7 Deep Impact[①]

7.1 竞赛题

For some time, the National Aeronautics and Space Administration (NASA) has been considering the consequences of a large asteroid impact on the earth.

As part of this effort, your team has been asked to consider the effects of such an impact were the asteroid to land in Antarctica. There are concerns that an impact there could have considerably different consequences than one striking elsewhere on the planet.

You are to assume that an asteroid is on the order of 1 000 m in diameter, and that it strikes the Antarctic continent directly at the South Pole.

Your team has been asked to provide an assessment of the impact of such an asteroid. In particular, NASA would like an estimate of the amount and location of likely human casualties from this impact, an estimate of the damage done to the food production regions in the oceans of the southern hemisphere, and an estimate of possible coastal flooding caused by large-scale melting of the Antarctic polar ice sheet.

① 1999年美国大学生数学建模竞赛A题。

赛题简析

本题的背景是备受人们关注的小行星撞击地球问题,要求对一颗小行星撞击到地球南极后所产生的后果进行评估。题目给出了评估重点:关于撞击下可能的人类人员伤亡的数量和所在地区的估计,对南半球海洋的食物生产的破坏的估计,以及由于南极洲极地冰岩的大量融化造成的可能的沿海岸地区的洪水的估计。

关于行星撞击地球问题,有很多资料可以查询,但是要确定行星撞击到地球南极后所产生的短期和长期的影响,却不是一件容易的事。这给参赛者提供了充分的发挥余地。研究这一问题时,在物理上和数值参数上的合理与适当的假设将是重要的。

7.2 参赛论文

The Assessment Method of Impact

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Summary

In this article we present a Mathematical Model to analyze the consequences of an asteroid on the order of 1 000 m in diameter impact at the South Pole.

The entire model has four departments.

• Part 1: In this part, we use the impact-dynamics theory to build the model of an asteroid impact on the earth. And describe the direct effects of the explosion.

• Part 2: It is a model of large-scale melting of the Antarctic polar ice sheet. In this part we calculate the mass of the melting ice in detail.

• Part 3: We develop a model to describe the rule of diffusing and distribution of the dust and smog. The assessment of polluted scope is obtained.

• Part 4: We analyze the effects on the biology brought by the impact. A model is built especially to estimate the damage done to the food production in the oceans of the Southern Hemisphere.

At last we integrate the four parts of the model and give out the whole effect of the impact.

The Background

From the birthday of the earth to nowadays, some naughty kids in the universe always like to throw the stones to the earth. Sometime they hit the objective; sometime they miss it. The asteroid impact will make effects in almost every field. Then suppose there is an asteroid on the order of 1 000 m in diameter impact at the South Pole. Let us see what will happen.

Constructing the Model

Part Ⅰ the Settlement of Impact Problem

Assumptions:

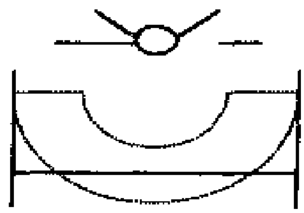
1. The asteroids impact on the South Pole vertically.

2. The asteroid's diameter is about 1 000 m after crossing the aerosphere.

3. The asteroid's velocity is about 25 km/s when it reaches the earth. The velocity of asteroid is different so we adopt the typical value.

4. The Antarctic can be seen as a plain because the asteroid's diameter is much smaller comparing to the earth's.

According to the assumptions, we can predigest the problem and make use of the high velocity impact theory. The asteroid can been seen as a pill and the Antarctic a butt. The pill with super high velocity rush to the butt. The material of the pill and the butt will be compressed because of the instant impact pressure. A hemisphere pot will appear.



We use the function of the pill and butt's density and Mash number to show the inrush depth and pit's capacity. We can get some important equations.

$$\frac{D_e}{d_p} = 1.96\left(\frac{\rho_p}{\rho_t}\right)^{\frac{1}{2}} \cdot \left(\frac{v_0}{C_t}\right)^{\frac{2}{3}} \qquad (1)$$

$$\frac{V_c}{V_p} = 30.25\left(\frac{\rho_p}{\rho_t}\right)^{\frac{3}{2}} \cdot \left(\frac{v_0}{C_t}\right)^{2} \qquad (2)$$

Here we list the symbols' meaning in the equations.

D_e: pit's depth;

V_c: pit's capacity;

V_p: pill's volume;

d_p: pill's diameter;

C_t: speed of sound in the butt;

v_0: muzzle velocity of pill;

ρ_p: density of pill;

ρ_t: density of butt;

We find the value of some symbols in the related reference.

$d_p = 1\ 000$ m; $C_t = 4\ 500$m/s;

$v_0 = 25 \times 10^3$ m/s;

$V_p = 0.52 \times 10^9$ m^3;

$\rho_p = 3.6 \times 10^3 \text{kg/m}^3$;

$\rho_t = 2.8 \times 10^3 \text{ kg/m}^3$;

Replace the symbols of equation (1)(2) with data, V_c, D_e would be

$$V_c = 0.52 \times 10^9 \times 30.25 \times \left(\frac{3.6 \times 10^3}{2.8 \times 10^3}\right)^{\frac{3}{2}} \times \left(\frac{25 \times 10^3}{4\ 500}\right)^2$$
$$= 0.713 \times 10^{12} (\text{m}^3)$$

$$D_e = 1\ 000 \times 1.96 \times \left(\frac{3.6 \times 10^3}{2.8 \times 10^3}\right)^{\frac{1}{2}} \times \left(\frac{25 \times 10^3}{4\ 500}\right)^{\frac{2}{3}} = 7\ (\text{km})$$

That is, the impact will form a pit with $14(2D_e)$ kilometer diameter, whose area $(s = \pi D_e^2)$ is 154km^2, $1/10^5$ of the Antarctic.

Part Ⅱ the Assessment of the Melting Ice

The energy released by the explosion equals to one million tons dynamite, which is 1 600 times as the energy of the most powerful nuclear weapon. Therefore, the ice in the explosion areas will be vaporized directly. The volume of ice in the Antarctic contains 90% of the total volume of the ice in the world, exceeding $24 \times 10^6 \text{km}^3$. The energy of the explosion melts a large -scale ice into vapor or water. The vapor ruches to the atmosphere and reach to supersaturation. Later it condensation and fall to ground, which make sea level rise. Now we will analyze it in detail.

In the part Ⅰ we have already constructed the model of pill and butt. According to the conservation of mass and the conservation of momentum, we have equations

$$\rho_t C_t = \rho_c (C_t - v_c) \tag{3}$$

$$(\rho_t C_t) C_t + P_0 = [\rho_c (C_t - v_c)](C_t - v_c) + P_c \tag{4}$$

where v_c, P_c stand for velocity and pressure of the particle after the surface of the shock wave; C_t is the velocity of shock wave in the butt; and P_0 is the pressure at the beginning in the butt. The other symbol is the same with which in part one. According the equation (3) and (4) we have equation to the butt

$$P_c - P_0 = \rho_t C_t v_c \tag{5}$$

A similar equation to pill is

$$P_c - P_0 = \rho_P C_P (v - v_c) \tag{6}$$

Expurgation v_c from (5) and (6), we get

$$P_c = \frac{\rho_P C_P - \rho_t C_t}{\rho_P C_P + \rho_t C_t} v \tag{7}$$

We suppose that material of the pill and the butt is almost the same in order to predigest the problem. So we have

$$P_c = \frac{1}{2} \rho_c v \tag{8}$$

where v is the velocity of pill and ρ_c is the density of butt under the super high pressure. Under the super high pressure, the relation between the deformation and pressure is not linear. The deformation will reduce with the increase of the pressure. We can describe that as:

$$\epsilon_v = \frac{\Delta v}{v} = \frac{1}{k} p - \mu p^2 \tag{9}$$

The typical value of the parameters in equation (3) to (9) are

$$v_0 = 25\ 000 \text{ m/s};\ \frac{1}{K} = 2 \times 10^{-11} \text{ m}^2/\text{N};$$

$$\mu = 0.4 \times 10^{-22} \text{ m}^4/\text{N}^2;\ \rho_p = 3.6 \times 10^3 \text{ kg/m}^3;$$

$$\rho_t = 2.8 \times 10^3 \text{ kg/m}^3;\ \rho_c = 1.9 \times 10^7 \text{ kg/m}^3;$$

where ρ_c is the density of the compacted part and $\frac{1}{K}$, μ is the distortion coefficient.

Then we can get:

$$P_c = \frac{1}{2} \rho_c v_0 = \frac{1}{2} \times 1.9 \times 10^7 \times 25 \times 10^3$$

$$= 2.4 \times 10^{11}(\mathrm{N/m^2})$$

then the power on the unite volume:

$$\omega = \frac{1}{2}P_c\varepsilon_v = \frac{1}{2}P_c\left(\frac{1}{K}P_c - \mu P_c^2\right)$$

$$= \frac{1}{2} \times 2.4 \times 10^{11} \times [2 \times 10^{-11} \times 2.4 \times 10^{11}$$

$$- 0.4 \times 10^{-22} \times (2.4 \times 10^{11})^2]$$

$$= 3 \times 10^{11}(\mathrm{J})$$

The total power:

$$W = \omega \cdot V_c$$

$$= 3 \times 10^{11} \times 0.713 \times 10^{12}$$

$$= 2.14 \times 10^{24}(\mathrm{J})$$

One-kilogram ice melting needs 334.7×10^3 J heat.

Then $\quad E = W = \rho_{\text{ice}} \cdot q \cdot V$

$$2.14 \times 10^4 = 0.917 \times 10^3 \times 334.7 \times 10^3 \cdot V$$

$$V = 7 \times 10^{15}(\mathrm{m}^3)$$

Assume the water distributes to the ocean, then the sea level will rise h (m) by average.

$$h \cdot S = V\frac{\rho_{\text{ice}}}{\rho_{\text{water}}} \tag{10}$$

$$h \cdot 3.61 \times 10^{14} = 7 \times 10^{15} \times \frac{0.917}{1.01}$$

$$h = 17.6\ (\mathrm{m})$$

That is to say, the large-scale melting of the Antarctic polar ice sheet will raise the sea level nearly 17.6 m.

Part Ⅲ The Estimate of Amount and Scope of the Dust and Smog

Analysis of the problem

When the asteroid impact on the earth, the explosion will bomb out a large scale of the dust to the stratosphere. The dust under the order of 0.001m in diameter will stay in the stratosphere for a long time but the large dust will fall to the earth quickly. Besides the dust , the high temperature of the explosion will produce vast scale of the NO_x. The dust, NO_x and the vapor mix into smog. The smog will diffuse to a certain scope by the movement of the atmosphere. That will lead to the change of the climate and the pollution of the atmosphere in the partial scope or the whole earth.

Assumptions

In order to simplify the model, we assume:

1. We think the location of the explosion as one point in the air. It releases the smog in all directions with the same intensity at a moment.

2. We aim at analyzing the distribution of the smog in the stratosphere. And assume the density of the smog in the vertical direction is the same. That is, we neglect the thickness of the stratosphere.

3. When the density of the smog is under 1 mg/m^2 then we can neglect it.

The first step: Neglect the wind.

1. The rule of the density change.

We assume the explosion time $t=0$ and let explosion location as the coordinate origin. $C(x,y,z,t)$ present the density of arbitrary point in the infinite space at time t. According to the assumption 2, the volume of flow q through per area in the normal direction during per time:

$$q = -k \cdot \mathrm{grad} C \tag{11}$$

where k stands for the coefficient of the diffusion, *grad*. stands for grads and the negative symbol present that the diffusion is from the space with high density to the low.

Search space Ω, V is the volume of the Ω, S is the curve surface around Ω, n is the normal-line out of the surface, then the volume of flow Q_1 through the Ω during time t to $t+\Delta t$:

$$Q_1 = \int_t^{t+\Delta t} \iint_S q \cdot n \, \mathrm{d}\delta \mathrm{d}t \tag{12}$$

However the increment Q_2 to the smog in Ω:

$$Q_2 = \iiint_V [C(x, y, z, t) - C(x, y, z, t + \Delta t)] \mathrm{d}v \tag{13}$$

According to the conservation of mass:

$$Q_1 = Q_2 \tag{14}$$

According to the Gauss theorem:

$$\iint_S q \cdot n \, \mathrm{d}\delta = \iiint_V \mathrm{div} q \cdot \mathrm{d}v \tag{15}$$

According to (11) to (15) and the integral median theorem we can get:

$$\begin{aligned} \frac{\partial C}{\partial t} &= k \cdot \mathrm{div}\ (\mathrm{grad} C) \\ &= k\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) \end{aligned} \tag{16}$$

According to the assumption 1, the point function at the beginning time:

$$C(x, y, z, 0) = Q\delta(x, y, z) \tag{17}$$

where Q means the total smog amount, $\delta(x, y, z)$ is the unit intensity point function.

The result of the equation (16) under the condition (17):

$$C(x, y, z, t) = \frac{Q}{(4\pi k t)^{\frac{3}{2}}} e^{\frac{x^2+y^2+z^2}{4kt}} \tag{18}$$

The result show that at the arbitrary time t, the equivalent value surface of the function $C(x, y, z, t)$ is spherical surface. We mark it as $x^2+y^2+z^2=R^2$. And the value of the density C will reduce with the increase of the diameter R. Only when time t tends to infinite, $C(x,y,z,t)$ tends to zero.

The second step: Consider the reality situation.

Because it is very cold in the Antarctic all the year, there is a high-pressure field. It is the low-pressure terrain around the South Antarctic continent. So wind always exits from the continent to the sea. The speed of wind is very high so as to 17－18 m/s. The effect of it makes the smog diffuse more quickly.

We take the scope of the dust instead of the polluted scope of the smog. We get the volume of the pit from the part one.

$$V_c = 0.713\times 10^{12}\ \mathrm{m}^3$$

$$\begin{aligned} M &= V_c \cdot \rho_t \\ &= 2.8\times 10^3 \times 0.713\times 10^{12} \\ &= 2\times 10^{15}(\mathrm{kg}) \end{aligned}$$

As there exists vast of vapor, we assume only 1/1 000 of total quality dust reaches the stratosphere.

$$\begin{aligned} m &= k\cdot M = 1/1\,000\times 2\times 10^{15} \\ &= 2\times 10^{12}(\mathrm{kg}) \end{aligned}$$

That is, there is 200 million ton dust floating in the stratosphere. According to the assumption 2 we can get the polluted area is $2\times 10^{11}\ \mathrm{km}^2$. It is round area on the order of 2.5×10^5 km in the diameter.

Part Ⅳ the Effect to the Organism

The impact will bring the disaster to the creatures on the earth. The consequences has three stage in general.

1. High temperature. The explosion causes the temperature rising in short time and infinite area.

2. Cold wave. It seems like nuclear winter: black clouds dense-

ly covered, no sunshine in area full of the dust, temperature descending and surface of the sea frosted.

3. Warm climate. The CO_2 produced by impact cause the greenhouse effect, which lead to the melting of the ice in the earth.

Effect done to biology in the sea

The change of temperature has a little effect on the organism near the Antarctic because it's very cold all along in the Antarctic. But vast of the nitrifier fall to the sea with rain and form nitric acid, which make the PH value of the seawater drops.

The little change of the PH value of the seawater will destroy plankton's procreate ability, so the amount of the plankton, the base of the food chain in the sea, will decrease, which lead directly to attenuation of the food production in the oceans.

The following model will analyze the problem.

Assume the biology in the oceans can be divided simply into two kinds of species. One is called A species that can live alone. The other species, which we call B species, can not live without A species. A species consist mainly of plankton that lives depending on CO_2 and photosynthesis. B species include mainly most fishes.

When A species exists alone in a natural environment, the evolution of the amount of A species obeys Logistic Law. $x_1(t)$ is the amount of A species, r is the increasing ratio, n_1 is it's max capacity. So we obtain the equation:

$$x_1(t) = r_1(t) \cdot x_1(t) \cdot \left(1 - \frac{x_1}{N_1}\right) \qquad (19)$$

where factor $\left(1 - \frac{x_1}{N_1}\right)$ reflect that A species' consuming finite resource have a retardant effect on increasing of itself. $\frac{x_1}{N_1}$ can be explain that the A species' consuming food relative to N_1. (Assume the total amount of food is 1.)

The B species will die without A species. Assume B species exists alone and it's death ratio is r_2, obviously,

$$\dot{x}_2(t) = -r_2 \cdot x_2 \tag{20}$$

A species supply food for B species, so we add the stimulative of A species to (20), thus

$$\dot{x}_2(t) = r_2 \cdot x_2 \cdot \left[-1 + \sigma_2 \cdot \frac{x_1}{N_1}\right] \tag{21}$$

The meaning of σ_2 is that the food supplied by the unitage of A species is as σ_2 times as those consumed by the unitage of B species. Obviously, only when $\sigma_2 \frac{x_1}{N_1} > 1$, the amount of B species will increase.

At the same time, the B species has a retardant effect on increasing of itself. So we must add a Logistic term to (21), the (21) would be:

$$\dot{x}_2(t) = r_2 \cdot x_2 \cdot \left[-1 + \sigma_2 \cdot \frac{x_1}{N_1} - \frac{x_2}{N_2}\right] \tag{22}$$

Here x_2 is amount of B species. N_2 is the max capacity of B species.

According to (22), when $\frac{\sigma_2 \cdot x_1}{N_1} < 1 + \frac{x_2}{N_2}$, then $\dot{x}_2(t) < 0$, that is, the B species will decrease. The bigger $|\dot{x}_2(t)|$ is, the faster B species decrease.

The model can explain that large-scale attenuation of food production in the oceans of the Southern Hemisphere.

Effect done to the human being

According to Part Ⅱ, the large-scale belting of ice make sea level ascend 17.6 m. The coastal city with altitude under 17.6 m will be flooded. Obviously, it causes serious human casualties.

Besides, the atmosphere pollution, soil acidification, the descending of the seafood and crops, and the destroying of the ozonosphere will do harm to human. The local famine will appears, many people will have skin cancer, the acidic smog will erode the human's trachea and the rot of animal's body cause spreading of disease.

Conclusion

When the asteroid on the order of 1 000m in diameter impacts on the earth, it will explode. The shockwave will kill many animals directly. Addition to that, it will cause the pollution in the atmosphere. So a great deal of the insects in the Antarctic will die out because of the weaken adaptability. As to the mammal such as penguin and seal, a large number of them will die also.

The asteroid with 1 000m diameter weighs about 2 kilomega tons. Assume the explosion will produce 2 billion tons NO_x. Utilize the result of Part Ⅲ, we assume that all the NO_x is distributed into the sea. Then the sea in the range of 2.5×10^5km will be polluted. The plankton will die or lose their reproduction ability. All these will make the animals in the sea lose their food and die. Consider that the coastal is polluted serious especially, there also has the major fishery. Then the food production in the oceans of the Southern Hemisphere will reduce 1/3. The exact number can be got by Part Ⅳ on the ground of getting the degree of the plankton decrease. It is clear that the fishery around the Antarctic will be damaged at all.

Because of the rising of the sea level, many coastal cities will be flooded. We can know that the sea will rise nearly 18 m. Assume the population density is 300 people/ km^2 and 1/ 10 000 000 area of the continent be flooded. Then the direct human casualties will be

50 million.

Consider the damage to the ecosystem, there exists more far harmful effects to the human being. It estimates innumerable people will die of various reasons.

Perhaps the earth will be ruined if this asteroid is larger than 1 000m in diameter.

Strengths and Weakness

Our model has four parts. They are independent and have the logic relation in the sequence. We simplify the complex course into four little models and use the concise language to describe the phenomenon. It improves that the simplifying is actually right in some range. As a fact, applying the model to the impact accident in the history, we can see its result accords to the records.

The train of our thought is clear. And the approximate consequences can be gotten easily.

However, our model also has many weaken aspects. As the complication of the nature, there are many details need to be considered. The model can't get the precise answer for example; we can't get the accurate amount of human casualties. Because of its simpleness, the consequence will have some distance to the reality.

The Reference

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2. Liu Desheng. The World Geography. High Education Publishing Company, 1986.

3. Xu Jinghua. The Depopulation. Life-Reader-Knowledge Publishing Company, 1997.

论 文 点 评

论文建立的模型基于如下的分析。

小行星撞击地球南极后引起的结果是连锁性的，因素众多，情况复杂。撞击中释放的能量会引起南极冰雪消融，还会导致大量尘埃的产生；冰雪融化会引起海平面的上升，危及沿海地区，造成人员伤亡，而大量尘埃会随大气循环，严重影响地球的温度及生态系统，从而长久影响人类生活。

作者认为建立一个整体的模型来实现对各种因素的评估是比较困难的，因此先建立一个模型去估计撞击产生的能量及尘埃，在得到这些数据的基础上，再用其他模型估计造成的影响。按照这样的思路，作者将问题分为四部分来建立模型：

1. 利用刚体中的撞击理论建立起小行星撞击地球的模型，以描述碰撞的直接影响；

2. 建立了南极冰雪融化的模型，由此计算出冰雪融化的数量；

3. 建立了估计尘埃分布的无风扩散模型，并得出了污染范围的估计。

4. 建立了估计对海洋生物链的影响的模型。

上述模型考虑了问题的各个因素，比较好地估计了结果。

本文的特点是将一个复杂的问题分解为几个问题进行讨论，得出各个子问题的结果后，再综合起来得出整个问题的结果。本文所建模型合理，表述清晰。

模型中没有给出人类伤亡的数量和所在地区的详细估计，这是该论文的缺陷。

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2. 建立了南极冰雪融化的模型，由此计算出冰雪融化的数量；

3. 建立了估计尘埃分布的无风扩散模型，并得出了污染范围的估计。

4. 建立了估计对海洋生物链的影响的模型。

上述模型考虑了问题的各个因素，比较好地估计了结果。

本文的特点是将一个复杂的问题分解为几个问题进行讨论，得出各个子问题的结果后，再综合起来得出整个问题的结果。本文所建模型合理，表述清晰。

模型中没有给出人类伤亡的数量和所在地区的详细估计，这是该论文的缺陷。

本篇论文获得 1999 年美国数学建模竞赛的二等奖。