Groovin' with the Big Band(width)

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Introduction

We have a planar surface divided into hexagonal cells with sides of length s. In **Figure 1**, all of the "first concentric" (striped) cells lie within $2s$ of the central cell, whereas all of the dotted cells and all the "second concentric" (dotted plus striped) cells lie within $4s$ of the central cell.

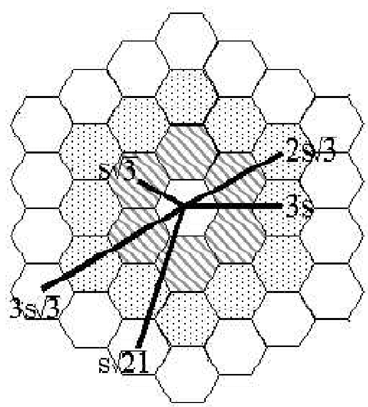

Figure 1. Diagram of interference regions.

The First Case

We analyze the case where transmitters within $4s$ must differ by at least 1 channel and those within $2s$ must differ by at least 2 channels. We show that the span of the network is 9 for both the finite grid in the problem statement and for the infinite plane; the answers to Requirements A and B are identical.

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We begin by considering the "first concentric," shown in **Figure 1** by the central cell and the ring of striped cells surrounding it. Since any cell of the first concentric is within $4s$ of all the others, each cell in it must be assigned a distinct integer; so the span cannot be 7 or less.

A more careful examination reveals that the span cannot be 8. Suppose that it were. Consider three adjacent hexagons that share a common vertex, we find that only one cell can be assigned a 1 and only one cell can be assigned an 8. Thus, the remaining cell must be assigned some numbers between 2 and 7. Consider this last cell as the center of a first concentric and assign it n. Then the $2s$ constraint dictates that the ring of 6 cells surrounding it cannot be assigned numbers $n-1$, n, or $n+1$. Their assignments must also be distinct from one another, since all cells within the first concentric are within $4s$ of each other. To make these cell assignments, we need six integers other than $n-1$, n, or $n+1$, or at least 9 numbers altogether; so the span must be at least 9.

Figure 2 shows a solution with 9, so the span is 9. The central column in gray is the sequence 1, 3, 5, 7, 9, 2, 4, 6, 8 repeated over and over. The column to the right of it (dotted) is the same sequence but shifted down 3 cells; the striped column to the left of center is the same sequence shifted up 3 cells. Repeat this process of shifting up or down indefinitely to the left and right. Look at each 1 in the pattern (in black). The column to the left of each 1 is always shifted up by 3, and the column to the right is always shifted down by 3. Therefore, each 1 must have the same neighbors. The cells within $2s$ of the 1s differ from it by at least 2, and those within $4s$ by at least 1; so the pattern meets the constraints. Checking the neighbors of the other numbers 2 through 9 shows that they meet the constraints also. This pattern can fill the grid supplied in the problem, or it can be extended arbitrarily far left and right and also up and down to cover the plane. This pattern is unique, not including rotations and reflections. [EDITOR'S NOTE: We omit the proof of this fact.]

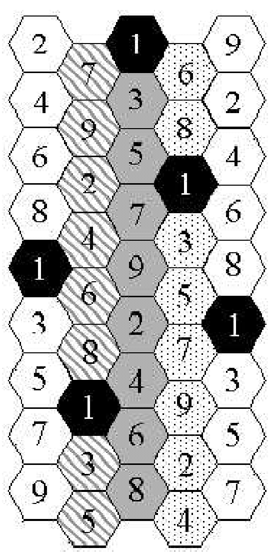

Figure 2. Solution with 9 channels.

Generalization: Differing k

In this section, we maintain the constraint that transmitters within a distance $4s$ of one another cannot use the same channel but generalize the second constraint, so that transmitters within a distance $2s$ of one another must have channels whose assignment numbers differ by k. The previous section treated the case $k = 2$.

We show that for $k = 1$, the span is 7; for $k = 3$, the span is 12; and for $k > 3$, the span is $2k + 7$.

First, we show that for all k, $2k + 5$ is a lower bound for the span. Suppose that we have a channel configuration that uses only 1 through $2k + 4$; this will lead to contradiction. Let $A = \{1, 2, \ldots, k\}$ and $B = \{k + 5, k + 2, \ldots, 2k + 4\}$. All numbers in A are within k of each other, as are all numbers in B. Consider three adjacent hexagons that share a common vertex. At most one of these three can be assigned an element of A and at most one can be assigned an element of B, so the third must be assigned some channel n between $k + 1$ and $k+4$. Consider a first concentric in which the central cell has been assigned this integer n. The $2s$ constraint dictates that the 6 adjoining cells cannot be assigned numbers $n-k+1, n-k+2, \ldots, n+k-2, n+k-1$. Their assignments must also be distinct from one another, since all cells within the first concentric are within $4s$ of each other. To make these cell assignments, we need six integers other than $n - k + 1$ through $n + k - 1$. This means that we need $6 + (2k - 1) = 2k + 5$ integers. Therefore, we cannot make proper channel assignments using only the integers 1 through $2k + 4$.

$k = 1$

When $k = 1$, the $2s$ constraint is subordinate to the $4s$ constraint. The span must be at least $2k + 5 = 7$. In fact, we can complete the grid using a span of exactly 7. As in **Figure 2**, the central column is a sequence of numbers repeated over and over, in this case the sequence 1, 2, 3, 4, 5, 6, 7. Also as in **Figure 2**, the adjacent column on the right contains the same sequence shifted down 3 cells, and the adjacent column on the left contains the same sequence shifted up 3 cells. For example, the 1 in the column to the right is between the 3 and 4 of the central column. As in the $k = 2$ constraint, every occurrence of each integer would have identical neighbors. Using this pattern, we can construct a satisfactory network. Moreover, since we have proved that the span must be greater than 6, our construction demonstrates that the span is exactly 7.

$k = 3$

We show that no assignment exists that uses only 1 through 11 and provide an example that works for 12, thereby demonstrating that the span is 12.

Assertion A: *The span must be greater than 11.*

Proof of A: By contradiction. Suppose that the span is 11. We show that several channel numbers cannot appear and use these facts for our final contradiction.

Case A1: Suppose that some transmitter is assigned channel 3. Consider a first concentric about a central cell assigned channel 3. No transmitters in the first concentric can use channels 1, 2, 3, 4, or 5, because they are all within $2s$ of the center transmitter operating on 3. We are left with six viable channels, 6, 7, 8, 9, 10, and 11, all of which must be used to provide distinct assignments to the cells surrounding the center cell. Clearly, channel 8 must be used somewhere in the first concentric. We must then use two of the five remaining channels (6, 7, 9, 10, 11) in two empty cells of the first concentric lying to either side of 8. However, this is not possible, since placing either 6, 7, 9, or 10 in either of these cells would violate the $2s$ requirement (as 6, 7, 9, 10 are all within 3 of 8). It follows that no transmitter can be assigned channel 3.

Case A2: Suppose that some transmitter is assigned channel 9. Since channels n and m within a distance $2s$ of each other must differ by at least k, we have that $|n - m| \geq k$. If we flip all channel numbers m to (span $+ 1 - m$), then $|(\text{span} - n) - (\text{span} - m)| = |m - n| = |n - m| \geq k$. Thus, the set of new channels functions identically under the $2s$ and $4s$ constraints. The channel numbers remain between $12 - 11 = 1$ and $12 - 1 = 11$, so we have a correct channel assignment, and the span remains 11. So if some transmitter is assigned channel 9, a flip produces a configuration with a channel $12 - 9 = 3$, which Case A1 shows is impossible. Therefore, no transmitter can be assigned channel 9.

Case A3: Assume that some transmitter is assigned channel 10. Consider the first concentric around a channel 10 (in gray), as shown in **Figure 3**.

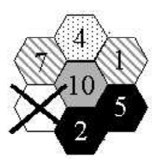

Figure 3. Case A3.

No transmitters in these cells can use the channels 8, 9, 10, 11, since the cells are within $2s$ of the center transmitter operating on 10; and none can be assigned channel 3, as we showed in Case A1. We are left with six usable channels, 1, 2, 4, 5, 6, 7, all of which we must use, since six distinct channels are required to fill the concentric. Channel 4 must be assigned to one of the cells, as in the dotted cell in **Figure 3**, and the striped cells neighboring it must contain channels 1 and 7. The $2s$ constraint requires that the cell with channel 5 can be adjacent only to the cells using channels 1 and 2, so the 5 and 2 must be added as shown in the figure (in black). However, we cannot assign channel 6 to the remaining cell, because that would violate the $2s$ requirement (since $7 - 6 = 1 < k$). It follows that no transmitter may be assigned channel 10.

We interrupt the flow of the argument to establish a claim that we need.

Claim: Any network of transmitters can be renumbered so that some transmitter operates on channel 1.

Proof: Suppose that there is a set of channels where no transmitter operates on channel 1. Let a be the smallest channel. As in Case 2, we renumber every channel m, this time as $m-a+1$. This new numbering preserves differences between channel assignments, so it still satisfies the difference constraints. Moreover, it contains channel 1 and all numbers in it are positive integers.

So we can assume that some transmitter is assigned channel 1. Consider the first concentric around this transmitter. No transmitters in these cells can use channels 1, 2, or 3, because the cells are within $2s$ of the transmitter operating on 1, and none can be assigned channels 9 or 10, as we showed in Cases A2 and A3. We are left with six usable channels, 4, 5, 6, 7, 8, 11, all of which we must use since six distinct channels are required to fill the first concentric. Channel 6 must be assigned to one of the cells, but there are not two numbers remaining in the list (4, 5, 7, 8, 11) that differ from 6 by more than $k = 3$. Therefore, it is impossible to complete the concentric in a way that satisfies the $2s$ constraint. This contradicts our supposition that we could assign channels using numbers between 1 and 11. Hence, when $k = 3$, the span must be greater than 11. □

Assertion B: *For $k = 3$, the span is 12.*

Proof B: The span is at least 12; we construct a solution that realizes 12. As was the case for $k = 1$ and $k = 2$, there exists a sequence of integers that, when applied in a series of adjacent, offset columns, produces a satisfactory network on an infinite plane (as in **Figure 2** for $k = 2$). The central column for $k = 3$ is the sequence 1, 8, 3, 10, 5, 12, 7, 2, 9, 4, 11, 6 repeated over and over. The adjacent column to one side is the same sequence shifted down 4, and the adjacent column to the other side is the sequence shifted up 4. As before, this pattern can be repeated indefinitely, and since the neighborhood of each number is exactly the same, the conditions are met. Therefore, for $k = 3$, the span is 12. □

$k > 3$

We prove that for $k > 3$, the span is $2k + 7$. We must first prove that no assignment with the channels 1 through $2k + 6$ satisfies the constraints. This proof requires a detailed analysis. [EDITOR'S NOTE: We omit the details of this analysis.] We must also show that there is a configuration of channels with span $2k + 7$ satisfying the constraints; **Figure 4** shows our solution. The same rhombus pattern is repeated over and over, tiling the plane (for example, the dotted, striped, and gray parallelograms are all identical copies). This rhombus consists of the numbers 1, 2, 3, $k+3$, $k+4$, $k+5$, $2k+5$, $2k+6$, and $2k+7$. As before, the neighborhood of each 1 is identical, and we can see that it satisfies the constraints, as do the neighborhoods of the other 8 cells. Therefore, $2k+7$ is the span for all $k > 3$. □

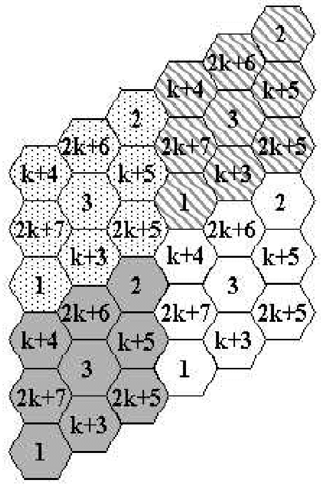

Figure 4. Configuration that tiles the plane to show that $2k+7$ channels will work.

More Generalizations

We generalize the $4s$ constraint to: Transmitters $4s$ apart must have channels m apart, $m \leq k$. While we do not determine the span in this general case, we deduce bounds on it: It must lie between $1+2k+4m$ and $1+2k+6m$.

First, suppose that we can make correct assignments using only channels $1, \ldots, 2k+4m$. Let $A = \{1, 2, \ldots, k\}$ and $B = \{k+4m+1, k+4m+2, \ldots, 2k+4m\}$. All numbers in A are within k of each other, as are all numbers in B. Consider three cells that share a common vertex. At most one of these three can be assigned an element of A, and at most one can be assigned an element of B; so the third must be assigned a channel n between $k+1$ and $k+4m$. Consider the first concentric about this central cell with channel n. We need 7 numbers to make enough assignments to fill this first concentric (including the central cell, n). We label these in increasing order: $x_1 < \cdots < x_7$.

Case 1: n one of x_2, x_3, x_4, x_5, x_6. Since all of these transmitters are within $4s$ of each other, each of the gaps between x_1 and x_2, between x_2 and x_3, and so on must contain at least $m-1$ numbers; two of these six gaps (the two around n), must contain at least $k-1$ numbers. Summing up the seven channels in the first concentric and the channels in the gap, we need $7+4(m-1)+2(k-1) = 1+2k+4m$ channels, which contradicts our earlier assumption that we could make the assignments using only $2k+4m$.

Case 2: $n = x_1$. This means that n is the smallest of the numbers. We still have one gap of size $k-1$ (between n and x_2), and the rest are of size $m-1$. Furthermore, since n is chosen so that it is at least $k+1$, there are k channels below it. Therefore, we need $k+7+1(k-1)+5(m-1) = 2k+5m > 2k+4m$

channels, which contradicts our assumption.

Case 3: $n = x_7$. This is the same as $n = x_1$, except that n was chosen to be at most $k + 4m$. Therefore, we need at least a span of $1 + 2k + 4m$ channels to make correct assignments.

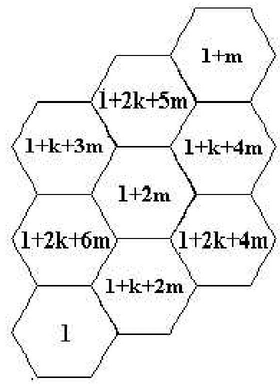

Figure 5. A rhombus that tiles the plane and uses only integers between 1 and $1 + 2k + 6m$.

A generalized network using only the integers between 1 and $1+2k+6m$ is shown in **Figure 5**. The situation is analogous to the $k > 3$ case discussed earlier. This time, we have a rhombus that tiles the plane, with channels assignments $1, 1+m, 1+2m, 1+k+2m, 1+k+3m, 1+k+4m, 1+2k+4m, 1+2k+5m$, and $1+2k+6m$. As before, we can check the neighborhood of each channel to make sure that it satisfies the constraints across all values of m and k.

To assess whether $1+2k+6m$ is a good upper bound for the span, consider how much smaller the span could be. There is a lower bound of $1 + 2k + 4m$, so our upper bound is not more than $2m$ greater than the span. Furthermore, for $m = 1$, we have $1 + 2k + 6m = 2k + 7$, which is exactly the span for $k > 3$.

Most important, the pattern we offer in this section provides a surprisingly efficient way to generate assignments for any sized grid, based on k and m: One need simply construct a rhombus of nine hexagons and tile the grid. In summary, though we have not proven that the span is $1 + 2k + 6m$, that expression appears to be a close approximation.

More Layers of Interference

We consider what happens if there are three levels of interference. We construct a method for deriving assignments that satisfy all the conditions:

- $2sk$ **constraint:** Channel assignments for transmitters within a distance of $2s$ of each other must differ by k.
- $4sm$ **constraint:** As above, but within a distance $4s$ they must differ by m.
- $6sn$ **constraint:** As above, but within a distance $6s$ they must differ by n.

We require that $n \leq m \leq k$.

We build up this assignment from a 2-level interference assignment. **Figure 6** shows a triangular lattice that results from drawing lines between centers of adjacent hexagons, while **Figure 7** shows a triangular lattice that connects only some of the cells. **Figure 7** looks identical to **Figure 6** but on a larger scale. In **Figure 7**, the dotted cells are within $4s$ of the central gray cell but more than $2s$ away, while the striped cells are within $6s$ of the central cell but more than $4s$ away.

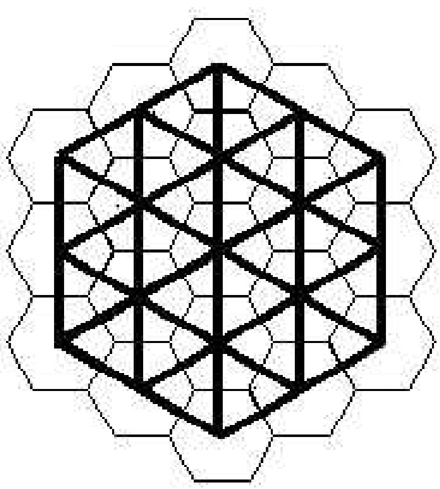

Figure 6. Triangular from drawing lines between centers of adjacent hexagons.

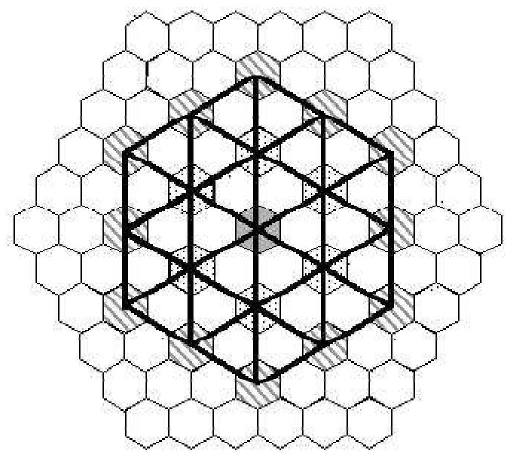

Figure 7. Triangular lattice that connects only some of the cells.

Suppose that an assignment satisfies both the $2sm$ and the $4sn$ constraints. If we assign these channels to the vertices of the lattice of **Figure 7**, they will meet the $4sm$ and $6sn$ constraints. **Figure 8** shows how we can overlap three lattices (light gray, dark gray, and black) such that every hexagon is centered on a vertex of one of the three lattices.

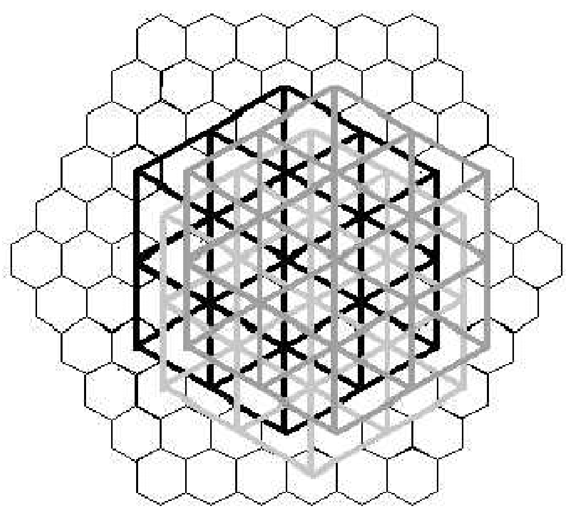

Figure 8. How the three lattices can be overlapped so that every hexagon is centered on a vertex of one of the three lattices.

We give the details. Suppose that the assignments on the light gray lattice use 1 through L and satisfy the $2s$ and $4s$ constraints. We label the cells on the dark gray lattice with $k+L$ to $k+2L-1$ (by simply adding $k+L-1$ to each channel, following the same assignment). We label the cells on the black lattice with $2k+2L-1$ to $2k+3L-2$ (by adding $2k+L-2$ to each channel).

Cells on different lattices have channels at least k apart. Cells on the same lattice are more than $2s$ apart; if they are less than $4s$ apart, their channels differ by m; and if they are less than $6ns$ apart, their channels differ by n. Thus, the assignment meets all the constraints, with maximum channel $2k+3L-2$.

Figure 9 gives a practical example of this method. Suppose that we seek a configuration for which channel assignments for transmitters within $2s$ of one another must differ by at least 3, those within $4s$ of one another must differ by at least 2, and those within $6s$ of one another must differ by at least 1. We use the configuration derived earlier (using the $2s2$ and $4s1$ constraints), which uses 9 integers. The gray vertices in **Figure 9** use the integers 1 through 9, the dotted vertices use 12 through 20, and the striped vertices use 23 through 31.

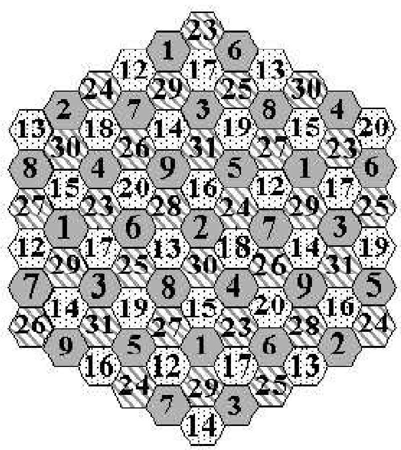

Figure 9. Example of labeling constructed to satisfy prescribed constraints.

We can apply this process to get an assignment configuration that satisfies the $2sk$, $4sm$, and $6sn$ constraints, for arbitrary k, m, and n. We have already shown that there exists an assignment configuration satisfying the $2sm$ and $4sn$ requirements whose maximum integer is $1+2m+6n$. Using the above method, we obtain a configuration that satisfies the $2sk$, $4sm$, and $6sm$ requirements, and its largest integer (substituting $1+2m+6n$ for L) is $2k+3(1+2m+6n)-2 = 1+2k+6m+18n$.

Does this method produces efficient configurations? That is, is the maximum integer that it obtains close to the actual span? While we have no proof, we suggest why it is an efficient method. We use the method to move from the 2-layer interference to the 3-layer interference, but we could have used it to move from 1 layer to 2 layers. So let's use this method to generate an assignment configuration with $2sk$ and $4s1$ constraints. We begin by finding the span when the only constraint is the $2s1$ constraint (i.e., that adjacent cells must have different channels). This is clearly accomplished by the sequence 1, 2, 3 repeated in a central column, shifted down two in the adjacent column to the right, and shifted up two in the adjacent column to the left.

If we use our method to construct a configuration with $2sk$ and $4s1$ constraints, its maximum channel assignment (substituting 3 for L) would be $2k+3\times3-2=2k+7$. This is the span for $k>3$. Therefore, our method generates a 2-layer interference from a 1-layer interference efficiently. It is reasonable, therefore, to expect that it also generates 3 layers from 2 fairly efficiently.

It is possible to expand to even higher layers of interference using our method. For example, in **Figure 10**, the striped dots are all between $9s$ and $6s$ of the gray cell. A 3-layer interference assignment configuration on the lattice gray, dotted, and striped cells can produce an assignment configuration on the whole grid, with constraints for $2s$, $4s$, $6s$, and $9s$.

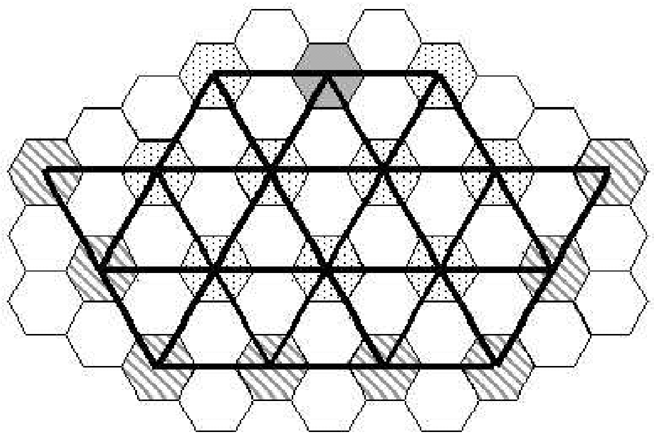

Figure 10. Example of labeling satisfying higher layers of interference.

Students Clamor for Bandwidth Optimization

WINSTON-SALEM, Feb. 30 — A team of three college students testified today before the Congressional Subcommittee on Bandwidth Regulation, revealed key research findings that may unleash a flood of proposed legislation designed to boost the efficiency of the information economy.

Several months ago, the Subcommittee issued a challenge to the world's mathematicians to find a method by which the United States can conserve its bandwidth efficiently. Yesterday, the three students, all from Wake Forest University here in Winston-Salem, stunned the world with their solution to how to assign radio channel frequencies. They found patterns of frequency assignments that maximize efficiency while maintaining the quality of the signals. The new discovery may be a breakthrough for frequency assignments for TV, radio, wireless modems, and cellular phones.

Furthermore, the students devised methods that could help the government determine the optimum number of channels for a given area, based on the likelihood of interference between channels. These models may have far-reaching implications, possibly affecting how the Federal Communication Commission assigns radio channels in the future.

Since its inception, the Congressional Subcommittee on Bandwidth Regulation has sought to make sure that all available parts of the spectrum are conserved for government, commercial, and private use. "We minimize interference by careful regulation of station licensing," commented Subcommittee Chairwoman Jane Doe (D–NY). "And this new discovery will help. Simply put, if we don't waste what we have, there will be more left over to sell, which could mean lower taxes."

In spite of the potential benefits of the discovery, there was some dissension about implementing policies based on it. "While I agree that the patterns that these kids have generated are quite beautiful from a mathematical standpoint," commented Sen. Laissez Fair (R–TX), "government regulation is not necessary in an industry that tends to regulate itself. After all, radio stations tend to space themselves out naturally."

Others at the hearing disagreed with Mr. Fair, noting that all popular stations seemed to have converged inexplicably to the high side of the FM band, leaving large portions of the FM spectrum unused ("except for that useless government-welfare National Public Radio down at the bottom of the band somewhere," retorted Mr. Fair). Ms. Doe responded that "The economic consequences are important. We are certainly going to recommend legislation to optimize channel assignments on portions of the bandwidth that are already in use. And future assignments should follow the patterns that these bright young mathematicians have discovered."

However, industry sources signaled that they are vehemently opposed to radio and TV stations (including satellite TV) being be forced to change frequencies to comply with any new efficiency standards. To this industry reaction, one of the student researchers commented, "Our goal isn't to force our model of efficiency on a market that has been functioning for decades. We're just trying to help the government plan for future expansion."

The Subcommittee on Bandwidth Regulation was initially formed in response to the public's concerns surrounding the notorious HDTV "bandwidth heist," which became a popular issue for the Bob Dole campaign in the 1996 presidential election. With widespread support from his party, Dole promised to auction off the new HDTV broadcast spectrum rather than give it away to TV networks interested in converting to HDTV. Sen. John McCain (R–AZ), a front-runner in this year's Republican presidential primary and Chairman of the Senate Commerce, Science and Transportation Committee, has estimated that such an auction would bring in over $70 billion that could help to reduce taxes. "After all, the public airwaves are owned by the American people and managed by our government," said a McCain spokesperson.