PS 5

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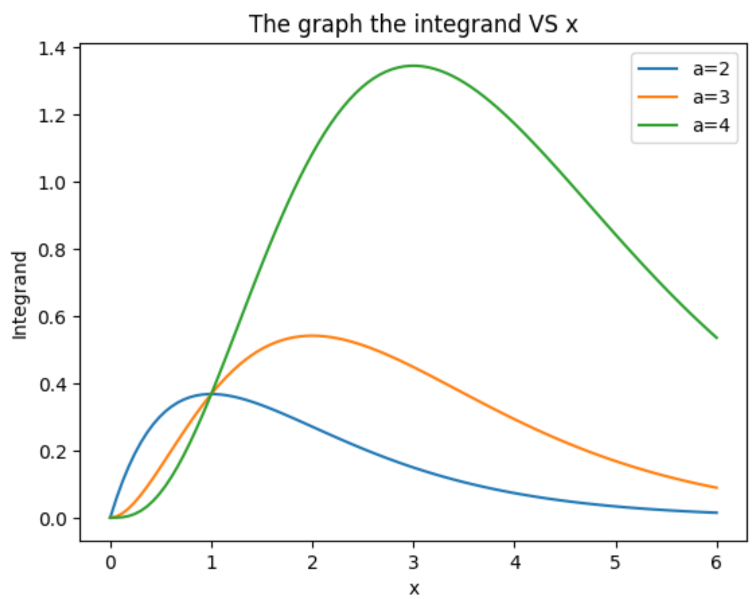
1 Exercise 5.17

a) Here is the graph of our integrand as a function of x. All the curves start at zeros, rises to the maximum, and then decays again. When the parameter a increases, the maximum value of function will be higher and the position x where the peak falls also shift to the right.

Figure 1: The graph of integrand of the gamma function versus x



b) When the integrand reaches its peak, the derivative function of the integrand function should be zero. Based on the calculation shown below, we proved that the maximum falls at $x = a - 1$. The integrand is

$$I = x^{a-1}e^{-x}$$
$$\frac{dI}{dx} = e^{-x}x^{a-2}(a-1-x)$$

Therefore, if $\frac{dI}{dx} = 0$, then $a - 1 - x = 0$, hence $x = a - 1$.

c) Based on our calculation below, $x = c$ gives the change of variable value $z = 1/2$. Therefore, $c = a - 1$ puts the peak of the integrand for the gamma function.

$$z = \frac{x}{c+x} = \frac{1}{2}$$
$$x = c$$

At $z = 1/2$, $x = c$, so the peak value falls at

$$x = a - 1 = c$$

d) The new expresssion will be:

$$e^{(a-1)\ln x} = e^{\ln x^{a-1}} = x^{a-1}$$
$$Integrand = x^{a-1}e^{-x} = e^{(a-1)\ln x}e^{-x} = e^{(a-1)\ln x - x}$$

When x is very large, the term e^{-x} is very small, and the term $(a-1)\ln x$ is also relatively small, so the term $(a-1)\ln x$ can limit the growth of the function when the value of x is very large as $\ln x$ tends to be small. When x is very small, then term $e^{-}x$ can limit the reduction of the function. Hence, numerical overflow or underflow can be avoided.

e) First, we need calculate the new $f(x)$ and the expression of dx by dz. We know $c = a - 1$ from part (c)

$$z = \frac{x}{c+x} = \frac{x}{a-1+x}$$
$$\text{so } x = \frac{(a-1)z}{1-z}$$
$$f(x) = f(\frac{(a-1)z}{1-z}) = e^{(a-1)\ln x - x} = e^{(a-1)\ln\frac{(a-1)z}{1-z} - \frac{(a-1)z}{1-z}}$$

Next, we need to get the expression dx by dz:

$$\frac{dz}{dx} = \frac{a-1}{(a-1+x)^2}$$

Plug $x = \frac{z(a-1)}{1-z}$ into the equation above, we get

$$\frac{dz}{dx} = \frac{a-1}{(a-1+\frac{z(a-1)}{1-z})^2} = \frac{(1-z)^2}{a-1}$$
$$dx = \frac{(a-1)}{(1-z)^2}dz$$

Therefore, the final integral is

$$\int_0^\infty f(x)\,dx = \int_0^1 \frac{a-1}{(1-z)^2} f(\frac{z(a-1)}{1-z})\,dz = \int_0^1 \frac{a-1}{(1-z)^2} e^{(a-1)\ln\frac{(a-1)z}{1-z} - \frac{(a-1)z}{1-z}}\,dz$$

Then, I wrote a function in python using the new integral, and get the result $\Gamma(3/2) = 0.8862$, which is extremely close to the real value 0.886, so my function is very accurate.

f) The results I got are:

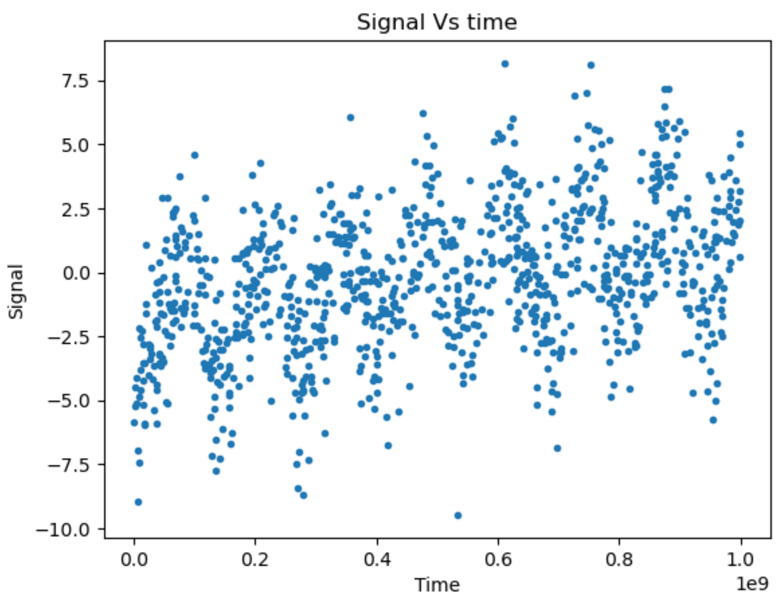
$$\Gamma(3) = 2$$
$$\Gamma(6) = 119.99999999999997$$
$$\Gamma(10) = 362879.9999999999$$

My answers are quite close to 2!, 5!, and 9!.

2 Problem 2

a) After extracting data from the file, we made the graph below. It shows a slightly positive relation between the value of data and time.

Figure 2: The graph of data versus time



b) The method of SVD is used to solve $Ax = b$, where b contains signals, A contains time and its values to the power of a, a extending from 0 to the degree. x is the coefficient to fit the data. For example, if degree is 3, $A = (1, T, T^2, T^3)$, where 1, T, T^2, and T^3 here are all columns. x is a 3×1 matrix of coefficient which multiplies each polynomial, and add them together to get the estimated value of b whose distances between real value are minimized. Therefore, in my code, I firstly calculate the value U, S, and V^T in SVD decomposition such that

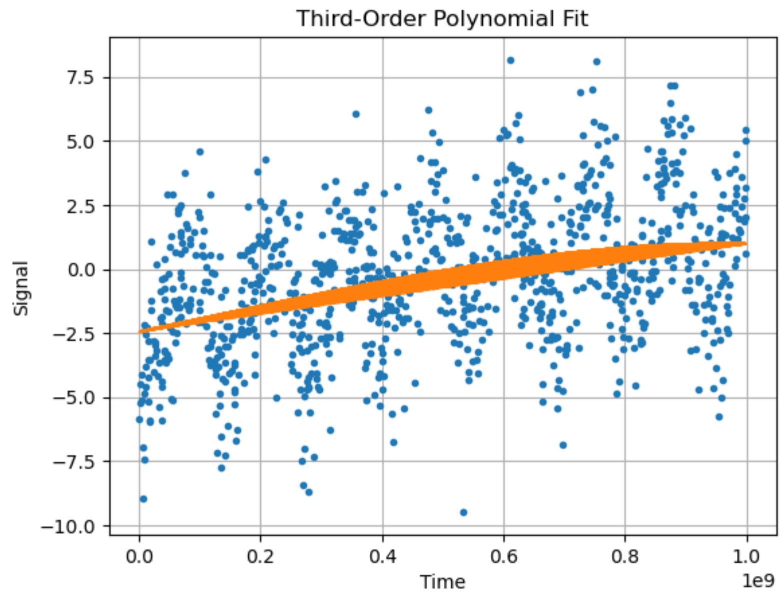
$$USV^T = A$$

then I try to get the value of coefficients by

$$x = A^{-1}b = VS^{-1}U^Tb$$

using U, S, and V^T we've got above, where b is the signal. Then, we put the value of x (coefficient value) into the function "poly1d" to generate a equation which estimated a value of signal. The equation is $Ax^3 + Bx^2 + Cx + D$, where $x^T = (D, C, B, A)$. Then, we plot the result of the equation when each value time plugged versus the time to get the fit line. The value of coefficient we got is $x^T = (-2.24196278 \times 10^{-27}, 7.31252316 \times 10^{-19}, 4.95111993 \times 10^{-9}, -2.44865508)$ The graph of the fitted curve is

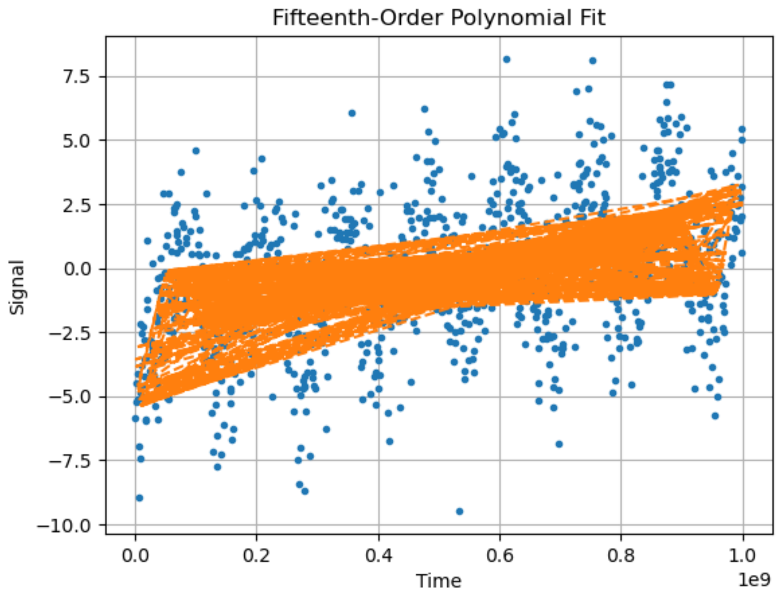
Figure 3: The graph of 3rd-order fit versus time



c) The residue equales $||Ax - b||$, where b is the real value of the signal, Ax is the estimated value of b which can minimize $||Ax - b||$ as I mentioned. We find that there are some extreme values in the data, which can be considered as outliers. Residuals can be affected by outliers in the data. Outliers will decrease the accuracy of the model. Also, the uncertainty of the data is too big, so the residuals are scattered across a wider range, which makes it more difficult to find a clear trend of data.

d) This time we tried the degree of 15, and here is the graph we got:

Figure 4: The graph of 15th-order fit versus time



we find the range of curve gets bigger. The condition number we calculate gets bigger than that in degree of 3. It is much higher than 10^{15}, indicating that the matrix is ill-conditioned, which means the model with degree of 15 is more unreliable than that with degree of 5. I found at degree of 3, the condition number has reach 10^{27}, and it gets bigger and bigger when the degree increases, so it is unlikely that there exists a reasonable polynomials in that case.

e) I put sine and cosine into the matrix to create the model. It did a better than previous one because the data is periodic, which corresponds to the periodic nature of Sin and cos function. The period should be 0.135.