习题 2.1

2.1.2.(2) $y=\sqrt{2px}$

$$y'=\lim_{\Delta x\to 0}\frac{\sqrt{2p(x+\Delta x)}-\sqrt{2px}}{\Delta x}=\lim_{\Delta x\to 0}\sqrt{2p}\cdot\frac{\Delta x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\frac{\sqrt{2}}{2}\cdot\sqrt{\frac{p}{x}}$$

2.1.3

$y=2^x$. $y'=2^x\ln 2$ $y'|_{x=0}=\ln 2$

$l: y=(\ln 2)x+1$

2.1.7

(4) $y=\frac{9x+x^2}{5x+6}$ $y'=\frac{(9+2x)(5x+6)-5(9x+x^2)}{(5x+6)^2}=\frac{5x^2+12x+54}{(5x+6)^2}$

(9) $y=x\cos x+\frac{\sin x}{x}$ $y'=\cos x-x\sin x+\frac{x\cos x-\sin x}{x^2}$

2.1.10

$$f'(0)=\lim_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0^+}\frac{f(-\Delta x)-f(0)}{\Delta x}=-f'(0)$$

故 $f'(0)=0$

习题 2.2

2. 记 $f'(g(x))=f'(u)|_{u=g(x)}$. 现设 $f(x)=x^2+1$.

(1) 求 $f'(x)$, $f'(0)$, $f'(x^2)$, $f'(\sin x)$;

$f'(x)=2x$, $f'(0)=0$, $f'(x^2)=2x^2$, $f'(\sin x)=2\sin x$

2.2.3.(4)

$y=\sin^3x\cdot\cos 3x$ $y'=3\sin^2x\cos x\cos 3x-3\sin 3x\sin^3x$ $3\sin^2x\cos 4x$

(9)

$y=\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$ $y'=\frac{1}{\tan(\frac{x}{2}+\frac{\pi}{4})}\cdot\frac{1}{\cos^2(\frac{x}{2}+\frac{\pi}{4})}\cdot\frac{1}{2}=\frac{1}{\sin(x+\frac{\pi}{2})}=\frac{1}{\cos x}$ $(x\neq k\pi+\frac{\pi}{2}, k\in Z)$

2.2.4

(6) $y=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\frac{x+\sqrt{x^2+a^2}}{a}$ $(a>0)$;

$y'=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x^2}{2\sqrt{x^2+a^2}}+\frac{a^2}{2}\cdot\frac{1+\frac{x}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}$

$= \frac{2x^2+a^2}{2\sqrt{x^2+a^2}} + \frac{1}{2}\cdot(1+\frac{x}{\sqrt{x^2+a^2}})(\sqrt{x^2+a^2}-x)$

$= \sqrt{x^2+a^2}$

(8) $y=\frac{2}{\sqrt{a^2-b^2}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$ $(a>b\geqslant 0)$;

$y' = \frac{2}{\sqrt{a^2-b^2}}\cdot\frac{1}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}\cdot\sqrt{\frac{a-b}{a+b}}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}$

$= \frac{1}{(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}}$

$= \frac{1}{a+b\cos x}$

(14) $y=(x-1)\sqrt[3]{(3x+1)^2(2-x)}$;

$y=(x-1)(3x+1)^{\frac{2}{3}}(2-x)^{\frac{1}{3}}$

$y' = (3x+1)^{\frac{2}{3}}(2-x)^{\frac{1}{3}} + 2(x-1)(3x+1)^{-\frac{1}{3}}(2-x)^{\frac{1}{3}} - \frac{1}{3}(x-1)(3x+1)^{\frac{2}{3}}(2-x)^{-\frac{2}{3}}$

习题 2.5

2.5.1 (2)

$y = (\sqrt{x+2}-\sqrt{2})\sin x$

$\lim_{x\to 0}\frac{y}{x^2} = \lim_{x\to 0}\frac{x\cdot\sin x}{x^2(\sqrt{x+2}+\sqrt{2})} = \frac{1}{2\sqrt{2}}$

故当 $x\to 0$ 时，y 是 x 的二阶无穷小量

2.5.3

3. 设 $\alpha(x)=o(x)$ $(x\to 0)$，$\beta(x)=o(x)$ $(x\to 0)$. 试证明：$\alpha(x)+\beta(x)=o(x)$.
上述结果有时可写成：$o(x)+o(x)=o(x)$.

证：由题意：$\lim_{x\to 0}\frac{\alpha(x)}{o(x)} = \lim_{x\to 0}\frac{\beta(x)}{o(x)} = 0$，极限都存在

故 $\lim_{x\to 0}\frac{\alpha(x)+\beta(x)}{o(x)} = \lim_{x\to 0}\frac{\alpha(x)}{o(x)} + \lim_{x\to 0}\frac{\beta(x)}{o(x)} = 0+0=0$

故 $\alpha(x)+\beta(x)=o(x)$

4. 计算下列函数在指定的点 x_0 处的微分：
(1) $x\sin x$，$x_0=\pi/4$；

$d\,x\sin x\,\big|_{x_0=\frac{\pi}{4}} = (\sin x + x\cos x)\,dx\,\big|_{x_0=\frac{\pi}{4}} = \frac{\sqrt{2}}{2}(1+\frac{\pi}{4})\,dx$

5. 求下列各函数的微分：
(1) $y=\frac{1-x}{1+x}$ $(x\neq -1)$；

$dy = d\frac{1-x}{1+x} = -\frac{1}{(1+x)^2}\,dx$

2.5.8

(3) $\arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}$；

左右求导：$\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{y'x-y}{x^2} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot(2x+2y\cdot y')$

$$\Rightarrow \frac{x \cdot y'}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{x}{x^2+y^2} + \frac{y \cdot y'}{x^2+y^2}$$

$$\Rightarrow y' = \frac{x+y}{x-y}$$

2.5.9 (2) $e^{xy} - 5x^2 y = 0$, $M\left(\frac{e^2}{10}, \frac{20}{e^2}\right)$.

$$\frac{de^{xy}}{dxy} \cdot \frac{dxy}{dx} - \frac{d5x^2y}{dx} = 0$$

$$e^{xy}(y + y') - 10xy - 5x^2 y' = 0$$

代入 $x = \frac{e^2}{10}$, $y = \frac{20}{e^2}$ 得：$e^2\left(\frac{20}{e^2} + y'\right) - 20 - \frac{e^4}{20} y' = 0$

$$\Rightarrow y' = 0$$

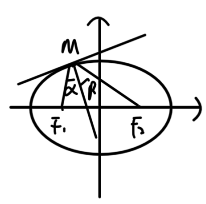
2.5.10 (2) $\begin{cases} x = t\ln t, \\ y = e^t; \end{cases}$

$$y' = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{1+\ln t}{e^t}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t}{1+\ln t}$$

2.5.11

11. 试求椭圆周 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 上一点 $M_0(x_0, y_0)$ 处的切线方程与法线方程，并证明：从椭圆的一个焦点向椭圆周上任一点 M 发射的光线，其反射线必通过椭圆的另一个焦点。



记 $x = a\cos\theta$. $y = b\sin\theta$

$$y' = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b^2}{a^2}\frac{x}{y}$$

$l_{切}$: $y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0)$

$l_{法}$: $y - y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0)$

$k_{MF_1} = \frac{y_0}{x_0 + c}$ $\quad k_{MF_2} = \frac{y_0}{x_0 - c}$

原命题 $\Leftrightarrow \tan\alpha = \tan\beta$

$$\Leftrightarrow \frac{\frac{a^2 y_0}{b^2 x_0} - \frac{y_0}{x_0+c}}{1 + \frac{a^2 y_0^2}{b^2(x_0+c)x_0}} = \frac{\frac{y_0}{x_0-c} - \frac{a^2 y_0}{b^2 x_0}}{1 + \frac{a^2 y_0^2}{b^2 x_0(x_0-c)}}$$

$$\Leftrightarrow \frac{c^2 x_0 y_0 + a^2 c y_0}{a^2 b^2 + b^2 c x_0} = \frac{a^2 c y_0 - c^2 x_0 y_0}{a^2 b^2 - b^2 c x_0}$$

$$\Leftrightarrow a^2 b^2 c^2 x_0 y_0 + a^4 b^2 c y_0 - b^2 c^3 x_0^2 y_0 - a^2 b^2 c^2 y_0 x_0 = a^4 b^2 c y_0 - b^2 c^3 x_0^2 y_0$$

即证