Giving Queueing the Booth

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Contents

1 Introduction

Delays at tolling systems are ubiquitous in major road networks throughout the western world. In fact traffic has created problems for urban centers of civilization throughout the ages. "Vehicular traffic, except for chariots and official vehicles, was prohibited from entering Rome during the hours of daylight" for a period in the first century [3, Page 3]. "At least one (Roman) emperor was forced to issue a proclamation threatening the death penalty to those whose chariots and carts blocked the way" of traffic [1, Section 1.2] Most toll booth systems involve a major highway with a certain number of lanes, which suddenly increases to a larger number of lanes. Each of these lanes then passes through a toll booth where a tolling system charges the motorist the required toll. Traditionally this has involved the exchange of actual currency, which delayed traffic, although modern electronic payment systems are eliminating this feature (see [15]). The roadway then squeezes back to the initial number of lanes at the end of the toll area. Increasing the number of booths so that the ratio of booths to initial lanes is much larger than 1 will result in delays at the actual booth being virtually eliminated, however as the large flow of traffic is squeezed back down to the initial number of lanes results in major congestion, especially at "rush hour". If the number of toll booths equals the initial number of lanes, then there is no squeezing of traffic, however large tailback occur at the toll booths themselves. The challenge (faced by traffic authorities throughout the western world) is to find an intermediate value that minimizes congestion. Our aim was that the delay time for a rush hour trip through our system in comparison with a trip through with no traffic be minimized

1.1 Assumptions

We made a number of assumptions in order to simplify our system and allow us to construct our model.

- We assume all vehicles entering our system are identical. We ignore differences between cars, vans, trucks, lorries, etc.
- We assume that all vehicles move at the same speed. We assume traffic is slowed in the toll plaza relative to the previous section of road, perhaps by imposing a lower speed limit speed limit. Of course in this scenario we now have to assume no-one breaks the speed limit

- We assume that the vehicles are evenly distributed throughout the lanes, i.e. there is no preference for any lane.

- We allow vehicles to be in one of two states, either moving at a fixed speed or stopped.

- If the number of vehicles arriving at a particular section (namely plaza, toll booths or squeeze point) of our system is greater than the maximum capacity of our system, then that capacity of vehicles pass through, and the remainder form a "virtual queue" based on a "first-come, first-served" basis.(It should be noted that in this report plaza will refer to the wide section before the toll booth and the whole area to be modelled referred to as the system). If vehicles arrive at a section and a queue has already formed, those vehicles join the back of the queue. This is essentially, in the nomenclature of queueing theory, a blocked customers delayed system with parallel servers, see [10, Chap. 3] and [7, Pg. 38] respectively

- All toll booths are identical. In reality almost all modern toll roads employ an electronic system that allows motorists to pass through without stopping, e.g. [15]. we are also ignoring the presence of specialist lanes, e.g. lanes devoted exclusively to heavy goods vehicles.

- Our toll plazas are of large enough length that congestion at the squeeze point does not back up as far as the toll booths, which would delay passage of vehicles through the booths and similarly congestion at the toll booths does not affect traffic entering the plaza.

- We divide the lanes into discrete intervals of constant length. This length is the length of the vehicle plus the length between successive vehicles, which we assume to be constant. We then model vehicles to be points moving between successive intervals. Since we have assumed constant vehicle speed this means that there is a characteristic time, θ, associated with the passage of vehicles from one interval to the next.

- We assume all vehicles arriving at an empty toll booth pass through in the same time, τ. We ignore all possible technical malfunctions, customer delay, etc. For ease of modelling we assume τ is an integer multiple of θ, $\tau = p\theta$, $p > 1$

- We assume that if a vehicle arrives at the toll booths and there is a free booth the vehicle will pass through that booth.

- We ignore all external factors other than the incoming traffic for the toll plaza. For example during the transition from domestic currencies to the single European currency, most European toll booths experienced delays, see, e.g. [14]. We ignore these such factors.

- We assumed the traffic flow would have the general shape shown in Figure 1.1 . This is a justified assumption for a radial route in an

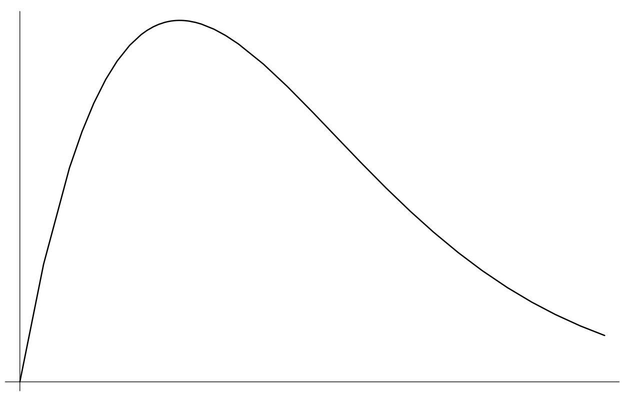


Figure 1: The general trend for traffic flow

urban centre, see [1, Chap. 6]. The accepted time span for the peak in Figure is of the order 2 hours and we will assume it is exactly 2 hours when considering explicit examples in the results section.

- In order to measure congestion we decided to measure the average delay time of the vehicles who arrive in the system during the period of congestion. In other words we decided to measure the extra time spent waiting during congestion compared to the time it would take to pass through the system if there was absolutely no other traffic present, and average this over all vehicles entering the system while there is congestion. We defined an optimal system to be one which minimizes this average delay time.

2 Model

We began by considering a function $h(t)$ that gives the instantaneous rate of traffic incoming on our system.

While this continuous approach may seem to contradict our assumptions that give a discrete nature to our approach, this issue will be addressed shortly. We then multiplied our function by $1 + \gamma$ where γ is a random real number between -0.01 and 0.01. This gives us a function $f(t)$ which contains an inherent randomness which we believe is an integral part of traffic flow. We now consider κ, the maximum rate at which cars can leave our system. By the definition of θ this will be

$$\kappa = \frac{n}{\theta}$$

where n is the number of lanes after the squeeze point (and before the toll plaza). Analogously we will have the maximum rate at which vehicles can pass through the toll booth to be

$$\lambda = \frac{m}{\tau}$$

where m is the number of toll booths. Since we are assuming the same speed and lane-interval length in all parts of our system we will have a maximum rate in the plaza of

$$\mu = \frac{m}{\theta}$$

. If we denote by r the peak of our function $h(t)$ then the behavior of our system can be characterized by the relative values of r, κ, λ and μ. We note that for any traffic approaching our system the max rate that will ever approach the squeeze point is λ since the toll booths will delay any traffic flowing at a greater rate. This means that in order for congestion to occur at the squeeze point in our system we require

$$\kappa < \lambda$$

Equivalently

$$\frac{n}{\theta} < \frac{m}{\tau}$$

$$\Leftrightarrow \frac{m}{n} > \frac{\tau}{\theta}$$

$$\frac{m}{n} > p \tag{1}$$

Condition (1) will be crucial to the analysis of our model. It is a necessary but not sufficient condition for congestion to occur at the squeeze point.

2.1 $r < \kappa, \lambda, \mu$

In this case the flow of traffic into our system is less than the maximum capacity of each individual section of the system. Consequently there is no congestion in our system and so corresponds to light traffic. We need not concern ourselves with such behavior since varying $\frac{m}{n}$ has no effect on congestion.

2.2 $\kappa < r < \lambda < \mu$

It is clear that $\tau = p\theta > \theta$ and so

$$\lambda = \frac{m}{\tau} \leq \frac{m}{\theta} = \mu$$

in general. Suppose congestion occurs at the squeeze point but not at the toll booth. This means condition (1) must hold. If this congestion is to occur we also require $\kappa < r \leq \lambda$. This behavior is shown in Figure 2 . We now

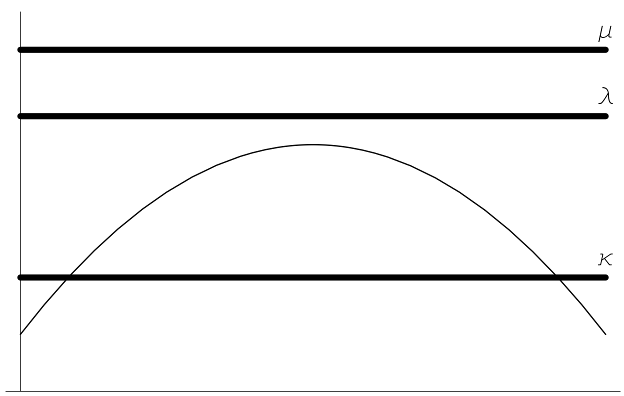


Figure 2: The relationship between r, κ, λ and μ

proceed as follows. We define

$$\Delta\rho(t) = \begin{cases} \lceil f(t) \rceil & \text{for } f(t) < \lambda \\ \lambda & \text{for } f(t) \geq \lambda \end{cases}$$

The second condition may seem redundant since $r < \lambda$, but it should be noted that r is the peak of our function h and not of our function f. It is entirely possible that our "noise" term γ would cause $f(t)$ to exceed λ and thus $\Delta\rho$ to exceed λ. However $\Delta\rho$ refers to a discrete arrival rate at the squeeze point, and we cannot have this exceeding λ so we apply the above careful conditional definition. Based on this we calculate the number of vehicles arriving at the squeeze point at time t as

$$\Delta c(t) = \theta \Delta\rho(t)$$

We are now ready to apply our model. We set up the following difference equations for the function $c(t)$ which refers to the number of vehicles trying to go through the squeeze point. It includes those who will get through at time t. It can be thought of as the number of people in the (virtual) queue. In order to simplify matters we take $t = 0$ when $h(t) = \kappa$, which is valid since we have assumed $h(t)$ is monotonic in the region before the peak and so by the discussion earlier in this section there will be no congestion for $t < 0$. Our initial condition is

$$c(0) = \Delta c(0)$$

This simply says that the number of vehicles trying to pass through the squeeze point at time 0 is those that arrive at time 0, since there is no congestion before this, and consequently no (virtual) queue. This now gives

$$c(\theta) = c(0) - n + \Delta c(\theta)$$

since we have $c(0) - n$ remaining in the (virtual) queue from time 0 and $\Delta c(\theta)$ arrive at time θ. Repeating this gives

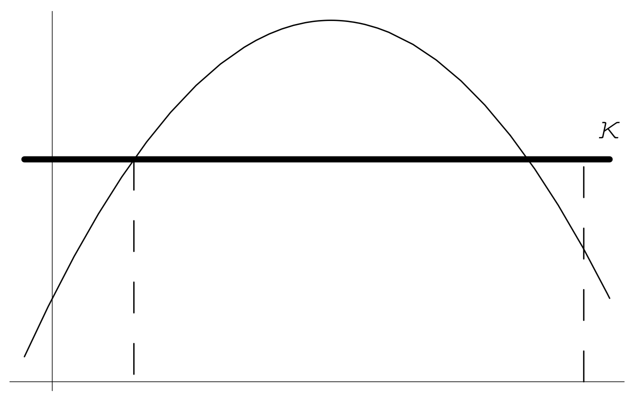
$$c(i\theta) = c((i-1)\theta) - n + \Delta c(i\theta)$$

where $i = 1, 2, \ldots$ Since our function f peaks at some value less than λ and then falls below κ this means $c(t)$ will eventually begin to fall and at some point become ≤ 0. At this point we define $c(t)$ to be zero and our model is finished since there is now no more congestion

$$c(N\theta) \leq 0, \; c(N\theta) := 0$$

This means that our model gives the following difference equation with specified initial and terminating conditions

$$\begin{aligned} &c(0) = \Delta c(0) \\ &c(i\theta) = c((i-1)\theta) - n + \Delta c(i\theta) \\ &c(N\theta) \leq 0, \; c(N\theta) := 0 \end{aligned} \tag{2}$$



This can be solved recursively. The above approach highlights the discrete nature of our approach alluded to earlier. In effect we are incrementing time in discrete intervals of θ and allowing each vehicle to either move forward one space increment or to remain in situ.

During each time interval of size θ, $c((i-1)\theta) - n$ vehicles are delayed by amount θ, which gives us

$$T_{delay} = \sum_{i=0}^{N} (c(i\theta) - n)\theta$$

for the total delay time. The total number of vehicles arriving during congestion is, clearly, given by

$$V_{total} = \sum_{i=0}^{N} \Delta c(i\theta)$$

Hence our "optimization parameter" is

$$\overline{t_{delay}} = \frac{T_{delay}}{V_{total}} = \frac{\sum_{i=0}^{N}(c(i\theta) - n)}{\sum_{i=0}^{N} \Delta c(i\theta)} \tag{3}$$

where the overbar represents mean.

It would now seem to simply be a computing problem to vary the ratio $\frac{m}{n}$ to minimize the mean delay time as given by (13). However this is not quite the case. Suppose we are faced with the challenge of designing an optimal toll plaza for an already existing highway. This means n, τ, θ, f and r are determined by factors outside our control (pre-existing values, efficiency of booths, speed of cars in plaza, incoming traffic and incoming traffic respectively). This means that m is our only free parameter, which influences λ. The crucial point is that λ only affects whether or not this particular version of our model is valid. To generalize, what our assumptions say is that if the rate of influx of traffic is too low to result in congestion at the toll booths, then the congestion at the squeeze point is independent of m, the number of toll booths. While this is clearly not a perfect model of reality, it is not as far removed from the truth as one might think. Intuitively, there should be a correlation between these two quantities. However the overall number of vehicles "squeezing in" and the number of lanes into which they squeeze are constants so the dependence may not be that strong. If this is the case then our model is a good approximation, especially when we consider that in later sections, the model is much better.

2.3 $\kappa < \lambda < r < \mu$

Take the case where, congestion occurs at both the squeeze point and the toll booths. Again condition (1) must hold as well as $\kappa < \lambda < r < \mu$. We now define temporarily

$$\Delta\rho(t) = \begin{cases} \lceil f(t) \rceil & \text{for } f(t) < \lambda \\ \lambda & \text{for } f(t) \geq \lambda \end{cases}$$

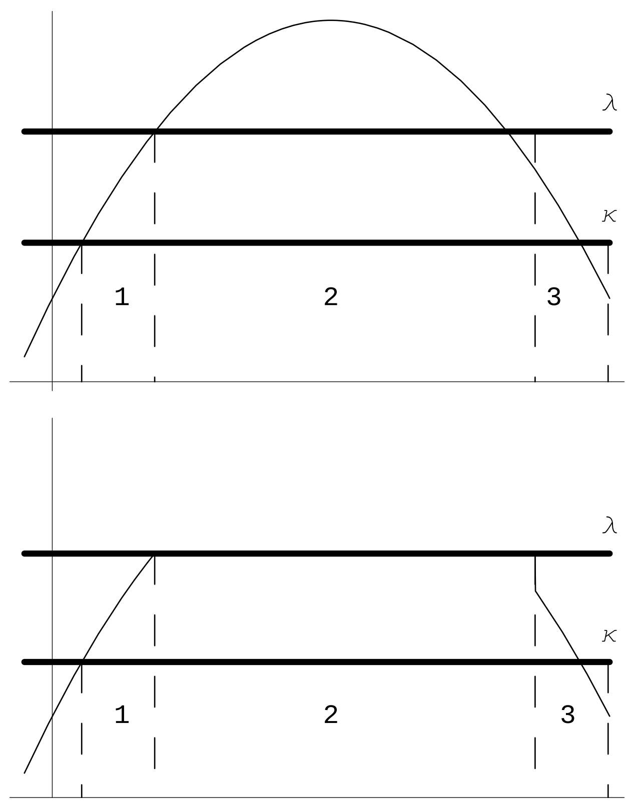
which is exactly the same as the previous section,and

$$\Delta\beta(t) = \begin{cases} \lceil f(t) \rceil & \text{for } \lambda < f(t) < \mu \\ \mu & \text{for } f(t) \geq \mu \end{cases}$$

refereing to a discrete arrival at the toll booths this time. To model this in a clear and concise way, the model was broken into three separate sections.

2.3.1 Section 1

This starts at the point where $f(t)$ first intersects the line κ and is set to be $t = 0$. Before this point, $f(t)$ is always below κ so no congestion can occur. Above this point however, congestion occurs at the squeeze point. The conditions in this section are the same as the beginning of the model in



2.2, so it is reasonable to take the difference equation for congestion at the squeeze point to be

$$\begin{aligned} c(0) &= \Delta c(0) \\ c(i\theta) &= c((i-1)\theta) - n + \Delta c(i\theta) \end{aligned} \tag{4}$$

with the terminating conditions removed for the moment. Section 1 ends, however as $f(t)$ increases to the point where it intersects λ. This is labelled $t = t_o$.

2.3.2 Section 2

The second section intuitively begins at the point where congestion begins to occur at the toll booths. A difference equation is constructed here much in the same way as for the squeeze point in 2.2. The number of vehicles at time t arriving at the booths is given in time intervals of τ though

$$\Delta b(t) = \tau \Delta \beta(t)$$

We now define $b(t)$ to be the number of vehicles trying to go through the toll booths or the number in the (virtual) queue. There will be no congestion for $f(t) < \lambda$ so we can assume the initial condition:

$$b(t_0) = \Delta b(t_0)$$

The general difference equation then becomes:

$$b(t_0 + i\tau) = b(t_0 + (i-1)\tau) - m + \Delta b(t_0 + i\tau)$$

again with $i = 1, 2, \ldots.$ The function f peaks at a value $r < \mu$ and then decreases below λ, meaning that at some value of t, $b(t) \leq 0$. After this point, there will be no more congestion at the toll booths, and so is the terminal condition for $b(t)$ and is defined as 0 to avoid negative numbers:

$$b(t_0 + M\tau) \leq 0, b(t_0 + M\tau) := 0$$

The difference equation with initial and terminal conditions for the toll booth is

$$\begin{aligned} &b(t_0) = \Delta b(t_0) \\ &b(t_o + i\tau) = b(t_0 + (i-1)\tau) - m + \Delta b(t_o + i\tau) \\ &b(t_o + M\tau) \leq 0,\ b(t_o + M\tau) := 0 \end{aligned} \tag{5}$$

The effect of the congestion at the toll booths on the squeeze point is that the rate of influx of vehicles is constantly at λ. This gives a general difference equation for the section of

$$c(t_0 + i\theta) = c(t_0 + (i-1)\theta) - n + \lambda\theta$$

2.3.3 Section 3

The third and last section begins where section 2 ends, at $t = t_0 + M\tau$. Here, there is no more queuing at the toll booths, but queuing continues at the squeeze point.

(As can be seen from the graph above, there is a sharp drop at the start of section 2 until $f[t]$ is obtained again. Though this seems counter-intuitive, it was noticed, by chance, while sitting over coffee during a break that someone who joined a long queue at the end of a busy period, still had no one behind when he reached the counter!)

The general difference equation for the squeeze point reverts back to that of 2.2, since $f[t]$ is now below λ again. The terminating condition can also be taken from the model in 2.2 giving

$$c(N\theta) \leq 0,\ c(N\theta) := 0 \tag{6}$$

We can now redefine $c(t)$ properly over all three sections

$$\Delta\rho(t) = \begin{cases} \lceil f(t) \rceil & \text{for } t < t_o, t_o + M\tau < t \\ \lambda & \text{for } t_o < t < t_o + M\tau \end{cases}$$

The total delay time is then given by:

$$T_{delay} = [\sum_{i=0}^{N}(c(i\theta) - n)]\theta + [\sum_{i=0}^{M}(b(t_o + i\tau) - m)]\tau$$

and with the total number of cars delayed given by

$$V_{total} = \sum_{i=0}^{N} \lceil f(i\theta) \rceil$$

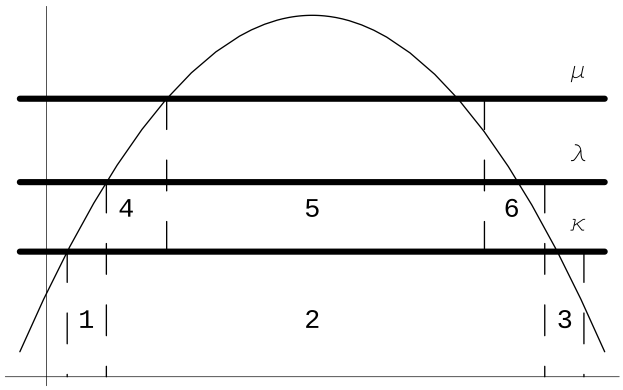
we get our "optimization parameter" to be

$$\overline{t_{delay}} = \frac{T_{delay}}{V_{total}} = \frac{[\sum_{i=0}^{N}(c(i\theta) - n)]\theta + [\sum_{i=0}^{M}(c(t_o + i\tau) - m)]\tau}{\sum_{i=0}^{N} \Delta c(i\theta)} \tag{7}$$

2.4 $\kappa < \lambda < \mu < r$

This is perhaps the most complicated part of our model, with more traffic than the plaza can handle. The intuitive thinking behind this is that, in the event of unusually large traffic, for example a sports event or concert, the maximum rate for the plaza itself is too low for the rate of traffic. In effect, a queue forms before the plaza itself.

The model for this is simply an extra tier onto the model for 2.3. Much the same as 2.3 is an extra tier on top of 2.2. Here, the graph of $f[x]$ is split first into three sections as in part B.



2.4.1 Section 1

This is the same as section 1 in 2.3 (also the same as the beginning of 2.2) and as such, the start point is set to $t = 0$ and end point set to $t = t_o$, the rest of the difference equation also follows from 2.3

$$\begin{aligned} c(0) &= \Delta c(0) \\ c(i\theta) &= c((i-1)\theta) - n + \Delta c(i\theta) \end{aligned} \tag{8}$$

2.4.2 Section 2

In order to simplify the model, section 2 was taken to be the same as in part B, i.e. from the end point of section 1 to the terminating point of the queuing at the toll booth. As a result of this, the behavior of the queuing at the squeeze point is exactly the same as in part B:

$$c(t_0 + i\theta) = c(t_0 + (i-1)\theta) - n + \lambda\theta$$

The queuing at the booth and plaza are entirely contained in section 2 but will be broken down into further sections later.

2.4.3 Section 3

This is again the last section in terms of chronological order, but labelled in this way, is exactly the same as 2.3.3, and as such the difference equation, by the same arguments is

$$\begin{aligned} &c(i\theta) = c((i-1)\theta) - n + \Delta c(i\theta) \\ &c(N\theta) \leq 0,\ c(N\theta) := 0 \end{aligned} \tag{9}$$

2.4.4 Section 4

Sections 4 to 6 are now analogous to 2.3 with μ instead of λ and λ instead of κ. There is also, however a time shift, with $t = 0$ going to $t = t_o$. Hence the equations for the queuing at the toll booth become

$$\begin{aligned} &b(t_0) = \Delta b(t_0) \\ &b(t_o + i\tau) = b(t_0 + (i-1)\tau) - m + \Delta b(t_o + i\tau) \end{aligned} \tag{10}$$

These apply until $f(t) > \mu$, when the congestion at the plaza kicks in. This time is labelled $t = t_1$ and is the starting point for section 5.

2.4.5 Section 5

The queuing at the plaza is described by similar difference equations to the queuing at the squeeze point in that it depends on the characteristic time θ. However, because there are m lanes instead of n, m vehicles get through to the plaza in each time interval θ. There is also a time scale based on the congestion at the plaza beginning at $t = t_1$. The finished equation reads

$$\begin{aligned} &\Delta\alpha(t) = \lceil f(t) \rceil \\ &\Delta a(t) = \theta \Delta\alpha(t) \\ \\ &c(t_1) = \Delta a(t_1) \\ &a(t_1 + i\theta) = a(t_1 + (i-1)\theta) - m + \Delta a(t_1 + i\theta) \\ &a(t_1 + Q\theta) \leq 0,\ a(t_1 + Q\theta) := 0 \end{aligned} \tag{11}$$

The traffic now arriving at the booth is at a constant rate of μ. This has the same effect as on the squeeze point in both 2.3.2 and 2.4.2 . The congestion is modelled in the following way until there is no more congestion at the plaza

$$b(t_o + i\tau) = b(t_0 + (i-1)\tau) - m + \mu\tau$$

There is one major difference though between the model for 2.3 and the whole of 2.4.2 here. Because the queuing at the plaza is measured in time intervals of θ and the queuing at the booth in intervals of $\tau = p\theta$, the congestion at the plaza may not end on an integer number of τ for the booth. Intuitively, if θ and τ are out of sync, there will be a slight delay in the effect on τ. As a result, the next integer value of $t_1 + i\tau$ after the termination of $a(t)$ is taken as the end of 2.4.2 for the booth

$$b(t_1 + D\theta) \text{ s.t. } D = \lceil \frac{Q\theta}{\tau} \rceil$$

2.4.6 Section 6

Here, we have no more congestion at the plaza, and we aren't considering the queuing at the squeeze point (already covered as section 2), so the only concern is the toll booth itself. Hence this section starts at $t_1 + D\tau$. It then follows the basic difference equation for the booth, with the appropriate time scaling, terminating in the usual way.

$$b(t_o + M\tau) \leq 0, \;\; b(t_o + M\tau) := 0$$

Now the three different functions for the arriving traffic at each section need to be defined properly

$$\Delta\rho(t) = \begin{cases} \lceil f(t) \rceil & \text{for } t < t_o, t_o + M\tau < t \\ \lambda & \text{for } t_o < t < t_o + M\tau \end{cases}$$

$$\Delta\beta(t) = \begin{cases} \lceil f(t) \rceil & \text{for } t < t_1, t_1 + D\tau < t \\ \mu & \text{for } t_1 < t < t_1 + D\tau \end{cases} \tag{12}$$

$$\Delta\alpha(t) = \lceil f(t) \rceil$$

The summations of these functions over their respective time span

$$T_{delay} = [\sum_{i=0}^{N}(c(i\theta) - n)]\theta + [\sum_{i=0}^{M}(b(t_o + i\tau) - m)]\tau + [\sum_{i=0}^{Q}(a(t_1 + i\theta) - m)]\theta$$

Total number of cars delayed

$$V_{total} = \sum_{i=0}^{N} \lceil f(i\theta) \rceil$$

"optimization parameter"

$$\overline{t_{delay}} = \frac{T_{delay}}{V_{total}} = \frac{T_{delay} = [\sum_{i=0}^{N}(c(i\theta) - n)]\theta + [\sum_{i=0}^{M}(b(t_o + i\tau) - m)]\tau + [\sum_{i=0}^{Q}(a(t_1 + i\theta) - m)]\theta}{\sum_{i=0}^{N} \Delta c(i\theta)} \tag{13}$$

2.5 $\frac{m}{n} < p$

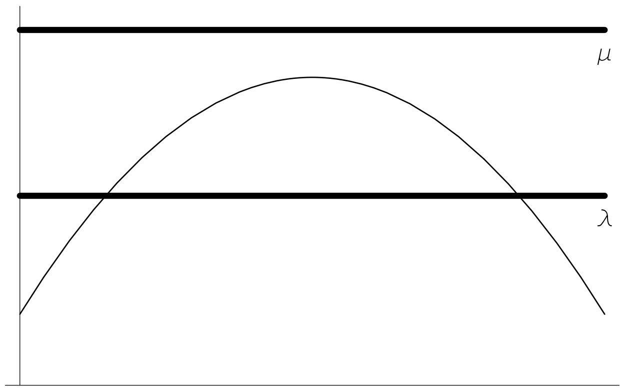
Here are the last remaining variations of the model, where the condition 1 doesn't hold, i.e. $\lambda < \kappa$. In this scenario, maximum rate of vehicles that leave the booth will never be enough to cause congestion at the squeeze point.

The case where $m = n$ is a special case of this scenario.

2.5.1 $\lambda < r < \mu$

In this situation, traffic congestion occurs at the toll booths, but nowhere else. It is very similar to the model for 2.2but with a number of differences. Instead of $c(t)$ we use $b(t)$ because the congestion is at the toll booths and not the squeeze point. θ also changes to τ every where and κ and λ go to λ and μ respectively. This gives the difference equations

$$\begin{aligned} &b(0) = \Delta b(0) \\ &b(i\tau) = b((i-1)\tau) - m + \Delta b(i\tau) \\ &b(N\tau) \leq 0,\ b(N\tau) := 0 \end{aligned} \tag{14}$$



Total delay time for system

$$T_{delay} = [\sum_{i=0}^{N}(b(i\tau) - m)]\tau$$

The total number of vehicles arriving during congestion

$$V_{total} = \sum_{i=0}^{N} \Delta b(i\tau)$$

Our "optimization parameter" is

$$\overline{t_{delay}} = \frac{T_{delay}}{V_{total}} = \frac{[\sum_{i=0}^{N}(b(i\tau) - m)]\tau}{\sum_{i=0}^{N} \Delta b(i\tau)} \tag{15}$$

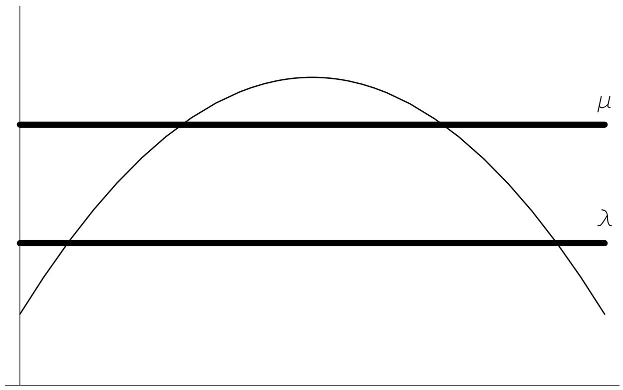
where the overbar represents mean.

2.5.2 $\lambda < \mu < r$

In this section, the peak rate of traffic influx is greater than plaza can deal with as well and so congestion builds up there as well as at the booths. This is analogous to 2.3 with a few minor adjustments again. c and b go to b and a respectively θ and τ swap and κ and λ go to λ and μ respectively.

The final average comes out as

$$\overline{t_{delay}} = \frac{T_{delay}}{V_{total}} = \frac{[\sum_{i=0}^{N}(b(i\tau) - m)]\tau + [\sum_{i=0}^{M}(a(t_o + i\theta) - m)]\theta}{\sum_{i=0}^{N} \Delta b(i\tau)} \tag{16}$$



3 Strengths and Weaknesses of our Model

3.1 Strengths

- The model we have chosen is easily programmed and the program we wrote (in Mathematica 5.0) is included as an Appendix.
- None of the inputs for our model are specific to certain types of toll booths. Parameters such as $h(t)$, τ, etc can be easily measured by a team of statistical consultants.
- If our data is as good an approximation of reality as we think it is, then the behavior of our model should be stable for any realistic variations.
- Our model contains a random nature ensured by the presence of γ, which makes it more realistic of real-life.
- Our model will in theory give an optimal ratio of m to n

3.2 Weaknesses

- While we have a random element to our data, repeated studies and standard probability or queueing theory have established the need to consider a quasi-random Poisson arrival model for this situation (see, for example, [6, Section 3.5])
- Almost all of the assumptions we have made for the nature of the movement of vehicles through our system are highly unrealistic. For

example, vehicles will not be homogenous, will not move in discrete lengths over discrete time intervals, will not arrive in continuous time.

- Even if our discrete model is valid, there is absolutely no justification for assuming $p = \frac{\tau}{\theta}$ is an integer if we are considering a general toll plaza.

- If 1 person experiences a 1 hour delay and 99 people experience no wait, then, by our definition of optimal, this is better than 100 people waiting for half a minute. This is not an ideal scenario, and while this example is highly extreme, it does indicate that we may not be optimizing what we think we are.

3.3 Possible Improvements

- Clearly from above, a poisson arrival would be the obvious generalization of our model

- Another added feature that would bring our model closer to reality would be to correlate the likelihood of a vehicle going through a booth to the distance from the lane the vehicle is on to the nearest free tollbooth

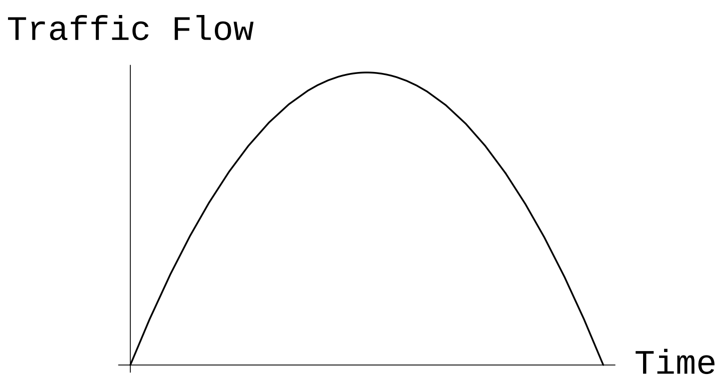
- While there any many more possibilities, these seem to be the two key aspects of reality that we are ignoring.

4 Results

In order to obtain real data to test our model, we chose the Westlink M50 toll plaza in Dublin, Ireland. Some of our team members had some limited personal experience of this particulartoll plaza.

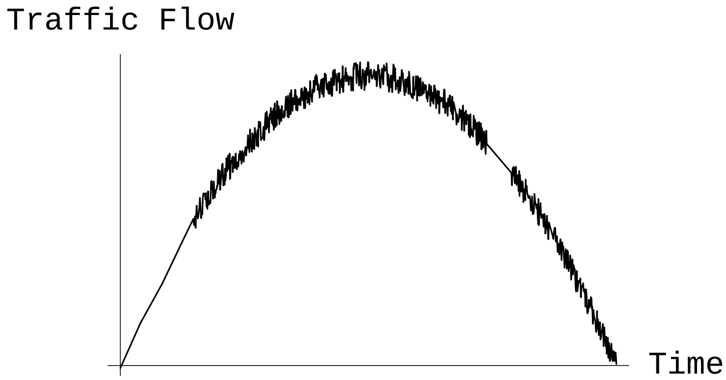
4.1 Estimation of Parameters

As already mentioned in the assumptions, we restrict our attention to the two busiest hours, and approximate this as the parabola shown.



Of course the change of flow will never be as uniform as this, so adding randomness to the model gives the following distribution for our rate of flow.

On the Westlink M50, 2 lanes fan out into 5 for the toll plaza, so in our notation, $m = 5$ and $n = 2$. From before, we require $p < \frac{m}{n}$ in order for congestion to occur at the constriction after the toll booths. Therefore, since $\frac{m}{n} = 2.5$, we must take $p = 1$ or 2 so as to be able to apply our model. We have $\tau = p\Theta$, i.e. p relates how long one spends passing through the constriction to the amount of time to go through the toll booth, and so $p = 2$ is the more realistic value to take (intuitively, the time spent driving through the booth will be longer, since, for example, the car must stop to pay the toll).
The length a particular vehicle could range anywhere, from 3.6 meters for a Toyota Yaris for instance (see [12]), up to 25 meters for large trucks(see



[13]). $\Theta = \frac{l}{v}$, where l is the average vehicle length and v the average speed of a vehicle at the constriction, so taking $\Theta = 4$ seconds or $\frac{1}{900}$ hours is reasonable. Using statistics relating to the number of vehicles which use the M50, we approximated the maximum traffic flow during the busy two hour period to be, on average, 4000 vehicles per hour.

4.2 Results from Program

The Mathematica programs in the appendix calculate the average waiting time $\overline{t_{delay}}$ (in minutes) for a vehicle travelling through the toll system. We fixed $n = 2$, $\Theta = \frac{1}{900}$, and $p = 2$, and then varied m and r, the peak traffic flow, to get values for the waiting time. m took values from 2 to 14, while $r = 2000, 3000, 4000, 5000, 6000$. We obtained the data shown in the following tables:

r	m	$\overline{t_{delay}}$
2000	2	2.97507
	3	12.1533
	4	1.5802
	5	1.56333
	6	1.58727
	7	1.58521
	8	1.55829
	9	1.55829
	10	1.55829
	11	1.55829
	12	1.55829
	13	1.55829
	14	1.55829

r	m	$\overline{t_{delay}}$
3000	2	74.7801
	3	3.02861
	4	18.1931
	5	13.9078
	6	2.26458
	7	18.2488
	8	18.2488
	9	18.2488
	10	18.2488
	11	18.2488
	12	18.2488
	13	18.2488
	14	18.2488

r	m	$\overline{t_{delay}}$
4000	2	220.114
	3	42.3599
	4	2.96445
	5	62.3328
	6	32.2738
	7	13.5456
	8	3.01919
	9	37.7419
	10	37.7419
	11	37.7419
	12	37.7419
	13	37.7419
	14	37.7419

r	m	$\overline{t_{delay}}$
5000	2	430.195
	3	116.371
	4	29.773
	5	110.105
	6	86.7116
	7	52.9446
	8	29.3646
	9	13.6021
	10	3.70615
	11	0.0242454
	12	58.3183
	13	58.3183
	14	58.3183

r	m	$\overline{t_{delay}}$
6000	2	706.507
	3	217.076
	4	74.6198
	5	196.88
	6	111.355
	7	113.582
	8	74.6693
	9	48.494
	10	28.113
	11	13.9387
	12	4.46381
	13	0.268646
	14	79.3435

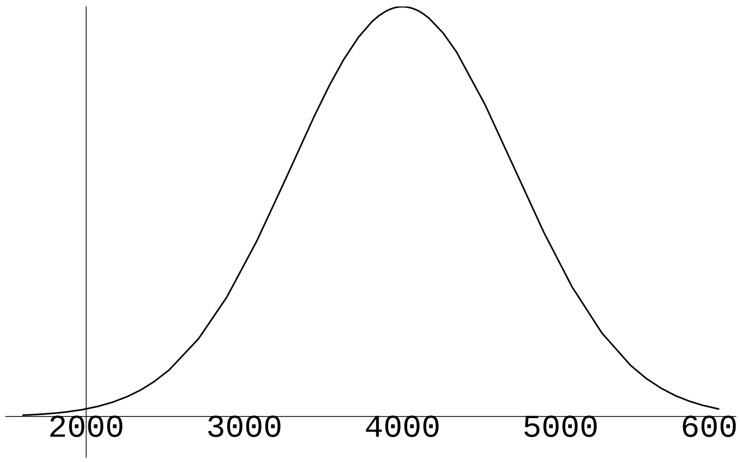
We now had five sets of fourteen data points $(m, \overline{t_{delay}})$. We had reckoned r to be 4000 on average, and so we applied the following weightings to the values for r, based on how likely that level of traffic flow would occur:

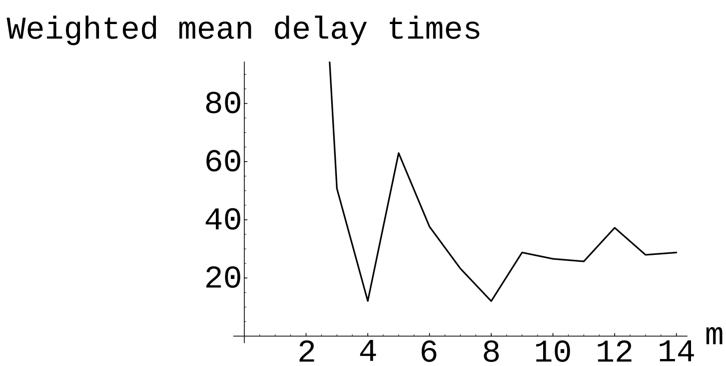
r	weighting
2000	0.01
3000	0.2
4000	0.58
5000	0.2
6000	0.01

We felt that the distributions of flows would be normal with mean 4000, and that those weights chosen would approximate this:

In each of the five sets of data points corresponding to different values of r, the $\overline{t_{delay}}$ was multiplied by the the respective weight. Finally, we summed the $\overline{t_{delay}}$ components of corresponding points in all five data sets to give one set of fourteen points. When plotted (see below), we get a graph of m versus the mean weighted delay time:

Clearly, for our model of the Westlink M50 with $n = 2$ roads, $m = 4$ and $m = 8$ minimize the delay time. The time difference is so small, and will fluctuate with the randomness in the model, so either number of roads will minimize delays.

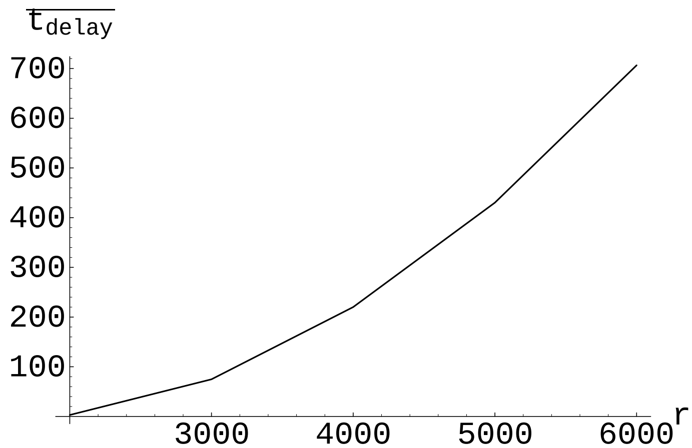




4.3 When $m = n$

The case where $m = n = 2$ is worth considering on its own. Referring back to the tables from earlier, we get the following data and resulting graph:

r	$\overline{t_{delay}}$
2000	2.97507
3000	74.7801
4000	220.114
5000	430.195
6000	706.507



We see a nonlinear increase in the delay time as the traffic flow increases. Obviously, the only acceptable situation here is for $r = 2000$, as the other values give delay times of over an hour per vehicle. This corresponds to very light traffic coming to the toll plaza. Any moderate to heavy traffic will cause appalling delays.

5 Conclusions

With our model, we analyzed the case of $n = 2$ lanes coming into and out of the Westlink M50 toll plaza in Dublin. We found that the delay time per vehicle was minimized for $m = 4$ or $m = 8$ lanes in the plaza. Neither of these were significantly faster or slower than the other, though both are more efficient than the current 5 lanes in the plaza. At this point, we must consider what we mean by "optimal". Closing one lane and booth, thereby reducing m to 4 may easily lead to public outrage, as drivers perceive that services are being withdrawn. On the other hand, $m = 8$ has huge and obvious economic disadvantages, given the cost of widening the entire plaza and constructing the 3 extra lanes. Simply closing one booth would be a much less expensive and surely more desirable way of optimizing on the toll booth system and reducing queueing times.

5.1 $n = m$

From the data collected from our model, we see that $n = m$ can be a very efficient system, with delay times of less than 3 minutes, but only in very light traffic. In heavier traffic, this is the least successful system for reducing queueing, as the delays can stretch to nearly 12 hours. These results agree with our intuition and experience. With $n = m$, of course there is no constriction after the booths and hence no congestion. However with no lanes to fan out into when approaching the booth, heavy traffic will clearly build up and lead to huge delays.

References

[1] M Slinn, P Matthews & P Guest, "Traffic Engineering Design", *Arnold.*

[2] C A O' Flaherty, "Transport Planning and Traffic Engineering", *Butterworth-Heinemann.*

[3] R J Salter, "Highway Traffic and Analysis Design (Revised Edition 1976)", *The MacMillan Press Ltd.*

[4] D R Cox and Walter L Smith, "Queues", *Chapman and Hall.*

[5] Søren Asmussen, "Applied Probability and Queues", *John Wiley and Sons.*

[6] Joseph A Panico, "Queueing Theory", *Prentice-Hall, Inc.*

[7] Ng Chee Hock, "Queueing Modelleing Fundamentals", *John Wiley and Sons.*

[8] Donald Gross & Karl M Harris, "Fundamentals of Queueing Theory (3rd edition)", *The MacMillan Press Ltd.*

[9] Brian D Bunday, "Basic Queueing Theory", *Edward Arnold.*

[10] Robert B Cooper, "Introduction to Queueing Theory", *The MacMillan Company.*

[11] Daniel T Finkbeiner II & Wendell D Lindstrom, "A Primer of Discrete Mathematics", *W H Freeman and Company.*

[12] http://fr.toyota.be/showroom/yaris/specs.asp?body=T.

[13] http://www.highways.gov.sk.ca/docs/trucking/M006-SkHiwayseightDim11.5.pdf.

[14] http://www.straughan.com/italy/living/euro.html

[15] http://www.eazypass.ie/