2020

1. (1) $(y_1\ y_2\ y_3) = (x_1\ x_2\ x_3)\,C$

即 $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} C = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ -1 & -1 & -1 \end{pmatrix}$, $AC = B$

$$(A|E) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right) = (E|A^{-1})$$

$\therefore$ $C = BA^{-1} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ -1 & -1 & -1 \end{pmatrix} \left(\cdot\frac{1}{2}\right) \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 4 & -4 & 1 \\ 5 & -6 & 1 \\ -1 & 0 & -1 \end{pmatrix}$

(2) T在基 $x_1 x_2 x_3$ 下的矩阵为 A, 则

$T(x_1 x_2 x_3) = (x_1\ x_2\ x_3)\,A$

$$T(x_1,x_2,x_3) = (y_1,y_2,y_3) = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} A = x \cdot C$$

可见过渡矩阵C与A相等. 设T在y基下矩阵为B;

$$T(y_1,y_2,y_3) = (y_1,y_2,y_3) B$$

$$T(x_1,x_2,x_3) C = T(x_1,x_2,x_3) \cdot B$$

$$\therefore C \cdot \cancel{\ } = \cancel{\ } B$$

$$\Rightarrow B = C \qquad \text{算出来…}$$

$$\therefore \cancel{\ }$$

$$= \frac{1}{4}\begin{pmatrix} 4 & -4 & 1 \\ 5 & -6 & 1 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & -4 & 1 \\ 5 & -6 & 1 \\ -1 & 0 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} +2 & 8 & -1 \\ -9 & 17 & -2 \\ -3 & 4 & 0 \end{pmatrix}$$

2.(1) 令 $Y = (y_{ij})_{2\times 2} \mid y_{11}+y_{22}=0$

$X+Y = (x_{ij}+y_{ij})_{2\times 2} \mid x_{11}+x_{22}$ 且 $y_{11}+y_{22}=0$

$x_{11}+y_{11}+x_{22}+y_{22} = x_{11}+x_{22}+y_{11}+y_{22}=0$

故 $X+Y \in V_1 \subset V$

$kX = (kx_{ij})_{2\times 2} \mid kx_{11}+kx_{22} = k(x_{11}+x_{22})=0$

故 $kX \in V_1 \subset V$

即 V_1 是 V 的线性子空间;

取一个基为: $\overset{A_1}{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}, \overset{A_2}{\begin{pmatrix}0&0\\1&0\end{pmatrix}}, \overset{A_3}{\begin{pmatrix}0&1\\0&0\end{pmatrix}}$, 三者线性无关,

任意一个矩阵 $(a_{ij})_{2\times2}$ 均可由以上三者线性组合。

且使 $a_{11}+a_{22}=0$

故 $A = a_{11}A_1 + a_{21}A_2 + a_{12}A_3$

$\therefore A_1\ A_2\ A_3$ 即 V 的一个基，且 $\dim V=3$.

3C、$|\lambda E-A| = \begin{vmatrix}\lambda-2&-2&2\\-2&\lambda-5&4\\2&4&\lambda-5\end{vmatrix} = \begin{vmatrix}\lambda-2&-2&2\\-2&\lambda-5&4\\0&\lambda-1&\lambda-1\end{vmatrix} = \begin{vmatrix}\lambda-2&-4&2\\-2&\lambda-9&4\\0&0&\lambda-1\end{vmatrix}$

$=(\lambda-1)\left[\lambda^2-11\lambda+18-8\right] = (\lambda-1)(\lambda^2-11\lambda+10) = (\lambda-1)^2(\lambda-10)$

当 $\lambda=1$ 时: $(1E-A) = \begin{pmatrix}-1&-2&2\\-2&-4&4\\2&4&-4\end{pmatrix} \to \begin{pmatrix}1&2&-2\\0&0&0\\0&0&0\end{pmatrix}$

$x_1=(-2\ 1\ 0)^T \quad x_2=(2\ 0\ 1)^T$

当 $\lambda=10$ 时 $(10E-A) = \begin{pmatrix}8&-2&2\\-2&5&4\\2&4&5\end{pmatrix} \to \begin{pmatrix}4&-1&1\\2&4&5\\-2&5&4\end{pmatrix} \to \begin{pmatrix}-2&4&5\\0&9&9\\0&9&9\end{pmatrix}$

$\to \begin{pmatrix}-2&4&5\\0&1&1\\0&0&0\end{pmatrix} \quad x_3 = (\frac{1}{2}\ -1\ 1)^T$

将 $x_1\, x_2$ 正交化

$y_1 = x_1 = (-2, 1, 0)^T$

$$y_2 = x_2 - \frac{(x_2, y_1)}{(y_1, y_1)} y_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{-4}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}^T = \begin{pmatrix} \frac{2}{5} \\ -\frac{4}{5} \\ 1 \end{pmatrix}$$

$y_3 = x_3$

分别单位化有：

$p_1 = \frac{1}{\sqrt{5}}(-2, 1, 0)^T \quad p_2 = \frac{\sqrt{5}}{3}\left(\frac{2}{5}, -\frac{4}{5}, 1\right)^T \quad p_3 = \sqrt{\frac{2}{3}}\left(\frac{1}{2}, -1, 1\right)$

则
$$P = \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{2\sqrt{5}}{15} & \frac{1}{2}\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{5}} & -\frac{4\sqrt{5}}{15} & -\sqrt{\frac{2}{3}} \\ 0 & \frac{\sqrt{5}}{3} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

令
$$P^{-1}AP = \Lambda = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

则
$$P^T A P = \Lambda$$

与A的相似对角化一样，而相对角化.

4. $|\lambda E - A| = \begin{vmatrix} \lambda+3 & -4 & -2 \\ 2 & \lambda-3 & -1 \\ 2 & -2 & \lambda-2 \end{vmatrix} = \begin{vmatrix} \lambda-1 & -4 & -2 \\ \lambda-1 & \lambda-3 & -1 \\ 0 & -2 & \lambda-2 \end{vmatrix} = \begin{vmatrix} \lambda-1 & -4 & -2 \\ 0 & \lambda+1 & 1 \\ 0 & -2 & \lambda-2 \end{vmatrix}$

$= (\lambda-1)(\lambda^2-\lambda-2+2) = (\lambda-1)(\lambda^2-\lambda) = \lambda(\lambda-1)^2$

令 $f(\lambda) = e^{t\lambda}$　且 $f(A) = a + bA + cA^2$

则 $f(0) = e^0 = a = 1$

$f(1) = e^{t\cdot 1} = a+b+c = 1+b+c$

$f'(\lambda)\big|_{\lambda=1} = t\,e^{t\lambda}\big|_{\lambda=1} = te^t = b+2cA\big|_{A=1} = b+2c$

$\therefore b = (2-t)e^t - 2, \quad c = (t-1)e^t$

$A^2 = \begin{pmatrix} -3 & 4 & 2 \\ -2 & 3 & 1 \\ -2 & 2 & 2 \end{pmatrix}\begin{pmatrix} -3 & 4 & 2 \\ -2 & 3 & 1 \\ -2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 4 & 2 \\ -2 & 3 & 1 \\ -2 & 2 & 2 \end{pmatrix} = A$

$\therefore f(A) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + (b+c)A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + (e^t-2)\begin{pmatrix} -3 & 4 & 2 \\ -2 & 3 & 1 \\ -2 & 2 & 2 \end{pmatrix}$

$= \begin{pmatrix} -3e^t+7 & 4(e^t-2) & 2(e^t-2) \\ -2(e^t-2) & 3e^t-5 & e^t-2 \\ -2(e^t-2) & 2(e^t-2) & -2e^t-3 \end{pmatrix}$

5、A中各列线性无关，记为 $a_1 a_2 a_3 a_4$，将其正交化

$b_1 = a_1 = (3\ 0\ 0\ 4)^T$

$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = a_2 - \frac{0}{\ } b_1 = a_2 + 0 b_1 = (0\ 6\ 0\ 0)^T$

$b_3 = a_3 - \frac{(a_3\ b_1)}{(b_1\ b_1)} b_1 - \frac{(a_3\ b_2)}{(b_2\ b_2)} b_2 = a_3 - \frac{11}{25} b_1 - \frac{12}{36} b_2 = (-\frac{8}{25}, 0, 0, \frac{6}{25})^T$

$b_4 = a_4 - \frac{(a_4\ b_1)}{(b_1\ b_1)} b_1 - \frac{(a_4\ b_2)}{(b_2, b_2)} b_2 - \frac{(a_4 b_3)}{(b_3 b_3)} b_3$

$= a_4 - \frac{-24}{25} b_1 - \frac{24}{36} b_2 - \frac{\frac{32}{25} - \frac{18}{25}}{\frac{102}{25^2}} b_3 = a_4 + \frac{24}{25} b_1 - \frac{2}{3} b_2 - \frac{7}{2} b_3$

$= (0, 0, 5, 0)^T$

再单位化

$q_1 = \frac{1}{\sqrt{25}} (3, 0, 0, 4)^T = \frac{1}{5} (3\ 0\ 0\ 4)^T = (\frac{3}{5}\ 0\ 0\ \frac{4}{5})^T$

$q_2 = \frac{1}{6} (0\ 6\ 0\ 0)^T = (0\ 1\ 0\ 0)^T$

$q_3 = \frac{5}{2} (-\frac{8}{25}, 0\ 0, \frac{6}{25}) = (\frac{-4}{5}, 0\ 0\ \frac{3}{5})^T$

$q_4 = \frac{1}{5} (0\ 0\ 5\ 0)^T = (0\ 0\ 1\ 0)^T$

则 $Q = \begin{pmatrix} \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \end{pmatrix}$ 且 $\begin{cases} a_1 = b_1 \\ a_2 = 0 b_1 + b_2 \\ a_3 = \frac{11}{25} b_1 + \frac{1}{3} b_2 + b_3 \\ a_4 = -\frac{24}{25} b_1 + \frac{2}{3} b_2 + \frac{7}{2} b_3 + b_4 \end{cases}$

则 $R=\begin{pmatrix} 5 & & & \\ & 6 & & \\ & & \frac{2}{5} & \\ & & & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & \cancel{\frac{11}{15}} & \cancel{\frac{26}{15}} \\ & \cancel{1} & \frac{1}{3} & \frac{2}{3} \\ & & 1 & \frac{7}{2} \\ & & & 1 \end{pmatrix}$

使得：$A=QR$

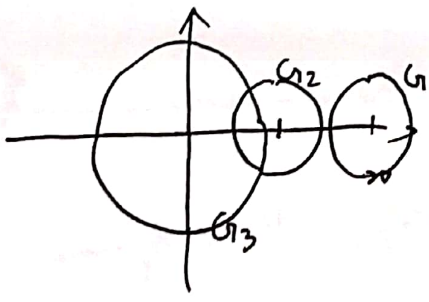
6、$|B|\neq 0$，B可逆　　$Ax=\lambda Bx$　　$B^{-1}Ax=\lambda x$

即为$B^{-1}A$的广义特征值

$(B|E)=\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 1 \end{array}\right)\to\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right)\to\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right)$

$B^{-1}A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 20 & 3 & 1 \\ 5 & 5.\cancel{8} & 1 \\ 4 & 0.5 & 0 \end{pmatrix}=\begin{pmatrix} 20 & 3 & 1 \\ 2 & 10 & 2 \\ 8 & 1 & 0 \end{pmatrix}$

$G_1: |z-20|\leq 4$　　$G_2: |z-10|\leq 4$　　$G_3: |z-0|\leq 9$



欲以缩小G_3，则令

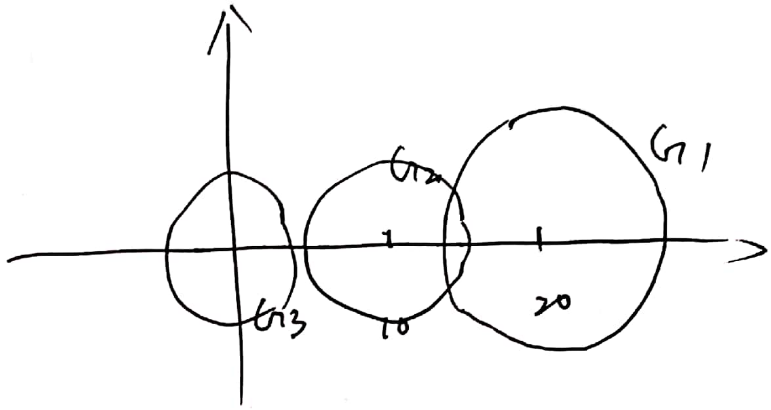
$D=\begin{pmatrix} 2 & & \\ & 1 & \\ & & 7 \end{pmatrix}$

$$B = DAD^{-1} = \begin{pmatrix} 20 & 6 & 2 \\ 1 & 10 & 4 \\ 4 & 0.5 & 0 \end{pmatrix}$$

$G_1: |z-20| \le 8$

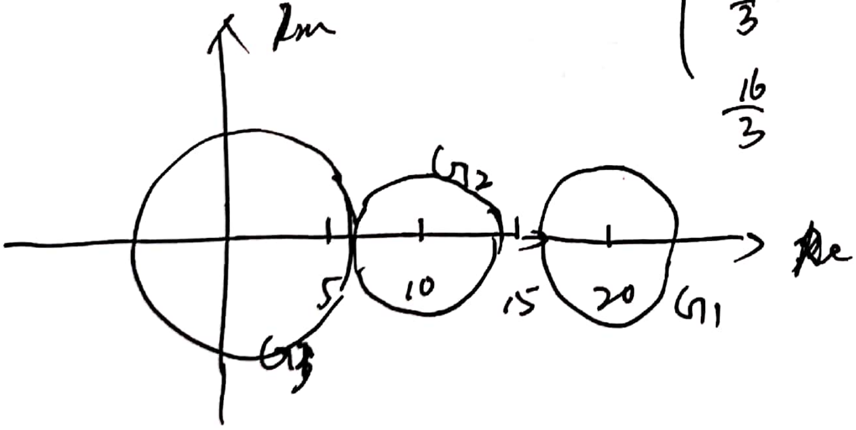
$G_2: |z-10| \le 5$

$G_3: |z-0| \le 4.5$



貌似还有问题

取 $D = \begin{pmatrix} \frac{3}{2} & & \\ & 1 & \\ & & 1 \end{pmatrix} \Rightarrow B = \begin{pmatrix} 20 & 4.5 & \frac{3}{2} \\ \frac{4}{3} & 10 & 3 \\ \frac{16}{3} & \frac{2}{3} & 0 \end{pmatrix}$



终于隔离了！

7. $\because A^H A = AA^H$，且 $A = (A^H A)^+ A^H$

$= A^H (AA^H)^+$

$\therefore$

$A^+A = (A^H A)^+ A^H A$

$= (AA^H)^+ AA^H$

$= (A^+A)^H$

$= AA^H((AA^H)^+)^H$

$= AA^H (AA^H)^+$

$= AA^+$

证毕；

8. $A = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 3 & 7 & -1 \end{pmatrix}$ $b = (0,1,2)^T$ $Ax = b$

$|A| = \begin{vmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 3 & 7 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 0$ A^{-1} 不存在，只能讨论

广义逆求解；

$A \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

令 $G=\begin{pmatrix}1&0&2\\0&1&-1\end{pmatrix}$ $F=\begin{pmatrix}1&3\\1&2\\3&7\end{pmatrix}$ 则 $A=FG$

$$A^{+}=G^{H}(F^{H}AG^{H})^{-1}F^{H}$$

$$F^{H}AG^{H}=\begin{pmatrix}1&1&3\\3&2&7\end{pmatrix}\begin{pmatrix}1&3&-1\\1&2&0\\3&7&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\2&-1\end{pmatrix}=\begin{pmatrix}6&28\\16&57\end{pmatrix}=B$$

$$(B|E)\rightarrow\left(\begin{array}{cc|cc}1&&\frac{1}{6}&\frac{14}{277}\\&1&0&-\frac{3}{277}\end{array}\right)$$

$$A^{+}=\begin{pmatrix}1&0\\0&1\\2&-1\end{pmatrix}\begin{pmatrix}\frac{1}{6}&\frac{14}{277}\\0&-\frac{3}{277}\end{pmatrix}\begin{pmatrix}1&1&3\\3&2&7\end{pmatrix}$$

问 ① $AA^{(1)}b=AA^{+}b=b=(0,1,2)^{T}$

故 $Ax=b$ 相容 $\Rightarrow$ $x=A^{(1,4)}b=A^{+}b$：极小范数解

② $AA^{(1)}b\neq b$ 则 $Ax=b$ 不相容，则

$x=A^{(1,3)}b=A^{+}b$：最小二乘解

9. 显然 $x = A^{(1,3)}b$

$$\varepsilon = Ax - b = AA^{(1,3)}b - b$$

$$\|\varepsilon\|_2^2 = \|b - AA^{(1,3)}b\|_2^2$$

$$= \|b\|_2^2 - 2\|b\|_2 \|AA^{(1,3)}b\|_2 + \|AA^{(1,3)}b\|_2^2$$

$$= \|b\|_2^2 - (2\|b\|_2^2 - I)\|AA^{(1,3)}b\|_2^2$$

$$= \|b\|_2^2 - (2\|b\|_2^2 - I)\|P_{R(A)}b\|_2^2$$

假如 $\|b\|_2^2 = I$，则

$$\|\varepsilon\|_2^2 = \|b\|_2^2 - \|P_{R(A)}b\|_2^2$$

证毕;

哪个大佬证一下？

扫描全能王 创建