# Design and Stack the Cardboard Boxes

## Introduction

We present a model to minimize the total surface area of all the cardboard boxes with four constraints ：

① the motorcycle's final velocity to crash against should be smaller than the safe velocity;

② he total height of all the floors is out of the range the camera could see;

③ the height of a box should be more than its maximal amount of deformation;

④ the impulsive force should be less than that a normal man could endure.

We approach the modeling problem by beginning with Kinematics Equation：for an obstacle with given height，the optimal velocity exists. Then we calculate the stunt person and the motorcycle's loss of energy approximately with deformation–energy by integration and the impulsive force by the Law of Kinetic Energy. Considering the randomness of the amount and direction of the initial velocity, we obtain the landing area. We also do simulation and design the layout of boxes. Finally, we analyze the height of the obstacle and the mass's effect on the size and layout of boxes.

We plus data from various Web and published sources，then numerical methods to obtain solution from the model. We did some validation of the model for sensitivity, strengths and weaknesses.

A major difficulty is obtaining reliable data，sources differed or even contradicted one another .The rang of the data that we could find was insufficient，jeopardizing the accuracy of our results.

## Analysis of the Problem

Our purpose is to determine what size cardboard box, how many boxes to use and how they are stacked. Meanwhile，we must make sure that the stunt person is safe and the cost of the boxes is the least，Namely ,the total surface square of the boxes designed is the least，and the total height of stacking boxes is out of the range the camera could see.

During the course，the stunt person have two kinds of movement：one is that，when he jumps over the elephant，he and the motor's movement looks like a lead ball is cast to the sky；after that，during the course of landing in a pile of cardboard boxes to cushion his fall，it is a collision–cushion movement. The boxes is used to absorb kinetics energy，and it is difficult to calculate the energy . Although in the collusion –cushion movement the boxes must be bent, we cannot calculate the energy like a spring. In this paper, we design a approximate algorithm to calculate the energy.

First, our task is determine an optimal initial velocity to make the motor jump over the obstacle (namely, the elephant). The height of the obstacle determines that the initial velocity must be enough to jump over it. Second, we design and stack the box in the line of the velocity.

# Assumption and Hypothesis

•The price of unite square cardboard box is fixed.

•During the course, the stunt person and the motor would never be separated, and they can be seen as a point.

•When the motor lands on the cardboard, it is seen as a point.

•The motor has the ability to cushion, and it only make the stunt person comfortable, but it doesn't affect the cardboard boxes.

•Each side of a cardboard box doesn't produce a cushioning effect.

•When the motor collides a box, the box couldn't be bumped out of the way and collapsed, and the pile wouldn't collapse, but the box could be penetrated.

•When the box is pressed, it doesn't have damping effect and its sides don't deform.

•The motorcycle won't break down.

•We ignore the inside structure of the cardboard and the thickness.

# Modeling

## The Velocity the Motorcycle Landing on a Box

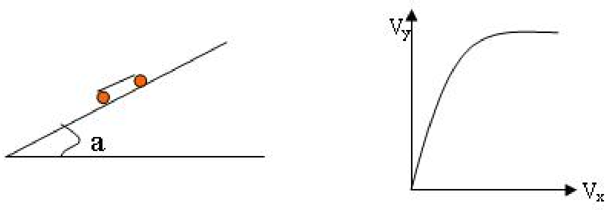



**Figure 1**

As Figure 1 shown, the motorcycle and the stunt person jump over the obstacle at some velocity. That course likes a lead ball is cast to the sky. According to the Kinematics Equation, we derive:

$$\begin{cases} v_x = v_0 \cdot \cos a \\ v_y = v_0 \cdot \sin a - g \cdot t \end{cases} \tag{1}$$

Where, $v_x$, $v_y$ is a sub-velocity respectively in the direction of $x$-axis and $y$-axis, $v_0$ is an initial velocity, $\alpha$ is an angle the initial velocity against the ground , $g$ is the gravitational acceleration and $t$ is the jump-time.

Therefore, the displacement respectively in the direction of $x$-axis and $y$-axis is:

$$\begin{cases} s_x = \int_0^t v_x dt = v_0 t \cos a \\ s_y = \int_0^t v_y dt = v_0 t \sin a - \frac{1}{2} g t^2 \end{cases} \tag{2}$$

where $s_x$, $s_y$ are respectively the displacement in the direction of $x$-axis and $y$-axis.

When the motorcycle reach the highest place, the sub-velocity becomes 0 (Because of gravity, the acceleration is $-g$) .The largest height it can reach is:

$$s_{\max Y} = \frac{{v_0}^2 \sin^2 a}{2g} \tag{3}$$

by formula (2), we can get the function between $s_x$ and $s_y$ as follows:

$$s_y = s_x \tan a - \frac{1}{2} g \frac{s_x^2}{{v_0}^2 \cos^2 a} \tag{4}$$

we can sign: $s_y = f(s_x)$.

From formula (4), we know that the jump-curve is a parabola. As Figure 2 shown.

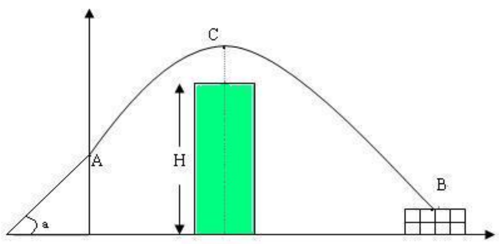



**Figure 2**

Generally speaking, if the obstacle is sharply below the highest place, it can reduce the velocity $v_0$ and make sure the chance the motorcycle crashes with the obstacle is the least.

The motorcycle can jump over the obstacle, the following constraints should be met:

$$\begin{cases} f(l - \frac{L}{2}) \geq H + \varepsilon \\ f(l + \frac{L}{2}) \geq H + \varepsilon \end{cases} \tag{5}$$

Where $H$ is the height of the obstacle, $L$ is the width of the obstacle, $l = \frac{v_0^2 \cdot \sin 2\alpha}{2g}$, $\varepsilon$ is a safe height above the obstacle (for a given obstacle, it is a constant).

If the obstacle is not high, like an elephant, it could be considered as a wall. So, the two constraints can be converted to only one:

$$s_{Y\max} = \frac{{v_{0y}}^2}{2g} = \frac{{v_0}^2 \sin^2 a}{2g} \geq H + \varepsilon \qquad (6)$$

Where $v_{0y}$ is the sub-velocity of $v_0$ in the direction of $y$-axis. For a given $H$ and $\varepsilon$, the optimal $v_{0y}$ is definite. In fact, $\alpha$ is also given, so the optimal $v_0$ is also definite. Because there is no air resistance but only gravity affects, we can easily obtain the velocity $v_0^{'}$, when the motorcycle is landing on the top of the cardboard boxes by the Law of Kinetic Energy:

$$\frac{1}{2}M \cdot v_0^{'2} - \frac{1}{2}M \cdot v_0^2 = M \cdot \Delta h \qquad (7)$$

where, $\Delta h$ is the height difference between A and B.

The relationship between $h$ and $v_0^{'}$ is shown as Figure 3.

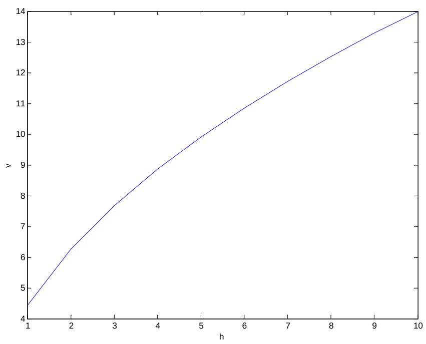



Figure 3

## Calculate the Energy Absorbed by Boxes

A box is used to absorb part kinetic energy of the motorcycle and the stunt person, and then the energy is converted to potential and internal energy of the box . When the motorcycle collides with a box, if its energy is big enough, it can pierce through the box. In this way, most of the kinetic energy can be absorbed floorer by floorer. Therefore, it can cushion the motorcycle and the stunt person. In this course, the box produces some reacting force against the motorcycle, and the maximal reacting force must be within the range a normal man can endure.

When the motorcycle crashes against the box, the energy absorbed by cardboard can't be exactly calculated[Burden and Faires 1997].Therefore, we can calculate it approximately. We consider a cardboard to be composed of millions of cardboard-strips and each cardboard-strip can absorb a small amount of energy. Because the force caused by the wheel presses on the surface, and when it presses on the surface-center of the box, the deformation is the most .Therefore, it is the easiest to be pierced through .When the motorcycle crashes against it , it is the most probable that the stunt person is injured .Thus, we only think about the force pressing on the surface-center of the box.. (as Figure 4 shown)

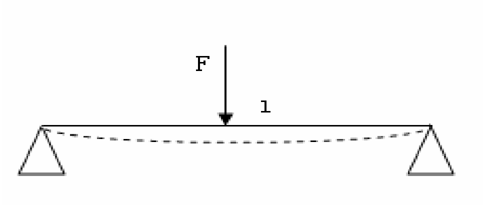



**Figure 4**

If a cardboard-strip is deformed and broken, the energy absorbed is as follows:

$$e = \mu l_1^3 \qquad (8)$$

Where, $\mu$ is a constant, $l_2$ is the length of the cardboard.

Thus, the energy absorbed by the whole cardboard is:

$$E = \int_0^{l_2} e \cdot dl = \mu l_1^3 l_2 \qquad (9)$$

Where, $l_2$ is the width of the cardboard.

## The Force on the Stunt Person

When the motorcycle crashes against a box, the resistance the box gives the motorcycle [Green,1950] is:

$$f = -kv^2 \qquad (10)$$

Where, $k$ is a constant coefficient, $v$ is a velocity the motorcycle relative to the cardboard.
For the motorcycle, according to Newton's Second Law, we have:

$$M\frac{dv}{dt} = f \qquad (11)$$

Where, $m$ is the total mass of the motorcycle and the stunt person, $\frac{dv}{dt}$ is their acceleration.

For formula (7), we integrate:

$$\frac{1}{v} = \frac{k}{M}t + \frac{1}{v_1} \qquad (12)$$

where, $v_1$ is the initial velocity.

For each floorer of all the boxes, we derive the energy, $E_i$, absorbed for each floorer by Law of

Kinetic Energy：

$$E_i = \frac{1}{2}Mv_{i1}^2 - \frac{1}{2}Mv_{i2}^2 \qquad (13)$$

Where， $v_{i1}$ is the velocity ， So， $v_{i2} = \sqrt{v_{i1}^2 - \frac{2E}{M}}$ ， with formula（8），we have：

$$\frac{k}{M}t = \frac{1}{\sqrt{v_{i1}^2 - \frac{2E_i}{M}}} - \frac{1}{v_{i1}} \qquad (14)$$

Additionally，the deformation of a box is shown as Figure 5.

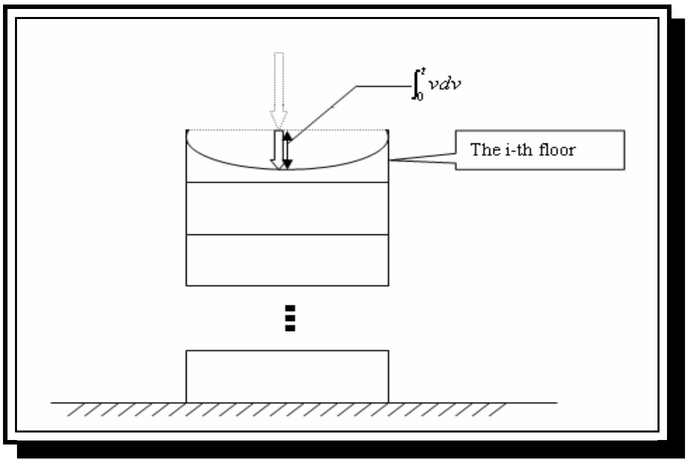



Figure 5

By formula（8），the amount of deformation can be integrated：

$$d = \int_0^t vdt \quad = kM\ln\frac{kv_{i1}t + M}{M} \qquad (15)$$

Because the time is very short, the force changes little, so it could be seen as unchanged, by Impulse law, we have:

$$(Mg - F_i)\cdot t = M\cdot v_{i2} - M\cdot v_{i1} \qquad (16)$$

Therefore , we get：

$$F_i = -\frac{M(\sqrt{v_{i1}^2 - \frac{2E}{M}} - v_{i1})}{\frac{1}{\sqrt{v_{i1}^2 - \frac{2E}{M}}} - \frac{1}{v_{i1}}}\cdot\frac{M}{k} + Mg \qquad (17)$$

# Basic Model

Generally, the boxes are cuboid. Therefore，we design cuboid boxes.

The mathematical model is：

$$\min \quad \sum 2(a\cdot b + b\cdot c + c\cdot a)$$

$$s.t\begin{cases}\frac{1}{2}Mv_0^2 - \sum E \le \frac{1}{2}Mv^{*2} \text{ .............1)},\\ \sum c \le c_0 \text{ ...................................2)},\\ c \ge \int_0^t v\cdot dt \text{ ...............................3)},\\ F_i \le F_{\max} \text{...................................4)},\end{cases}$$

Where ， a----------the length of a box

b----------the width of a box

c-----------the height of a box

n--------the number of floors

$c_0$--------the critical height the camera can't see

$\int_0^t vdt$ -----the descending height when the motorcycle crashes against the box

$F_{max}$----------the maximal force a normal man can endure

$v^*$———————— the safe velocity

The version of the constraints:

1）reduce the Kinetic Energy of the motor and person

2）the height of stacked boxes should not be seen by camera

3）the height of one box should be above its camber.

4）the impulsive force constraint.

# Calculate the results

## the Size of Boxes

We choose the initial parameters ： $h=2\ m$，$\varepsilon=\frac{1}{2}h$，$c_0=1.5\ m$，$v^*=3m/s$，$M=200\ kg$

The size: $a=3.975\ m$，$b=0.225\ m$，$c=0.112\ m$ ，

and the number of floors : $n= 12$.

## the Number of all Boxes and How to Stack

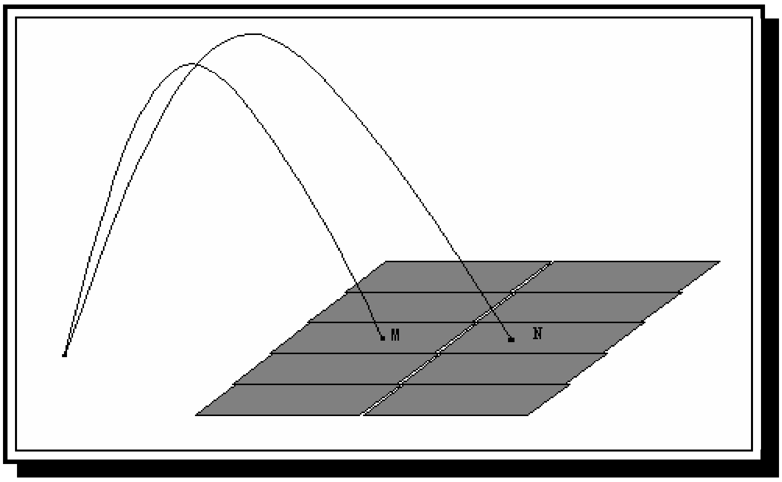
According to our analysis, if we need the least boxes to cushion, all boxes should be clinker-built .

Before solving the problems2， we give a definition.

**Ideal landing point:** Supposed that the stunt person doesn't make any mistake, we calculate the landing point of the jumping.

Actually, it is those boxes broken by the motorcycle, that cushion the stunt person and the motorcycle's fall. Since the landing point is centre of top surface of one of the top boxes and that we ignore the motorcycle's sub-velocity which is vertical to its main moving direction, the cushioning box should be placed vertically.

Now considering air tolerance and other conditions, we must need some protective boxes around the ideal cushioning point.

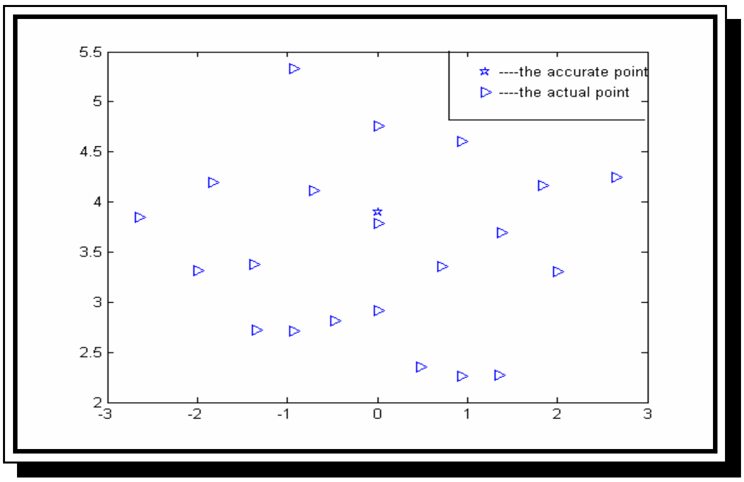



the point of the motorcycle landing on the boxes
**M** is ideal landing point, **N** is actual landing point.

**Figure 6**

Considering the contingency in the actual running, the velocity and direction will be vibrated (within the permitted range), so the motorcycle's actual landing point is indefinitely. All the probable landing points form an area. Therefore, when staking the boxes, the area must be covered by boxes. Meanwhile, if the motorcycle lands at the edge of the area, it may be toppled. So it is required to add protective cardboard boxes along the area. The whole area is just the shape of cardboard boxes stacked and determines how many boxes should be stacked.

Considering the actual conditions, we permit the direction and the velocity change a little and simulate it .The parameter are : $v_x = 18$ m/s, We get the following result:



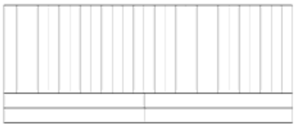

Simulating the point of the motorcycle landing on the top of boxes

**Figure 7**

The whole safe area is: 24.37 $m^2$

The length of the safe area is 5.8 *m*
The width of the safe area is 4.30 *m*

Because we know all boxes should be clinker- built and the number of floors.
Therefore ，we only need to know how to orient boxes of every floor，of which numbers is the least and cover all of the safe area
We obtain the number of every floor is 31 through the search , whose orientation is as fallow ：



the orientation of the boxes in every floor

**Figure 8**

So the total number of the boxes is ：12×31=372.

## Modification of the Boxes

On one hand, we know the more floors, the more boxes can absorb the Kinetic Energy of the motor and person's .So the floors of the boxes are the most sensitive to cushion. On the other hand, the more floors, the more numbers we need boxes. The more floors don't mean to we cost more money. But we think the floors should help us to design and stack boxes, the most reason is the height of the boxes is rather low.

## Generalization

In order to get an outcome, which is correspondent with the real situation, we choose the following parameters：

M＝（150～600）*kg*
h＝（1～20）*m*

We obtain the following tables and figures.

**Table－1**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| M | 150 | 160 | 170 | 180 | 190 | 200 | 210 | 220 | 260 | 300 | 360 | 500 |
| A | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 |
| B | 0.100 | 0.725 | 0.225 | 0.6 | 0.85 | 0.225 | 0.35 | 0.225 | 0.35 | 0.225 | 0.6 | 0.475 |
| C | 0.022 | 0.140 | 0.0423 | 0.102 | 0.138 | 0.036 | 0 052 | 0.033 | 0.043 | 0.025 | 0.051 | 0.031 |
| N | 20 | 3 | 10 | 4 | 3 | 12 | 8 | 13 | 10 | 18 | 8 | 14 |

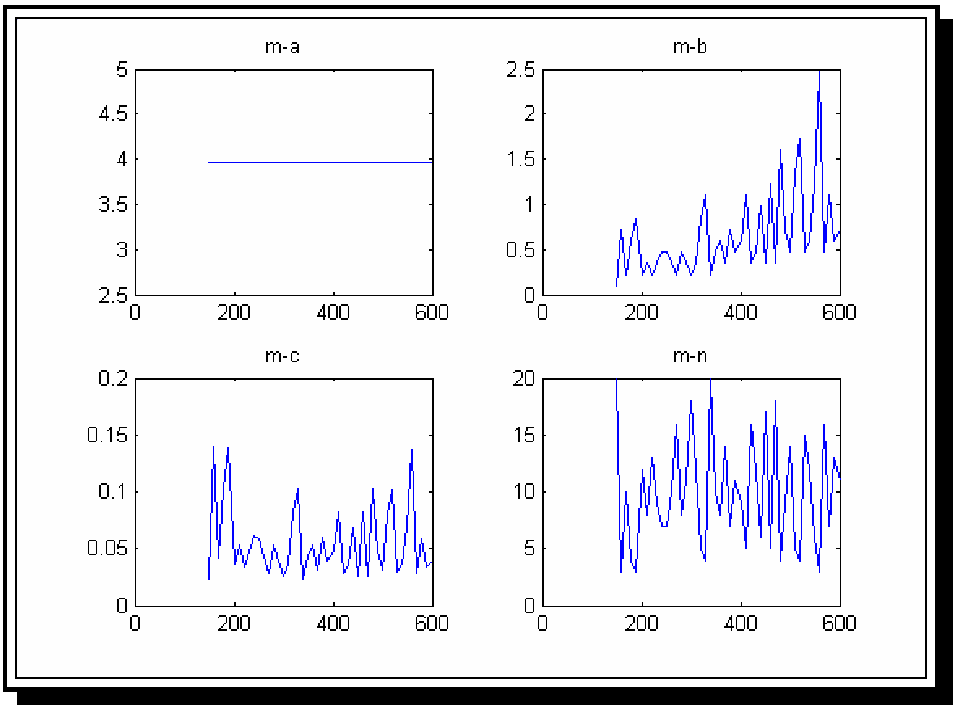



**Figure 9**

Table-1 and Figure 9 stand for the change of box argument with the variation of motorcycle and stunt person's total weight, while the box's height is unchangeable.

From the table and figure, we draw the following conclusion below:

①the length of a box hardly changes;

②the width of a box and the number of floors change most obviously;

③no matter how the total weight changes, a box's is at very low height.

**Table—2**

| h | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 10 | 12 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 | 3.975 |
| b | 0.100 | 0.225 | 0.850 | 0.725 | 1.225 | 0.6 | 2.1 | 1.1 | 0.85 | 2.1 | 1.225 | 3.975 |
| c | 0.042 | 0.036 | 0.075 | 0.044 | 0.054 | 0.025 | 0.022 | 0.058 | 0.026 | 0.017 | 0.018 | 0.015 |
| n | 11 | 12 | 5 | 8 | 6 | 15 | 5 | 11 | 16 | 15 | 20 | 14 |

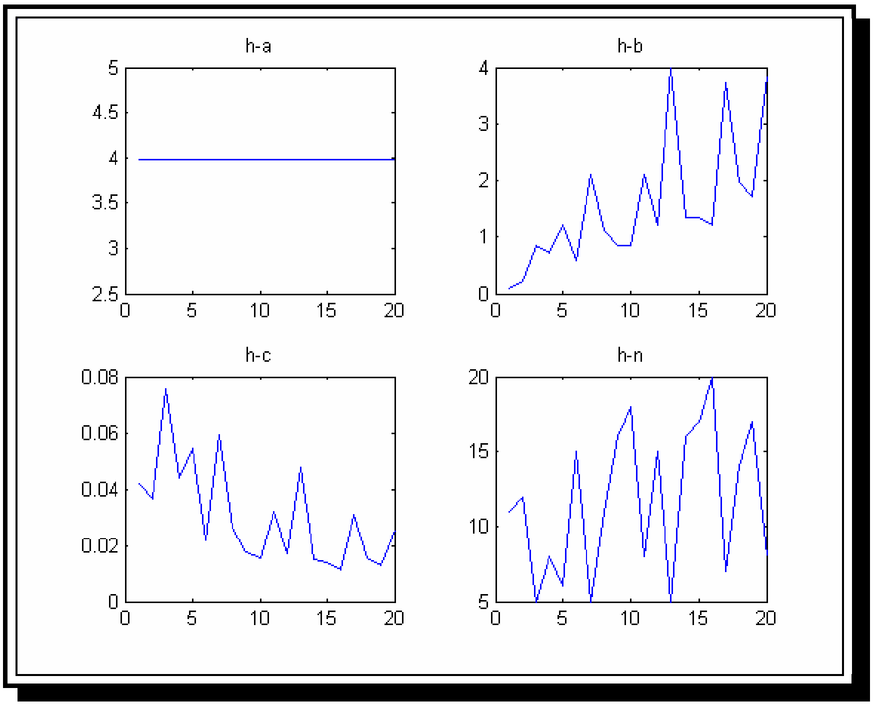



Figure 10

Table-2 and figure 10 stand for the change of box argument with the variation of objective height, when motorcycle and stunt person's total weight is a unchangeable.

From the table and figure, we draw the following conclusions:

①the length of a box hardly changes;

②the width of a box and the number of floors change most obviously;

③no matter how the total weight changes, a box's is at very low height.

Thus, no matter whether the boxes' height or flying stuff's total weight varies, a box's height must be designed relatively low, since from the formal hypothesis and analysis, we know, the amount of energy absorbed by the cardboard is directly associated with floorers of cardboards. The energy absorbed by the cardboard varies with the boxes' height and total weight, so it is the floorers of the boxes not the geometric size of them that produce the main cushioning function.

# Sensitivity Analysis

The factors, that affect the ability of the boxes that absorb the kinetic energy of motorcycle and stunt person, are the boxes' floorer and their size.

The floorers of the boxes are the most sensitive to the $E$（the boxes absorb the kinetic energy of motorcycle and stunt person）

Changing motorcycle and stunt person's weight, as well as the height of the object they flying over, will affect the minimum amount of energy the boxes must absorb to assure security. The height of object plays much more important role than motorcycle and stunt person's total weight in determining the amount of energy absorbed by the boxes.

We have:

$$\frac{W_{\Delta h}}{W_{\Delta m}} \approx \frac{m}{H}. \quad (18)$$

Where, $W_{\Delta h}$ represents energy variation stemmed from object's height.

$W_{\Delta m}$ represents energy variation stemmed from total weight.

Hence we come to the conclusion that the height of object affects the designation of cushioning box the most.

## Strengthens of the Model

Our greatest success is that we have abstracted a complicated non-linear elastic problem into a mathematical programming problem. Supposed that we obtained accurate data and did some relevant experiments, such as, the maximum punch an ordinary guy could stand and the most awful collision by a hard stuff man can bear etc. Then we were able to design the boxes and the way to floor them. All in all, we have a very good mathematical model to describe the course of movement and the collision or something like that.

When deposing the cushioning course, we just consider the energy each floorer of cardboard absorbs and the maximum reacting force from the boxes to man. Therefore the model is simpler, more intuitional, easier understood, more effectively.

## Weaknesses of the Model

Firstly, we have done a lot of simplified work, ignored many factors. Actually, in the course of flying the motorcycle and the stunt person are in air resistance. That is the motorcycle and stunt person will lose some kinetic energy during the performance, so the velocity when the motorcycle lands on the cardboard will be slower.

Secondly, the sides of boxes will deform at the collision. That is the actual energy the cardboard absorbs is bigger than our calculation. But if we consider the energy the boxes' sides absorb, the problem will be much more complicated.

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# The Original Code

```
Function draw           % draw the figure
AA1=[ ];
AA2=[ ];
AA3=[ ];
AA4=[ ];
AA5=[ ];
for h=1:1:20;
m=200;%%     the mass of the person and motor
Fmax=5000;
func=1000000;
c0=1.5;
k1=30;
k2=0.1;
e1=0.5*h;
v1=sqrt(2*9.8*(h+e1));
v3=3;
for n=1:20
    for a=0.1:0.125:4
        for b=0.1:0.125:4
            E=k1*a^3*b;
            v2=sqrt(v1^2-2*E/m);
            t=(m/k2)*(v1*v2)/(v1-v2);
            F=(m*(v1-v2))/t-m*9.8;
           c=(m/(k2*t))*log((k2*t/m)+1/v1)-(m/(k2*t))*log(1/v1);
            E1=(0.5*m*v1^2-n*E);
            E2=0.5*m*v3^2;
            if F>Fmax      %
                continue;
            end
            if n*c>c0
                continue;
           end

           if E1>E2
               continue;
          end
              func1=n*(a.*b+b.*c+c.*a)./2;
              if func1<func
                  func=func1;
                  a1=a;
                  b1=b;
                  c1=c;
```

```
                    n1=n;
                end
            end
        end
    end
AA1=[AA1 [h;a1]];
AA2=[AA2 [h;b1]];
AA3=[AA3 [h;c1]];
AA4=[AA4 [h;n1]];
AA5=[AA5 [h;func]];
end
subplot(2,2,1);
plot(AA1(1,:),AA1(2,:));
title('h-a');
subplot(2,2,2);
plot(AA2(1,:),AA2(2,:));
title('h-b');
subplot(2,2,3);
plot(AA3(1,:),AA3(2,:));
title('h-c');
subplot(2,2,4);
plot(AA4(1,:),AA4(2,:));
title('h-n');
```