"We're Sorry, You're Outside the Coverage Area"

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Our Approach

We assume that the physical transmission properties do not result in penetration or interference varying from channel to channel. Since the channels occupy a continuous portion of the frequency spectrum, they can be numbered with integers from 1 up to the number n of channels; n represents the bandwidth of the portion of the spectrum. The minimum possible value for the bandwidth that achieves all the requirements for a given transmitter arrangement we call the *span* of that arrangement.

Our problem is to provide a method for arranging channels among the transmitters that achieves as low a bandwidth as possible. We analytically establish a lower limit for the span and we find the best solutions that we can to the given problem.

The bandwidth of a feasible solution is an upper limit on the span. We seek to raise (through further analysis) the lower limit and to lower (through further construction) the upper limit. If our upper limit meets our lower limit, then we have completely determined the span.

Requirements A and B are special instances of the more general problem posed in Requirement C; we treat A, B, and C as one main problem and solve them together. Requirement D asks us to consider generalizations of the problem; we examine how to treat some of the weaknesses in our main problem. Finally, Requirement E asks us to write an article for the local newspaper, which appears at the end.

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The Main Problem

At any point in the plane, a receiver can obtain a clear signal from transmitters within some maximum range, without interference from the signals broadcast by neighboring transmitters.

Assumptions

- The region of interest is partitioned into adjacent regular hexagons of the same size.
- The length of the side any hexagon is s.
- Each hexagon represents the area serviced by one transmitter, which is located in the center of its hexagon.
- Each transmitter broadcasts a single channel.
- To minimize interference, any two transmitters occupying the same channel must be at least $4s$ apart.
- To minimize sideband interference, transmitters less than $2s$ apart must use channels that differ by at least k channels. (In Requirements A and B, $k = 2$.)
- The region of interest is a field of indefinite size and shape. A specific region is given for Requirement A and for a special case of Requirement C.

Definitions

- **Cell:** The area serviced by a given tower—in this case, hexagons.
- **Distance:** The minimum number of edges that must be crossed to move from one cell to another. Any hexagon in an infinite field is surrounded by six cells of distance 1, twelve of distance 2, and so on.
- **Separation:** The absolute value of the difference between the integers assigned to two channels.

The Model

Each hexagon is labeled with the integer channel of its transmitter. To satisfy the $2s$ requirement, adjacent hexagons must have a channel separation of at least k. To satisfy the $4s$ requirement, neither adjacent hexagons nor hexagons that are a distance of 2 away can be labeled with the same integer; they must have a separation of at least 1. In **Figure 1**, the inner ring of 6 unmarked hexagons cannot assume the same label as the center hexagon or labels with a separation of less than k. The surrounding ring of 12 barred hexagons cannot be labeled with the same value as the center.

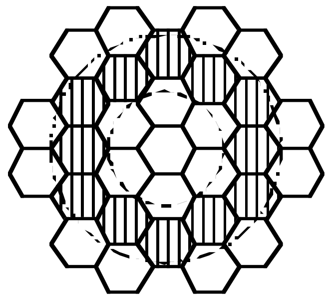

Figure 1. The dashed circle has radius $2s$, the dash-dotted circle has radius $4s$.

We establish a minimal value for the span:

Theorem. *The span of any solution is at least 7.*

Proof: Given any hexagon X, there are 6 additional hexagons whose centers are within a distance of $2s$ from the center of X. Any two of these 7 hexagons have centers that are less than $4s$ apart. Therefore, no 2 of the 7 hexagons can have the same label. □

Theorem. *For $k = 1$, the span is 7.*

Proof: The span is at least 7. A solution with 7 as the largest label used is illustrated in **Figure 2**. A solution for the infinite region of interest can be constructed by tiling the plane with the center group of 7 cells. □

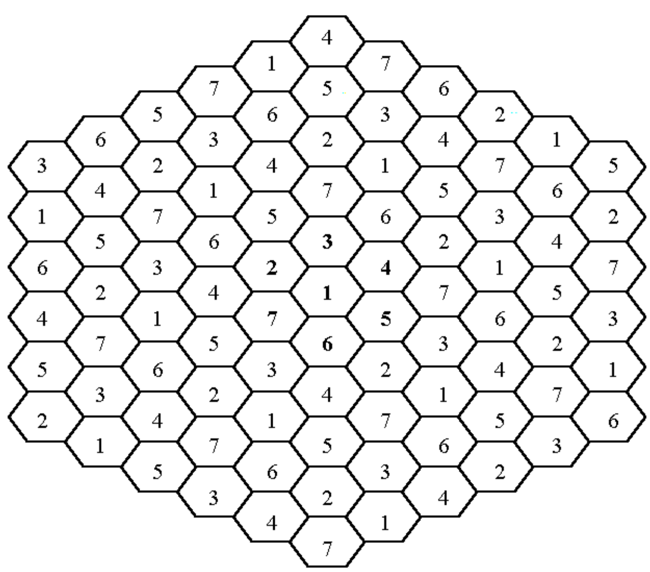

Figure 2. Solution showing that the span is 7.

Theorem. *For any $k \geq 1$, the span is at least $2k+5$.*

Proof: Select three cells such that each is adjacent to the others and no cell is on the boundary of the region of interest. No two of these three cells can assume the same label. Thus, there must be a maximum, a middle, and a minimum value: A, B, and C respectively. Since two adjacent cells' labels cannot be separated by less than k, we know that $A - B \geq k$ and $B - C \geq k$.

Consider the six hexagons adjacent to B. The minimum separation between B and any adjacent hexagon is k, so the labels of the cells adjacent to B cannot be any of B, $B \pm 1$, $B \pm 2, \ldots, B \pm (k-1)$. Therefore, there are $2(k-1)+1$ channels between A and C that cannot be adjacent to B. There must be at least 6 distinct channels adjacent to B; thus, there must be at least $2(k-1)+7$ channels, so the span is at least $2k+5$. □

Theorem. *For $k = 2$, the span is 9.*

Proof: By the preceding theorem, the span is at least 9. A solution for Requirement A with 9 as the largest label used is illustrated in **Figure 3**. □

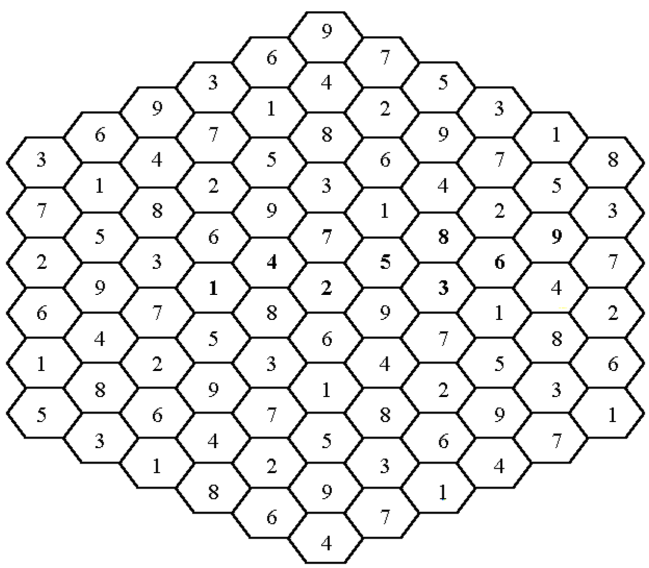

Figure 3. Solution to Requirement A with span 9.

The solution contains a center repeating group of 9 cells that tiles the plane, hence providing a solution to the case in Requirement B. An additional method of generating this arrangement is to add 2 whenever you move from one cell to its neighbor up and to the left, add 3 when you move up and to the right, and add 4 when you move down; if the result is greater than 9, subtract 9. Label each cell with the result and continue in all directions. If you start at a cell labeled x and move once in each direction, you will return to the starting cell. You will have added 2, 3, and 4 to x and subtracted 9 once, giving a net label of x. Therefore, this labeling algorithm is consistent.

At this point, the span has been completely specified for the cases $k = 1, 2$.

Theorem. *For an infinite region of interest and for $k \geq 2$, the span is no more than $3(k+1)$.* (This extends the spanning solution for $k = 2$.)

A solution for $k = 3$ is displayed in **Figure 4**; the group of 12 cells numbered in bold tiles the plane. We do not prove this theorem for general k, because we improve on it shortly.

We summarize in **Table 1** what we know about the span as a function of k.

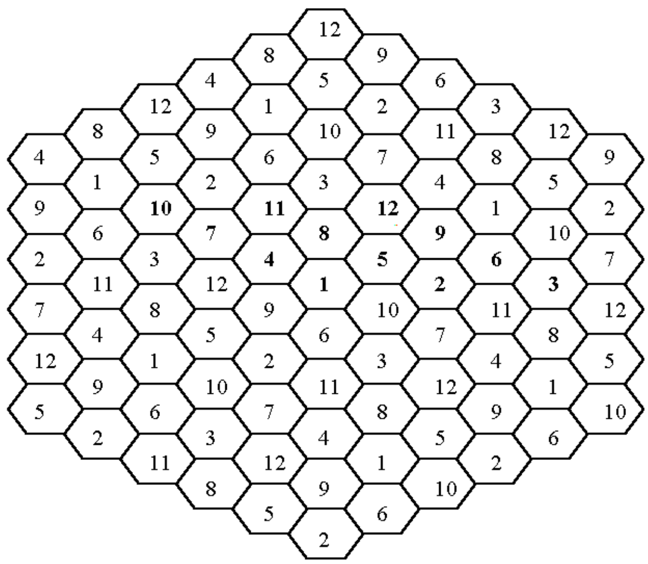

Figure 4. Solution for $k = 3$, with span 12.

Table 1.
Minimum and maximum values for the span by value of k.

k	Minimum	Maximum
1	7	7
2	9	9
3	11	12
4	13	15
5	15	18
k	$2k+5$	$3k+3$ (not proven)

We develop some further theory to help us determine the span for $k = 3$.

Theorem (The Symmetry Argument). *If a solution with span s uses label x, then there is a solution with the same span that uses label $s+1-x$.*

Proof: The cells of the given solution are labeled with values between 1 and s. Relabel each cell y with the label $s+1-y$. Since $1 \leq y \leq s$, then $1 \leq s+1-y \leq s$. The difference $|a-b|$ between the labels a and b of any two cells does not change, because $|(s+1-a)-(s+1-b)| = |a-b|$. Therefore, the new labeling is also a solution, because solutions depend on only the differences between the labels of the cells. We know that at least one cell was originally labeled x; that cell is now labeled $s+1-x$. □

Contrapositive of the Symmetry Argument: *If there are no solutions of span s that include label x, then there are no solutions of span s that include label $s+1-x$.*

Theorem. *For $k = 3$, the span is 12.*

Proof: The span is at least 11. Suppose that there is a solution that uses label 3. Labels 1, 2, 3, 4, 5 cannot be adjacent to 3, so the six adjacent hexagons must be labeled 6, 7, 8, 9, 10, and 11. The only label that could be adjacent to both 3 and 9 would be 6; but we need at least 2 channels that are adjacent to both (**Figure 5**). Hence, 3 cannot be used in a solution with span 11. By the contrapositive of the symmetry argument, no solutions of span 11 include label 9, either.

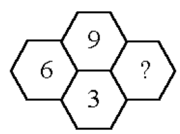

Figure 5. Situation that arises if label 3 is used in a span of 11.

Now suppose that label 4 is used. Then labels 2, 3, 4, 5, 6, and 9 cannot be adjacent to 4, leaving only labels 1, 7, 8, 10, and 11. We need 6 distinct labels adjacent to 4 but only 5 remain. Thus, 4 cannot be used in a solution with span 11. Again by symmetry, no solutions of span 11 include label 8.

An identical argument excludes labels 5 and 7. Only labels 1, 2, 6, 10, and 11 remain; but at least 7 labels are required for a solution. Thus, a solution for $k = 3$ with a span of 11 is not possible. We have already constructed a solution of bandwidth 12 for $k = 3$, so the span is 12. □

We turn to $k \geq 4$.

Theorem. *For $k \geq 4$, the span is no more than $2k + 7$.*

Proof: A solution satisfying all constraints may be constructed as follows. Consider **Figure 6**. Let $A_1, A_2, A_3 = 1, 2, 3$; $B_1, B_2, B_3 = k+3, k+4, k+5$; and $C_1, C_2, C_3 = 2k+5, 2k+6, 2k+7$.

Such an assignment guarantees that the difference between any two channels that are in different groups is at least k. Every channel X is surrounded by channels that are in the other two groups, so there are no adjacent channels in the same group as X. Therefore, all channels surrounding X have labels that differ from X's label by at least k. In addition, no two cells with the same label are closer together than $4s$. These two properties establish this arrangement as a solution. The highest label used is $2k + 7$, so the bandwidth is $2k + 7$. This establishes a new upper limit on the span. □

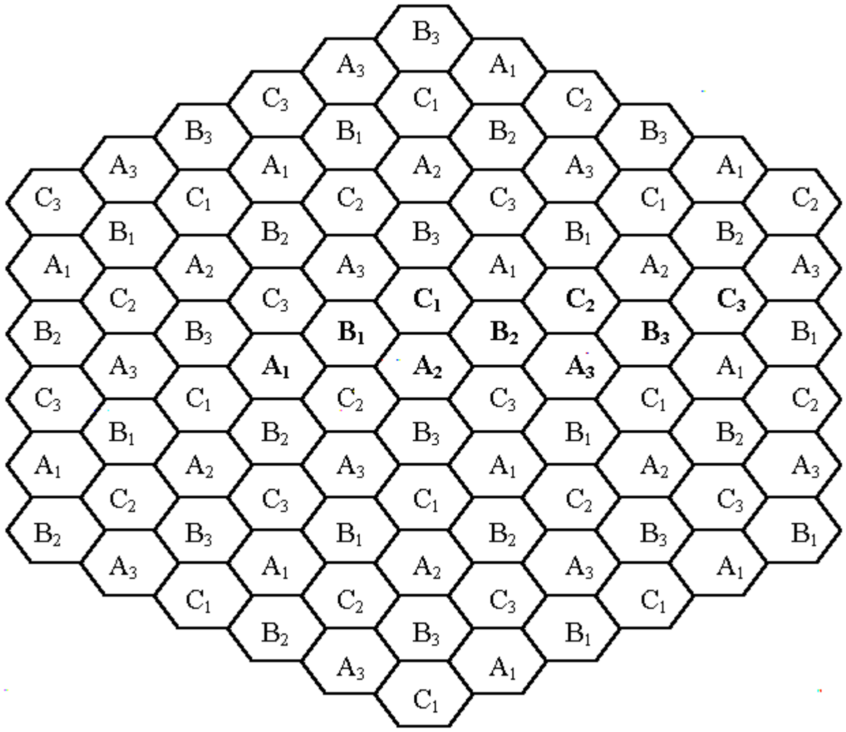

Figure 6. Design of a solution.

So, for $k \geq 4$, the span is either $2k+5$, $2k+6$, or $2k+7$. We need more building blocks to help us:

Lemma 1. *If there is no solution with bandwidth b, than the span is greater than b.*

Proof: Assume that there is a solution with bandwidth $m < b$. Replace the largest channel used with b. Since the relative spacing between labels either stays the same or grows, this also must be a solution, with bandwidth b. $\square$

Lemma 2. *For any $k \geq 4$, given 6 consecutive labels, any selection of 5 of these cannot all be adjacent to a given cell X.*

Proof: Five hexagons all adjacent to a given hexagon include four edges between pairs (**Figure 7**).

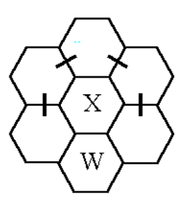

Figure 7. Situation of **Lemma 2**.

But for the six labels y, $y+1$, $y+2$, $y+3$, $y+4$, and $y+5$, there are at most three ways in which they can be adjacent (y with $y+4$, y with $y+5$, or $y+1$ with $y+5$, for $k=4$). Since no label can be used more than once, there is no way to label all five hexagons. □

Lemma 3. *Any 4 hexagons that are all adjacent to a given hexagon must include at least 2 whose labels differ by more than k.*

Proof: Assume that there are 4 hexagons around a given hexagon such that the lowest label of the 4 is x and the highest label is $y \leq x+k$. At least 2 edges are shared among the four hexagons (**Figure 8**). But there is at most one way in which the labels between x and y can be adjacent: x with $x+k$. Since no labels can be used more than once, there is no way to label all 4 hexagons. □

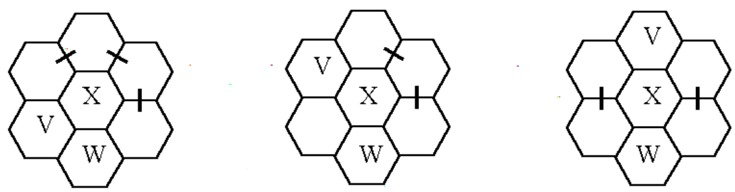

Figure 8. Situation of **Lemma 3**.

Theorem. *For $k \geq 4$, the span is at least $2k+7$.*

Proof: Suppose that the span is $2k+6$ and the label $k+1$ is used. The possible labels adjacent to $k+1$ are $1, 2k+1, \ldots, 2k+6$, which include 6 consecutive channels. But a selection of 5 out of 6 consecutive labels cannot all be adjacent to $k+1$ (**Lemma 2**). So any possible solution uses neither $k+1$ nor, by the contrapositive of the symmetry argument, $k+6$.

Next, suppose that the label $k+2$ is used. The possible labels adjacent to $k+2$ are $1, 2, 2k+2, \ldots, 2k+6$, which include 5 consecutive channels. But a selection of 4 out of 5 consecutive labels cannot all be adjacent to $k+2$ (**Lemma 3**). So any solution uses neither $k+2$ nor $k+5$ (by symmetry).

There are now 3 groups of labels remaining: $\{1, \ldots, k-1\}$ (group A), $\{k+3, k+4\}$ (group B), $\{k+8, \ldots, 2k+6\}$ (group C). No two channels in the same group can be adjacent, since they differ by less than k.

Suppose that a label X from group A is used. The possible adjacent labels must be from groups B and C. Since group B only has 2 channels, there must be at least 4 channels from C adjacent to X. But by **Lemma 3**, this is impossible; so no label from group A is used. By symmetry, no label from group C is used.

This leaves only the 2 channels in group B. Since every solution includes at least 7 distinct channels, there is no solution with a bandwidth of $2k+6$, so by **Lemma 1** the span is at least $2k+7$. □

We have specified the span for all k: 7 for $k=1$, 9 for $k=2$, 12 for $k=3$, and $2k+7$ for $k \geq 4$.

Evaluation of the Model

Our model finds the span exactly for the constraints in requirements A, B, and C. This model still is limited by the assumptions of regular placement of transmitters, equal coverage around transmitters, and equal channel size. These constraints, however, may be too limiting or erroneous in real-world applications. We therefore examine three generalizations of the model.

Multiple Levels of Interference

Typically, in wireless communication the strength of a signal from a transmitter declines with the distance between receiver and transmitter. So interference caused by transmitters occupying the same or close frequencies also decreases with distance. In our main model, transmitters on the same frequency must be at least $4s$ apart, while transmitters on close frequencies must be $2s$ apart. We consider a generalization.

The distance between any two cells is 1 more than the minimum number of cells you must go through to go from the first cell to the second (a distance of 1 corresponds to $2s$). We consider a model where the amount of interference decreases linearly with distance. The needed separation f between any two cells as a function of distance d is $f = k(n - d)$, for $d \leq n$.

Theorem. *The span for $n = 1$ is $2k + 1$.*

Proof: For any solution, there is a smallest label z that is used. Select two cells that are adjacent to this cell and adjacent to each other; these two cells must have different labels, x, y, with $x < y$. We know that $x \geq k + z$. But then $y \geq x + k$, so $y \geq 2k + z$. Since $z \geq 1$, we have $y \geq 2k + 1$. This proves that the span is at least $2k + 1$.

By setting $z = 1$, $x = k+1$, and $y = 2k+1$, and arranging them as indicated in **Figure 9**, we achieve $2k + 1$ as the bandwidth, so the span is $2k + 1$. □

Theorem. *For $k = 1$ and $n \geq 2$, the span is no more than $n^3 + n^2 - n + 1$.*

Proof: We demonstrate our construction first for the case $n = 3$ in **Figure 10**. The general construction is shown in **Figure 11**, where $X, a, b = 1 + (a-1)(n-1) + (b-1)n^2$. The largest channel is $X, (n+1), (n+1) = 1 + (n+1-1)(n-1) + (n+1-1)n^2 = n^3 + n^2 - n + 1$. □

Theorem. *The span for any k is at least*

$$\begin{cases} 3k\left(\dfrac{n^2}{4} + \dfrac{n}{2}\right), & \text{for even } n; \\[2ex] \dfrac{3k(n^2-1)}{4} + 1, & \text{for odd } n. \end{cases}$$

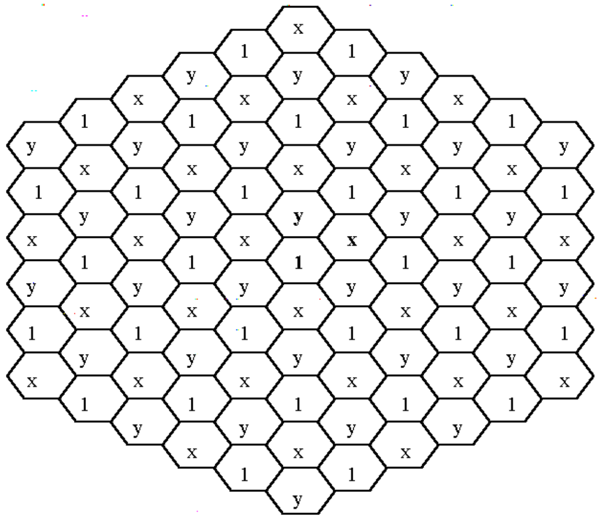

Figure 9. Solution with span $2k+1$.

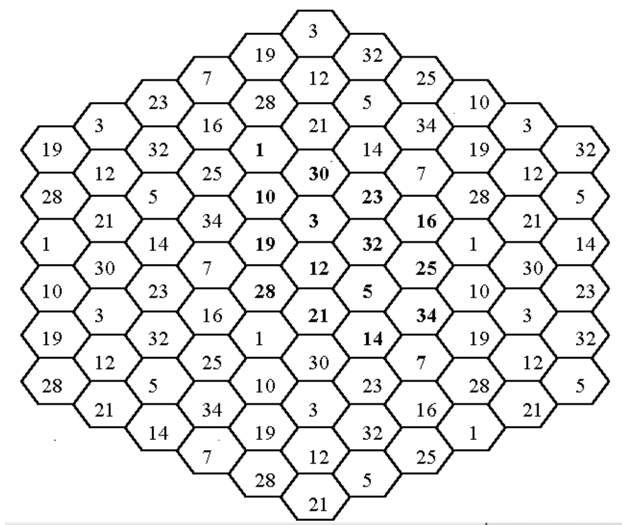

Figure 10. Construction for the case $n = 3$.

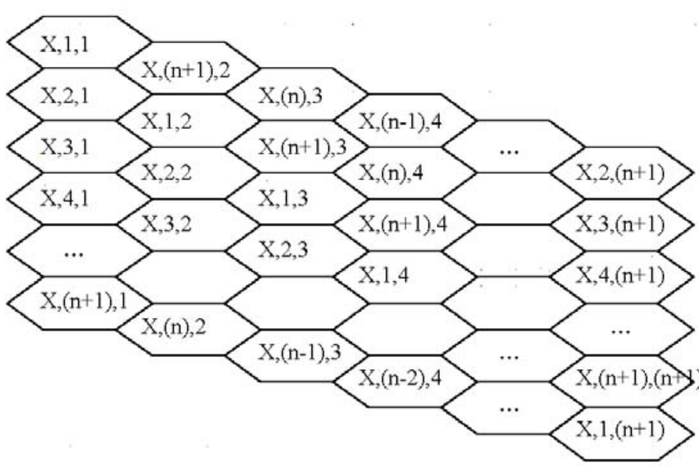

Figure 11. Construction for general n.

Proof: Case n even: Consider an arbitrary cell and the first $n/2$ rings of cells around it; all of the cells in this region are within a distance n of each other. Therefore, they must all be distinct, and they all must differ from each other by at least k. There are

$$3\left(\frac{n^2}{4}+\frac{n}{2}\right)+1$$

cells in this region, so the bandwidth must be at least as large.

Case n odd: Any solution for n is also a solution for $n-1$. Therefore the same formula applies with $n-1$ in place of n, yielding the value stated. □

Theorem. *Given a solution for some n and with bandwidth b, a solution exists for n and any k with bandwidth $kb-k+1$.*

Proof: Multiply all labels by k; this increases the separation between any two cells by a multiple of k. Subtract $k-1$ from all channels. This returns the lowest label to 1 and does not affect separation. □

We have established that the lower bound of the span for this modified model increases linearly with k and with n^2; we have established an upper bound that increases linearly with k and with n^3.

Weaknesses

This modification still assumes uniform arrangement of the transmitters. In addition, there is a discrepancy error between distance as we have defined it for hexagons and actual linear distance; however, an increase in hexagonal distance occurs if and only if there is an increase in actual distance as long as the hexagonal distance is less than 7.

Freedom of Transmitter Placement

We consider how far from the center of its cell a transmitter may be placed without violating any of the original constraints.

We assume that all transmitters can be displaced from the centers of their cells by an equal amount. The two interference constraints still apply: Transmitters within $2s$ of each other must still have a separation of at least k, and transmitters within $4s$ of each other must have a separation of at least 1. We consider two questions:

- What is the maximum freedom of displacement that can be allotted to transmitters before a solution developed by using our main model ceases to be a solution?
- What is the maximum freedom that can be allotted before a solution ceases to be a minimum solution?

Theorem. *If transmitters are displaced by less than $0.29s$, a solution developed by assuming that they are in the centers of the cells is still a solution.*

Proof: In the uniform arrangement of transmitters, the nearest transmitter that is more than $2s$ away from a given transmitter is at a distance of $s\sqrt{7} \approx 2.64s$. Their channel separation could potentially be less than 2, so we assume that it is. If both are moved toward each other by equal amounts, the $2s$ rule is violated when each has moved $0.32s$.

The nearest transmitter that is more than $4s$ away is at a distance of $s\sqrt{21}$. The two could have the same label. If they both move toward each other by equal amounts, the $4s$ rule is violated when they have both moved $0.29s$.

Therefore, as long all transmitters move less than $0.29s$, neither rule will be violated. $\square$

Theorem. *If transmitters are displaced by less than $0.13s$, a minimum solution developed by assuming that they are in the centers of the cells is still a minimum.*

Proof: In the uniform arrangement of transmitters, the farthest transmitter that less than $2s$ away from a given transmitter is at a distance of $s\sqrt{3} \approx 1.73s$. If both are moved away from each other by equal amounts, they are more than $2s$ apart when they have each moved $0.13s$.

The farthest transmitter that is less than $4s$ away is at a distance of $2s\sqrt{3} \approx 3.46s$. If they both move away from each other equal amounts, they will become more than $4s$ apart when they have both moved $0.27s$.

Therefore, as long all transmitters move less than $0.13s$, none of the constraints is affected. $\square$

Weaknesses

If transmitters are displaced from the centers, regions of their cells are more than s away from the transmitter. In addition, this approach does not consider the relative strength of signals as a function of the relative distance to the transmitters.

Rectilinear Constraints on Transmitter Placement

Our main model assumes that transmitters are arranged in a honeycomb pattern. However, due to city streets and township and county lines, a cellular service provider may be constrained to arrange transmitters rectilinearly. Therefore, we attempt to find the span under the original $2s$ and $4s$ constraints when the infinite plane is tiled with squares, each containing a transmitter in the center.

So, we consider a generalization of the $k = 2$ case (as specified in Requirements A and B) with an infinite grid of squares instead of hexagons. This requires a new definition of s, which now becomes the distance from the center of a square to one of its corners, so that s is still the maximum distance between a transmitter and a point in its cell. The cells considered within $2s$ and $4s$ from any given cell are shown in **Figure 12**. The inner four unmarked squares must have a separation of at least two from the center square. The outer ring of barred squares must not have the same label as the center (representing a separation of at least 2). The cells with a distance of exactly $2s$ and $4s$ are not included in the corresponding regions. We make this choice because when the cells are hexagons, each cell contains regions whose points are simultaneously within a distance of s of two transmitters; with square cells, there is no region of nonzero area where a point in one cell is within s of the transmitter in a diagonally adjacent cell.

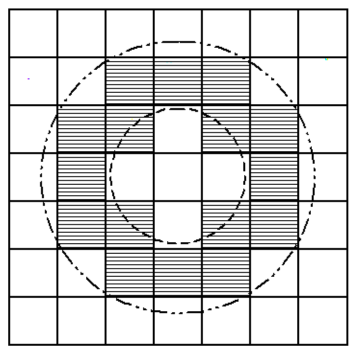

Figure 12. Square lattice of transmitters, with radii of $2s$ and $4s$ indicated.

Theorem. *The span is at least 8.*

Proof: In **Figure 13**, by the $4s$ condition, all of the boldfaced cells must be labeled distinctly. Therefore, there must be at least 7 distinct labels. Assume that there is a solution with a bandwidth of 7. Then all of the cells that are not in boldface must be labeled as indicated. However, all 7 of the labels are within $4s$ of the cell indicated with a question mark. Therefore, 7 labels are insufficient. □

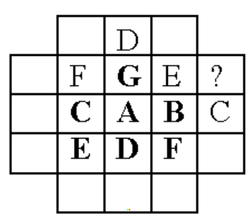

Figure 13. Seven labels are not enough . . .

Theorem. *The span is 8.*

Proof: The construction of a solution with a bandwidth of 8 is shown in **Figure 14**. □

7	4	1	6	3	8	5	2	7	4	1	6	3	8
3	8	5	2	7	4	1	6	3	8	5	2	7	4
5	2	7	4	1	6	3	8	5	2	7	4	1	6
1	6	3	8	5	2	7	4	1	6	3	8	5	2
7	4	1	6	3	8	5	2	7	4	1	6	3	8
3	8	5	2	7	4	1	6	3	8	5	2	7	4
5	2	7	4	1	6	3	8	5	2	7	4	1	6
1	6	3	8	5	2	7	4	1	6	3	8	5	2

Figure 14. . . . but eight are.

The block of repeating channels in **Figure 14** is reflected about a horizontal axis and then tiled to the right and left. In this fashion a solution for an infinite array of squares can be constructed.

Weaknesses

This modification still assumes that transmitters are placed regularly, that channels with a separation of more than 2 never interfere with each other, and that there is no interference beyond a radius of $4s$.

News Release

Let's say that you work for a company that provides cellular telephone service. Your job is to decide how the company should go about expanding into a new area. You decide where transmission towers will be placed and what channels will be needed.

You know that gaps in coverage must be avoided. There must not be any places in the new service area where a customer's cell telephone reads "We're sorry, you're outside the coverage area."

One way to ensure that there are no "holes" or gaps in coverage is simply to put up lots and lots of towers. "Put up a tower every half-mile," you tell your boss. Your boss says that the idea is interesting, and would certainly guarantee complete coverage, but that another essential goal is to minimize the number of towers needed. Each tower costs tens of thousands of dollars.

So you return to your cubicle and do some research. You learn that the signals from a transmission tower get weaker as a customer's phone gets farther and farther from the tower. You find out that there is a distance, say, five miles, beyond which clear reception can no longer be guaranteed. So you take a map of the region and start drawing circles whose radius is this distance (reduced to the scale of the map, of course). Right away you realize that the circles will have to overlap a bit. There are some quarters sitting on your desk, and you quickly see that one quarter can be surrounded by six other quarters, so that all the quarters are pushed together as close as possible. But there are little triangular holes in between them, so you have to overlap them a bit so that the area that they cover has no holes.

This, in fact, turns out to be the most efficient way to cover a plane with circles—there is as little overlap as possible. So you cover your map with circles in this fashion and decide to place a transmission tower in the center of each circle. Since the two neighboring circles overlap a bit, to eliminate those pesky triangular holes you split the overlap area between them. Since every circle is surrounded by six other circles, you do this for all six overlapping areas. Now the region (cell) that is serviced by any transmission tower is a hexagon. In fact, the hexagons form a honeycomb pattern on your map, and you wonder why you didn't think of that before. The distance from the center to the corner of any hexagon is as far as you can get from the center without leaving the hexagon, so this distance must be the five miles that you determined earlier is as far as a customer can get from a tower and still be assured of coverage.

Quite excited with your discovery, you run to your boss shouting, "Eureka! Hexagons!" Your boss smiles, a bit condescendingly, and informs you that they already knew that hexagons were the best way to cover a large area. What your boss really needs you to determine is how many channels the company will have to purchase from the FCC in order to have enough to for all the towers. "That's easy," you respond, still not realizing that nothing is as easy as it seems, "just buy one channel for each tower!" Your boss sighs, silently wondering if you are the right person for this job. Then your boss reminds you that not spending large amounts of unnecessary money is also an essential part of doing business. The company needs to know: What is the smallest number of channels that they will have to buy? And how should the channels be assigned to the towers?

Having finally learned to be cautious, you suppress the urge to blurt out, "Just put the same channel on all the towers!" Instead, you begin asking questions. Your manager refers you to the engineering department. This is where the problem gets interesting.

First, you are told that if a transmitter in some cell uses a particular channel, then none of the cells in the first ring of six hexagons around it can share that channel. This is because when you move from one cell to another, the way that the cell phone stops talking to one tower and starts talking to the tower in the new cell is by changing channels. If two cells right next to each other used the same channel, a phone in between would not know which one to listen to!

Next, you learn that none of the cells in second ring of 12 hexagons can use the same channel either. In fact, the closest that two cells that use the same channel can be is three hexagons apart. So the same channel can be used over and over again, but the cells that use it must be spread out so that they are all at least three cells apart from each other.

Finally, you find out that cells that are right next to each other cannot use channels that are too close together in frequency. When you ask why, the engineers begin enthusiastically telling you about something called "spectral spreading." You decide that it is better to not know why it is true, only that neighboring cells must use channels that are separated by some number of channels. For instance, suppose neighboring channels must be separated by at least four. Then if one cell uses channel 10, then perhaps all its neighbors must use channels that are at least four away, that is, none of the neighbors can use 7, 8, 9, 11, 12, or 13. You are told that for this new region, it has not been determined yet just how far apart the channels in neighboring cells must be from each other. Perhaps it depends on the expected call volume and on atmospheric conditions in the new area. But as soon as this channel separation is determined, you will be expected to say immediately how many channels are needed and how they should be distributed among the towers.

When you return to your desk, you find a message on your chair. It seems that your boss neglected to mention an important detail: It does not matter how many channels the company actually uses, but what is crucial is *how big a block of consecutive channels* in the frequency spectrum the company occupies. For example, if you use only channels 11 and 20, you still have to pay for the block of ten channels between 11 and 20.

This is the problem that our mathematical modeling team recently solved. We determined that if neighboring cells must be at least two channels apart, then a region of any size can be covered with a block of 9 consecutive channels. If a separation of three channels is required, any region can be covered with 12 channels. If a separation of four or more channels is required between neighboring cell, then the necessary number of channels can be found by doubling the separation and adding 7. For example, if neighboring cells must be 5 channels apart, then 17 channels must be purchased from the FCC. We also determined that these are the best possible solutions. That is, there is no way to cover a large region with fewer channels, without breaking some of the rules that the engineering department requires.

But what is interesting about our result is not the number of channels required but how the channels are distributed among the towers. An example will illustrate this point. Suppose again that neighboring cells must be five channels apart; then our model calls for 17 channels to be purchased, say channels 1 through 17. But the only channels that our solution uses are 1, 2, 3, 8, 9, 10, 15, 16, and 17! We spread out the first three channels among the cells so that none of them are adjacent to each other. This covers one-third of the all the cells. We repeat this with channels 8, 9, and 10. Now two-thirds are covered. The rest are covered with 15, 16, and 17. We proved that this is the best possible solution. It is surprising that the best solution has 8 channels in the middle of its block that are not even used, but this is indeed the case. What the unused channels do is divide the remaining channels into three groups. The groups are far enough apart that any channel in one group can be surrounded by channels in the other two groups.