The Sweet Spot: A Wave Model of Baseball Bats

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Abstract

We determine the sweet spot on a baseball bat. We capture the essential physics of the ball–bat impact by taking the ball to be a lossy spring and the bat to be an Euler-Bernoulli beam. To impart some intuition about the model, we begin by presenting a rigid-body model. Next, we use our full model to reconcile various correct and incorrect claims about the sweet spot found in the literature. Finally, we discuss the sweet spot and the performances of corked and aluminum bats, with a particular emphasis on hoop modes.

Introduction

Although a hitter might expect a model of the bat–baseball collision to yield insight into how the bat breaks, how the bat imparts spin on the ball, how best to swing the bat, and so on, we model only the sweet spot.

There are at least two notions of where the sweet spot should be—an impact location on the bat that either

- minimizes the discomfort to the hands, or
- maximizes the outgoing velocity of the ball.

We focus exclusively on the second definition.

The velocity of the ball leaving the bat is determined by

- the initial velocity and rotation of the ball,
- the initial velocity and rotation of the bat,

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- the relative position and orientation of the bat and ball, and
- the force over time that the hitter's hands applies on the handle.

We assume that the ball is not rotating and that its velocity at impact is perpendicular to the length of the bat. We assume that everything occurs in a single plane, and we will argue that the hands' interaction is negligible. In the frame of reference of the center of mass of the bat, the initial conditions are completely specified by

- the angular velocity of the bat,
- the velocity of the ball, and
- the position of impact along the bat.

The location of the sweet spot depends not on just the bat alone but also on the pitch and on the swing.

The simplest model for the physics involved has the sweet spot at the *center of percussion* [Brody 1986], the impact location that minimizes discomfort to the hand. The model assumes the ball to be a rigid body for which there are *conjugate points*: An impact at one will exactly balance the angular recoil and linear recoil at the other. By gripping at one and impacting at the other (the center of percussion), the hands experience minimal shock and the ball exits with high velocity. The center of percussion depends heavily on the moment of inertia and the location of the hands. We cannot accept this model because it both erroneously equates the two definitions of sweet spot and furthermore assumes incorrectly that the bat is a rigid body.

Another model predicts the sweet spot to be between nodes of the two lowest natural frequencies of the bat [Nathan 2000]. Given a free bat allowed to oscillate, its oscillations can be decomposed into fundamental modes of various frequencies. Different geometries and materials have different natural frequencies of oscillation. The resulting wave shapes suggest how to excite those modes (e.g., plucking a string at the node of a vibrational mode will not excite that mode). It is ambiguous which definition of sweet spot this model uses. Using the first definition, it would focus on the uncomfortable excitations of vibrational modes: Choosing the impact location to be near nodes of important frequencies, a minimum of uncomfortable vibrations will result. Using the second definition, the worry is that energy sent into vibrations of the bat will be lost. This model assumes that the most important energies to model are those lost to vibration.

This model raises many questions. Which frequencies get excited and why? The Fourier transform of an impulse in general contains infinitely many modes. Furthermore, wood is a viscoelastic material that quickly dissipates its energies. Is the notion of an oscillating bat even relevant to modeling a bat? How valid is the condition that the bat is free? Ought the system be coupled with hands on the handle, or the arm's bone structure, or possibly even the ball? What types of oscillations are relevant? A cylindrical structure can support numerous different types of modes beyond the transverse modes usually assumed by this model [Graff 1975].

Following the center-of-percussion line of reasoning, how do we model the recoil of the bat? Following the vibrational-nodes line of reasoning, how do we model the vibrations of the bat? In the general theory of impact mechanics [Goldsmith 1960], these two effects are the main ones (assuming that the bat does not break or deform permanently). Brody [1986] ignores vibrations, Cross [1999] ignores bat rotation but studies the propagation of the impulse coupled with the ball, and Nathan [2000] emphasizes vibrational modes. Our approach reconciles the tension among these approaches while emphasizing the crucial role played by the *time-scale* of the collision.

Our main goal is to understand the sweet spot. A secondary goal is to understand the differences between the sweet spots of different bat types. Although marketers of bats often emphasize the sweet spot, there are other relevant factors: ease of swing, tendency of the bat to break, psychological effects, and so on. We will argue that it doesn't matter to the collision whether the batter's hands are gripping the handle firmly or if the batter follows through on the swing; these circumstances have no bearing on the technique required to swing the bat or how the bat's properties affect it.

Our paper is organized as follows. First, we present the Brody rigid-body model, illuminating the recoil effects of impact. Next we present a full computational model based on wave propagation in an Euler-Bernoulli beam coupled with the ball modeled as a lossy spring. We compare this model with others and explore the local nature of impact, the interaction of recoil and vibrations, and robustness to parameter changes. We adjust the parameters of the model to comment on the sweet spots of corked bats and aluminum bats. Finally, we investigate the effect of hoop frequencies on aluminum bats.

A Simple Example

We begin by considering only the rigid recoil effects of the bat–ball collision, much as in Brody [1986]. For simplicity, we assume that the bat is perfectly rigid. Because the collision happens on such a short time-scale (around 1 ms), we treat the bat as a free body. That is to say, we are not concerned with the batter's hands exerting force on the bat that may be transferred to the ball.

The bat has mass M and moment of inertia I about its center of mass. From the reference frame of the center of mass of the bat just before the collision, the ball has initial velocity v_i in the positive x-direction while the bat has initial angular velocity ω_i. In our setup, v_i and ω_i have opposite signs when the batter is swinging at the ball as in **Figure 1**, in which arrows point in the positive directions for the corresponding parameters.

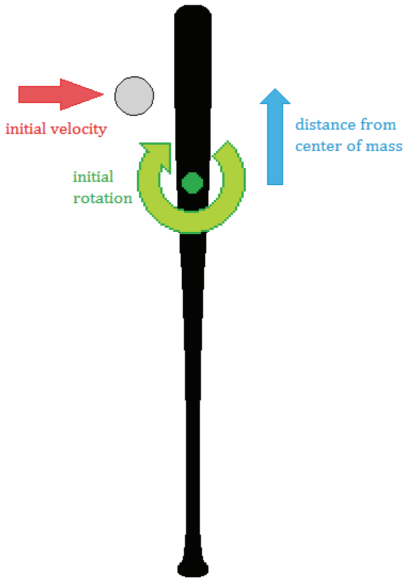

Figure 1. The collision.

The ball collides with the bat at a distance l from the center of mass of the bat. We assume that the collision is head-on and view the event such that all the y-component velocities are zero at the moment of the collision. After the collision, the ball has a final velocity v_f and the bat has a final linear velocity V_f and an angular velocity ω_f at the center of mass.

When the ball hits the bat, the ball briefly compresses and decompresses, converting kinetic energy to potential energy and back. However, some energy is lost in the process, that is, the collision is inelastic. The ratio of the relative speeds of the bat and the ball before and after the collision is known as the *coefficient of restitution*, customarily designated by e: $e = 0$ represents a perfectly inelastic collision, and $e = 1$ means a perfectly elastic one. In this basic model, we make two simplifying assumptions:

- e is constant along the length of the bat, and
- e is constant for all v_i.

Given our pre-collision conditions, we can write:

Conservation of linear momentum:

$$MV_f = m(v_i - v_f)$$

Conservation of angular momentum:

$$I(\omega_f + \omega_i) = ml(v_i - v_f),$$

Definition of the coefficient of restitution:

$$e(v_i - \omega_i l) = -v_f + V_f + w_f l.$$

Solving for v_f gives

$$v_f = \frac{-v_i(e - \frac{m}{M^*}) + \omega_i l(1 + e)}{1 + \frac{m}{M^*}},$$

where

$$M^* = \frac{M}{1 + \frac{Ml^2}{I}}$$

is the effective mass of the bat.

For calibration purposes, we use the following data, which are typical of a regulation bat connecting with a fastball in Major League Baseball. The results are plotted in **Figure 2**.

m	0.145 kg	5.1 oz
M	0.83 kg	29 oz
L	0.84 m	33 in
I	$0.039\ \mathrm{kg \cdot m^2}$	
v_i	67 m/s	150 mph
ω_i	−60 rad/s	
e	0.55	

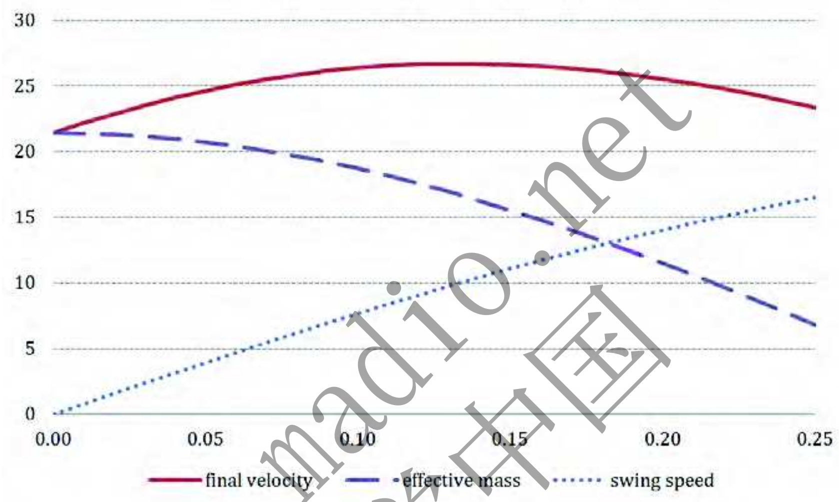

Figure 2. Final velocity v_f (solid arc at top), swing speed $\omega_i l$ (dotted rising line), and effective mass (dashed falling curve) as a function of distance l (in meters) from center of mass.

The maximum exit velocity is 27 m/s, and the sweet spot is 13 cm from the center of mass. Missing the sweet spot by up to 5 cm results in at most 1 m/s difference from the maximum velocity, implying a relatively wide sweet spot.

From this example, we see that the sweet spot is determined by a multitude of factors, including the length, mass, and shape of the baseball bat; the mass of the baseball; and the coefficient of restitution between bat and ball. Furthermore, the sweet spot is not uniquely determined by the bat and ball: It depends also on the incoming baseball speed and the batter's swing speed.

Figure 2 also shows intuitively why the sweet spot is located somewhere between the center of mass and the end of the barrel. As the point of collision moves outward along the bat, the effective mass of the bat goes up, so that a greater fraction of the initial kinetic energy is put into the bat's rotation. At the same time, the rotation in the bat means that the barrel of the bat is moving faster than the center of mass (or handle). These two effects work in opposite directions to give a unique sweet spot that's not at either endpoint.

However, this model tells only part of the story. Indeed, some of our starting assumptions contradict each other:

- We treated the bat as a free body because the collision time was so short. In essence, during the 1 ms of the collision, the ball "sees" only the local geometry of the bat, not the batter's hands on the handle. On the other hand, we assumed that the bat was perfectly rigid—but that means that the ball "sees" the entire bat.
- We also assumed that e is constant along the length of the bat and for different collision velocities. Experimental evidence [Adair 1994] suggests that neither issue can be ignored for an accurate prediction of the location of the sweet spot.

We need a more sophisticated model to address these shortcomings.

Our Model

We draw from Brody's rigid-body model but more so from Cross [1999]. One could describe our work as an adaptation of Cross's work to actual baseball bats. Nathan [2000] attempted such an adaptation but was misled by incorrect intuition about the role of vibrations. We describe his approach and error as a way to explain Cross's work and to motivate our work.

Previous Models

Brody's rigid-body model correctly predicts the existence of a sweet spot not at the end of the bat. That model suffers from the fact that the bat is not a rigid body and experiences vibrations. One way to account for vibrations is to model the bat as a flexible object. Beam theories (of varying degrees of accuracy and complication) can model a flexible bat. Van Zandt [1992] was the first to carry out such an analysis, modeling the beam as a *Timoshenko beam*, a fourth-order theory that takes into account both shear forces and tensile stresses. The equations are complicated and we will not need them. Van Zandt's model assumes the ball to be uncoupled from the beam and simply takes the impulse of the ball as a given. The resulting vibrations of the bat are used to predict the velocity of the beam at the impact point (by summing the Brody velocity with the velocity of the displacement at the impact point due to vibrations) and thence the exit velocity of the ball from the equations of the coefficient of restitution [van Zandt 1992].

Cross [1999] modeled the interaction of the impact of a ball with an aluminum beam, using the less-elaborate Euler-Bernoulli equations to model the propagation of waves. In addition, he provided equations to model the dynamic coupling of the ball to the beam during the impact. After discretizing the beam spatially, he assumed that the ball acts as a lossy spring coupled to the single component of the region of impact.

Cross's work was motivated by both tennis rackets and baseball bats, which differ importantly in the *time-scale* of impact. The baseball bat's collision lasts only about 1 ms, during which the propagation speed of the wave is very important. In this local view of the impact, the importance of the baseball's coupling with the bat is increased.

Cross argues that the actual vibrational modes and node points are largely irrelevant because the interaction is localized on the bat. The boundary conditions matter only if vibrations reflect off the boundaries; an impact not close enough to the barrel end of the bat will not be affected by the boundary there. In particular, a pulse reflected from a free boundary returns with the same sign (deflected away from the ball, decreasing the force on the ball, decreasing the exit velocity), but a pulse reflected from a fixed boundary returns with the opposite sign (deflected towards the ball, pushing it back, increasing the exit velocity). Away from the boundary, we expect the exit velocity to be uniform along a non-rotating bat. Cross's model predicts all of these effects, and he experimentally verified them. In our model, we expect similar phenomena, plus the narrowing of the barrel near the handle to act somewhat like a boundary.

Nathan's model also attempted to combine the best features of Van Zandt and Cross [Nathan 2000]. His theory used the full Timoshenko theory for the beam and the Cross model for the ball. He even acknowledged the local nature of impact. So where do we diverge from him? His error stems from an overemphasis on trying to separate out the ball's interaction with each separate vibrational mode.

The first sign of inconsistency comes when he uses the "orthogonality of the eigenstates" to determine how much a given impulse excites each mode. The eigenstates are *not* orthogonal. Many theories yield symmetric matrices that need to be diagonalized, yielding the eigenstates; but Timoshenko's theory does not, due to the presence of odd-order derivatives in its equations. Nathan's story plays out beautifully if only the eigenstates were actually orthogonal; but we have numerically calculated the eigenstates, and they are not even approximately orthogonal. He uses the orthogonality to draw important conclusions:

- The location of the nodes of the vibrational modes are important.
- High-frequency effects can be completely ignored.

We disagree with both of these.

The correct derivation starts with the following equation of motion, where k is the position of impact, y_i is the displacement and F_i is the external force on the ith segment of the bat, and H_{ij} is an asymmetric matrix:

$$y_k''(t) = H_{kj}y_j(t) + F_k(t).$$

We write the solutions as $y_k(t) = \Phi_{kn}a_n(t)$, where the rows of Φ_{kn} are eigenmodes with eigenvalues $-\omega_n^2$. Explicitly, $H_{jk}\Phi_{kn} = -\omega_n^2\Phi_{jn}$, and Φ_{kn}

indicates the kth component of the nth eigenmode. Then we write the equation of motion:

$$\Phi_{kn}a_n''(t) + \Phi_{kn}\omega_n^2 a_n(t) = F_k = \Phi_{kn}\Phi_{nj}^{-1}F_j,$$
$$a_n''(t) + \omega_n^2 a_n(t) = \Phi_{nk}^{-1}F_k.$$

In the last step, we used the fact that the eigenmodes form a complete basis.

Nathan's paper uses on the right-hand side simply $\Phi_{kn}F_k$ scaled by a normalization constant. At first glance, this seems like a minor technical detail, but the physics here is important. We calculate that the $\Phi_{nk}^{-1}F_k$ terms stay fairly large for even high values of n, corresponding to the high-frequency modes (k is just the position of the impact). This means that there are significant high-frequency components, at least at first. In fact, the high-frequency modes are necessary for the impulse to propagate slowly as a wave packet. In Nathan's model, only the lowest standing modes are excited; so the entire bat starts vibrating as soon as the ball hits. This contradicts his earlier belief in localized collision (which we agree with), that the collision is over so quickly that the ball "sees" only part of the bat. Nathan also claims that the sweet spot is related to the nodes of the lowest mode, which contradicts locality: The location of the lowest-order nodes depends on the geometry of the entire bat, including the boundary conditions at the handle.

While the inconsistencies in the Nathan model may cancel out, we build our model on a more rigorous footing. For simplicity, we use the Euler-Bernoulli equations rather than the full Timoshenko equations. The difference is that the former ignore shear forces. This should be acceptable; Nathan points out that his model is largely insensitive to the shear modulus. We solve the differential equations directly after discretizing in space rather than decomposing into modes. In these ways, we are following the work of Cross [1999].

On the other hand, our model extends Cross's work in several key ways:

- We examine parameters much closer to those relevant to baseball. Cross's models focused on tennis, featuring an aluminum beam of width 0.6 cm being hit with a ball of 42 g at around 1 m/s. For baseball, we have an aluminum or wood bat of radius width 6 cm being hit with a ball of 145 g traveling at 40 m/s (which involves 5,000 times as much impact energy).
- We allow for a varying cross-section, an important feature of a real bat.
- We allow the bat to have some initial angular velocity. This will let us scrutinize the rigid-body model prediction that higher angular velocities lead to the maximum power point moving farther up the barrel.

To reiterate, the main features of our model are

- an emphasis on the ball coupling with the bat,

- finite speed of wave propagation in a short time-scale, and
- adaptation to realistic bats.

These are natural outgrowths of the approaches in the literature.

Mathematics of Our Model

Our equations are a discretized version of the Euler-Bernoulli equations:

$$\rho \frac{\partial^2 y(z,t)}{\partial t^2} = F(z,t) + \frac{\partial^2}{\partial z^2}\left(YI\,\frac{\partial^2 y(z,t)}{\partial z^2}\right),$$

where

ρ is the mass density,

$y(z,t)$ is the displacement,

$F(z,t)$ is the external force (in our case, applied by the ball),

Y is the Young's modulus of the material (a constant), and

I is the second moment of area ($\pi R^4/4$ for a solid disc).

We discretize z in steps of Δ. The only force is from the ball, in the negative direction to the kth segment. Our discretized equation is:

$$\rho A\Delta \frac{d^2 y_i}{dt^2} = -\delta_{ik}F(t) - \frac{Y}{\Delta^3}\Big[I_{i-1}(y_{i-2} - 2y_{i-1} + y_i) \\ -2I_i(y_{i-1} - 2y_i + y_{i+1}) + I_{i+1}(y_i - 2y_{i+1} + y_{i+2})\Big].$$

Our dynamic variables are y_1 through y_N. For a fixed left end, we pretend that $y_{-1} = y_0 = 0$. For a free left end, we pretend that

$$y_1 - y_0 = y_0 - y_{-1} = y_{-1} - y_{-2}.$$

The conditions on the right end are analogous. These are the same conditions that Cross uses.

Finally, we have an additional variable for the ball's position (relative to some zero point) $w(t)$. Initially, $w(t)$ is positive and $w'(t)$ is negative, so the ball is moving from the positive direction towards the negative. Let $u(t) = w(t) - y_k(t)$. This variable represents the compression of the ball, and we replace $F(t)$ with $F\big(u(t), u'(t)\big)$. Initially, $u(t) = 0$ and $u'(t) = -v_{\text{ball}}$. The force between the ball and the bat takes the form of hysteresis curves such as the ones shown in **Figure 3**.

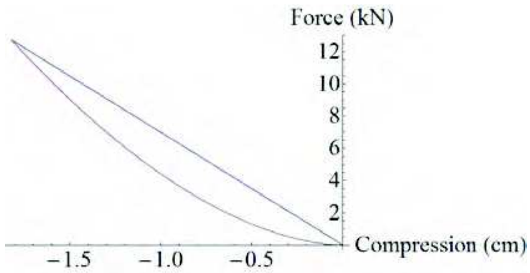

Figure 3. A hysteresis curve used in our modeling, with maximum compression 1.5 cm.

The higher curve is taken when $u'(t) < 0$ (compression) and the lower curve when $u'(t) > 0$ (expansion). When $u(t) > 0$, the force is zero. The equation of motion for the ball is then

$$w''(t) = u''(t) + y_k''(t) = F\big(u(t), u'(t)\big).$$

We have eliminated the variable w.

We have yet to specify the function $F\big(u(t), u'(t)\big)$. As can be seen in videos [Baseball Research Center n.d.], the ball compresses significantly (often more than 1 cm) in a collision. The compression and decompression is lossy. We could model this loss by subtracting a fraction of the ball's energy after the collision; that approach is good enough for many purposes, but we instead follow Nathan and use a nonlinear spring with hysteresis.

Since $W = \int F dx$, the total energy lost is the area between the two curves in **Figure 3**. A problem with creating hysteresis curves is that one does not know the maximum compression (i.e., where to start drawing the bottom curve) until after solving the equations of motion. In practice, we solve the equation in two steps.

The main assumptions of our model derive from the main assumptions of each equation:

- The first is the exact form of the hysteresis curve of the ball. Cross [1999] argues that the exact form of the curve is not very important as long as the duration of impact, magnitude of impulse, maximum compression of the ball, and energy loss are roughly correct.
- Both the Timoshenko and Euler-Bernoulli theories ignore azimuthal and longitudinal waves. This is a fundamental assumption built into all of the approaches in the literature. Assuming that the impact of the ball is transverse and the ball does not rotate, the assumption is justified.

The assumptions of our models are the same as those in the literature, so they are confirmed by the literature's experiments.

Simulation and Analysis

Simulation Results

Our model's two main features are wave propagation in the bat and nonlinear compression/decompression of the ball. The latter is illustrated by the asymmetry of the plot in **Figure 4a**. This plot also reveals the time-scale of the collision: The ball leaves the bat 1.4 ms after impact. During and after collision, shock waves propagate through the bat.

In this example, the bat was struck 60 cm from the handle. What does the collision look like at 10 cm from the handle? **Figure 4b** shows the answer: The other end of the bat does not feel anything until about 0.4 ms and does not feel significant forces until about 1.0 ms. By the time that portion of the bat swings back (almost 2.0 ms), the ball has already left contact with the bat. This illuminates an important point: We are concerned only with forces on the ball that act within the 1.4 ms time-frame of the collision. Any waves taking longer to return to the impact location do not affect exit velocity.

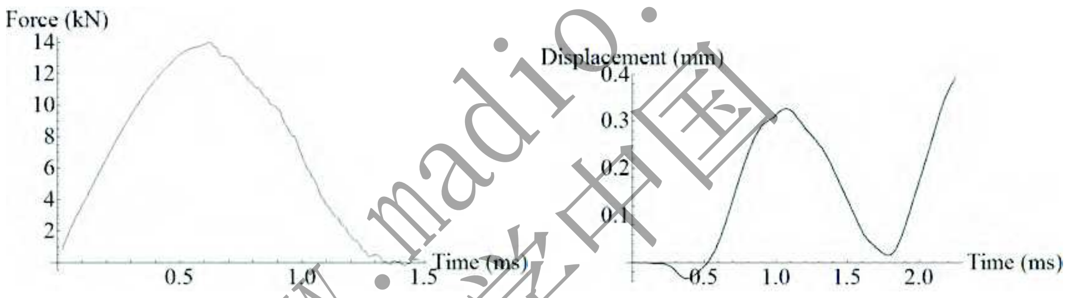

Figure 4.

a. Left: The force between the ball and the bat as a function of time; the impulse lasts 1.4 ms.
b. Right: The waveform of $y_{10}(t)$ when the bat is struck at 60 cm. The impulse reaches this chunk at around 0.4 ms but does not start moving significantly until later.

Having demonstrated the basic features of our model, we now replicate some of Cross's results but with baseball-like parameters. In **Figure 5a**, we show that the effects of fixed- vs. free-boundary conditions are in agreement with Cross's model.

As we expected, fixed boundaries enhance the exit velocity and free boundaries reduce them. From this result, we see the effect of the shape of the bat. The handle does indeed act like a free boundary. The distance between the boundaries is too small to get a flat zone in the exit velocity vs. position curve. If we extend the barrel by 26 cm, a flat zone develops (**Figure 5b**; notice the change in axes). Intuitively, this flat zone exists because the ball "sees" only the local geometry of the bat and the boundaries are too far away to have a substantial effect.

From now on, we use an 84-cm bat free on both ends, where position zero denotes the handle end. In this base case, the sweet spot is at 70 cm. We

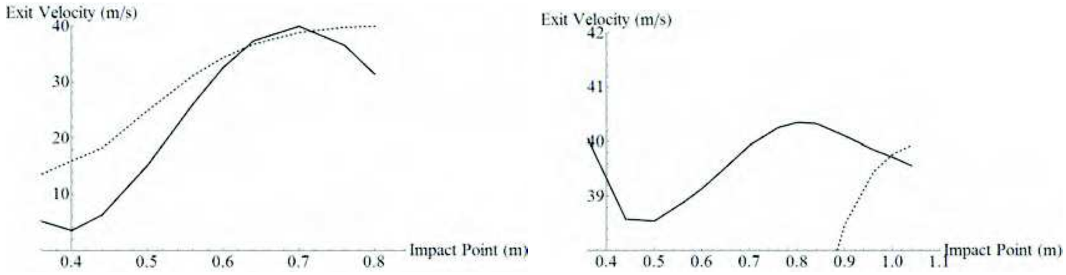

Figure 5.

a. Left: Exit velocity vs. impact position for a free boundary (solid line) and for a fixed boundary (dashed line), with barrel end fixed but handle end free, for an 84-cm bat
b. Right: The same graph for a free 110-cm bat.

investigate the dependence of the exit speed on the initial angular velocity. According to rigid-body models, the sweet spot is exactly at the center of mass if the bat has no angular velocity. In **Figure 6**, we present the results of changing the angular velocity. Our results contrast greatly with the simple example presented earlier. While the angular-rotation effect is still there, the effective mass plays only a negligible role in determining the exit speed. In other words, the bat is not a rigid body because the entire bat does not react instantly. The dominating effect is from the boundaries: the end of the barrel and where the barrel tapers off. These free ends cause a significant drop in exit velocity. Increasing the angular velocity of the bat increases the exit velocity, in part just because the impact velocity is greater (by a factor of ω_i times the distance from the center of mass of the bat).

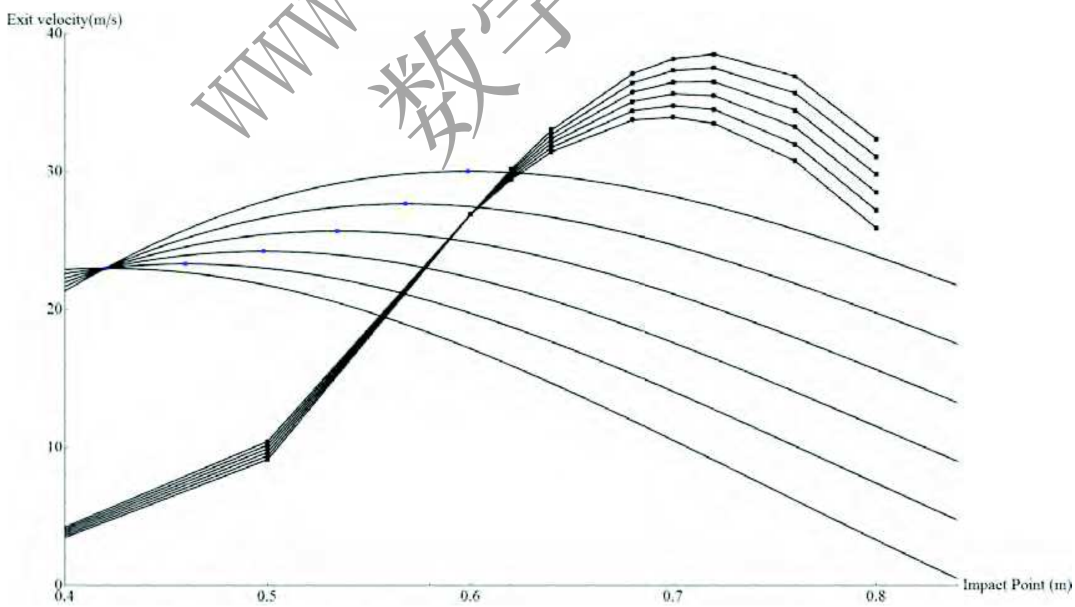

Figure 6. Exit velocity vs. impact position at various initial angular velocities of the bat. Our model predicts the solid curves, while the dashed lines represent the simple model. The dots are at the points where Brody's solution is maximized.

In **Figure 7a**, we show that near the sweet spot (at 0.7 m), increasing angular velocity actually decreases the excess exit velocity (relative to the impact velocity). We should expect this, since at higher impact velocity, more energy is lost to the ball's compression and decompression. To confirm this result, we also recreate the plot in **Figure 7b** but without the hysteresis curve—in which case this effect disappears. This example is one of the few places where the hysteresis curve makes a difference, confirming experimental evidence [Adair 1994; Nathan 2003] that the coefficient of restitution decreases with increasing impact velocity.

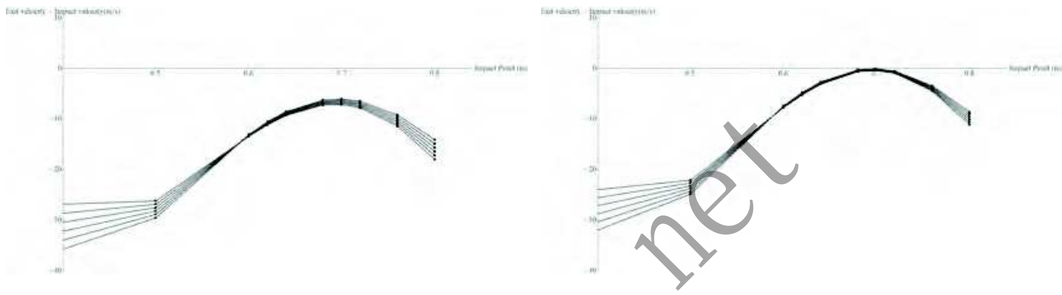

Figure 7.
Exit velocity minus impact velocity vs. impact position, for initial angular velocities of the bat.
a. Left: Near the center of mass, higher angular velocity gives higher excess exit velocity, but towards the sweet spot the lines cross and higher angular velocity gives lower excess exit velocity.
b. Right: The same plot without a hysteresis curve. The effect disappears.

The results for angular velocity contrast with the simple model. As evident from **Figure 8**, the rigid-body model greatly overestimates this effect for large angular velocities.

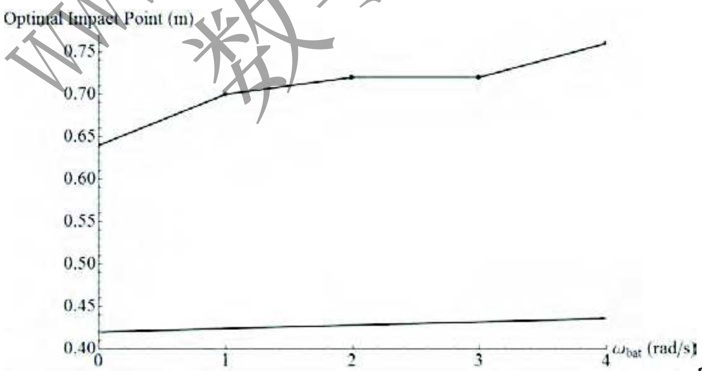

Figure 8. Optimal impact position vs. angular velocity. The straight line is the rigid-body prediction, while the points are our model's prediction.

Parameter Space Study

There are various adjustable parameters in our model. For the bat, we use density $\rho = 649\,\mathrm{kg/m^3}$ and Young's modulus $Y = 1.814 \times 10^{10}\,\mathrm{N/m^2}$.

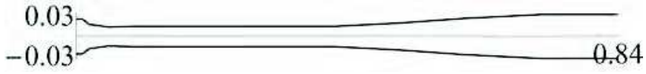

Figure 9. The profile of our bat.

These values, as well as our bat profile (**Figure 9)**, were used by Nathan as typical values for a wooden bat. While these numbers are in good agreement with other sources, we will see that these numbers are fairly special. As a result of our bat profile, the mass is 0.831 kg and the moment of inertia around the center of mass (at 59.3 cm from the handle of our 84 cm bat) is 0.039 kg·m^2. We let the 145-g ball's initial velocity be 40 m/s, and set up our hysteresis curve so that the compression phase is linear with spring constant 7×10^5 N/m.

- We vary the density of the bat and see that the density value occupies a narrow region that gives peaked exit-velocity curves (see **Figure 10**).

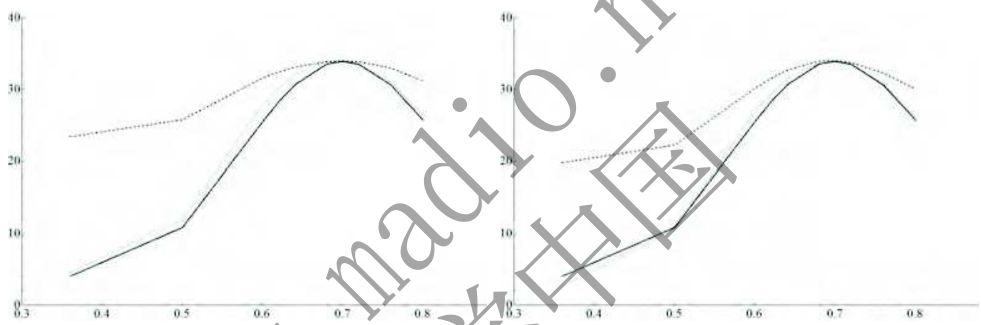

Figure 10.

Exit velocity vs. impact position for various densities. The solid line is the original $\rho = 649\text{kg/m}^3$. **a.** Left: Dotted is $\rho = 700$, dashed is $\rho = 1000$. **b.** Right: Dotted is $\rho = 640$, dashed is $\rho = 500$.

- We also vary the Young's modulus and shape of bat to similar effect (see **Figure 11**). The fact that varying any of Nathan's parameters makes the resulting exit velocity vs. location plot less peaked means that baseball bats are specially designed to have the shape shown in **Figure 9** (or else the parameters were picked in a special way).
- Finally, we vary y, the speed of the ball (see **Figure 12**). The exit velocity simply scales with the input velocity, as expected.

Alternatives to Wooden Bats

Having checked the stability of our model for small parameter changes, we now change the parameters drastically, so as to model corked and aluminum bats.

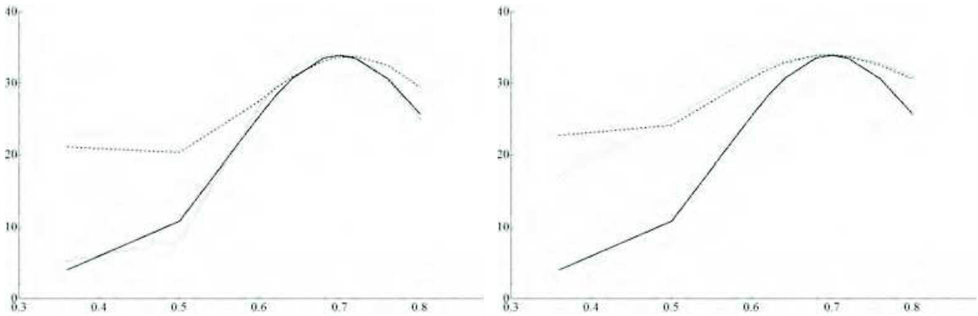

Figure 11.

a. Left: Varying the value of Y. Solid is $Y = 1.1814 \times 10^{10}\mathrm{N/m}^2$; dashed is 1.25 times as much, while dotted is 0.8 times.

b. Right: Varying the shape of the bat. Solid is the original shape; dashed has a thicker handle region, while dotted has a narrower handle region.

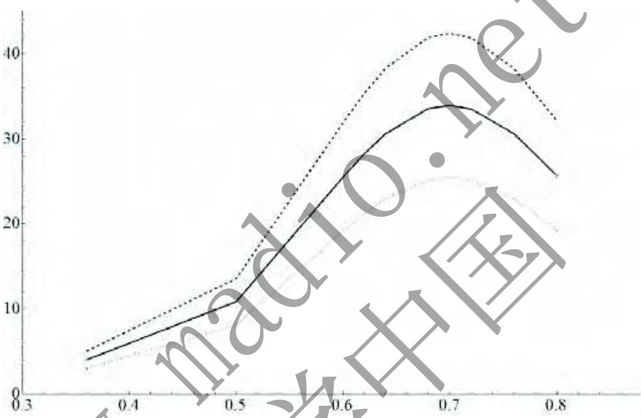

Figure 12. Varying the speed of the ball. Solid is the original 40 m/s, dashed is 50 m/s, while dotted is 30 m/s.

Corked Bat

We model a corked bat as a wood bat with the barrel hollowed out, leaving a shell 1 cm or 1.5 cm thick. The result is shown in **Figure 13a**. The exit velocities are higher, but this difference is too small to be taken seriously. This result agrees with the literature: The only advantages of a corked bat are the changes in mass and in moment of inertia.

Aluminum Bat

We model an aluminum bat as a 0.3 cm-thick shell with a density of 2700 kg/m^3 and Young's modulus 6.9×10^{10} N/m^2. The aluminum bat performs much better than the wood bat (**Figure 13b**). It has the same sweet spot (70 cm) and similar sweet-spot performance, but the exit velocity falls off more gradually away from the sweet spot.

To gain more insight, we animated the displacement of the bat vs. time; we present two frames of the animation in **Figure 14**. The aluminum bat is

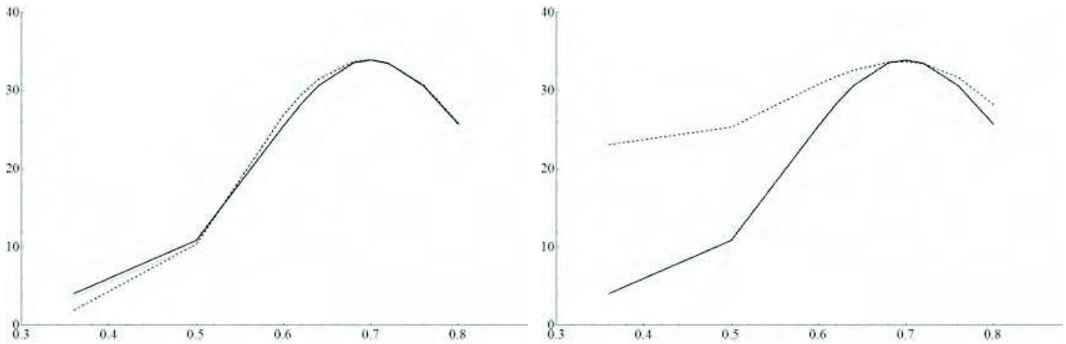

Figure 13. Exit velocity vs. distance of point of collision on the bat from the handle end.

a. Corked bat. **b.** Aluminum bat.

displaced less (absorbing less energy). More importantly, in the right-hand diagram of **Figure 14**, the curve for the wood bat is still moving down and left, while the aluminum bat's curve is moving left and pushing the ball back up. The pulse in the aluminum bat travels faster and returns in time to give energy back to the ball. By the time the pulse for the wood bat returns to the impact location, the ball has already left the bat.

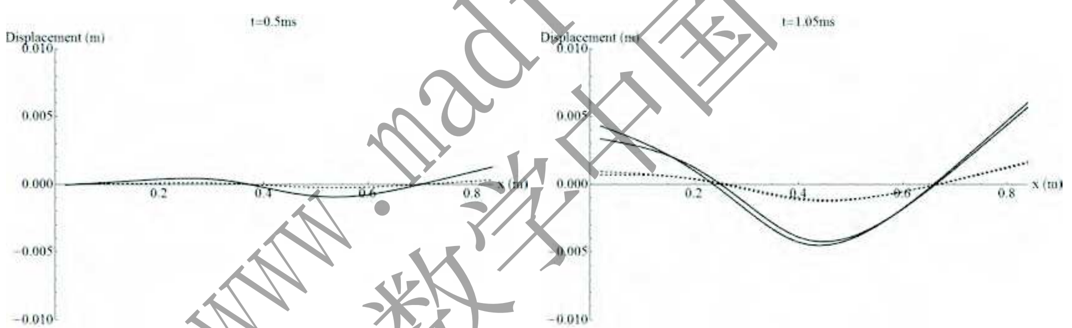

Figure 14. Plots of the displacement of an aluminum bat (dashed) and wood bat (solid) being hit by a ball 60 cm from the handle end. The diagram on the right shows two frames superimposed ($t = 1.05$ ms and $t = 1.10$ ms) so as to show the motion. The rigid translation and rotation has been removed from the diagrams.

In the literature, the performance of aluminum bats is often attributed to a "trampoline effect," in which the bat compresses on impact and then springs back before the end of the collision [Russell 2003]. This effect would improve aluminum-bat performance further. The trampoline effect involves exciting so-called "hoop modes," modes with an azimuthal dependence, which our model cannot simulate directly. For an aluminum bat, one could conceivably use wave equations for a cylindrical sheet (adjusting for the changing radius) and then solve the resulting partial differential equations in three variables. Analysis of such a complex system of equations is beyond the scope of this paper.

Instead, we artificially insert a hoop mode by hanging a mass from a spring at the spot of the bat where the ball hits. We expect the important modes to be the ones with periods near the collision time (1.4 ms, corresponding to 714 Hz). We find that this mode does affect the sweet spot, although the exact change does not seem to follow a simple relationship with the frequency. Our results, as shown in **Figure 15**, show that hoop modes around 700 Hz do enhance the exit velocity. They not only make the sweet spot wider but also shift it slightly toward the barrel end of the bat.

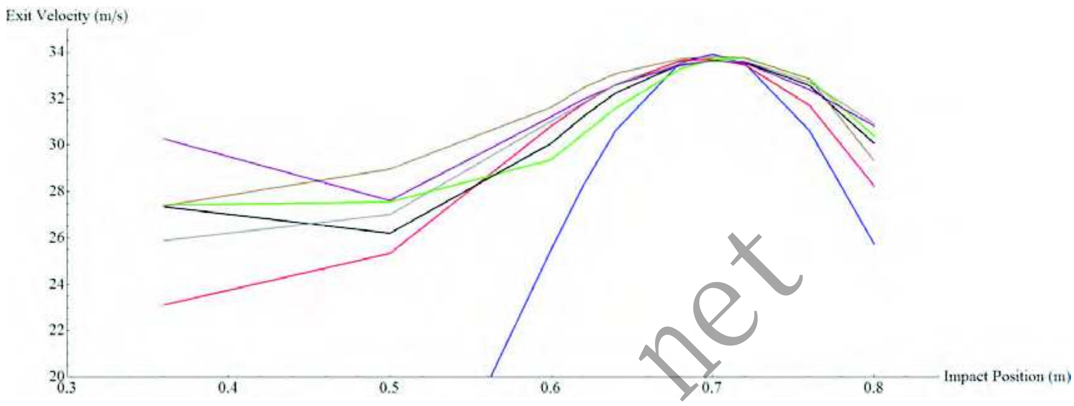

Figure 15. Exit velocity vs. impact position at different hoop frequencies. The lines from bottom to top at the left edge (color) are: (blue, starts off the chart) wood bat, (next higher, red) no hoop mode, (gray) 2000 Hz, (black) 500 Hz, (green) 300 Hz, (brown) 800 Hz, and (purple) 1250 Hz.

Conclusion

We model a ball–bat collision by using Euler-Bernoulli equations for the bat and hysteresis curves for the baseball. By doing so, we reconcile the literature by emphasizing the role of the time-scale of the collision and how the ball "sees" only a local region of the bat because of the finite speed of wave propagation. As a result, the sweet spot is farther out in our model than the rigid-body recoil model predicts.

We vary the input parameters and show that the effects are in line with intuition and key results in previous experimental work.

Finally, we show that aluminum bats have wider sweet spots than wooden bats.

We offer several suggestions for improvements and extensions:

- The ball is assumed to be non-rotating with head-on impact; rotating balls and off-center collisions excite torsional modes in the bat that we ignore and make the problem nonplanar.
- We neglect shear forces in the bat.
- Our analysis of hoop modes is rather cursory.

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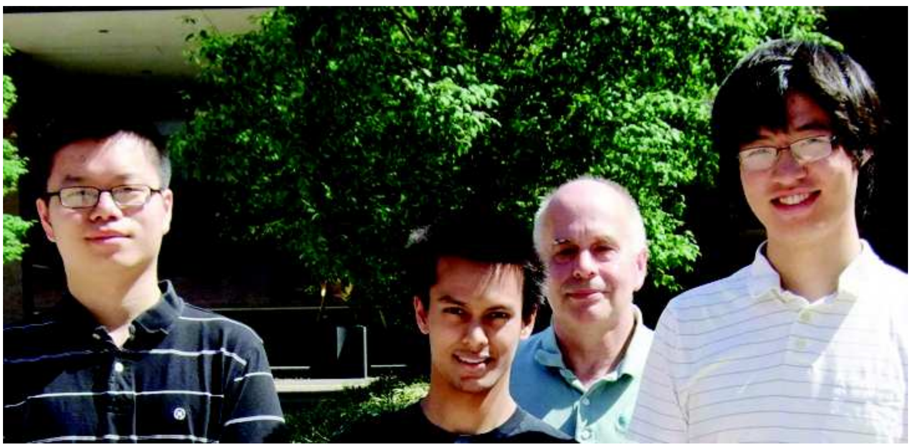

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