Radio Channel Assignments

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Summary

We use mainly combinatorial methods to estimate and prove bounds for various cases, concentrating on two levels of interference. We use the concept of a *span*, the minimum largest channel among assignments that satisfy the constraints.

For Requirements A and B, the span is 9. For Requirement C, the span is 7 when $k = 1$, 9 when $k = 2$, 12 when $k = 3$, and $2k + 7$ for $k \geq 4$.

For Requirement D, we present the results in a table (**Table 2**). Some of our results improve on upper bounds in Shepherd [1998].

Only regular transmitter placement needs to be considered; irregular placement can be accommodated by making the hexagons so small that the transmitters are in regular placement, with the bounds adapted correspondingly.

For Requirement E, we discuss both the limitations of our model and its ability to produce an upper bound for any situation.

Definitions

- Let s denote the length of a side of a hexagon. Then the distance from the center of one hexagon to the center of an adjacent hexagon is $s\sqrt{3}$.
- A *region* is a collection of hexagons, finite or otherwise.
- For u and v hexagons in a region $\mathcal{X}$, let $D(u, v)$ be the minimum number of hexagons (including the first but not including the last) that one must pass through to move from u to v in region $\mathcal{X}$. Set $D(u, u) = 0$. So, for example, the stipulation in the problem that any two different transmitters

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(in hexagons u and v) are within distance $2s$ is equivalent to $D(u,v) \leq 1$. Similarly, two hexagons are within distance $4s$ if and only if $D(u,v) \leq 2$.

- Let T be the the portion of a plane that includes a hexagon u along with all hexagons v such that $D(u,v) \leq 3$ (**Figure 1**).

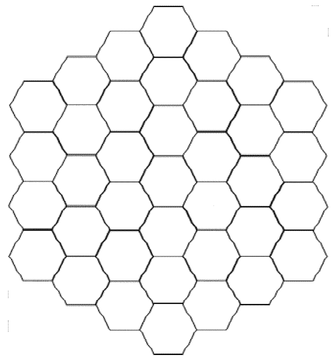

Figure 1. Region T.

- Let R be any planar hexagonal grid that contains T.

- Let k_i be the minimum allowed difference in channels of two hexagons u and v in a region R that have $D(u,v) = i$. For example, if $k_1 = 2$ and $k_2 = 1$, then transmitters in hexagons u and v that are adjacent must have channels that differ by at least 2. If the transmitters in hexagons u and v are two hexagons apart (i.e., $D(u,v) = 2$), then their channels must not be the same.

- Let C be a function from the hexagons in a region R to the positive integers. Given a set of constraints, call C a *channel assignment* to R under those constraints if C maps the hexagons to an allowed set of frequencies.

- The *width of the interval* of the frequency spectrum in region R is the largest channel used. The minimum width over all channel assignments of a region R is the *span*.

- Let the function $S(l_1, l_2, \ldots, l_n)$ of a region R be the span under the restrictions that $k_i = l_i$ for all i from 1 to n.

- For a given $k_1 \geq 4$, define the set

$$\mathcal{N}_k = \{1, 2, 3, k+3, k+4, k+5, 2k+5, 2k+6, 2k+7\}$$

as the *channel assignment set*. That is, for a region R, $C(R) \subseteq \mathcal{N}_k$.

Solution

We are concerned with planar regions that expand out in every direction infinitely or else are finite. First, we prove some general results.

Lemma 1. *Let M be any positive integer. If $S(k_1, k_2, \ldots, k_n) = L$, then*

$$S(k_1, k_2, \ldots, k_{i-1}, k_i + M, k_{i+1}, \ldots k_n) \leq L + M\left(\left\lceil \frac{L}{k_i} \right\rceil - 1\right).$$

Proof: Let C_1 be an assignment of channels on the region R with span L and satisfying the given constraints. We construct an assignment that satisfies the new constraints, with the desired largest channel. Define a new channel arrangement C_2 as follows:

$$C_2(u) = C_1(u) + M\left(\left\lceil \frac{C_1(u)}{k_i} \right\rceil - 1\right).$$

To see that the new constraints are satisfied, notice that

$$|C_2(u) - C_2(v)| \geq |C_1(u) - C_1(v)|;$$

so all the constraints for k_j, $j \neq i$ are still satisfied. Furthermore, if

$$|C_1(u) - C_1(v)| \geq k_i,$$

then

$$|C_2(u) - C_2(v)| = |C_1(u) - C_1(v)| + M.$$

This is because if $|C_1(u) - C_1(v)| \geq k_i$, then

$$\left\lceil \frac{C_1(u)}{k_i} \right\rceil \neq \left\lceil \frac{C_1(v)}{k_i} \right\rceil.$$

This demonstrates that the constraint for the new value of k_i is now satisfied. Thus, the only channels used are of the form

$$C_1(u) + M\left(\left\lceil \frac{C_1(u)}{k_i} \right\rceil - 1\right) \leq L + M\left(\left\lceil \frac{L}{k_i} \right\rceil - 1\right).$$

Therefore, the channel assignment that we have constructed is valid, and we have further shown the desired feature that

$$S(k_1, k_2 \ldots k_{i-1}, k_i + M, k_{i+1}, \ldots k_n) \leq L + M\left(\left\lceil \frac{L}{k_i} \right\rceil - 1\right). \qquad \square$$

Lemma 2. *On any region R containing T, $S(4,1) > 14$.*

[EDITOR'S NOTE: We omit the proof.]

Lemma 3. *On any region R containing T, $S(3,1) > 11$ and $S(2,1) > 8$.*

Proof: If $S(3,1) = L \leq 11$, then by **Lemma 1** we know that

$$S(4,1) \leq L + \left\lceil \frac{L}{3} \right\rceil - 1 \leq 11 + \left\lceil \frac{11}{3} \right\rceil - 1 = 14,$$

which is a contradiction to **Lemma 2**. Similarly, if $S(2,1) = L \leq 8$, then by **Lemma 1** we have

$$S(3,1) \leq L + \left\lceil \frac{L}{2} \right\rceil - 1 \leq 8 + \left\lceil \frac{8}{2} \right\rceil - 1 = 11,$$

which violates what we just proved. □

Lemma 4. *If $l > 4$, then for region T, $S(l,1) > 2l + 6$.*

[EDITOR'S NOTE: We omit the proof.]

The proof works for any region, finite or infinite.

$k_2 = 1$

For any two hexagons u and v, if $D(u,v) = 1$, then their channels differ by at least k ($k_1 = k$) for any positive integer k, and $k_2 = 1$. With this generalization, we would like to see how the span relates to k_1.

Lemma 5. *For $k_1 \geq 4$, a width of the interval of the frequency spectrum in region R is less than or equal to $2k_1 + 7$.*

Proof: By induction. We use the set defined in **Definition 9**. First we show that $k_1 = 4$ satisfies **Lemma 5**. If $k_1 = 4$, then for all u, v such that $D(u,v) \leq 1$ we have $|C(u) - C(v)| \geq 4$, by definition. By **Lemma 2**, we have $S(4,1) > 14$. To see that for $k_1 = 4$ a frequency width is $2(4)+7 = 15$, use the channel assignment set $\mathcal{N}_4 = \{1,2,3,7,8,9,13,14,15\}$. As shown in **Figure 2**, the channel assignment set satisfies the constraints.

Also, the channel assignments tessellate and the resulting pattern always meets the constraints. To see this, translate the channel assignments from A to B. After translation, we have a repeated pattern with no gaps and the constraints still hold. Now, instead translate from A to C, and again we have a repeated pattern with no gaps while keeping all constraints. Since these are the only two possible kinds of translation, we have shown that the pattern is a tessellation. Since the maximum channel assigned is 15, the width of the frequency spectrum is $15 = 2(4) + 7$.

Next, let k be any integer such that $k \geq 4$, assume that **Lemma 5** holds true for $k_1 = k$ with channel assignment set $\mathcal{N}_k$. This generates a tessellation as illustrated in **Figure 3**. It is easy to see that this pattern tessellates and meets the constraints.

Now we need to prove that **Lemma 5** holds for $k_1 = k + 1$. To do so, we generate a tessellation pattern from $\mathcal{N}_{k+1}$ in region R that satisfies the constraints. In our hypothesis that **Lemma 5** holds for k, we replace all k with $k + 1$ in **Figure 3**. The result is the tessellation in **Figure 4**, which meets all the constraints. The maximum frequency used is $2k + 9 = 2(k+1) + 7$; that is, for $k + 1$ the width of the interval of the frequency spectrum is $2(k+1) + 7$. Since this matches our inductive hypothesis for $k + 1$, we have proven the lemma. □

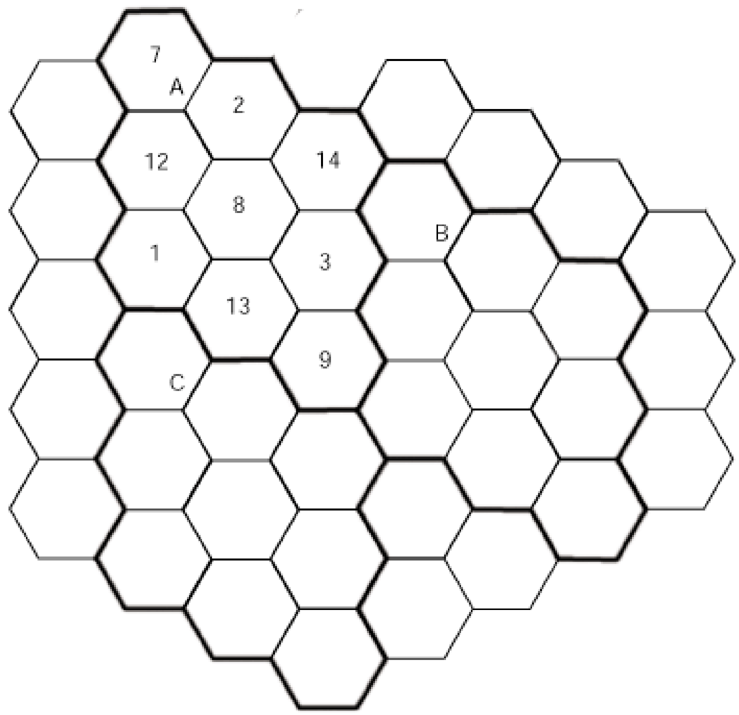

Figure 2. Channel assignment for $k_1 = 4$.

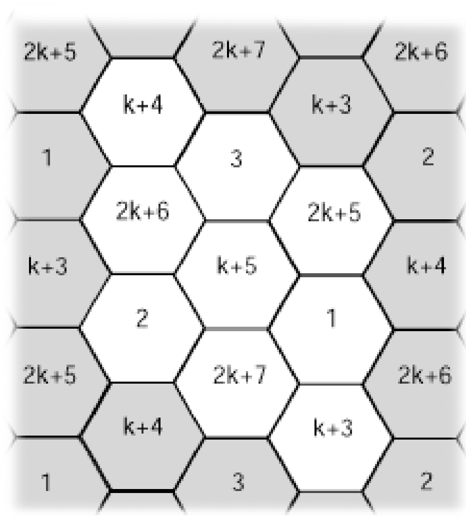

Figure 3. Channel assignment for $k_1 = k$.

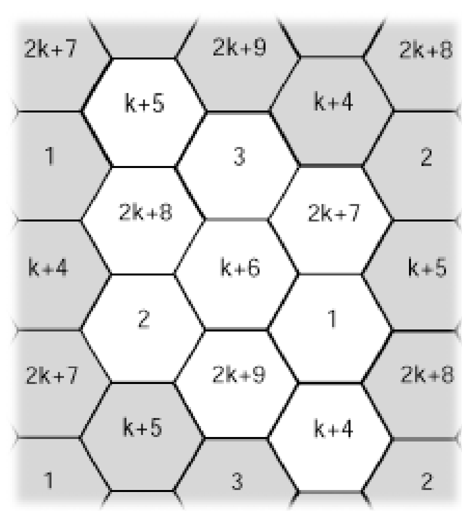

Figure 4. Channel assignment for $k_1 = k + 1$.

Lemma 5 is a very nice result. It gives a way of constructing a tessellation under the constraints that $k_1 \geq 4$ and $k_2 = 1$, and we can make this pattern using the channel assignment set $\mathcal{N}_{k_1}$. Most important, we can assign the channels with a frequency width of $2k_1 + 7$. Next, we prove that this width is a lower bound for any $k_1 \geq 4$.

Theorem 1. *Let $k_1 \geq 4$. Then $S(k_1, 1)$ of a region R is $2k_1 + 7$.*

[EDITOR'S NOTE: We omit the proof.]

With **Theorem 1** and **Lemma 5**, we know how to form a repeating pattern for the given constraints, and we also know the span over the region R. A very nice outcome from these results is that for *any* $k_1 \geq 4$, we can choose nine connected hexagons and produce a channel assignment with $S(k_1, 1) = 2k_1 + 7$. By looking at **Figure 3**, we can see that for large k values we have a larger spread in frequencies; that is, for larger k_1 we have a more efficient system of transmitters in terms of interference because the frequency width is large.

We now attend to $k_1 = 2$ and $k_1 = 3$.

Theorem 2. *For $k_1 = 2$, the channel assignment set is*

$$\mathcal{N}_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

with $S(2, 1) = 9$.

For $k_1 = 3$, the channel assignment set is

$$N_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\},$$

with $S(3, 1) = 12$.

Proof: By **Lemma 3**, we know that $S(2, 1) > 8$ and $S(3, 1) > 11$. In **Figure 5**, we have a tessellation pattern for $k_1 = 2$ with channel assignment set as C_2. By inspection, $S(2, 1) = 9$, the lowest possible value. For $k_1 = 3$, we have a similar argument, only we use the channel assignment set C_3. By inspection of **Figure 6**, $S(3, 1) = 12$, the lowest possible value. □

$k_1 = k$ and $k_2 = 0$

The values in **Figure 7** meet the constraints. Therefore, the span over a region R for this case is $2k + 1$. To see this, try for $k - 1$; then the channel assignment set is $\{1, k, 2k - 1\}$, but k and 1 must be at least k apart. Hence, $2k + 1$ is the span.

$k_1 = k_2 = k$

The values in **Figure 8** meet the constraints. Hence, the span over a region R is $6k + 1$. To see this, as above try for $k - 1$; then we have a contradiction in **Figure 7** with the hexagons containing 1 and $(k - 1) + 1 = k$. Therefore, $2k + 1$ is the span.

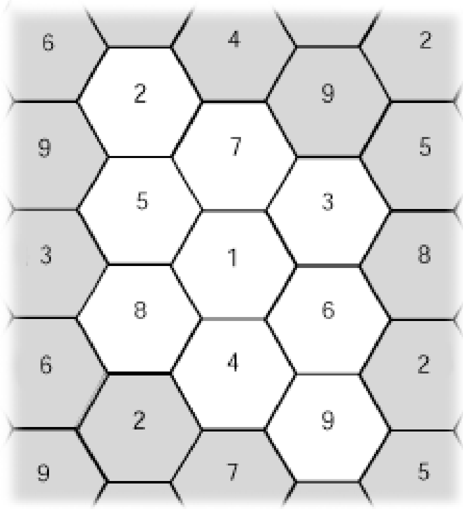

Figure 5. Channel assignment for $k_1 = 2$.

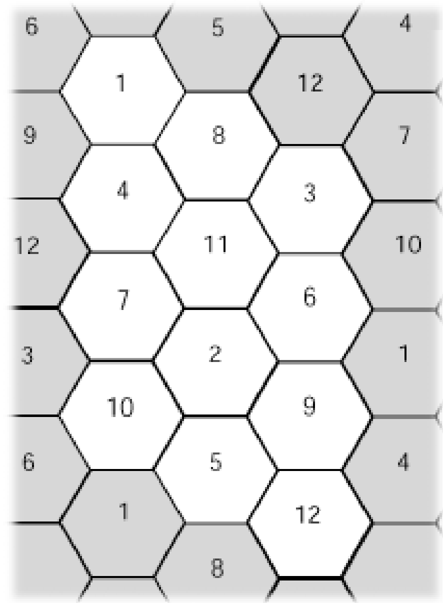

Figure 6. Channel assignment for $k_1 = 3$.

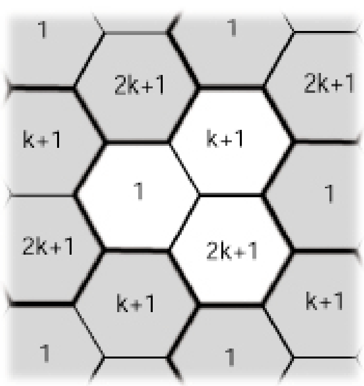

Figure 7. Channel assignment for $k_1 = k$ and $k_2 = 0$.

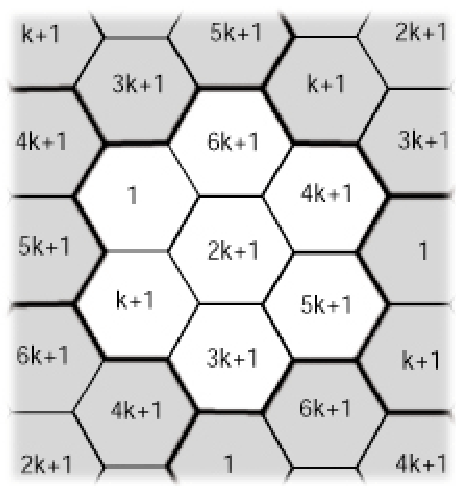

Figure 8. Channel assignment for $k_1 = k_2 = k$.

General Case

In this section, k_1 and k_2 can be any positive integers.

Theorem 3. *Let R be a region that contains region T and let $k_1 \geq 4k_2$. Then*

i) *If k_2 divides k_1, then $S(k_1, k_2) = 2k_1 + 6k_2 + 1$.*

ii) *If $k_1 > 6k_2 + 1$, then $S(k_1, k_2) = 2k_1 + 6k_2 + 1$.*

[EDITOR'S NOTE: We omit the proofs.]

Theorem 4. *Let $3k_2 \leq k_1 \leq 4k_2$. Then $S(k_1, k_2) \leq 3k_1 + 2k_2 + 1$.*

Proof: By construction. Consider the tiling in **Figure 9**. As long as $3k_2 \leq k_1 \leq 4k_2$, the channel assignment holds. Then by construction,

$$S(k_1, k_2) \leq 3k_1 + 2k_2 + 1.$$

As shown by the highlighted tiles, this tiling works only if $2k_2 + 1$ and $k_1 + 1$ differ by at least k_2 (by definition of k_2).

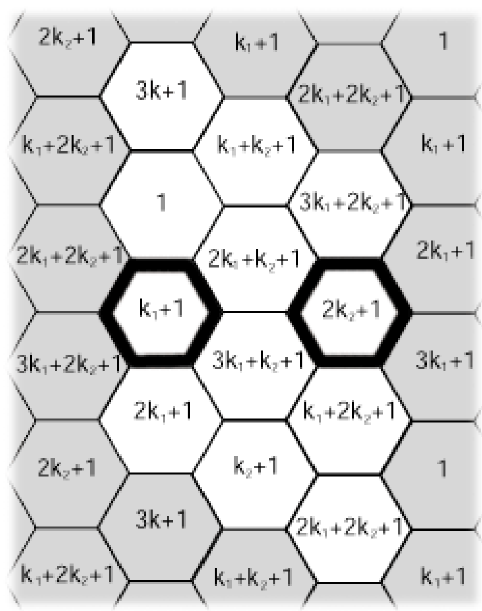

Figure 9. Channel assignment for **Theorem 4**.

It follows that

$$\begin{aligned} (k_1 + 1) - (2k_2 + 1) &\geq k_2, \\ k_1 - 2k_2 &\geq k_2, \\ k_1 &\geq 3k_2. \end{aligned}$$

Yet we know from **Theorem 3** that for $k_1 \geq 4k_2$ we have a strict lower bound; therefore, we must have a strict upper bound, that is

$$k_1 \leq 4k_2.$$

Hence, we have that if $3k_2 \leq k_1 \leq 4k_2$, then $S(k_1, k_2) \leq 3k_1 + 2k_2 + 1$. □

Conclusion

We summarize specific proved results in **Table 1**. For the cases $k_2 = 2$ and $k_1 = 9, 11, 13$, we are unable to determine $S(k_1, k_2)$. However, we find bounds for those values by **Lemma 2**.

Table 1.

Compilation of spans for different values of k_1 and k_2.

k_1	k_2	$S(k_1, k_2)$
1	1	7
2	1	9
3	1	12
4	1	15
5	1	17
$l > 5$	1	$2l + 7$
2	2	13
3	2	17
4	2	17
5	2	21
6	2	23
7	2	26
8	2	29
9	2	30, 31 or 32
10	2	33
11	2	34, 35 or 36
12	2	37
13	2	39 or 40
$l > 13$	2	$2l + 13$

Table 2.

General results for values of k_1 and k_2.

Constraints	Span
any k_1 , $k_2 = 0$	$2k_1 + 1$
$k_1 = k_2$	$6k_1 + 1$
$k_1 = 2$, $k_2 = 1$	9
$k_1 = 3$, $k_2 = 1$	12
$k_1 \geq 4$, $k_2 = 1$	$2k_1 + 7$
$k_1 \geq 4k_2$	$\leq 2k_1 + 6k_2 + 1$
$3k_2 \leq k_1 \leq 4k_2$	$\leq 3k_1 + 2k_2 + 1$
$k_1 > 4$, $k_2 = 1$	$> 2k_1 + 6$
$3k_2 \geq 2k_1$	$4k_1 + 3k_2$

Table 2 has our general results. The last row was proven not by us but by Mark Shepherd [1998]. For selected values of k_1 and k_2, we establish the span of an arbitrary planar hexagonal region that includes T. For all combinations, we can find a pattern that repeats—that is, we can find a tessellation of frequencies. This is a major result, because we know how to construct a frequency assignment based on the values of k_1 and k_2 through a simple formula, as shown in **Figure 4** for $k_1 \geq 4$ and $k_2 = 1$.

News Release: Mathematicians Help Clear Out Airwaves

Last week mathematical researchers at Washington University in St. Louis announced that they have improved the current method of assigning radio frequencies, such as the channel of your favorite station. With the increase in wireless communication, it has become more important than ever to assign frequencies efficiently while avoiding interference as much as possible.

The mathematicians made use of a certain type of pattern, made popular by the contemporary artist M.C. Escher, called a tessellation. These patterns are carefully constructed to cover an entire region without leaving any gaps. The new results show how to assign channel frequencies to regions in a tessellation so as to minimize several kinds of interference from nearby stations. The work extends efforts currently in progress at Oxford University in England.

"Our work is quite general," commented one of the researchers. "It applies regardless of geographical situations, such as differences in altitude or other natural phenomena."

Radio listeners have nothing to fear from these new developments; the frequency of your favorite station is unlikely to change. The new results will help long-term planning by engineers, operators of cell-phone services, and government regulators. "In the future you won't have the kind of interference that causes someone to flip to Rush Limbaugh's channel but end up instead with Howard Stern. We're just making sure that listeners get to hear Rush say his whole two cents worth."

Reference

Shepherd, Mark. 1998. Radio channel assignment. Ph.D. thesis, Merton College, Oxford University. `http://www.maths.ox.ac.uk/combinatorics/thesis.html` .