An Assignment Model for Fruitful Discussions

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Introduction

We present a boolean programming model to solve a practical problem of giving assignments for a meeting.

Because the objective function of the model is nonlinear and because there are too many variables in the model, the problem is quite difficult to solve by means of general methods from integer programming. We use a greedy algorithm to get an initial feasible solution, then we optimize locally and iterate to approach the optimal solution.

We believe that our algorithm solves the given problem quite well. For the possibility that some board members will cancel at the last minute or that some not scheduled will show up, we give an adjustment method that makes the fewest necessary changes in assignments.

Our ideas admit of generalization. The parameters, such as the number of members, the number of types of attendees, and the number of different levels of participation can be varied, and the model and the algorithm always give a good solution. The model has the following advantages:

- It solves the given problem successfully, and it can generate a group of quite optimized solutions quickly.
- The model is general; it can give quite good solutions for different parameter values.
- It has lots of applications.

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Assumptions

- Three kinds of members attend the meeting:
  - senior officers (6),
  - in-house members (9),
  - other members (20).
- The whole meeting is divided into two stages:
  - A.M.: The morning meeting includes 3 sessions, each session consists of 6 discussion groups, and each group is led by a senior officers.
  - P.M.: The afternoon meeting includes 4 sessions, each session consists of 4 discussion groups, and no senior officer attends.
- The assignments should satisfy the following two criteria:
  - For the morning sessions, no board member should be in the same senior officer's discussion group more than once.
  - No discussion group should contain a disproportionate number of in-house members.
- No member changes groups during the meeting.

Table 1.

Description of the variables.

X	strategy vector; x_{ijk} means member i is or is not in group k in session j
P	dividing matrix
P_j	the dividing matrix of session j
Q	acquaintance matrix
Q_j	the acquaintance matrix of session j
Q_{sum}	the summary acquaintance matrix, $Q_{\text{sum}} = \sum_{j=1}^{7} Q_j$
$f(X)$	objective function, the number of 0s in matrix Q_{sum}
$g(X)$	objective function, the square of the norm of the matrix $$T = Q_{\text{sum}} - k \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix},$$ where k is a constant.

Analysis and Model Design

Preparation Knowledge

Divide the set $S = \{s_1, \dots, s_M\}$ into n groups $G_1, \dots, G_n$ and represent the division by the $m \times n$ matrix $P = (p_{ij})$, where $p_{ij} = 1$ if $s_i \in G_j$ and 0 otherwise. We call P the *dividing matrix*. We also consider the $m \times m$ matrix $Q = (q_{ij})$, where $q_{ij} = 1$ if s_i and s_j are in the same group ($i \neq j$, $1 \leq i, j \leq m$) and 0 otherwise (in particular, 0 on the diagonal). We call this the *acquaintance matrix*. We have a basic theorem that relates the dividing matrix and acquaintance matrix.

Theorem. *Let P be a dividing matrix. Then the corresponding acquaintance matrix is $Q = PP^T - E$, where E is a matrix of all 1s.*

Proof: Because each s_i can be in only one group, only one element of each row of P is 1 and the others are 0. We can easily calculate the elements of $Q = (PP^T - E)$:

$$q_{ij} = \begin{cases} \sum_{k=1}^{m} p_{ik}p_{jk}, & i \neq j; \\ \sum_{k=1}^{m} p_{ik}^2 - 1, & i = j. \end{cases}$$

If P_{ik} and P_{jk} are not both 1, then s_i and s_j and are not in the same group, so $q_{ij} = 0$.

If P_{ik} and P_{jk} are both 1, then s_i and s_j are in the same group, so $q_{ij} = 1$. □

Set

$$x_{ijk} := \begin{cases} 1, & \text{if member } i \text{ is assigned to group } k \text{ in session } j; \\ 0, & \text{otherwise.} \end{cases}$$

For our problem, i ranges from 1 to 29 (let $i = 1, \dots, 9$ be the in-house members), j from 1 to 7, and k from 1 to 6. For each assignment (session) j, there is a dividing matrix P_j and a corresponding acquaintance matrix Q_j.

Constraints

- Each member is assigned to only one group in each session:

$$\sum_{k=1}^{6} x_{ijk} = 1, \qquad i = 1, \dots, 29; j = 1, \dots, 3;$$

$$\sum_{k=1}^{4} x_{ijk} = 1, \qquad i = 1, \dots, 29; j = 4, \dots, 7.$$

- Each discussion group should contain a proportionate number of in-house members in each session:

$$1 \le \sum_{i=1}^{9} x_{ijk} \le 2, \qquad k = 1, \ldots, 6; j = 1, \ldots, 3;$$
$$2 \le \sum_{i=1}^{9} x_{ijk} \le 3, \qquad k = 1, \ldots, 4; j = 4, \ldots, 7.$$

- In the morning session, each of six discussion groups is led by a senior officer and no board member should be in the same senior officer's discussion group more than once:

$$0 \le x_{i1k} + x_{i2k} + x_{i3k} \le 1, \qquad i = 1, \ldots, 29; k = 1, \ldots, 6.$$

We seek a 0–1 three-dimensional matrix X that satisfies all these constraints. How can we judge whether it is good for our purposes?

Objective Function

The problem requires that the assignments should

- mix all the board members well,
- have each board member with each other board member in a discussion group the same number of times while minimizing common membership of groups for the different sessions.

Consider an extreme situation: If the 29 members are divided into just one group, the goal of mixing well is satisfied but the number of repetitions (session after session) is greatest. That is, when the number of the dividing groups is small, more sessions will increase the repetitions. At the same time, we must avoid too many people (thereby discouraging productive discussion) and a group being controlled or directed by a dominant personality. *Therefore, during the course of model design and solution, we consider only the situation of the 29 board members divided as equally as possible into groups.* We think that such a plan should minimize the number of repetitions, though we can't prove this claim. So we divide each morning session into 6 groups of 5, 5, 5, 5, 5, and 4, and each afternoon session into 4 groups of 7, 7, 7, and 8. And how can we describe or judge whether the results of the seven divisions are good mathematically? The number of times that each member meets may be calculated by following formula:

$$Q_{\text{sum}} = \sum_{j=1}^{7} Q_j = (q_{ij}^{\text{sum}})_{29\times 29} \qquad s, t = 1, \ldots, 29,$$

where

$$q_{ij}^{\text{sum}} = \sum_{l=1}^{3} \sum_{k=1}^{6} x_{ilk} x_{jlk} + \sum_{l=4}^{7} \sum_{k=1}^{4} x_{ilk} x_{jlk}, \qquad i, j = 1, \dots, 29.$$

The matrix Q_{sum} is called the *summary acquaintance matrix* of the division. Considering the goals, we think that the final summary acquaintance matrix should have as few zero elements as possible. It would be ideal for each nondiagonal element to be 1, with each main diagonal element 0.

Altogether, the assignments provide

$$3 \times \left(4 \times \binom{5}{2} + \binom{4}{2}\right) + 4 \times \left(3 \times \binom{7}{2} + \binom{8}{2}\right) = 532$$

chances for two individuals to meet each other. On the other hand, there are only $\binom{29}{22} = 406$ pairs of members. So every pair meets $K = 532/406 \approx 1.31$ times on average. Let

$$T = (t_{ij})_{29 \times 29} = Q_{\text{sum}} - K \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 0 \end{pmatrix}$$

and let

$$f(X) = \text{the number of 0 elements in } Q_{\text{sum}}, \quad \text{and} \quad g(X) \ = \|T\|^2 = \sum_{i=1}^{29} \sum_{j=1}^{29} t_{ij}^2,$$

where

$$t_{ij} = \begin{cases} \sum_{l=1}^{3} \sum_{k=1}^{6} x_{ilk} x_{jlk} + \sum_{l=4}^{7} \sum_{k=1}^{4} x_{ilk} x_{jlk} - K, & i, j = 1, \dots, 29,\ i \neq j; \\ \sum_{l=1}^{3} \sum_{k=1}^{6} x_{ilk}^2 + \sum_{l=4}^{7} \sum_{k=1}^{4} x_{ilk}^2, & i = j = 1, \dots, 29. \end{cases}$$

When the function $f(X)$ is minimized, the goal of mixing well will be satisfied best; and we think that an assignment would have each board member in a discussion with each other board member the same number of times when the function $g(X)$ attains a minimum.

Model

We have two objective functions to minimize, $f(X)$ and $g(X)$, and constraints as indicated earlier. This is a standard 0–1 integer programming and multiobjective programming problem. The task now is to solve this model.

Model Solution

Analysis

Although the constraints are linear, the objective functions are nonlinear. For this kind of integer programming problem, there is no general method to get the optimal solution efficiently; in fact, it is an NP-complete problem. For our problem instance, with 986 variables, the solution space has size at least least 6^{986}, so the method of exhaustion is infeasible. We must devise an efficient algorithm.

We find a good feasible solution and adjust it iteratively to approach the optimal solution, arriving at an acceptable approximately optimal solution.

Initial Feasible Solution

We use the greedy heuristic to get an initial feasible solution. The heuristic assigns the 29 members into each group one by one in each session. Before each member is assigned into one of groups, we examine which would be the the best group for the member.

Iteration

Because there are two objective functions, we use the strategy of multiobjective programming. We program the problem first using the function $f(X)$ until its value cannot be reduced. We can always get assignments that minimize $g(X)$ when $f(X)$ is minimized.

In the first step, we first adjust the vector for the seventh division to reduce the value of $f(X)$, then we perform exchanges (see below) to reduce $g(X)$ without affecting $f(X)$.

Then we similarly adjust the vector for the sixth division, then the fifth, and so on. Repeating the procedure until the values of the objective functions cannot be reduced further in this way, we obtain an approximate optimal solution.

Permutation

We still have more than 10^{80} different combinations, regardless of whether the members are divided into 6 or 4 groups in one session. It is still impossible for a computer to investigate each possibility. So we give a simple strategy. We exchange the seats of two members who are not assigned to the same group. If the exchange reduces the value of the objective function, it is acceptable; otherwise, we refuse to accept it. We repeat the procedure over all such pairs, thereby minimizing the value of the objective function.

Steps of the Algorithm

[EDITOR'S NOTE: For space reasons, we omit the detailed pseudocode of the algorithm.]

Solution

Using the greedy heuristic gives the plan of **Table 2** as the initial feasible solution, which has $f(X) = 209$ and $g(X) = 28.25$. Applying our iteration yields the result in **Table 3**, with $f(X) = 81$ and $g(X) = 19.44$.

Table 2.

Assignment table for initial feasible solution from greedy heuristic.

Morning.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Session 1	1 7 13 19 25	2 8 14 20 26	3 9 15 21 27	4 10 16 22 28	5 11 17 23 29	6 12 18 24
Session 2	4 11 14 21 26	5 10 15 23 28	6 12 16 20 29	1 8 17 24 27	2 7 18 22	3 9 13 19 25
Session 3	2 9 15 22 27	1 11 16 21 29	4 7 17 24 28	3 12 18 23	6 8 13 19 25	5 10 14 20 26

Afternoon.

	Group 1	Group 2	Group 3	Group 4
Session 4	3 7 11 14 20 22 27	1 5 9 16 18 24 26	4 8 12 15 19 23 28	2 6 10 13 17 21 25 29
Session 5	2 7 10 16 19 23 27	3 6 11 15 17 22 26	4 8 12 13 18 21 28 29	1 5 9 14 20 24 25
Session 6	1 6 10 14 18 21 27	2 5 11 13 19 22 26 29	3 7 12 16 20 24 28	4 8 9 15 17 23 25
Session 7	1 6 11 13 18 22 25	2 7 9 15 20 23 28	3 5 12 16 19 24 27	4 8 10 14 17 21 26 29

Table 3.

Assignment table after application of iteration.

Morning.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Session 1	1 6 13 19 24	2 3 14 25 26	5 8 10 15 27	7 16 20 22 28	9 11 21 23 29	4 12 17 18
Session 2	3 10 11 26 28	5 12 19 22 23	6 16 20 21 29	1 2 17 24 27	4 7 15 18	8 9 13 14 25
Session 3	5 7 14 17 29	4 13 15 16 21	1 9 19 26 28	3 11 18 23	6 8 12 25 27	2 10 20 22 24

Afternoon.

	Group 1	Group 2	Group 3	Group 4
Session 4	4 6 11 14 20 23 27	7 8 9 16 18 24 26	2 5 17 19 21 25 28	1 3 10 12 13 15 22 29
Session 5	1 4 10 14 16 23 25	5 6 11 15 17 22 26	2 8 13 18 19 27 28 29	3 7 9 12 20 21 24
Session 6	6 9 10 18 21 22 27	1 2 11 12 14 15 16 19	7 8 13 17 20 23 26	3 4 5 24 25 28 29
Session 7	1 5 11 13 18 20 25	2 6 9 15 23 24 28	3 7 10 16 17 19 27	4 8 12 14 21 22 26 29

We compare these two plans in **Table 4**. The initial feasible solution satisfies the criterion of balanced assignment. The final solution as optimized by the iterative method not only keeps the balance of the initial solution but also reduces the objective functions $f(X)$ and $g(X)$. We believe that the final solution satisfies the criteria of the problem. It attains the two goals of the problem, and it reduces not only the number of pairs that fail to meet (to 26) but also the number of multiple meetings.

Table 4.

Comparison of initial feasible solution with result after iteration.

Number of times a pair meets	0	1	2	3	4	5
Initial feasible solution from greedy heuristic	90	154	119	33	9	1
Solution after iteration	26	253	102	25	0	0

Adjusting Assignments

Let's consider how to adjust the assignments when some board members cancel at the last minute or some not scheduled show up. We could just renumber the attendees and solve the problem again. But in real life, we would not like to adjust the assignments too much. So we give another strategy for adjustment.

Case 1

When some members not scheduled show up at the last minute, we consider only how to assign without changing the given assignments. We use an adjustment method that is similar to the greedy heuristic. The additional members are assigned into the groups one by one. We always try to find the best group that the member should be in, according to the given constraint of keeping the assignments balanced and making the attendees mixed well.

Case 2

When some board members cancel, we delete one absentee at a time. We select one of the board members of the same type from the original assignments whose absence will lead to best mixing, that is, whose absence would reduce the value of the objective functions most. Then we let that member replace one of the absentees and delete the absentee in the list of assignments.

Case 3

When some board members cancel and some not scheduled show up at the same time, we classify them according to their types. Let a stand for the number of absentees and b stand for the number of additional members. For the members of the same type, we do the following operation:

- If $a = b$, then use the additional members to replace the absentees.
- If $a < b$, then first replace all of the absentees by some of the additional members. Then assign the remaining additional members using the method of Case 1.
- If $a > b$, then replace all of the additional members by some of the absentees, keeping the balance of the assignments. Then delete the remaining absentees using the method of Case 2.

Table 5 compares the results of such adjustment with the greedy heuristic and with the iterative algorithm, for several cases of absentees and additional members.

Table 5.

Comparison of results for the greedy heuristic (G), the iterative algorithm (I), and the adjustment algorithm (A) for various cases of absentees and additional members.

Situation			Method	Number of times that a pair meets					
Senior	In-house	Other		0	1	2	3	4	5
6	9	20	G	90	154	119	33	9	1
			I	26	253	102	25		
			A	no adjustment needed					
6	9	21	G	88	171	132	39	5	
			I	33	253	128	21		
			A	33	264	108	28	2	
6	9	19	G	76	145	126	29	2	
			I	24	236	100	16	2	
			A	26	235	93	24		
6	10	20	G	87	164	146	36	2	
			I	35	247	134	19		
			A	33	263	104	28	2	
6	8	20	G	74	140	140	24		
			I	26	231	103	17	1	
			A	23	237	95	23		

The adjustment strategy and the iterative algorithm are both satisfactory. It seems that the solution from the adjustment strategy should be as good as that from the iterative method; but in several cases (e.g., for 8 in-house members), the solution from the adjustment strategy is better.

Extension of the Model

Our model and solution method are completely general and apply to number of members, kinds of members, and levels of participation.

Assume that

- there are d types of attendees,
- the whole meeting is divided into w stages,
- there are S_i sessions for the ith stage, and
- each session consists of G_i discussion groups for the ith stage.

The assignments should also satisfy the following requirements:

- Each member can be assigned to only one group in each session.
- The attendees of type α are to be divided equally in a session.
- The whole assignment must always be balanced.
- An attendee of type L is not allowed to meet a particular senior officer more than c times in stage r.

Assume that there are b_i members for the ith type of attendee. Then there are $m = \sum_{i=1}^{d} b_i$ attendees, and we number them from 1 to m. The whole meeting involves w stages with S_i sessions in the ith stage, for a total of $S = \sum_{i=1}^{w} S_i$ sessions. We assume that each session in the ith stage is divided into discussion groups. We define all variables analogously to the simpler setting of the original problem, and we arrive at a generalized boolean programming problem.

Strengths and Weakness of the Model

The model has quite good practicality, and the given algorithm has little time complexity. For the given problem size, our C program for the greedy algorithm and iterative method runs in less than 5 min on a Pentium–100 computer. That means the model can give a list of assignments quickly when the number of attendees is not too large. The assignments produced are close to the optimal solution.

The model is quite easy to extend to different numbers of attendees, numbers of groups, types of attendees, and levels of participation.

We can adjust to last-minute changes either by re-running our program or (to minimize effect on assignments already made) by using our adjustment procedure.

The weakness of the model is that there is some difference between the real optimal solution and the solution obtained from the model.

References

Churchman, C.W. 1957. *Introduction to Operations Research.* New York: Wiley.

Hwang, C.L., and K.S. Yoon. 1981. *Multiple Attribute Decision Making.* New York: Springer-Verlag.

3.7.1 会议分组的优化(AMCM—97B 题)

在讨论重要问题尤其是长远规划问题时,越来越普遍的做法是召开小组讨论会,因为人们相信与会人数过多会有碍富有成效的讨论。而且权威人士通常能控制并操纵会议的讨论。因此,在举行全体会议之前,将以小组形式来讨论问题,即便如此,这些比较小的小组仍有被权威人士控制的危险,为了减少这种危险,通常将会议安排为几个场次,每次会有不同的人参加。

Tostal 公司将举行一个由 29 个公司董事参加的会议,其中有 9 位是在职董事(即,公司雇员)。会议要开一天,上午分三场,下午分四场,每场 45 分钟,开会时间从上午 9:00 到下午 4:00,中午 12:00 午餐。上午的每一场分 6 个小组开会,每个小组由公司派出 6 名资深官员之一作组长。这 6 名资深官员中没有一位是公司董事,并且他们不参加下午的会议。下午的每一场分为 4 个小组开会。

公司董事长要一份公司董事参加 7 场次的每个小组讨论会的分配名单。该名单要尽可能多地把董事均匀搭配。理想的搭配应为每一位董事和另一位董事一起参加讨论会的次数相同,与此同时要使不同场次的小组中在一起开过会的董事数目达到最小。

名单搭配满足下面两个准则:

① 上午的开会场次中，不能有一个董事参加过两次由同一个资深官员主持的小组会；

② 每个场次的分组讨论不应有不均匀数目的在职董事参加。

给出一张1～9号在职董事、10～29号董事、1～6号资深高级职员的搭配名单，说明该名单在多大程度上满足了前面提出的各种要求和准则。

因为有些董事可能临时不能到会，或者有些未被安排的人员将出席会议，因此，最好能设计一个算法，能够对将来不同参与类型的人参加和各种不同参与水平的会议进行安排则更好。

关于这个问题，在AMCM－97B题中一共收到175篇论文，其中有4篇被列为优秀论文，我们在此介绍其中的一篇，材料取自《数学的实践与认识》，*Vol*. 27，*No*. 4，*pp*：335～347，1997（华东理工大学 施政，杨辉，曹瀚，指导教师：鲁习文）。

假设条件

① 每种类型与会者地位相同；

② 与会者坚决服从会议组织者的安排；

③ 在整个会议进行过程中，不允许与会者变动。

变量与符号说明

$\boldsymbol{X}$——决策向量；其元素x_{ijk}表示在第j场会议中，成员i是否在第k组；

$\boldsymbol{P}$——分组矩阵；

$\boldsymbol{P}_j$——第j场会议的分组矩阵；

$\boldsymbol{Q}$——相遇矩阵；

$\boldsymbol{Q}_j$——第j场会议的相遇矩阵；

$\boldsymbol{Q}_{\text{sum}}$——总相遇矩阵：$Q_{\text{sum}}=\sum_{j=1}^{7}Q_j$；

$f(X)$——目标函数，定义为Q_{sum}中0元素的个数；

$g(X)$——另一个目标函数，定义为矩阵$\boldsymbol{T}$的范数平方。

$$
T = Q_{sum} - k\begin{bmatrix} 0 & 1 & 1 & \cdots & & 1 \\ 1 & 0 & 1 & \cdots & & 1 \\ 1 & 1 & 0 & \cdots & & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}
$$

其中 k 是常数，表示平均重复相遇次数。

问题分析和模型建立

(1) 预备知识

首先我们对分组作一个数学描述，令与会者集合 $S=\{s_1,s_2,\cdots,s_m\}$，将之分为 n 组：$G_1,G_2,\cdots,G_n$. 于是我们得到矩阵

$$
P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix}
$$

其中

$$
p_{ij} = \begin{cases} 1, & \text{如果 } s_i \in G_j (1 \leqslant i \leqslant m,\quad 1 \leqslant j \leqslant n), \\ 0, & \text{否则。} \end{cases}
$$

因为矩阵 P 清楚地表达了分组情况，我们定义它为分组矩阵。理想的情况是，通过分组矩阵所给出的信息，我们建立另一个矩阵，用它来判断元素 s_i 与 s_j 是否曾在同一个小组中，这个新的矩阵为

$$
Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1m} \\ q_{21} & q_{22} & a_{23} & \cdots & q_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ q_{m1} & q_{m2} & q_{m3} & \cdots & q_{mm} \end{bmatrix}
$$

其中

$$
q_{ij} = \begin{cases} 1, & \text{如果 } s_i \text{ 和 } s_j \text{ 在同一组中} (i \neq j, 1 \leqslant i,j \leqslant m), \\ 0, & \text{否则。} \end{cases}
$$

我们定义它为相遇矩阵，并由此得到一个关于分组矩阵和相遇矩阵的基本定理。

定理 若$\boldsymbol{P}$为一个分组矩阵，则其对应的相遇矩阵为$\boldsymbol{Q}=\boldsymbol{P}\cdot\boldsymbol{P}^{\mathrm{T}}-\boldsymbol{I}$（$\boldsymbol{I}$为单位阵）。

证 对于每一位与会者$s_i(i=1,2,\cdots,m)$来说，每次只能分在n个小组的某一个中，因此矩阵$\boldsymbol{P}$每一行只有一个元素为1，其余均为0，考虑矩阵$\boldsymbol{PP}^{\mathrm{T}}-\boldsymbol{I}$的元素$r_{ij}$直接计算得

$$r_{ij}=\begin{cases}\sum\limits_{k=1}^{n}p_{ik}\cdot p_{jk}, & i\neq j,\\ \sum\limits_{k=1}^{n}(p_{ik})^2-1, & i=j.\end{cases}$$

① 若s_i与s_j在同一组，则$q_{ij}=1$，此时意味着存在唯一的$k\in\{1,2,\cdots,n\}$，使$p_{ik}=p_{jk}=1$，而对其它l均有$p_{il}=0$，$p_{jl}=0$，因此$r_{ij}=1=q_{ij}$.

② 若s_i与s_j不在同一组，则$q_{ij}=0$，另一方面，对任意$k\in\{1,2,\cdots,n\}$，p_{ik}与p_{jk}不同时为1，因此有$r_{ij}=0=q_{ij}$.

总之有

$$\boldsymbol{Q}=\boldsymbol{PP}^{\mathrm{T}}-\boldsymbol{I}$$

(2) 约束条件

结合问题中的条件和要求，用人工变量x_{ijk}表示第i个人在第j场次会议中被分于第k组，

$$x_{ijk}=\begin{cases}1, & \text{若第 } j \text{ 个场次中会员 } i \text{ 被安排在第 } k \text{ 个讨论组},\\ 0, & \text{其它。}\end{cases}$$

其中，$i=1\sim29$代表29个参加会议的董事，而$i=1\sim9$代表9个在职董事，$i=10\sim29$代表其余非在职董事，$j=1\sim7$代表7个场次的会议，$k=1\sim6$（或4）代表每个场次分6(4)个小组。

于是，对每一场次的分组来说，存在一个分组矩阵$\boldsymbol{P}_j$满足

$$\boldsymbol{P}_j=\begin{bmatrix} x_{1j1} & x_{1j2} & \cdots & x_{1jN} \\ x_{2j1} & x_{2j2} & \cdots & x_{2jN} \\ \vdots & \vdots & & \vdots \\ x_{29j1} & x_{29j2} & \cdots & x_{29jN} \end{bmatrix},$$

其中 $N=6$ 或 4。

再根据问题的条件和要求，得到下列约束条件：

① 在每一次分组中，每人只能分在一组中

$$\begin{cases} \sum\limits_{k=1}^{6} x_{ijk}=1, & i=1,2,\cdots,29, \quad j=1,2,3; \\ \sum\limits_{k=1}^{4} x_{ijk}=1, & i=1,2,\cdots,29, \quad j=4,5,6,7; \end{cases}$$

② 每次分组时，每组中在职董事尽量均衡

$$\begin{cases} 1\leqslant \sum\limits_{i=1}^{9} x_{ijk}\leqslant 2, & k=1,2,\cdots,6, \quad j=1,2,3; \\ 2\leqslant \sum\limits_{i=1}^{9} x_{ijk}\leqslant 3, & k=2,3,4, \quad j=4,5,6,7; \end{cases}$$

同时保证非在职董事尽量均衡，

$$\begin{cases} 3\leqslant \sum\limits_{i=10}^{29} x_{ijk}\leqslant 4, & k=1,2,\cdots,6, \quad j=1,2,3; \\ 5\leqslant \sum\limits_{i=10}^{29} x_{ijk}\leqslant 6, & k=1,2,3,4, \quad j=4,5,6,7; \end{cases}$$

③ 上午阶段每场会议，6 个小组每一个都要有一名资深官员主持，每个参加会议的董事都不能与任一名资深官员重复见面。

$$0\leqslant x_{i1k}+x_{i2k}+x_{i3k}\leqslant 1, \quad i=1,2,\cdots,29, \quad k=1,2,\cdots,6.$$

(3) 目标函数

$\boldsymbol{P}_j(j=1,2,\cdots,7)$代表了 7 次分组的分组矩阵，那么我们如何判断 7 次分组的效果是好还是坏呢？

对于这个问题，我们可以参考问题的要求，问题要求模型给出的分组方案应使所有与会董事混合得最好，并且在每个场次中保

证董事们尽可能相互认识的基础上，使委员们重复见面的次数尽量平均。先考虑一种极端情况：29 名全分在一个组中，这样，充分混合的目标是最好的得到满足了，但是重复见面的次数也达到最大。因此，当分组数小或者各组人数不均衡时，开会场次越多，与会者重复见面的次数就越大。同时，我们还必须注意避免人数太多影响讨论的效果，或者被权威人士所影响和操纵。在模型建立和求解过程中，只考虑 29 个委员均分为 6 组或 4 组的情况，此时，重复见面次数总和将达到最小。因此，将参加上午会议的 29 个委员分为 5,5,5,5,5,4，共 6 组；而将下午的分为 7,7,7,8，共 4 组。

现在我们用数学语言描述 7 次分组的优劣。对于第 j 次分组而言，有分组矩阵 P_i，又存在相遇矩阵 Q_i，7 次分组后，董事 1～29 相互见面的总次数可通过以下公式来计算

$$Q_{\text{sum}} = \sum_{j=1}^{7} Q_j = (q_{ij}^{\text{sum}})_{29\times 29}$$

其中

$$q_{ij}^{\text{sum}} = \sum_{l=1}^{3}\sum_{k=1}^{6} x_{ilk}\,x_{jlk} + \sum_{l=4}^{7}\sum_{k=1}^{4} x_{ilk}x_{jlk},\quad i,j = 1,2,\cdots,29$$

矩阵 Q_{sum} 称为分组的总和相遇矩阵。考虑到充分交流与任意两个会员重复见面次数尽量相同这两个目标，应该使总的相遇矩阵 Q_{sum} 中 0 元素的个数达到最小。理想状态是：除了主对角线上元素为 0 以外，Q_{sum} 中别的元素均相同（即任意两个与会成员之间见面次数相同）。根据前述均匀分组的原则，与会者相互见面的次数为 $3\times(5\times c_5^2+c_4^2)+4\times(3\times c_7^2+c_8^2)=532$，另一方面只需要 $c_{29}^2=406$ 次相互见面的机会可以使 29 个会员两两相遇，因此，任意两个与会成员平均见面

$$k = 532/406 = 1.31\text{ 次},$$

令

$$T = (t_{ij})_{29\times 29} = Q_{\text{sum}} - k\begin{bmatrix} 0 & 1 & 1 & \cdots & & 1 \\ 1 & 0 & 1 & \cdots & & 1 \\ 1 & 1 & 0 & \cdots & & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \end{bmatrix}$$

$$f(x) = \sum_{i=1}^{29}\sum_{j=1}^{29} I_{ij},$$

$$g(X) = \| T \|^2 = \sum_{i=1}^{29}\sum_{j=1}^{29} t_{ij}^2$$

其中

$$I_{ij} = \begin{cases} 1, & \text{如果 } q_{ij}^{\text{sum}} = 0, \\ 0, & \text{否则}, \end{cases} \qquad i,j = 1,2,\cdots,m,$$

$$t_{ij} = \begin{cases} \sum_{l=1}^{3}\sum_{k=1}^{6} x_{ilk}x_{jlk} + \sum_{l=4}^{7}\sum_{k=1}^{4} x_{ilk}x_{jlk} - k, \\ i,j = 1,\cdots,29, \quad i \neq j \\ \sum_{l=1}^{3}\sum_{k=1}^{6} x_{ilk}^2 + \sum_{l=4}^{7}\sum_{k=1}^{4} x_{ilk}^2, \\ i = j = 1,2,\cdots,29. \end{cases}$$

可以认为，$g(X)$达到最小时，任意两个成员重复见面次数达到尽量均衡。而当 $f(x)$达到最小时，充分见面之目标将得以最好地满足。

（4）建立模型

根据前面的讨论，我们建立如下的模型：

目标函数

$$\min f(X) = \sum_{i=1}^{29}\sum_{j=1}^{29} I_{ijk},$$

$$\min g(X) = \| T \|^2 = \sum_{i=1}^{29}\sum_{j=1}^{29} t_{ij}^2.$$

约束条件

$$\sum_{k=1}^{6} x_{ijk} = 1, \quad i = 1,2,\cdots,29, \quad j = 1,2,3,$$

$$\sum_{k=1}^{4} x_{ijk} = 1, \quad i = 1,2,\cdots,29, \quad j = 4,\cdots,7$$

$$1 \leqslant \sum_{i=1}^{9} x_{ijk} \leqslant 2, \quad k = 1,2,\cdots,6, \quad j = 1,2,3;$$

$$2 \leqslant \sum_{i=1}^{9} x_{ijk} \leqslant 3, \quad k = 1,2,3,4 \quad j = 4,\cdots,7$$

$$3 \leqslant \sum_{i=10}^{29} x_{ijk} \leqslant 4, \quad k = 1,2,\cdots,6, \quad j = 1,2,3;$$

$$5 \leqslant \sum_{i=10}^{29} x_{ijk} \leqslant 6, \quad k = 1,2,3,4, \quad j = 4,\cdots,7;$$

$$0 \leqslant x_{i1k} + x_{i2k} + x_{i3k} \leqslant 1,$$
$$i = 1,2,\cdots,29, \quad k = 1,2,\cdots,6.$$

其中

$$x_{ijk} \in \{0,1\},$$

$$\boldsymbol{I}_{ij} = \begin{cases} 1, \text{ 如果 } q_{ij}^{\text{sum}} = 0, \\ 0 \text{ 否则}, \end{cases} \quad i,j = 1,2,\cdots,29.$$

$$\boldsymbol{T} = (t_{ij})_{29\times 29}$$

$$= \boldsymbol{Q}_{\text{sum}} - k \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

这是一个标准的整数规划和多目标规划问题，下面讨论其求解方法。

模型求解

不难看出，上述模型是一个标准的多目标0—1整数规划问题，且是一个NP完全问题。由于模型含变量个数为986个，用穷举法搜索求最优是不现实的，因此只能探讨有效的近似算法，其想法是先找出一个满足所有条件的初始可行解，在这组解基础上逐步迭代，逐步逼近最优解。

现在我们对迭代法的思想作一个简单描述，首先通过贪婪算法找到一组初始可行解，然后运用坐标轮换法固定其中六组向量(每一组向量均代表一个场次的分组情况)，调整剩下的那一组向量使目标函数减少，下一步，我们再调整代表第六场次分组情况的向量组使目标函数值减小，不断重复该过程，便可以获得一组近似最优解了。由于目标函数有两个，可以使用多目标规划级别优先的策略：首先利用 $f(X)$ 进行迭代，直到不能再减少为止，然后在 $f(X)$ 不变的基础上对 $g(X)$ 迭代，使其尽量可能小。算法具体描述如下：

算法 A 求出一组初始可行解

① 令 ses＝0，num＝0(“ses”代表会议的不同场次，“num”代表各成员的序号)；

② ses：＝ses＋1；

③ num：＝num＋1；

④ 如果 ses≤3，就必须考虑每一位董事不得与资深官员重复见面的限制条件；

⑤ 如果 num≤9，就必须考虑到公司委员被均分的条件；

⑥ 设每一位资深官员见过的与会成员保持均衡；

⑦ 计算目标函数 $f(X)$ 和 $g(X)$，找出一个最佳位置使成员插入后目标函数 $f(X)$ 和 $g(X)$ 尽可能小；

⑧ 当有多个组可选择时，选择序号较小的组；

⑨ 如果 num≤28，回到③，否则转⑩；

⑩ 如果 ses＜7，回到②，否则转算法 B.

算法B　迭代算法得近似最优解

① ses:=7;

② 输入初始可行解;

③ 调整分组矩阵 P_{ses},在保证 $P_j(j\neq ses)$不变的基础上交换满足下列条件的每一对成员的位置:两个成员必须是同一类型的参与者,必须在不同的组中,在上午的三次分组中不得与资深官员重复见面,目标函数 $f(X)$的值在交换后必须变小;

④ 交换满足下列条件的每一对成员:两个成员必须是同一类型的参与者、必须在不同的组中、在上午的三次分组中不得与资深官员重复见面、目标函数 $f(X)$的值保持不变,目标函数 $g(X)$的值在交换后必须变小;

⑤ ses:=ses−1;

⑥ 若 ses>0,则回到③;

⑦ 若一轮循环(ses=7 到 ses=1)后无一对相交换便退出,否则以当前解回到②。

利用算法A,可以算出如下初始可行方案:见表 3-3～3-4

表 3-3　上午分配表

	第1组	第2组	第3组	第4组	第5组	第6组
第一场	1,7,13,19,25,	2,8,14,20,26,	3,9,15,21,27,	4,10,16,22,28,	5,11,17,23,29,	6,12,18,24,
第二场	4,11,14,21,26,	5,10,15,20,28,	6,12,16,20,29,	1,8,17,24,27,	2,7,18,22,	3,9,13,19,25,
第三场	2,9,15,22,27,	1,11,16,21,29,	4,7,17,24,28,	3,12,18,23,	6,8,13,19,25,	5,10,14,20,26,

表 3-4　下午分配表

	第 1 组	第 2 组	第 3 组	第 4 组
第四场	3,7,11,14, 20,22,27,	1,5,9,16, 18,24,26,	4,8,12,15, 19,23,28,	2,6,10,13, 17,21,25,29,
第五场	2,7,10,16, 19,23,27,	3,6,11,15, 17,22,26,	4,8,12,13, 18,21,28,29,	1,5,9,14, 20,24,25,
第六场	1,6,10,14, 18,21,27,	2,5,11,13, 19,22,26,29,	3,7,12,16, 20,24,28,	4,8,9,15, 17,23,25,
第七场	1,6,11,13, 18,22,25,	2,7,9,15, 20,23,28,	3,5,12,16, 19,24,27,	4,8,10,14, 17,21,26,29,

$$f(X) = 209$$

且

$$g(X) = 28.35.$$

根据算法 A 得到的初始可行解，由算法 B 得到如下近似最优方案：见表 3-5～3-6。

表 3-5　上午分配表

	第 1 组	第 2 组	第 3 组	第 4 组	第 5 组	第 6 组
第一场	1,6,13, 19,24,	2,3,14, 25,26,	5,8,10, 15,27,	7,16,20, 22,28,	9,11,21, 23,29,	4,12,17, 18,
第二场	3,10,11, 26,28,	5,12,19, 22,23,	6,16,20, 21,29,	1,2,17, 24,27,	4,7,15, 18,	8,9,13, 14,25,
第三场	5,7,14, 17,29,	4,13,15, 16,21,	1,9,19, 26,28,	3,11,18, 23,	6,8,12, 25,27,	2,10,20, 22,24,

$$f(X) = 81$$

且

$$g(X) = 19.44.$$

表 3-6　下午分配表

	第 1 组	第 2 组	第 3 组	第 4 组
第四场	4,6,11,14, 20,23,27,	7,8,9,16, 18,24,26,	2,5,17,19, 21,25,28,	1,3,10,12, 13,15,23,29,
第五场	1,4,10,14, 16,23,27,	5,6,11,15, 17,22,26,	2,8,13,18,19, 27,28,29,	3,7,9,12, 20,21,24,
第六场	6,9,10,18, 21,22,27,	7,8,13,17, 20,23,26,	1,2,11,12,14, 15,16,19,	3,4,5,24, 25,28,29,
第七场	1,5,11,13, 18,20,25,	3,7,10,16, 17,19,27,	2,6,9,15, 23,24,28,	4,8,12,14, 21,22,26,29,

在算法 B 中不难得到相互不见面、见 1 次、2 次、3 次面的个数分别为 26,253,102,25 次,见 4 次以上面的情况没有出现。将资深官员分别插入 6 个不同组中即可得到完整的最佳会议安排表。

调整算法

现在,我们考虑在最后一刻,一些计划参加的董事未能参加和一些未计划参加的董事将出席会议。

一种最简单的方法是将与会成员重新编号然后按照算法求解模型,但这样做不太经济,希望能利用已有的安排作局部调整使之达到新的近似最优。

第一种情况:最后一刻,一些未计划参加的董事将出席会议。我们使用一种贪婪算法在不改变原有安排,同时使额外参加会议的董事能保持平衡和使与会者最大可能的相见的前提下,得到一种新的安排方案。

第二种情况:最后一刻,一些计划参加会议的董事未能参加,此时可考虑将这些会员从计划中删除,删除后的方案作为初始可行解,启动算法 B 得新的最优方案。

模型推广

不难看出，在建模和求解过程中对参数诸如：会员的总数、会议的场次、会员的类型和参与的层次均没有特殊要求。因此模型容易推广到更一般情形。

设有 d 类会员，整个会议被分成 w 段，第 i 个阶段有 S_i 个场次且每个场次由 G_i 个讨论组组成，安排也应满足以下要求：

① 在每个场次中，每一个会员仅能安排到一个组中；

② 第 β 个场次中，α 类型的会员要在每个场次中平衡的分布；

③ 整个安排情况也要保持平衡；

④ 在第 r 阶段中，L 类型的会员不允许超过 c 次与资深官员相见。

假定第 i 种类型的会员有 b_i 个，那么总的会员数为 $m=\sum_{i=1}^{d} b_i$，我们将其从 1 到 m 进行编号。例如：第 k 类型的第 i 个会员编号为 $\sum_{j=1}^{k-1} b_j+i$，整个会议包括 w 个阶段，第 i 阶段有 $S_i(i=1,2,\cdots,w)$ 个场次。记 $S=\sum_{i=1}^{w} S_i$，代表全部场次之和。我们假定在第 i 个阶段每个场次都将会员划分为 $G_i(i=1,2,\cdots,w)$ 个讨论组。

现在我们定义下列决策变量 X_{ijk}.

$$X_{ijk}=\begin{cases}1, & \text{如果在第 } j \text{ 个场次会员 } i \text{ 被安排在第 } k \text{ 组，} \\ 0, & \text{否则}\end{cases}$$

$(i=1,2,\cdots,m;\ j=1,2,\cdots,s;\ k=1,2,\cdots,\max(G_j))$。

则可按照前面的思想方法建立与求解更一般的模型。

模型评价

① 模型结构清晰、层次分明，数学表达式含义直观、明确、易懂；

② 模型推广性很好，能针对不同情况设计出一般参数，得到满意的效果；

③ 目标函数的建立独具匠心，充分利用优化思想，很完善地解决了多变量非线性 0—1 整数规划问题，具有较好的应用价值。