A Polynomial Model for Estimating Water Flow

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Summary

We graph the given data in Figure 1.

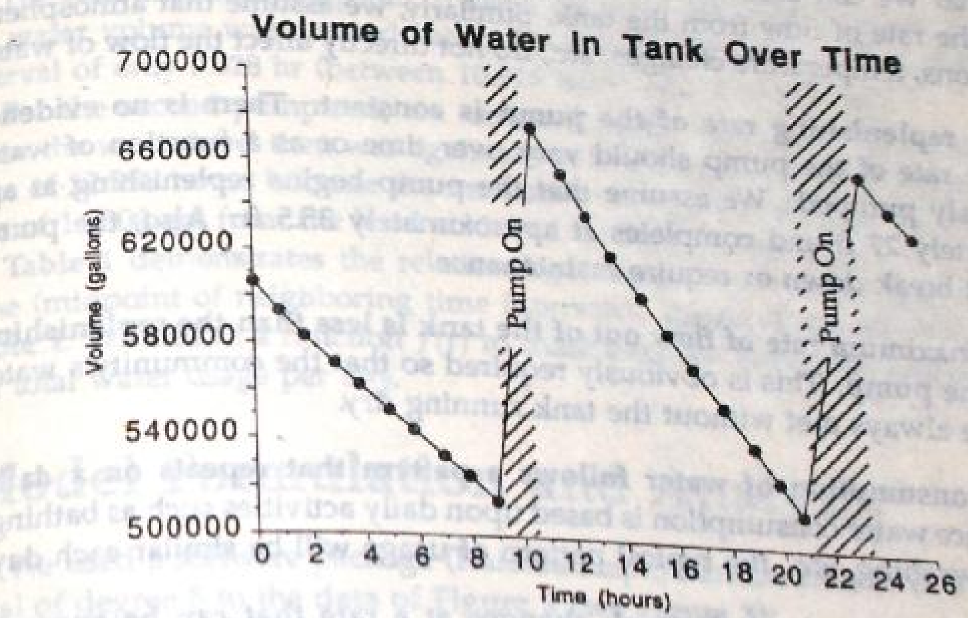

Figure 1. Given data for the volume of water in the tank as a function of time.

To model the rate of flow from the tank, we first manipulated the data to obtain the change in volume in the tank per unit time, and each quotient was then assigned a time instant.

Because of the nature of the consumption of water, we assumed that the transformed data could be estimated by a smooth curve. We fitted a polynomial of degree 8, with a regression coefficient of 0.971. This polynomial, although seemingly complex, is readily evaluated and easily integrable.

The model predicts a constant pumping rate, and we find that the total water consumed in a 24-hour period is 338,000 gallons, with a mean flow rate of 14,100 gal/hr and a pumping rate of 96,100 gal/hr. The predictions of the model also correlate well with data governing related utilities and known consumption patterns.

Assumptions

The only factor affecting the flow of water out of the tank is the typical demand for water by the community. The given data given are assumed to reflect a typical day and not to include any unusual circumstances such as leakage from the tank, water-main breaks, major fires, natural disasters, etc.

By Torricelli's law, the maximum flow rate from the tank is proportional to the square root of the water height. For the given data, the maximum height is 35.5 ft and the minimum is 27 ft, so the ratio of the maximal flow rates for the two heights is $\sqrt{35.5/27} \approx 1.15$. This figure is near enough to 1 so that we can assume that the level of the water in the tank does not affect the rate of flow from the tank. Similarly, we assume that atmospheric conditions, temperature changes, etc., do not directly affect the flow of water.

The replenishing rate of the pump is constant. There is no evidence that the rate of the pump should vary over time or as a function of water previously pumped. We assume that the pump begins replenishing at approximately 27 ft and completes at approximately 35.5 ft. Also, the pump does not break down or require maintenance.

The maximum rate of flow out of the tank is less than the replenishing rate of the pump. This is obviously required so that the community's water needs are always met without the tank running dry.

The consumption of water follows a pattern that repeats on a daily basis. Since water consumption is based upon daily activities such as bathing, cooking, washing, etc., the typical pattern of usage will be similar each day.

The water flow from the tank changes at a rate that can be approximated by some smooth curve. Each user's demand is so small compared to the whole community's demand, and it is highly unlikely that the entire community's needs would increase or decrease simultaneously.

Notation

V, V_t volume of water (gal), volume of water (gal) at time t

t time (hr)

$f(t)$ model estimate of flow out of the tank as a function of time (gal/hr)

p replenishing rate of the pump (gal/hr)

T_0 time of day at which initial data point for model fit occurs

T time of day (in 24-hr time)

Problem Analysis

The average flow rate out of the tank during the second pumping interval must be greater than during the first pumping period, since the pump runs much longer. However, the times at which the pump actually starts and stops are not known. The time of the next measurement after the pump's second operation shows a level of water different from when the pump would have ceased operation. Hence, the pump must have stopped between the two measurements, in the righthand shaded region of **Figure 1**. We must try to approximate the start and stop times of the replenishing pump.

The first time the pump ran, it came on soon after 8.968 hours, because the water volume was approximately 514,800 gal. The pump was off for an interval of only 0.028 hr (between 10.926 hours and 10.954 hours).

For the second pump run, we assigned a start time of 20.839 hours, because the water level then was again about 514,800 gal. We chose a stopping time of 22.958 hours because the next observation (at 23.880 hours) shows a loss of 14,300 gal from the level achieved by the first pump run.

Table 1 demonstrates the relationship between water flow (gal/hr) and time (midpoint of neighboring time intervals). **Figure 2** graphs the data of **Table 1**. We will fit a function $f(t)$ to these data and integrate it to calculate the total water usage per day.

Model Formulation and Analysis

We used a software package (KaleidaGraph, Version 2.02) to fit a polynomial of degree 8 to the data of **Figure 2** (see **Figure 3**):

$$\begin{aligned} f(t) = 17575 - 9624t + 4478t^2 - 1684t^3 + 268.9t^4 - 23.31t^5 \\ + 1.106t^6 - 0.02700t^7 + 0.0002654t^8, \end{aligned}$$

with regression coefficient $r = 0.971$, and with t defined by

$$t = \begin{cases} 24 + (T - T_0) + 0.4606, & \text{for } T < T_0; \\ (T - T_0) + 0.4606, & \text{for } T \geq T_0. \end{cases}$$

At first glance, it may seem excessive to use a polynomial of degree 8. However, the fluctuations in the data suggest that a polynomial of high degree is needed to model accurately the pattern of the data. We chose degree

Table 1.

Midpoints between observations vs. average flow during the interval.

Midpoint of time interval (hours)	Average flow rate ($\times 10^3$ gal/hr)
0.4606	14.0
1.382	12.0
2.396	10.0
3.411	9.6
4.425	9.6
5.439	8.9
6.453	9.6
7.468	8.9
8.448	10.0
9.474	no data
10.45	no data
10.94	no data
11.49	18.6
12.49	20.0
13.42	19.0
14.43	16.0
15.44	16.0
16.37	16.0
17.38	14.0
18.49	14.0
19.50	16.0
20.40	15.0
21.43	no data
22.49	no data
23.42	no data
24.43	14.0
25.45	12.0

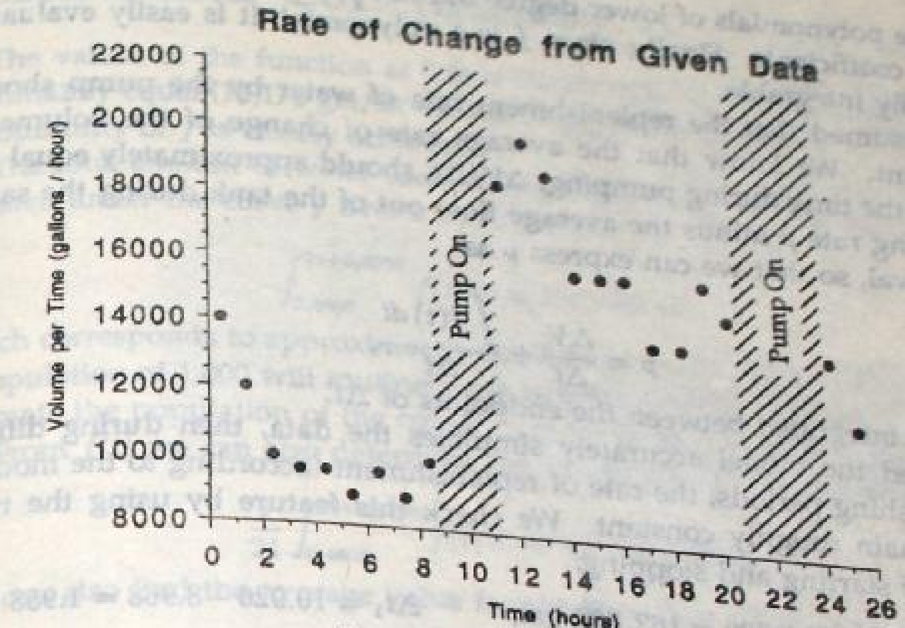

Figure 2. Rate of flow vs. time.

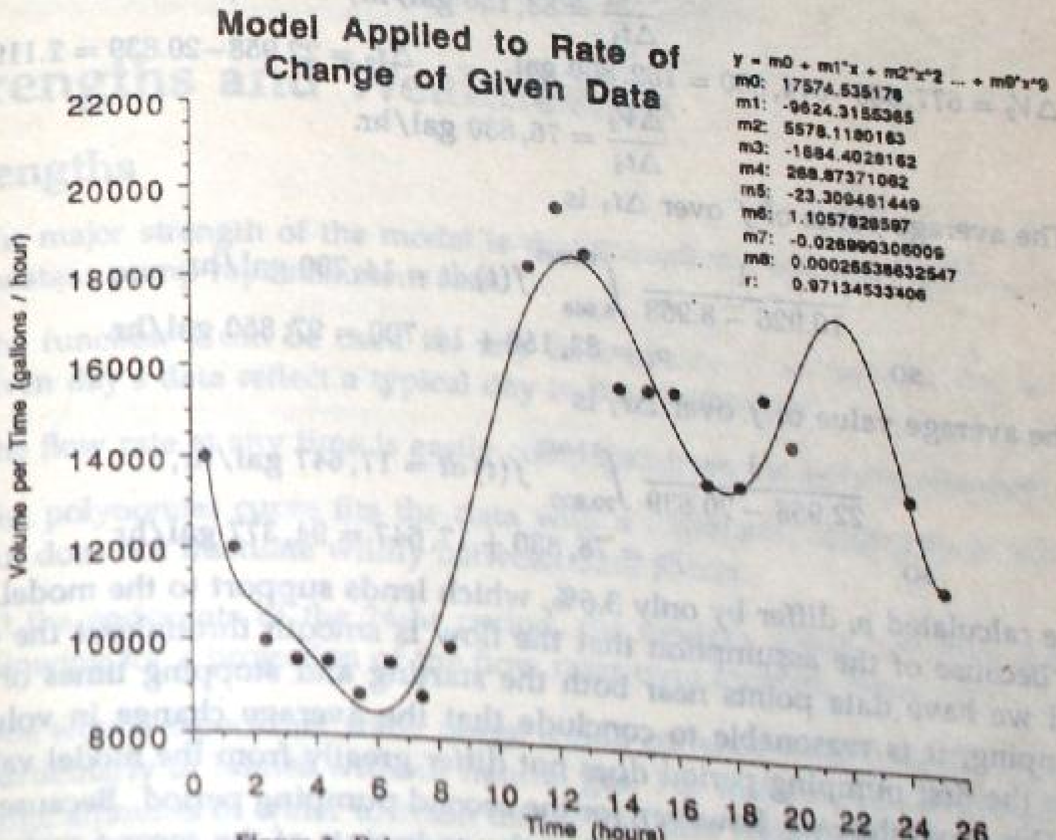

Figure 3. Polynomial fit to the data of Figure 2.

8 because polynomials of lower degree did not produce sufficiently high regression coefficients. Finally, since f is a polynomial, it is easily evaluated and readily integrable.

We assumed that the replenishment rate of water by the pump should be constant. We know that the average rate of change of the volume of water for the time during pumping, $\Delta V/\Delta t$, should approximately equal the replenishing rate p minus the average flow out of the tank during the same time interval, so that we can express p as

$$p = \frac{\Delta V}{\Delta t} + \frac{\int f(t)\,dt}{\Delta t},$$

where f is integrated between the endpoints of Δt.

If indeed the model accurately simulates the data, then during different replenishing periods, the rate of replenishment (according to the model) should remain roughly constant. We check this feature by using the two instances of starting and stopping:

$$\Delta V_1 = 677,600 - 514,800 = 162,800 \text{ gal}, \qquad \Delta t_1 = 10.926 - 8.968 = 1.958 \text{ hr},$$

$$\frac{\Delta V_1}{\Delta t_1} = 83,150 \text{ gal/hr};$$

$$\Delta V_2 = 677,600 - 514,800 = 162,800 \text{ gal}, \qquad \Delta t_2 = 22.958 - 20.839 = 2.119 \text{ hr},$$

$$\frac{\Delta V_2}{\Delta t_2} = 76,830 \text{ gal/hr}.$$

The average value of f over Δt_1 is

$$\frac{1}{10.926 - 8.968} \int_{8.968}^{10.926} f(t)\,dt = 14,700 \text{ gal/hr},$$

so

$$p_1 = 83,150 + 14,700 = 97,850 \text{ gal/hr}.$$

The average value of f over Δt_2 is

$$\frac{1}{22.958 - 20.839} \int_{20.839}^{22.958} f(t)\,dt = 17,647 \text{ gal/hr},$$

so

$$p_2 = 76,830 + 17,647 = 94,377 \text{ gal/hr}.$$

The calculated p_i differ by only 3.6%, which lends support to the model.

Because of the assumption that the flow is smooth throughout the day, and we have data points near both the starting and stopping times of the pumping, it is reasonable to conclude that the average change in volume over the first pumping period does not differ greatly from the model value.

This is not the case, however, for the second pumping period. Because the estimated flow rates (from Table 1) at the endpoints of the second pumping period are within 1,000 gal/hr of each other, and we do not have data points near the stopping time of the pumping, we are much less certain of the behavior of the actual flow during the second pumping period.

Results and Conclusions

The values of the function at 0.4606 hours and at 24.4606 hours are approximately equal (14,170 gal/hr vs. 13,730 gal/hr, a difference of about 3%), so continuity of f is closely achieved.

The total amount of water used during the day can be approximated by the area under the curve f over a 24-hr period:

$$\int_{0.4606}^{24+0.4606} f(t)\,dt = 338{,}000 \text{ gal},$$

which corresponds to approximately one-half of the total tank volume. Since a population of 1,000 will routinely use 105,000 gal/day [Todd 1970], we can estimate the population of the community to be about 3,200 people.

From $f(t)$ we can also determine the average flow rate out of the tank:

$$\frac{1}{24}\int_{0.4606}^{24+0.4606} f(t)\,dt \approx 14{,}100 \text{ gal/hr}.$$

We can also find the average value for the flow rate of the pump:

$$(97{,}850 + 94{,}377)/2 \approx 96{,}100 \text{ gal/hr}.$$

Strengths and Weaknesses

Strengths

- The major strength of the model is that it confirms our expectation of a constant pump-replenishment rate.
- The function f can be used for any time of day, if we assume that the given day's data reflect a typical day in the community.
- The flow rate at any time is easily computed from the polynomial model.
- The polynomial curve fits the data with a regression coefficient of 0.971 but does not fluctuate wildly between data points.
- At the endpoints of the 24-hr period, the model's values are very close, allowing for a projection of the flow rates over multiple days.
- One would expect that water usage would correlate with electricity usage, particularly in homes without natural gas. For instance, cooking requires large amounts of water to wash dishes and also requires electricity to run ovens, lights, etc.; bathing is another large consumer of water accompanied by electrical consumption (water heaters, hair dryers, etc.). A comparison of Figure 1 with Figure 4 shows a dramatic resemblance between

the modeled demand for water and a typical daily demand pattern for electricity.

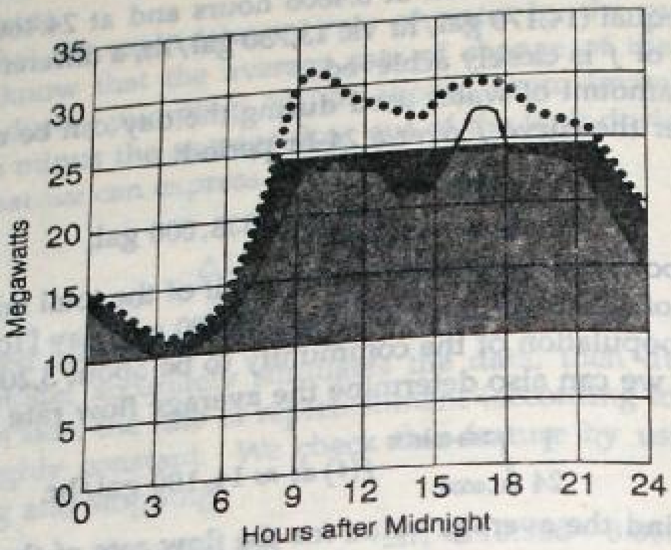

Figure 4. Electricity demand on a winter day (from [Overman 1969, 117]). The dotted (top) line indicates demand on a very cold day, and the solid line indicates demand on an average winter day. The shading indicates the sources of electricity (coal, hydro, etc.).

Weaknesses

- A major weakness of the model is its reliance on a single day's data. In modeling any type of phenomenon, one would like varied data, i.e., data taken over many days under differing conditions.
- If we knew the actual rate of the replenishing pump, then we could estimate better the flow rate during the times when the pump is on and model more accurately the flow out of the tank.
- The model is produced by considering the difference in volume measurements, which involves some imprecision.

References

Overman, Michael. 1969. *Water: Solutions to a Problem of Supply and Demand.* Garden City, NY: Doubleday.

Todd, David Keith, ed. 1970. *The Water Encyclopedia.* Manhasset Isle, Port Washington, NY: Water Information Ctr.