The existence for solution of the minimization problem

Based on the proof in [1], the existence for solution the minimization problem is given as follows.

First, for proof's sake, a tiny value Δ is added with the observed image S to avoid it equals to 0 when S in the denominator. Note that S is actually $S = S + \Delta$ and this operation does not affect the final result.

Let S is defined on Ω. The solution set is defined as:

$$\Lambda = \left\{ (R, I) \,\middle|\, (R, I) \in BV(\Omega) \times W^{1,2}(\Omega), S \leq I \right\}.$$

The energy minimization problem is:

$$\min_{(R,I)\in\Lambda} E(R, I) = \min_{(R,I)\in\Lambda} \|R \cdot I - S\|_2^2 + \alpha \|\partial I\|_2^2 + \beta\|\partial R\|_1 + \gamma \|I - I_0\|_2^2 \quad s.t. \;\; S \leq I. \tag{A-1}$$

Theorem: Let $S \in L^{\infty}(\Omega)$, the problem (A-1) has at least one solution.

Proof. Let I and R be constants, the energy will be finite. Assume (R_t, I_t) is a minimizing sequence of problem (A-1), then a constant $M > 0$ exists that

$$E(R_t, I_t) \leq M.$$

This inequality can be written as:

$$\|R_t \cdot I_t - S\|_2^2 + \alpha \|\partial I_t\|_2^2 + \beta\|\partial R_t\|_1 + \gamma \|I_t - I_0\|_2^2 \leq \mathrm{M}.$$

The boundedness of $\|\partial I_t\|_2^2$ and $\|I_t - I_0\|_2^2$ guarantees that $\{I_t\}$ is uniformly bounded in $W^{1,2}(\Omega)$. Note that $W^{1,2}(\Omega)$ is embedded in $L^2(\Omega)$, deducing that up to a subsequence, $\{I_t\}$ converges to $I_* \in W^{1,2}(\Omega)$, i.e.,

$$I_t \xrightarrow[L^2(\Omega)]{} I_* \text{ and } I_t \rightharpoonup I_* \in W^{1,2}(\Omega). \tag{A-2}$$

Meanwhile, the sequence $\{R_t\}$ satisfies

$$\beta\|\partial R_t\|_1 \leq \mathrm{M}.$$

and

$$\|R_t \cdot I_t - S\|_2^2 \leq M.$$

Note that $I_t \geq S$ and S is the observed image thus can be seen as constant in every pixel, we have

$$\begin{aligned}
\|R_t\|_2^2 &= \left\| R_t - \tfrac{S}{I_t} + \tfrac{S}{I_t} \right\|_2^2 \\
&= \left\| \tfrac{1}{I_t}(R_t \cdot I_t - S + S) \right\|_2^2 \\
&\leq \left\| \tfrac{1}{S}(R_t \cdot I_t - S + S) \right\|_2^2 \\
&\leq \left\| \tfrac{1}{S} \right\|_2^2 \{ \|(R_t \cdot I_t - S)\|_2^2 + \|S\|_2^2 \}.
\end{aligned}$$

Because $S \in L^{\infty}(\Omega)$ and S is actually not equal to 0 as described before, both $\left\|\frac{1}{S}\right\|_2^2$ and $\|S\|_2^2$ are upper boundedness. Meanwhile $\|R_t \cdot I_t - S\|_2^2 \leq M$, we can deduce that $\{R_t\}$ is uniformly bounded in $L^2(\Omega)$, which means in $L^1(\Omega)$. Combining it with the boundedness of TV, $\{R_t\}$ is uniformly bounded in $BV(\Omega)$. Therefore, $R_* \in BV(\Omega)$ such that, up to a subsequence,

$$R_t \xrightarrow[L^1(\Omega)]{} R_* \text{ and } R_t \rightharpoonup R_* \in L^2(\Omega). \tag{A-3}$$

Note that (A-2) holds for I_t, which corresponds to R_t; therefore, deducing that, up to a subsequence, $\{(R_t, I_t)\}$ satisfies (A-2) and (A-3). As a consequence of the lower semicontinuity for the $W^{1,2}(\Omega)$ norm,

$$\liminf_{t\to\infty} \left(\alpha \|\partial I_t\|_2^2 + \gamma \|I_t - I_0\|_2^2 \right) \geq \alpha \|\partial I_*\|_2^2 + \gamma \|I_* - I_0\|_2^2.$$

Since $R_t I_t \to R_* I_*$ in $L^2(\Omega)$ and recalling the lower semicontinuity for the $L^2(\Omega)$ norm, we have

$$\liminf_{t\to\infty} \|R_t \cdot I_t - S\|_2^2 \geq \|R_* \cdot I_* - S\|_2^2.$$

Noting the lower semicontinuity of $BV(\Omega)$

$$\liminf_{t\to\infty} \beta\|\partial R_t\|_1 \geq \beta\|\partial R_*\|_1,$$

we have

$$\min_{(R,I)\in\Lambda} E(R, I) = \liminf_{t\to\infty} E(R_t, I_t) \geq E(R_*, I_*).$$

Meanwhile, $I_* \geq S$, the proof is completed.

References

[1] M. K. Ng and W. Wang, "A total variation model for retinex", *SIAM Journal on Imaging Sciences*, vol. 4, no. 1, pp. 345-365, 2011.