Problem 1

一个质量为 m1 的小球被轻质硬杆链接在一个固定点上，质量为 m2 的小球被轻质硬杆链接在 m1 上，杆长分别为 l1，l2。链接处可活动，无摩擦。求解 2d 情况下的运动方程。并尝试探讨不同初态下的时间演化,以及 3d 的情况。

答：

令固定点为原点，用球坐标(l_1,θ_1,ϕ_1)表示 m1 位置，对应直角坐标为

$$\mathrm{r}_1=(l_1\sin\theta_1\cos\phi_1\,,l_1\sin\theta_1\sin\phi_1\,,l_1\cos\theta_1)$$

用球坐标(l_2,θ_2,ϕ_2)表示 m2 相对 m1 的位置，m2 在直角坐标下位置为

$$r_2=(l_1\sin\theta_1\cos\phi_1\,,l_1\sin\theta_1\sin\phi_1\,,l_1\cos\theta_1)+(l_2\sin\theta_2\cos\phi_2\,,l_2\sin\theta_2\sin\phi_2\,,l_2\cos\theta_2)$$

系统拉格朗日量为

$$\mathrm{L}=\frac{1}{2}m_1(\dot{r}_1)^2+\frac{1}{2}m_2(\dot{r}_2)^2-V$$

拉格朗日方程

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)=\frac{\partial L}{\partial q_i}$$

1). 对二维情况

令$\phi_1=\phi_2=0$

$$r_1=(l_1\sin\theta_1\,,l_1\cos\theta_1)$$
$$r_2=(l_1\sin\theta_1\,,l_1\cos\theta_1)+(l_2\sin\theta_2\,,l_2\cos\theta_2)$$

$$\dot{r}_1=\left(l_1\dot{\theta}_1\cos\theta_1\,,-l_1\dot{\theta}_1\sin\theta_1\right)$$
$$\dot{r}_2=\left(l_1\dot{\theta}_1\cos\theta_1\,,-l_1\dot{\theta}_1\sin\theta_1\right)+\left(l_2\dot{\theta}_2\cos\theta_2\,,-l_2\dot{\theta}_2\sin\theta_2\right)$$

若考虑重力势能，拉氏量和其偏导为

$$\mathrm{T}=\frac{1}{2}m_1l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2l_2^2\dot{\theta}_2^2+m_2l_1l_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)$$
$$\mathrm{V}=g(\mathrm{m}_1+m_2)l_1\cos\theta_1+g\mathrm{m}_2l_2\cos\theta_2$$
$$\mathrm{L}=\mathrm{T}-\mathrm{V}$$
$$\frac{\partial L}{\partial\dot{\theta}_1}=m_1l_1^2\dot{\theta}_1+m_2l_1^2\dot{\theta}_1+m_2l_1l_2\dot{\theta}_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)$$
$$=(m_1+m_2)l_1^2\dot{\theta}_1+m_2l_1l_2\dot{\theta}_2\cos(\theta_2-\theta_1)$$
$$\frac{\partial L}{\partial\theta_1}=m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)+g(\mathrm{m}_1+m_2)l_1\sin\theta_1$$
$$\frac{\partial L}{\partial\dot{\theta}_2}=m_2l_2^2\dot{\theta}_2+m_2l_1l_2\dot{\theta}_1\cos(\theta_2-\theta_1)$$
$$\frac{\partial L}{\partial\theta_2}=-m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)+g\mathrm{m}_2l_2\sin\theta_2$$

由$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)=\frac{\partial L}{\partial q_i}$

$$\left(\frac{m_1}{m_2}+1\right)\frac{l_1}{l_2}\ddot{\theta}_1\quad+\ddot{\theta}_2\cos(\theta_2-\theta_1)-\dot{\theta}_2^2\sin(\theta_2-\theta_1)=\frac{g}{l_2}\left(\frac{m_1}{m_2}+1\right)\sin\theta_1$$

$$\frac{l_2}{l_1}\ddot{\theta}_2+\ddot{\theta}_1\cos(\theta_2-\theta_1)+\dot{\theta}_1^2\sin(\theta_2-\theta_1)=\frac{g}{l_1}\sin\theta_2$$

令$\left(\frac{m_1}{m_2}+1\right)\frac{l_1}{l_2}=p,\frac{l_2}{l_1}=q,\theta_2-\theta_1=\theta_3$

$$p\ddot{\theta}_1+\ddot{\theta}_2\cos\theta_3=\dot{\theta}_2^2\sin\theta_3+\frac{g}{l_2}\left(\frac{m_1}{m_2}+1\right)\sin\theta_1$$

$$q\ddot{\theta}_2+\ddot{\theta}_1\cos\theta_3=-\dot{\theta}_1^2\sin\theta_3+\frac{g}{l_1}\sin\theta_2$$

$$\begin{bmatrix} p & \cos\theta_3 \\ \cos\theta_3 & q \end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}=\begin{bmatrix}\dot{\theta}_2^2\sin\theta_3+\frac{g}{l_2}\left(\frac{m_1}{m_2}+1\right)\sin\theta_1\\ -\dot{\theta}_1^2\sin\theta_3+\frac{g}{l_1}\sin\theta_2\end{bmatrix}$$

$$\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}=\frac{\sin\theta_3}{pq-\cos^2\theta_3}\begin{bmatrix} q & -\cos\theta_3 \\ -\cos\theta_3 & p \end{bmatrix}\begin{bmatrix}\dot{\theta}_2^2\sin\theta_3+\frac{g}{l_2}\left(\frac{m_1}{m_2}+1\right)\sin\theta_1\\ -\dot{\theta}_1^2\sin\theta_3+\frac{g}{l_1}\sin\theta_2\end{bmatrix}$$

令

$$X=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}\theta_1\\ \theta_2\\ \dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix} x_3 \\ x_4 \\ \frac{\sin(x_2-x_1)}{pq-\cos^2(x_2-x_1)}\begin{bmatrix} q & -\cos(x_2-x_1) \\ -\cos(x_2-x_1) & p \end{bmatrix}\begin{bmatrix}\dot{\theta}_2^2\sin\theta_3+\frac{g}{l_2}\left(\frac{m_1}{m_2}+1\right)\sin\theta_1\\ -\dot{\theta}_1^2\sin\theta_3+\frac{g}{l_1}\sin\theta_2\end{bmatrix}\end{bmatrix}$$

这个形式可用 ODE45 函数求解：

Matlab code:

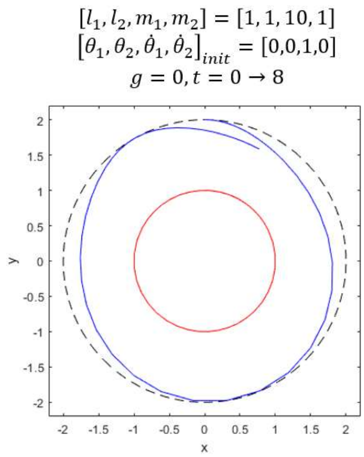
```
l1=1;
l2=1;
m1=1;
m2=1;
g=1;


t1_init=pi/2;
t2_init=pi/2;
dt1_init=0;
dt2_init=0;
```

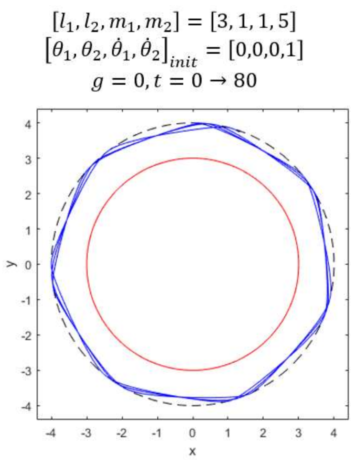
```
tspan=[0 60];
% tspan=linspace(0,5,51);
% tspan=0:.05:5;
init=[t1_init;t2_init;dt1_init;dt2_init];


p=(m1/m2+1)*l1/l2;
q=l2/l1;
lm=(l1+l2);
p1=(m1/m2+1)*g/l2;
p2=g/l1;


func=@(t,x)[x(3);x(4);1/(p*q-cos(x(2)-x(1))^2)*...
    [q -cos(x(2)-x(1));-cos(x(2)-x(1)) p]*...
    [sin(x(2)-x(1))*x(4)^2+p1*sin(x(1));-sin(x(2)-
x(1))*x(3)^2+p2*sin(x(2))]];


[T,Y]=ode45(func,tspan,init);


theta1=Y(:,1);
theta2=Y(:,2);


x1=l1*sin(theta1);
y1=l1*cos(theta1);
x2=x1+l2*sin(theta2);
y2=y1+l2*cos(theta2);


figure();
hold on;


ang=linspace(0,2*pi,100);
plot(lm*sin(ang),lm*cos(ang),'k--');


plot(x1,y1,'r');
plot(x2,y2,'b');


% plot([x1*0 x1]',[y1*0 y1]','r','linewidth',.5);
% plot([x1 x2]',[y1 y2]','b','linewidth',.5);
```

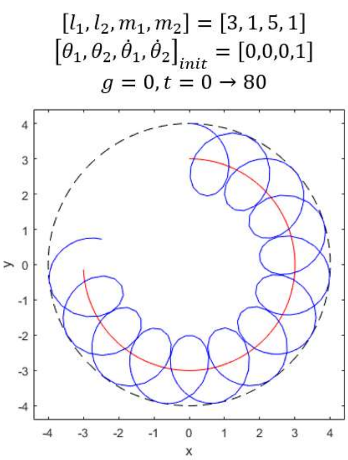
```
hold off;



axis(1.1*[-lm lm -lm lm]);
axis square;
box on;
xlabel('x');ylabel('y');
```

以下为部分结果，红线为 m1 轨迹，蓝线为 m2 轨迹，

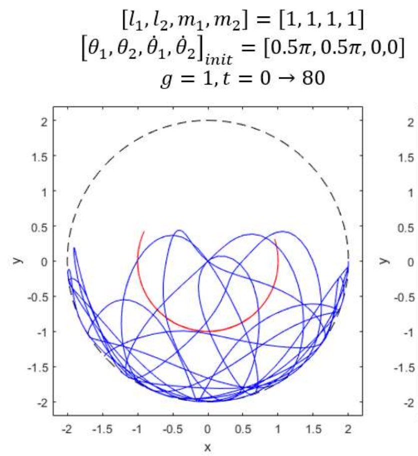
无重力时（g=0）：

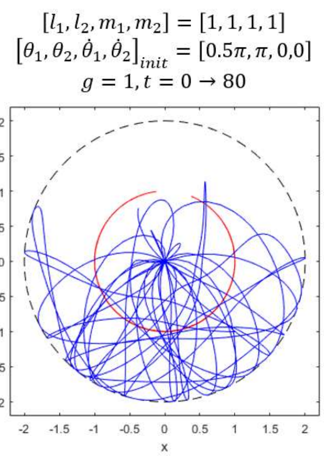


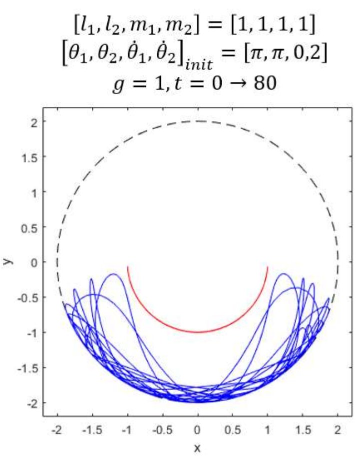




有重力时（g=1）：







2) 对三维情况

3D 情况下在球坐标下求解形式太复杂了，这里改用直角坐标下的拉格朗日乘子法求解

令两球的坐标为$\mathrm{r}_1, \mathrm{r}_2$

$$\mathrm{r}_1 = (x_1, y_1, z_1)$$
$$r_{12} = (x_2, y_2, z_2)$$
$$\mathrm{r}_2 = r_1 + r_{12}$$

拉格朗日量为

$$\mathrm{T} == \frac{1}{2}(m_1 + m_2)(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) + m_2(\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 + \dot{z}_1\dot{z}_2)$$
$$\mathrm{V} = g(m_1 + m_2)z_1 + gm_2z_2$$
$$\mathrm{L} = \mathrm{T} - \mathrm{V}$$

约束为

$$F_1 = \mathrm{x}_1^2 + y_1^2 + z_1^2 - l_1^2 = 0$$
$$F_2 = \mathrm{x}_2^2 + y_2^2 + z_2^2 - l_2^2 = 0$$

由拉格朗日乘子法

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \lambda_1\frac{\partial F_1}{\partial q_i} + \lambda_2\frac{\partial F_2}{\partial q_i}$$

$$(m_1 + m_2)\ddot{x}_1 + m_2\ddot{x}_2 = 2\lambda_1 x_1$$
$$(m_1 + m_2)\ddot{y}_1 + m_2\ddot{y}_2 = 2\lambda_1 y_1$$
$$(m_1 + m_2)\ddot{z}_1 + m_2\ddot{z}_2 + g(m_1 + m_2) = 2\lambda_1 z_1$$
$$m_2\ddot{x}_2 + m_2\ddot{x}_1 = 2\lambda_2 x_2$$
$$m_2\ddot{y}_2 + m_2\ddot{y}_1 = 2\lambda_2 y_2$$
$$m_2\ddot{z}_2 + m_2\ddot{z}_1 + gm_2 = 2\lambda_2 z_2$$

令$p = \left(\frac{m_1}{m_2} + 1\right), \lambda_3 = \frac{2\lambda_1}{m_2}, \lambda_4 = \frac{2\lambda_2}{m_2}$

$$p\ddot{x}_1 + \ddot{x}_2 = \lambda_3 x_1$$
$$p\ddot{y}_1 + \ddot{y}_2 = \lambda_3 y_1$$
$$p\ddot{z}_1 + \ddot{z}_2 + gp = \lambda_3 z_1$$
$$\ddot{x}_2 + \ddot{x}_1 = \lambda_4 x_2$$
$$\ddot{y}_2 + \ddot{y}_1 = \lambda_4 y_2$$
$$\ddot{z}_2 + \ddot{z}_1 + g = \lambda_4 z_2$$

对 F1,F2 求导得到约束

$$x_1\ddot{x}_1 + \dot{x}_1^2 + y_1\ddot{y}_1 + \dot{y}_1^2 + z_1\ddot{z}_1 + \dot{z}_1^2 = 0$$
$$x_2\ddot{x}_2 + \dot{x}_2^2 + y_2\ddot{y}_2 + \dot{y}_2^2 + z_2\ddot{z}_2 + \dot{z}_2^2 = 0$$

联立得到

$$\begin{bmatrix} p & & & 1 & & & -x_1 & \\ & p & & & 1 & & -y_1 & \\ & & p & & & 1 & -z_1 & \\ 1 & & & 1 & & & & -x_2 \\ & 1 & & & 1 & & & -y_2 \\ & & 1 & & & 1 & & -z_2 \\ x_1 & y_1 & z_1 & & & & & \\ & & & x_2 & y_2 & z_2 & & \end{bmatrix} \times \frac{d}{dt}\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \\ \int \lambda_3 \\ \int \lambda_4 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ gp \\ 0 \\ 0 \\ g \\ \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 \\ \dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2 \end{bmatrix}$$

这个形式稍作处理后可用 ODE45 函数求解：

Matlab code:

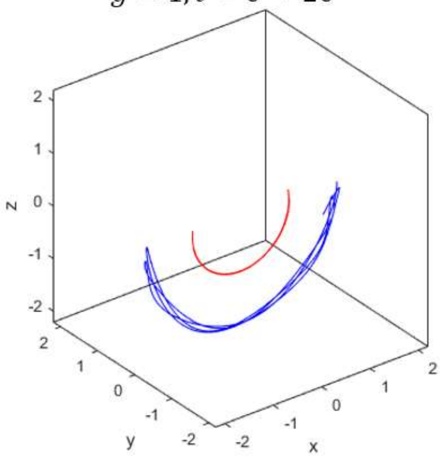
```
l1=1;
l2=1;
m1=1;
m2=1;
g=1;


x1_init=1*l1;
y1_init=0*l1;
z1_init=sqrt(l1^2-x1_init^2-y1_init^2);
x2_init=0*l2;
y2_init=1*l2;
z2_init=sqrt(l2^2-x2_init^2-y2_init^2);


dx1_init=0;
dy1_init=0;
dz1_init=0;
dx2_init=0;
dy2_init=0;
dz2_init=0;


tspan=[0 20];
% tspan=linspace(0,5,51);
% tspan=0:.05:5;
init=[x1_init;y1_init;z1_init;x2_init;y2_init;z2_init;...
    dx1_init;dy1_init;dz1_init;dx2_init;dy2_init;dz2_init;0;0];


p=(m1/m2+1);
lm=l1+l2;


func3DD=@(t,x)func3D(t,x,p,g);
[T,Y]=ode45(func3DD,tspan,init);


x1=Y(:,1);
y1=Y(:,2);
z1=Y(:,3);
x2=Y(:,1)+Y(:,4);
y2=Y(:,2)+Y(:,5);
```

```
z2=Y(:,3)+Y(:,6);


figure();
hold on;
plot3(x1,y1,z1,'r');
plot3(x2,y2,z2,'b');
% plot3([x1*0 x1]',[y1*0 y1]',[z1*0 z1]','r','linewidth',.5);
% plot3([x1 x2]',[y1 y2]',[z1 z2]','b','linewidth',.5);
hold off;
axis(1.1*[-lm lm -lm lm -lm lm]);
axis square;
box on;
view(3);
xlabel('x');ylabel('y');zlabel('z');


function [y] = func3D(t,x,p,g)
r1=[x(1);x(2);x(3)];
r2=[x(4);x(5);x(6)];
dr1=[x(7);x(8);x(9)];
dr2=[x(10);x(11);x(12)];
lmb1=x(13);
lmb2=x(14);


A11=[p*eye(3) eye(3);
    eye(3) eye(3)];
A12=-[r1 zeros(3,1);zeros(3,1) r2];
A=[A11 A12;-A12' zeros(2,2)];
B=-[0;0;g*p;0;0;g;sum(dr1.^2);sum(dr2.^2)];


y=zeros(14,1);
y(1:6)=x(7:12);
y(7:14)=A^(-1)*B;
end
```

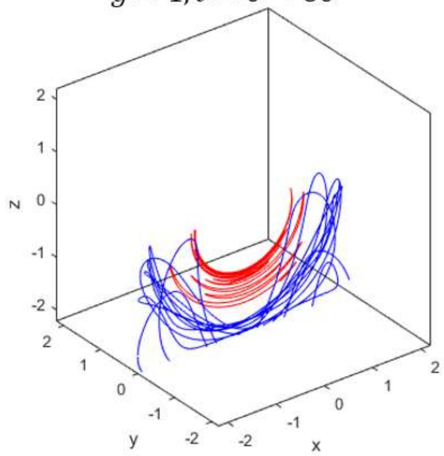
以下为部分结果，红线为 m1 轨迹，蓝线为 m2 轨迹，取重力 g=1

考虑初始位置在 XOZ 平面，初速度为 0，双摆应始终在 XOZ 平面内运动，模型实际上是 2D 的。从结果上看 20s 内结果与(1)中 2D 模型接近，80s 时的轨迹出现较大偏差。也许是因为用到了更多维度，拉格朗日乘子法得到的方程误差积累较快。

$[l_1, l_2, m_1, m_2] = [1, 1, 1, 1]$
$[r_1, r_{12}]_{init} = [1,0,0,1,0,0]$
$\frac{d}{dt}[r_1, r_{12}]_{init} = [0,0,0,0,0,0]$
$g = 1, t = 0 \rightarrow 20$

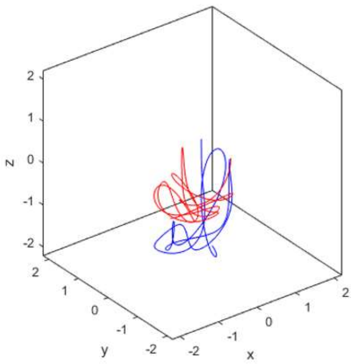


$[l_1, l_2, m_1, m_2] = [1, 1, 1, 1]$
$[r_1, r_{12}]_{init} = [1,0,0,1,0,0]$
$\frac{d}{dt}[r_1, r_{12}]_{init} = [0,0,0,0,0,0]$
$g = 1, t = 0 \rightarrow 80$

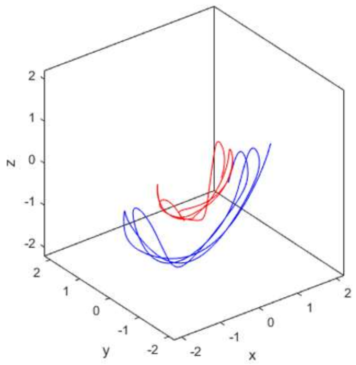


以下是一些在三维空间中运动的轨迹

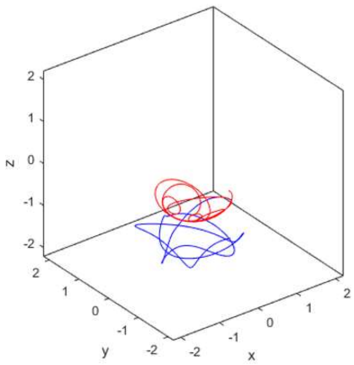
$[l_1, l_2, m_1, m_2] = [1, 1, 1, 1]$
$[r_1, r_{12}]_{init} = [1,0,0,0,1,0]$
$\frac{d}{dt}[r_1, r_{12}]_{init} = [0,0,0,0,0,0]$
$g = 1, t = 0 \rightarrow 20$



$[l_1, l_2, m_1, m_2] = [1, 1, 1, 1]$
$[r_1, r_{12}]_{init} = [1,0,0,1,0,0]$
$\frac{d}{dt}[r_1, r_{12}]_{init} = [0,0,0,0,.1,0]$
$g = 1, t = 0 \rightarrow 20$



$[l_1, l_2, m_1, m_2] = [1, 1, 1, 1]$
$[r_1, r_{12}]_{init} = [0,0,-1,1,0,0]$
$\frac{d}{dt}[r_1, r_{12}]_{init} = [0,0,0,0,1,0]$
$g = 1, t = 0 \rightarrow 20$



Problem 2

求解扩散方程：

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

数值解与严格解作比对，体会空间时间间隔对精度的影响。

答：

考虑初值为 delta 函数的演化

$$u(x, t = 0) = \delta(x)$$

其严格解为

$$u(x,t) = \left(\frac{1}{4\pi a^2 t}\right)^{1/2} \exp\left(-\frac{x^2}{4a^2 t}\right)$$

将函数离散化，令$u_\mathrm{n} = u(nh), n = -\frac{L}{2h}, \dots, \frac{L}{2h}$，方程为

$$u(t+\delta t) = \left(I + \frac{a^2 \delta t}{h^2}\begin{bmatrix} -2 & 1 & \dots \\ 1 & -2 & \\ \dots & & \dots \end{bmatrix}\right) u = Tu$$

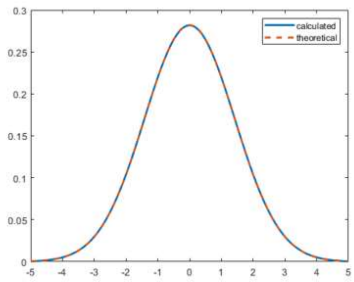
delta 函数对应的初值为

$$\mathrm{u_i} = \frac{1}{h}\delta(i, 0)$$

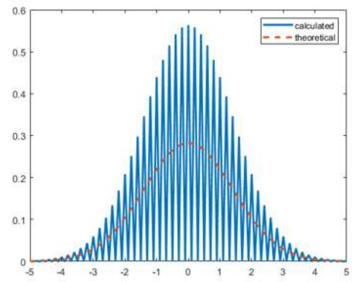
令 a=1, L=10, 比较 t=1 时数值解与严格解的偏差

以下为$\frac{a^2\delta t}{2h^2} = 0.8, 1, 1.2$时的解，可见要让解有较高精度，应该保证$\frac{a^2\delta t}{2h^2} < 1$，即 T 正定

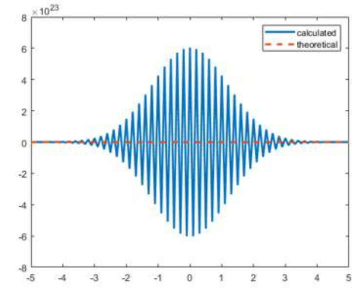
$\delta t = 0.004, \mathrm{h} = .1$



$\delta t = 0.005, \mathrm{h} = .1$



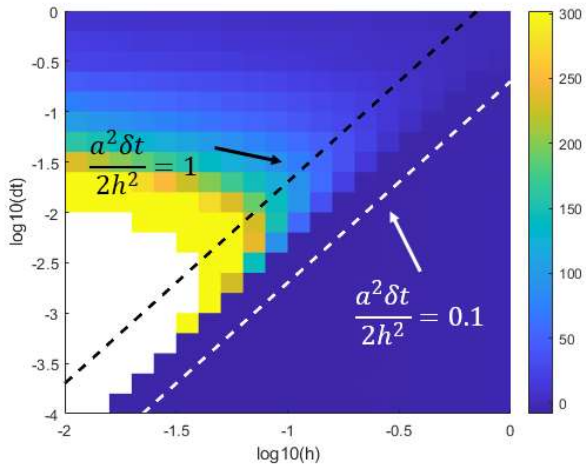
$\delta t = 0.006, \mathrm{h} = .1$



定义偏差为

$$\mathrm{ERR} = \log_{10}\left(\frac{1}{N}\sum\left(u_i - u(nh)\right)^2\right)$$

偏差 ERR 随δt（取 0.0001~1）和 h（取 0.01~1）变化情况如下图



可见取$\frac{a^2\delta t}{2h^2} = 0.1$足够保证精度