# Computer Projects: Applied Stochastic Analysis

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There are several choices for the computer projects of my course "Applied Stochastic Analysis". Each student is required to choose *2 projects* to make the study. The final project reports must be carefully written with LaTeX to include the following points:

- The detailed setup of the problem.
- The procedure you take to do the computation and analysis of the numerical results.
- The issues you encounter and how you overcome.
- Possible discussion about the results and further thinking.

Please submit the hardcopy to our TA. The reports could be composed in either Chinese or English.

1. Ising model: Basic methods.

   *Problem.* Consider the 2D Ising model on the $N \times N$ square lattice with periodic boundary condition. The Hamiltonian of the system is defined as

   $$H(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i, \quad i = 1, 2 \dots, N^2$$

   where $\sigma = (\sigma_i)$ is the configuration of the spin system, $\sigma_i = \pm 1$ is the state of spin on the $i$th site, $\langle i, j \rangle$ means the nearest neighbor interaction, $h$ is the external magnetic strength. Apply Monte Carlo simulations to investigate the following questions.

   (a) Take $J = 1$, $k_B = 1$ and $h = 0$. Plot the internal energy $u$

   $$u = \frac{U}{N^2} \quad \text{where} \quad U = \langle H \rangle = \frac{1}{N^2} \frac{1}{Z} \sum_{\sigma} H(\sigma) \exp(-\beta H(\sigma))$$

   and the specific heat

   $$c = \frac{C}{N^2} \quad \text{where} \quad C = k_B \beta^2 \mathrm{Var}(H)$$

   as the function of temperature $T$, where $\beta = (k_B T)^{-1}$ and $Z = \sum_\sigma \exp(-\beta H(\sigma))$ is the partition function. Identify the critical temperature $T_*$ of the phase transition when $N$ is sufficiently large.

(b) For different temperature $T$, plot the magnetization

$$m = \frac{M}{N^2} \quad \text{where} \quad M = \Big\langle \sum_i \sigma_i \Big\rangle$$

as the function of $h$. Can you say something from these plots?

(c) Define the spatial correlation function

$$\Gamma(r) = \langle \rho(0)\rho(r) \rangle$$

and the correlation length $\xi$ as the characteristic length that $\Gamma(r)$ decays to 0. Here $\rho(r)$ is the spatial configuration field of $\sigma$. You can approximate $\Gamma(r)$ by considering the spin correlations in the horizontal, or vertical direction (or any other direction). The starting point $\rho(0)$ may be chosen arbitrarily or by taking average with respect to all possible sites.

Study the correlation length $\xi$ as the function of $T$ when $h = 0$, and compare your results obtained by different choices.

(d) When $h = 0$, investigate the behavior of $c, m$ and $\xi$ around the critical temperature $T_*$ if we assume the limiting behavior

$$m \sim m_0 \epsilon^{\alpha}, \;\; c \sim c_0 \epsilon^{-\gamma} \text{ and } \xi \sim \xi_0 \epsilon^{-\delta},$$

where $\epsilon = |1 - T/T_*|$. That is, you need to numerically find the scaling exponents $\alpha$, $\gamma$ and $\delta$.

(e) (*optional*) Study the above problems in the 3D case.

2. Ising model: Advanced methods.

   *Problem.* Apply the kinetic Monte Carlo method, or Swendsen-Wang algorithm, or multi-level sampling methods to study the 2D Ising model if you have already the experience to compute with standard algorithm. Compare the efficiency with the standard algorithm at different temperature.

3. Potts model.

   *Problem.* Apply the Monte Carlo simulations to study the phase transition behavior of the Potts model. The Hamiltonian of the $q$-state Potts model is defined as

$$H(\sigma) = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}$$

   where $\sigma_i = 1, 2, \ldots, q$. Take $q = 3$ or $q = 10$ as concrete example to explore similar problems listed in the Ising model.

4. Clock model.

*Problem.* Apply the Monte Carlo simulations to study the phase transition behavior of the clock model. The Hamiltonian of the $q$-state clock model is defined as

$$H(\sigma) = -J \sum_{\langle i,j \rangle} S_i \cdot S_j$$

where

$$S_i = \left( \cos \frac{2m\pi}{q}, \sin \frac{2m\pi}{q} \right), \; m = 0, 1, \ldots, q-1$$

are $q$ discrete directions on a clock. It is equivalent to Ising model when $q = 4$. Take $q = 6$ as concrete example to explore similar problems listed in the Ising model.

5. Traveling Salesman Problem (TSP).

   *Problem.* Apply the simulated annealing or genetic algorithm to solve the CHN144 problem, i.e. the TSP problem

   $$\min_{x \in \chi} H(x) = \sum_{i=1}^{N} l_{x_i x_{i+1}}, \quad x_{N+1} := x_1,$$

   where $\chi$ is the set of all possible permutations of $\{1, 2, \ldots, N\}$. The CHN144 data can be downloaded from here.

6. Numerical SDE: Multilevel Monte Carlo method.

   *Problem.* Perform the standard Euler-Maruyama, Milstein scheme and the multilevel Monte Carlo method to the simple SDE:

   $$dS_t = rS_t dt + \sigma S_t dW_t, \; t \in [0, 1]$$

   with $S_0 = 1$, $r = 0.05$ and $\sigma = 0.2$. The interested function $P$ of $S$ has the form

   $$P = e^{-r} \max\{0, S_1 - 1\}.$$

   Compare the numerical efficiency of different methods.

7. Numerical SDE: Exit problem.

   *Problem.* Numerically solve the following boundary value problem via the simulation of SDEs

   $$\begin{cases} b \cdot \nabla u + \frac{1}{2} \Delta u = f(x, y), & (x, y) \in B_1(0), \\ u = \frac{1}{2} \text{ on } (x, y) \in \mathbb{S}^1, \end{cases}$$

   where $b = (x, y)$, $f(x, y) = x^2 + y^2 + 1$. We have exact solution $u(x, y) = (x^2 + y^2)/2$ for the model problem. Utilize the standard Euler-Maruyama scheme to do the simulation and check the numerical convergence order in time.

   Investigate the multi-level Monte Carlo methodology to solve the above exit problem$^{(optional)}$.

8. Random Matrix Theory.

*Problem.* Explore the spectral property of random matrices through numerical experiments and analytical study.

(a) Form the GOE($N$) ensemble by generating $N \times N$ matrices $H$ with $H = H^T$, $H_{ij} \sim N(0,1)$ and $H_{ii} \sim \sqrt{2}N(0,1)$. The upper triangular elements are independent.

(b) Take $N = 2, 4, 10$ or other choices. Compute the eigenvalues $\lambda_n$ and sort them in the increasing order. Define the normalized spacing

$$s = (\lambda_{n+1} - \lambda_n)/\langle s \rangle$$

where $\langle s \rangle = \langle \lambda_{n+1} - \lambda_n \rangle$ is the empirical mean spacing. Plot the histogram of the eigenvalue spacings.

(c) Take $N = 2$. Try to derive the distribution of the eigenvalue spacing analytically. Compare this analytical result with numerical experiments.

(d) Try to write down the probability density $\rho(H)$ for GOE($N$) ensemble and prove that is invariant under orthogonal transformation, namely $\rho(Q^T HQ) = \rho(H)$, where $Q$ is any orthogonal matrix. What can you infer from this property?

(e) Perform similar task for the GUE($N$) ensemble, i.e. the $N \times N$ complex matrices $H$ with $H = H^*$ such that $H_{ij} \sim N(0, 1/2) + iN(0, 1/2)$ and $H_{ii} \sim N(0,1)$.

(f) Numerically investigate the distribution of eigenvalues of real Wigner matrices $X$ with $X = X^T$ and $X_{ij} \sim N(0,1)$ independently when $N$ is sufficiently large. How should you rescale the eigenvalues to observe their empirical distribution in a suitable window? What do you find? Perform similar task to the GOE, GUE ensemble and complex Wigner matrices $X$, where $X = X^*$ and $X_{ij}$ has mean 0 and $\mathbb{E}|X_{ij}|^2 = 1$.