Author/Judge's Commentary: The Outstanding Channel Assignment Papers

Jerrold R. Griggs
Dept. of Mathematics
University of South Carolina
Columbia, SC 29208
griggs@math.sc.edu
homepage: http://www.math.sc.edu/~griggs/

Background

This 2000 MCM Problem, which I wrote, is based on a subject of considerable current interest to mathematicians and communications engineers. The original "channel assignment problem" has a long history. The problem is to assign an integer channel to each transmitter in a network, with the condition that the absolute difference between channels for two nearby transmitters must not belong to a certain set T that arises from interference considerations (see Hale [1980] for motivation). A feasible assignment can be obtained with channels far apart, but this is highly inefficient. Typically, a frequency band that spans the assigned channels is allocated to the network; the wider the band, the more it costs. The problem, then, is to minimize the "span" of the assignment, which is the difference between the maximum channel and the minimum channel.

This problem is modeled nicely with graph theory by letting each transmitter correspond to a vertex, with edges corresponding to pairs of nearby transmitters. The problem becomes a special vertex-coloring problem, owing to the set T of forbidden differences [Cozzens and Roberts 1982]. Among the methods that come into play are number theory (in the case of complete graphs [Griggs and Liu 1994]) and the complexity of graph homomorphisms.

The UMAP Journal 21 (3) (2000) 379–386.

History of the Problem

In 1987, I heard from Fred Roberts [1987] about a variation of the T-coloring problem in which channels for two nearby transmitters, say within distance s, must differ by at least two, while those within distance $2s$ must differ by at least one. The network may have thousands of transmitters. Again we seek to minimize the span of a feasible channel assignment.

In considering this problem, I realized that it no longer translates directly into a graph theory problem by assigning vertices to transmitters and edges to pairs at distance at most s: Although two transmitters at distance at most s correspond to adjacent vertices, a pair of transmitters at distance between s and $2s$ corresponds to a pair of vertices at distance two in the graph only when some third transmitter is within distance s of both of them. (Note that the *distance* between two vertices in a graph is the number of edges in a shortest path between them.) In fact, the vertices for two transmitters at distance between s and $2s$ may not even be connected in the graph.

Nonetheless, it is clear that for the real problem it is useful to understand the natural graph analogue, which is to find the minimum span for the integer labelings of a graph such that labels for vertices at distance one (resp., two) differ by at least two (resp., one). For the transmitter networks in Parts A and B of the contest problem this year, the associated graph problem is precisely of this type with one change: The span in the contest problem is one more than in the labeling problems in the literature.

Griggs and Yeh [1996] introduce this graph labeling problem and pose some fundamental questions about it. Included is the natural generalization of the graph problem in which there are multiple levels of spectral spreading interference: Given integers $d_1, \ldots, d_r$ we seek minimum span labelings such that for all i, the labels for any pair of vertices at distance i differ by at least d_i. Such a labeling is called an $L(d_1, \ldots, d_r)$*-labeling.*

Applications

While such problems are mathematically interesting, they have taken on greater importance in recent years due to their potential applicability to the design of mobile radio networks. Large areas are often covered by a network of regularly spaced transmitters such that the associated graph labeling problem exactly models the network problem. The most common design places the transmitters in a triangular lattice, so that the whole region can be tessellated by a honeycomb of hexagons, with each transmitter in the center of a hexagonal region that it covers. An early reference considering such a model is Gamst [1982]; and evidently MacDonald [1979], cited by contest teams, also does this. A group led by Robert Leese at Oxford has been prominent in this program in recent years [Leese 1999].

The Outstanding Papers

Requirement A

Part A of this year's contest problem is a basic instance of this application, a sizable array of transmitters with $d_1 = 2, d_2 = 1$. Feasible solutions are trivial to find, but working down to an optimal one requires some cleverness. We had expected many teams would enjoy working on this and that most would achieve the minimum span; this was indeed the case.

Requirement B

Part B extends the network of Part A to the whole plane. While one can solve Part A by trial-and-error or by a computer search (to obtain an optimal assignment and rule out smaller ones), Part B requires a method to keep going forever labeling this infinite network. Successful teams for Part B usually found a pattern (a strip or a tile of numbers) that could be repeated indefinitely and achieve the same span as the bounded array in Part A. One way is to label a strip by an appropriate ordering, say

$$1\,3\,5\,7\,9\,2\,4\,6\,8\,1\,3\,\ldots,$$

and then use the same strip shifted appropriately for the next row, and so on. Another perspective is to construct an appropriate tile of nine hexagons and replicate it. Judges were pleased to see papers, such as that of the team from California Polytechnic State University, that test a variety of heuristics to assign channels in Parts A and B, since such methods are needed for more general arrays and distance parameters. At least one paper, by the team from Wake Forest University, makes the interesting observation (with proof) that the optimal labeling for Parts A and B is essentially unique!

Requirement C

What is most remarkable is that several teams were able to solve Part C, in which the channel spread parameters for Parts A and B are extended to $d_1 = k, d_2 = 1$. One can give decent labelings for the bounded array in Part A and for the full plane in Part B that are not far off from the lower bounds that one can quickly derive. However, we had not completely solved the problem for general k before the contest. It seems to be a new result.

Requirement D

Part D is the open-ended generalization of the problem to general array configurations and multiple levels of interference. It has the most room for creativity and for model design. This part was expected to be the main point of differentiation among the entries. Judges were disappointed that most entries did not do much here—perhaps they ran out of time working on Parts A–C, where the assumptions and model are explicit. Weaker papers only considered, say, what happens if one transmitter is not at the center of its hexagon. But stronger papers gave this part considerable thought. Some considered general conditions with two levels of interference, such as the impressive results contained in the paper by the team from Washington University that nearly solve it. (In part, they built on the thesis of Mark Shepherd [1998].) The paper from the Wake Forest University team analyzes an assignment method for multiple levels of interference. Judges wanted to see teams use the real problem of wireless communication as motivation, such the choice of multiple-level distance parameters analyzed by the team from Lewis & Clark University. Some considered how to adapt their hexagonal lattice approach to other configurations of transmitters.

Requirement E

For Part E, judges wanted to see an article that conveys to the public the sense of the problem and the team's ideas on how to attack it. A particularly amusing article was crafted by a team from Harvey Mudd College whose entry received Honorable Mention.

General Remarks

Several teams located related results in the literature or the Web, particularly for the problem of *cyclic* labelings, where integers $\{1, 2, \ldots, n\}$ are used but the distance between two labels is measured modulo n, that is, by the shortest path on the circle labelled 1 through n. This approach can be used when a large number of channels must be assigned to each location: When a vertex receives label i, it is given all channels congruent to i modulo n. For two levels of interference $(L(d_1, d_2))$, this cyclic problem is solved in van den Heuvel et al. [1998]. However, this does not immediately solve the contest problem. A solution for general $L(d_1, d_2)$ of the (linear) contest problem remains to be found.

Teams typically found good labelings for the bounded array in Part A by trial and error or by exhaustive computer search, for small values of k, and identified patterns or tiles that could be extended to general k to yield good labelings for the bounded and unbounded arrays.

One cannot be certain that a labeling is optimal without proving that there is no labeling of smaller span. Also, it is by no means clear that there exist optimal labelings using a repeating pattern, although many teams seemed to assume this. Thus, it is not sufficient to check just labelings from a repeating pattern. (In fact, it would be very interesting if one could show that for all sets of distance parameters d_i that there is an optimal labeling of the plane built from a repeating pattern. This seems to be an open question.)

Judges favored papers that provide a *clear proof* that their labelings are optimal for general k. The best proofs that we read were impressive, such as the one by the team from the National University of Defence Technology. That paper is among those that made the interesting observation that for general k there is an optimal labeling for the arrays in Parts A and B that uses only nine different channels—which could be useful in some applications!

Related Research

Chang and Kuo [1992] made noteworthy progress on the original graph labeling problems for $d_1 = 2, d_2 = 1$ posed by Griggs and Yeh. Griggs and Yeh conjectured that every graph of maximum degree $\Delta \geq 2$ has an $L(2,1)$-labeling of span at most Δ^2; this bound is achieved by cycles. Their conjecture remains open, even for $\Delta = 3$. For the famous Petersen graph, in which every vertex has degree 3, the minimum span is 9, the conjectured maximum.

Georges and Mauro [1995] showed how the $L(2,1)$-labeling problem for general graphs G is equivalent to a path covering problem for the complement of G. Such problems are known to be difficult (consider the problem of whether a graph has a Hamilton path, for instance), and, indeed, it has been recently shown [Fiala et al., to appear] that determining whether a graph has a labeling with span at most k is NP-complete for all $k \geq 4$. A good general upper bound on the span in the case $L(p,q)$ has been given recently [van den Heuvel and McGuinness 1999] for general planar graphs of maximum degree Δ, by applying the methods of the proof of the Four Color Theorem.

Leese [1997] considers channel assignments for the hexagonal array of our problem that are obtained by tilings (periodic labelings). A wide range of applied references is provided in this paper. McDiarmid and Reed [1997] and Fitzpatrick et al. [2000] discuss algorithms for channel assignments for the hexagonal array of our problem in which each location must v must receive a specified number w_v of channels. Many papers seem to be emerging that employ familiar methods of discrete optimization to produce channel assignments of low span (not necessarily optimal). Contest teams discovered work that we were not aware of, by Hurley [n.d.] and by Smith and Hurley [1997], that uses heuristics and search methods, including tabu search and genetic programming; Hurley developed software for these problems. A new project by Leese [2000] tests a linear programming method based on column generation.

Conclusion

Returning to the contest problem, judges had hoped to see more entries employ such methods of discrete optimization on Part B. We also hoped that more effort would be spent considering the open-ended modeling Part D of the problem. It may simply be that teams found more direct analysis to be successful on the specific problem instances in Parts A, B, and C, and most of their energy was spent on tackling these parts. The Outstanding papers published here are among the very few that accomplished much with the extension to Part D.

Judges raised fewer concerns than in past years about specific missing elements in entries; but again this is likely because the model for Parts A, B, and C, which teams focused on, is clear-cut. In general, what judges particularly sought in winning papers was clarity, both in explaining their approach and in proofs, especially for the lower bound in Part C. Since space permits, I reproduce below the discussion in my Judge's Commentary last year [Griggs 1999] of crucial elements in an outstanding contest entry.

Crucial Elements in an Outstanding Entry

Here are some general tips that the judges feel apply to any contest problem.

- *Teams should attempt to address all major issues in the problem.* Projects missing several elements are eliminated quickly.
- *A thorough, informative summary is essential.* Papers that are strong otherwise are often eliminated in early judging rounds due to weak summaries. Don't merely restate the problem in the summary, but indicate how it is being modeled and what was learned from the model. The summary should not be overly technical.
- *Develop a model that people can use!* The model should be easy to follow. While an occasional "snow job" makes it through the judges, we generally abhor a morass of variables and equations that can't be fathomed. Well-chosen examples enhance the readability of a paper. It is best to work the reader through any algorithm that is presented; too often papers include only computer code or pseudocode for an algorithm without sufficient explanation of why and how it works.
- *Supporting information is important.* Figures, tables, and illustrations are very helpful in selling your model. A complete list of references is essential—document where your ideas come from.

References

Chang, G., and D. Kuo. 1992. Labelling graphs with a condition at distance 2. *SIAM Journal of Discrete Mathematics* 5: 586–595.

Cozzens, M.B., and F.S. Roberts. 1982. *T*-colorings of graphs and the channel assignment problem. *Congressus Numerantium* 35: 191–208.

Fiala, J., T. Kloks, and J. Kratochvíl. To appear. Fixed-parameter complexity of λ-labellings. *Discrete Applied Mathematics.*

Fitzpatrick, S., J. Janssen, and R. Nowakowski. 2000. Distributive online channel assignment for hexagonal cellular networks with constraints. Preprint, January 2000.

Gamst, A. 1982. Homogeneous distribution of frequencies in a regular hexagonal cell system. *IEEE Transactions on Vehicular Technology* VT-31: 132–144.

Georges, J.P., and D.W. Mauro. 1995. Generalized vertex labelings with a condition at distance two. *Congressus Numerantium* 109: 141–159.

Griggs, Jerrold R. 1999. Judge's commentary: The outstanding lawful capacity papers. *The UMAP Journal* 20 (3): 331–333.

________, and D. D.-F. Liu. 1994. The channel assignment problem for mutually adjacent sites. *Journal of Combinatorial Theory A* 68: 169–183.

________, and R.K. Yeh. 1996. The $L(2,1)$-labeling problem on graphs. *SIAM Journal of Discrete Mathematics* 9: 309–316.

Hale, W.K. 1980. Frequency assignment: Theory and applications. *Proceedings of the IEEE* 68: 1497–1514.

van den Heuvel, J., R.A. Leese, and M.A. Shepherd. 1998. Graph-labeling and radio channel assignment. *Journal of Graph Theory* 29: 263–283.

van den Heuvel, J., and S. McGuinness. 1999. Colouring the square of a planar graph. Preprint.

Hurley, S. n.d. Frequency assignment research. `http://www.cs.cf.ac.uk/user/Steve.Hurley/freq.htm`. Accessed August, 2000. Software description included.

Leese, R.A. 1997. A unified approach to the assignment of radio channels on a regular hexagonal grid. *IEEE Transactions on Vehicular Technology* 46: 968–980.

________. 1999. The mathematics of radio spectrum management. Research program report at `http://www.radio.gov.uk/busunit/research/mathsp/mathsp.htm`.

________. 2000. A linear programming approach to radio channel assignment in heavily loaded, evolving networks. Preprint.

MacDonald, V.H. 1979. Advanced mobile phone service: The cellular concept. *Bell System Technical Journal* 58: 968–980.

McDiarmid, C., and B. Reed. 1997. Channel assignment and weighted coloring. Preprint.

Roberts, F.S. 1987. Personal communication.

Shepherd, Mark. 1998. Radio channel assignment. Ph.D. thesis, Merton College, Oxford University. `http://www.maths.ox.ac.uk/combinatorics/thesis.html`.

Smith, D.H., and S. Hurley. 1997. Bounds for the frequency assignment problem. *Discrete Mathematics* 167/168: 571–582.

Acknowledgment

The author's research was supported in part by NSF grant DMS–0072187.

About the Author

Jerry Griggs is a graduate of Pomona College and MIT, where he earned his Ph.D. in 1977. Since 1981, he has been at the University of South Carolina, where he is Professor of Mathematics and a member of the Industrial Mathematics Institute. He received the 1999 award at the University for research in science and engineering.

His research area is combinatorics and graph theory, both fundamental theory and applications to database security, communications, and biology. He has published more than 60 papers and supervised 11 doctoral and 9 master's students. He serves on the Board of the Mathematics Foundation of America, which oversees the Canada/USA Mathcamp. He has been an MCM judge since 1988.