Stochatic elliptic Partial Differectial Equation: exiting problem

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1 Steps of solution of elliptic PDE by stochastic process

a reference can be found at exit problem for PDE
the source code in github page XShinHe/simple_SPDEs
for general elliptic PDE:

$$
\begin{aligned}
Lu(x) - c(x)u(x) &= g(x), \quad x \in \Omega \\
u(x) &= f(x), \quad x \in \partial\Omega
\end{aligned}
$$

where the operator (Einstern summation convenstion)

$$
\begin{aligned}
L &= b_i \partial_i + \frac{1}{2} a_{ij} \partial_i \partial_j \\
a_{ij} &= \sigma_{ik} (\sigma^T)_{kj} = \sigma_{ik} \sigma_{jk}
\end{aligned}
$$

the wiener integral path gives:

$$
u(x) = \mathbb{E}_x^L \Big[\int_0^{t_{\partial\Omega}} \{ \frac{f(X^x(t_{\partial\Omega}))}{t_{\partial\Omega}} - g(X^x(t)) \} e^{-\int_0^t c(X^x(s)) ds} dt \Big]
$$

with the stochastic path:

$$
\begin{aligned}
dX^x(t) &= b(X^x, t)dt + \sigma(X^x, t)dW \\
X^x(0) &= x
\end{aligned}
$$

2 The concrete example (HW7. Numerical SDE: Exit problem.)

for problem:

$$
\begin{aligned}
x\partial_x u + y\partial_y u + \frac{1}{2}\Delta u &= x^2 + y^2 + 1 \overset{def}{=} g(x, y) \\
u(\partial\Omega) = 1/2 &\overset{def}{=} f(x, y), \quad \partial\Omega : x^2 + y^2 = 1
\end{aligned}
$$

the exact solution is

$$u(x,y) = \frac{1}{2}(x^2 + y^2)$$

so

$$u(x,y) = \mathbb{E}^L_{x,y}[f(X^{x,y}(t_{\partial\Omega})) - \int_0^{t_{\partial\Omega}} g(X)dt]$$

so the stochastic of elliptic PDE needs:

- from start point (x,y) start, simulate process by (with Eular-Maruyama method)

$$dX_t = bX_t + dW_t, \quad X \overset{def}{=} (X_x, X_y), \quad b = (X_x, X_y)$$
$$X_{n+1} - X_n = (X_{nx}, X_{ny})^T X_n \delta t + \delta W_t$$

- evaluate u(x,y) from

$$\begin{aligned} u(x,y) &= \mathbb{E}^L_{x,y}[f(X^{x,y}(t_{\partial\Omega})) - \int_0^{t_{\partial\Omega}} g(X)dt] \\ &= \mathbb{E}^L_{x,y}[\frac{1}{2} - \int_0^{t_{\partial\Omega}} (X^2_{tx} + X^2_{ty} + 1)dt] \end{aligned}$$

- repeat above until converge with error threshold.

But, note the implement of stochastic is only dependent the polar radius in Ω, irrelative to the rangle, so we solution must to be (**cylindrically symmetry**), this symmetry can be used to reduce the calculation of stochastic process.

3 Following gives the python version

python version is only an implement, and for faster calculation, the results actually are collected from fortran version, seeing my github project simple_SPDEs

```
In [0]: import numpy as np

        def check_if_exit(v):
            # Checks if v has intersected with the boundary of D = S(1)
            return (np.linalg.norm(v,2) >=1)

        def em(x,dt):
            # b = (x,y), sigma = 1
            x += x * dt + np.random.normal(size=2,scale=np.sqrt(dt))

        # nonhomogeneous term
        g = lambda x : 1+x[0]*x[0]+x[1]*x[1]
        # boundary term
        f = lambda x : 0.5

        def simulate_exit_time(v):
```

```python
    # Simulates exit time starting at v=(x,y), returns exit position
    delta_t = 0.0001
    exit = False

    # Copy because simulation modifies in place
    if hasattr(v,'copy'): # For NumPy arrays
        x = v.copy()
    else:
        x = np.array(v) # We input a non-NumPy array

    esti_val = 0
    while not check_if_exit(x):
        em(x, delta_t)
        esti_val -= g(x) * delta_t
    return esti_val + f(x)

v=np.array((0.5,0.5)) # The origin
print(v)

u = lambda x : 0.5*(x[0]**2+x[1]**2)

def get_exp_f_exit(starting_point, n_trials):
    return np.mean([ simulate_exit_time(v) for k in range(0,n_trials)])

exp_f_exit = get_exp_f_exit(v,2000) # Expected value of f(Exit(x,d))
print('The value u(v) = %s\nThe value of Exp(f(Exit))=%s' %(u(v), exp_f_exit))
```

```
[ 0.5  0.5]
The value u(v) = 0.25
The value of Exp(f(Exit))=0.233134289226
```

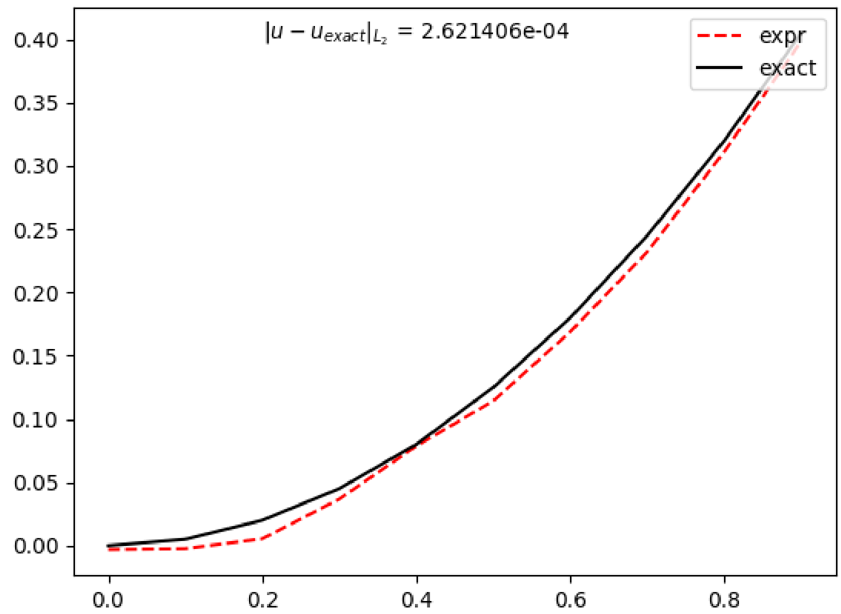
4 Result analysis

4.1 first we show calculations along the radius

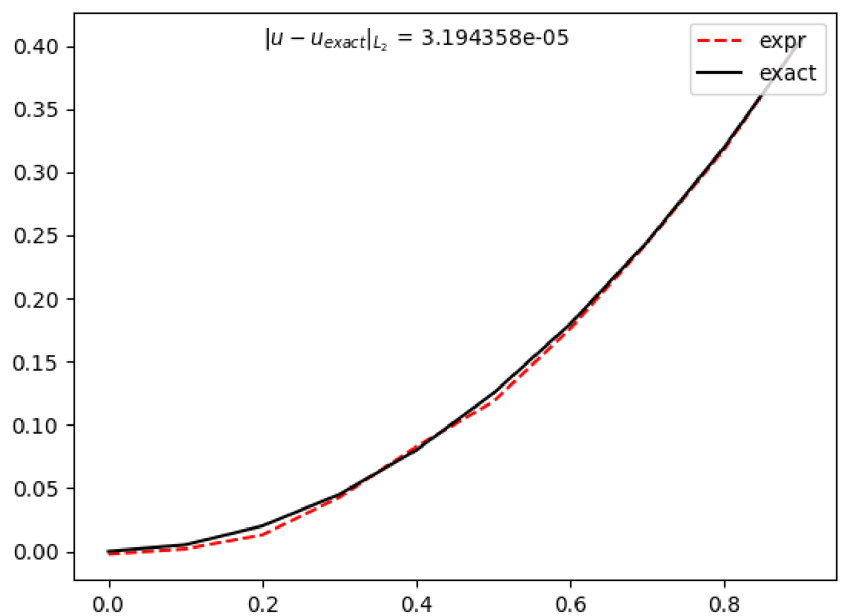
here the error defined as:

$$Err = \int_0^1 2\pi r dr |u_{expr}(r) - u_{exact}(r)|^2$$

1. the simulation choose $dt = 0.0001$ and $N_{MC} = 3000$



2. the simulation choose $dt = 0.00002$ and $N_{MC} = 3000$



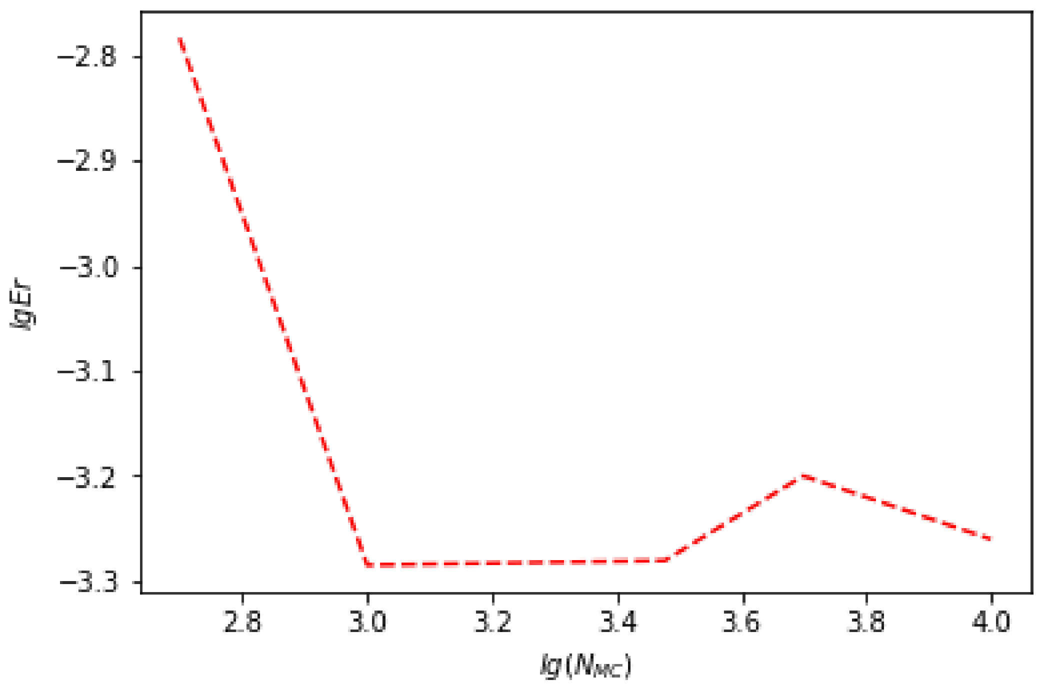
4.2 the relation of error with N_{MC}

here $dt = 0.0005$, it show that $N_{MC} \geq 1000$ is enough to reach the limit of error.

```
In [14]: import numpy as np
         import pandas as pd
         import matplotlib.pyplot as plt
         %matplotlib inline

         # the dt=0.0005
         a=pd.read_csv('collect2.dat',header=None,sep='\s+').values
         logN = np.log10(a[:,0])

         logerr = np.log10(a[:,1])
         plt.plot(logN,logerr,'r--')
         plt.xlabel(r'$lg(N_{MC})$')
         plt.ylabel(r'$lg Er$')
         plt.show()
```



4.3 the relation of error with dt

all $N_{MC} = 3000$, $dt = 0.01, 0.005, 0.002, 0.001, 0.0005, 0.0002, 0.0001, 0.00005, 0.00002$
the result show that Eular-Maruyama is 1-order in weak convergence.

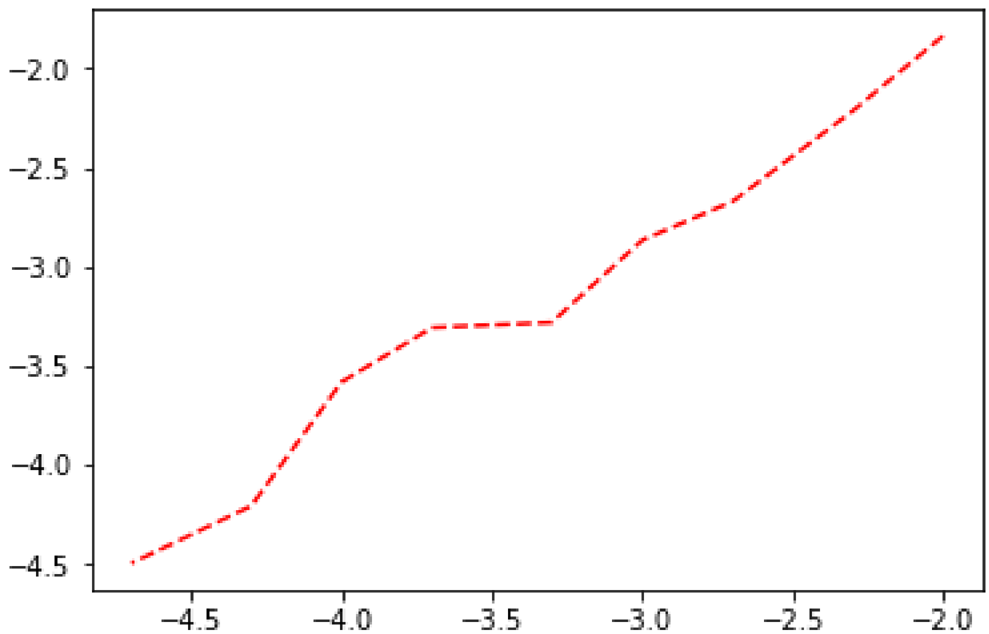
```
In [17]: import numpy as np
         import pandas as pd
         import matplotlib.pyplot as plt
         import scipy.optimize as optimization
```

```
%matplotlib inline

a=pd.read_csv('collect.dat',header=None,sep='\s+').values
logdt = np.log10(a[:,0])
logerr = np.log10(a[:,1])

plt.plot(logdt,logerr,'r--')
plt.show()

def func(x, k, m):
    return k*x + m
sigma = np.zeros(len(logdt)) + 0.5
args = optimization.curve_fit(func, logdt, logerr , np.array([0.0,0.0]), sigma)[0]
print('the order is: %f '%(args[0]))
```




```
the order is: 0.937673
```