Exact Solution of a Three Dimensional Hyperbolic Positioning System

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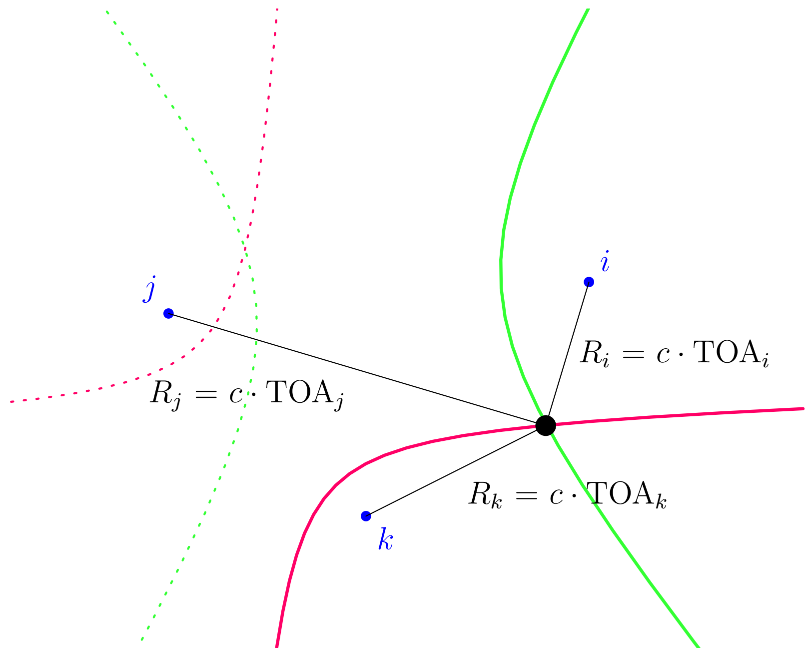
1 Introduction

Multilateration is a method of determining the position of an object of interest. It could be used in systems like Global Positional System (GPS) or E911 (the FCC mandated requirement to locate wireless phones in emergencies).

Given four satellites i, j, k, and l at positions $\langle x_i, y_i, z_i \rangle$, etc., in three-dimensional space, the problem is to find an unknown position $\langle x, y, z \rangle$ given the time t_i, t_j, t_k, and t_l, in nanoseconds it takes to reach the satellites. Since the speed of light (exactly 299 792 458 meters/second) is known, knowing the time is the same as knowing the distance.

TOA (time of arrival) is the time it takes the signal to travel from the transmitter to the receiver. TDOA (time-difference of arrival) is the difference in time it takes signals from two different transmitters to reach the receiver. With four satellites there are three (independent) TDOAs. The advantage of an approach using TDOA is that the difference between the arrival times can be determined by an accurate clock which need not be synchronized with the other clocks. (The satellites still have to send the signal in the same frame of reference, so they have to be synchronized.)

The locus of points whose difference in distance from two fixed points (the foci) is the constant c·TDOA forms a hyperbola. The fundamental mathematical principle in determining the position of the target is the intersections of hyperbolas. This is illustrated below in the two-dimensional case.



In two-dimensional space, given the fixed locations $\langle x_i, y_i \rangle$, $\langle x_j, y_j \rangle$ and the TDOA between them, called TDOA_{ij}, the set of all points $\langle x, y \rangle$ satisfying the data is given by the equation for the hyperbola:

$$R_i - R_j = \sqrt{(x - x_i)^2 + (y - y_i)^2} - \sqrt{(x - x_j)^2 + (y - y_j)^2} = c \cdot \mathrm{TDOA}_{ij}$$

This set of points is represented by the green curves in the previous diagram (point i and j are the foci).

This set of points is represented by the red curves is the hyperbola with foci j and k. The intersection of the two hyperbola results in two distinct points on the two-dimensional plane. The object has now been located (to one of two possible locations).

We now describe the solution of hyperbolic positioning system in three dimensions.

2 Preliminary Definitions

Given the x, y, z coordinates of the four satellites i, j, k, and l, let $x_{ij} = x_i - x_j$, $y_{jk} = y_j - y_k$ etc. Let R_i, R_j, R_k, and R_l be the distances from the satellites to the target. These four positions and the four distances are the inputs.

For notational and programming convenience we need a large number of additional definitions. Let R_{ij} be the differences $R_i - R_j$, R_{ik} be $R_i - R_k$, and so on. We use only four of the six possible differences: R_{ij}, R_{ik}, $R_{k,j}$, and R_{kl}.

	i	j	k	l
i		✓	✓	✗
j	–		–	✗
k	–	✓		✓
l	✗	✗	–	

It is just as good to know the four corresonding TDOAs as from them the differences R_{ij}, R_{ik}, and so on, can be computed using the speed of light.

We define the next set of quantities using the differences.

$$X_{ijy} = R_{ij} y_{ki} - R_{ik} y_{ji} \tag{1}$$
$$X_{ikx} = R_{ik} x_{ji} - R_{ij} x_{ki} \tag{2}$$
$$X_{ikz} = R_{ik} z_{ji} - R_{ij} z_{ki} \tag{3}$$
$$X_{kjy} = R_{kj} y_{lk} - R_{kl} y_{jk} \tag{4}$$
$$X_{klx} = R_{kl} x_{jk} - R_{kj} x_{lk} \tag{5}$$
$$X_{klz} = R_{kl} z_{jk} - R_{kj} z_{lk} \tag{6}$$

These definitions are used in equations (15) through (20).

The next four definitions:

$$Si2 = x_i^2 + y_i^2 + z_i^2 \tag{7}$$
$$Sj2 = x_j^2 + y_j^2 + z_j^2 \tag{8}$$
$$Sk2 = x_k^2 + y_k^2 + z_k^2 \tag{9}$$
$$Sl2 = x_l^2 + y_l^2 + z_l^2 \tag{10}$$

are used to define the next quantities:

$$Rij2xyz = R_{ij}^2 + Si2 - Sj2 \tag{11}$$
$$Rik2xyz = R_{ik}^2 + Si2 - Sk2 \tag{12}$$
$$Rkj2xyz = R_{kj}^2 + Sk2 - Sj2 \tag{13}$$
$$Rkl2xyz = R_{kl}^2 + Sk2 - Sl2 \tag{14}$$

3 Equations

Define A, B, C, and D by the following equations.

$$X_{ijy}A = X_{ikx} \tag{15}$$
$$X_{ijy}B = X_{ikz} \tag{16}$$
$$X_{kjy}C = X_{klx} \tag{17}$$
$$X_{kjy}D = X_{klz} \tag{18}$$

Define E, and F by the following equations.

$$2XijyE = R_{ik}Rij2xyz - R_{ij}Rik2xyz \tag{19}$$

$$2XkjyF = R_{kl}Rkj2xyz - R_{kj}Rkl2xyz \tag{20}$$

Define G and H by the following equations.

$$(A - C)G = (D - B) \tag{21}$$

$$(A - C)H = (F - E) \tag{22}$$

Define I and J by the following:

$$I = AG + B \tag{23}$$

$$J = AH + E \tag{24}$$

Define K and L by the following equations.

$$K = Rik2xyz + 2x_{ki}H + 2y_{ki}J \tag{25}$$

$$L = 2(x_{ki}G + y_{ki}I + z_{ki}) \tag{26}$$

Define M, N, and O by the following:

$$M = 4R_{ik}^2[G^2 + I^2 + 1] - L^2 \tag{27}$$

$$N = 8R_{ik}^2[G(x_i - H) + I(y_i - J) + z_i] + 2LK \tag{28}$$

$$O = 4R_{ik}^2[(x_i - H)^2 + (y_i - J)^2 + z_i^2] - K^2 \tag{29}$$

The roots of the quadratic equation $Mz^2 - Nz + O = 0$ are the z coordinate of the target. The numerically best way to compute the roots is this way:

$$Q = N + \operatorname{sgn}(N)\sqrt{N^2 - 4MO} \tag{30}$$

The sign function extracts the sign of a real number:

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

Now we compute the two possible locations of the object starting with the two possible values of the z coordinate. The x and y each are computed depending on which z coordinate is choosen.

$$z_1 = Q/2M \quad \text{or} \quad z_2 = 2O/Q \tag{31}$$

$$x = Gz + H \tag{32}$$

$$y = Iz + J \tag{33}$$

Given only the difference in arrival times of the satellite signals, it is not possible to know which solution is correct. If, for example, it may be assumed that the position is near the surface of the earth, then that can be used to decide which position is the true one.

4 Example

Suppose the locations of four satellites in three-dimensional space are the following:

```
15562569.98 -10009671.82  8102646.83  meters (Satellite i)
 4895787.68  18128508.45 -7445028.17  meters (Satellite j)
 7344421.11 -15419735.80 10785695.45  meters (Satellite k)
18265492.16   1545680.89 -8486616.93  meters (Satellite l)
```

Suppose the arrival times as measured in some highly precise time scale are the following:

```
13520053.4651 0.0000 19267255.6042 13292290.4092 (nanoseconds)
```

The intermediate values are:

```
Rij, Rik, Rkj, Rkl: 4.05e+06  -1.72e+06   5.78e+06   1.79e+06
a=  1.95e+00, b=  5.99e-01
c= -1.78e+00, d=  2.08e+00
e= -7.60e+05, f=  5.44e+05
g=  3.96e-01, h=  3.50e+05, i=  1.37e+00, j= -7.87e+04
k= -1.93e+12, l= -1.60e+07
m= -2.19e+14, n=  7.39e+19, o=  4.70e+27
```

Then the two possible positions of the object to the nearest meter are:

```
x=   -1552077, y=   -6658281, z=   -4801313
x=    2117912, y=    6038289, z=    4463760
```

References

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