f(x)|\le\left|\int_0^1f(t)dt\right|+\int_0^1|f'(t)|dt$

即证 $|f(x)|-\left|\int_0^1f(t)dt\right|\le\int_0^1|f'(t)|dt$

即证 $\left|f(x)-\int_0^1f(t)dt\right|\le\int_0^1|f'(t)|dt$

由积分中值定理: $\exists m\in[0,1]$ 使 $f(m)=\int_0^1f(t)dt$

即证 $|f(x)-f(m)|\le\int_0^1|f'(t)|dt$

由微积分基本定理

即证 $\left|\int_m^xf'(t)dt\right|\le\int_0^1|f'(t)|dt$

即证 $\begin{cases}\left|\int_m^xf'(t)dt\right|\le\int_x^m|f'(t)|dt & (m\ge x时)\\\left|\int_m^xf'(t)dt\right|\le\int_m^x|f'(t)|dt & (m<x时)\end{cases}$

这显然成立

$\therefore$ 原不等式得证

7 先证: $\lim\limits_{x\to\infty}\sin x$ 不存在

取 $x_1=2k\pi$ $(k\in N^*)$. $k\to\infty$ 时, $x_1\to\infty$

$\sin x_1=0$

取 $x_2=2k\pi+\frac{\pi}{2}$ $(k\in N^*)$ $k\to\infty$ 时, $x_2\to\infty$

$\sin x_2=1\ne0$

$\therefore\lim\limits_{x\to\infty}\sin x$ 不存在, 同理, $\lim\limits_{x\to\infty}\cos x$ 不存在

$x\ne0$ 时, $f'(x)=mx^{m-1}\sin\frac{1}{x}-x^{m-2}\cos\frac{1}{x}$

$f''(x)=m(m-1)x^{m-2}\sin\frac{1}{x}-mx^{m-3}\cos\frac{1}{x}-(m-2)x^{m-3}\cos\frac{1}{x}$
$+x^{m-4}\sin\frac{1}{x}$

$\lim\limits_{x\to0}\frac{f(\Delta x)-f(0)}{\Delta x}=\lim\limits_{x\to0}(\Delta x)^{m-1}\sin\frac{1}{\Delta x}$

$m\ge2$ 时, $\lim\limits_{\Delta x\to0}(\Delta x)^{m-1}\sin\frac{1}{\Delta x}=0$ $\therefore f'(0)=0$

$m=1$ 时, $\lim\limits_{\Delta x\to0}\sin\frac{1}{\Delta x}$ 不存在

若 $f''(x)$ 连续, $f'(0)$ 一定存在. $\therefore m\ge2$.

$m=2$ 时, $\lim\limits_{x\to0}f'(x)=\lim\limits_{x\to0}(2x\sin\frac{1}{x}-\cos\frac{1}{x})$ 不存在.

$m\ge3$ 时. $\lim\limits_{x\to0}f'(x)=0$. $f'(x)$ 在R上连续

$\lim\limits_{\Delta x\to0}\frac{f'(\Delta x)-f'(0)}{\Delta x}=\lim\limits_{\Delta x\to0}\left[m(\Delta x)^{m-2}\sin\frac{1}{\Delta x}-(\Delta x)^{m-3}\cos\frac{1}{\Delta x}\right]$

$m=3$ 时. $\lim\limits_{\Delta x\to0}\cos\frac{1}{\Delta x}$ 不存在. 故上式极限不存在

$m\ge4$ 时. $\lim\limits_{\Delta x\to0}\left[m(\Delta x)^{m-2}\sin\frac{1}{\Delta x}-(\Delta x)^{m-3}\cos\frac{1}{\Delta x}\right]=0$

此时. $\lim\limits_{x\to0}f''(x)$

$=\lim\limits_{x\to0}\left[m(m-1)x^{m-2}\sin\frac{1}{x}-mx^{m-3}\cos\frac{1}{x}-(m-2)x^{m-3}\cos\frac{1}{x}+x^{m-4}\sin\frac{1}{x}\right]$

$m=4$ 时. $\lim\limits_{x\to0}x^{m-4}\sin\frac{1}{x}$ 不存在

$m\ge5$ 时. $\lim\limits_{x\to0}f''(x)=0$. $\therefore f''(x)$ 连续

综上, $m\ge5$.