

# Electoral Redistricting with Moment of Inertia and Diminishing Halves Models

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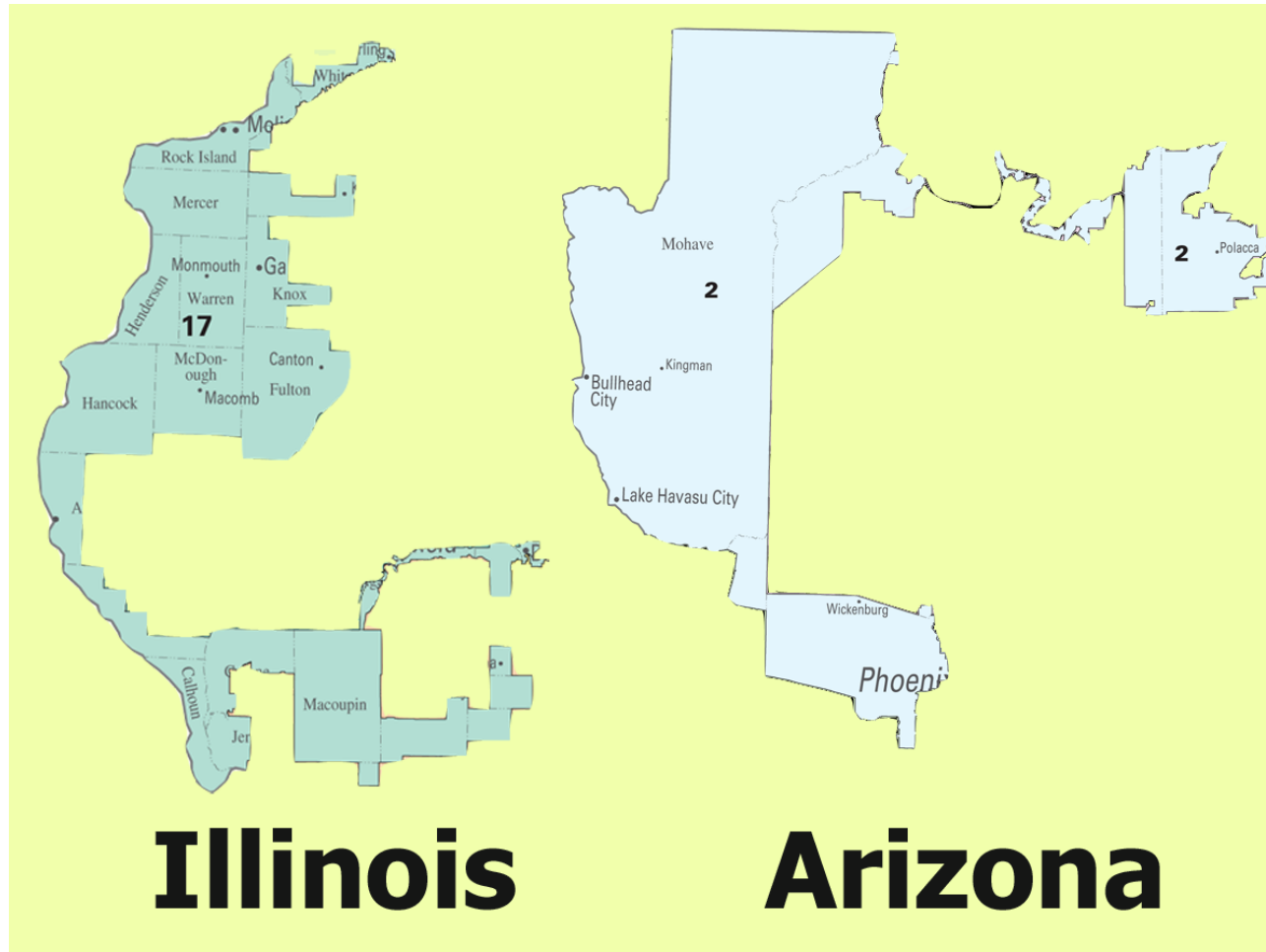
# Outline

1. Introduction
2. Motivation
3. Simplifying Assumptions
4. Moment of Inertia Method
5. Diminishing Halves Methods
6. Quantitative Compactness Analysis
7. Conclusion

## Problem Statement: Congressional Apportionment

- We wish to draw congressional districts for a state.
- Goal: Algorithm that avoids Gerrymandering.
- Want to create “simplest” shapes.
- Definition of “simple” left to problem solvers.
- Only rule is that districts have equal population.

# Gerrymandering Examples



Adapted from *National Atlas of the United States*.

# Motivation

Many possible criteria suggested in literature.

- Equality of district size
- Compactness
- Contiguity
- Similarity to existing borders
- Targeted homogeneity/heterogeneity

Instead of specifying many properties, we wish to explicitly specify as few as possible. Additional properties become an **emergent behavior**.

# Motivation

Our algorithms will use only the criteria of

1. **Equal population districts**
2. **Compactness**

Examine map afterwards to determine emergent properties.

## Simplifying Assumptions

We make the following simplifying assumptions in our model:

- A 2% error tolerance from the mean in size of district population is acceptable.
- Euclidean geometry: variations in longitudinal spacing negligible.
- County borders not sacred (ratio of counties:districts in many states necessitates cutting between borders).

## Extracting test data

- Perl script extracts US Census data at the census tract level.
- Discretizes problem into points with a latitude, longitude, and population.
- For New York, 6398 tracts of nonzero population, median 2518 people.



## Test Data

State	Population	Districts	Non-empty Census Tracts
TX	20,851,820	32	7530
NY	18,976,457	29	6398
IL	12,419,293	19	8078
AZ	5,130,632	8	1934

# Algorithms for Fair Apportionment

We will now compare two methods for apportioning districts

1. Moment of Inertia Method
2. Diminishing Halves Method (Recursive Splitting)

## Moment of Inertia Method

Intuition: Minimize the expected squared distance between all pairs of people in a district.

- Equivalent to physics concept of moment of inertia.
- One of the first methods to appear in literature (*Hess 1965*)
- Results in districts that are not only connected, but also convex, which limits the possibilities of oddly shaped districts

## Moment of Inertia Method

- Moment of Inertia given by  $\sum_i P_i (X_i - \bar{X})^2$ .
- Equals  $\min_Y \sum_i P_i (X_i - Y)^2$ .
- Define districts and centers.
- Given centers optimize districts by computing “size” parameters.
- Given districts optimize centers as centers of mass.
- Gives iterative method to converge to near optimal solution.

# Diminishing Halves Methods

Intuition: **Divide the state into two halves of equal population. Then divide those halves recursively.**

- Suggested by *Forrest 1964*.
- Many different ways to divide state in half.
- After experimentation, we choose the following:
  1. Treat the population data as regression coordinates.
  2. Compute the best fit squared distance (bfsd) line to the data.
  3. Use a line with slope perpendicular to this bfsd line to divide the state.

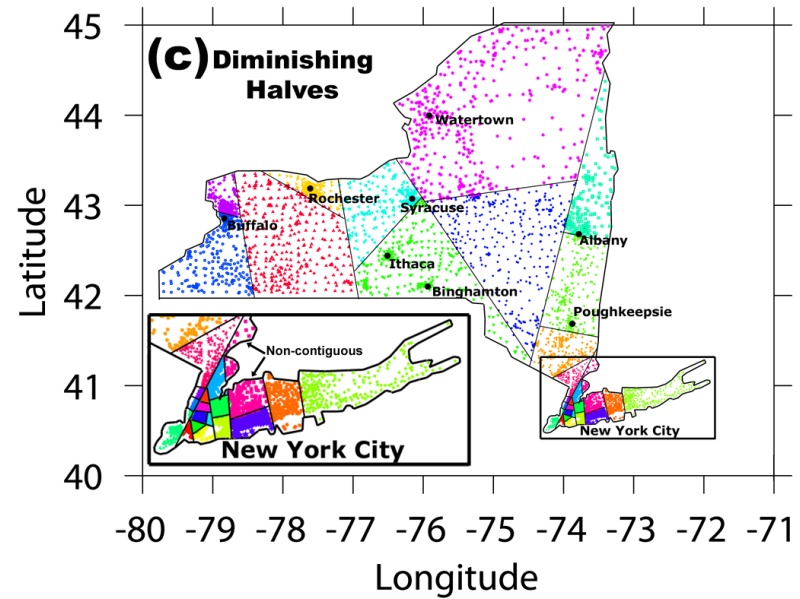
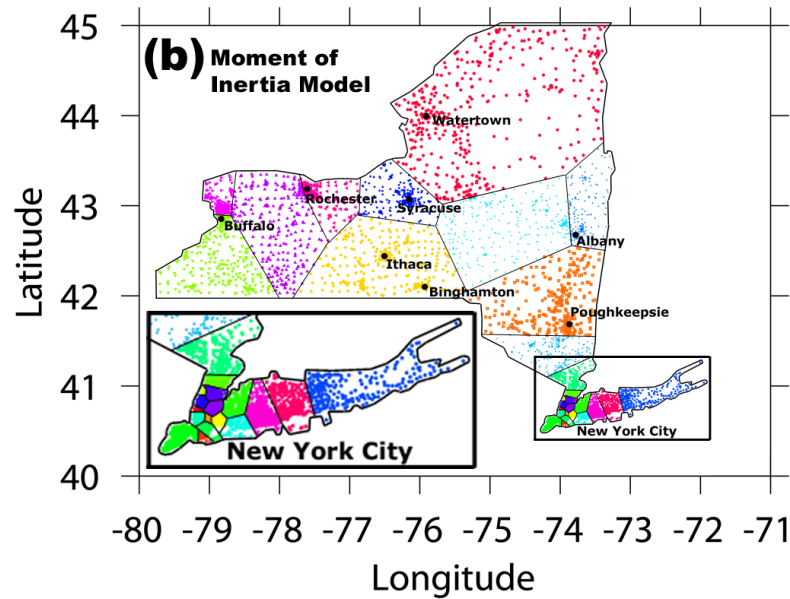
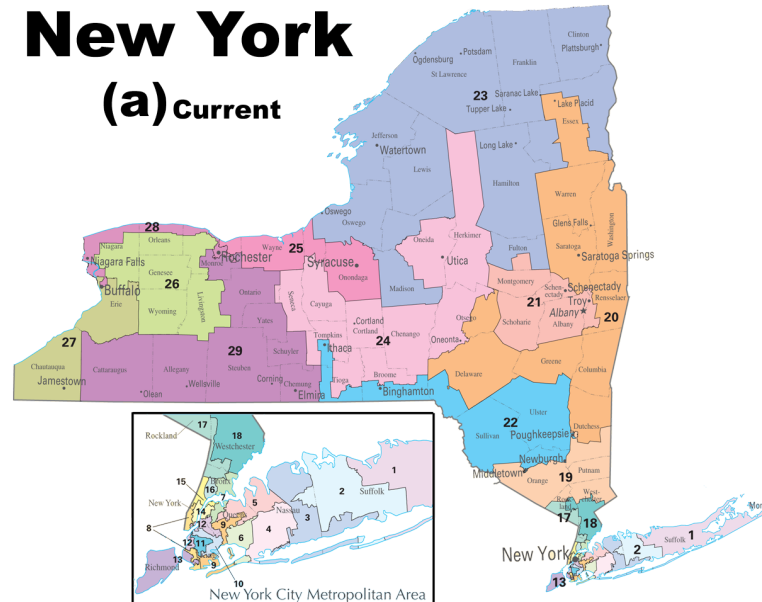
## Diminishing Halves Methods

Compute best fit squared distance line based on orthogonal distance to data.

- Best fit line for given slope contains center of mass.
- Line of form  $(X - \bar{X}) \sin \theta + (Y - \bar{Y}) \cos \theta = 0$ .
- Line minimizes
$$E \left[ \left( (X - \bar{X}) \sin \theta + (Y - \bar{Y}) \cos \theta \right)^2 \right] = \sin^2 \theta \text{Var} [X] + 2 \sin \theta \cos \theta \text{Cov} (X, Y) + \cos^2 \theta \text{Var} [Y].$$
- Optimal angle satisfies  $\tan(2\theta) = \frac{-2\text{Cov}(X,Y)}{\text{Var}[X] - \text{Var}[Y]}.$

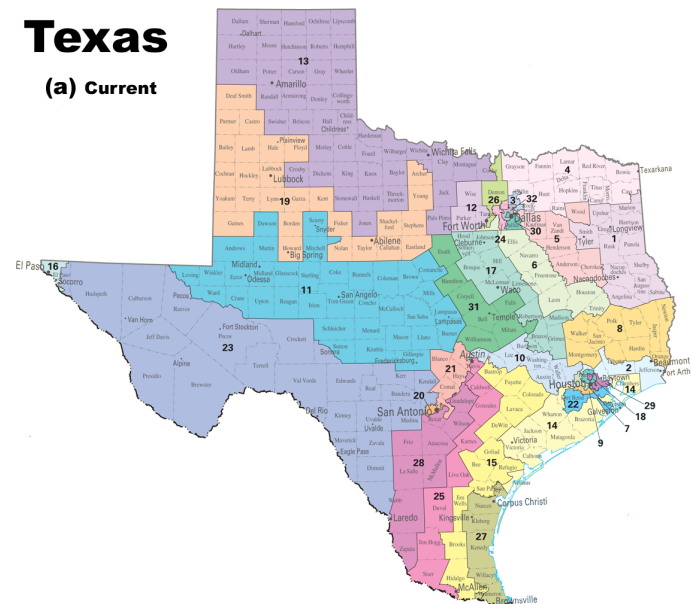
# New York

(a) current

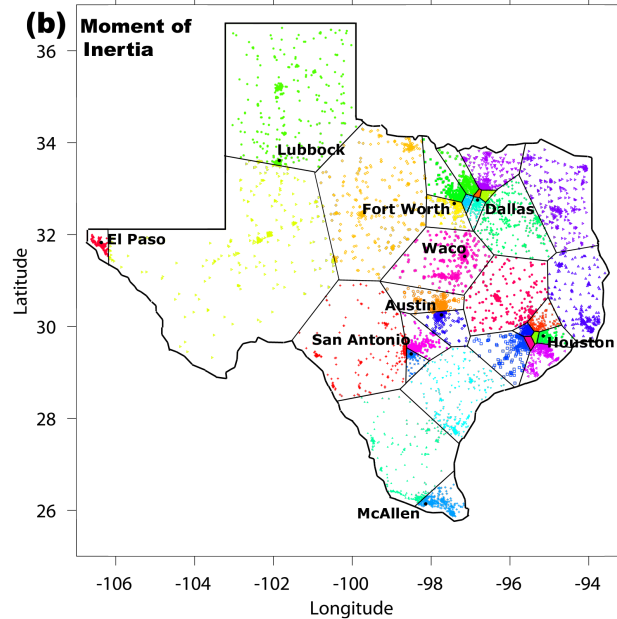


# Texas

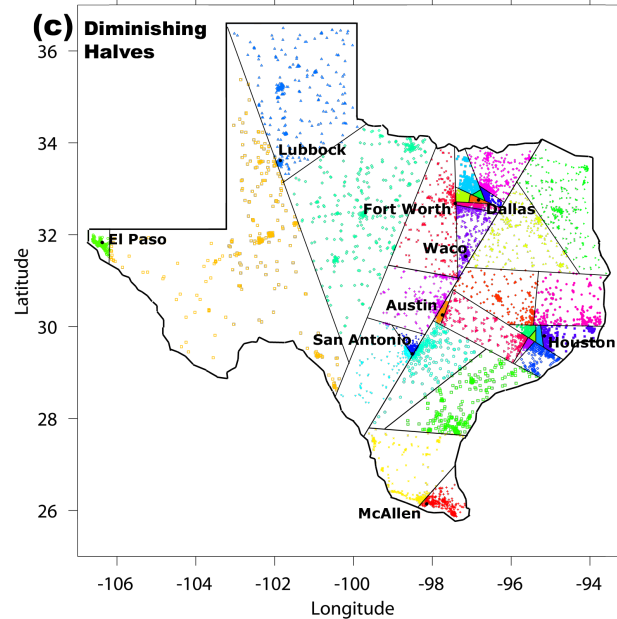
(a) Current



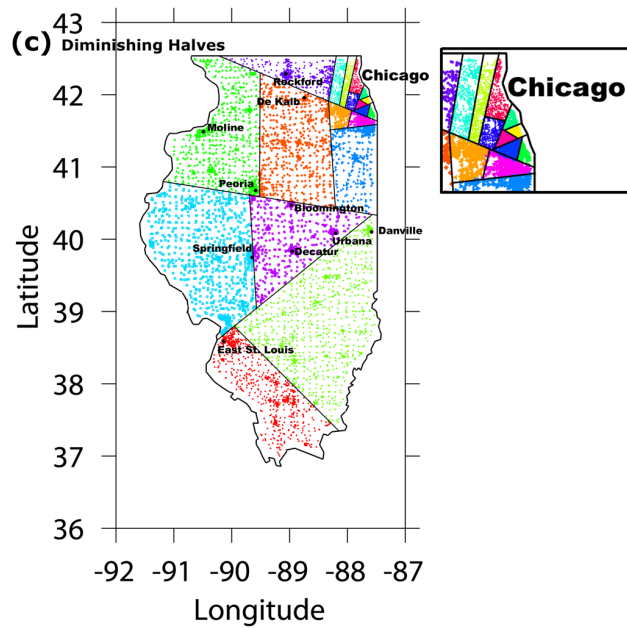
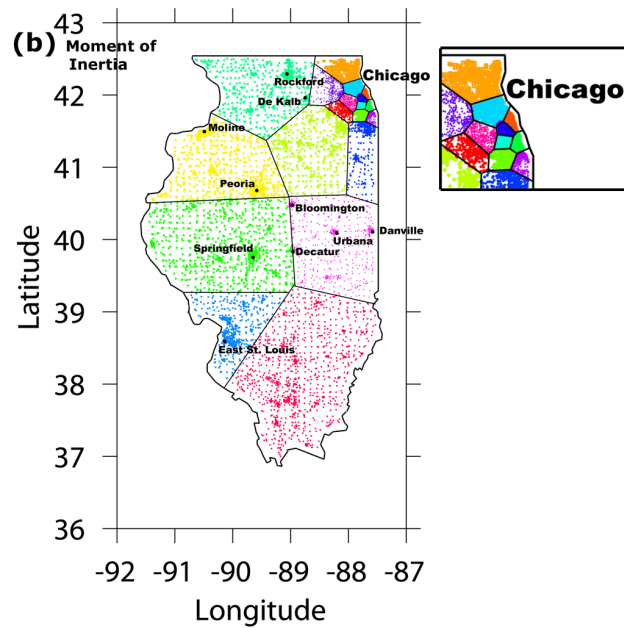
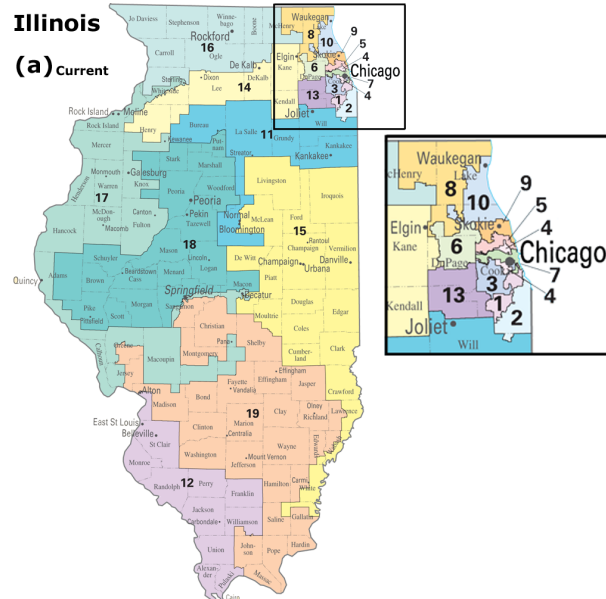
(b) Moment of Inertia

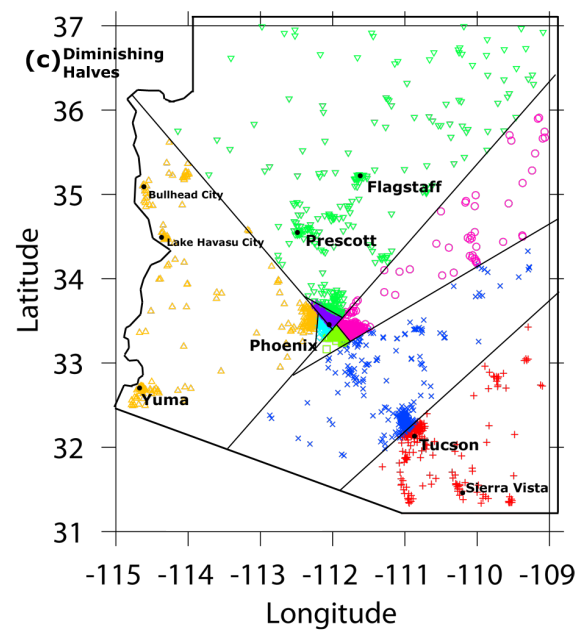
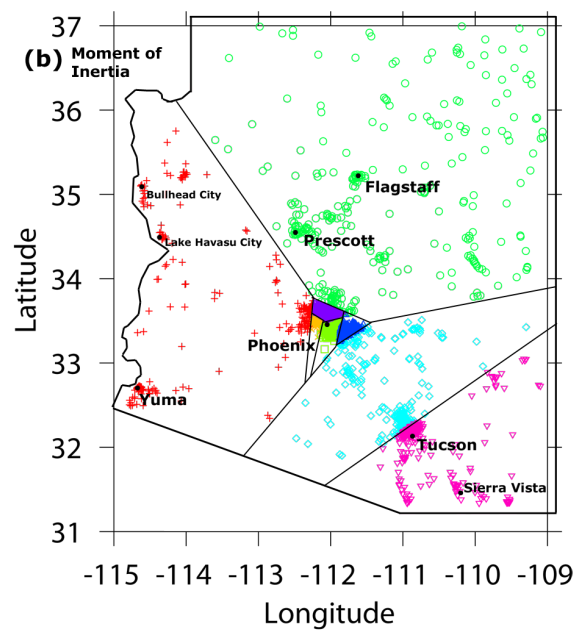
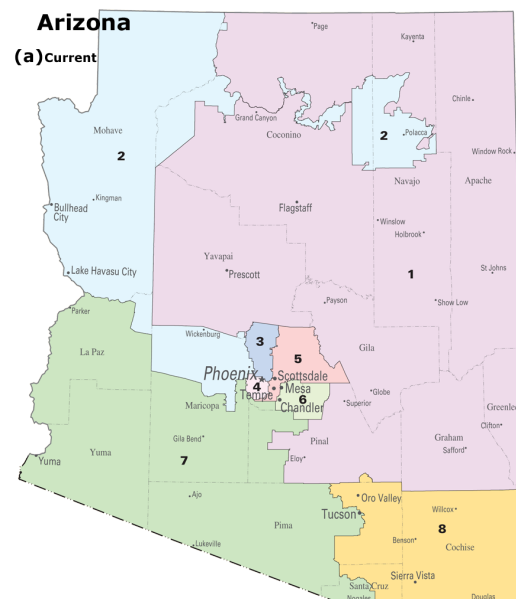


(c) Diminishing Halves









## Quantitative Measures of Compactness

- **Inverse Roeck Test.** Let  $C$  be the smallest circle containing the region,  $R$ . Measure  $\frac{Area(C)}{Area(R)}$ .
- **Schwartzberg Test.** Compute  $\frac{Perimeter}{\sqrt{4\pi Area}}$ .
- **Length-Width Test.** Inscribe the region in the rectangle with largest length-to-width ratio.  
Calculate  $\frac{Length\ of\ Rectangle}{Width\ of\ Rectangle}$ .

Smaller numbers are better. Circles give optimal value 1.

See *Young 1988* for a review.

## Compactness Results

Districts	Inv. Roeck	Schwartzberg	Length-Width
NY (MoI)	$2.29 \pm 0.66$	$1.64 \pm 0.62$	$1.91 \pm 0.61$
NY (DH)	$2.50 \pm 0.87$	$1.74 \pm 0.69$	$1.91 \pm 0.77$
TX (MoI)	$2.04 \pm 0.64$	$1.14 \pm 0.09$	$1.72 \pm 0.57$
TX (DH)	$2.76 \pm 1.66$	$1.27 \pm 0.20$	$2.30 \pm 1.73$
IL (MoI)	$1.90 \pm 0.36$	$1.28 \pm 0.26$	$1.55 \pm 0.39$
IL (DH)	$2.49 \pm 0.99$	$1.35 \pm 0.24$	$2.01 \pm 0.96$
AZ (MoI)	$2.18 \pm 0.56$	$1.17 \pm 0.08$	$1.77 \pm 0.51$
AZ (DH)	$2.69 \pm 0.91$	$1.29 \pm 0.15$	$2.07 \pm 0.79$

MoI = Moment of Inertia, DH = Diminishing Halves.  
 Smaller numbers correspond to more compact districts.

## Conclusion

- Obtain nice districts despite few specified criteria.
- Both methods give regions that are not only connected but also **convex**.
- Processing data at the census tract level is feasible.
- Moment of Inertia method gives better results than our Diminishing Halves.
- Moment of Inertia derived from physics intuition.