A Real Snow Job

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Summary

We were asked to design optimal routes for two plow-trucks to clear the county roads in a snow removal district of Wicomico County, MD. We felt that an optimal route would minimize the time spent driving over already-plowed roads and highways. We also felt that an optimal route would end where it had started.

Drawing analogies to graph theory, we found a subgraph, which was a tree, of the graph representation of the snow-removal district. Traversing this tree in preorder yielded an Euler circuit. This led us to a route where neither plow-truck ever drove over highways or county roads that had already been cleared. This route also enabled both plow-trucks to finish where they started. Therefore this route satisfies our definition of an optimal route.

The model we present has several strengths. First, it produces an optimal route for the two plow-trucks. Second, it is easily adaptable to changes in the snow-removal district. Third, it is easy to compensate for changes in the relative capabilities of the plow-trucks or the number of the plow-trucks. Lastly, it provides a simple algorithm for the plow-truck drivers to follow.

Assumptions

1. Plow-trucks do not break down or get stuck.
2. Neither intersections nor dead ends require special snow removal techniques.
3. Plow-trucks travel in the right lane of the road.
4. The plow-trucks enter the district on the roads directly east of the starred locations on the map.
5. Snowfall is uniform throughout the district.
6. Both plow-trucks are identical in their snow-removal capabilities.

7. Both plow-trucks travel at the same speed.
8. A plow-truck plows exactly one lane at a time.

Justification of Assumptions

1. The problem statement says that plow-trucks do not break down or get stuck.
2. The problem statement also says that intersections do not require special plowing techniques. It seems reasonable to include dead ends in this assumption, because there will be no traffic at the ends of roads.
3. Plow-trucks must obey traffic laws.
4. Since the plow-trucks begin 4 mi west of the starred locations on the map, off the map, we assume that the plow-trucks can reach the district in the shortest time on Dagsboro Road and Ocean City Road.
5. Because of the small size (approximately 5.6 mi by 12.9 mi) of the snow-removal district and the relatively large size of weather fronts, we assume that snowfall is uniform throughout the district.
6. Since the problem states nothing about the relative capabilities of the plow-trucks, we had to make some assumption. We assume equality to simplify the exposition of the model; however, the model can easily compensate for differences in the capabilities of plow-trucks.
7. In at least one state, all county roads have the same speed limit. There is nothing on the map which would indicate that this is not the case in Wicomico county. Therefore, assume that any plow-truck in the district will be able to travel at that maximum speed unless excessive snow makes this impossible. If there is excessive snow, it will affect all plow-trucks in the district in the same way, because we assume uniform snowfall (assumption (5) above).
8. It is unreasonable to assume that a plow-truck would plow less than one lane, because to do so would require the plow-truck to traverse each road more than twice. It would be unsafe for a plow-truck to plow more than one lane at a time, because doing this would be hazardous to oncoming traffic. We assume, therefore, that a plow-truck plows exactly one lane at a time.

Analysis and Design of the Model

The problem is to find the most efficient way for the two plow-trucks to clear the county roads. Our model measures the efficiency by the amount of snow that the plow-trucks are able to clear in a given period of time spent in the district. This time is affected only by the speed of the plow-trucks and the amount of time spent clearing county roads that have not yet been plowed. Thus, plow-trucks driving on highways and county roads that have already been cleared would reduce efficiency. Since the model cannot affect the speed of the plow-trucks, efficiency will be maximized when the ratio of time spent clearing county roads in the district to the total time spent in the snow-removal district is maximized. Therefore, the best possible result, which this model obtains, is for both plow-trucks to spend all of their time clearing roads not previously plowed.

In keeping with the desire to spend as much time as possible clearing new roads, we decided that turning around is preferable to driving on highways or already-cleared roads. It is beneficial to end the route where we enter the district, for two reasons. Primarily, it may be necessary to redo the route if snow has continued to fall. If not, the plow-trucks will need to return to their garages 4 mi west of the starred locations on the map.

Our model requires that the snow-removal district be divided into two areas, with one plow-truck assigned to each area. To accomplish this, the roads should be divided so that each plow-truck will have the same number of miles of roads to clear. The alternative, one plow-truck finishing before the other, requires more time to clear the entire district. Furthermore, both areas should be connected. If not, the plow-truck assigned to an area would need to travel on roads that the other plow-truck would be clearing anyway.

To split the district evenly, we measured the distances of all the county road segments. We enlarged the provided map with a photocopier and scaled distances from the enlargement. Our method was to conform a pliable string to the shape of a map road segment; the length of the string when taut determined the map length of the road segment. The map length was then scaled to give the length of the road segment. Having determined the lengths of all county road segments, we then split the district into two connected areas, with one having 0.06 more miles of county roads than the other (see **Figure 1**), a negligible difference compared to the 125.29 total miles of road.

The problem is now reduced to finding an optimal route for each plow-truck to cover its area of the district. To solve this problem, it was helpful to view the map in the context of graph theory. Each area was represented by a directed graph, with each intersection and dead end a vertex, and each lane an edge. Both graphs are connected, since the roads in the snow district are connected and each of the areas are connected. An optimal route for a plow-truck becomes an Euler circuit for the graph of that plow-truck's area.

If one of the graphs, $\mathcal{A}$, has a tree structure, finding an Euler circuit would be simple. The associated undirected graph of $\mathcal{A}$ would be a rooted tree, $\mathcal{G}$. A simple Euler circuit for $\mathcal{A}$ could then be obtained by following the preorder traversal [Grimaldi 1989, 487] of $\mathcal{G}$ and then returning to the root of $\mathcal{G}$.

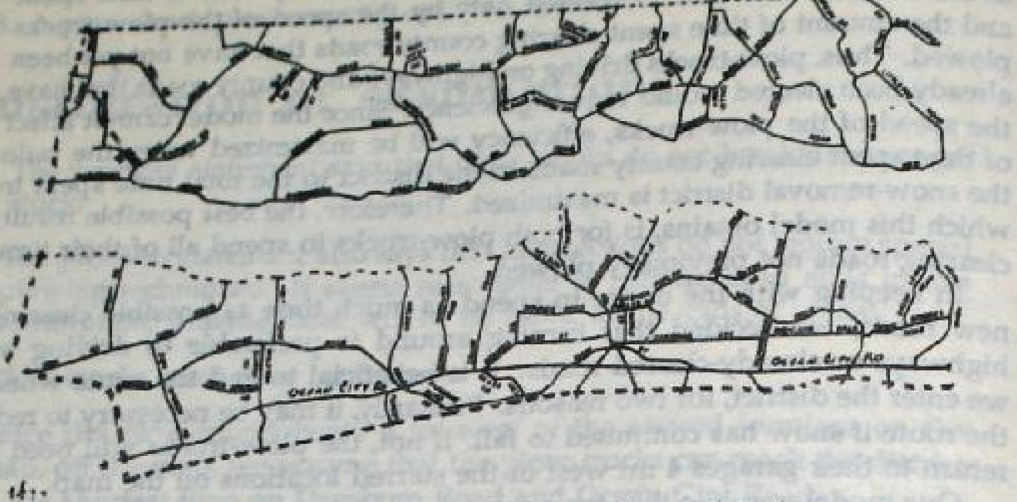

Figure 1. The snow-removal district split into northern and southern areas.

If A is not a tree graph, it has a subgraph that includes all the vertices of A and is a directed tree graph. To find a subgraph, S, of this form, one edge is eliminated from each loop until there no loops remain. Since only edges from loops have been removed, S will still be connected; therefore, S is a tree. **Figure 2** shows an example of a graph A and such a subgraph S.

Let T be the undirected tree associated with S. For example, the graph in **Figure 3a** is the undirected tree associated with **Figure 2b**. The preorder traversal of **Figure 3a** is I, F, G, H, E, A, B, C, and D. The reader should note, for example, that to travel from I to F, it is necessary to pass through, D, E, H, and G.

Let T' denote the pseudotree formed by adding to T edges that were in A but not in S. These edges will be added such that they are branches only of the first of their vertices to appear in the preorder. It is also necessary to require that these new edges be added only to original vertices of T. To finish constructing T', each edge is made into two directed edges between the two vertices, so that T' will be a directed graph. **Figure 3b** shows the directed pseudotree for **Figure 3a**.

T' may be traversed by following the preorder of T and agreeing that any edge in T' connecting vertices that are not connected in T will be traversed at the first opportunity, to be followed immediately by returning on the other edge connecting those two vertices (see **Figure 4**).

Call this completed circuit $\mathcal{P}$. Thus $\mathcal{P}$ is also an Euler circuit for A, because there is a one-to-one correspondence between the edges in A and the edges in T'. Thus every edge in A is traversed the one time it is traversed in T' by following $\mathcal{P}$.

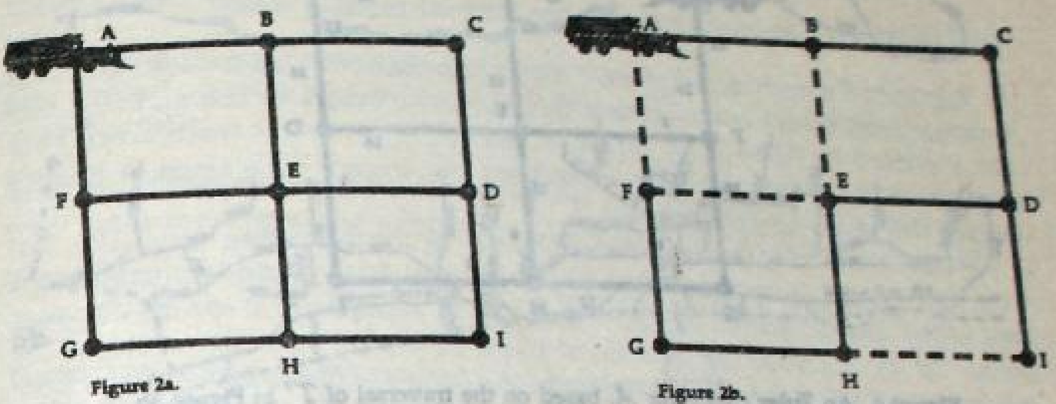

Figure 2a. A graph $\mathcal{A}$. **Figure 2b.** A subgraph $\mathcal{S}$ of $\mathcal{A}$ that includes all the vertices of $\mathcal{A}$ and is a tree.

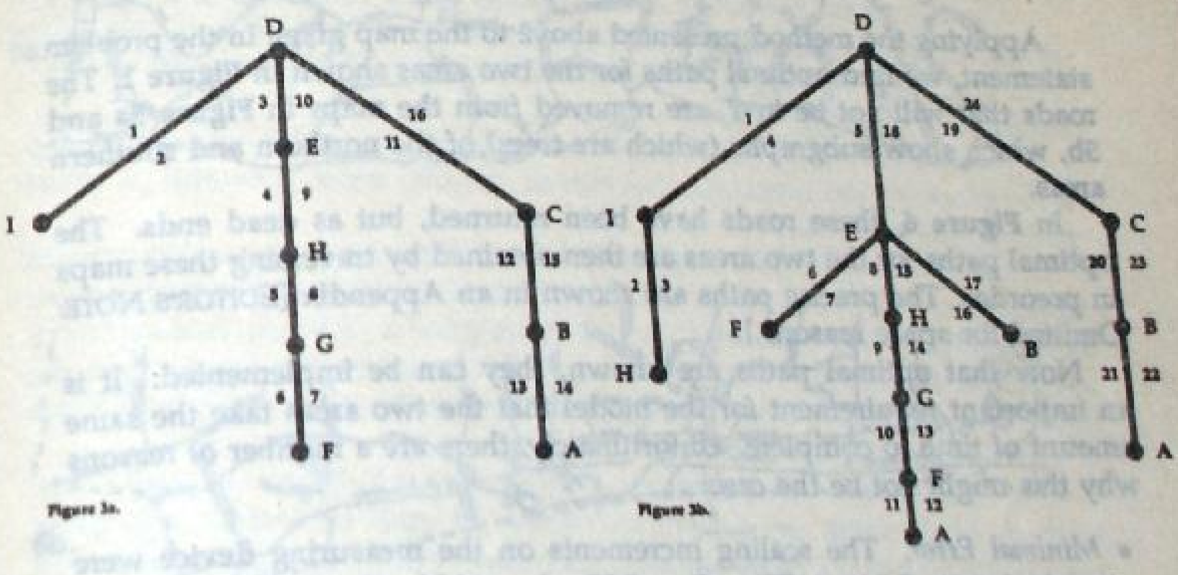

Figure 3a. An undirected tree $\mathcal{T}$ associated with the tree-subgraph $\mathcal{S}$ of Figure 2b, with preorder traversal indicated. **Figure 3b.** A directed pseudotree $\mathcal{T}'$ for the undirected tree $\mathcal{T}$ of Figure 3a, formed by adding edges that were in $\mathcal{A}$ but not in $\mathcal{S}$; the numerals denote the preorder traversal that extends the preorder traversal of $\mathcal{T}$.

Figure 4. An Euler circuit for $\mathcal{A}$, based on the traversal of $\mathcal{T}'$ in Figure 3b.

Applying the Model and Analyzing Errors

Applying the method presented above to the map given in the problem statement, we find optimal paths for the two areas shown in **Figure 1**. The roads that will not be in $\mathcal{T}$ are removed from the maps in **Figure 5a** and **5b**, which show subgraphs (which are trees) of the northern and southern areas.

In **Figure 6**, these roads have been returned, but as dead ends. The optimal paths for the two areas are then obtained by traversing these maps in preorder. The precise paths are shown in an Appendix [EDITOR'S NOTE: Omitted for space reasons.].

Now that optimal paths are known, they can be implemented. It is an important requirement for the model that the two areas take the same amount of time to complete. Unfortunately, there are a number of reasons why this might not be the case:

- *Minimal Error.* The scaling increments on the measuring device were equivalent to 0.04 mi. Therefore, 0.02 mi would be the minimal error on any measurement.
- *Errors in Measuring Roads.* The map length of the road was approximated with pliable string. This string was then measured with the measuring device, as was the scale of the enlargement. The inaccuracies involved make it is doubtful that the minimal error was achieved.

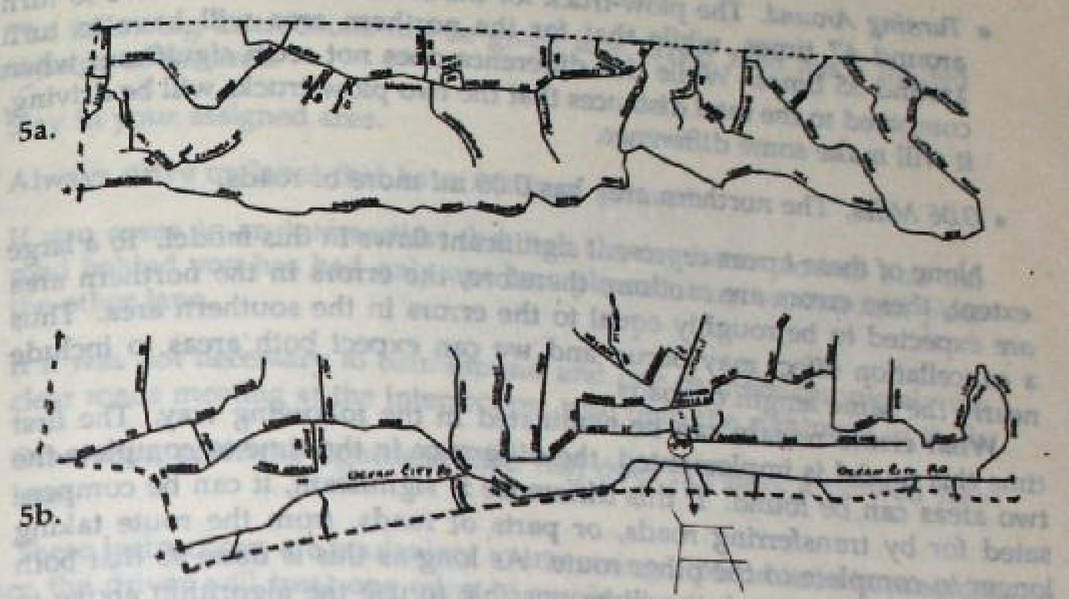

Figure 5. Tree-subgraphs for the northern and southern areas, with roads removed that will not be in the pseudotrees for the areas.

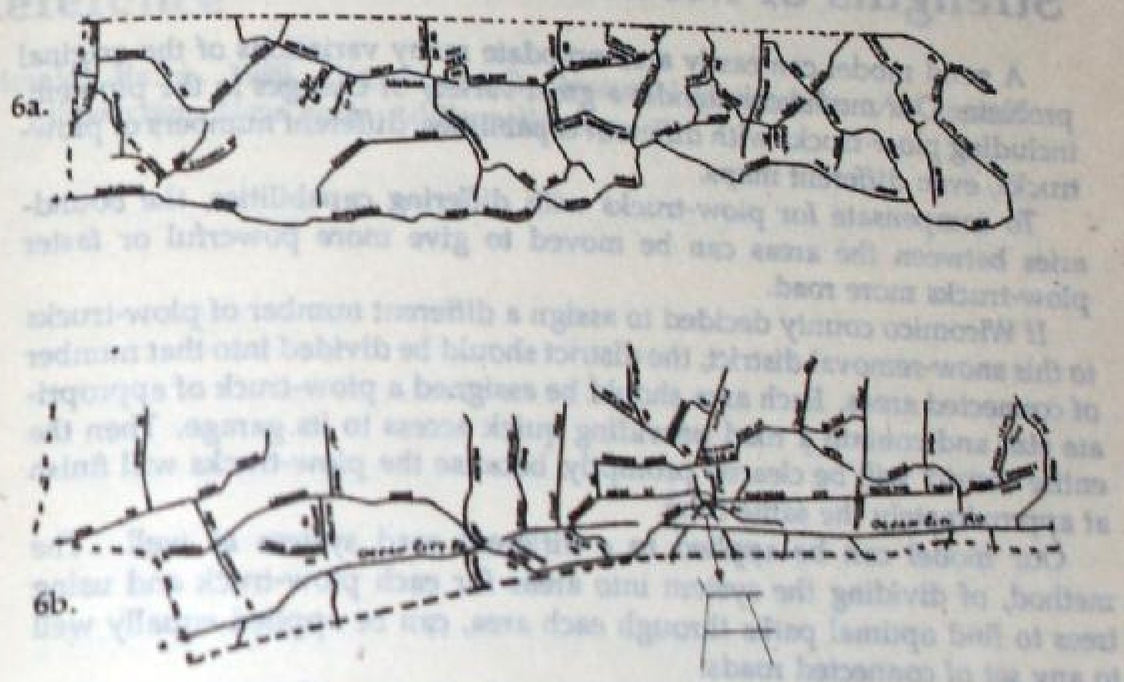

Figure 6. Subgraphs for the northern and southern areas, with roads not in their pseudotrees reincluded as dead ends.

- *Turning Around.* The plow-truck for the southern area will have to turn around 47 times, while that for the northern area will have to turn around 35 times. While this difference does not seem significant when compared to the total distances that the two plow-trucks will be driving, it will make some difference.
- *0.06 Miles.* The northern area has 0.06 mi more of roads.

None of these errors represent significant flaws in this model. To a large extent, these errors are random; therefore, the errors in the northern area are expected to be roughly equal to the errors in the southern area. Thus a cancellation effect may occur, and we can expect both areas to include nearly the same length of roads.

What errors remain can be eliminated in the following way. The first time this model is implemented, the difference in the time to complete the two areas can be found. If this difference is significant, it can be compensated for by transferring roads, or parts of roads, from the route taking longer to complete to the other route. As long as this is done so that both routes remain connected, it will be possible to use the algorithm above to produce two new optimal routes.

Strengths of Model

A good model can easily accommodate many variations of the original problem. Our model can handle a great variety of changes in the problem, including plow-trucks with different capabilities, different numbers of plow-trucks, even different maps.

To compensate for plow-trucks with differing capabilities, the boundaries between the areas can be moved to give more powerful or faster plow-trucks more road.

If Wicomico county decided to assign a different number of plow-trucks to this snow-removal district, the district should be divided into that number of connected areas. Each area should be assigned a plow-truck of appropriate size and contain a road providing quick access to its garage. Then the entire district will be cleared promptly, because the plow-trucks will finish at approximately the same time.

Our model can be applied to a different road system as well. The method, of dividing the system into areas for each plow-truck and using trees to find optimal paths through each area, can be applied equally well to any set of connected roads.

Furthermore, the model has several inherent strengths that arise from the underlying graph-theoretic structure. The drivers can follow simple instructions to complete optimal paths covering their routes; the plow-trucks finish their routes near their respective garages; a minimum amount of fuel is used, as there is no unnecessary driving.

Driver's Instructions

The following instructions can be given to a driver who has a limited knowledge of the area:

1. Stay in your assigned area.
2. Always drive on lanes that have not been plowed.
3. If you come to an intersection that has already been plowed and the road behind you has had only one lane plowed, turn around and plow the other lane.
4. If it was not necessary to turn around and one or more completely unclear roads meeting at the intersection, turn onto the rightmost of these.
5. Otherwise, plow the rightmost road that does not have a plowed right lane.

These instructions are equivalent to traversing a pseudotree in preorder, since the driver will treat one edge of any loop as an edge unique to the pseudotree, by turning around at the end of the road representing that edge and then clearing the other lane of that road.

Reference

Grimaldi, Ralph. 1989. *Discrete and Combinatorial Mathematics: An Applied Introduction*. New York: Addison-Wesley.