Simulating a Fountain

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Introduction

We establish the mathematical behavior of water droplets emitted from a fountain and apply this behavior in a computer model to predict the amount of splash and spray produced by a fountain under given conditions.

We combine height and volume of the fountain spray, making both functions of the speed at which water exits the fountain nozzle. We simulate water droplets launched from the fountain, using basic physics to model the effects of drag, wind, and gravity. The simulation tracks the flight of droplets in the air and records their landing positions, for wind speeds from 0 to 15 m/s and water speeds from 5 to 30 m/s. It calculates the amount of water spilled outside of a pool around the fountain, for pool radii from 0 to 40 m.

We design an algorithm for a programmable logic controller, located inside an anemometer, to do a table search to find allowable water speeds for given pool radius, acceptable water spillage, and wind velocity. We simulated subjecting a fountain with a 4-m pool radius to wind speeds from 0 to 3 m/s with an allowable spillage of 5%. We tested the model for accuracy and sensitivity to changes in the base variables.

Problem Analysis

Wind

The anemometer measures two main wind factors that affect the fountain: speed, which affects the force exerted on the water, and direction.

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Fountain

The main components of the fountain are the pool and the nozzle. The factors associated with the pool are its radius, which remains constant within a trial, and the acceptable level of spillage, which describes the percentage of water that may acceptably fall outside of the fountain.

Nozzle

Major aspects of the nozzle are the radius of the opening, the angle relative to the vertical axis (normal axis), and the spread and speed of the water passing through it. The angle of the nozzle relative to the vertical axis determines the initial trajectory of the water. The spread, described in standard deviations from the angle of the nozzle, determines the extent to which the initial trajectory of droplets differs from the angle of the nozzle. For a given water speed and nozzle radius, the flow of water through the nozzle may be determined from

$$f = \pi r^2 v,$$

where f is flow, v is the water launch speed, and r is the radius of the nozzle. The radius is constant, so the flow and consequent volume are functions of the speed, the dominant controllable factor affecting the height of the stream.

Assumptions

. . . about Fountains

- The fountain is composed of a single nozzle located at the center of a circular pool.
- The ledge of the pool is sufficiently high to collect the splatter produced by particles impacting the surface of the water.
- Fountains with higher streams are more attractive than those with lower streams.

. . . about the Nozzle

- The nozzle has a fixed radius, but the speed of the water through it can be controlled.
- The nozzle is perpendicular to the ground.
- The nozzle responds rapidly to input from the anemometer.
- The nozzle produces a normally distributed spread of droplets with a low standard deviation.

. . . about Water Droplets

- Because the droplets are small and roughly spherical, they may be treated as spherical.
- The radii of droplets are normally distributed.
- The density of water is unaffected by conditions and therefore remains constant among and within droplets.
- The only outside forces exerted on a water droplet are gravity and the force exerted by the surrounding air, including drag and wind.
- Acceleration due to gravity is the same for all droplets.
- The effect of air perturbations produced by droplets on other droplets is insignificant.
- All droplets share the same constant drag coefficient.
- Droplet interactions and collisions do not increase the overall energy of the system or increase the distance traveled by droplets.

. . . about the Anemometer and Control System

- The anemometer and control system can rapidly evaluate the wind speed, apply a basic formula, and adjust the nozzle in changing wind conditions.

. . . about the Wind

- The wind speed is uniform regardless of altitude.
- Wind blows parallel to the ground without turbulence or irregularities.

Basic Description of Model

Water droplets are emitted from the nozzle and follow trajectories affected by wind and drag. The particles are tracked until they land, including recalculations of trajectories in case of changes in conditions, such as wind. The landing distance from the center of the fountain is recorded. Since the fountain pool is circular, only radial distance is important.

The model ignores wind direction (does not affect a circular fountain pool) and turbulence (insignificant and too complicated to model accurately).

We tested droplet collisions and found that they do not greatly affect the distance that droplets land from the center of the pool; so we ruled out incorporating complex interactions into the model. Further physical analysis supported that decision: Because of conservation of energy and momentum, a droplet could not travel significantly farther after a collision.

Finally, we combined fountain height and volume into speed of the water out of the nozzle, because they are directly determined by the speed.

Our simulation tries all combinations of 11 different water speeds, from 5 to 30 m/s (at intervals of 0.5 m/s), with 16 wind speeds, from 0 to 15 m/s (at intervals of 1 m/s). Each combination is run for five trials of 10,000 droplets. Spillage is logged for radii from 0 to 40 m (at intervals of 0.1 m). The five trials are then averaged to construct an entry in a three-dimensional reference table.

The Underlying Mathematics

The simulation uses basic physics equations to model the flight of water droplets through the air.

Each droplet is acted on by three forces: gravity, drag, and wind. Drag is calculated from the following equation [Halliday et al. 1993]:

$$D = \tfrac{1}{2} C \rho A v^2,$$

where

D is the drag coefficient, an empirically-determined constant dependent mainly on the shape of an object;

ρ is the density of the fluid through which the object is traveling, in this case air;

A is the cross-sectional area of the object; and

$v = |\vec{v}|$ is the speed of the object relative to the wind.

The drag coefficient of a raindrop is 0.60 and the density of air is about 1.2 kg/m^3 [Halliday et al. 1993]. Drag acts directly against velocity, so the acceleration vector from drag can be found from Newton's law $\vec{F} = m\vec{a}$ as

$$\vec{a} = \frac{-D}{m}\frac{\vec{v}}{|v|} = \frac{\frac{1}{2}C\rho A|\vec{v}|^2}{m}\frac{\vec{v}}{|\vec{v}|} = \frac{\frac{1}{2}C\rho A|\vec{v}|}{m}\,\vec{v},$$

where $\vec{a}$ is the acceleration vector and m is mass.

We factor in gravity by subtracting the acceleration g of gravity at Earth's surface, 9.8 m/s^2, from the vertical component of the acceleration vector:

$$\vec{a}_z = -\frac{\frac{1}{2}C\rho A|\vec{v}|}{m}\,\vec{v}_z - g.$$

Next, we use the acceleration to find velocity, beginning with the expression

$$\frac{d\vec{v}}{dt} = -\frac{\frac{1}{2}C\rho A|\vec{v}|}{m}\,\vec{v} = \vec{a}.$$

To circumvent the difficulties of solving a differential equation for each component of the velocity vector, we use Euler's method to approximate the velocity at a series of discrete points in time:

$$\frac{d\vec{v}}{dt} = \vec{a}, \qquad \Delta\vec{v} \approx \Delta t\vec{a}, \qquad \vec{v}_1 \approx \vec{v}_0 + \Delta t\vec{a}_0.$$

We use a similar process to find the position of the droplet, resulting in

$$\vec{x}_1 \approx \vec{x}_0 + \Delta t\vec{v}_0.$$

With $\Delta t = 0.001$ s, error from the approximation is virtually zero.

Now that we have equations for describing the droplet in flight, we generate its initial position and velocity. First, we randomly select a value z from a standard Gaussian (normal) distribution (mean 0, standard deviation 1). We calculate the angle from a set mean μ and standard deviation σ of the distribution of possible angles as

$$\phi = z\sigma + \mu.$$

We randomly select another angle θ between 0 and 2π radians to be the angle between the velocity vector and the x-axis.

Thus, the initial velocity vector of the droplet in spherical coordinates is (ρ, θ, ϕ), where ρ is the magnitude of the velocity. Conversion to rectangular coordinates yields $(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$.

We also randomly select a starting location within the nozzle (whose diameter is 1 cm) and create a radius for the droplet using a similar sampling from a normal distribution. The mass of the droplet is then

$$m = \tfrac{4}{3}\pi r^3 \rho,$$

where ρ is the density of water, 998.2 kg/m^3 at 20° C [Lide 1995]. In the basic simulation, the ϕ distribution has a mean of 0 and a standard deviation of $\pi/60$ radians, and the radius distribution has a mean of 0.0015 m and a standard deviation of 0.0001 m.

In the basic simulation, the nozzle points straight up; however, we also test the effect of tilting the nozzle away from the wind. The program first rotates the nozzle a set angle away from z-axis ($\pi/18$, $\pi/9$, or $\pi/6$ radians). The initial position and velocity vectors are changed by the formula for rotating a point t radians about the x-axis, from z towards negative y [Dollins 2001]:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Next, the program rotates the nozzle around the z-axis to point directly away from the wind (in spherical coordinates, the θ of the nozzle is equal to that of the wind vector). The formula to rotate a point t radians about the z-axis, from x towards y [Dollins 2001] is

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Design of Program

We developed a program to simulate the fountain. The program component `Simulator.class` manages interactions among the other components of the program. `Particle.class` describes a water droplet in terms of position, velocity, radius, and mass. `Vector3D.class` creates and performs functions with vectors, including setting vector components, adding and subtracting vectors, multiplying vectors by scalars, finding the angle between vectors, and finding the magnitude of a vector.

`Emitter.class` creates a fountain by spraying droplets. It considers the nozzle radius, direction, and angle orientations and generates launch angle ϕ and launch location on the nozzle according to the prescribed distributions.

Launch speed is determined by `Anemometer.class`, which takes the wind-speed reading from the anemometer and sends that plus fountain radius and tolerable spillage percentage to `FindingVelocity.class`. This latter class does a table lookup and returns the maximum droplet speed for the spillage percentage. `Anemometer.class` then sets the droplet emission speed.

Once a droplet is emitted, its trajectory is updated every iteration using `Physics.class`, which checks `Wind.class` (which contains a vector of the current wind) in each iteration in calculating an updated trajectory. Then `Physics.class` iterates through the entire collection of particles and computes new velocities and positions based on the forces acting on them.

The `Analyzer.class` checks to see if any particles have hit the ground; their locations are recorded and they are removed from consideration. It then relays this information back to `Simulator.class`, where it is written to disk.

Results

A program run takes 5 min to model 2 sec of spray (10,000 droplets).

Scatterplots showing where droplets land appear uniform and radially symmetric (**Figure 1**); a side profile of the points appears uniformly distributed along a line and bilaterally symmetric (**Figure 2**).

We then introduced wind in the positive x-direction. As expected, the landing plot and the side profile plot are skewed horizontally (**Figure 3**).

Figures 1–3 conform very well to the actual appearance of fountains, indicating that our model creates an accurate portrait of a real fountain.

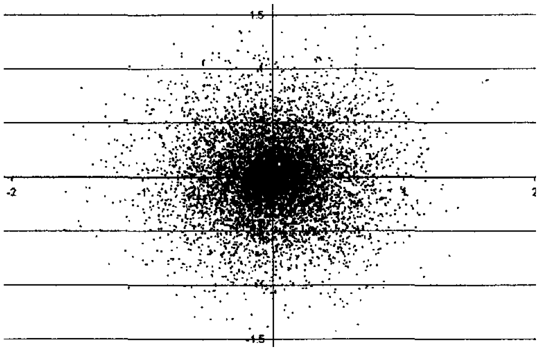

Figure 1. Fountain from overhead: launch speed 10 m/s, no wind.

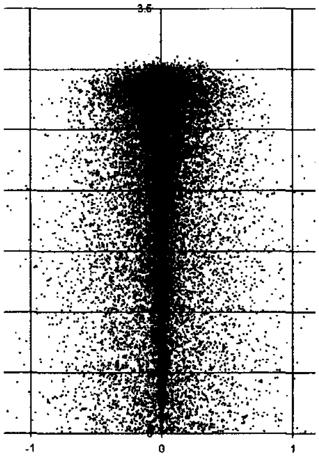

Figure 2. Fountain from the side: launch speed 10 m/s, no wind.

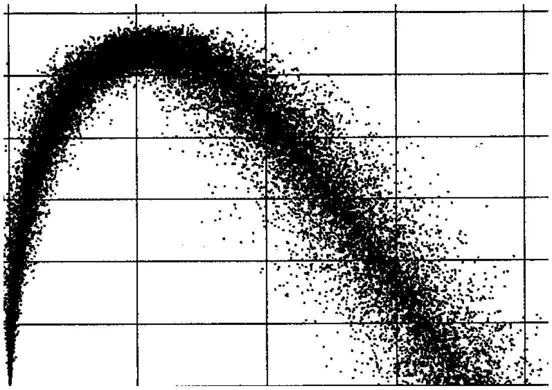

Figure 3. Fountain from the side: launch speed 10 m/s, wind of 5 m/s

We used a pool radius of 4 m and an acceptable spillage of 5% to generate a table of water speeds. We then simulated control of the fountain by a theoretical anemometer using the table. The anemometer was subjected to sinusoidal wind ranging from 0 to 3 m/s. There was 7.6% spillage; the extra loss is from droplets carried farther by an increase of wind after launch.

Analysis of Results

We tested the model for accuracy and sensitivity. We did some useful analysis of the physics of the model by creating a miniature version of the simulation on an Excel spreadsheet to track the trajectory of a single particle.

Our first test was of the accuracy of the Euler's method approximation. Continuous equations for the motion of a flying droplet can be easily developed if drag and wind are ignored, so we chose this scenario to test our approximation. We considered a particle with a speed of 10 m/s and a launch angle of $\pi/60$

radians. We calculated its trajectory using

$$x = (v_i \sin\phi)t, \qquad y = (v_i \cos\phi)t - \tfrac{1}{2}gt^2,$$

where

x is the position along the horizontal axis,

y is the position along the vertical axis,

v_i is the magnitude of the initial velocity,

t is time,

g is the acceleration of gravity, and

ϕ is the launch angle, following our previous convention of measuring from the vertical axis towards the horizontal.

We compared that trajectory with the one calculated Euler's method. The two were indistinguishable, showing that the Euler's method approximation results in virtually no error.

We also used the spreadsheet model to examine the effects of wind and drag on individual particle trajectories. **Figure 4** compares trajectories of particles with and without drag; and **Figure 5** compares the trajectories of two droplets, one with a 5 m/s wind and the other with no wind. Drag has a major effect and cannot be ignored.

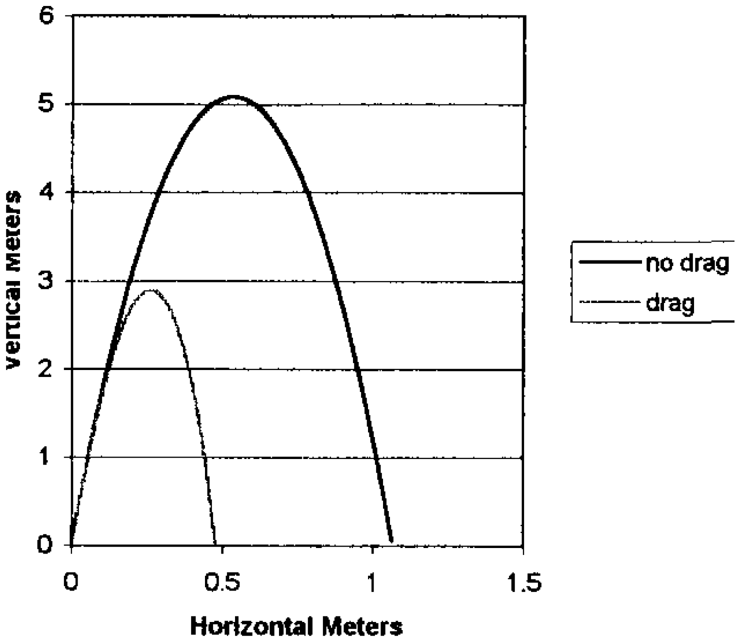

Figure 4. Droplet trajectories with and without drag.

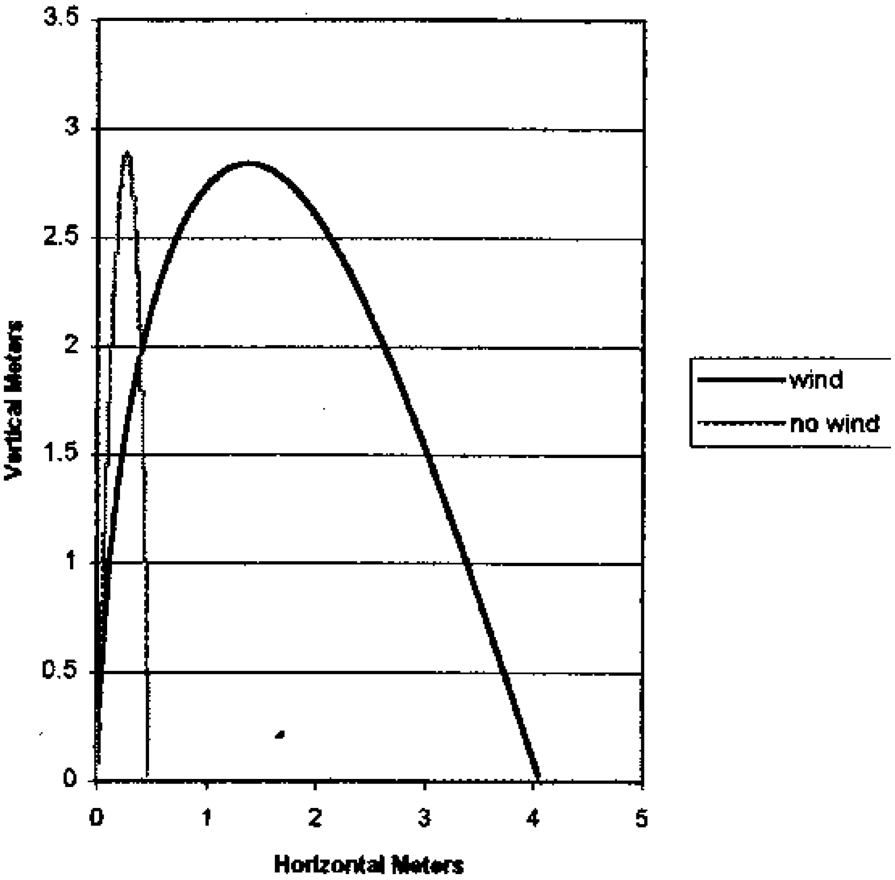

Figure 5. Droplet trajectories with and without wind.

Sensitivity

We tested the effect of changing some base factors in the model, using an initial water speed of 10 m/s. Fountain pool radii were chosen to highlight general trends in the data, either stability or sensitivity.

Nozzle angle

We ran the simulation at a wind speed of 5 m/s with the nozzle tilted 0, $\pi/18$, $\pi/9$, or $\pi/6$ radians in the same direction as the wind vector. For a pool with a radius of 6 m, no water fell outside when the nozzle was pointed straight up and virtually none with a tilt of $\pi/18$ radians. With a tilt of $\pi/9$ radians, 47% of the water fell outside; for $\pi/6$ radians, 99.9% fell outside. The data suggest that tilting the nozzle into the wind could be used to prevent spillage.

Nozzle radius

With no wind and a pool radius of 2 m, virtually no water was spilled for nozzle radii of 0.25, 0.5, or 1 cm. With a 5 m/s wind, virtually all of the water was spilled at all three radii. The radius of the nozzle thus has virtually no effect on the percentage spilled, supporting our decision to use a percentage measure so as to allow the model to apply to fountains with different flow rates.

Water droplet size

In a fountain with a pool radius of 3.5 m, droplet radii of 0.75, 1.5, and 3 mm resulted in 94%, 53%, and 6% percent spillage. The sensitivity to droplet radius is a reflection of real-world behavior rather than a weakness of the model: Small particles, because of their low mass, are greatly affected by wind and drag.

Variability of launch angle

With a 3.5 m pool, a 5 m/s wind produced 15%, 45%, and 49% spillage for standard deviations of$\pi/180$, $\pi/20$, and $\pi/12$ radians. Thus, results are sensitive to the launch angles of the droplets, dictating that the angle be measured carefully before the model is used.

Graphical Examination

A final test of the model's accuracy was to create pictures of the droplets in flight and a scatterplot of where the droplets landed. These pictures (

Strengths

As intended, the model controls the fountain height and volume according to conditions. It creates the largest and therefore most interesting fountain possible while maintaining the set spillage level. For low spillage levels, no passersby get drenched nor is much water wasted.

The model is easy to adapt by changing parameters, including nozzle size, mean droplet size, mean launch angle and standard deviation, and mean droplet size and standard deviation.

Graphs of the droplets in midair show that the programmed fountain accurately depicts a real fountain.

Use of a table means that the radius or spill percentage can be changed without requiring recalculations. Since the control system does not do any calculation, it is fast.

Weaknesses

A major problem occurs when wind speed increases quickly: Water droplets already emitted cannot be slowed down and will be carried away on the wind. However, any fountain system will suffer from this dilemma. To give the fountain a small buffer, the radius of the fountain can be set lower than the radius of the pool.

We model the wind as moving parallel to the ground with uniform speed. Real wind may vary with altitude and may blow from above or below the droplets. We also neglect wind turbulence.

We ignore droplet collisions. Some droplets may combine and then separate, causing slightly more splatter or mist; or the droplets' collisions may cause more of them to fall short of their expected trajectories, reducing spillage.

References

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