# On the similarity of Tensors\*

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#### Abstract

In this paper, we characterize all similarity relations when  $m \geq 3$ , obtain some interesting properties which are different from the case m = 2, and show that the results of matrices about the Jordan canonical form cannot be extended to tensors.

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### 1 Introduction

Since the work of Qi [5] and Lim [3], the study of tensors and the spectra of tensors (and hypergraphs) and their various applications have attracted much attention and interest.

An order m dimension n tensor  $\mathbb{A} = (a_{i_1 i_2 \dots i_m})_{1 \leq i_j \leq n}$  ( $j=1,\dots,m$ ) over the complex field  $\mathbb{C}$  is a multidimensional array with all entries  $a_{i_1 i_2 \dots i_m} \in \mathbb{C}(i_1,\dots,i_m \in [n] = \{1,\dots,n\})$ . The majorization matrix  $M(\mathbb{A})$  of the tensor  $\mathbb{A}$  is defined as  $(M(\mathbb{A}))_{ij} = a_{ij\dots j}, (i,j \in [1,n])$  by Pearson [4]. The unit tensor of order m and dimension n is the tensor  $\mathbb{I} = (\delta_{i_1,i_2,\dots,i_m})$  with

$$\delta_{i_1,i_2,\dots,i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \dots = i_m; \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\mathbb{A}$  (and  $\mathbb{B}$ ) be an order  $m \geq 2$  (and  $k \geq 1$ ), dimension n tensor, respectively. Recently, Shao [6] defined a general product  $\mathbb{AB}$  to be the following tensor  $\mathbb{D}$  of order (m-1)(k-1)+1 and dimension n:

$$d_{i\alpha_1...\alpha_{m-1}} = \sum_{i_2,...,i_m=1}^n a_{ii_2...i_m} b_{i_2\alpha_1} \dots b_{i_m\alpha_{m-1}} \quad (i \in [n], \ \alpha_1, \dots, \alpha_{m-1} \in [n]^{k-1}).$$

The tensor product possesses a very useful property: the associative law ([6], Theorem 1.1). With the general product, the following definition of the similarity relation of two tensors was proposed by Shao [6].

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**Definition 1.1.** ([6], Definition 2.3) Let  $\mathbb{A}$  and  $\mathbb{B}$  be two order m dimension n tensors. Suppose that there exist two matrices P and Q of order n with  $P\mathbb{I}Q = \mathbb{I}$  such that  $\mathbb{B} = P\mathbb{A}Q$ , then we say that the two tensors are similar.

It is easy to see that the similarity relation is an equivalent relation, and similar tensors have the same characteristic polynomials, and thus they have the same spectrum (as a multiset). For example, the permutation similarity and the diagonal similarity (see also [6, 7, 8]) are two special kinds of the similarity of tensors.

**Definition 1.2.** ([6]) Let  $\mathbb{A}$  and  $\mathbb{B}$  be two order m dimension n tensors. We say that  $\mathbb{A}$  and  $\mathbb{B}$  are permutational similar, if there exists some permutation matrix P of order n such that  $\mathbb{B} = P\mathbb{A}P^T$ , where  $\sigma \in S_n$  is a permutation on the set [n] and  $P = P_{\sigma} = (p_{ij})$  is the corresponding permutation matrix of  $\sigma$  with  $p_{ij} = 1 \Leftrightarrow j = \sigma(i)$ .

**Definition 1.3.** ([6], Definition 2.4) Let  $\mathbb{A}$  and  $\mathbb{B}$  be two order m dimension n tensors. We say that  $\mathbb{A}$  and  $\mathbb{B}$  are diagonal similar, if there exists some invertible diagonal matrix D of order n such that  $\mathbb{B} = D^{-(m-1)}\mathbb{A}Q$ .

About the matrices P and Q in Definition 1.1, [6] showed the following proposition.

**Proposition 1.4.** ([6], Remark 2.1) If P and Q are two matrices of order n with  $P\mathbb{I}Q = \mathbb{I}$ , where  $\mathbb{I}$  is the order m dimension n unit tensor, then both P and Q are invertible matrices.

In this paper, we characterize all similarity relations when  $m \geq 3$  in Section 2, and in Section 3, we obtain some interesting properties which are different from the case m = 2, and show that the results of matrices about the Jordan canonical form cannot be extended to tensors.

### 2 Main results

In this section, we will prove Theorem 2.3. We first prove the following two lemmas.

**Lemma 2.1.** Let  $P = (p_{ij})$  and Q be two matrices of order n with  $P\mathbb{I}Q = \mathbb{I}$ ,  $\mathbb{I}$  the order m dimension n unit tensor. Take  $\mathbb{I}Q = \mathbb{A} = (a_{i_1 i_2 ... i_m})$ , then we have

- (i)  $a_{i_1 i_2 \dots i_m} = 0$  when  $(i_2, i_3, \dots, i_m) \neq (i_2, i_2, \dots, i_2);$
- (ii)  $PM(\mathbb{A}) = I$ , where  $M(\mathbb{A})$  is the majorization matrix of  $\mathbb{A}$  and I is the unit matrix of order n.

*Proof.* For the proof of (i), we let  $\alpha = i_2 i_3 \dots i_m \neq i_2 i_2 \dots i_2$ . By  $P \mathbb{A} = \mathbb{I}$ , we have

$$\delta_{i\alpha} = \sum_{j=1}^{n} p_{ij} a_{j\alpha}, i = 1, 2, \dots, n.$$

Since  $\alpha \neq i_2 i_2 \dots i_2$ , then  $\delta_{i\alpha} = 0$ , it follows that

$$P(a_{1\alpha}, a_{2\alpha}, \dots, a_{n\alpha})^T = (0, 0, \dots, 0)^T.$$

Observe that P is an invertible matrix by Proposition 1.4, thus  $a_{j\alpha} = 0, j = 1, 2, ..., n$ , and the conclusion of (i) follows.

For the proof of (ii), by  $PA = \mathbb{I}$ , we have

$$\delta_{ij...j} = \sum_{u=1}^{n} p_{iu} a_{uj...j} = \sum_{u=1}^{n} p_{iu} (M(\mathbb{A}))_{uj} = (PM(\mathbb{A}))_{ij}.$$

Hence  $PM(\mathbb{A}) = I$ , this completes the proof of (ii).

**Lemma 2.2.** Let  $\mathbb{I}$  be the order  $m \geq 3$  dimension n unit tensor. Suppose that  $Q = (q_{ij})$  is a matrix of order n such that

$$(\mathbb{I}Q)_{i\alpha} = 0$$

for all  $i \in [n]$  and all  $\alpha \neq j \dots j, j \in [n]$ , then there is at most one nonzero element in every row of Q.

*Proof.* Since

$$(\mathbb{I}Q)_{i_1i_2...i_m} = \sum_{j_2,...,j_m=1}^n \delta_{i_1j_2...j_m} q_{j_2i_2} \dots q_{j_mi_m} = q_{i_1i_2} q_{i_1i_3} \dots q_{i_1i_m},$$

by the assumption, for every  $\alpha = i_2 i_3 \dots i_m \neq i_2 i_2 \dots i_2$ , we get

$$q_{i_1 i_2} q_{i_1 i_3} \dots q_{i_1 i_m} = 0. (2.1)$$

If  $q_{i_1t} \neq 0$  for some  $t \in [n] \setminus \{i_2\}$ , then we take  $i_3 = \ldots = i_m = t$ . By (2.1) we have  $q_{i_1i_2}q_{i_1t}^{m-2} = 0$ , hence  $q_{i_1i_2} = 0$ . Since the choice of  $i_2$  is arbitrary, then we have proved that for any  $i \in [n]$ , there is at most one nonzero element in  $\{q_{i1}, q_{i2}, \ldots, q_{in}\}$ . This completes the proof of the lemma.

**Theorem 2.3.** Let  $\mathbb{I}$  be the unit tensor of order  $m \geq 3$  and dimension  $n \geq 2$ . Suppose that P and Q are two matrices of order n with  $P\mathbb{I}Q = \mathbb{I}$ , then there exist a permutation matrix  $R_{\sigma}$  and an invertible diagonal matrix D, such that  $Q = DR_{\sigma}$  and  $P = R_{\sigma}^T D^{1-m}$ , where  $\sigma \in S_n$  is a permutation on the set [n].

*Proof.* Since  $\mathbb{I} = P\mathbb{I}Q = P(\mathbb{I}Q)$ , combining Lemmas 2.1 and 2.2, we obtain that there is at most one nonzero element in every row of Q. Note that Q is an invertible matrix by Proposition 1.4, hence there is precisely one nonzero element, say,  $q_{i\sigma(i)} \neq 0, 1 \leq i \leq n$  in every row of Q. Obviously,  $\sigma(1), \sigma(2), \ldots, \sigma(n)$  is a permutation of  $\{1, 2, \ldots, n\}$ , hence

$$Q = DR_{\sigma}$$

is the product of an invertible diagonal matrix  $D = diag(d_{11}, \ldots, d_{nn})$ , where  $d_{ii} = q_{i\sigma(i)}$  and a permutation matrix  $R_{\sigma} = (r_{ij})$ , where  $r_{ij} = 1$  if  $j = \sigma(i)$  and  $r_{ij} = 0$  otherwise.

Now by the general tensor product, we get

$$(M(\mathbb{I}Q))_{ij} = (\mathbb{I}Q)_{ij\dots j} = q_{ij}^{m-1} = \begin{cases} q_{i\sigma(i)}^{m-1} & \text{if } j = \sigma(i); \\ 0 & \text{otherwise.} \end{cases}$$

Hence  $M(\mathbb{I}Q)$  is the product of the invertible diagonal matrix  $D^{m-1} = diag(d_{11}^{m-1}, \ldots, d_{nn}^{m-1})$  and the permutation matrix  $R_{\sigma} = (r_{ij})$ . By Lemma 2.1 (ii), we have  $PM(\mathbb{I}Q) = I$ , it follows that  $P = R_{\sigma}^T D^{1-m}$ , is the product of the permutation matrix  $R_{\sigma}^T$  and the invertible diagonal matrix  $D^{1-m} = diag(d_{11}^{1-m}, \ldots, d_{nn}^{1-m})$ .

**Remark 2.4.** By the proof of Theorem 2.3, we see that Q and P are closely related with  $QP = D^{2-m}$  and  $PQ = R_{\sigma}^T D^{2-m} R_{\sigma}$  are invertible diagonal matrices. Note that when m = 2, P,Q are invertible matrices and PQ = QP = I. It implies that the case of  $m \geq 3$  is completely different from the case m=2.

Let  $\mathbf{P}_n$  be the set of all permutation matrices of order n,  $\mathbb{P}_n$  be the set of all matrices which have the same zero patterns with some permutation matrix of order n. Clearly,  $\mathbf{P}_n \subseteq \mathbb{P}_n$ . For example,  $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbf{P}_2, T = \begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix} \in \mathbb{P}_2$ , where S and T have the same zero pattern.

**Remark 2.5.** Let  $\mathbb{I}$  be the unit tensor of order  $m \geq 3$  and dimension  $n \geq 2$ . Suppose that P and Q are two matrices of order n with  $P\mathbb{I}Q = \mathbb{I}$ , then  $P,Q \in \mathbb{P}_n$ .

**Remark 2.6.** By Definition 1.1, we know that Theorem 2.3 gives a characterization for the similarities of tensors with order  $m \geq 3$  dimension n, i.e., as for the similarity of tensors, we need only consider the permutation similarity, the diagonal similarity and their compositions.

**Theorem 2.7.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be two order m > 3 dimension n tensors. If the tensors  $\mathbb{A}$  and  $\mathbb{B}$  are similar, then there exists a tensor  $\mathbb{C}$  such that  $\mathbb{A}$  and  $\mathbb{C}$  are diagonal similar, and  $\mathbb{B}$ and  $\mathbb{C}$  are permutational similar.

*Proof.* By Definition 1.1 and Theoren 2.3, there exist a permutation matrix R and an invertible diagonal matrix D of order n such that Q = DR,  $P = R^T D^{1-m}$  and  $\mathbb{B} = P \mathbb{A} Q$ .

Take  $\mathbb{C} = D^{1-m} \mathbb{A}D$ , then  $\mathbb{B} = R^T \mathbb{C}R = R^T \mathbb{C}(R^T)^T$ . By Definitions 1.2 and 1.3, the results hold. 

#### 3 Some applications

Let  $Z(\mathbb{A})$  be the tensor obtained by replacing all the nonzero entries of  $\mathbb{A}$  by one. Then  $Z(\mathbb{A})$  is called the zero-nonzero pattern (or simply the zero pattern) of A. Let a be a complex number, we define Z(a) = 1 if  $a \neq 0$  and Z(a) = 0 if a = 0.

**Lemma 3.1.** Let  $\mathbb{A} = (a_{i_1 i_2 \dots i_m})_{1 \leq i_j \leq n} \ (j=1,\dots,m)$  and  $\mathbb{B} = (b_{i_1 i_2 \dots i_m})_{1 \leq i_j \leq n} \ (j=1,\dots,m)$  be two order  $m \geq 3$  dimension n tensors. If the tensors  $\mathbb{A}$  and  $\mathbb{B}$  are diagonal similar, then  $Z(\mathbb{A}) = Z(\mathbb{B})$ .

*Proof.* By Definition 1.3, there exists an invertible diagonal matrix  $D = diag(d_{11}, \ldots, d_{nn})$ of order n such that  $\mathbb{B} = D^{1-m} \mathbb{A}D$ . Then

$$b_{i_1 i_2 \dots i_m} = (D^{1-m} \mathbb{A} D)_{i_1 i_2 \dots i_m}$$

$$= \sum_{j_1, \dots, j_m = 1}^{n} (D^{1-m})_{i_1 j_1} a_{j_1 j_2 \dots j_m} d_{j_2 i_2} \dots d_{j_m i_m}$$

$$= a_{i_1 i_2 \dots i_m} d_{i_1 i_1}^{1-m} d_{i_2 i_2} \dots d_{i_m i_m}.$$
wherefore  $b_{i_1 i_2 \dots i_m} \neq 0 \Leftrightarrow a_{i_1 i_2 \dots i_m} \neq 0$  and thus  $Z(\mathbb{A}) = Z(\mathbb{R})$ 

Therefore  $b_{i_1i_2...i_m} \neq 0 \Leftrightarrow a_{i_1i_2...i_m} \neq 0$ , and thus  $Z(\mathbb{A})=Z(\mathbb{B})$ .

**Theorem 3.2.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be two order  $m \geq 3$  dimension n tensors. If the tensors  $\mathbb{A}$  and  $\mathbb{B}$  are similar, then  $Z(\mathbb{A})$  and  $Z(\mathbb{B})$  are permutational similar.

*Proof.* By Theorem 2.7, there exists a tensor  $\mathbb{C}$  such that  $\mathbb{A}$  and  $\mathbb{C}$  are diagonal similar, and  $\mathbb{B}$  and  $\mathbb{C}$  are permutational similar. It is easy that  $Z(\mathbb{A}) = Z(\mathbb{C})$  by Lemma 3.1. Now we show that  $Z(\mathbb{B})$  and  $Z(\mathbb{C})$  are permutational similar.

Let R be a permutation matrix of order n such that  $\mathbb{B} = R\mathbb{C}R^T$ . Then

$$(Z(\mathbb{B}))_{i_{1}i_{2}...i_{m}} = (Z(R\mathbb{C}R^{T}))_{i_{1}i_{2}...i_{m}}$$

$$= Z(\sum_{j_{1},...,j_{m}=1}^{n} r_{i_{1}j_{1}}c_{j_{1}j_{2}...j_{m}}(R^{T})_{j_{2}i_{2}}...(R^{T})_{j_{m}i_{m}})$$

$$= Z(\sum_{j_{1},...,j_{m}=1}^{n} r_{i_{1}j_{1}}c_{j_{1}j_{2}...j_{m}}r_{i_{2}j_{2}}...r_{i_{m}j_{m}})$$

$$= Z(c_{\sigma(i_{1})\sigma(i_{2})...\sigma(i_{m})})$$

$$= \sum_{j_{1},...,j_{m}=1}^{n} r_{i_{1}j_{1}}Z(c_{j_{1}j_{2}...j_{m}})r_{i_{2}j_{2}}...r_{i_{m}j_{m}}.$$

So  $Z(\mathbb{B}) = RZ(\mathbb{C})R^T$ , and thus  $Z(\mathbb{A})$  and  $Z(\mathbb{B})$  are permutational similar.

**Corollary 3.3.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be two order  $m \geq 3$  dimension n tensors,  $N(\mathbb{A})$  the number of the nonzero entries of tensor  $\mathbb{A}$ . If the tensors  $\mathbb{A}$  and  $\mathbb{B}$  are similar, then  $N(\mathbb{A}) = N(\mathbb{B})$ .

*Proof.* By Theorem 3.2, there exists a permutation matrix  $R = R_{\sigma} = (r_{ij})$  of order n such that  $Z(\mathbb{B}) = RZ(\mathbb{A})R^T$ , where  $\sigma \in S_n$  is a permutation on the set [n] and  $r_{ij} = 1 \Leftrightarrow j = \sigma(i)$ . Then similar to the proof of Theorem 3.2, we have

$$(Z(\mathbb{B}))_{i_1 i_2 \dots i_m} = (RZ(\mathbb{A})R^T)_{i_1 i_2 \dots i_m}$$

$$= \sum_{j_1, \dots, j_m = 1}^n r_{i_1 j_1}(Z(\mathbb{A}))_{j_1 j_2 \dots j_m} (R^T)_{j_2 i_2} \dots (R^T)_{j_m i_m}$$

$$= \sum_{j_1, \dots, j_m = 1}^n r_{i_1 j_1}(Z(\mathbb{A}))_{j_1 j_2 \dots j_m} r_{i_2 j_2} \dots r_{i_m j_m}$$

$$= (Z(\mathbb{A}))_{\sigma(i_1)\sigma(i_2)\dots\sigma(i_m)}.$$

Therefore  $(Z(\mathbb{B}))_{i_1 i_2 \dots i_m} \neq 0 \Leftrightarrow (Z(\mathbb{A}))_{\sigma(i_1)\sigma(i_2)\dots\sigma(i_m)} \neq 0$ , and thus  $N(\mathbb{A})=N(\mathbb{B})$ .

**Remark 3.4.** Note that the result of Corollary 3.3 does not hold when m=2. For example, let  $P=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $Q=\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $A=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ,  $B=PAQ=\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ . It is clear PIQ=QIP=I, then the matrix A and B are similar, but  $N(A)=1\neq N(B)=4$ .

**Definition 3.5.** ([1]) Let  $\mathbb{A} = (a_{i_1 i_2 \dots i_m})_{1 \leq i_j \leq n}$  be an order  $m \geq 3$  dimension n tensor. If  $a_{i_1 i_2 \dots i_m} \equiv 0$  whenever  $\min\{i_2, \dots, i_m\} < i_1$ , then  $\mathbb{A}$  is called an upper triangular tensor. If  $a_{i_1 i_2 \dots i_m} \equiv 0$  whenever  $\max\{i_2, \dots, i_m\} > i_1$ , then  $\mathbb{A}$  is called a low triangular tensor. If  $\mathbb{A}$  is either upper or low triangular tensor, then  $\mathbb{A}$  is called a triangular tensor. If  $a_{i_1 i_2 \dots i_m} \equiv 0$  whenever  $i_1 i_2 \dots i_m \neq i_1 i_1 \dots i_1$ , then  $\mathbb{A}$  is called a diagonal tensor.

Clearly, a triangular tensor is both an upper and a low triangular tensor.

**Corollary 3.6.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be two order  $m \geq 3$  dimension n tensors. Suppose that  $\mathbb{A}$  and  $\mathbb{B}$  are similar, and  $\mathbb{A}$  is a diagonal tensor, then  $\mathbb{B}$  is a diagonal tensor.

*Proof.* We only need to show  $(Z(\mathbb{B}))_{i_1 i_2 \dots i_m} = 0$  (and thus  $b_{i_1 i_2 \dots i_m} = 0$ ) for any  $i_1 i_2 \dots i_m \neq i_1 i_2 \dots i_1$  where  $i_1 \in [n]$ .

By Theorem 3.2, there exists a permutation matrix  $R = R_{\sigma} = (r_{ij})$  of order n such that  $Z(\mathbb{B}) = RZ(\mathbb{A})R^T$ , where  $\sigma \in S_n$  is a permutation on the set [n] and  $r_{ij} = 1 \Leftrightarrow j = \sigma(i)$ . Then by the proof of Corollary 3.3, we have  $(Z(\mathbb{B}))_{i_1i_2...i_m} \neq 0 \Leftrightarrow (Z(\mathbb{A}))_{\sigma(i_1)\sigma(i_2)...\sigma(i_m)} \neq 0$ . Note that  $i_1i_2...i_m \neq i_1i_1...i_1$  if and only if  $\sigma(i_1)\sigma(i_2)...\sigma(i_m) \neq \sigma(i_1)\sigma(i_1)...\sigma(i_1)$ , therefore  $(Z(\mathbb{B}))_{i_1i_2...i_m} = 0$  when  $i_1i_2...i_m \neq i_1i_1...i_1$  since  $\mathbb{A}$  is a diagonal tensor.  $\square$ 

**Remark 3.7.** Note that the result of Corollary 3.6 does not hold when m = 2. For example, all real symmetric matrices are similar to diagonal matrices, but symmetric matrices are not diagonal matrices in general. Another example, if a real matrix A of order n has n distinct eigenvalues, then A is similar to diagonal matrices, but A is not necessarily a diagonal matrix.

**Definition 3.8.** A Jordan block  $J_k(\lambda)$  is an upper triangular matrix of order k of the form

$$P = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}.$$

There are k-1 terms 1 in the superdiagonal; the scalar  $\lambda$  appears k times on the main diagonal. All other entries are zero, and  $J_1(\lambda) = [\lambda]$ .

**Theorem 3.9.** ([2], Jordan canonical form theorem) Let A be a given complex matrix of order n. There is a nonsingular matrix S of order n such that  $A = SJS^{-1}$ , where

$$J = \begin{pmatrix} J_{n_1}(\lambda_1) & & & 0 \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{n_k}(\lambda_k) \end{pmatrix}$$

is a Jordon matrix with  $n_1 + n_2 + \ldots + n_k = n$ , and J is unique up to permutations of the diagonal Jordan blocks, called the Jordan canonical form of A. The eigenvalues  $\lambda_i$ ,  $i = 1, \ldots, k$  are not necessarily distinct. If A is a real matrix with only real eigenvalues, then the similarity matrix S can be taken to be real.

Clearly, the Jordon matrix J is an upper triangular matrix.

Remark 3.10. The results of matrices about the Jordan canonical form cannot be extended to tensors since not all tensors are similar to some upper triangular tensor by Corollary 3.3.

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