The Geometry and the Game Theory of Chases

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Introduction

We investigate the hunting strategies of predators and the fleeing strategies of their prey in a chase of finite time. For the velociraptor and thescelosaur, there is no overwhelming advantage for either animal; the velociraptor is faster, but the thescelosaur is more agile. We find optimal strategies for both the hunter and the hunted, for a velociraptor hunting a thescelosaur, as well as for a pair of velociraptors hunting a thescelosaur.

This original problem actually has a low probability of having occurred, as fossil remains of the velociraptor have been found only in Mongolia, while fossil remains of the thescelosaur have been found only in the Midwestern region of the United States and Canada [Weishampel et al. 1990, 270, 500]. However, this model can be useful in the study of a wide range of such problems, simply by varying the parameters. In studying these particular creatures, we may come to understand the tradeoff between speed and maneuverability.

Assumptions and Preliminary Calculations

We are given that the velociraptor moves at a speed of v_v = 60 km/h (16.7 m/s), the thescelosaur moves at a speed of of v_t = 50 km/h (13.9 m/s), and the velociraptor's hip height is 0.5 m. It is estimated that a velociraptor's turning radius is three times its hip height; thus, the velociraptor can turn in a minimum radius of r_v = 1.5 m, while the thescelosaur's minimum turning radius is r_t = 0.5 m. We assume that both always find it more advantageous

The UMAP Journal 18 (3) (1997) 225–242.

to turn with this wide radius rather than to decelerate, stop, change direction, and re-accelerate.

Hunts are limited in time, e.g., by the maximum endurance (or patience) of the predator, or by the onset of night or day. In this case, the limit is the pitiful endurance of the otherwise fearsome velociraptor. After a burst of speed for T = 15 s, the velociraptor must stop to rest, while the thescelosaur can run for a comparatively infinite length of time. We make the additional assumption that the velociraptor must rest for more than $T(v_v - v_t)/v_t$ = 3 s, i.e., more than the time required for the thescelosaur to run as far as the maximum distance that the velociraptor could close in 15 s. Thus, the velociraptor must catch the thescelosaur in the first 15 s after the thescelosaur senses the velociraptor.

The thescelosaur first detects the velociraptor at a distance D, with 15 m < D < 50 m, while the velociraptor can detect its prey farther than 50 m away. We assume that D is not dependent upon angle; i.e., if each of a pair of velociraptors approaches a thescelosaur from a different angle, the thescelosaur detects each when its distance is less than D.

We assume that because of the position of the eyes on opposite sides of the dinosaurs' heads, their vision is virtually 360°; thus, independent of its own orientation, each is aware of the position of the opponent.

The average human reaction time is $\approx$ 0.1 s; the thescelosaur can turn around 180° in this amount of time. We assume that animals with the speed and agility of these dinosaurs have a considerably smaller effective reaction time, which we vary between 0.005 s and 0.05 s in our study. Furthermore, we assume that the additional burden on the senses of the thescelosaur of the presence of two velociraptors instead of one does not change its effective reaction time.

From a picture of the velociraptor [Czerkas and Olson 1987, 28, compared with Paul 1988, 363], we deduce approximate measurements: a body length of 3 m, foreclaw length 0.5 m, hip-to-foreclaw distance 0.6 m, and hip-to-head distance 1.2 m. Moreover, a running bipedal dinosaur, because of its long tail, has a center of gravity close to the hips [Alexander 1989, 69]. Based on these measurements, we assume that at top speed the velociraptor will catch anything that comes within a distance δ_v = 0.6 m of its position, which we define to be the place on its torso from which the foreclaws extend. At the widest point of its torso, the velociraptor is only 0.4 m wide; we can ignore this thickness, as it is contained well within the reach of its foreclaws. Note that the location of the center of this reach is not at the hips, which is the point from which we assume the turning radius was calculated by the scientists. However, this slight incongruity does not qualitatively change our approach to the problem.

The thescelosaur is a biped of similar size. For the velociraptor to catch it, we assume that the velociraptor must be able to grab it at the torso, as the head and tail are too thin to grab easily at 60 km/h. So, we represent the thescelosaur as a circle of radius δ_t = 0.2 m over its hips. If the grabbing region of the velociraptor intersects this circle, the thescelosaur is caught.

To facilitate calculation, we assume that both predator and prey move at full speed for the entire time of the hunt T, even when they are moving in curves, with radii of curvature no less than r_v and r_t. This assumption is not entirely reasonable, as one can calculate the centripetal accelerations to be 19 g's and 39.4 g's, respectively. Given time for further investigation, it would be appropriate to model the dinosaurs with a maximum acceleration up to a top speed.

For the second part of the problem, with two velociraptors, we assume that the velociraptors work perfectly together:

- A velociraptor has just as much incentive to let its companion catch the thescelosaur as to catch the prey itself.
- The velociraptors are perfectly coordinated and can communicate their plans.
- The velociraptors allow each other space to move; we assume that this is equivalent to preventing their grabbing regions from intersecting.

Analysis: One Velociraptor

Approach

We begin with the simple case of the velociraptor initially chasing the thescelosaur along a straight line, separated by a distance d significantly larger than the turning radii of the dinosaurs. The thescelosaur's goal is to evade the velociraptor for however much time $(T - t)$ remains before the velociraptor runs out of endurance. Thus, if $d > (v_v - v_t)(T - t)$, the thescelosaur can run directly away from the velociraptor and the velociraptor cannot close the distance in the time remaining.

But what if $d < (v_v - v_t)(T - t)$? (Certainly this will be the case if the velociraptor can approach the thescelosaur undetected to a distance closer than $(v_v - v_t)T$ = 42 m.) In this scenario, the thescelosaur must make use of its superior maneuverability if it is to survive. For sufficiently large d, no matter how the thescelosaur turns, it is easy for the velociraptor to adjust its course to keep heading directly toward its prey.

Encounter Strategies

The thescelosaur must now make some decisions: When has the velociraptor come near enough for the thescelosaur to make use of its superior agility (while not getting eaten), and how should it let the velociraptor approach? We consider two representative strategies.

Encounter Strategy A

The thescelosaur initially runs directly away from the velociraptor. This costs the velociraptor time, since it can close the relative distance at a rate of only $(v_v - v_t)$. Once the velociraptor has closed to within a distance k, the thescelosaur uses its superior maneuverability to "dive" out of the way. It turns at its minimum turning radius; the velociraptor then turns at its minimum turning radius to intercept, but it is too late (see **Figure 1**). The distance k must be chosen with great care: If it is too large, the velociraptor can adjust its angle and close on the thescelosaur; if it is too small, the thescelosaur will not be able to get out of the way of the velociraptor's grabbing radius.

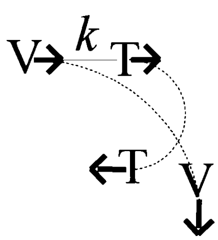

Figure 1. Encounter Strategy A.

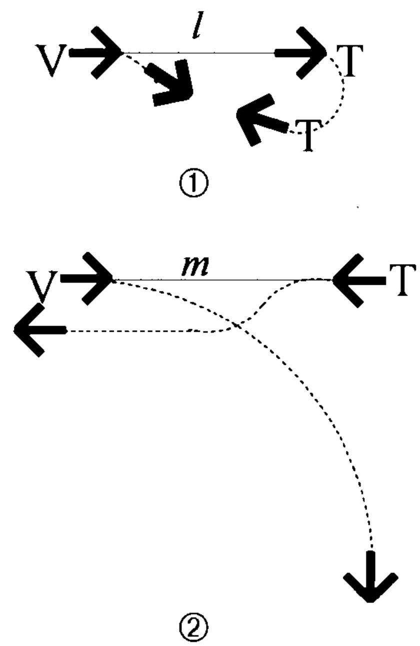

Figure 2. Encounter Strategy B.

Encounter Strategy B

The thescelosaur allows the velociraptor to close only to a distance l considerably greater than the k in Strategy A. At this point, the thescelosaur turns around and heads directly toward the velociraptor (see ① in **Figure 2**). The velociraptor, of course, continues to close; the distance between them now shrinks at a rate $(v_v + v_t)$. At an appropriate distance m, the thescelosaur again dives out of the way (see ② in **Figure 2**). Compared to Strategy A, however, the thescelosaur will be even more successful at dodging the velociraptor, as it need only change its course by a small amount to fly by the velociraptor at a relative velocity of approximately $(v_v + v_t)$. The value for m must be chosen with great care: if it is too small, the thescelosaur will not be able to stay outside the reach of the velociraptor, while if it is too large, the velociraptor will be able to compensate and intercept the thescelosaur.

Endgame

If the thescelosaur survives the encounter, the velociraptor will attempt to turn around and once again close in on its prey. The thescelosaur then has two endgame strategies.

Endgame Strategy A

Run away! If the distance between the two is greater than $(v_v - v_t)(T - t)$, the thescelosaur escapes unscathed as the velociraptor runs out of endurance.

Endgame Strategy B

This is a more daring maneuver but will take a big chunk of the velociraptor's time. Instead of running away from the velociraptor, the thescelosaurus should try to curve around it, ending up directly behind it. The velociraptor must turn around to come at the thescelosaur; because of its superior agility, the thescelosaur may be able to remain in this position relative to the velociraptor for some time. If the velociraptor starts to turn left, the thescelosaur also starts to turn left, attempting to remain 180° behind it. Because of its superior speed, however, the velociraptor will eventually outdistance the thescelosaur, and the thescelosaur will no longer be able to stay directly behind it. At this point, the thescelosaur should resort to Endgame Strategy A, as the velociraptor will soon turn around and chase it.

At the end of this post-encounter "endgame," the velociraptor will once again be chasing the thescelosaur, and we return to the "approach" phase.

Modeling the Chase: One Velociraptor

The Velociraptor Metric

How does the velociraptor get from point A to point B? More precisely: If the velociraptor is at the origin of the plane, facing in the positive y-direction, how would it get to the point (x, y)? Since we are considering the velociraptor to have constant speed, it should simply take the shortest path from the origin to the point. Unfortunately for the velociraptor, this distance is not the Euclidean metric, since the velociraptor has a limited turning radius! It cannot take a Euclidean straight-line path. So what is the appropriate path?

In **Figure 3a**, the velociraptor is at the origin facing upward. The two circles to either side represent the path of minimum turning radius. For points (such as A and B) outside these circles, the choice of minimum distance path is fairly clear: The velociraptor turns around the circle of minimum radius until it is heading directly toward the destination point; it then leaves the circle and heads straight toward the point. Representative paths are shown at right as dashed lines. Note that it is always advantageous to turn toward the side of the plane on which the destination point lies. (For the calculation of this length, refer to the **Appendix**.)

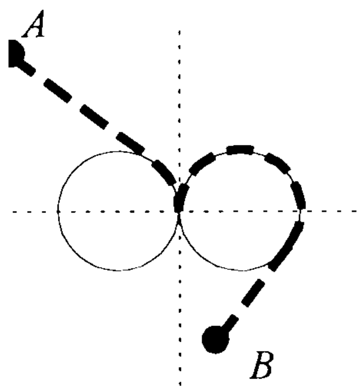

Figure 3a. How a velociraptor at the origin and facing upward gets to points A (at upper left) or B (at lower right).

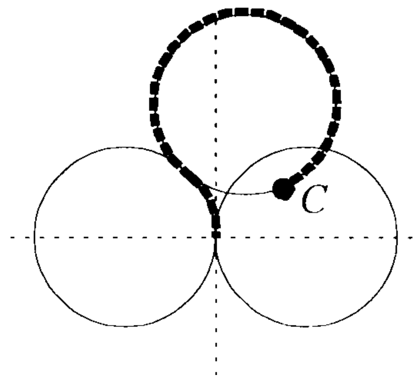

Figure 3b. How it gets to a point C inside the circle of minimum turning radius to its right.

Points inside the circles of minimum turning radius are more difficult for the velociraptor to access. It must somehow move such that destinations points inside these circles (e.g., point C in **Figure 3b**) are on or outside the circles of minimum curvature. The shortest way to do this is to turn away (in this case, to the left) from the destination point, following the other circle of minimum radius. Once the destination point lies outside the circles of minimum radius, the velociraptor follows the circles to the point. (For the calculation of this length, refer to the **Appendix**.)

We now define a new metric (velociraptor metric 1) on the plane: the distance along the curve from the origin (the location of the velociraptor, with the velociraptor facing the positive y-direction) to the point that the velociraptor follows. This metric is represented in **Figure 4** as a density plot with contour lines superimposed. Darker regions correspond to shorter distances for the velociraptor. The circles of minimum turning radius are easy to see, because of the discontinuity of the metric on the portions of the circles above the x-axis.

Thus far, we have considered the velociraptor as a point, though it has a grabbing radius of δ_v. To access a point, it need only reach any point a distance δ_v away from the desired point. We assume that the velociraptor minimizes how far it has to travel. Therefore, we replace the value of the metric at each destination point with the minimum value of the original metric on a disk of radius δ_v surrounding the destination point, yielding **Figure 5** (velociraptor metric 2). Note that the only parameters on which metrics 1 and 2 depend are the grabbing radius and minimum turning radius of the velociraptor, and that metric 1 is simply metric 2 with a grabbing radius of zero.

Now we treat a subtlety that we alluded to in the previous section. The origin of the coordinate system (the velociraptor's center of gravity) is actually

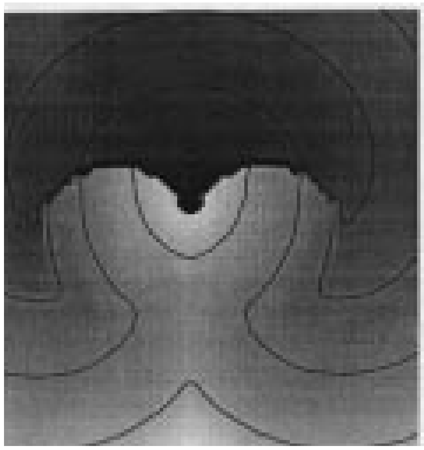

Figure 4. Velociraptor metric 1, depicted as a density plot.

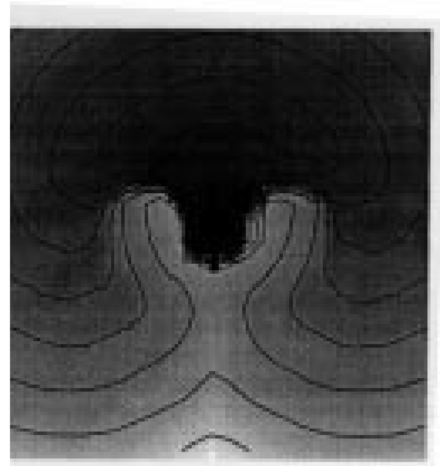

Figure 5. Velociraptor metric 2, depicted as a density plot.

0.6 m behind the center of the grabbing radius. However, one can see from **Figure 6** that given the circular and straight-line motions discussed above, the model of the situation remains exactly the same if we shift the origin to the center of the velociraptor-ball and simply change the effective minimum turning radius to $\sqrt{1.5^2 + 0.6^2}$ = 1.6 m.

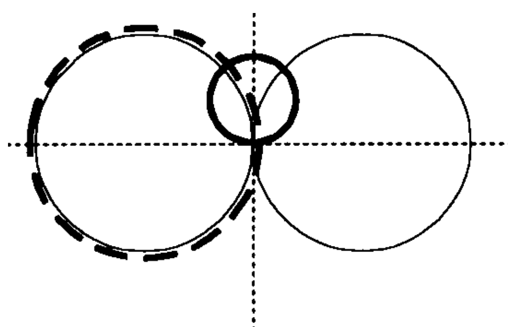

Figure 6. Result of shifting the origin to take into account the fact that the velociraptor is not a point.

We can further simplify our model in the following manner: We know that the velociraptor has caught the thescelosaur if their effective regions (circles of radius δ_v and δ_t) overlap. This is equivalent to saying that the centers of the two circles are separated by a distance less than $\delta = \delta_v + \delta_t$, or that the velociraptor has a grabbing radius of δ and the thescelosaur is a point. We thus define our metrics as described above, with effective turning radius 1.6 m and effective grabbing radius δ = 0.8 m.

Dinosaurs Have Peanut-Sized Brains

We first assume that the velociraptor acts to minimize, and the thescelosaur to maximize, the value of the velociraptor metric. We will see later that this is not sufficient to encompass all strategies (in particular, Encounter Strategy B, in which the thescelosaur heads straight for the velociraptor until the last possible moment), so we will soon make our dinosaurs a bit more sophisticated.

To evolve the system in time at a given time t, each dinosaur considers the location and heading of its opponent. It then chooses how to move during the next time step Δt. (Note that Δt is approximately equal to the effective reaction time, as it is only after the time Δt that the dinosaur can next evaluate the movement of its opponent.) As a range of options, the dinosaur chooses from a selection of arcs with length $v\Delta t$ and radii between the minimum turning radius and infinity (a line segment), shown in **Figure 7**. The dinosaur may choose the path with the most advantageous endpoint, or it may extrapolate each path several more time steps and choose from among those based on the metric evaluated at their endpoints. We vary this choice of strategy in our analysis, to optimize success rates of both predator and prey.

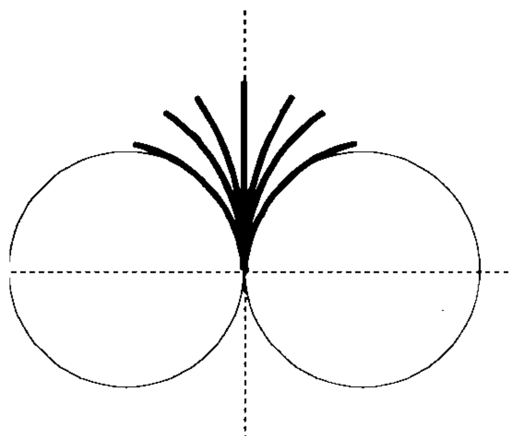

Figure 7. Possible strategies for the velociraptor.

We developed a computer simulation of the dinosaurs' behavior. Using the strategy of the velociraptor attempting to minimize the metric and the thescelosaur attempting to maximize it, we observed several phenomena discussed in the previous section. Most important, the dinosaurs chose paths similar to those used in determining the metric.

When the dinosaurs were separated by a distance larger than approximately 3 m, we observed the "approach" phase of the chase. The thescelosaur would run directly away from the velociraptor, while the velociraptor would adjust its course to trail directly behind, closing the distance. Under most circumstances, the thescelosaur would attempt to "shake" the velociraptor; but since the velociraptor was a sufficient distance behind, it was easy for it to adjust its course appropriately. Thus, we observed a rapid (on the order of a time-step of $\approx$ 0.01 s) small-amplitude oscillation of the thescelosaur's direction in the approach phase. In the simulation, once the velociraptor gets close enough to the thescelosaur, the thescelosaur adopts Encounter Strategy A. If it survives, it adopts one of the endgame strategies. In **Figures 8** and **9**, we show a hunt in which the thescelosaur successfully evades the velociraptor by using Encounter Strategy A followed by Endgame Strategy B.

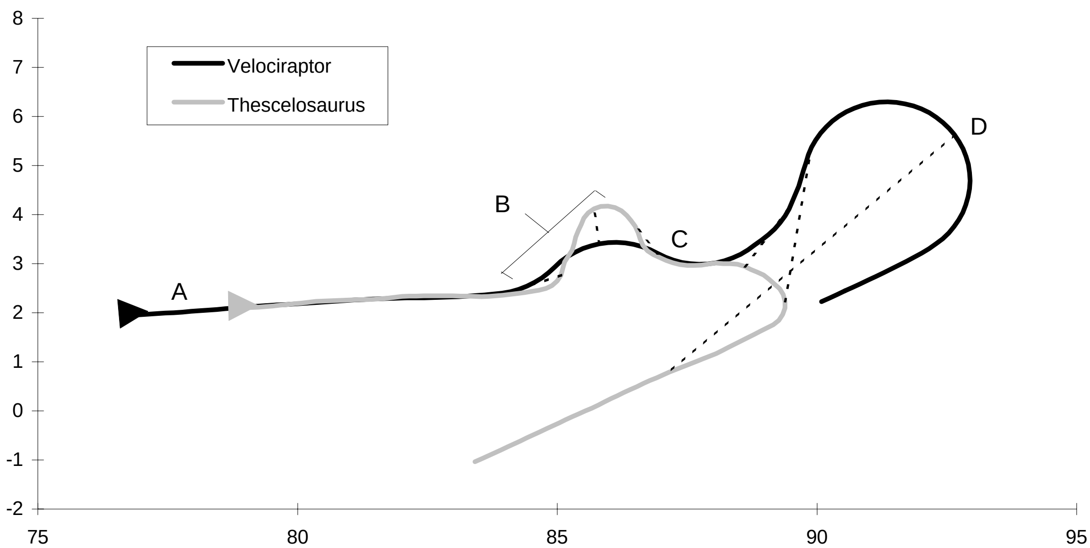

Figure 8. Encounter Strategy A, close up. Dotted lines connect points on the two curves that correspond to the same time. The velociraptor makes the big loop. This encounter strategy allows the agile thescelosaur to escape virtually every time if the velociraptor's grabbing radius is below a critical value (0.45 m). Unfortunately, the strategy fails with equal certainty if the velociraptor's grabbing radius is above that critical value. The velociraptor in this figure has a grabbing radius of 0.4 m. A few key stages are:

A. The thescelosaur runs away from the velociraptor.

B. When the velociraptor gets too close, the thescelosaur quickly turns out of the way.

C. The velociraptor cannot respond to this sudden turn fast enough, letting the thescelosaur duck behind it. Now, the velociraptor must loop around to continue chasing its meal.

D. The thescelosaur escapes before the velociraptor completes its loop.

We further found that the thescelosaur performed better using metric 2 looking only one time step ahead. Metric 2 is clearly advantageous for the prey, because this metric teaches it to stay out of the path of the predator's grabbing radius, rather than simply avoiding its center. The thescelosaur relies on its ability to maneuver quickly; thus it constantly adjusts its heading, rendering it useless for it to estimate several time steps into the future.

The velociraptor performed optimally using metric 1, looking 5 time steps ahead. We originally programmed the velociraptor to use metric 2, but it turned out to be a bit too cocky; the velociraptor was constantly disappointed as the thescelosaur barely slipped out of reach. When we changed its strategy to employ metric 1, this problem was eliminated. In a future investigation, it may be useful to make the velociraptor use metric 2 with a nonzero grabbing radius smaller than the actual value.

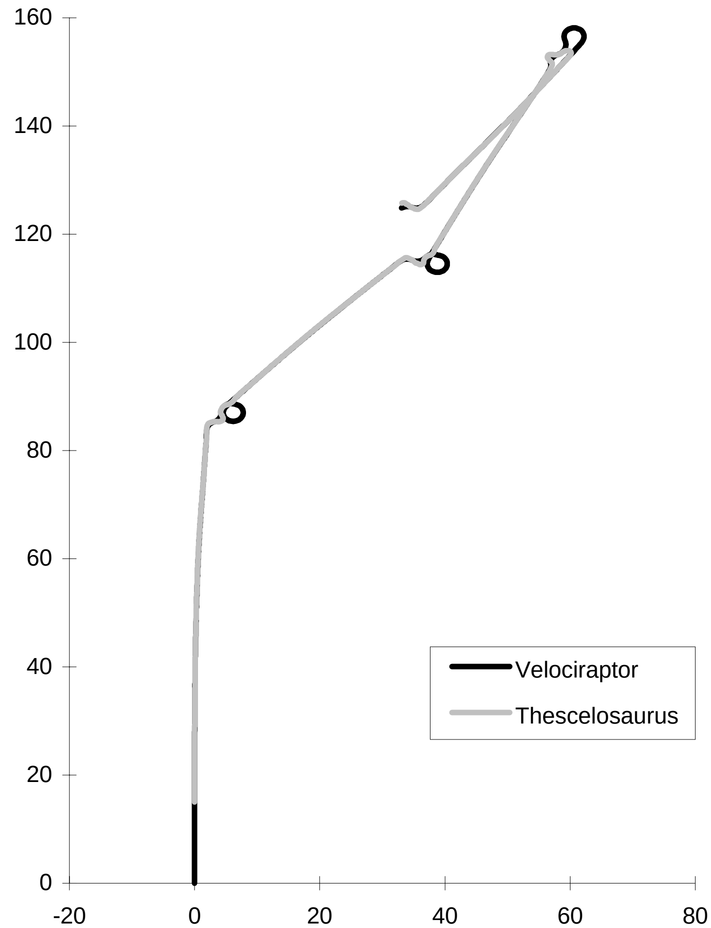

Figure 9. Encounter Strategy A, far away. After a round of Encounter Strategy A, the velociraptor soon catches up with the thescelosaur, creating a new encounter every 3.2 s. This figure shows a typical series of encounters for a 15-s chase when the thescelosaur detects the velociraptor at the minimum detection radius (15 m). Notice that the angle between the chase and escape paths is highly variable and sensitive to initial conditions. After 15 s, the velociraptor gets tired and the thescelosaur can simply run away.

We now ask what parameters allow the thescelosaur to survive. For the given speeds and minimum turning radius, the thescelosaur will always survive for values of the effective grabbing radius $\delta < 0.4$ m and is always captured for $\delta > 0.5$ m. In the region in between, the outcome is highly sensitive to initial conditions. Unfortunately for the thescelosaur, the given value of δ is actually 0.8 m. Thus, the thescelosaur should try Encounter Strategy B.

The Thescelosaur Learns to Play "Chicken"

Encounter Strategy B requires the thescelosaur to head directly toward the velociraptor, which is incompatible with the looking forward a few time steps to see which path will maximize its distance from the velociraptor (based on metric 2). Thus, we must modify our simulation to study this strategy.

We assume that the thescelosaur has sufficient time and distance to turn around and head directly toward the velociraptor when the encounter begins. We therefore consider only part 2 of Encounter Strategy B, in which the thescelosaur dives out of the path of the velociraptor just before collision.

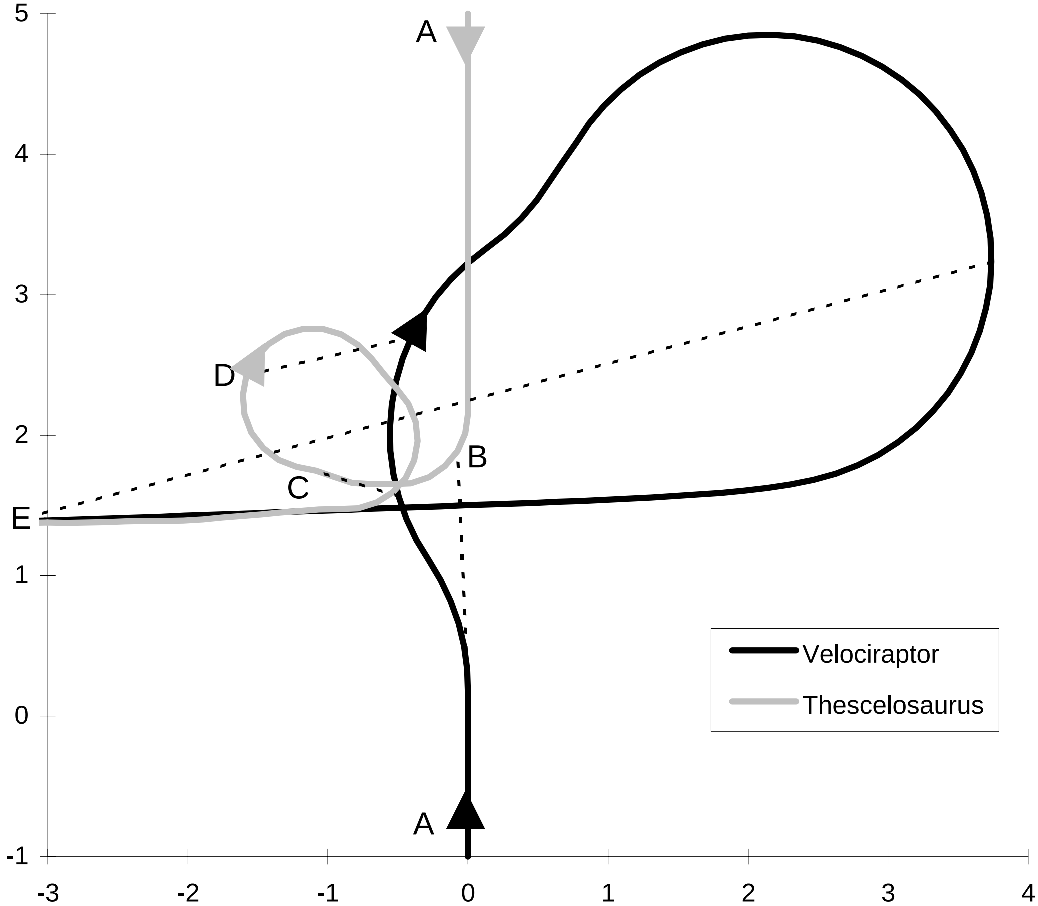

Figure 10. Encounter Strategy B. The thescelosaur must follow Encounter Strategy B if the velociraptor has a grabbing radius above 0.45 m (at which Encounter Strategy A no longer works). The velociraptor in this figure has a grabbing radius of 0.6 m. A few key stages are:

A. The thescelosaur runs towards the velociraptor.

B. When the velociraptor is too close, the thescelosaur dodges off to a side and the velociraptor follows.

C. The thescelosaur is now to the velociraptor's left, but the velociraptor curves to the right because it knows that it cannot make a sharp enough left turn to catch its meal.

D. The thescelosaur has sent the velociraptor on a huge loop while it swiftly makes a much tighter turn to come in behind the velociraptor.

E. The thescelosaur is far away even before the velociraptor completes its loop. Soon after this, the thescelosaur must turn around and run towards the velociraptor again.

We assume that once the thescelosaur begins to dodge, it is simply resuming its original strategy of maximizing distance (according to metric 2) between the two dinosaurs (see **Figure 10**). Thus, we can simulate this strategy using the original simulation, with the initial condition that the dinosaurs are heading straight toward each other separated by a small distance m, as shown in the second part of **Figure 3**.

We found that if the thescelosaur runs towards the velociraptor and starts turning when it is 2.15 m from the velociraptor, it will escape every time from a grabbing radius of 0.6 m, even if the other parameters are changed slightly. Thus Encounter Strategy B is a clear improvement over Encounter Strategy A.

A rough calculation reassures us that such a strategy indeed works for multiple passes, confirming our original assumption. After one such pass, the thescelosaur has a 7 m lead, and in the time it takes for the thescelosaur to turn around 180°, the velociraptor will gain about 1.9 m; this leaves the thescelosaur a bit of maneuvering before the critical 2.15 m turning point.

The Velociraptor Takes a Gamble, or, The Rational Dinosaurs

If the thescelosaur can escape using Encounter Strategy B, what is the velociraptor to do? We found that for a grabbing radius of 0.6 m, the 2.15 m critical distance had very little room for error—if the thescelosaur dodges too early or too late by 0.1 m, it will be caught. Thus, the velociraptor knows exactly when the thescelosaur will make its dodge.

If the velociraptor pursues its until-now optimal strategy of minimizing its metric, it will lose its dinner every time. Therefore, it should try to anticipate the movement of the thescelosaur; if the velociraptor guesses correctly which way the thescelosaur will swerve, and correct its own course accordingly, it can gain valuable time and thus catch its prey. However, if it does not guess correctly, it loses even more time than if it had merely gone straight. Moreover, the thescelosaur pursued by such a decision-oriented velociraptor, for its part, also wants to anticipate the movement of its predator.

We can model this as a game-theory problem. Consider the last possible moment before the thescelosaur must swerve. Under our original Encounter Strategy B, the thescelosaur swerves either left or right. The velociraptor, knowing this, should arbitrarily choose to swerve either left or right in this instance, giving it a 50% chance of guessing correctly and catching its prey. However, the thescelosaur knows this! So, if it is sure that the velociraptor will swerve, it should keep going straight past the critical point, and once the velociraptor swerves it can dodge the the other way an instant later. The velociraptor, knowing this, realizes it is not always to its advantage to anticipate the movement of the thescelosaur; perhaps the thescelosaur will anticipate its anticipation. In this situation, the velociraptor's optimal strategy is to keep going straight! The thescelosaur, then past the critical point, will be eaten.

Thus, it may be reasonable for the thescelosaur to move left (L), right (R), or stay straight toward the center of the velociraptor (C). The velociraptor can choose to anticipate these moves; we thus denote the velociraptor's strategy by

L, C, or R. If the velociraptor's guess is correct, we assume that it catches the thescelosaur, receiving a normalized payoff of 1, and the thescelosaur receives a payoff of 0. If the velociraptor guesses incorrectly, the thescelosaur survives the encounter, and the game will be played again at the next encounter, and so on until the velociraptor's endurance runs out.

If the thescelosaur swerves one way and the velociraptor anticipates the other (a "large miss"), then there will be a decent interval of time before the velociraptor catches up to the thescelosaur for the next encounter. If, however, one of the dinosaurs goes straight and the other swerves (a "small miss"), it will take less time for the velociraptor to catch up. Thus, there will be fewer encounters for the remainder of the hunt following a large miss than after a small miss, and thus the probability p that the thescelosaur survives the hunt after a small miss is less than the probability q that it survives after a large miss. In the small-miss case, the thescelosaur's payoff is therefore p, and that of the velociraptor is $1-p$. In the large miss case, the thescelosaur's payoff is q, and that of the velociraptor is $1-q$.

In this analysis, we have simplified the payoffs so all small misses result in the same payoffs and all large misses result in the same payoffs. This may not be entirely correct, as small misses come in two different forms: those in which the velociraptor goes straight, and those in which the thescelosaur goes straight. We have also assumed that the dinosaurs are symmetric and do not prefer one side to the other.

We would like to find any Nash equilibrium of this payoff matrix. It is clear that there is no pure equilibrium, so we look for a mixed strategy. Let a and b be the respective probabilities that the velociraptor and thescelosaur choose L. Since there is no difference between right or left, the probability of a dinosaur picking L equals the probability that it picks R. Thus, the probabilities that the dinosaurs choose strategy C are $1-2a$ and $1-2b$, respectively.

Finding the mixed Nash equilibrium is now easy. As in the elementary game-theory problem, each dinosaur wants to maximize its own expected payoff, and minimize that of the other. This occurs when the expected payoffs of the opponent are equal for any of its strategies.

Let $P_t(V|L)$ be the expected payoff for the thescelosaur if the velociraptor chooses L. Thus we have $P_t(V|L) = P_t(V|R) = (1-2b)q+bp$ and $P_t(V|C) = 2qb$. Setting these thescelosaur payoffs equal, we find $b = q/(4q-p)$.

Similarly, we have $P_v(V|L) = P_v(V|R) = a + (1-q)(1-2a) + a(1-p)$ and $P_v(V|C) = 2a(1-q) + (1-2a)$. Setting these equal, we find that a is also $q/(4q-p)$.

To determine the probabilities a and b, we must determine p and q. If there is time remaining in the chase for only one encounter, then $p = q = 1$, as the thescelosaur will not have another chance. Thus, $a = b = 1/3$, each dinosaur chooses each of its three strategies with equal probabilities, and the thescelosaur escapes 2/3 of the time! Thus, if both dinosaurs know that only one encounter remains, their expected payoffs from that encounter are 1/3 (for the velociraptor) and 2/3 (for the thescelosaur).

Now suppose (for example) that if there is a miss, there will be time for one more encounter if it is a small miss but not if it is a large miss. Thus, $p = 1$, since a large miss means that the thescelosaur survives the chase, and $q = 2/3$, since the probability of the thescelosaur surviving the next encounter is 2/3, from the previous paragraph. Therefore, $a = b = 2/5$.

Given more time for this study, time-dependent values of p and q could be determined, and we could determine the subgame-perfect Nash equilibrium of the dinosaurs for the entire chase.

Two Velociraptors: What Changes?

Approach

It is to the thescelosaur's advantage to run directly away from the velociraptors when the distance from them is large. With two velociraptors, this translates to the thescelosaur running so that the distance between it and one velociraptor remains the same as the distance between it and the other velociraptor. In this large-distance limit, the strategy for the velociraptors is also clear: They should run towards the thescelosaur using the same strategies as above.

There is one substantial difference between this case and that of one velociraptor: At the very beginning of the chase, the initial configuration is specified not only by D (which suffices for the single velociraptor and prey), but also by the angle made by the two velociraptors with the thescelosaur as the vertex. It is to the velociraptors' advantage to start out 180° apart, assuming that the thescelosaur moves in a straight line and the velociraptors continually adjust their directions to intercept it (a reasonable assumption in the far distance limit). This arrangement produces the maximum possible initial approach velocity: the velociraptor's maximum velocity v_v.

Encounter

By the time the velociraptors close in on the thescelosaur to the point that the prey will will have to start to curve, the configuration of the two velociraptors plus thescelosaur approaches one of only two cases. In the first case, both velociraptors run side by side and hence act roughly as one velociraptor with rather large turning and grabbing radii. In the second case, the velociraptors and the thescelosaur form a straight line but one velociraptor is behind another.

There are, then, two main strategies for the velociraptors: either to run side by side, or for one to run behind the other and pounce as soon as the thescelosaur starts turning. The side-by-side strategy is quite easy to model; as the two predators act as one, this case is a simple variant of the one-predator case. As one might expect, the critical grabbing radius for switching from

Encounter Strategy A to Encounter Strategy B turns out to be half that of the one-predator case.

The consecutive-velociraptor strategy, on the other hand, is a bit trickier. This strategy is meant for very small critical radii (0.3 m or less). The idea is for one velociraptor to "corral" the thescelosaur by curving toward it, even though according to the distance metric this actually makes the thescelosaur farther away. This maneuver restricts the movement of the thescelosaur by a great deal; the other velociraptor, which has been cruising behind the corraling velociraptor during this maneuver, can then circle in for the kill.

Preliminary studies using our simulation show that this strategy shows a great deal of merit. However, the simulator breaks down; the would-be corraling velociraptor curves the wrong way. Although we did not program the simulator to allow the animals to work together as the preceding paragraph implies, we are confident that this strategy will work for small radii if such experiments are done. Note also that it becomes much harder for the thescelosaur to dive between the predators, although this is probably possible for small enough critical radius.

Sensitivity of the Model

We tested the model by varying the parameters of the simulator, most notably the reaction time and the grabbing radius. Changing the curving radii and the relative velocities, while undoubtedly having a pronounced effect on the outcome, does not exhibit counterintuitive, extremely sensitive, or chaotic behavior.

On the other hand, Encounter Strategy A is rather sensitive to the reaction times of the dinosaurs. For instance, the critical grabbing radius varies from 0.4 m to 0.6 m over the range of reaction times that we tested. Changes in reaction time, along with small changes in initial position and velocity, significantly affected behavioral parameters, such as the angle between chase and escape paths. These effects have a biological interpretation: Escaping from a velociraptor using Strategy A involves exploiting a small window of opportunity during which the thescelosaur can duck out of the way before the velociraptor has time to respond. The thescelosaur must respond boldly to the slightest opportunity, resulting in a sensitivity that ensures that no two encounters are alike.

Encounter Strategy B exhibits a different kind of unpredictability. Although this strategy is less sensitive to initial conditions and reaction times, considerations of game theory require that each dinosaur choose randomly the direction in which to swerve during the encounter.

Thus, as expected, both models exhibit some unpredictable behavior. After all, the chase that we are modeling is a mortal battle of wits, not a preplanned ritual.

Strengths and Weaknesses of the Model

Our model has many strengths, perhaps the greatest of which is that the model is easy to understand: Minimization and maximization of a metric is a simple concept, and one that is not hard to implement.

Another prime strength of our model is the extreme robustness of the simulator. Not only can the simulator handle a wide range of similar scenarios simply by changing the parameters involved, but it can also handle a variety of different strategies simply by adjusting the initial conditions accordingly, as we did with Strategy B. However, we were not able to reprogram the simulator to deal with the cooperation between two predators.

Moreover, we feel that that the part of our model incorporating strategy B has many virtues to recommend it. Its robustness, as discussed in the previous section, lends it credibility as a feasible strategy. In addition, its applicability to a relatively large subset of turning radii makes it a better strategy than any other we could find. Furthermore, the game theory presented can be applied to most such finite games. However, we have a caveat in that in an actual situation involving live creatures, the prey would almost certainly not have the presence of mind to realize that running straight towards its predator would be the optimum strategy.

One of the main weaknesses of our model is the assumption that the dinosaurs go at top speed even when they are turning. More realistically, they should slow down to go around the curves at a reasonable centripetal acceleration.

Addendum: Realism Rears Its Ugly Head

The assumption that the dinosaurs are going at their top speeds even on curves is a rather poor one, due to the tremendous centripetal accelerations that would be involved. A better approximation is to model the dinosaurs' velocity on a circular arc as related to the radius of that circle. Since for centripetal acceleration, $a = v^2/4$, a first approximation is to assume that the dinosaurs can sustain an acceleration of, say, 3 g's, and can keep that maximum acceleration no matter what the radius of curvature is. Then we can model the velocity as a function of the radius as $v(r) = \min(\sqrt{a/r}, v_{\max})$.

The natural question we must ask is: Does this change the optimal strategy for the velociraptor?

A second approximation is to take into account the tangential acceleration and deceleration to the curved paths when the radius of curvature is changing.

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Appendix: The Velociraptor Metric

Outside the Minimum Turning Radius

Let the velociraptor (currently considered to be a point) be located at the origin, facing in the positive y-direction. Let the destination point be A, a distance l from the velociraptor, at an angle θ from the velociraptor's heading; thus, $A = (l\sin\theta, l\cos\theta)$. We abbreviate the minimum turning radius r_v as r. Let $B = (0, r)$ be the center of the circle of minimum turning radius, and let C be the point at which the velociraptor leaves the circle and moves along a straight line to point A. (Thus line AC is tangent to the circle.) Let $\alpha = \angle ABC$ and $\beta = \angle OBC$ (see **Figure 11**).

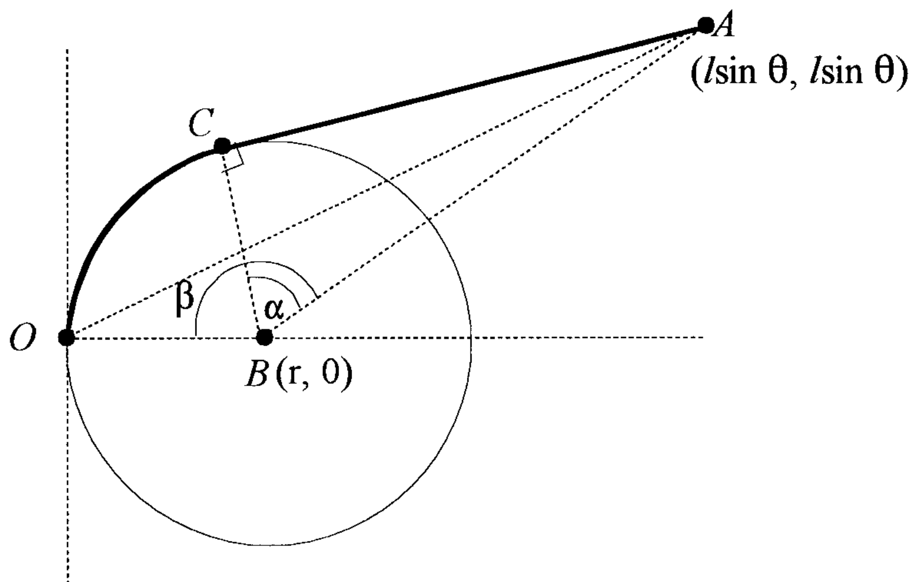

Figure 11. Diagram for the velociraptor metric outside the minimum turning radius.

The total distance that the velociraptor must travel is the length AC and the length of the arc OC, which is $r(\beta - \alpha)$. From the given coordinates, we have $AC = \sqrt{(l\sin\theta - r)^2 + (l\cos\theta)^2}$ and $BC = r$, so the length $AC = \sqrt{(l\sin\theta - r)^2 + (l\cos\theta)^2 - r^2} = \sqrt{l(l - 2r\sin\theta)}$. The law of cosines tells us that for a triangle with sides a, b, and c, the angle opposite the side with length c is $\cos^{-1}[(a^2 + b^2 - c^2)/2ab]$. We found AB above; we know $OB = r$ and $OA = l$; thus, we have angle β. The angle α is one of the acute angles in right triangle ABC, thus we can say that $\alpha = \tan^{-1}(AC/BC)$. Thus, the total length is

$$\sqrt{l(l - 2r\sin\theta)} + r\cos^{-1}\left(\frac{r - l\sin\theta}{\sqrt{l^2 - 2lr\sin\theta + r^2}}\right) - r\tan^{-1}\frac{l(l - 2r\sin\theta)}{r}.$$

Inside the Minimum Turning Radii

Let A be the destination point, P and Q the centers of the circles of minimum turning radius, and B the center of the right-hand circle of minimum turning radius after the velociraptor moves through arc OC, as shown in **Figure 12**. Let angles ϕ, α, and β be as shown. The velociraptor must move through minor arc OC and major arc AC, for a total length of $(2\pi - \phi + \alpha + \beta)$. Let A have coordinates (x, y). The Pythagorean theorem yields that $AP = \sqrt{(x + r)^2 + y^2}$ and $AQ = \sqrt{(x - r)^2 + y^2}$. The lengths PB, PQ, and AB are $2r$, $2r$, and r. Thus, we may apply the law of cosines as described in the subsection above and find the angles in question.

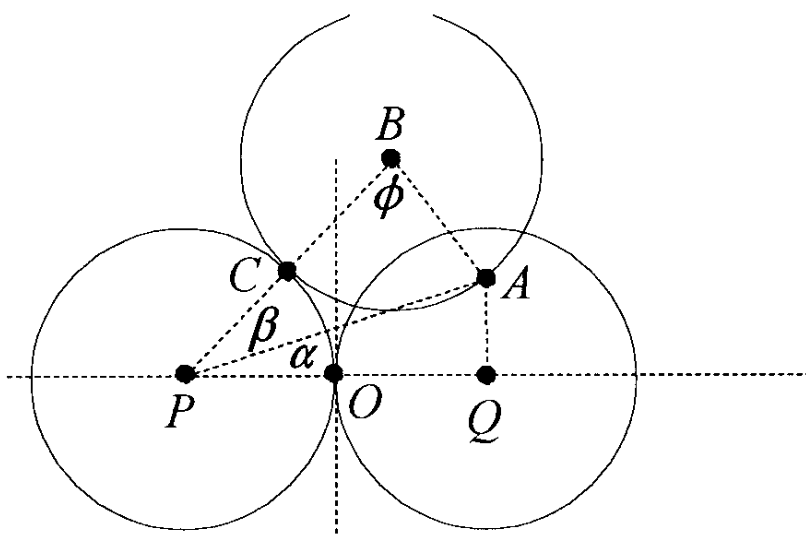

Figure 12. Diagram for the velociraptor metric inside the minimum turning radius.