习题 1.4

1.4.1 (4)

$\forall \varepsilon>0\ \exists\ \delta<\frac{\varepsilon}{2}$. s.t 当 $x\in U_0(\varepsilon,\delta)$

$|\cos a-\cos x| = |2\sin\frac{a+x}{2}\sin\frac{a-x}{2}| < |2\sin\frac{a-x}{2}| < |2\cdot\frac{a-x}{2}| = |a-x| < |\delta| < \varepsilon$

故 $\lim\limits_{x\to a}\cos x=\cos a$

1.4.2

$\because \lim\limits_{x\to a} f(x)=l$ $\therefore$ 对 $\varepsilon>0$, $\exists\ \delta>0$, 当 $x\in U_0(a,\delta)$, 有 $|f(x)-l|<\delta$

即 $l-\delta<f(x)<l+\delta$ 即 $f(x)$ 在 $U_0(a,\delta)$ 为有界函数

1.4.3

(2) $\lim\limits_{x\to0}\frac{1-\cos x}{x^2} = \lim\limits_{x\to0}\frac{2\sin^2\frac{x}{2}}{4\cdot(\frac{x}{2})^2} = \frac{1}{2}\lim\limits_{x\to0}(\frac{\sin\frac{x}{2}}{\frac{x}{2}})^2 = \frac{1}{2}$

(12) $\lim\limits_{x\to0}(a_0x^m+a_1x^{m-1}+\dots+a_m)=a_m$ $\lim\limits_{x\to0}(b_0x^n+b_1x^{n-1}+\dots+b_n)=b_n\neq0$

原式 $= \frac{\lim\limits_{x\to0}(a_0x^m+a_1x^{m-1}+\dots+a_m)}{\lim\limits_{x\to0}(b_0x^n+b_1x^{n-1}+\dots+b_n)} = \frac{a_m}{b_n}$

(16) $\lim\limits_{x\to a^+}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{(\sqrt{x-a})\cdot(\sqrt{x^2+ax+a^2})} = \lim\limits_{x\to a^+}\frac{\frac{x-a}{\sqrt{x-a}\cdot(\sqrt{x}+\sqrt{a})}+1}{\sqrt{x^2+ax+a^2}} = \lim\limits_{x\to a^+}\frac{\frac{\sqrt{x-a}}{\sqrt{x}+\sqrt{a}}+1}{\sqrt{x^2+ax+a^2}} = \frac{1}{3a}$

1.4.4

(3) $\lim\limits_{x\to0}\frac{\tan3x-\sin2x}{5x}\cdot\frac{5x}{\sin5x} = \lim\limits_{x\to0}(\frac{3\cdot\tan3x}{5\cdot3x}-\frac{2}{5}\cdot\frac{\sin2x}{2x})\cdot\lim\limits_{x\to0}\frac{5x}{\sin5x} = (\frac{3}{5}-\frac{2}{5})\cdot1=\frac{1}{5}$

(5) $\lim\limits_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} = \lim\limits_{x\to a}\cos\frac{x+a}{2}\cdot\lim\limits_{x\to a}\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} = \cos a$

(8) $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x\cdot\lim\limits_{x\to\infty}(1+\frac{1}{x})^{100} = e\lim\limits_{x\to\infty}(1+\frac{1}{x})^{100} = e\cdot1^{100}=e$

习题 1.5 (9.27)

1.5.2 设函数 $y=f(x)$ 在 x_0 处连续且 $f(x_0)>0$. 证明存在一个 $\delta>0$ 使得

$$f(x)>0,\quad 当\ |x-x_0|<\delta.$$

证: $\because f(x)$ 在 x_0 处连续 $\therefore$ 对 $\varepsilon=\frac{f(x_0)}{2}$ $\exists\ \delta_1>0$ s.t. $x\in U_0(x_0,\delta_1)$ 时 $|f(x)-f(x_0)|<\varepsilon$

$\therefore 0<\frac{f(x_0)}{2}<f(x)<\frac{3}{2}f(x_0)$ 即取 $\delta=\delta_1$ 即可

1.5.5 $\lim\limits_{x\to2}x^{\sqrt{x}}=2^{\sqrt{2}}$

$\lim\limits_{x\to\infty}\sqrt{(\sqrt{x^2+1}-\sqrt{x^2-2})|x|} = \lim\limits_{x\to\infty}\sqrt{\frac{3|x|}{\sqrt{x^2+1}+\sqrt{x^2-2}}} = \lim\limits_{x\to\infty}\sqrt{\frac{3}{\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{2}{x^2}}}} = \sqrt{\frac{3}{2}}$

1.5.7

(2) $\lim_{x\to 0^+} sgn(\sin x) = 1$ $\lim_{x\to 0^-} sgn(\sin x) = -1$ $\neq \lim_{x\to 0^+} sgn(\sin x) \neq f(0)$

第一类间断点

(5) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{2-x} = 1 = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2$ $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{1-x} = -1$ $x=2$ 为第一类间断点

习题1.6 (9.27)

1.6.1 不妨设 $f(x) = x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1x + a_0$

$= x^{2n+1}(1 + \frac{a_{2n}}{x} + \cdots + \frac{a_0}{x^{2n+1}})$

$\lim_{x\to -\infty} f(x) = -\infty \cdot 1 < 0$ 故 $\exists x_1$，当 $x \leq x_1$ 时，$f(x) < 0$

$\lim_{x\to +\infty} f(x) = +\infty \cdot 1 > 0$，故 $\exists x_2$，当 $x \geq x_2$ 时，$f(x) > 0$

$f(x_1) < 0 < f(x_2)$ 又 $f(x)$ 连续，故 $\exists x_0$，$f(x_0) = 0$

1.6.3 $g(x) = f(x) - \frac{m_1 f(x_1) + m_2 f(x_2)}{m_1 + m_2}$

$g(x_1) = \frac{m_2(f(x_1) - f(x_2))}{m_1 + m_2}$，$g(x_2) = \frac{-m_1(f(x_1) - f(x_2))}{m_1 + m_2}$

不妨令 $f(x_1) > f(x_2)$ 则 $g(x_1) > 0 > g(x_2)$

由 $f(x)$ 在 (a,b) 上连续，则 $g(x)$ 在 (a,b) 上连续

由介值定理知 $\exists \xi \in (x_1, x_2)$（若 $x_1 < x_2$）$g(\xi) = 0$

即 $f(\xi) = \frac{m_1 f(x_1) + m_2 f(x_2)}{m_1 + m_2}$

思考题

1. 知 $\lim(a_{n+1} - a_n) = l$

$\lim \frac{a_n}{n} = \lim \frac{a_{n+1} - a_n}{n+1-n} = l$

$\lim \frac{\sum^n a_p}{n^2} = \lim \frac{\sum^{n+1} a_p - \sum^n a_p}{(n+1)^2 - n^2} = \lim \frac{a_{n+1}}{2n+1} = \lim \frac{a_{n+1} - a_n}{2n+1-(2n-1)} = \frac{l}{2}$

2. $x_{n+1} = \sqrt{x_n + 2}$

先 $n=1$. $x_n > 2$ 成立，设 $n=k$ 时，$x_n > 2$ 成立，

(1) $n=k+1$ 时，$x_{k+1} = \sqrt{x_k + 2} > \sqrt{2+2} = 2$ 故 $x_n > 2$

再证 $\{x_n\}$ 单减：

$x_{n+1} - x_n = \sqrt{x_n + 2} - x_n = \frac{x_n + 2 - x_n^2}{\sqrt{x_n+2} + x_n} = \frac{\frac{9}{4} - (x_n - \frac{1}{2})^2}{\sqrt{x_n+2} + x_n}$

又 $x_n>2$，故 $(x_n-\frac{1}{2})^2>\frac{9}{4}$，故 $x_{n+1}-x_n<0$

∴ $\{x_n\}$ 单减有下界

∴ $\lim x_n$ 存在，设为 A

$\lim x_{n+1}=\lim\sqrt{x_n+2} \Rightarrow A=\sqrt{A+2}$　解得 $A=2$ 或 -1（舍）

(2) 求 $\lim 4^n(x_n-2)=[2\ln(2+\sqrt{3})]^2$