Hexagonal Unpacking

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Abstract

We present a model for movement within crowded structures, tessellating a room with hexagons and using a waiting-time function based on the harmonic mean of closest neighbors. We determine the maximum time required for all persons to exit, comparing this time to a target time based on the size of the structure. Our model is very general and its parameters can be modified for several types of buildings. We consider various specific cases, giving the maximum occupancy for each.

Assumptions

- *When several people jockey for a vacated position, the probability that one of them occupies it is independent of how long each has spent in his current location.* A person who has been waiting longer would seem to have an easier time, but this effect is compensated by the tendency, even in groups of people gathered around an exit, to move in lines. The forward momentum of moving into a new position gives an extra advantage, as the person may well be drafting behind someone else, forming a miniature line weaving through the crowd. Whichever advantage prevails, we posit that it is small enough to ignore.
- *People exiting a building generally move so as to decrease their overall expected exit time.* Years of selecting optimum shopping lines and struggling to get out of crowded theaters, along with a human's natural ability to see where holes are forming in the crowd, constitute a natural tendency to select paths that minimize the time to exit. Even if humans can't see instantly what course

The UMAP Journal 20 (3) (1999) 287–296.

will have the least resistance, they certainly can ascertain whether any given step will ultimately shorten their expected wait.

- *The average person can quickly accelerate to a speed of least 6 ft/s.* Normal walking speed is 4 ft/s, so a quick acceleration to 6 ft/s is feasible. The value of this parameter can be changed for kindergarten auditoriums, retirement homes, and other structures housing those likely to have less robust locomotion.

- *When people clump together in attempting to leave a structure, they are packed loosely enough to assign each a cell 1.4 ft in diameter.* While one could theoretically line people up in columns and pack them, standing still, into cells a bit smaller, greater space must be allowed for moving chaotic masses. The mechanics of the model depend very little on the size of the cells.

- *Movable furniture does not block an exit, though it may be in the immediate vicinity of the exit and thus affect the rate of egress.* We allow in the model for tables and other objects to be very close to doors. The safety code provides that doors cannot be blocked by such items, as time for their removal is so prohibitively high as to seriously depreciate the maximum occupancy. We do treat the possibility that one exit (in a multi-exit facility) can become blocked.

Practical Considerations

It is unclear what the target exit time should be. The bulk of our modeling determines exit times based on the parameters of a structure. We then give the maximum occupancy for various exit times.

Points That Must Be Considered

- The number of people exiting a facility during a crowding action is not necessarily the same as the maximum number of people who can leave through doors in orderly lines. To arrive at the total time for evacuation, one cannot simply divide the number of people in a room by how many can go through a door in a given time.

- The movements of individuals leaving a building are made individually, based on the position of the person and openings available.

Definitions

We tessellate the room with regular hexagons, each 1.5 ft along the diagonal (making them 1.299 ft from side to side, with side length 0.75 ft). These represent cells that people occupy while they are in a mass attempting to leave a building.

Our model has a single exit but is easily extendable to more exits. For simplicity, we assume a rectangular room with the door on the north wall (the top wall in all figures).

The orientation of the tessellation makes very little difference in the time calculation, as it is only an abstraction allowing for algorithmic-based movement toward positions of greater desirability.

We define several terms:

- The *neighbors* of a hexagon are those six hexes (or fewer in the case of border hexes) with which it shares sides.
- An *allowed movement* is a movement from a hex to any of its neighbors.
- The *radius* of a hex is the minimal number of allowed movements to take a person at the hex to the door.
- A *level curve* for a radius R is the collection of all hexagons with radius R.
- Two hexes are *isoradial* if they have the same radius.
- A *good neighbor* for a given hex is a neighbor with a smaller radius.
- The *good-neighbor number* of a hex is its number of good neighbors.
- A *desirable neighbor* for a hex is either a neighbor with a smaller radius (a good neighbor) or a neighbor on the same level curve with more good neighbors. The inherent geometry makes some hexes on the same level curve worse than others in terms of waiting time. If we design our model so that people want to go only to hexes of smaller radius, then they will not move toward these better hexes of the same radius. Giving each hex a radius and good-neighbor number accommodates this kind of move.
- The *inherent waiting time* of a hex is how long it takes to traverse it. This is independent of the vacancy or occupancy of neighboring cells. This parameter can be varied to model situations where the terrain makes moving difficult or has a tendency to cause accidents.
- The *actual waiting time* of a hex is the expected amount of time one spends in the hex given the inherent waiting time and the waiting times and competition for neighboring hexes.
- The *equivalent waiting time* of one hex with respect to another is the actual waiting time multiplied by the number of people competing for the hex.
- The *expected exit time* of a hex is the sum of the waiting times of the hexes that form the minimal path to the exit.
- A *click* is the basic unit of time for people fleeing a room. It is based on the type of door being used. The click is the time it takes for a single person to leave a single hex next to a door. Thus, if a door were 3 hexes wide and could let out 6 people/s, a click would equal 0.5 s.

Constructing the Model

We assign each hex an inherent waiting time based on the expected time to traverse it. When a position becomes available, the time to fill the opening is certainly nonnegligible. The base inherent waiting time of a hex that is otherwise free of obstacles and of danger of accident is set to 0.25 s, the time it takes to move 1.5 ft (the width of the hex) at the standard pace of 6 ft/s. Tables and other obstacles can be modeled as hexagons with higher waiting times. (One can jump over a row of seats in a theater, but it takes longer, there is increased chance of tripping, and so on)

To illustrate this step in the model, see **Figure 1**, our representation of a theater. The chairs are represented as hexes with a waiting time of 1 s. **Figure 2** illustrates level curves and good-neighbor numbers for a set of hexes.

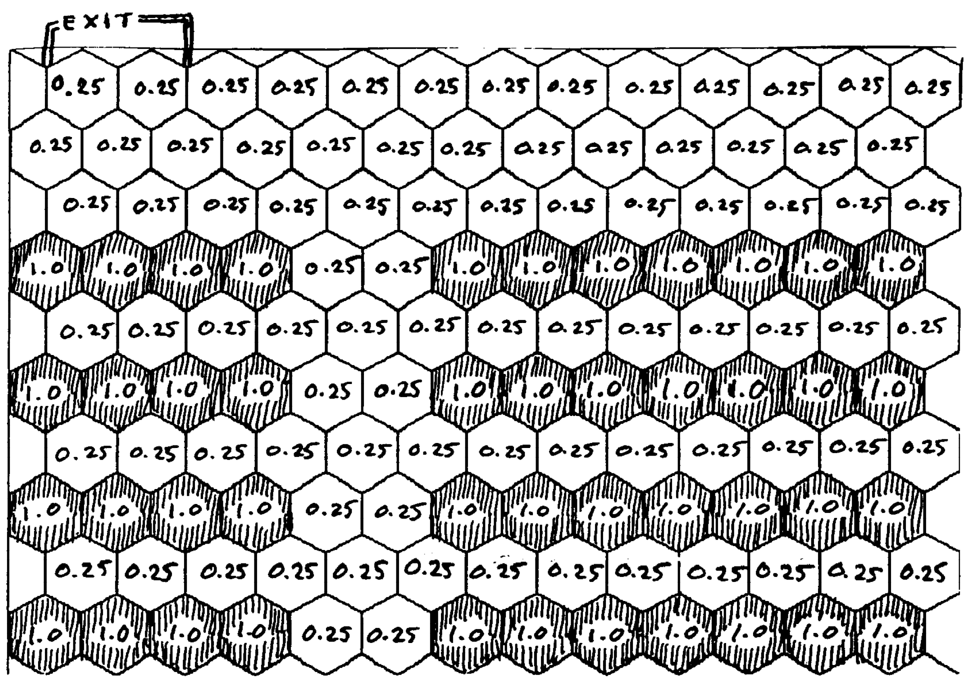

Figure 1. Intrinsic waiting times (in seconds) for cells in a hypothetical theater environment. Cells with value 0.25 represent free space; those with value 1.0 (shaded) correspond to the (fixed) seating.

After inherent waiting times are assigned, we determine how much time to assign to one click. A standard 7 ft $\times$ 2.5 ft door takes up two hexes and can exit 3 people/s, so its click time would be 0.67 s. In general,

$$\text{click time} = \frac{\text{width of door}}{\text{total outflux}},$$

where the width is in hexes and the outflux in people/s. This click time is the maximum speed of egress.

Hexes adjacent to the doors (those with radius 1) are assigned an actual waiting time of 1 click; this represents the expected actual waiting time of someone right next to the door. Some hexes with radius 2 have only one hex

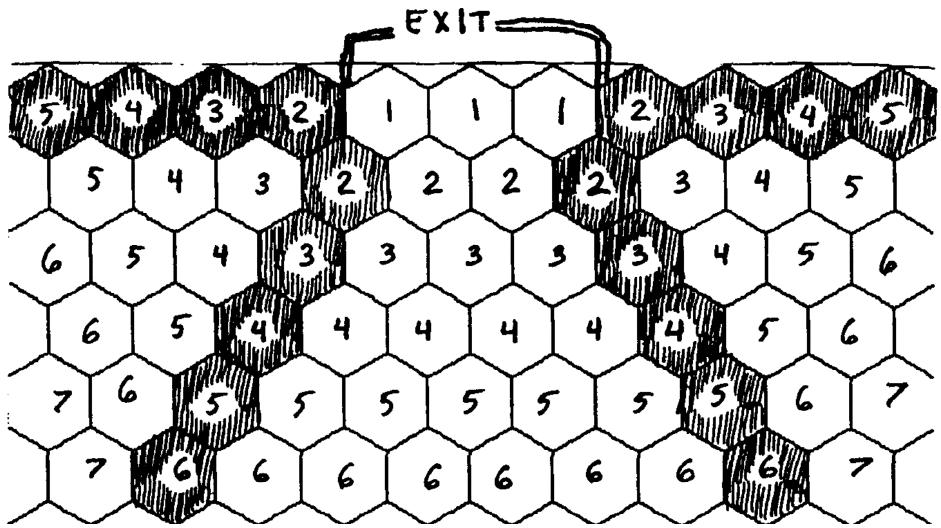

Figure 2. Cells numbered by their level. Cells with one good neighbor are shaded; all other cells are either adjacent to the exit or have two good neighbors.

next to them with a smaller radius, while others have two; the hexes with smaller radii are exit hexes. Hexes with only one neighbor of smaller radius have a good-neighbor number of 1 and are less preferred than those with two neighbors of smaller radius (they have a good-neighbor number of 2). The general rules of movement can be summarized as follows:

- Of two hexes with different radii, the hex of smaller radius is preferred.
- Among isoradial hexes, those with more good neighbors are preferred.

We consider all the hexes of greatest preference and determine the actual waiting time of these. Then we compute the actual waiting time of the next most preferred set of hexes. We thus work our way out from the door (since radius takes precedence over good-neighbor number).

Since the actual waiting time of a hex is based only on the actual waiting time of its desirable neighbors, the competition for these hexes, and its own inherent waiting time, we never need to know the actual waiting time of a hex less desirable than the one that we are working on.

Computing the Actual Waiting Time

Significant Factors

For each desirable neighbor, we compute an equivalent waiting time by multiplying the actual waiting time of the neighbor by the number of hexes vying for the desirable neighbor. If three people are all attempting to get a certain hex, then the equivalent waiting time for all three is three times the actual waiting time of the hex, since each of the vying hexes has a one-third chance of gaining the desirable hex each time it empties. Thus, competition for hexes tends to slow down a person's progress. However, many hexes have more than one desirable neighbor, just as there are typically more than one near position that someone in a crowd would occupy if given the opportunity. We model the effect of multiple desirable neighbors via the *reduced harmonic mean* (i.e., the harmonic mean divided by the number of desirable neighbors):

$$\mathrm{RHM}(A,B) = \frac{AB}{A+B} = \frac{1}{\frac{1}{A}+\frac{1}{B}};$$
$$\mathrm{RHM}(A,B,C) = \frac{ABC}{AB+BC+AC} = \frac{1}{\frac{1}{A}+\frac{1}{B}+\frac{1}{C}}.$$

Justifying Use of the Harmonic Mean

- We can model the actual waiting times of the desirable neighbors as resistors. The higher the actual waiting time, the longer it takes to shove the same current through a wire. When we have two (or more) desirable neighbors to use as conduit, they combine as resistors in parallel. This is precisely the same function as the reduced harmonic mean.
- All that concerns us is the amount of time that we expect to stay in the given hex. If one hex is open to us once every A clicks, and another is open once every B clicks, then every AB clicks there are B openings from the first and A from the second, so the average number of openings per click is $AB/(A+B)$, which is just the reduced harmonic mean.

The Final Factor

After the reduced harmonic mean of the equivalent waiting time of the various desirable neighbors is computed, the inherent waiting time assigned to the hex is added. This final value is the actual waiting time for the hex. **Figures 3** and **4** illustrate this computation applied to a simple model.

The Time Factor

For our model to determine maximum occupancy, it must have a target time. We decided to trust the actual posted occupancy maximums on simple structures with very few obstacles. A curve that fits these numbers well is the power curve $T = 0.4A^{3/4}$, where T is the target time and A is the area of the building in square feet.

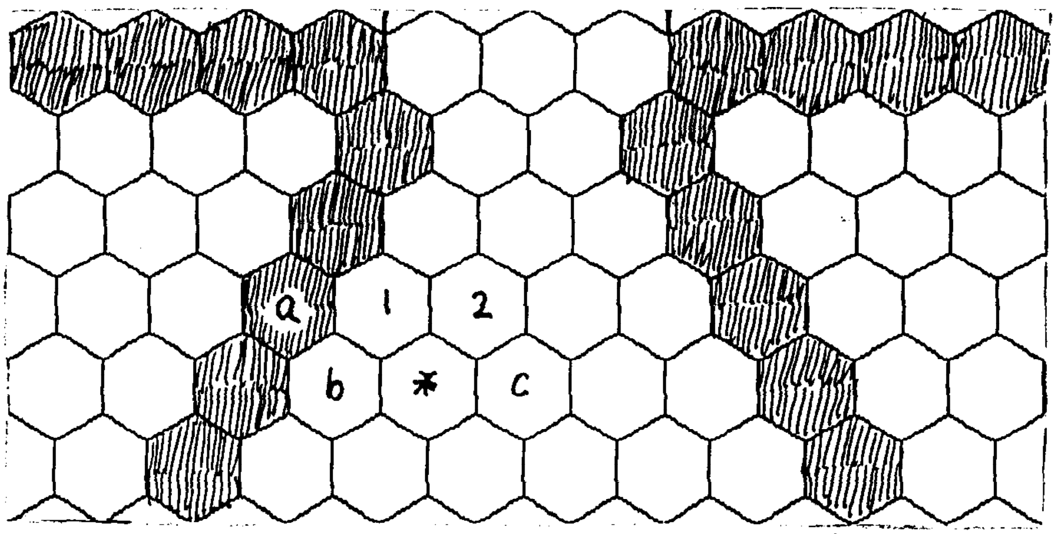

Figure 3. Illustration of the algorithm to compute waiting time of cell (*) in a rectangular room with a single exit at the top three cells wide. We first identify its good neighbors (cells (1) and (2)) and any neighbors on its level curve with a greater number of good neighbors (none in this case). For each of the cells for which (*) vies, we determine the total number of cells vying for that same spot. In the case of (1), this number is 3 (due to (a), (b), and (*)); for (2), it is 2 ((*) and (b)). The waiting time of (1) is then multiplied by the number of cells vying for it, and similarly for (2). The harmonic mean of these equivalent waiting times is divided by the total number of spots for which (*) vies; this reduced harmonic mean, plus the inherent waiting time of (*), gives the actual waiting time for (*).

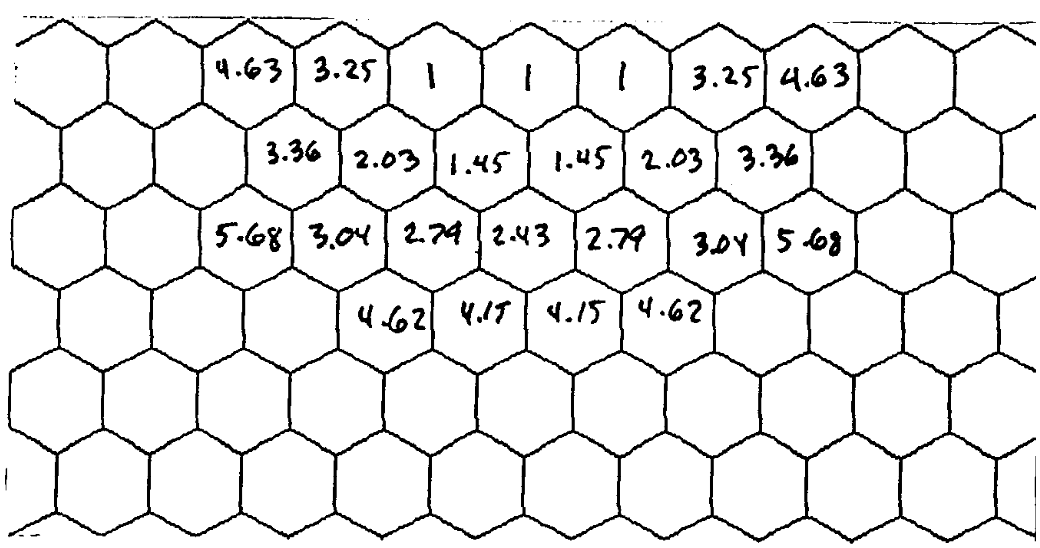

Figure 4. Waiting times for the room of **Figure 3**.

Testing the Model

Consider a bare room, say a gymnasium, with only one exit. If our model is accurate, the time prescribed by the model should be a bit higher than the ratio of number of occupants to greatest number able to leave in a given time—higher because of time lost due to people competing for positions, etc.

A standard gymnasium is built as a full-size basketball court (94 ft long) with one full-size volleyball court turned sideways per half (60 ft wide). With buffer space, the size is 80 ft $\times$ 110 ft, corresponding to a tiling of 84 $\times$ 53 hexagons. At the center of a longer side we place the exit: two sets of double doors, 110 in total width, or approximately 6 hexes. At the standard rate of 2 people/s per door hex, we get a maximum egress rate of 12 people/s.

Assume that there are 875 people in the building (fewer than the maximum capacity of about 1350 listed on such gymnasiums). The formation of a clump of people takes a while. We use a very elementary dynamical-systems approach to finding how long it takes. Consider a person as far as possible away from the door, $\sqrt{55^2 + 80^2} = 97.1$ ft away. Let T be the time for the aggregate to form when there are initially P people in the gymnasium; this is how long it takes the farthest person to walk freely to the back of the accumulated people. The number of hexes within radius R of the door is a quadratic expression in R with leading coefficient 1.5; the radius of the farthest one out is approximately $\sqrt{P/\pi} \approx P/3$. Prior to aggregation, people leave at the maximum rate of 12 people/s, so $P(t) = 875 - 12t$. Thus, noting that each hex has a diameter of 1.4 ft (the average of its diagonal and its side-to-side lengths), we have

$$T = \frac{97.1 - 1.4\sqrt{P(T)/3}}{6}.$$

Substituting and solving gives $T = 10.9$ s. Thus, after 10.9 s, 131 people have left, leaving 744 clumped around the exit.

We tessellate the gymnasium and compute the actual waiting times for the various tiles. We then compute for each hex the shortest expected exit time by finding the shortest path (where shortest here is the path that minimizes the sum of the actual waiting times). We sort these times and find the 744th. This is the time that it should take 744 people to leave the gymnasium. Our model gives approximately 221 s. This added to the approximately 11 s gives a total time of 232 s for leaving the building. This total time is three times as long as the 73 s predicted by simply divided the number of people in the room (875) by the number of people able to go through the doors per second (12). We feel that the longer estimate from our model is much more realistic.

Results of the Model

Table 1 gives the intrinsic waiting times that we assigned to various entities in rooms, and **Table 2** gives our results.

We ran the model on several structures, giving an occupancy for various times. Each building was created by modifying the appropriate intrinsic wait times. We give a value for the maximum capacity and an estimated time for the evacuation of the building. Some special cases that we feel could not be accurately modeled included structures that do not have discrete doors: swimming pools, open fields, and so on. A similar algorithm described later can deal with most of these. Since there are reasons other than emergencies for wanting to monitor the number of people in a room, for each situation we have provided two other times and their expected total number of exitable people.

One added use of this model for leaving time is to estimate how long it takes a group of people to get out of a maze or a hall of mirrors at a funhouse. This can be simulated by setting the intrinsic waiting time of the hexes to higher levels. As can be seen in the chart, this greatly reduces the number of people who can exit in a given time.

This model also gives a nice way to compare various furniture orientations. We give models of a theater with its door in the center of the back, along with a model of a theater with its door in the back corner. Similarly, large and small classrooms are modeled with two different desk configurations. One model used long tables while the other used individual desks. As one would expect, the long tables produce a longer expected exit time for given number of people; consequently, the maximum allowed occupancy is less.

Table 1.

Inherent waiting times for various objects.

Object	Time (sec)
Free air	0.25
Theater chair	1
Maze in hall of mirrors	3
Table	0.7
Desk	1.2
Stall	20
Sink	20

Strengths and Weaknesses

The strength of this model lies in its general utility to model a broad range of structures by simply varying a few parameters. It demonstrates well how seemingly minor changes such as door position or furniture configuration can change the overall expected exit time.

Another strength of the model is that it gives more than simply the maximum safe occupancy level: It gives a specific time for any occupancy level, so that a user can estimate how long an exiting should take.

Table 2.
Results from the model.

Room Description	Area (sq ft)	Time (sec)	Max. cap.	Time	No.	Time	No.
Theater, 15 rows, door at corner	900	63	98	30	54	120	153
Theater, 15 rows, door at back center	900	63	98	30	54	120	174
Dance room	375	34	83	10	27	60	125
Elevator	60	6	21	3	10	10	26
Large classroom, 4 long tables across	750	57	99	30	57	120	187
Large classroom, 7 rows of 7 desks	750	57	96	30	55	120	186
Small classroom, 3 long tables across	300	25	51	15	27	45	75
Small classroom, 5 rows of 4 desks	300	25	44	15	27	45	67
Bathroom, 5 stalls and 4 sinks	200	21	25	10	15	30	33
Hall of mirrors	1200	82	32				

Implementation requires only a modest microcomputer, and run-time of our program is polynomial in the area of the tessellated region. The code we used took less than one minute of run-time on our computer for each building.

Limitations of the model include restriction to rooms with only one door, but we address this limitation below. Our implementation is confined to rectangular structures, but this is not a limitation of the model itself. The model does not account for individuals who wish to keep up with certain others in a crowd (family members, etc.). The model also does not use in any way the height of the ceiling; for certain emergencies, the volume of a room may well be more important than the area.

Possibly the greatest limitation of the model is basing the target exit-time function on existing codes for certain structures. If the simplest buildings cannot be trusted to have an accurate maximum capacity figure, then the target time function must be changed.

An Improved Implementation

A better implementation of the model, though more computationally complex, allows any number of doors (thus allowing for structures such as swimming pools, where the entirety of the border is a door). A hex has several different radii, one for each door, and the smallest radius determines which door to assign it to. The assignment of actual waiting times to hexes starts at the various doors and flows outward from each equally.

This implementation can be used to model such situations as doors becoming blocked. To model a door becoming blocked at time t_0, we count how many hexes have total exit times less than t_0, subtract this number from the total number, and then remodel the room without the door, using the new (reduced) number.