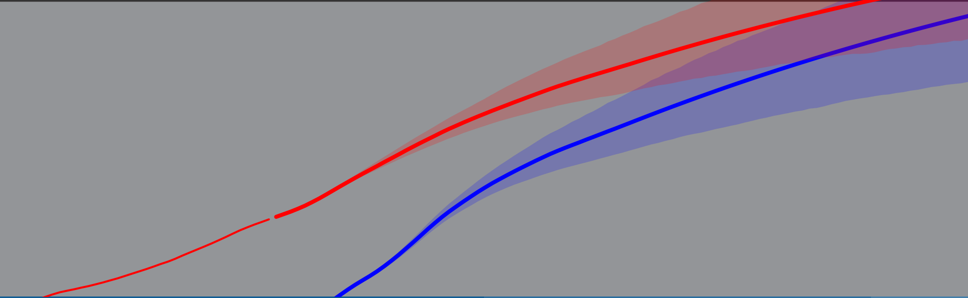
MONASH University

Rob J Hyndman

Coherent mortality forecasting using functional time series models



Mortality rates

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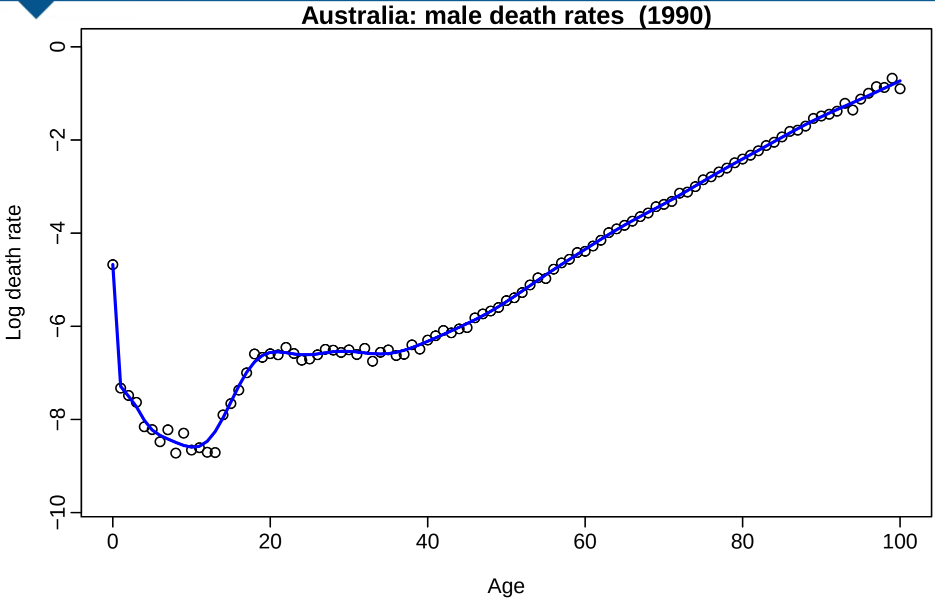

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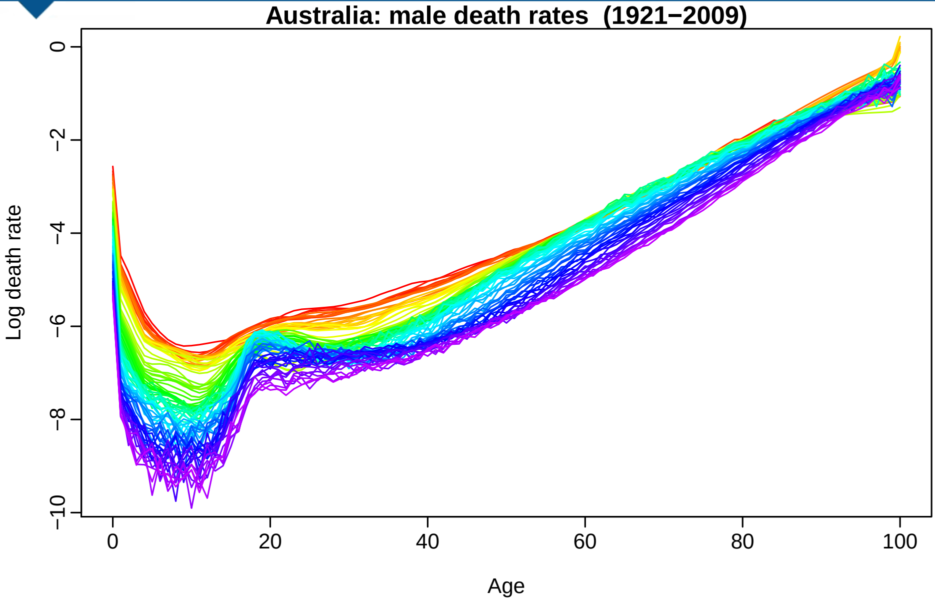

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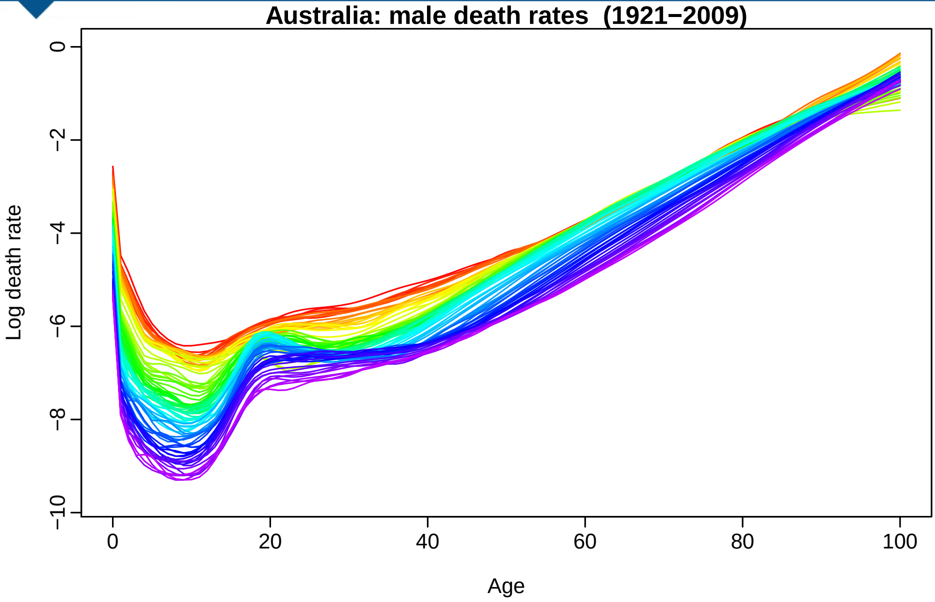

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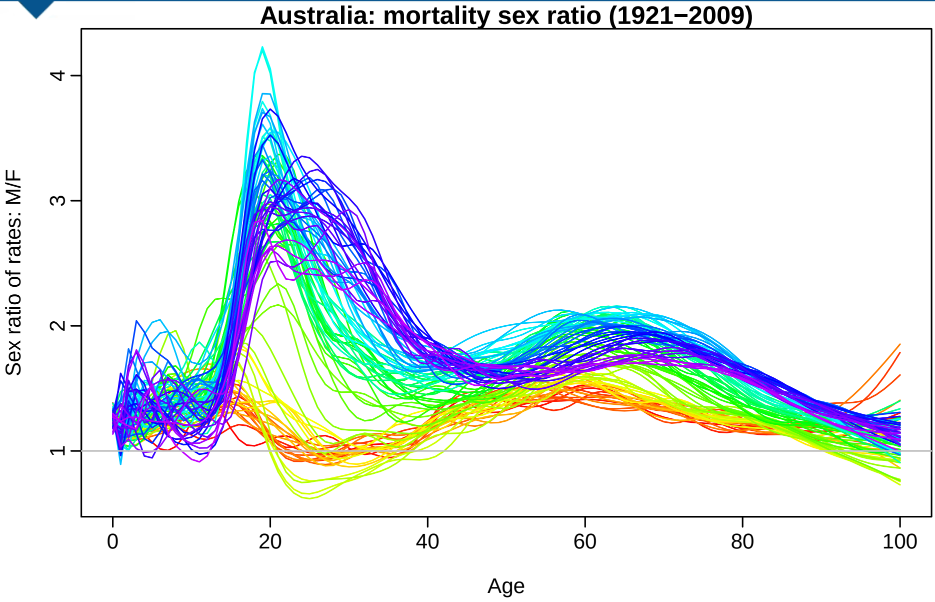

Outline

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Some notation

Let $y_{t,x}$ be the observed (smoothed) data in period t at age x, $t = 1, \dots, n$.

$$y_{t,x} = f_t(x) + \sigma_t(x)\varepsilon_{t,x}$$

$$f_t(x) = \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + e_t(x)$$

- Estimate $f_t(x)$ using penalized regression splines.
- Estimate $\mu(x)$ as mean $f_t(x)$ across years.
- Estimate $\beta_{t,k}$ and $\phi_k(x)$ using functional principal components.
- $\varepsilon_{t,x} \overset{iid}{\sim} N(0,1)$ and $e_t(x) \overset{iid}{\sim} N(0, v(x))$.

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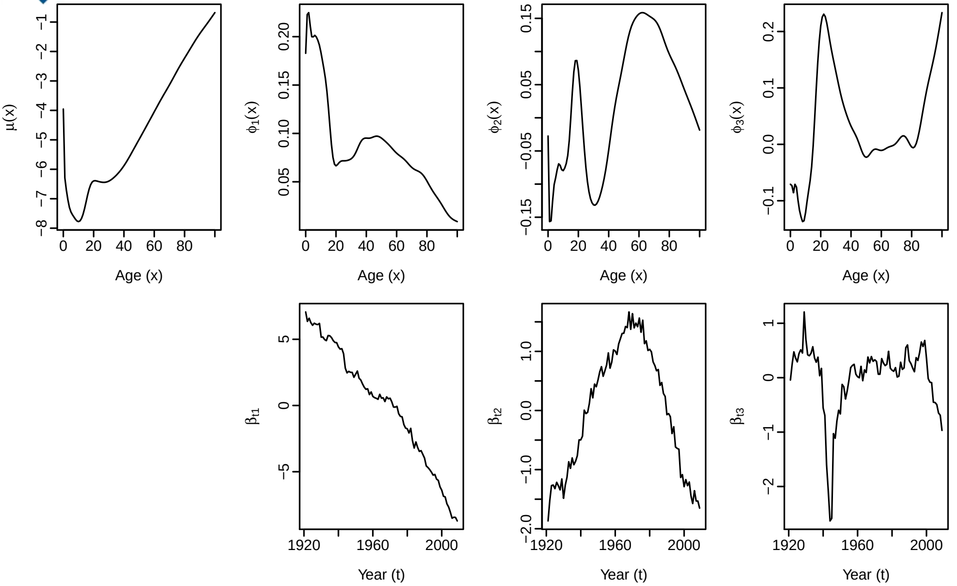
Australian male mortality model



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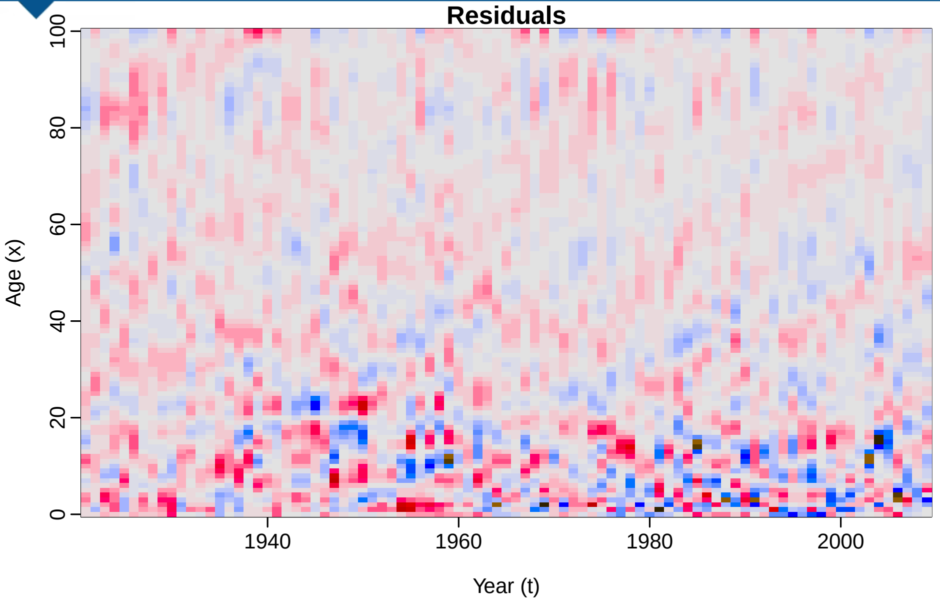

Functional time series model

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- The eigenfunctions $\phi_k(x)$ show the main regions of variation.
- The scores $\{\beta_{t,k}\}$ are uncorrelated by construction. So we can forecast each $\beta_{t,k}$ using a univariate time series model.
- Univariate ARIMA models can be used for forecasting.

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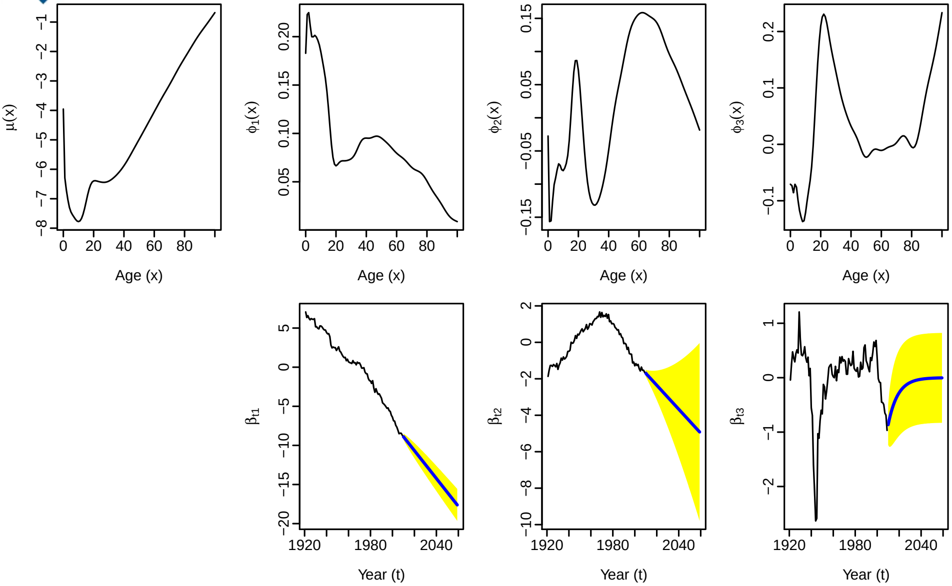
$$\text{E}[y_{n+h,x} \mid \boldsymbol{y}] = \hat{\mu}(x) + \sum_{k=1}^{K} \hat{\beta}_{n+h,k}\,\hat{\phi}_k(x)$$

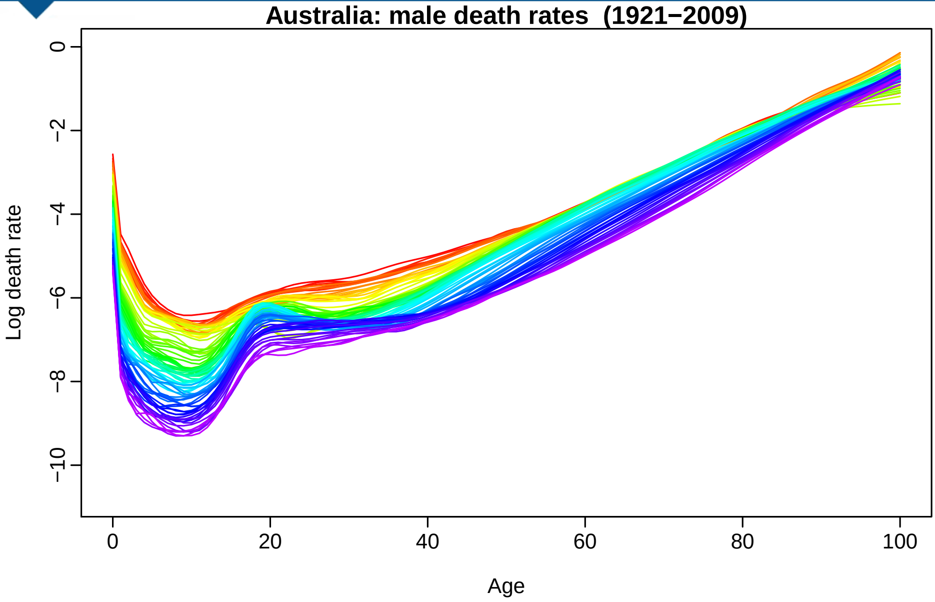
$$\text{Var}[y_{n+h,x} \mid \boldsymbol{y}] = \hat{\sigma}_\mu^2(x) + \sum_{k=1}^{K} v_{n+h,k}\,\hat{\phi}_k^2(x) + \sigma_t^2(x) + v(x)$$

where $v_{n+h,k} = \text{Var}(\beta_{n+h,k} \mid \beta_{1,k}, \dots, \beta_{n,k})$
and $\boldsymbol{y} = [y_{1,x}, \dots, y_{n,x}]$.

Forecasting the PC scores



Forecasts of $f_t(x)$



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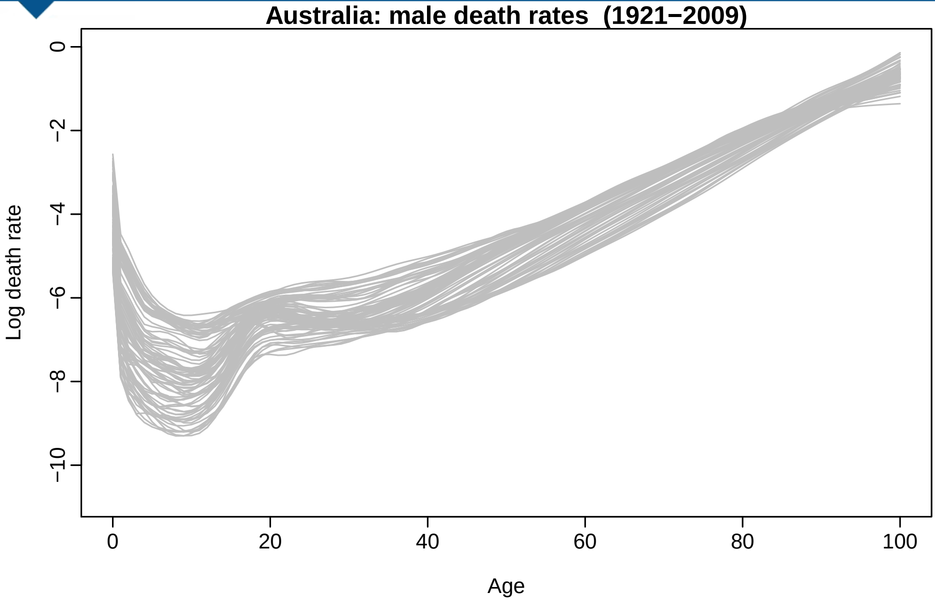

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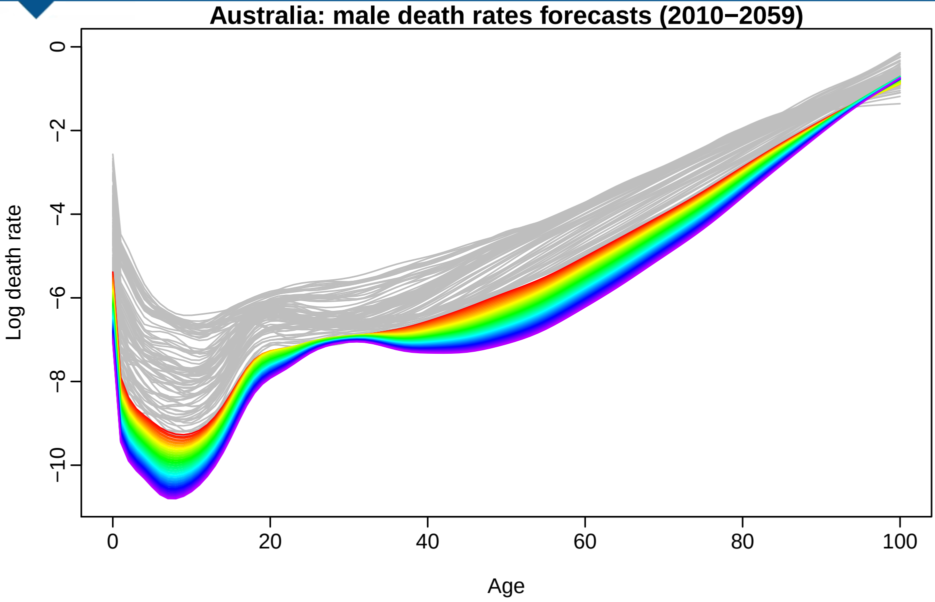

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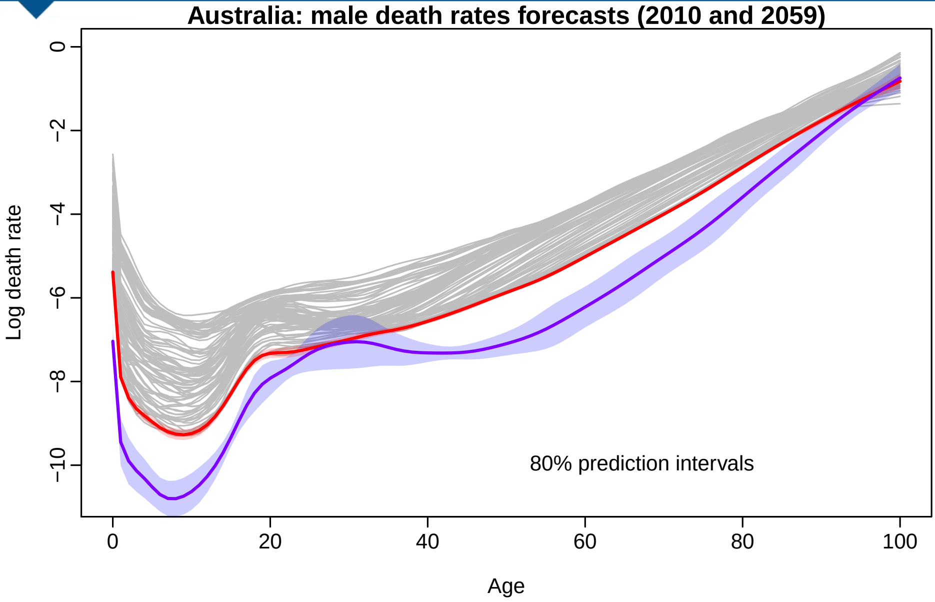
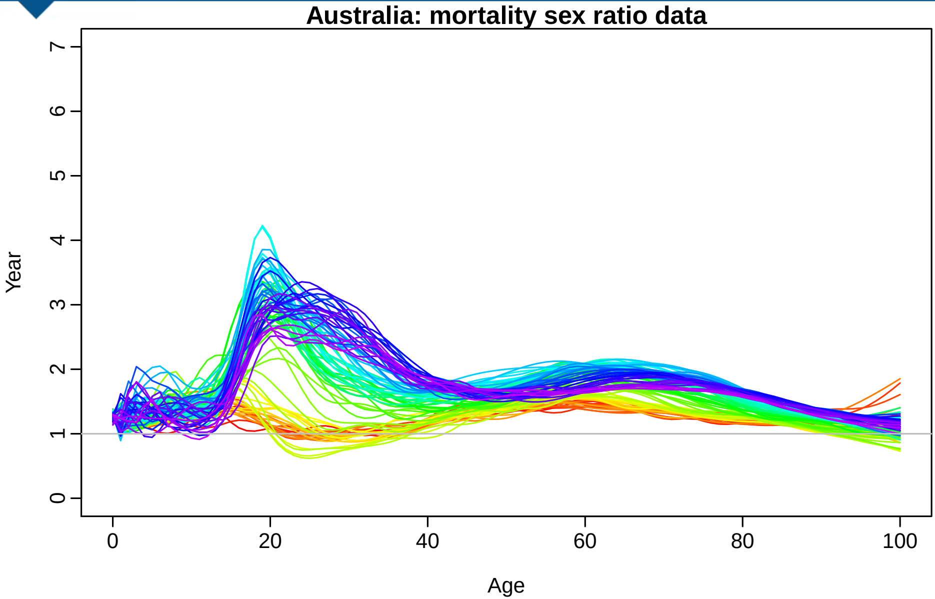


Forecasts of mortality sex ratio



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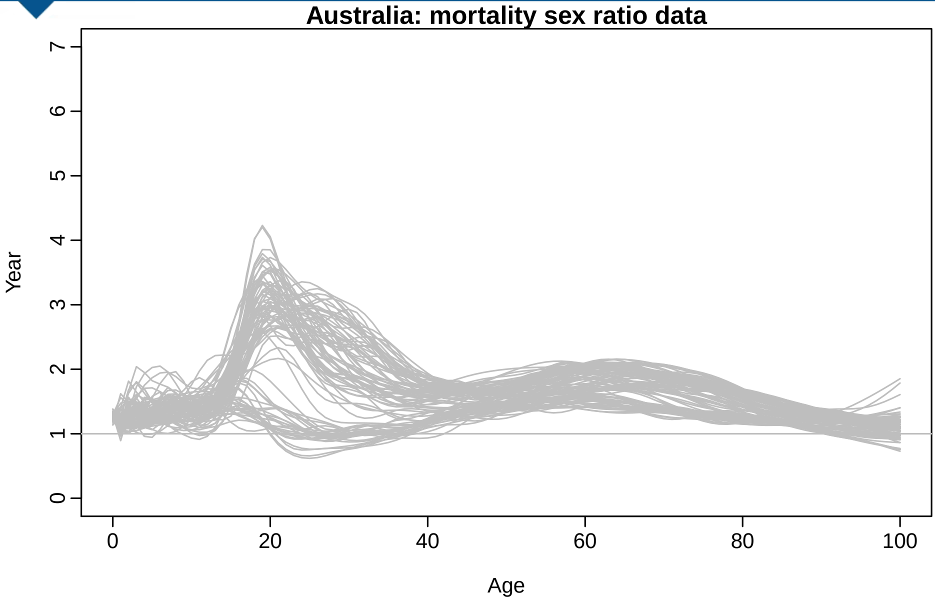

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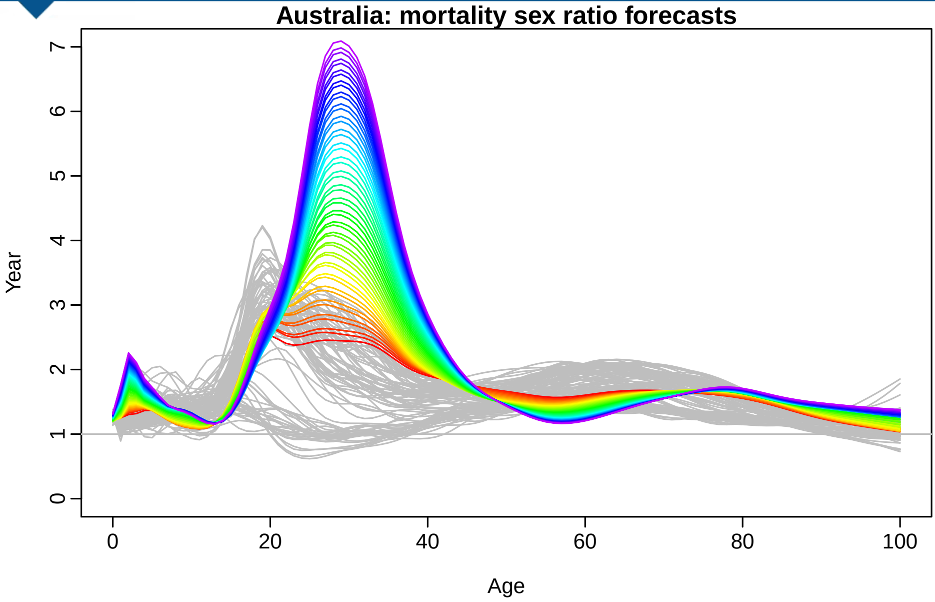

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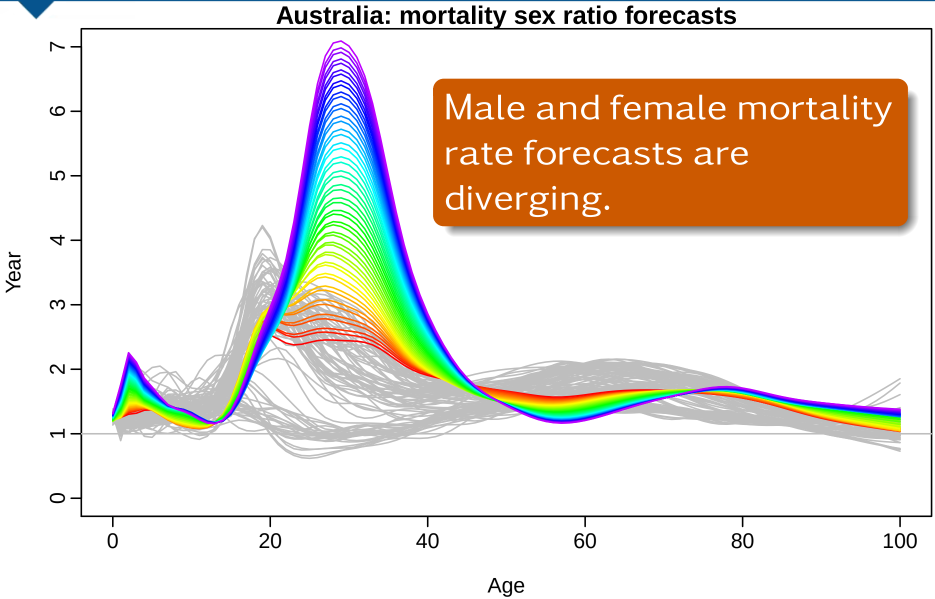

Male and female mortality rate forecasts are diverging.

Outline

The problem

Let $f_{t,j}(x)$ be the smoothed mortality rate for age x in group j in year t.

- Groups may be males and females.
- Groups may be states within a country.
- Expected that groups will behave similarly.
- Coherent forecasts do not diverge over time.
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Forecasting the coefficients

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$$f_t(x) = \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + e_t(x)$$

- We use ARIMA models for each coefficient $\{\beta_{1,j,k}, \dots, \beta_{n,j,k}\}$.
- The ARIMA models are non-stationary for the first few coefficients ($k = 1, 2$)
- Non-stationary ARIMA forecasts will diverge. Hence the mortality forecasts are not coherent.

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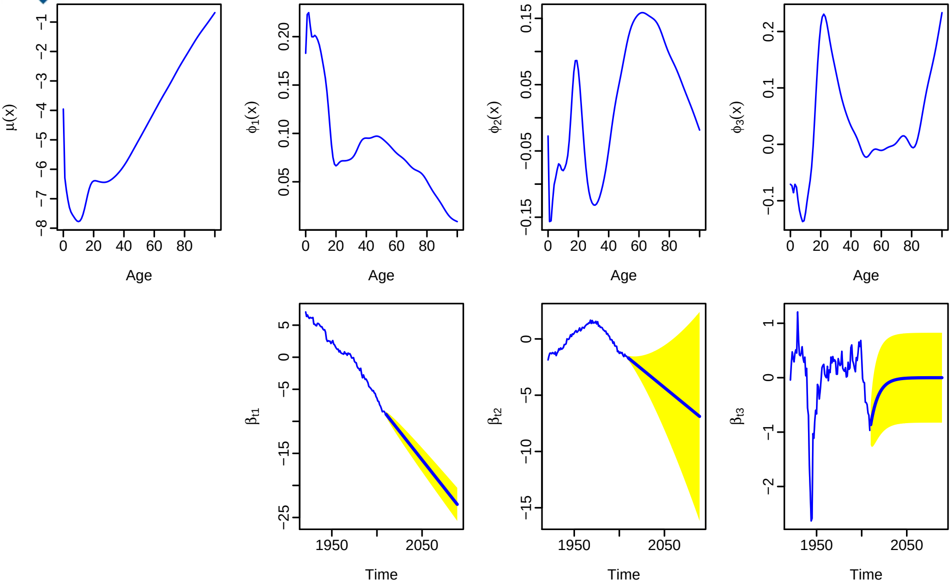
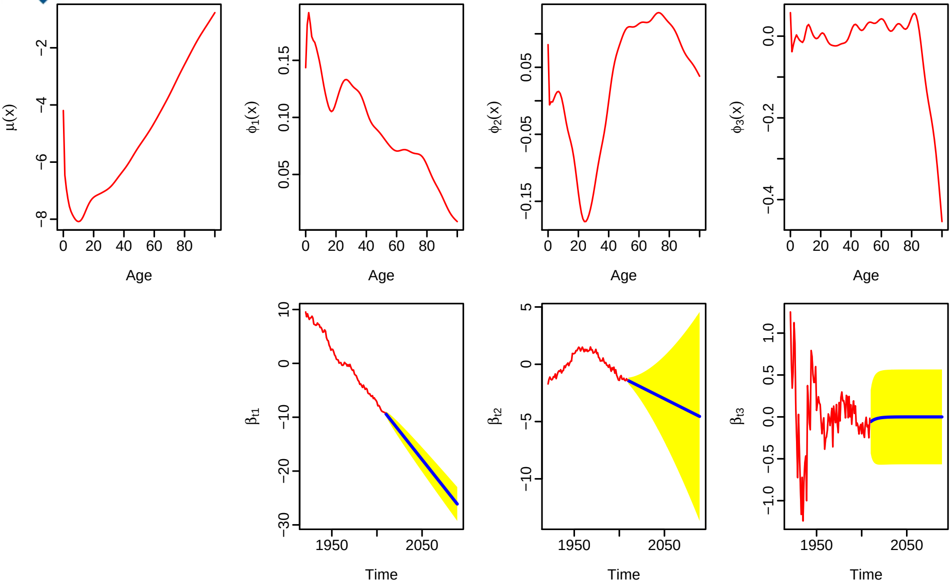
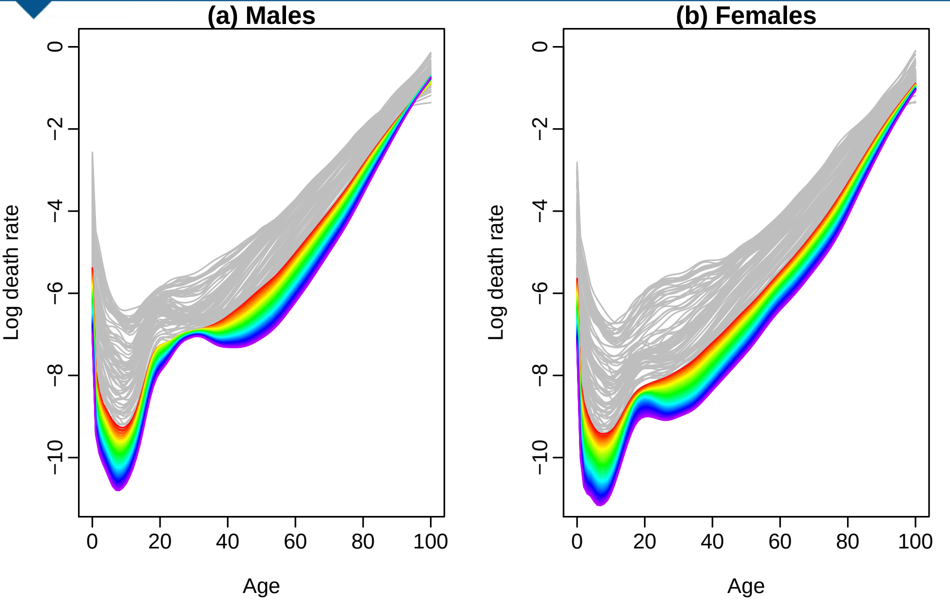


Female fts model



Australian mortality forecasts



Mortality product and ratios

Key idea

Model the geometric mean and the mortality ratio instead of the individual rates for each sex separately.

$$p_t(x) = \sqrt{f_{t,M}(x) f_{t,F}(x)} \quad \text{and} \quad r_t(x) = \sqrt{f_{t,M}(x) \big/ f_{t,F}(x)}.$$

- Product and ratio are approximately independent
- Ratio should be stationary (for coherence) but product can be non-stationary.

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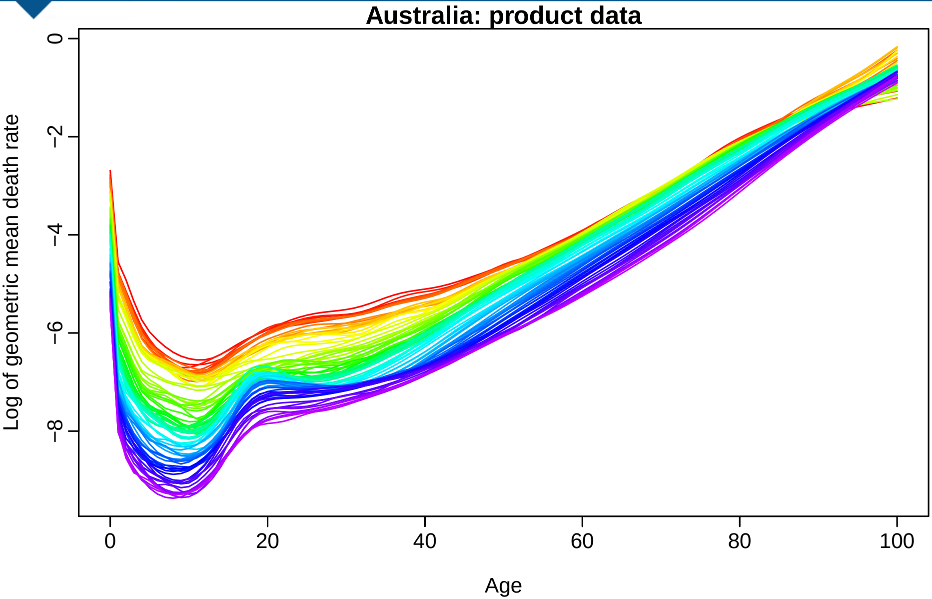
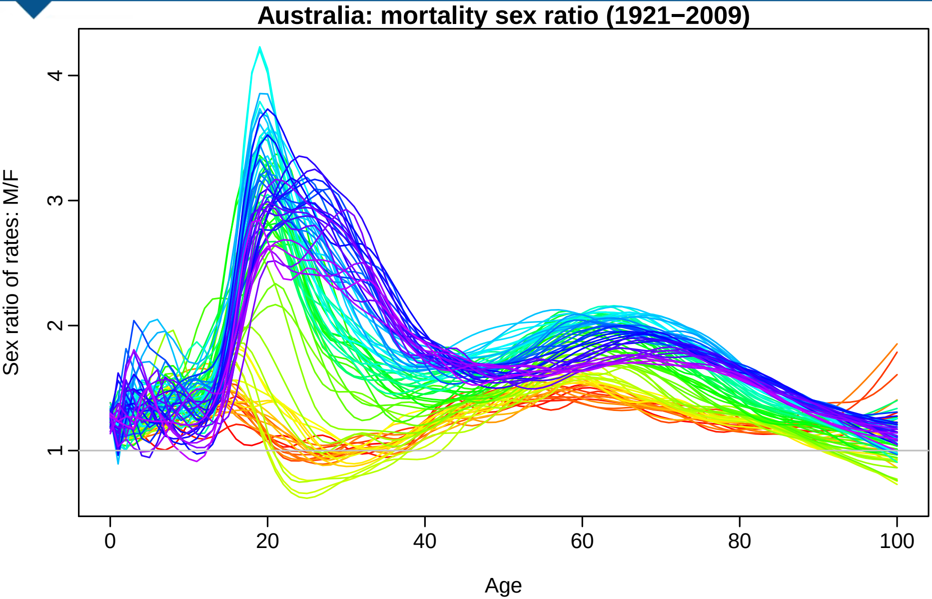


Ratio data



Australia: mortality sex ratio (1921−2009)
Sex ratio of rates: M/F
1
2
3
4
0
20
40
60
80
100
Age

Model product and ratios

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$$\log[r_t(x)] = \mu_r(x) + \sum_{\ell=1}^{L} \gamma_{t,\ell} \psi_\ell(x) + w_t(x).$$

- $\{\gamma_{t,\ell}\}$ restricted to be stationary processes: either ARMA(p,q) or ARFIMA(p,d,q).
- No restrictions for $\beta_{t,1}, \dots, \beta_{t,K}$.
- Forecasts: $f_{n+h|n,M}(x) = p_{n+h|n}(x) r_{n+h|n}(x)$

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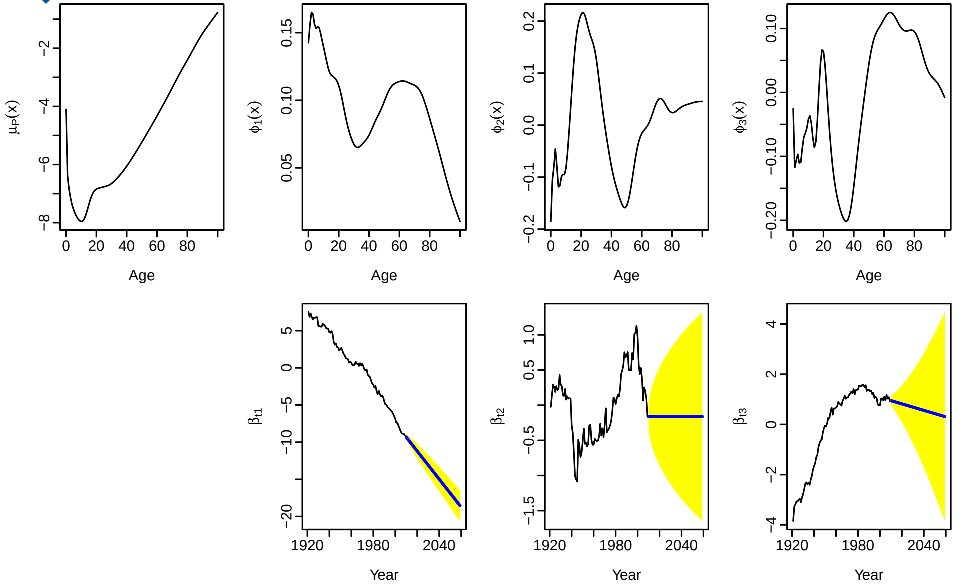
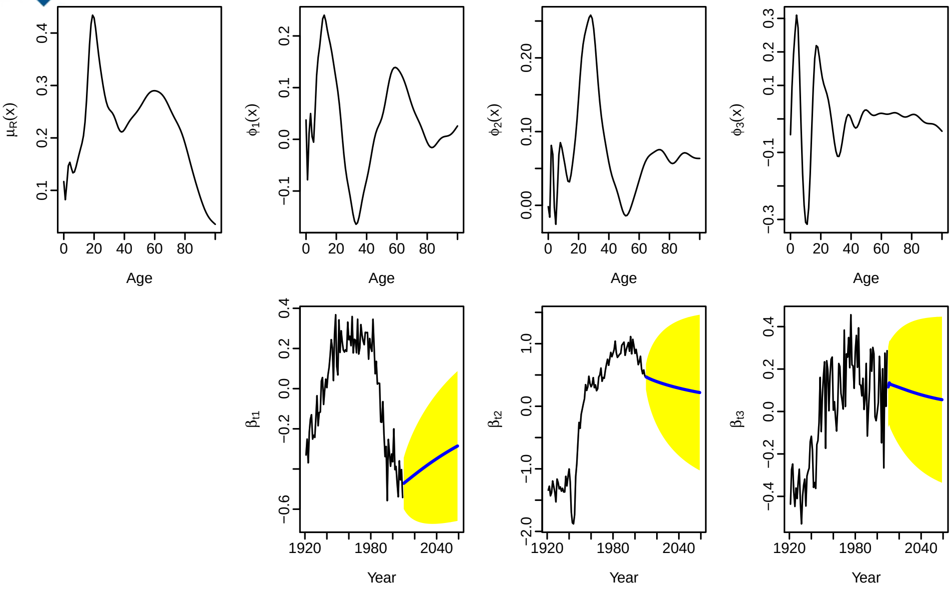
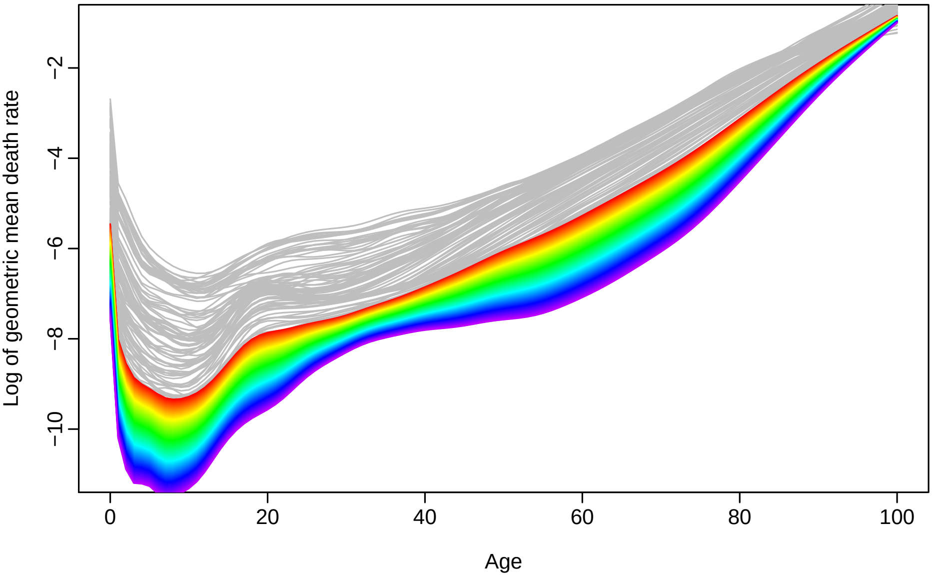


Ratio model



Product forecasts



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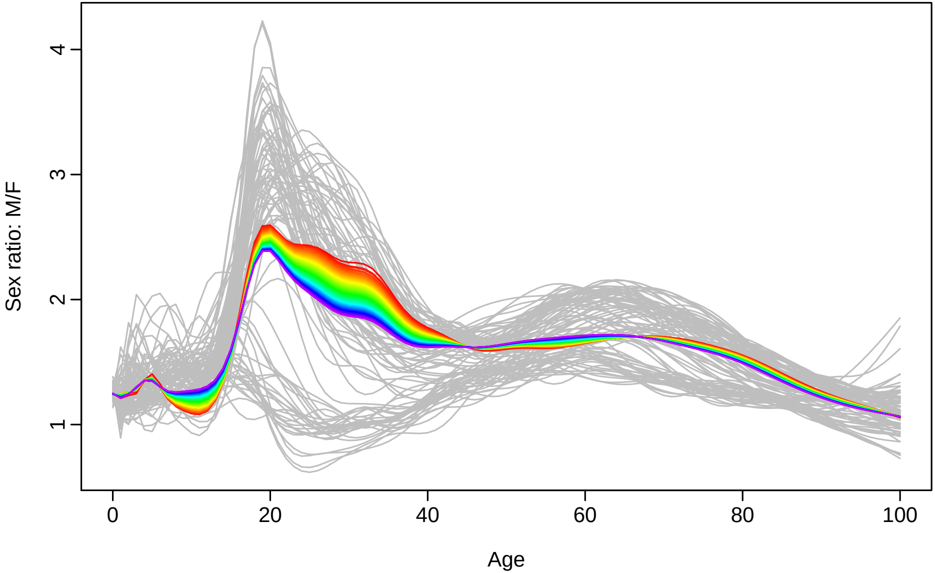
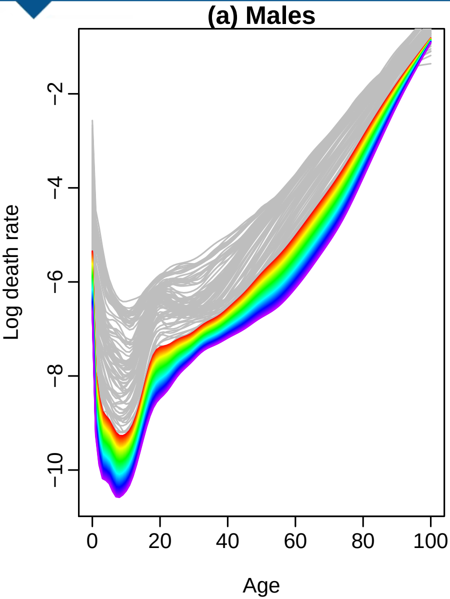
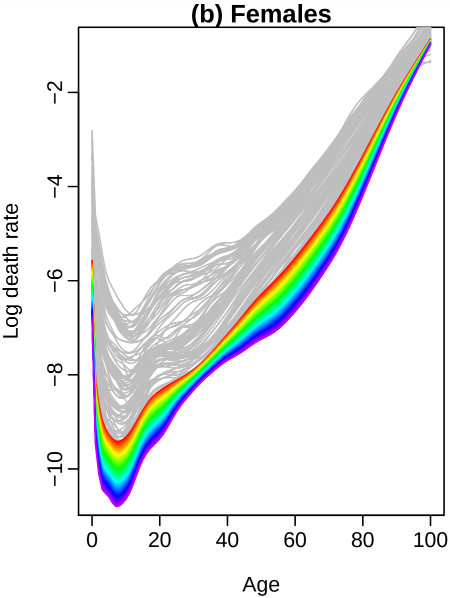


Coherent forecasts





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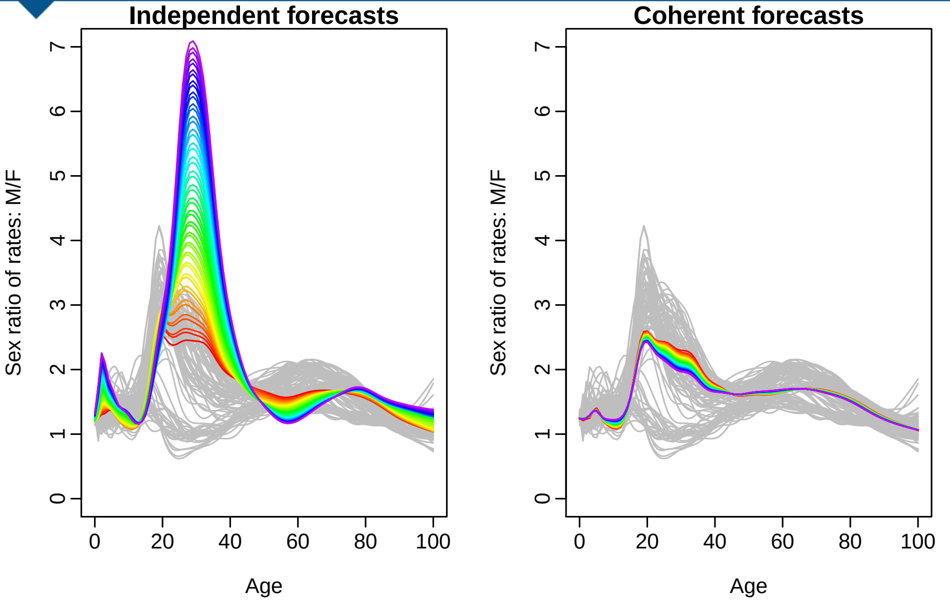
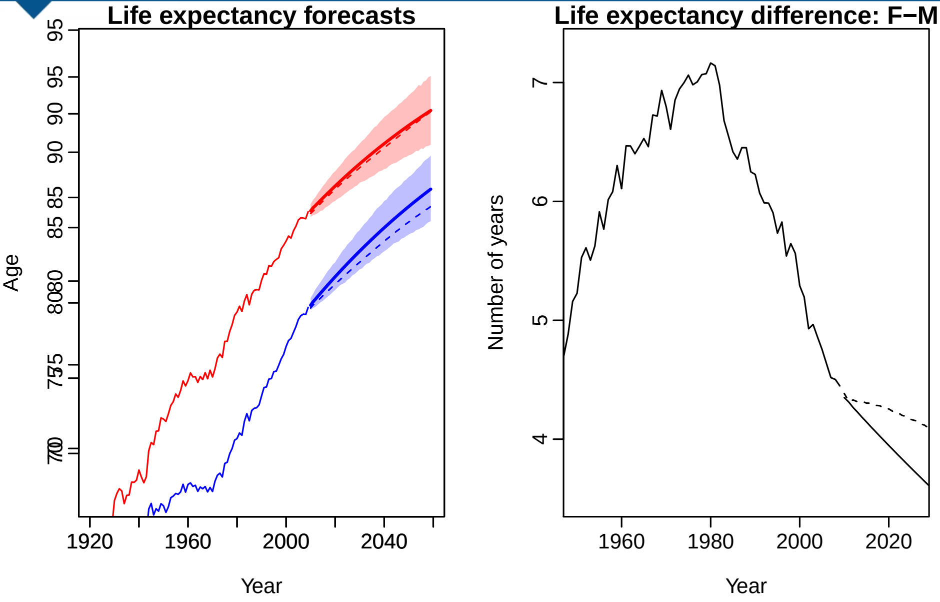


Life expectancy forecasts



Coherent forecasts for J groups

$$p_t(x) = [f_{t,1}(x) f_{t,2}(x) \cdots f_{t,J}(x)]^{1/J}$$

and $$r_{t,j}(x) = f_{t,j}(x) \big/ p_t(x),$$

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Li-Lee method

Li & Lee (*Demography*, 2005) method is a special case of our approach.

$$f_{t,j}(x) = \mu_j(x) + \beta_t \phi(x) + \gamma_{t,j} \psi_j(x) + e_{t,j}(x)$$

where f is *unsmoothed* log mortality rate, β_t is a random walk with drift and $\gamma_{t,j}$ is AR(1) process.

- No smoothing.
- Only one basis function for each part,
- Random walk with drift very limiting.
- AR(1) very limiting.
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Outline

Life expectancy calculation

Using standard life table calculations:

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\text{For } x = 0, 1, \dots, \omega - 1: \qquad
\begin{aligned}
q_x &= m_x/(1 + (1 - a_x)m_x) \\
\ell_{x+1} &= \ell_x(1 - q_x) \\
L_x &= \ell_x[1 - q_x(1 - a_x)] \\
T_x &= L_x + L_{x+1} + \cdots + L_{\omega-1} + L_{\omega+} \\
e_x &= T_x/L_x
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where $a_x = 0.5$ for $x \geq 1$ and a_0 taken from Coale et al (1983).
$q_{\omega+} = 1$, $L_{\omega+} = l_x/m_x$, and $T_{\omega+} = L_{\omega+}$.

- Period life expectancy: let $m_x = m_{x,t}$ for some year t.
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- Because we can forecast $m_{x,t}$ we can estimate the mortality rates for each birth cohort (using actual values when they are available).
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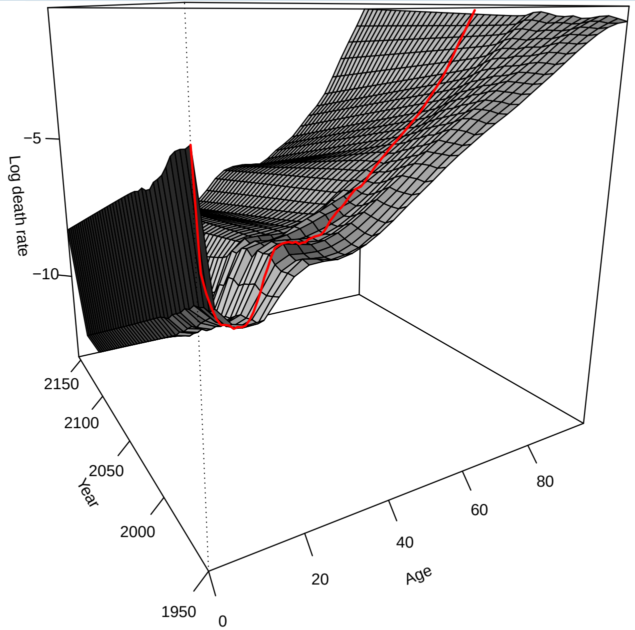

Simulate future mortality rates

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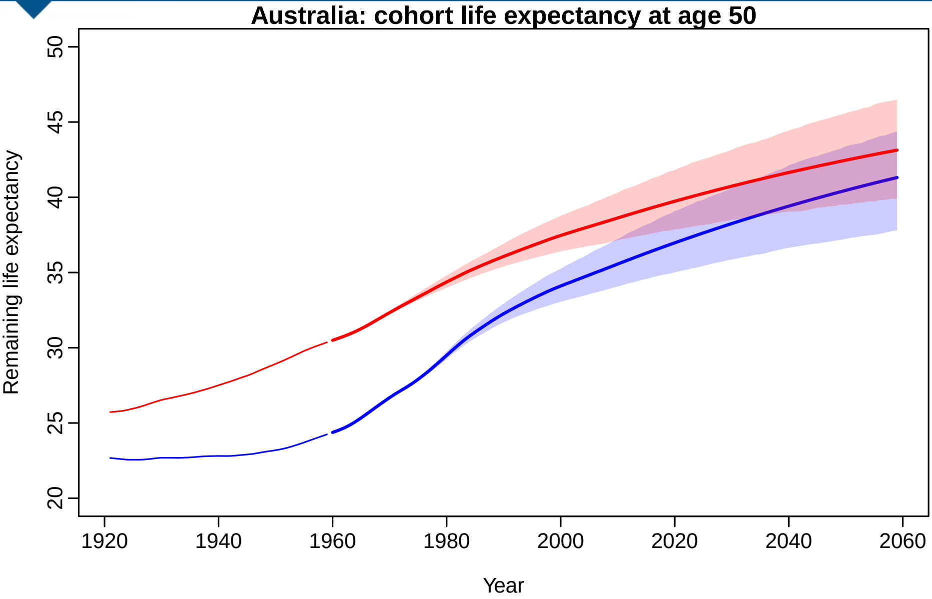

Complete code

```
library(demography)

# Read data
aus <- hmd.mx("AUS","username","password","Australia")

# Smooth data
aus.sm <- smooth.demogdata(aus)

#Fit model
aus.pr <- coherentfdm(aus.sm)

# Forecast
aus.pr.fc <- forecast(aus.pr, h=100)

# Compute life expectancies
e50.m.aus.fc <- flife.expectancy(aus.pr.fc, series="male",
  age=50, PI=TRUE, nsim=1000, type="cohort")
e50.f.aus.fc <- flife.expectancy(aus.pr.fc, series="female",
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```
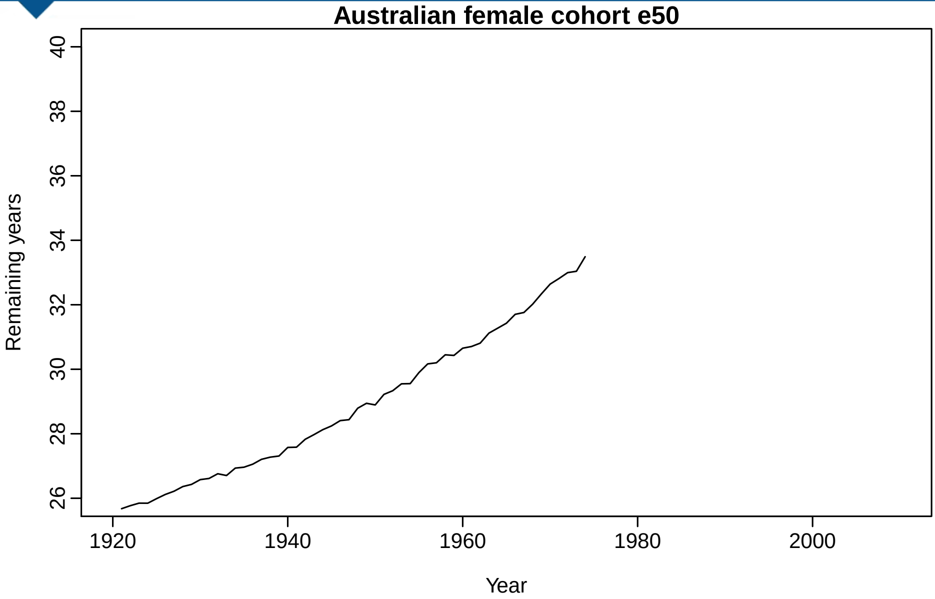

Forecast accuracy evaluation



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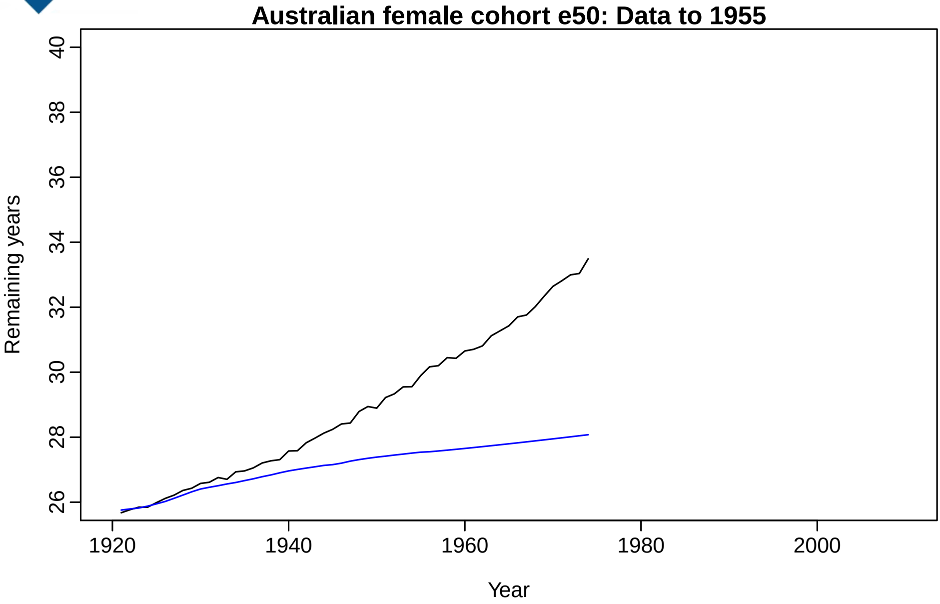

Forecast accuracy evaluation

- Compute age 50 remaining cohort life expectancy with a rolling forecast origin beginning in 1921.
- Compare against actual cohort life expectancy where available.
- Compute 80% prediction interval actual coverage.

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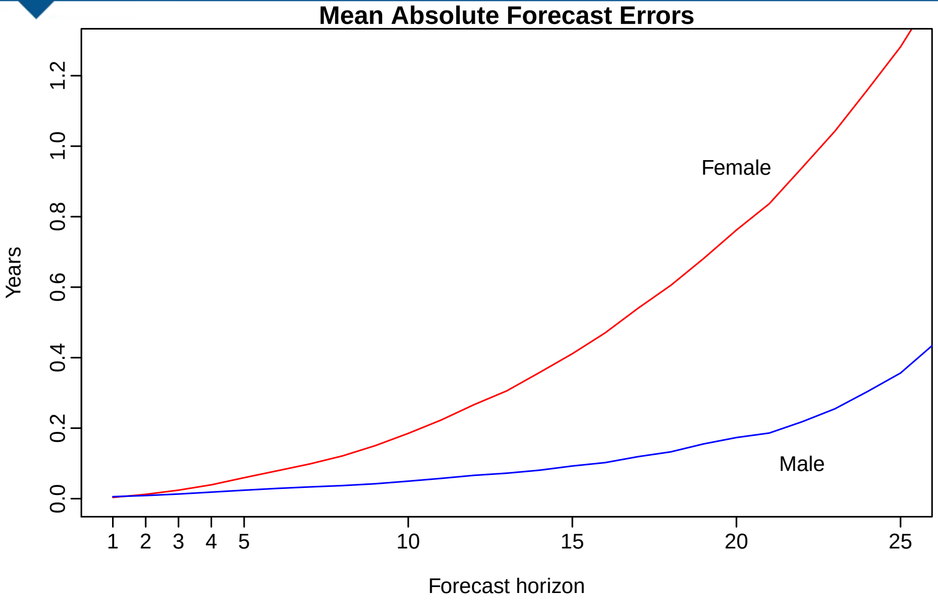

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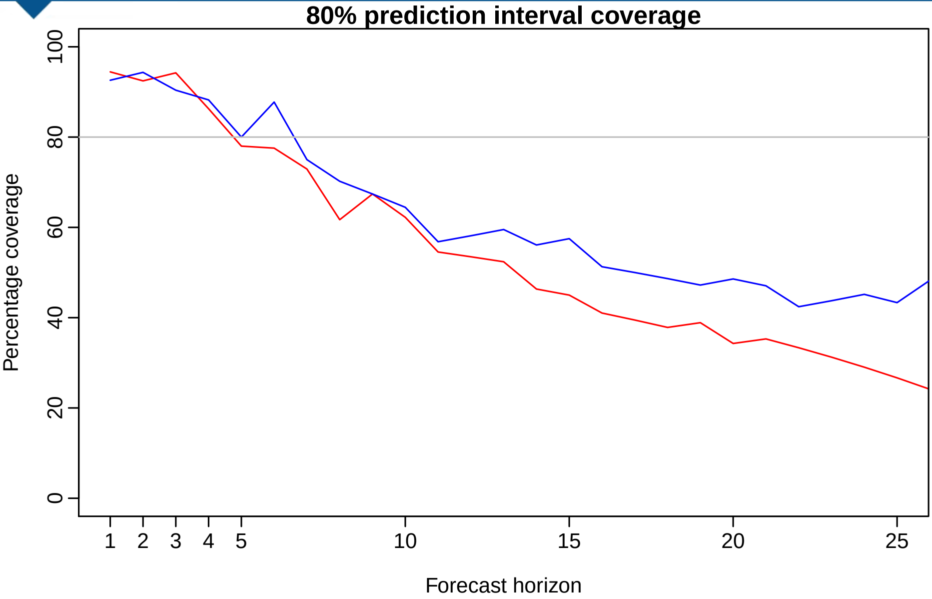

Outline

Some conclusions

- New, automatic, flexible method for coherent forecasting of groups of functional time series.
- Suitable for age-specific mortality.
- Based on geometric means and ratios, so interpretable results.
- More general and flexible than existing methods.
- Easy to compute prediction intervals for any computable statistics.

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- Hyndman, Shang (2009). "Forecasting functional time series (with discussion)". *Journal of the Korean Statistical Society* **38**(3), 199–221
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