A Time-Independent Model of Box Safety for Stunt Motorcyclists

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Abstract

We develop a knowledge of the workings of corrugated fiberboard and create an extensive time-independent model of motorcycle collision with one box, our Single-Box Model. We identify important factors in box-to-box and frictional interactions, as well as several extensions of the Single-Box Model.

Taking into account such effects as cracking, buckling, and buckling under other boxes, we use the energy-dependent Dual-Impact Model to show that the "pyramid" configuration of large 90-cm cubic boxes—a configuration of boxes in which every box is resting equally upon four others—is optimal for absorption of the most energy while maintaining a reasonable deceleration. We show how variations in height and weight affect the model and calculate a bound on the number of boxes needed.

General Assumptions

- The temperature and weather are assumed to be "ideal conditions"—they do not affect the strength of the box.
- The wind is negligible, because the combined weight of the motorcycle and the person is sufficiently large.
- The ground on which the boxes are arranged is a rigid flat surface that can take any level of force.
- All boxes are cubic, which makes for the greatest strength [Urbanik 1997].

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Table 1.

Variables.

Variable	Definition	Units
l	Box edge length	cm
$V(t)$	Velocity as a function of time (v_0 is velocity of impact)	cm/s
A_T	Energy change due to the top of the box	kg-cm^2/s^2
A_B	Energy change due to the buckling of the box	kg-cm^2/s^2
A_{NET}	Total energy including gravity for top	kg-cm^2/s^2
$A_{\text{CD-EDGE}}$	Energy absorbed by CD springs in modelled buckling	kg-cm^2/s^2
$A_{\text{MD-EDGE}}$	Energy absorbed by MD springs in modelled buckling	kg-cm^2/s^2
x_{NET}	Total depression of the top	cm
Δx	Change in a distance	cm
x_{DOWN}	Component of edge's depression in the z-direction	cm
$x(t)$	Downward displacement of the top of the box ($x_0 = 0$)	cm
x_F	Final depression before top failure.	cm
δL	Offset from top center.	cm
M	Motorcycle and stuntman combined mass	kg
P_{CD}	Tensile strength in the cross-machine direction	kg/(s^{2}cm)
P_{MD}	Tensile strength in the machine direction	kg/(s^{2}cm)
P_{ECT}	Maximum strength as measured with the Edge Crush Test	kg/s^2
P_{ML}	Maximum strength as measured with the Mullen Test	kg/(s^{2}cm)
F_{CD}	Force in the cross-machine direction	kg-cm/s^2
F_{CDMAX}	Maximum force in the cross-machine direction	kg-cm/s^2
F_{MD}	Force in the machine direction	kg-cm/s^2
F_{MDMAX}	Maximum force in the machine direction	kg-cm/s^2
F_{UP}	Net dampening force the box exerts on the motorcycle	kg-cm/s^2
F_{UPMAX}	Maximum dampening force the box exerts on the motorcycle	kg-cm/s^2
F_{ECT}	Force box exerts on the frame	kg-cm/s^2
F_{ECTMAX}	Maximum force the box exerts on the frame before yielding	kg-cm/s^2
F_{NET}	Total force	kg-cm/s^2
x_{MD}	Depression at which MD tensile strength is exceeded	cm
x_{CD}	Depression at which CD tensile strength is exceeded	cm
x_{ECT}	Depression at which a buckle occurs	cm
x_{ML}	Depression at which a puncture occurs	cm
V_x	Velocity in the x-direction	cm/s
V_y	Velocity in the y-direction	cm/s
V_{ix}	Initial velocity in the x-direction	cm/s
V_{iy}	Initial velocity in the y-direction	cm/s
V_{fx}	Final velocity in the x-direction	cm/s
V_{fy}	Final velocity in the y-direction	cm/s
M_B	Mass of boxes displaced	kg
t	Time	s
A_x	Energy in the x-direction	kg-cm^2/s^2
A_y	Energy in the y-direction	kg-cm^2/s^2
A_z	Energy in the z-direction	kg-cm^2/s^2
d	Distance	cm

Table 2.
Constants.

Variable	Definition	Value
E_{MD}	Young's modulus in the machine direction	3000000 kg/(s^2-cm)
E_{CD}	Young's modulus in the cross-machine direction	800000 kg/(s^2-cm)
E	Sum of E_{MD} and E_{CD}	3800000 kg/(s^2-cm)
L_{MD}	Tire rect. length in the machine direction	7cm front, 10cm back
L_{CD}	Tire rect. length in the cross-machine direction	10 cm
P_{CD}	Tensile strength in the cross-Machine direction	2000 kg/(s^2-cm)
P_{MD}	Tensile strength in the machine direction	2500 kg/(s^2-cm)
P_{ECT}	Max. strength as measured with the edge crush test	10000 kg/s^2
P_{ML}	Max. strength as measured with the Mullen test	25000 kg/(s^2-cm)
g	Gravitational Constant	980 cm/s^2
μ	Coefficient of kinetic friction	0.4
w	Cardboard thickness	0.5 cm
δv	Speed variation	200 cm/s
$\delta\phi$	Angle variation away from y-axis leaving the ramp	$\pi/36$
M	Mass of rider and motorcycle	300 kg
v_i	Initial velocity leaving ramp	1500 cm/s
θ	Ramp angle of elevation	$\pi/6$

Definitions and Key Terms

- **Buckling** is the process by which a stiff plane develops a crack due to a stress exceeding the yield stress.
- **Compressive strength** is the maximum force per unit area that a material can withstand, under compression, prior to yielding.
- **Corrugation** is the style found in cardboard of sinusoidal waves of liner paper sandwiched between inner and outer papers. We use boxes with the most common corrugation, *C-flute corrugation* (see below for definition of a flute).
- **Cracking** is a resulting state when the tensile force exceeds the tensile strength.
- **Cross-machine direction** is the direction perpendicular to the sinusoidal wave of the corrugation.
- **Depression** is when, due to a force, a section of a side or edge moves downwards.
- **ECT** is the acronym for a common test of strength, the *edge crush test*.
- **Fiberboard** is the formal name for cardboard.
- **Flute** is a single wavelength of a sinusoidal wave between the inner and outer portion papers that extends throughout the length of the cardboard. A C-flute has a height of 0.35 cm and there are 138 flutes per meter [Packaging glossary n.d.].

- **Machine direction** is parallel to the direction of the sinusoidal waves.
- **Motorcycle** We use the 1999 BMW R1200C model motorcycle for structural information; it has a 7.0-cm front wheel and a 10-cm back wheel, with radii 46 and 38 cm. It is 3 m long and has 17 cm ground clearance. The weight is 220 kg (dry) and 257 kg (fueled) [Motorcycle specs n.d.].
- **Mullen test** is a common measurement of the maximum force that a piece of cardboard can stand before bursting or puncturing.
- **Puncturing** is when a force causes an area to burst through the cardboard surface.
- **Pyramid configuration** is a configuration of boxes in which each box is resting equally on four others.
- **Strain** is the dimensionless ratio of elongation to entire length.
- **Stress** is the force per unit area to which a material is subject.
- **Tensile strength** is the maximum force per unit area which a material can withstand while under tension prior to yielding.
- **Young's modulus of elasticity** is the value of stress divided by strain and relates to the ability of a material to stretch.

Developing the Single-Box Model

Expectations

- Given sufficiently small impact area, the surface plane either punctures or cracks before the frame buckles.
- Given sufficiently large impact area, the frame should buckle and no puncturing occurs.
- During buckling, corners are more resistive to crushing than edges.

Preliminary Assumptions

- The force is exerted at or around the center of the top.
- The top of the box faces upward. This assumption allows us to use ECT (Edge Crush Test) results. This orientation also ensures that the flutes along the side are oriented perpendicular to the ground, so they serve as columns.

The Conceptual Single-Box Model

During the puncturing or cracking of the top of the box:

- The frame stays rigid.
- The cardboard can be modeled as two springs with spring constants equal to the length of board times its modulus of elasticity [Urbanik 1999] (**Figure 1**).

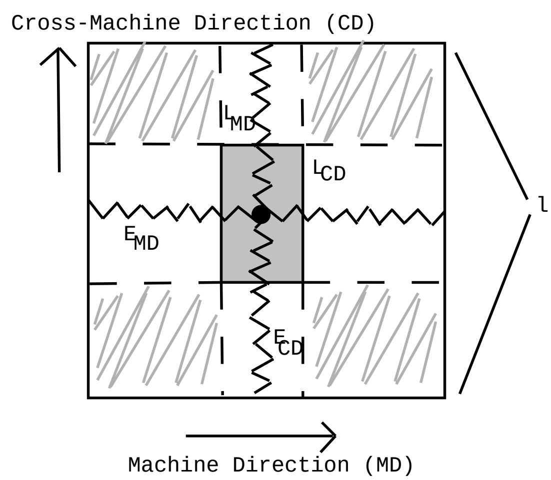

Figure 1. Our model for a tire hitting the top of one box. We treat the portions of the box directly vertical or horizontal from the edges of the motorcycle tire (the rectangle in the middle) as ideal springs, neglecting the effect of the rest of the box.

- The surface of the motorcycle tire that strikes the box is approximated as a rectangle with dimensions L_{MD} (the wheel's width) and L_{CD} (the length of the wheel in contact with the cardboard surface). We neglect the spin and tread of the tire.
- The part of the spring beneath the tire does not undergo any tension. In reality, this is not the case; but with this assumption, cracking and puncturing occur along the edges of the tire. The force still comes from the rigid frame, and the springs have the same constant; therefore, we believe this assumption affects only the position of the cracking and puncturing, not when it occurs or how much energy is dissipated.
- There is no torque on the box during this first process. This assumption can be made since the force is at the center of the top.

The top of the cardboard box can fail in several ways:

- If the resistive upwards force from the top, F_{UP}, exceeds $P_{\mathrm{ML}} \cdot L_{\mathrm{MD}} \cdot L_{\mathrm{CD}}$ (the Mullen maximum allowable pressure over this area), then puncturing occurs.

- If the force F_{MD} on the machine direction spring exceeds $P_{\text{MD}} \cdot w \cdot (3L_{\text{CD}} - l)/2$ (the tensile strength in the machine direction times cross-sectional area perpendicular to the force), then a crack occurs in the cross-machine direction. We assume that this force that causes cracking is evenly distributed over a section larger than the actual edge of the tire rectangle, because of the solid nature of cardboard.
- If the force F_{CD} on the cross machine direction spring exceeds $P_{\text{CD}} \cdot w \cdot (3L_{\text{MD}} - l)/2$, then a crack occurs in the machine direction, for the same reasons as above.
- If the force on the edges exceeds $P_{\text{ECT}} \cdot 4l$ (the compression strength of the edges times the total edge length), then buckling occurs first. Here we assume that even though we model the top as two springs, the force is evenly distributed over the edges, taking advantage of the high spring constant and solid behavior of cardboard.

There are several ways in which boxes can fail:

- Puncture: The wheel enlarges the hole, and only when the hull of the motorcycle hits the edge does buckling occur.
- Crack: The tire does not break through the material, and buckling eventually occurs.
- Buckling: Can occur without a puncture or crack first.

We assume that at most one puncture or crack occurs per box, followed inevitably by buckling.

Calculations for the Box Top

[EDITOR'S NOTE: We do not give all the details of the following calculations.] Refer to **Figure 2**. We solve for F_{CD}, getting

$$F_{\text{CD}} = E_{\text{CD}} L_{\text{MD}} \left(\sqrt{\left(\frac{l - L_{\text{CD}}}{2} \right)^2 + [x(t)]^2} - \frac{l - L_{\text{CD}}}{2} \right). \tag{1}$$

We apply the results to solve for F_{MD} and combine them to get F_{UP}. After obtaining the vertical component of F_{CD},

$$x(t) E_{CD} L_{MD} \left(\frac{\sqrt{\left(\frac{l - L_{CD}}{2}\right)^2 + [x(t)]^2} - \frac{l - L_{CD}}{2}}{\sqrt{\left(\frac{l - L_{CD}}{2}\right)^2 + [x(t)]^2}} \right),$$

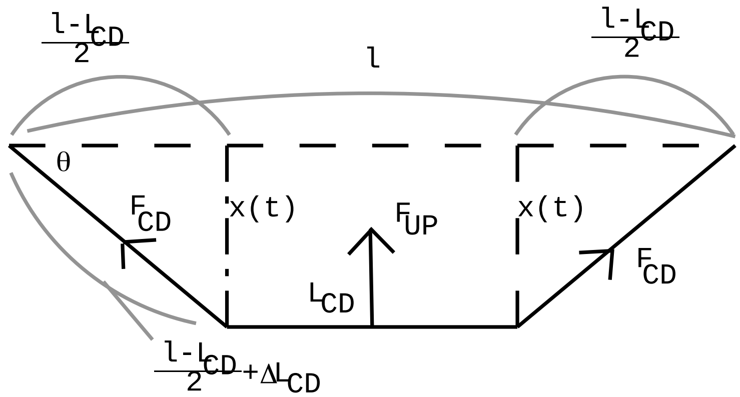

Figure 2. Side view of the depression of the motorcycle tire into the top of the box.

and the analogue by symmetry for F_{MD}, we have F_{UP}, namely:

$$F_{\mathrm{UP}} = x(t)\left[2E_{\mathrm{CD}}L_{\mathrm{MD}}\left(1-\frac{\frac{l-L_{\mathrm{CD}}}{2}}{\sqrt{\left(\frac{l-L_{CD}}{2}\right)^2+[x(t)]^2}}\right)+\right.$$
$$\left.2E_{\mathrm{MD}}L_{\mathrm{MD}}\left(1-\frac{\frac{l-L_{\mathrm{MD}}}{2}}{\sqrt{\left(\frac{l-L_{\mathrm{CD}}}{2}\right)^2+[x(t)]^2}}\right)\right]. \quad \textbf{(2)}$$

The force F_{UP} is the resistive force that the top exerts on the motorcycle's wheel. Balancing the force and taking into effect gravity (in the form of the normal force), we get the force equation of the motion of the motorcycle's wheel on the box prior to puncture, crack, or buckle:

$$F_{\mathrm{NET}} = F_{\mathrm{UP}} + mg.$$

We use this expression to calculate the energy as a function of depression into the box. We use our initial force calculation to determine the level of depression and the type of failure that the top incurs. This depression is the minimum depression for which any failure occurs.

If the force F_{CD} on the cross-machine direction spring (contributed by both sides of the spring) exceeds $2P_{CD}L_{MD}w$, then a crack occurs in the machine direction. Solving for the depression, we get

$$x_{\mathrm{CD}} = \sqrt{\left(\frac{P_{\mathrm{CD}}w}{E_{\mathrm{CD}}}\right)^2 + \frac{P_{\mathrm{CD}}w}{E_{\mathrm{CD}}}\left(l-L_{\mathrm{CD}}\right)}$$

Likewise, if the force F_{MD} on the machine direction spring (contributed by both sides of the spring) exceeds $2P_{\mathrm{MD}}L_{\mathrm{CD}}w$, then a crack occurs in the cross-machine direction, with the analogous formula for the depression.

If the resistive upwards force F_{UP} from the top exceeds $P_{ML}L_{\text{MD}}L_{\text{CD}}$, puncturing occurs. Similarly, if the net force on the edges, $2F_{\text{CD}} + 2F_{\text{MD}}$, exceeds $4P_{ECT}l$, buckling occurs first. We find the respective depressions in the next section.

We can use the x-position in energy calculations to determine the new speed of the motorcycle.

Energy is a distance integral of net force, so using **(2)** we can find energy A_T absorbed by the top:

$$A_T(x) = \int_0^x F_{NET}ds = E_{\text{CD}}L_{\text{MD}}\left(x^2 - (l - L_{\text{CD}})\sqrt{\left(\tfrac{l-L_{\text{CD}}}{2}\right)^2 + x^2} + \tfrac{(l-L_{\text{CD}})^2}{2}\right)$$
$$+E_{\text{MD}}L_{\text{CD}}\left(x^2 - (l - L_{\text{MD}})\sqrt{\left(\tfrac{l-L_{\text{MD}}}{2}\right)^2 + x^2} + \tfrac{(l-L_{\text{MD}})^2}{2}\right) + mgx.$$

Extensions of Top Model

Testing the model shows that not all the force and dissipative energy comes from the top of the box springs. We make the following further assumptions:

- Since the deflection is small compared to the edge lengths, $x \ll l - L_{\text{CM}}/2$ and $x \ll l/2$.
- The boxes have two layers coming together at the top that are corrugated in different directions, hence the cardboard on the top really is of width $2w$.
- Since the flutes in one top piece are perpendicular to the flutes of the other, the resulting combined modulus of elasticity is the sum of the two original values in both directions. This means that we can define $E = E_{\text{MD}} + E_{\text{CD}}$ and modify all equations accordingly.

The equations for the forces F_{CD} **(1)** and F_{MD} are of the form

$$f(x) = k(a^2 + x^2)^{1/2} - a,$$

with $a = (l - L_{\text{CD}})/2$. For small deflections x, such a function can be approximated well by its second-degree Taylor expansion around $x = 0$:

$$f(x) \approx \frac{x^2}{2a}.$$

The resulting equations are

$$F_{\text{CD}} = 2EL_{\text{MD}}\left(\frac{x^2}{l - L_{\text{CD}}}\right), \qquad F_{\text{MD}} = 2EL_{\text{CD}}\left(\frac{x^2}{l - L_{\text{MD}}}\right),$$
$$F_{\text{UP}} = x(t)\left(2EL_{\text{MD}}\frac{x(t)}{l - L_{\text{CD}}} + 2EL_{\text{CD}}\frac{x(t)}{l - L_{\text{MD}}}\right)$$

We deal briefly with the position of the rectangle on the box. The ECT deflection should not depend on the position of the rectangle, since ECT depends only on the net force. The other three deflections become less as the force moves away from the center, so we can model them with a standard linear decrease factor of $1 - \delta L(l/2)$, where δL is the distance radially from the center.

Now we solve equations for x, taking the δ factor into account, as well as $F_{\text{ECT}} = 2(F_{\text{CD}} + F_{\text{MD}})$, getting

$$x_{\text{CD}} = \left(1 - \tfrac{2\delta L}{l}\right)\sqrt{\tfrac{F_{\text{CDMAX}}(l-L_{\text{CD}})}{2EL_{\text{MD}}}}, \qquad x_{\text{MD}} = \left(1 - \tfrac{2\delta L}{l}\right)\sqrt{\tfrac{F_{\text{MDMAX}}(l-L_{\text{MD}})}{2EL_{\text{CD}}}},$$

$$x_{\text{ML}} = \left(1 - \frac{2\delta L}{l}\right)\sqrt{\frac{F_{\text{UPMAX}}}{\left(\frac{2EL_{\text{MD}}}{l-L_{\text{CD}}} + \frac{2EL_{\text{CD}}}{l-L_{\text{MD}}}\right)}}, \qquad x_{\text{ECT}} = \sqrt{\frac{F_{\text{ECTMAX}}}{\left(\frac{2EL_{\text{MD}}}{l-L_{\text{CD}}} + \frac{2EL_{\text{CD}}}{l-L_{\text{MD}}}\right)}}.$$

To obtain energy, we integrate the Taylor-expanded version of $F_{\text{UP}} + mg$:

$$A_T(x) = 2E\,\frac{x_F^3}{3}\left(\frac{L_{\text{MD}}}{l - L_{\text{CD}}} + \frac{L_{\text{CD}}}{l - L_{\text{MD}}}\right) + mgx_F,$$

where $x_F = \min\{x_{\text{CD}}, x_{\text{MD}}, x_{\text{ML}}, x_{\text{ECT}}\}$.

Now we can substitute the maximum values of the respective forces and solve for the x-values. We use the same values of $F_{\text{UP}} = P_{\text{ML}}L_{\text{MD}}L_{\text{CD}}$ and $F_{\text{ECT}} = 4P_{\text{ECT}}l$. For the cracking forces, we double their values to take into account both springs on the side of the tire. So this gives

$$F_{\text{CD}} = 2(P_{\text{CD}} + P_{\text{MD}})wL_{\text{CD}}, \qquad F_{\text{MD}} = 2(P_{\text{CD}} + P_{\text{MD}})wL_{\text{MD}}.$$

We make one last change. Energy is absorbed not only from the elasticity of the top but also from that of the edges. We determine the average force on this edge spring and the change in the depression x that occurs for this spring. It is reasonable to use average force, because this is over a small distance, and a parabolic function is reasonably linear in such an interval. Using modulus of elasticity E and first-order Taylor approximations, we estimate the displacement and energy absorbed.

We also assume that the edge does not fail before the top. The is reasonable because an edge has greater structural integrity.

So, assume that we are dealing with edge on which F_{CD} acts. To find the average force, integrate F_{CD} with respect to distance and divide by distance. Denote by x_F the depression at which failure occurs in our original top model. The average force is

$$\overline{F_{\text{CD}}} = EL_{\text{MD}}\left(\frac{x_F^2}{3(l - L_{\text{CD}})}\right).$$

To solve for Δx, observe that if the force is centered at the edge, then

$$\Delta x = \sqrt{\left(\frac{l}{2}\right)^2 + x_{\text{CD-EDGE}}^2} - \frac{l}{2}.$$

Taking a first-order Taylor approximation gives $\Delta x = x^2_{\text{CD-EDGE}}/l$. Using Hooke's law and $\overline{F_{\text{CD}}} = E\Delta x$, we solve for the depression of this spring $x_{\text{CD-EDGE}}$:

$$x_{\text{CD-EDGE}} = \sqrt{L_{\text{MD}} l \left(\frac{x_F^2}{3(l - L_{\text{CD}})} \right)}.$$

The corresponding energy is

$$A_{\text{CD}-EDGE} = \overline{F_{\text{CD}}}\Delta x = EL_{\text{MD}} x^2_{\text{CD-EDGE}} \left(\frac{x_F^2}{3(l - L_{\text{CD}})} \right) + mgx_{\text{CD-EDGE}}.$$

At this point, we assume that even though we have treated the two spring groups (the top and the edge) as separate systems—despite the fact that they move and affect each other dynamically—we can add the depression values x and energy absorption results A. This simplifying assumption is valid because the depressions are small enough that the approximations are the same as in the dynamic case. To obtain $x_{\text{MD-EDGE}}$ and $A_{\text{MD-EDGE}}$, interchange CD and MD.

So, reviewing, we have $x_F = \min(x_{\text{CD}}, x_{\text{MD}}, x_{\text{ML}}, x_{\text{ECT}})$ and total depression $x_{\text{NET}} = x_F + 2x_{\text{CD-EDGE}} + 2x_{\text{MD-EDGE}}$ and $A_{\text{NET}} = A_F + 2A_{\text{CD-EDGE}} + 2A_{\text{MD-EDGE}}$. We calculate

$$\overline{F_{\text{NET}}} = \frac{A_F + 2A_{\text{CD-EDGE}} + 2A_{\text{MD-EDGE}}}{x_{\text{NET}}}.$$

Finally, we take into account the fact that as the mass falls, it gains energy from gravity is countered by the energy gained. This fact leads to the important equation for A_T (the energy change during the top failures):

$$A_T = A_{\text{NET}} - mg(x_{\text{NET}}).$$

Single-Crack Buckling

We deal with energy dissipation for single-crack buckling, which occurs when a single side develops a crack from the top to a side. We model this as two sets of springs (**Figure 3**).

Once the top (side III) and a single side (side I) become weak, all corners but the two adjacent to the crack (C_1, C_2) remain strong. The only nonrigid corners, free to move, are C_1 and C_2. We assume that we are in the elastic range of the cardboard, so we can model this situation with two springs connected to the adjacent corners. We have springs connecting C_4 and C_1, C_5 and C_1, C_3 and C_2, C_6 and C_2. We apply the same methods as used in modeling the top to determine the energy as a function of how much the edges move in. [EDITOR'S NOTE: We do not give all the details.] The total force exerted by the box is

$$F_{\text{NET}} = 2E_{\text{MD}} \frac{l}{10} \left(\sqrt{l^2 + x^2} - l \right) + 2E_{\text{CD}} \frac{l}{10} \left(\sqrt{l^2 + x^2} - l \right)$$

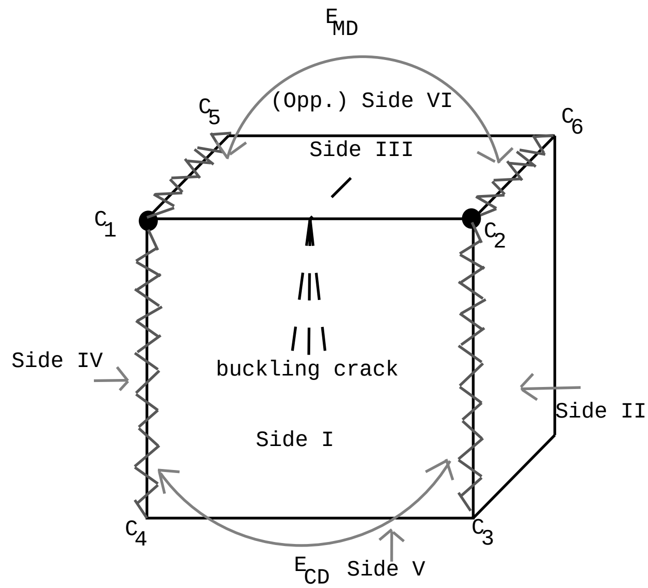

Figure 3. Our model of a box that is buckling.

and the buckling energy is

$$A_B = (E_{\text{MD}} + E_{\text{CD}}) \frac{l}{10} \left(l^2 \ln(\sqrt{l^2 + x^2} + x) + x\sqrt{l^2 + x^2} - 2lx - l^2 \ln l \right) + mgx_{\text{DOWN}}.$$

where x_{DOWN} is the component of x that points downwards from the edge.

Now we use methods similar to those used in modeling the top to determine the maximum x-displacement that can occur before the tensile strength is exceeded.

If the force F_{MD} on the machine direction spring (across the flutes) exceeds $P_{\text{MD}}wl/10$ (the tensile strength), then a crack occurs in the cross-machine direction, with

$$x_{\text{MD}} = \sqrt{\left(\frac{P_{\text{MD}}lw}{E_{\text{MD}}}\right)^2 + \frac{2P_{\text{MD}}wl^2}{E_{\text{MD}}}}.$$

Likewise, if the force F_{CD} on the cross-machine direction spring (along the flutes) exceeds $P_{\text{CD}}wl/10$, then a crack occurs in the machine direction, with

$$x_{\text{CD}} = \sqrt{\left(\frac{P_{\text{CD}}lw}{E_{\text{CD}}}\right)^2 + \frac{2P_{\text{CD}}wl^2}{E_{\text{CD}}}}.$$

The minimum of these x-displacements indicates when failure occurs. After failure, the box has less potential to continue to absorb energy. We also assume that the motorcycle has gone a significant distance, so there is little distance left to compress.

We make one last change, taking into account gravity's contribution. First, we solve for the depression of the tire during buckling, before failure. We have assumed that the buckled edge moves in towards the box's center. The total depression, x_{DOWN}, can be calculated, with a geometric argument, to be $\sqrt{lx - x^2}/\sqrt{2}$. Using this, we get a final value for A_B:

$$A_B = (E_{\text{MD}} + E_{\text{CD}})\,\frac{l}{10}\left(l^2 \ln(\sqrt{l^2 + x^2} + x) + x\sqrt{l^2 + x^2} - 2lx - l^2 \ln l\right) + \frac{mg\sqrt{lx - x^2}}{\sqrt{2}}$$

and

$$\overline{F_{\text{BNET}}} = \frac{(E_{\text{MD}} + E_{\text{CD}})\,\frac{l}{10}\left(l^2 \ln\left[\sqrt{l^2 + x^2} + x\right] + x\sqrt{l^2 + x^2} - 2lx - l^2 \ln l\right)}{\frac{\sqrt{lx - x^2}}{\sqrt{2}}} + mg.$$

With the same argument as for the top, we obtain

$$A_B = (E_{\text{MD}} + E_{\text{CD}})\,\frac{l}{10}\left(l^2 \ln(\sqrt{l^2 + x^2} + x) + x\sqrt{l^2 + x^2} - 2lx - l^2 \ln l\right).$$

Drawbacks of the One-Box Model

- The model does not take torque into account.
- The time-independence of all of our quantities makes it difficult to solve for quantities such as friction.
- The energy absorbed by the top of the box seems lower than desired.
- This model is difficult to extend to interactions between multiple boxes.

Stacking Two Identical Boxes

Consider two identical boxes, one stacked perfectly on the other, and suppose that the tire strikes the top of the higher box. We claim:

- The top box cracks, buckles, and/or punctures first; since it absorbs some of the force that acts on it, force it exerts on the second box is diminished.
- The lower box's top does not crack or puncture, it only buckles. Once the upper box has buckled, we may assume it is reasonably flattened. Then, the effective top of the second is at least two times—or maybe even three (if no punctures have occurred in the top of the first box) times—the thickness of a cardboard top. This effectively doubles or triples the tensile strength of the top, since tensile strength depends on width. Furthermore, the force from the motorcycle tire, now felt through extra cardboard, is spread over a larger area decreasing its ability to depress the top of the box.

The Effects of Friction and Adjacent Boxes

Preliminaries

For large box configurations, we predict that the following occur:

- As the motorcycle propels through boxes, it loses momentum through collisions.
- Each box experiences friction with other boxes and with the ground. Combined, these aid in slowing the motorcycle.

Average Friction Experienced

Suppose that the combined mass of the motorcycle and boxes that have "stuck" to it is m. Furthermore, let m_b be the mass of the box(es) that this system is about to strike horizontally. The frictional force is $\mu \cdot N$, where μ is the coefficient of friction and N is the force. For box-to-box interactions, $\mu \approx 0.4$; and for box-to-ground interactions, $\mu \approx 0.6$.

Let the motorcycle-boxes system have initial mass m and strike a box with initial vertical speed v_{iz} and horizontal y speed v_{iy}. By conservation of momentum, the new vertical speed is $v_y = mv_{iy}/(m + m_b)$. In our one-box model, we developed an expression for the average upwards force acting on a box, namely $\overline{F_{\text{NET}}}$ and $\overline{F_{BNET}}$. These forces are functions of x; for simplicity, we assume that the normal force felt equals the average force exerted upon the motorcycle by the box. We use the precalculated average forces due to buckling, cracking, and/or puncturing.

Thus, the average frictional force experienced is: $f_s = \mu\overline{F}$.

Horizontal Distance Travelled

To calculate the energy (and thus speed) that the motorcycle loses to friction, we determine the horizontal distance that the motorcycle travels while experiencing this frictional force. We make several approximations to estimate the horizontal energy lost during compression. Without loss of generality, suppose that we wish to calculate the horizontal distance travelled in the y-direction.

Since the vertical energy lost is ΔA_z, the final z velocity is

$$v_{fz} = \sqrt{2(A_z - \Delta A_z)/m};$$

and an approximate expression for the average z-velocity during this time is

$$\overline{v}_z \approx \frac{v_{fz} + v_{iz}}{2}.$$

This gives us the approximate time span of this event, $\bar{t} \approx x/\overline{v_z}$. Although the horizontal speed in the y-direction changes due to friction, for the purposes of calculating the horizontal distance d travelled we treat it as constant to obtain

$$d = v_y \bar{t} = \frac{m v_{iy}}{m + m_b} \frac{2x}{\sqrt{2(A_z - \Delta A_z)/m} + v_{iz}}.$$

Thus, the approximate energy lost to friction is

$$\Delta A_y = \mu \overline{F} d = \mu \overline{F} \cdot \frac{m v_{iy}}{m + m_b} \frac{2x}{\sqrt{2(A_z - \Delta A_z)/m} + v_{iz}},$$

and the new horizontal speed after this occurs is

$$v_{fy} = \sqrt{2\left(\frac{1}{2} m v_{iy}^2 - \Delta A_y\right)/(m + m_b)}.$$

Analogous equations are hold for frictional forces acting in the x-direction.

The Dual-Impact Model (DIM)

Since it simplifies computer modeling immensely while maintaining approximate accuracy, our multi-box model consists of a large three-dimensional array of points, where each point represents a 10 cm $\times$ 10 cm $\times$ 10 cm cube. The boxes have side-length 90 cm, 70 cm, or 50 cm and occupy "cubes" of these points in space. This has immediate simplifying consequences:

- Since boxes are modeled within a discrete set, their representations do not have full freedom of movement in the set. Therefore, while calculations involving buckling, cracking, and the like are done for each box, the box is modeled in the set as either existing or being flattened. Furthermore, movement will be rounded to the nearest 10 cm.
- This particular model does not allow boxes to be kept "off-angle," spinning, or modeled as flipping. It is the buckling that slows the motorcycle.

Which Configurations Are Desirable?

We prefer configurations that

- minimize the number of boxes used;
- minimize the magnitude of force acting upon the motorcycle—at most an acceleration of $5g$, if possible; and
- ensure that the motorcycle has no downward component of velocity by the time it reaches the ground (if it ever does) and that the motorcycle goes through as many boxes as possible without hitting the ground.

Preliminary Assumptions

- The front and rear motorcycle wheels have the same velocity.
- The motorcycle has the same angle of inclination when it lands as it did when it took off; that is, the back wheel lands on the cardboard boxes first.
- The motorcycle plus cyclist has sufficient velocity not to fall over.
- The motorcycle moves in the cross-machine direction of the top box.
- The boxes are in layers; boxes in a layer are all the same size.
- The mass of boxes m_b that the motorcycle with mass m strikes are sufficiently small that they can be neglected, i.e., $m + m_b \approx m$.

Step-by-Step Description of the Algorithm

Given a box configuration, the two inputs required by our simulation are the position that the bottom tire strikes the first box at and the velocity vector of the motorcycle.

The first interaction is the stress and failure of the top of the first box struck. Here our simulation uses the One-Box Model extensively. The distance of the tire from the center of the box determines which type of top failure occurs. This failure determines the change in vertical energy; our simulation calculates the energy loss for both horizontal directions.

Once the top fails, we model the buckling of the box, getting the change in vertical velocity and the distance/velocities with which the motorcycle has moved horizontally (taking friction into account).

This entire process is repeated for the front wheel of the motorcycle, as we assume that the front wheel pivots around the rear wheel to a box of its own with similar conditions. We refer to all of these combined interactions, which occur with the back and front wheels on their first box, including top failure and buckling, as the *primary impact*.

The *secondary impact* involves the motorcycle interacting with all boxes after the first. Although many more boxes are involved, the forces tend to be much more well-distributed. The inputs to the simulation are the velocity and position vector of the motorcycle. The only failure that can occur on a box below the first is buckling, when the force exerted exceeds the ECT yield strength. Frictional forces continue to affect the velocity, and some vertical velocity is absorbed in buckling the boxes. The DIM also allows for multiple boxes to buckle at once. In fact, this model automatically allows the box that is covered the most by the current box to buckle, as well as any other boxes whose tops are covered more than 15% by the current box. After this process is evaluated, the motorcycle's horizontal and vertical positions and recorded, and the process continues with the next layer.

Testing the Dual-Impact Model

We use a computer to simulate various patterns and sizes of boxes, including simple stacks of boxes on top of each other in columns, more-random configurations, and pyramidal configurations. For each configuration, we vary the speed at which the motorcycle leaves the ramp. We consider

- the maximum height, to determine whether the jump clears the elephant;
- variations in the speed and angle from which the motorcycle leaves the ramp;
- box-on-box interactions in transferring energy and dissipating heat through friction; and
- whether a configuration would stop the motorcycle's motion in all directions.

We find compelling evidence for using larger boxes in a pyramidal configuration.

The optimal configuration uses three layers of boxes 90 cm on a side stacked so that that every box (except those at the bottom) rest equally on four others. Also, smaller boxes at the bottom is a good strategy for absorbing energy. In fact, when height is an issue, having a large box with smaller boxes beneath it does better than all other smaller alternatives. The second-best configuration is a pyramid with 50 m boxes in the base, 70 cm boxes in the middle layer, and 90 cm boxes on top.

Our tests indicate that buckling is the main source of energy absorption: Buckling takes place over a much great distance than other failures and tends to last much longer.

Decelerations during energy absorption remain reasonable and within our bound of $5g$.

Why are pyramids such effective shapes? When a motorcycle hits a box and eventually interacts with the boxes bordering it below, the more boxes with large areas in contact with the box, the more energy dissipation. Every box buckles, not just the top one, as is the case for a column configuration. Pyramids also have a stability hardly found in standard columns of boxes, which individually collapse quite easily.

Conclusions and Extensions of the Problem

Solving the Problem

A motorcycle clearing a midsized elephant can be stopped effectively by 90 cm boxes but not as effectively by smaller ones, even in greater numbers. In addition, larger boxes have a higher chance of cracking, since the top of a large box has more give than a smaller box; this is useful, since additional energy is absorbed. Finally, one needs fewer large boxes.

Estimating how many boxes to use requires solving for the area in which the motorcycle could land and multiplying it by the number of layers of boxes. The approximate range in the y-direction is

$$\frac{(v_e+\delta v)\cos\theta}{g}\left[(v_e+\delta v)\sin\theta+\sqrt{(v_e+\delta v)^2\sin^2\theta+2g(h_r-h_b)}\right]$$
$$-\frac{(v_e-\delta v)\cos\theta\cos\delta\phi}{g}\left[(v_e-\delta v)\sin\theta+\sqrt{(v_e-\delta v)^2\sin^2\theta+2g(h_r-h_b)}\right], \tag{3}$$

where v_e is the expected velocity, δv is the deviation in velocity, $\phi = 0$, and $\delta\phi$ is the offset from the targeted position. The range in the x-direction is

$$2\,\frac{(v_e+\delta v)\cos\theta\sin\delta\phi}{g}\left[(v_e+\delta v)\sin\theta+\sqrt{(v_e+\delta v)^2\sin^2\theta+2g(h_r-h_b)}\right]. \tag{4}$$

The area to be covered is the product of **(2)** and **(3)**.

Each 90-cm box that buckles absorbs 4 million kg-cm^2/s^2. With the pyramid setup, looking at both wheels, 2 boxes buckle on first impact, 8 on the next level, and 15 on the third (assuming that the first two boxes were separated by one box). Instead of determining the number of boxes that intersect from both the front and back wheels, we assume an upper bound of $N(N+1)(2N+1)/6$, where N is the number of levels.

Knowing how much energy the boxes can absorb helps generalize the solution to motorcycles of any mass and jumps of any height.

Strengths of the Model

- All of our assumptions are well-grounded in the literature
- Our time-independent model allows for simple modeling without delving into the realm of differential equations.
- The values we obtain are reasonable.

Weaknesses of Model and Further Study

- Our ability to model certain interactions, such as friction and lateral velocity, is severely limited by the approximations that we are forced to make.
- Our model is discrete and thus cannot incorporate torques, slipping, and other such effects.

References

Abbreviations and definitions. n.d. `http://www.corrugatedprices.com/reference/definitions.html`.

Advantages of corrugated packaging. n.d. `http://www.fcbm.org/technology/advantages.htm`.

Benenson, Walter, et al. (eds.). 2002. *Handbook of Physics*. New York: Springer-Verlag.

ECT vs. # test. n.d. `http://www.papermart.com/packaging/info-ect.htm`.

Formed cardboard, formed hardboard parts. n.d. `http://www.diedrichs.de/uk/pappe_hartf.htm`.

Habitat utilisation in an elephant enclosure. n.d. `http://asio.jde.aca.mmu.ac.uk/new_gis/analysis/elephant.htm`.

Heritage Paper. n.d. `http://www.heritage-paper.net/packaging.html`.

Matcom: The properties of paper and cardboard. n.d. `http://www.enged.com/students/matcom/matcom89.html`.

Motorcycle specs. n.d. `http://home.pacbell.net/will207/motorcycle.htm`.

Motorcycle tire size conversion, speed ratings, load ratings chart. n.d. `http://www.webbikeworld.com/Motorcycle-tires/tire-data.htm`.

MSN learning and research—Elephant. *Encarta*. n.d. `http://encarta.msn.com/encnet/refpages/RefArticle.aspx?refid=761575386`.

Packaging glossary. n.d. `http://www.euro-corrugated-box.com/box_terms.htm`.

Party poopers?: NJ lawmakers limit roller coasters. 2002. WPVI.com Action News (15 September 2002). `http://abclocal.go.com/wpvi/news/09152002_nw_coasters.html`.

Urbanik, Thomas J. 1997. Linear and nonlinear material effects on postbuckling strength of corrugated containers. *Mechanics of Cellulosic Materials* 221: 93–99.

________. 1999. Nonlinear finite element modelling of corrugated board. *Mechanics of Cellulosic Materials* 231: 101–106.