A Single-Car Interaction Model of Traffic for a Highway Toll Plaza

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Making assumptions about the homogeneity of cars and tollbooths, we create the Single-Car Model, describing the motion of a car in the toll-plaza in terms of safety considerations and reaction time. The Multi-Car Interaction Model, a real-time traffic simulation, takes into account global car behavior near tollbooths and merging areas.

Drawing on data from the Orlando–Orange Country Expressway Authority, we simulate realistic conditions. For high traffic density, the optimal number of tollbooths exceeds the number of highway lanes by about 50%, while for low traffic density the optimal number of tollbooths equals the number of lanes.

The text of this paper appears on pp. 299–315.

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Definitions and Key Terms

- A **toll plaza** with n lanes is represented by the space $[-d, d] \times \{1, \ldots, n\}$, where members of the set $\{0\} \times \{1, \ldots, n\}$ are called **tollbooths** and d is called the **radius** of the toll plaza. Denote the tollbooth $\{0\} \times \{i\}$ by τ_i. The subspace $[-d, 0) \times \{1, \ldots, n\}$ is known as the **approach region** and $(0, d] \times \{1, \ldots, n\}$ is known as the **departure region**.

- A **highway/toll plaza pair** is represented by the space $H = (-\infty, d) \times \{1, \ldots, m\} \cup [-d, d] \times \{1, \ldots, n\} \cup (d, \infty) \times \{1, \ldots, m\}$, where the toll plaza is (as above) the subspace $[-d, d] \times \{1, \ldots, n\}$ and the stretches of highway are the subspaces $(-\infty, d) \times \{1, \ldots, m\}$ and $(d, \infty) \times \{1, \ldots, m\}$. Elements of the sets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ are **highway lanes** and **tollbooth lanes** respectively, and elements of $\mathbb{R}$ are **highway positions**. In practice, we take $m \geq n$.

- The **fork point** of a highway/toll plaza pair, given by the highway position $-d$, is the point at which highway lanes turn into toll lanes. Similarly, the **merge point** of a highway/toll plaza pair, given by the highway position d, is the point at which toll lanes turn back into highway lanes (**Figure 1**).
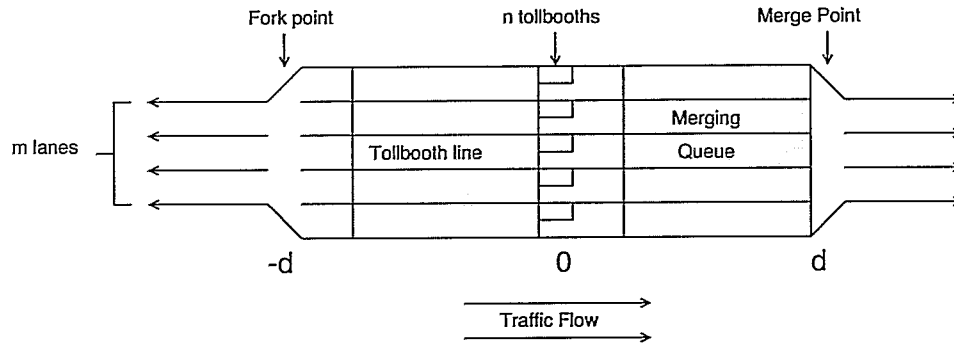

Figure 1. A depiction of the highway/toll plaza pair.

- A **car** C is represented by a 4-tuple $(L, a_+, a_-, a_{\text{brake}})$ and a position function $p = (x, k) : \mathbb{R} \to H$ where $x(t)$ is smooth for all t. Here, $x(t)$ gives the highway position of the front tip of C and $k(t)$ is the (tollbooth or highway) lane number of C. Let L be the length of C in meters, a_+ the constant comfortable positive acceleration, a_- the constant comfortable brake acceleration, and a_{brake} the maximum brake acceleration. At a fixed time, the region of H **in front** of C is the portion of H with greater highway position than C, while the **rear** of C is the region of H with highway position at most the position of C minus L.

- The **speed limit** v_{max} of H is the maximum speed at which any car in H can travel.

- The **traffic density** $\rho(t)$ of H at time t is the average number of cars per lane per second that would pass highway position 0 if there were no toll plaza.

- The **average serving rate** s of tollbooth τ_i is the average number of cars that can stop at τ_i, pay the toll, and leave, per second.

Table 1.
Variables, definitions, and units.

Variable	Definition	Units
n	Number of tollbooths	unitless
ρ	Traffic density	cars/s
T	Total delay time	s
x	Position	m
v	Velocity	m/s
x_o	Position of initial deceleration	m
t_o	Time of initial deceleration	s
x_f	Position upon returning to speed limit	m
t_f	Time upon returning to speed limit	s
x_1	Position of car C	m
x_2	Position of car C'	m
v_1	Velocity of car C	m/s
v_2	Velocity of car C'	m/s
x_1'	Position of car C after time step	m
x_2'	Position of car C' after time step	m
v_1'	Velocity of car C after time step	m/s
v_2'	Velocity of car C' after time step	m/s
G	Safety gap	m
G'	Safety gap after time step	m
t	Time	s
t'	Additional time	s
α_C	Compensation deceleration from car/safety gap overlap	m/s^2
α_O	Compensation deceleration from obstacle/safety gap overlap	m/s^2
x	Position	m
v	Velocity	m/s
c_i	Size of tollbooth line i	cars
l_i	Length of tollbooth line i	m
t_{serve}	Time C enters departure area	s
t_{merge}	Time C upon passing merge point	s
v_{out}	Velocity of a car C upon passing merge point	s

Table 2.
Constants, definitions, and units.

Constant	Meaning	Units
d	toll plaza radius	m
m	Number of highway lanes	unitless
a_+	Comfortable acceleration	m/s^2
a_+	Comfortable deceleration	m/s^2
a_{brake}	Hard brake deceleration	m/s^2
L	Car length	m
v_{max}	Speed limit	m/s
s	Mean serving rate	cars/s
σ	Standard deviation of serving time	s/car
Δt	Expected reaction time	s
γ	Unexpected reaction time	s
ϵ	Line spacing distance	m

General Assumptions

Time

- Time proceeds in discrete time steps of size Δt.

Geometry of the Toll Plaza

- The highway is straight and flat and extends in an infinite direction before and after the toll plaza. The highway is obstacle-free with constant speed limit $v_{\max}$. The assumption of infinite highway is based on toll plazas being far enough apart that traffic delays at one toll plaza don't significantly affect traffic at an adjacent one.

- A car's position is determined uniquely by a lane number and a horizontal position. Thus, on a stretch of road with m operating lanes, the position of a car is given by the ordered pair $(x, i) \in \mathbb{R} \times \{1, \ldots, m\}$.

Tollbooths and Lines

- A car comes to a complete stop at a tollbooth.

- The time required to accelerate and decelerate to move up a position in a waiting line is less than the serving time of the line. Thus, average time elapsed before exiting a line is simply a function of average serving time and line length measured in cars.

- A car cannot enter a tollbooth until the entire length of the car in front of it has left the tollbooth.

- All tollbooths have the same normally distributed serving time with mean $1/s$ and standard deviation σ.

Fork and Merge Points

- Transitions between the highway and tollbooth lanes are instantaneous.

- When transitioning at the fork point into a tollbooth lane, cars enter the lane with the shortest tollbooth lines.

- There is no additional delay associated with the division of cars into tollbooths.

- The process of transitioning at the merge points from the tollbooth lanes, called **merging**, does incur delay due to bottlenecking because we assume that there are at least as many tollbooth lanes as highway lanes.

Optimality

Measures of optimality for a toll include having minimal average delay, standard deviation of average delay, accident rate, and proportion of cars delayed [Edie 1954]. We assume that optimality occurs when cars experience minimal average delay. Specifically, for a car C, let x_0, t_0 be the position and time at which C first decelerates from the speed limit to enter the tollbooth line, and let x_f, t_f be time and position at which, having merged onto the highway once more, C reaches the speed limit. Then the delay T experienced by the car, or the time cost associated with passing through the toll plaza instead of travelling unhindered, is given by

$$T = t_f - t_0 - \frac{x_f - x_0}{v_{\max}}.$$ (1)

We secondarily prefer toll plaza configurations with minimal construction and operating cost, i.e., toll plaza configurations with fewer tollbooths. Specifically, for a given highway, if two values of n (the number of tollbooths) give sufficiently close average delay times (say, within 1 s), we prefer the lower n.

We rephrase the problem as follows:

Given a highway configuration with m lanes and a model of traffic density, what is the least number of tollbooth lanes n that minimizes the average delay (within 1 s) experienced by cars travelling through the tollbooth?

Expectations of Our Model

- For sufficiently low traffic density, the delay time per car is relatively constant and near the theoretical minimum, because the tollbooth line does not grow and there are no merging difficulties. *We expect that for low density the optimal number of tollbooths equals or slightly exceeds the number of lanes.*

- For high traffic density, the delay time per car is very large and continues to grow, because the tollbooth queue is unable to move fast enough to handle the influx of cars; waiting time increases approximately linearly in time. *We expect that for high density, the optimal number of tollbooths significantly exceeds the number of lanes.*

- An excessive number of tollbooths leads to merging inefficiency, causing great delay in the departure region.

The Single-Car Model

Additional Definitions and Assumptions

- An **obstacle** for a car C is a point in the highway/toll plaza pair which C must slow down to avoid hitting. The only obstacle that we consider is the merge point under certain conditions.

- At a fixed time, the **closest risk** to a car C is the closest obstacle or car in front of C.

- The **unexpected reaction time** γ is the amount of time a car takes between observing an unexpected occurrence (a sudden stop) and physically reacting (braking, accelerating, swerving, etc.). The **expected reaction time** Δt is the amount of time between observing an expected occurrence (light change, car brake, tollbooth) and physically reacting.

- Cars are homogeneous; that is, all have the same L, a_+, a_-, and a_{brake}.

- All cars move in the positive direction.

- All cars observe the speed limit $v_{\max}$. Moreover, unless otherwise constrained, a car travels at this speed or accelerates to it. In particular, outside a sufficiently large neighborhood of the toll plaza, all cars travel at $v_{\max}$.

- Cars accelerate and decelerate at constant rates a_+ and a_- unless otherwise constrained.

- Cars do not attempt to change lanes unless at a fork or merge point. That is to say, for a car C, $k(t)$ is piecewise constant, changing only at t such that $x(t) = -d$ or d.

- A car C prefers to keep a certain quantity of unoccupied space between its front and its closest risk, of size such that if C were to brake with maximum deceleration, a_{brake}, C would always be able to stop before reaching its closest risk [Gartner et al. 1992, §4]. We refer to this quantity as the **safety gap** G. Given the position of a car C, the position corresponding to distance G in front of C is the **safety position** with respect to C. If the safety position with respect C does not overlap the closest risk, we say C is **unconstrained**.

- A car can accurately estimate the position and velocity of itself and of the car directly in front of it and its distance from the merging point.

- If a car C comes within a sufficiently small distance ϵ of a stopped car, C stops. This minimum distance ϵ is constant.

- For each car, there is a delay, the reaction time, between when there is a need to adjust acceleration and when acceleration is actually adjusted. Green [2000] splits reaction times into three categories; the ones relevant to us are

expected reaction time Δt and unexpected reaction time γ, which are defined above. Although these times vary with the individual, we make the simplifying assumption that all cars have the same values, $\Delta t = 1$ s and $\gamma = 2$ s. Reaction times provide a motivation for discretizing time with time step Δt; drivers simply do not react any faster.

The Safety Gap

We develop an expression for the safety gap G of car C, which depends on the speed of the closest risk C'. Let the current speeds of C and C' be v_1 and v_2. Now suppose that C' brakes as hard as possible and thus decelerates at rate a_{brake}. In time v_2/a_{brake}, car C' stops; meanwhile it travels distance

$$v_2 \frac{v_2}{a_{\text{brake}}} - \tfrac{1}{2} a_{\text{brake}} \left(\frac{v_2}{a_{\text{brake}}} \right)^2 = \frac{v_2^2}{2a_{\text{brake}}}.$$

If C starts braking after a reaction time of γ, it takes total time $\gamma + v_1/a_{\text{brake}}$ to stop and travels distance

$$\gamma v_1 + \frac{v_1^2}{2a_{\text{brake}}}.$$

Thus, in the elapsed time, the distance between C and C' decreases by

$$\gamma v_1 + \frac{v_1^2 - v_2^2}{2a_{\text{brake}}}.$$

Therefore, this must be the minimum distance between the front of C and the back of C' in order to avoid collision. Accounting for the length of C', the minimum distance between C and C', and thus the safety gap, must be

$$G = L + \gamma v_1 + \frac{v_1^2 - v_2^2}{2a_{\text{brake}}}.$$

Now suppose that the closest risk is an obstacle, in particular the merge point. Rather than braking with deceleration a_{brake}, C will want to keep a safety gap that allows for normal deceleration of a_-. Because deceleration on approach is expected, C will opt to decelerate at a comfortable rate, a_-. Moreover, since C is reacting to an expected event, the reaction time is given by Δt. Since the length and velocity of the obstacle are both 0, the safety gap must be

$$G = \Delta t v_1 + \frac{v_1^2}{2a_-}.$$

Individual Car Behavior

An individual car C can be in one of several positions:

- No cars or obstacles are within its safety gap, that is, C is unrestricted. Consequently, C accelerates at rate a_+ unless it has velocity $v_{\max}$.

- The tollbooth line is within braking distance. Since this is an expected occurrence, the car brakes with deceleration a_-.

- Another car C' is within its safety gap, so C reacts by decelerating at some rate α_C so that in the next time step, C' is no longer within the safety gap. C chooses α_C based on the speeds v_1, v_2 and positions x_1, x_2 of both cars. If C assumes that C' continues with the same speed, then after one time step Δt the new positions and speeds are

$$x_1' = x_1 + v_1\Delta t - \tfrac{1}{2}\alpha_C(\Delta t)^2, x_2' = x_2 + v_2\Delta t,$$
$$v_1' = v_1 - \alpha_C\Delta t, v_2' = v_2,$$

and the new safety gap is

$$G' = \gamma v_1' + \frac{v_1'^2 - v_2'^2}{2a_{\text{brake}}}.$$

For the new position of C_2 to not be within the new safety gap, we must have

$$x_2' - x_1' - L = G'.$$

Substituting into this equation, we find:

$$x_2 + v_2\Delta t - v_1\Delta t + \tfrac{1}{2}\alpha_C(\Delta t)^2 - L = \gamma v_1 - \gamma\alpha_C\Delta t + \frac{(v_1 - \alpha_C\Delta t)^2 - v_2^2}{2a_{\text{brake}}}.$$

Solving this equation for α_C and taking the root corresponding to the situation that C trails C', we find that

$$c\alpha_C = \frac{1}{\Delta t}\left(\frac{\Delta t a_{\text{brake}}}{2} + v_1 + \gamma a_{\text{brake}}\right.$$
$$- \tfrac{1}{2}\left([(\Delta t)^2 - 4v_1\Delta t a_{\text{brake}} + 4\Delta t a_{\text{brake}}^2\gamma + (2\gamma a_{\text{brake}})^2\right.$$
$$\left.\left.+8(x_2 - x_1)a_{\text{brake}} + 8v_2\Delta t a_{\text{brake}} - 8La_{\text{brake}} + 4v_2^2\right]^{\frac{1}{2}}\right).$$

- The merge point is within its safety gap. The safety gap equation differs from the car-following case by using a_- instead of a_{brake} and Δt instead of γ and by leaving out the L. Therefore, by the same argument as in the previous paragraph, the deceleration is

$$\alpha_0 = \frac{1}{\Delta t}\left(v_1 + \frac{3\Delta t a_-}{2}\right.$$
$$\left.- \tfrac{1}{2}\sqrt{(\Delta t)^2 - 4v_1\Delta t a_- + 8(\Delta t)^2 a_-^2 + 8(x_2 - x_1) + 8v_2\Delta t a_- + 4v_2^2}\right).$$

Finally, once we have determined the new acceleration α of C, we can change its position and velocity for the next time step as follows (letting x, v and x', v' be the old and new position and velocity respectively):

$$v' = v + \alpha \Delta t, \qquad x' = x + v \Delta t + \tfrac{1}{2} \alpha \Delta t^2.$$

Calculating Delay Time

We calculate the delay time T for a car C moving through a toll plaza by breaking the process into several steps, tracing the car as soon as it starts slowing down before passing through the tollbooth, and until it merges back into a highway lane and accelerates to the speed limit.

Recalling our assumptions that cars do not change lanes, that they are evenly distributed among the lanes, and that there is no time loss associated with the distribution of cars into tollbooth lane at the fork point, we find that the period of approach to a tollbooth can be broken down as follows:

- **Deceleration from speed limit to stopping.** We assume that a car comes to a complete stop upon joining a tollbooth line as well as upon reaching the tollbooth. Therefore, the first action taken by a car approaching a toll plaza is to decelerate to zero; at constant deceleration a_-, it takes time $v_{\max}/a_-$ to go from the speed limit to zero, over distance $v_{\max}^2/2a_-$.

- **Line Assignment.** As a car approaches the toll plaza, it is assigned to the currently shortest line. Let c_i be the number of cars in line i. The cars are spaced equidistantly throughout the line with distance ϵ between cars. Thus, as long as the length of the line is less than d, we have that the length of the line is $l_i = c_i(L+\epsilon)$, where L is the length of a car. Now, if $c_i(L+\epsilon) > d$, then the line extends to before the fork area, where there are m lanes instead of n. Assuming that the line lengths are roughly the same, increasing the minimum line length by one car increases the total number of cars by about n, and therefore each of the m lanes has an additional n/m cars. It follows that

$$l_i = \begin{cases} c_i(L + \epsilon), & \text{if } c_i(L + \epsilon) < d; \\ d + \dfrac{n[c_i(L + \epsilon) - d]}{m}, & \text{otherwise.} \end{cases} \tag{2}$$

- **Movement through a Tollbooth Line.** A car C joins the tollbooth line that it was assigned if such a line exists, that is, if the line length l_i is positive. In this case, C must wait for the entire line ahead to be serviced before C reaches the tollbooth. Let t_{serve} be the time when C enters the departure area, after it has been serviced. If there is an overflow of cars from the merge line such that C cannot leave the tollbooth, t_{serve} is the time when the car actually leaves, after the line in front has advanced sufficiently.

- **Movement through the Departure Region.** Different scenarios can occur in the departure region.

- Once C enters the departure area, it accelerates forward until either another car or the merge point enters its safety gap.

- If another car C' enters the safety gap of C, C slows down and follows C' until C' merges, at which time the merge point will overlap the safety gap of C.

- When the safety position of C reaches the merge point, if C does not have right of way, C will slow down so as to prevent the merge point from overlapping the safety gap, treating the merge point as an obstacle. This is in order to allow other cars who have already begun to merge, to do so until C can merge.

- Upon having the right of way, C merges and accelerates unconstrained from the departure region until reaching the speed limit. Let t_{merge} be the time at which C merges and v_{out} be its speed at that time. Then

$$t_f = t_{merge} + \frac{v_{max} - v_{out}}{a_+},$$

$$x_f = d + v_{out}\frac{v_{max} - v_{out}}{a_+} + \frac{(v_{max} - v_{out})^2}{2a_+}.$$

Thus by (1), the delay experienced by C is

$$T = t_{merge} - t_{line} - \frac{l_i(t_{line}) + d}{v_{max}} + \frac{v_{max} - v_{out}}{a_+} - \frac{3v_{max}}{2a_-} - \frac{v_{out}(v_{max} - v_{out})}{a_+ v_{max}}.$$

The Multi-Car Interaction Model

We now determine the average delay time for a group of cars entering the toll plaza over a period of time. We simulate a group of cars arriving as per an arrival schedule and average their respective delay times. There are two complications: determining the arrival schedule (the distribution of individual cars over which to average) and the two variables t_{merge} and v_{out} (used in the delay-time formula above).

To determine computationally the average delay time, we must use the traffic density function $\rho(t)$ to produce a car arrival schedule. We create the arrival schedule by randomly assigning arrival times based on ρ. Using this schedule, we determine which cars begin to slow down for a given time step. Unfortunately this task is not as straightforward as determining whether a car's arrival time is less than the present time step. The arrival schedule provides the time a car reaches 0 (on the highway) if unconstrained. We wish to know when a car reaches a certain distance from the tollbooth line. Essentially given that a car would be at a set position (say 0 for the tollbooth) at time t, we seek the time t' when that car would have passed the front of the tollbooth line. This reduces to a question of Galilean relativity and we find that

$t' = t - l_i(t)/v_{\max}$. Now, up to knowing $l_i(t)$, we can exactly determine when cars join the tollbooth lines. We use **(2)** and the difference equation for car flow

$$\frac{\Delta c_i}{\Delta t} = \frac{m}{n}\rho\left(t - \frac{l_i}{v_{\max}}\right) - s_i$$

to keep track of the length of the tollbooth line, increasing it as cars join and decreasing it as cars are served.

As a car's arrival time (adjusted to the line length) is reached, we immediately assign it to the current shortest tollbooth line. We introduce normally distributed serving times with mean $\frac{1}{s}$ (where s is serving rate) and standard deviation σ that we assume to be $\frac{1}{6s}$.

The second consideration in simulating many cars is how to determine t_{merge} and v_{out} for each car. Our time-stepping model allows us to recursively update every car and thus to determine the actions of a single car at each time step. Following the rules in the previous section, we know exactly when and how much to accelerate (a_+) and decelerate (α_c, α_o). Furthermore, we observe that when a car that is first in its tollbooth lane approaches the merge point, it joins a merging queue (with at most n members). The only time when a car (on the merging queue) does not treat the queue as an obstacle (and consequently slow down) is when a highway lane clears and the car is taken from the queue and allowed to accelerate across the merge point and into free road. A lane is clear once the car in it accelerates $L + \epsilon$ passed the merge point.

With this model, we thus have a method, given a highway with m lanes, a certain traffic density function, and values for various constants, to calculate the optimal number of tollbooth lanes n. We can estimate a finite range of values of n in which the optimal number must lie. For each value of n we run our model, calculating the delay experienced by each car and averaging these to calculate average delay. We then compare our average delays for all n, choosing the minimal such n so that average delay is within 1 s of the minimum.

Case Study

We need reasonable specific values for our constants and density function for use in our tests. We take most of these from the Orlando–Orange Country Expressway Authority [2004] and a variety of reports on cars. We begin with a few simplifying assumptions about our traffic density function.

- To determine optimal average delay, it suffices to calculate the average delay over a suitably chosen day, as long as this day has periods of high and low density. This is reasonable because over most weekdays, traffic tends to follow similar patterns. Therefore, we limit the domain of ρ to the interval of seconds $[0, 3600 \times 24]$.

- The function $\rho(t)$ is piecewise constant, changing value on the hour. This

is reasonable: Since cars are discrete, $\rho(t)$ really is an average over a large amount of time and thus must already be piecewise constant.

* The length of the time interval between an arriving car and the next car is normally distributed.

The Orlando–Orange Country Expressway Authority's report on plaza characteristics [2004] allows us to construct a realistic traffic density function ρ for the purposes of testing. The report gives hourly traffic volume on several highways in Florida, which we use along with our assumption about normal arrival times to develop an arrival schedule for cars on the highway.

We assume several realistic values for constants defined earlier (Table 3).

Table 3.
Constant values used in testing.

Constant name	Symbol	Value
Comfortable acceleration	a_+	$2\,\mathrm{m/s^2}$
Comfortable deceleration	a_-	$2\,\mathrm{m/s^2}$
Hard braking deceleration	a_{brake}	$8\,\mathrm{m/s^2}$
Car length	L	$4\,\mathrm{m}$
Speed limit	v_{max}	$30\,\mathrm{m/s}$
Line spacing	ϵ	$1\,\mathrm{m}$

Our model assumes that every tollbooth operates at a mean rate of approximately s cars/s. But each type of tollbooth—human-operated, machine-operated, and beam-operated (such as an EZ-pass)—has a different service rate. We attempt to approximate the heterogenous tollbooth case by making s a composite of the respective services rates. According to Edie [1954], the average holding time (inverse of service rate) for a human operated tollbooth is 12 s/car, while according to the Orlando–Orange County Expressway Authority [n.d.], the average service rate for their beamoperated tollbooths, the E-Pass, is 2 s/car. Similarly, a report for the city of Houston [Texas Transportation Institute 2001] places the holding time for a human operated tollbooth at 10 s/car and a machine operated tollbooth at 7 s/car. Looking at these times, we find that a reasonable average holding time could be 5 s/car, giving us an average service rate $s = 0.2$ cars/s.

For verification, we consider hourly traffic volumes for six Florida highways, with from 2 to 4 lanes and varying traffic volumes [Orlando–Orange Country Expressway Authority 2004]. We use the data to obtain $\rho(t)$ and test various components of our model. After model verification, we use our model to determine the optimal tollbooth allocations.

We look at two typical cases. A toll plaza radius of $d = 250$ m [Orlando–Orange Country Expressway Authority 2004] is fairly standard. The hourly traffic densities of the six highways take a standard form; they differ mostly in amplitude, not in shape. Therefore, we model our density functions on two such standard highways, 4-lane Holland West (high density) and 3-lane Bee

Line (low density) (**Figure 2**). We extrapolate their traffic volume profiles to profiles for highways with 1 through 7 lanes. For m lanes, we scale the traffic volume by $m/4$ (Holland West) or $m/3$ (Bee Line). Doing so maintains the shape of the profile and the density of cars per lane while increasing the total number of cars approaching the toll plaza.
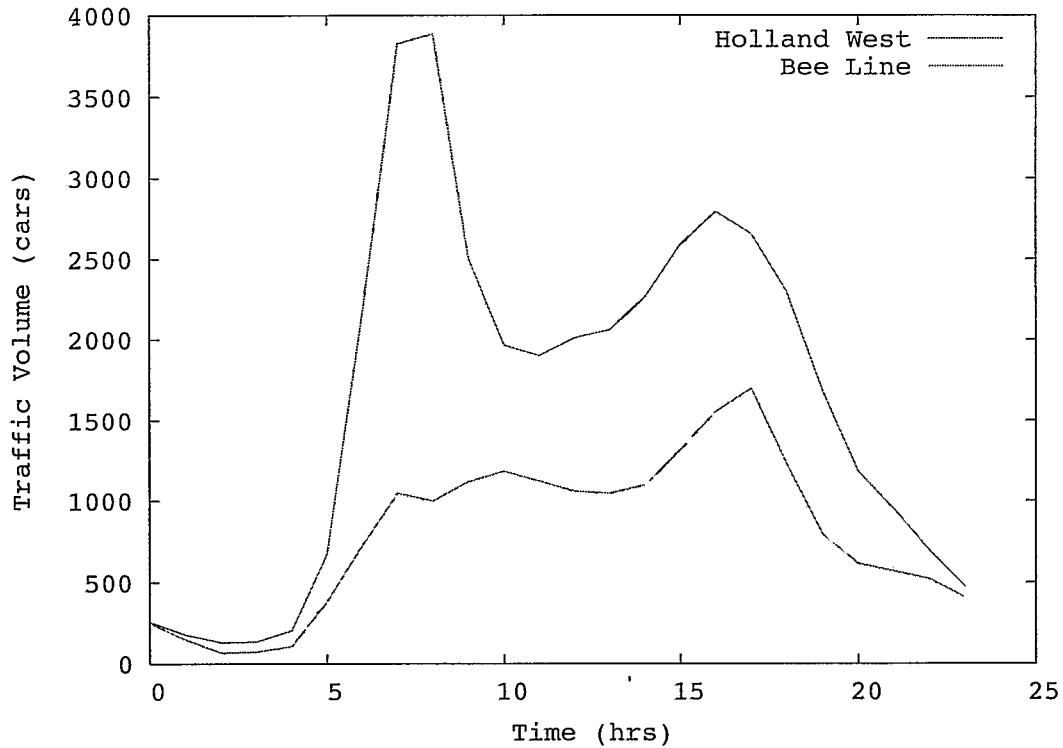

Figure 2. Traffic volume as a function of time for Holland West (top) and Bee Line (bottom).

Verification of the Traffic Simulation Model

Based on the optimality criteria, for various test scenarios we determine the minimal number of tollbooths with average delay time within 1 s of the minimal average delay. We show the results in **Table 4**.

Model results for three toll plaza match the actual numbers, and the other three differ only slightly. In the case of Dean Road, having 4 tollbooths (the actual case) instead of 5 leads to a significantly longer average delay time (70 s vs. 25 s). For Bee Line and Holland West, the difference is at most 1 s. These results suggest that our model agrees generally with the real world.

Table 4.

Comparison of model-predicted optimal number of tollbooths and real-world numbers for six specific highway/toll plaza pairs.

| Highway | Tollbooths | | Comparison |
	Optimal	Actual	
Hiawassee	4	4	same
John Young Parkway	4	4	same
Dean Road	5	4	mismatch
Bee Line	3	5	close
Holland West	7	6	close
Holland East	7	7	same

Results and Discussion

Using real-world data from the Orlando–Orange Country Expressway Authority [2004], we create 14 test scenarios: high and low traffic density profiles for highways with 1 to 7 lanes. For each scenario, we run our model for a number n of tollbooths n ranging from the number of highway lanes m to $2m + 2$ (empirically, we found it unnecessary to search beyond this bound) and determine at which n the average delay time is least—this is the optimal number of tollbooths. We present our optimality findings in **Table 5**. For high traffic densities and more than two lanes, the optimal number of tollbooths tends to exceed the number of highway lanes by about 50%, a figure that seems to match current practice in toll plaza design; for low densities, the optimal number of tollbooths equals the number of highway lanes.

Table 5.

The optimal number of tollbooths for 1 to 7 highway lanes, by traffic density.

	High density							Low density						
Lanes	1	2	3	4	5	6	7	1	2	3	4	5	6	7
Tollbooths	3	4	5	6	8	9	11	1	2	3	4	5	6	7

For high traffic density but only as many tollbooths as lanes, the average delay time is is roughly 500 s, almost 20 times as long as the average delay of 25 s for the optimal number of tollbooths. So we strongly discourage construction of only as many tollbooths as lanes if high traffic density is expected during any portion of the day. However, when there is low traffic density, this case is optimal, with an average delay time of 22 s.

Further Study

To simulate real-world conditions more accurately, we could

* consider the effect of heterogenous cars and tollbooths;

- allow for vehicles other than cars, each with their own size and acceleration constants;

- consider the effect of changing serving rates, since research shows that average serving time decreases significantly with line length [Edie 1954]; or

- vary the toll plaza radius.

Strengths of Model

The main strength of the Multi-Car Interactive Model stems from our comprehensive and realistic development of single-car behavior. The intuitive notion of a car's safety gap and its relation to acceleration decisions, as well the effects of reaction times associated with expected and unexpected occurrence all find validation in traffic flow theory [Gartner et al. 1992]. The idea of a merge point and a car's behavior approaching that point mimics the practices of yielding right-of-way as well as cautiously approaching lane merges. Our choice of time step realistically approximates the time that normal decision-making requires, allowing us to capture the complete picture of a toll plaza both on a local, small scale, but also on the scale of overall tendencies. Finally by allowing for certain elements of normally distributed randomness in serving time and arrival time we capture some of the natural uncertainty involved in traffic flow.

A great strength of our model lies in the accuracy of its results. Our model meets all of our original expectations and furthermore predicts optimal tollbooth line numbers very close to those actually used in the real world, suggesting that our model approximates real-world practice.

Finally, the Multi-Car Interactive Model provides a versatile framework for additional refinements, such as modified single-car behavior, different types of tollbooth, and nonuniform serving rates.

Weaknesses of Model

In the real world, a car in the center lane has an easier time merging into the center lanes than a car in a peripheral lane, but this behavior is not reflected in our model. We also disallow lane-changing except at fork and merge points, though cars often switch lanes upon realizing that they are in a slow tollbooth line. Our method of determining car arrival times may be flawed, since Gartner et al. [1992, §8] suggest that car volume is not uniformly distributed over a given time block but rather increases in pulses.

Perhaps the two greatest weaknesses of our model are that all cars behave the same and all tollbooth lanes are homogeneous. While we believe that we capture much of the decision-making process of navigating a toll plaza, we recognize that knowledge is imperfect, decisions are not always rational, and all tollbooth lanes, and not all cars (or their drivers) are created equal.

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