A Channel Assignment Model: The Span Without a Face

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Introduction

We were asked to design efficient assignment of radio channels to a symmetric network of transmitter locations over a large planar area, so as to avoid interference. Efficiency is based on the span, the minimum of the largest channel assigned.

We derive properties implied by the first set of constraints and by the geometry of the given figure, which we use to construct what we call "span theory." We prove upper and lower bounds for the span of the given figure. With the aid of a computer program, we narrow the bounds and prove that the span is 9. This is also the span of a network generated by extending the figure arbitrarily far in all directions.

We then consider slightly altered constraints, that the channels of neighboring transmitters cannot differ by less than k. We determine two distinct strategies for channel assignments and two associated formulas for the span; the span is $\min\{3k+3, 2k+7\}$ for both the figure and the generated plane.

Allowing a transmitter to be positioned irregularly in the hexagons changes the span by at most 1. Allowing all transmitters to be positioned irregularly—a worst-case scenario—gives a span of 18.

Assumptions and Justifications

- *Every hexagon in the field has a single transmitter at its center*. This can be assumed for Requirements A, B, and C from statements in the problem.

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- *Every transmitter is an ideal transmitter*, that is, it transmits radio signals equally well in all directions. No information is given suggesting any of the transmitters are less than ideal. According to our research, an ideal radio station would perform in this manner [Rorabaugh 1990, 134].
- *Every transmitter in the grid is assigned a single channel* (the problem so states).
- *Every positive integer works equally well as a transmission channel*.

Terms and Definitions

- **Neighbors:** Two polygons with a common side.
- **Network:** A finite or infinite group of connected, non-overlapping, maximally packed, regular polygons.
- **Span:** The minimum, over all assignments satisfying the constraints, of the largest channel used at any location.
- **Symmetric network:** A network that is symmetric about some axis.
- **Tessellate:** To repeat a geometric pattern over an infinite or finite plane.
- **Tessellation:** An arrangement of polygons that will fit together without overlapping or creating any gaps and cover an infinite plane.
- **Valid network:** A network that satisfies all the constraints of the given requirement.

Analysis

Hexagonal Geometry

For convenience, we rotate the figure given in the problem to obtain rows instead of columns, as shown in **Figure 1**.

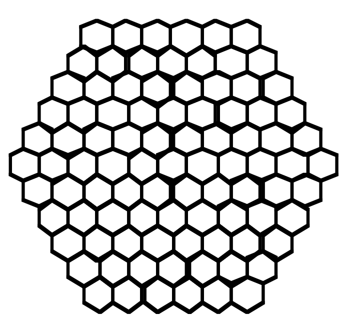

Figure 1. Rotated horizontal layout of hexagons.

We begin by analyzing the geometry of a regular hexagon with side length s. Drawing the diagonals as shown in **Figure 2** yields six regular triangles. The distance from the center to any corner is s, and the perpendicular distance from the center to any side is $s\sqrt{3}/2$.

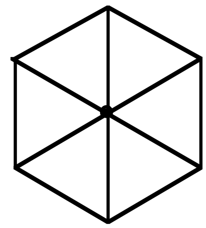

Figure 2. Detail of a regular hexagon.

We examine a small symmetric network to determine the effects of the spectral spreading constraint. **Figure 3** illustrates a circle of radius $2s$ drawn from the center hexagon. The centers of all six adjacent hexagons are within this circle, so no transmitter may neighbor a transmitter of an adjacent channel; but the circle does not spread beyond these six hexagons, so any transmitter beyond the six hexagons that surround the center may be assigned to an adjacent channel without interference.

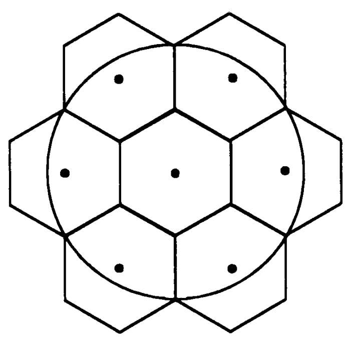

Figure 3. A symmetric network with seven hexagons and radius $2s$ marked from the center of the center hexagon.

For example, consider **Figure 4**, with two networks in which channels are assigned to each hexagon. The network on the left violates the constraint above, because channel 1 and channel 2 are neighbors; the network on the right is a valid network.

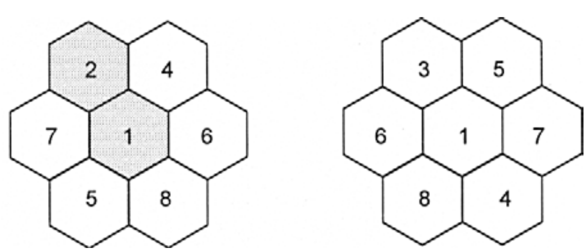

Figure 4. Two symmetric networks with seven hexagons and assigned channels.

We summarize the effect of spectral spreading with the following rule:

Adjacent Channel Principle: *No neighboring hexagons may be assigned adjacent channels.*

We now consider the requirement that no two transmitters with the same channel may be within $4s$ of each other.

We define two hexagons to be *n hexagons away from each other* if and only if one can construct a path of n straight line segments, and no fewer, with each segment having length $s\sqrt{3}$ and having both endpoints at centers of hexagons. For example, in **Figure 5**, hexagon A is 3 hexagons away from hexagon B.

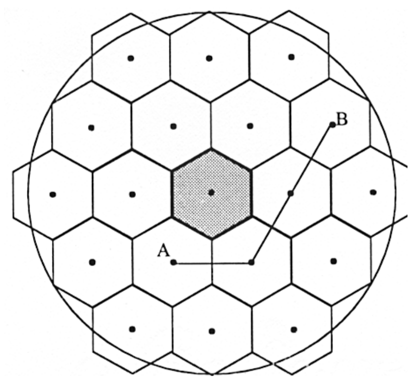

Figure 5. Hexagon A is 3 hexagons away from hexagon B.

From **Figure 5**, we observe that no two transmitters of the same channel can be 2 hexagons away from each other, but they can be 3 hexagons away from each other:

Same Channel Principle: *No two transmitters of the same channel may be less than three hexagons away from each other.*

Embedded Subgraph Method

We divide the hexagons into sets as indicated in **Figure 6**. The dotted lines indicate the embedded subgraph within the network. The vertices of the subgraph are indicated in the hexagons that are included in the subgraph, and those hexagons are marked as well. The vertices are intentionally drawn off-center so that the subgraph's edges do not coincide with any hexagon sides. The shaded hexagons alternate blue and green.[1]

The model works as follows. Any blue hexagon is at least 3 hexagons away from any other blue hexagon; so by the **Same Channel Principle**, all blue hexagons may be assigned the same channel. Similarly, all green hexagons may

[1] EDITOR'S NOTE: Like **Figure 6**, a number of the authors' original figures are in color, and readers of the electronic version of this article can see them in color. Expense prohibits reproducing the figures as color plates in the printed *Journal*.

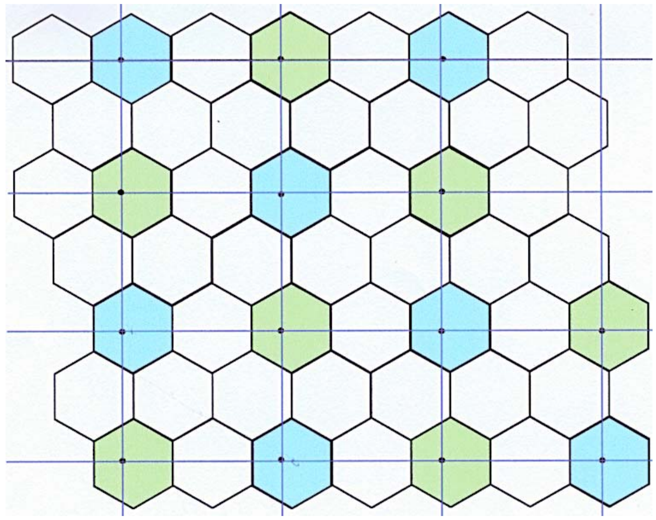

Figure 6. Embedded Subgraph Method setup.

be assigned the same channel. No blue hexagon neighbors any green hexagon; by the **Adjacent Channel Principle**, we may assign blue and green hexagons adjacent channels.

There are four such embedded subgraphs in the network. The hexagons on the same rows as the blue and green hexagons, but between them, create another embedded subgraph, which can also be assigned two adjacent channels. Next, we can place two more subgraphs to connect the rows that have not been assigned channels yet. The result is four embedded subgraphs that together contain all the hexagons in the network. **Figure 7**, in which the first and third rows alternate colors and the second and fourth rows alternate different colors, shows how every hexagon is assigned to exactly one subgraph. Hexagons of the same pattern belong to the same subgraph.

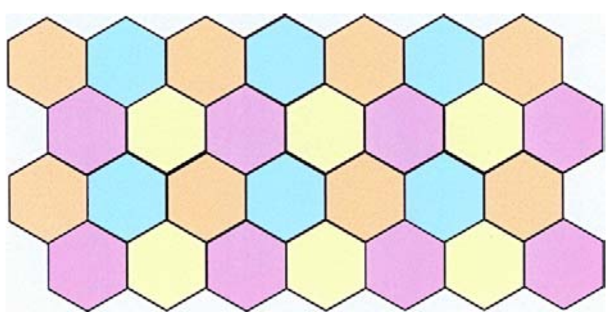

Figure 7. Embedded subgraphs indicated by shading.

The final step is to assign channels to all four subgraphs with two channels per subgraph, as described above. For the first subgraph, we use channels 1 and 2. At this point every hexagon in the other three subgraphs neighbors a channel 2 hexagon; thus no hexagon may be assigned channel 3, by the **Adjacent Channel Principle**. So we skip 3 and assign 4 and 5 to the hexagons of a second subgraph (it does not matter which subgraph). As before, each remaining hexagon neighbors a 5; so we skip channel 6 and assign 7 and 8 to a third subgraph. Finally, we skip channel 9 and assign channels 10 and 11 to the remaining subgraph. This pattern yields three channel values not being assigned. The resulting pattern is shown in **Figure 8**.

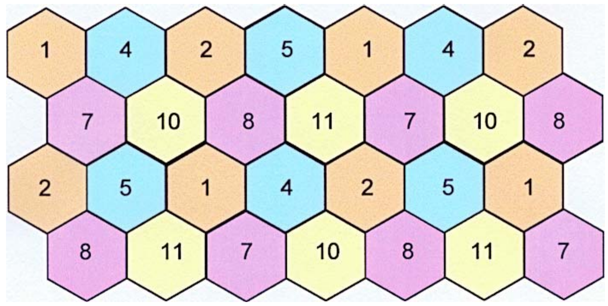

Figure 8. Channels assigned using the Embedded Subgraphs model.

For the sake of examining interference, the distance to transmitters of the same channel is at least $s\sqrt{21}$. The distance to adjacent channels is at least $3s$. Also, every channel has only one adjacent channel that appears in the network.

This pattern can be tessellated across any number of hexagons. Thus, we know that the span is at most 11 for **Figure 1** as well as a grid that spreads arbitrarily far in all directions.

Diagonals Method

We attempt to fill the network using diagonals. We want exactly three channels repeated along any diagonal. To avoid same-channel interference, each channel along the diagonal should be exactly three hexagons away from the closest hexagon with the same channel. This diagonal should then be repeated three diagonals away on either side. We call such repeated diagonals *same diagonals*. The result is shown in **Figure 9**. The channels x, y, and z are chosen so that no two of them are consecutive channels and spectral spreading is avoided. For example, x, y, and z may be assigned the values 1, 3, and 5.

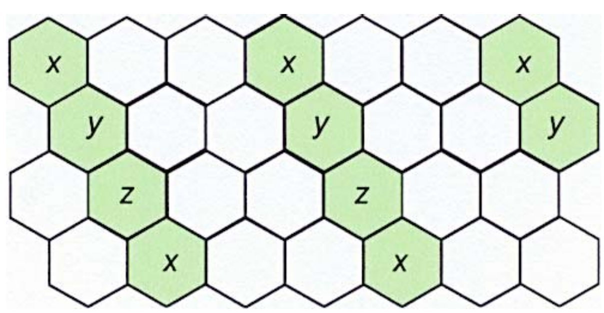

Figure 9. Diagonals Method setup.

After assigning channels to the first diagonal, we attempt to assign values to its neighbor diagonals without violating any constraints. With trial and error, we obtain the assignments shown in **Figure 10**.

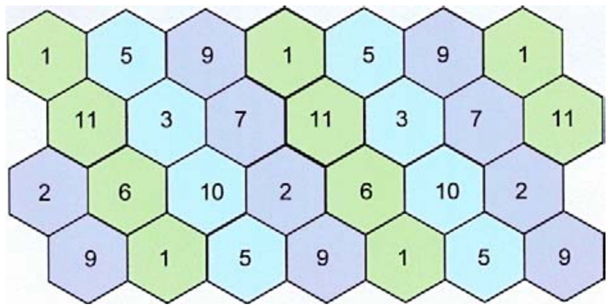

Figure 10. Channel assignments made using the Diagonals Method.

This tessellation also has a maximum channel of 11, and like the Embedded Subgraphs Method, it can be expanded infinitely in all directions. Also, the distance to the closest adjacent channel is $3s$. Unlike the Embedded Subgraphs Method, this pattern contains nine unique channels, leaving two unused (channels 4 and 8). Furthermore, the distance to the closest transmitter with the same channel is $3s\sqrt{3}$. Another difference from the Embedded Subgraphs Method is that three of the channels have two adjacent channels appearing in the network, while six of them have only one.

We explain why the assignment in **Figure 10** is valid. Consider every hexagon as belonging to one of three subgraphs, whose edges create a set of triangles. With the 1, 2, and 3 placed as they are, they violate no rules; but placing a 4 in any hexagon would violate the **Adjacent Channel Principle**. So we skip 4 and assign channels 5, 6, and 7 to a triangular subgraph with the same properties as the first subgraph. Finally, we skip 8 and assign 9, 10, and 11 to the remaining subgraph.

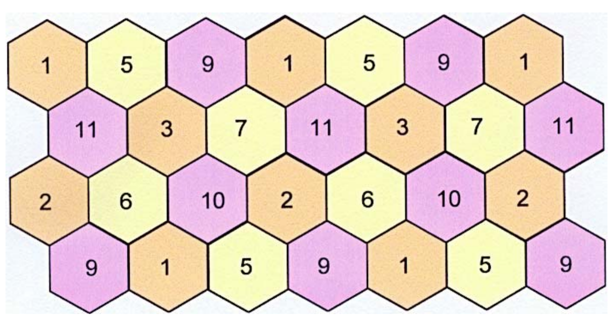

Figure 11. Embedded subgraph model setup.

Computer Program

We confirmed our results (maximum channel of 11) by writing a computer program to test every possible combination and determine the span. (Before writing the program, we knew that it would require 91 levels of recursion, but the speed and memory capacity of modern PCs makes this a feasible option.)

The program begins with an empty network, that is, a network with no channels assigned to any transmitter. It starts with the farthest left hexagon in the top row of the network and assigns it channel 1. It moves across the row and then similarly processes the following rows. Each time a new hexagon is encountered, the program attempts to assign a channel to the hexagon, starting with 1 and ending at some user-specified upper bound. After assigning a channel, it checks that the assignment does not violate the **Adjacent** and **Same Channel Principles** listed above. If there is a violation, it tries the next highest channel. If the channel assigned exceeds the upper bound, the program moves back to the previous hexagon and performs a reassignment before continuing. If the program successfully assigns a value to the last hexagon in the last row, it has successfully filled the network and displays the entire network for the user. If a network is never displayed, the upper bound was too low.

The program output various working configurations for an upper bound of 9 but no layouts for 8. The pattern that generates a span of 9 is shown in **Figure 12**. This pattern appears to be a variation of the Diagonals Method: the hexagons of any right diagonal consist of exactly three channels, which are repeated ad infinitum. However, as highlighted in the figure, for any entry in a diagonal, the corresponding entries of the closest same diagonals are on different rows. This does not change the fact that the **Same Channel Principle** is satisfied; any pair of same-channel transmitters are still at least three hexagons away from each other.

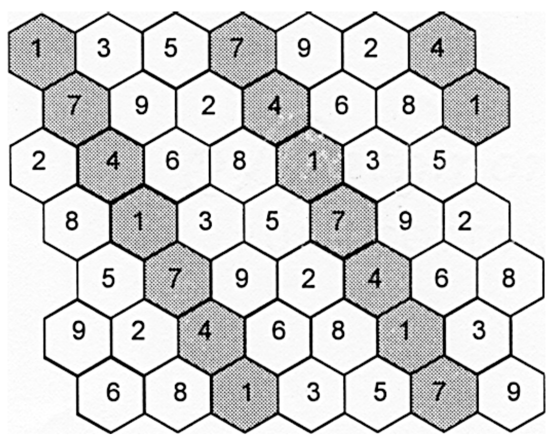

Figure 12. Computer-generated span of 9.

To verify that this is a valid pattern, we must also show that the **Adjacent Channel Principle** is not violated. The channel of any hexagon differs from its left and right neighbors by at least two. Every row is identical to the one above it but shifted three diagonals to the left. Due to the shift, neighbors above and below differ by at least three. Thus, the **Adjacent Channel Principle** is satisfied.

We develop "span theory" to prove that the span of **Figure 1**, as well as the span of an arbitrarily large grid, is 9.

Span Theory

Let S be the set of all networks constructed with regular hexagons of side length s, and let H_n be the symmetric network of regular hexagons with n regular hexagons on any side of it (**Figure 13**). Note that **Figure 1** is H_6; we denote by H_∞ the symmetric network that spreads arbitrarily far in all directions.

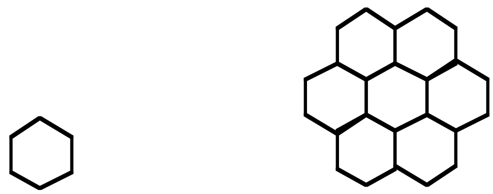

Figure 13. Symmetric networks H_1 and H_2.

Let $A \in S$. We define Span(A) as the minimum of the largest channel used at any location in A, over all channel assignments to the hexagons in A satisfying both the **Adjacent** and **Same Channel Principles**. From our previous results, we have:

Maximum Span Property: *For all $A \in S$, Span$(A) \leq 9$.*

Our goal is to show that Span(A) is also bounded below by 9 for $A = H_6$ and $A = H_\infty$.

Let $A, B \in S$. We define $A \subset B$ if A can be traced out entirely inside B, that is, B can be truncated by removing hexagons so that it is congruent to A.

Proposition 1. *If $A \subset B$, then Span$(A) \leq$ Span(B).*

[EDITOR'S NOTE: We omit the proof.]

Lemma 1. *If the channel of the center transmitter in an H_2 network is not 1 or 8, then there is a channel greater than or equal to 9 in that network.*

Proof: Suppose not. Let x be the channel of the center hexagon, $1 < x < 8$. Then there are at most 8 choices for the seven hexagons in the H_2 network. Let T be set of possible channels for the six neighbors of the center hexagon. By the **Same Channel Principle**, $x \notin T$. Since $x > 1$, we have $x - 1 \geq 1$, so by the **Adjacent Channel Principle**, $x - 1 \notin T$. Similarly, since $x < 8$, we have $x + 1 \leq 8$, and by the **Adjacent Channel Principle**, $x + 1 \notin T$. So $T \subset \{1, 2, 3, 4, 5, 6, 7, 8\} - \{x - 1, x, x + 1\}$, so $|T| \leq 5$. Since we must assign to the six hexagons at most five different channels, at least one channel is assigned to two hexagons. Doing so violates the **Same Channel Principle**, so we arrive at a contradiction. $\square$

Theorem 1. Span$(H_6) = 9$.

Proof: Let A be as shown in **Figure 14**; $A \in S$.

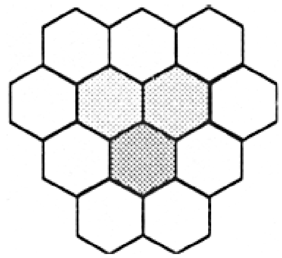

Figure 14. Network A of Theorem 1.

Note that A has three hexagons (shaded in the figure) that are the centers of three corresponding H_2 networks. By the **Same Channel Principle**, since these three hexagons are neighbors, they must be assigned three unique channels. Thus, at most one can be assigned channel 1, at most one can be assigned channel 8, and therefore at least one of the shaded hexagons must be assigned a number other than 1 or 8. Since the shaded hexagon with a channel other than 1 or 8 is the center of an H_2 network, that network has a channel greater than or equal to 9 in it, by **Lemma 1**. Since A contains this H_2 network, there must be a hexagon in A with a channel greater than or equal to 9 in it. Thus, $\text{Span}(A) \geq 9$. By the **Maximum Span Property**, $\text{Span}(H_6) \leq 9$. By **Proposition 1**, since $A \subset H_6$ we have $9 \leq \text{Span}(A) \leq \text{Span}(H_6) \leq 9$. Therefore, $\text{Span}(H_6) = 9$. $\square$

Corollary 1. $\text{Span}(H_\infty) = 9$.

Requirement C

We move to the case in which transmitters within distance $2s$ differ by at least some given integer k. The result of this new constraint is a modified principle for adjacent channels:

The k-Adjacent Channel Principle: *No neighboring hexagons may have channels that differ by less than k.*

Requirement C states that the distance to same-channel transmitters remains at least $4s$, so the **Same Channel Principle** applies as before.

We must modify our definition of $\text{Span}(A)$: Let $A \in S$ and $k > 1$. We redefine $\text{Span}(A, k)$ as the minimum of the largest channel used at any location in A, over all channel assignments to the hexagons in A that satisfy the **k-Adjacent** and **Same Channel Principles**. Thus, $\text{Span}(A) = \text{Span}(A, 2)$.

Note that the analogue of **Proposition 1** holds, that is, if $A \subset B$, then $\text{Span}(A, k) \leq \text{Span}(B, k)$.

We want to find $\text{Span}(H_6, k)$ for general k. Running our computer program to find networks satisfying the **k-Adjacent Channel Principle** with various values of k, we find

$$\begin{array}{lll} \text{Span}(H_6, 2) = \ \ 9 & \text{Span}(H_6, 7) = 21 & \text{Span}(H_6, 3) = 12 \\ \text{Span}(H_6, 8) = 23 & \text{Span}(H_6, 4) = 15 & \text{Span}(H_6, 9) = 25 \\ \text{Span}(H_6, 5) = 17 & \text{Span}(H_6, 10) = 27 & \text{Span}(H_6, 6) = 19 \end{array}$$

So we hypothesized that

$$\text{Span}(H_6, k) = \begin{cases} 3k+3, & 2 \leq k \leq 4; \\ 2k+7, & k \geq 4. \end{cases}$$

We found two distinct patterns, one with maximum channel $3k+3$ and the other with maximum channel $2k+7$ (**Figure 15**).

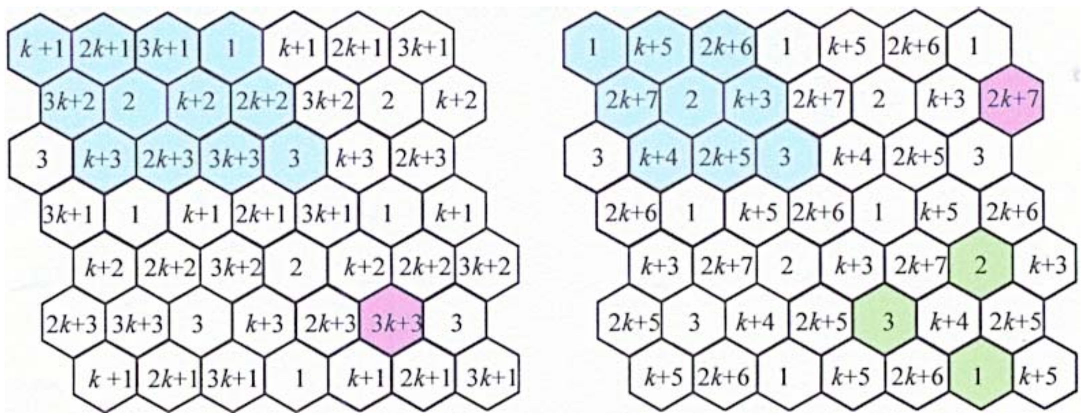

Figure 15. Patterns for channel assignments.

Highlighted with shading in the upper left-hand corner of each pattern is the tessellation that generates the network. Also shaded is the maximum channel used by each tessellation. The $2k+7$ pattern is actually an arrangement that fits the Diagonals Method; a sample triangular subgraph is highlighted in the lower right-hand corner.

The $2k+7$ pattern uses channels 1, 2, 3, $k+3$, $k+4$, $k+5$, $2k+5$, $2k+6$, and $2k+7$. It is valid because it follows the same rules as the Diagonals Method:

- Specifically, any given channel is three hexagons away from a hexagon with the same channel on the same row. Also, the nearest row with the same channel is three rows away. Thus the **Same Channel Principle** is satisfied.

- We note that any channel cannot be placed next to any other channel in its column for $k > 2$. However, it can be placed next to every entry in the other columns, since they will differ by at least k. Since every entry has six unique neighbors, placing the six channels from the other two columns around the entry results in a valid arrangement. Examining the network confirms this deduction: Every 1, 2, or 3 is surrounded by the six channels listed in the other two columns; the same also holds for these six channels. Thus, the **k-Adjacent Channel Principle** is satisfied.

The verification of the $3k + 3$ pattern is similar.

Comparing the two patterns, the minimum distance to a transmitter of the same channel is $3s\sqrt{3}$ in both patterns, and the minimum distance to the nearest adjacent channel is $3s$ in both. From these results, we can state a modified maximum span property:

Second Maximum Span Property. *If $A \in S$, then*

$$\mathit{Span}(A, k) \leq \min\{3k + 3, 2k + 7\}.$$

Theorem 2. *If $A \in S$, then $2k + 4 \leq \mathit{Span}(A, k) \leq \min\{3k + 3, 2k + 7\}$.*

Lemma 2. *If $k > 6$, then $\mathit{Span}(H_4, k) = 2k + 7$.*

Theorem 3. *If $A \in S$, $H_4 \subset A$, and $k > 6$, then $\mathit{Span}(A, k) = 2k + 7$.*

[EDITOR'S NOTE: We omit the proofs of these results.]

There are a few cases that our mathematical results do not cover; but since our program verifies all results that we obtain mathematically, we are confident that it can find Span(A, k) for any $A \in S$, $k > 1$.

Requirement D

We begin by considering the case of irregular transmitter placements and analyze two cases.

All Transmitters Except One Are in Hexagon Centers

The exception may be anywhere in its hexagon. How far from the center of a hexagon can a transmitter be and still be in the hexagon? Just s. We consider the constraints of Requirement A.

Adjacent-channel transmitters must be $2s$, and same-channel transmitters $4s$, away from each other. If we give one transmitter freedom to move up to s away from its center, then to avoid interference the distance between the center of the irregularly placed transmitter and the other transmitters must not be less than $2s + s = 3s$ for adjacent-channel transmitters, and $4s + s = 5s$ for same-channel transmitters.

The change to $3s$ has no effect, since there are no transmitters between $2s$ and $3s$ away from the center of a hexagon. Thus, the **Adjacent Channel Principle** does not change. However, there are 12 hexagons whose centers are $s\sqrt{21}$ away, which means they could have been same-channel transmitters before (by the **Same Channel Principle**) but now we cannot be sure. Thus, the **Same Channel Principle** no longer holds, as we have transmitters three hexagons away that cannot have the same channel. These 12 hexagons are the shaded ones in a ring in **Figure 16**. The center hexagon is the one that can be irregularly placed.

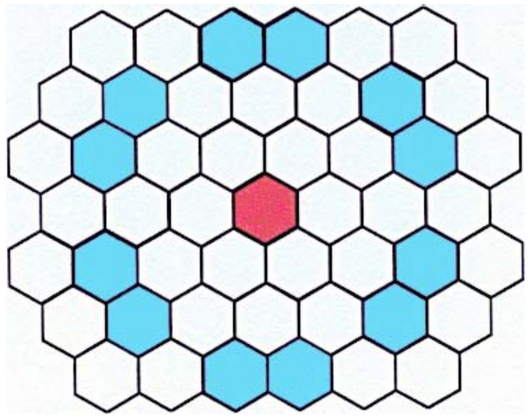

Figure 16. The channel of any shaded hexagon cannot be the same as that of any white hexagon.

Allowing one transmitter to be irregularly placed has minimal effect on the span, changing it by at most 1, a fact that we prove. Let $A \in S$, $k > 1$, and $n \geq 0$. We define $\text{Span}_n(A, k)$ as the minimum of the largest channel used at any location in A, over all channel assignments to the hexagons in A that satisfy the k**-Adjacent** and **Same Channel Principles**, with the additional allowance that up to n transmitters in A may appear anywhere within their respective hexagons. Note that $\text{Span}_0(A, k) = \text{Span}(A, k)$.

Theorem 4. *$Span_1(A, k) \leq Span(A, k) + 1$.*

Proof: Let $x = \text{Span}(A, k)$. We can assign channels to all hexagons in A so as to yield a span of x, with the irregularly placed hexagon assigned channel x. By the **Same Channel Principle**, all transmitters with channel x are at least three hexagons away from the irregularly placed transmitter. By the k**-Adjacent Principle**, the irregularly placed hexagon has no neighbors with channels greater than $x - k$. Change the assignment of the irregularly placed transmitter from x to $x + 1$. Since there are no other transmitters with channel $x + 1$, the **Same Channel Principle** is satisfied. Furthermore, since there are no hexagons neighboring the irregularly placed transmit with channels greater than $x - k$, the k**-Adjacent Channel Principle** is satisfied. Thus, we have constructed a valid pattern for A with highest channel $x+1$ and a single irregularly placed transmitter. So $\text{Span}_1(A, k) \leq x + 1 = \text{Span}(A, k) + 1$. $\square$

Any Transmitter Can Occupy Any Position within Its Hexagon

The farthest that a transmitter can be from its center is s. Suppose that two transmitters need to be at least $2s$ apart; to guarantee that they are $2s$ units apart, we require that their hexagons' centers be $2s + 2s = 4s$ apart. To guarantee that transmitters are $4s$ apart, we can place their hexagons' centers at least $4s + 2s = 6s$ apart.

Placing this scenario in our computer program, we find a pattern for the case of the large network (**Figure 17**).

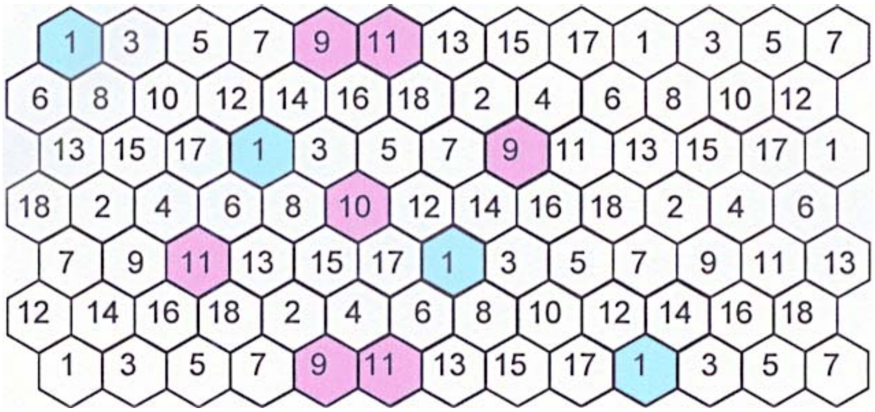

Figure 17. A valid network in which any transmitter can be located anywhere in its hexagon.

This network has a maximum channel of 18. Every channel is the minimum distance from a same-channel transmitter on two sides, as illustrated by the shaded hexagons assigned channel 1. Every transmitter is the minimum distance from each adjacent channel on three sides; that is, there is hexagon closer that could be assigned an adjacent channel and still satisfy both principles. If a channel is not 1 or 18, it has two adjacent channels, so it is the minimum distance from six transmitters with adjacent channels; this is illustrated by the shaded hexagon assigned channel 10, which is the minimum distance from the six shaded hexagons assigned channels 9 and 11.

Since all 18 channels are used, each transmitter is minimally close to at least five other transmitters, and since our program cannot find a pattern with maximum channel 17, we conjecture that $\text{Span}(H_6, 2) = \text{Span}(H_\infty, 2) = 18$.

Several Levels of Interference and Other Factors

We were asked to consider generalizations of the problem.

- A network with several levels of interference could imply a third level of interference, in addition to same-channel interference and spectral-spreading interference. It could also imply varying levels of spectral spreading, such as a rule that channels differing by 1 must be at least $3s$ apart, channels differing by 2 must be $2s$ apart, and same-channel interference remains unchanged.

- Interference levels may vary in different parts of the grid. For example, the top half of the network may satisfy the **2-Adjacent Channel Principle** while the bottom half requires the **3-Adjacent Channel Principle**. Our program could be modified to calculate the span of such a network.

- Transmitters could have non-repeated channels, or there may be certain channels that no transmitters may use.

- Perhaps a small amount of spectral spreading is acceptable, that is, a network might set n as the limit to the number of transmitters allowed within distance

$d \cdot s$ with channels differing by less than k, for some nonnegative integers d, k, and n. In Requirement A, spectral spreading would be described by $n = 0$, $d = 2$, $k = 2$. The **Same Channel Principle** would be defined by $n = 0$, $d = 4$, $k = 1$.

As the problem states, one basic approach is to partition the region into regular hexagons. Squares and triangles could also be used; these are the only regular polygons besides hexagons that tessellate a plane [Firby and Gardiner 1982, 151]. Our program could easily be modified to handle such networks.

Press Release

Radio Channel Assignments Problem Solved:

Robust Computer Program, Mathematical Theory Pave Path to Solution

SAN LUIS OBISPO, Feb. 30 — A team of three undergraduates students from CalPoly cracked the case of the radio channel assignments. The team had to assign channels to radio transmitters on a hexagonal grid in such a way as to prevent several levels of frequency interference.

The team first determined that no more than 11 channels would be needed; then a computer program suggested that a solution with 9 channels might be possible. To prove that 9 channels—but no fewer—would work, the team developed "span theory," a new mathematical theory of channel assignment.

The team also solved several more general problems accounting for wider channel separation or allowing transmitters to be moved around.

Strengths and Weaknesses

Through a series of models and some mathematical theory, we find and rigorously prove that the span of the given figure in Requirement A, and an arbitrarily large figure in Requirement B, is 9.

Our computer program verifies all of our results and is an invaluable tool for determining patterns. It is very robust in its ability to calculate spans for networks of almost any size, subject to constraints that can easily be modified. Execution is almost instant for networks with fewer than 100 hexagons. It would be very difficult to prove that the code is correct, but we develop a rigorous span theory to prove the values of spans.

We also present some early heuristics, the Embedded Subgraph Method and Diagonals Method, that provide near-span solutions and are easily shown to be valid without span theory. In some scenarios, these methods might be preferred for assigning channels, such as if certain channels are forbidden.

For Requirement C, aided by our computer program and span theory, we find two strategies and set upper and lower bounds on the span of a network. With span theory, we find the exact value of the span both for **Figure 1** and for an arbitrarily large network, as well as for hexagonal networks with side length exceeding 3, provided $k > 6$. Though we cannot rigorously determine the spans for $3 \leq k \leq 6$, the computer program calculates them.

We consider only extreme cases of irregular placement in Requirement D. We prove a formula for the span if only one transmitter is allowed to be irregularly placed. On the basis of our program, we also find a span, 18, for allowing every transmitter to be irregularly placed. Channel assignments are valid no matter where a transmitter is moved, provided that it stays within its hexagon.

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