Exponential smoothing and non-negative data

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Outline

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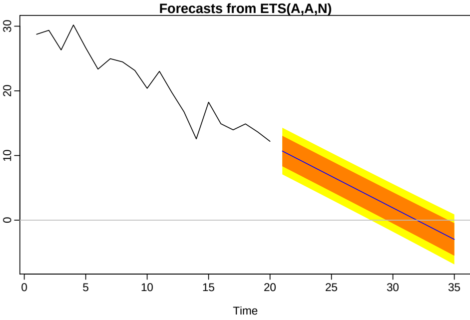

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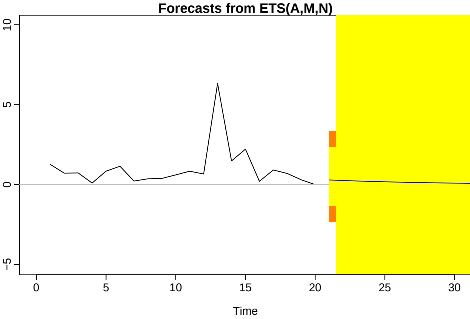

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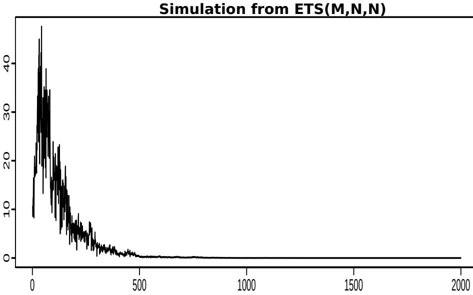
1. They can produce negative forecasts
2. They can produce infinite forecast variance
3. They can converge almost surely to zero.



Taxonomy of models

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
N	(None)	N,N	N,A	N,M
A	(Additive)	A,N	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
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ETS(A,N,N): Simple exponential smoothing with additive errors

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ETS(A,A,N): Holt's linear method with additive errors

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ETS(A,A,A): Additive Holt-Winters' method with additive errors

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ETS(M,A,M): Multiplicative Holt-Winters' method with multiplicative errors

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General notation **ETS**(*Error,Trend,Seasonal*)
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ETS(A,A_d,N): Damped trend method with additive errors

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N	(None)	N,N	N,A	N,M
A	(Additive)	A,N	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
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General notation **ETS**(*Error,Trend,Seasonal*)

Exponen**T**ial **S**moothing

There are 30 separate models in the ETS framework

Innovations state space model

No trend or seasonality and multiplicative errors

Example: ETS(M,N,N)

$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$
$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

$0 \leq \alpha \leq 1$

ε_t is white noise with mean zero.

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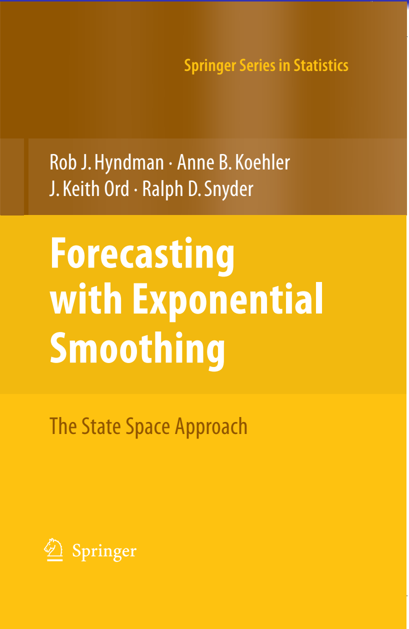
All exponential smoothing models can be written using analogous state space equations.

New book!



- State space modeling framework
- Prediction intervals
- Model selection
- Maximum likelihood estimation
- All the important research results in one place with consistent notation
- Many new results
- 375 pages but only US$39.95.

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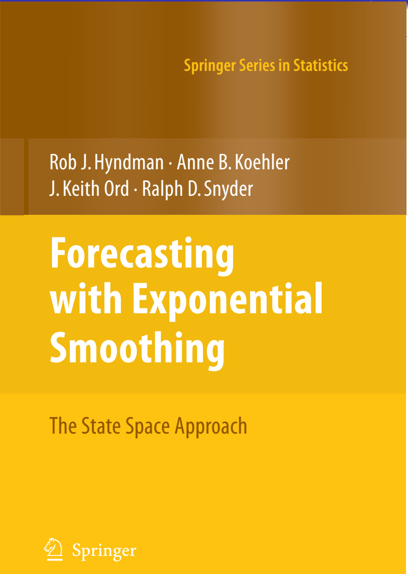

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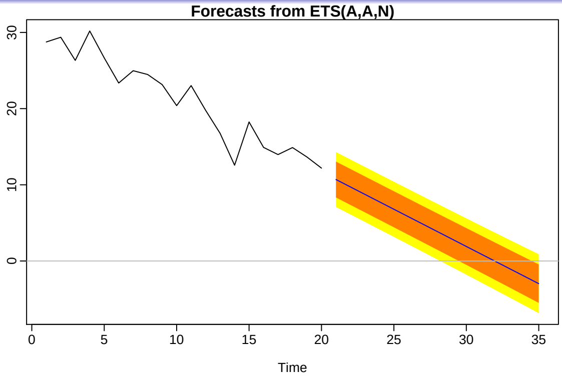
Negative forecasts



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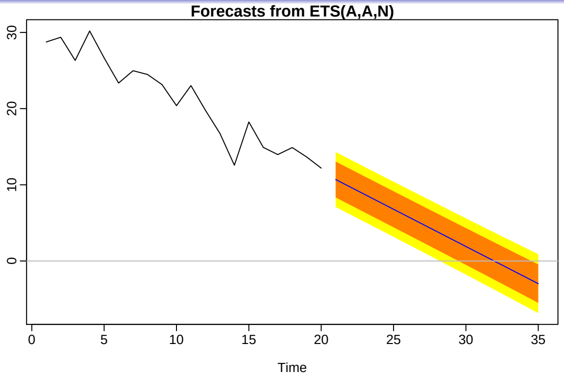
Negative forecasts



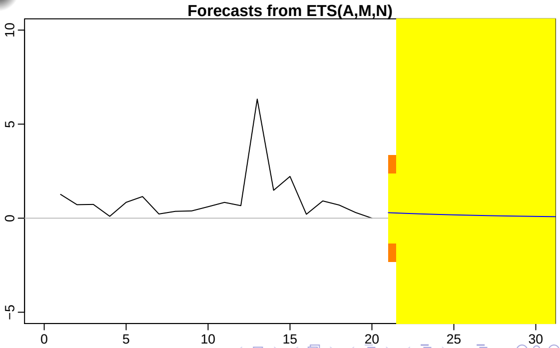
- Could solve by taking logs or some other Box-Cox transformation. However, this limits models to be additive in the transformed space.
- Could solve by only using multiplicative models. But these can have other problems.

Infinite forecast variances

ETS(A,M,N) model

$$y_t = \ell_{t-1} b_{t-1} + \varepsilon_t$$
$$\ell_t = \ell_{t-1} b_{t-1} + \alpha \varepsilon_t$$
$$b_t = b_{t-1} + \beta \varepsilon_t / \ell_{t-1}.$$

- $\ell_0 = 0.1$
- $b_0 = 1$
- $\alpha = 0.1$
- $\beta = 0.05$
- $\sigma = 1$



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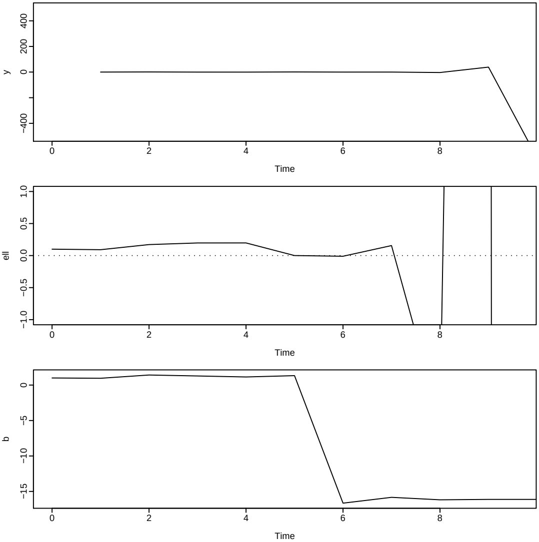

Infinite forecast variances

Suppose ε_t has positive density at 0

For ETS models (A,M,N), (A,M,A), (A,M_d,N), (A,M_d,A), (A,M,M), (A,M_d,M), (M,M,A) and (M,M_d,A):

- $\text{V}(y_{n+h} \mid \boldsymbol{x}_n) = \infty$ for $h \geq 3$;
- $\text{E}(y_{n+h} \mid \boldsymbol{x}_n)$ is undefined for $h \geq 3$.

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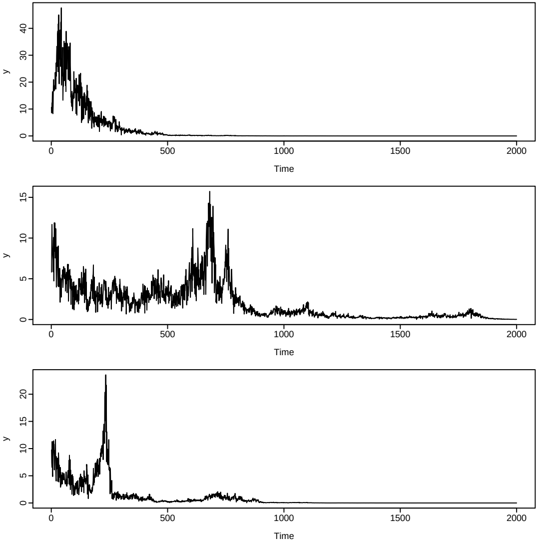
➥ **These problems occur regardless of the sample space of $\{y_t\}$.**

Convergence to zero

ETS(M,N,N) model

$$y_t = \ell_{t-1}(1+\varepsilon_t)$$
$$\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$$

- $\ell_0 = 10$
- $\alpha = 0.3$
- $\sigma = 0.3$ with truncated Gaussian errors



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- $\delta_t = 1 + \varepsilon_t$ has mean 1 and variance σ^2.

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$$\ell_t = \ell_0(1+\alpha\varepsilon_1)(1+\alpha\varepsilon_2)\cdots(1+\alpha\varepsilon_t) = \ell_0 U_t,$$

where $U_t = U_{t-1}(1+\alpha\varepsilon_t)$ and $U_0 = 1$. Thus U_t is a non-negative product martingale.

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- Consequently, all sample paths for y_t converge to 0 almost surely.
- Similar results follow for all purely multiplicative models: (M,N,N), (M,N,M), (M,M,N), (M,M,M), (M,M_d,N) and (M,M_d,M).

Four model classes

Class M: **Purely multiplicative models:** (M,N,N), (M,N,M), (M,M,N), (M,M,M), (M,M_d,N) and (M,M_d,M).

Class A: **Purely additive models:** (A,N,N), (A,N,A), (A,A,N), (A,A,A), (A,A_d,N) and (A,A_d,A).

Class X: **Mixed models:** (A,M,$*$), (A,M_d,$*$), (A,$*$,M), (M,M,A), (M,M_d,A). (11 models)

Class Y: **Mixed models:** (M,A,$*$), (M,A_d,$*$) or (M,N,A). (7 models)

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- Only Class M can guarantee a sample space restricted to the positive half-line.
- All Class M models converge to 0 if E$(\varepsilon) = 0$
- All Class X models have infinite forecast variance for $h \geq m + 2$ where m is the seasonal period.

Outline

New models

Let $\delta_t = (1 + \varepsilon_t)$ be a positive random variable.

METS(M,N,N) model

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$$\log(\ell_t) = \log(\ell_{t-1}) + \alpha \log(\delta_t)$$

Thus the log-transformed model is identical to Gaussian ETS(A,N,N) model if δ_t is logNormal with median 1.

Long term forecast behaviour

METS(M,N,N; LN) model

$$y_t = \ell_{t-1}\delta_t$$
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$$\delta_t \sim \text{logN}(\mu, \omega)$$

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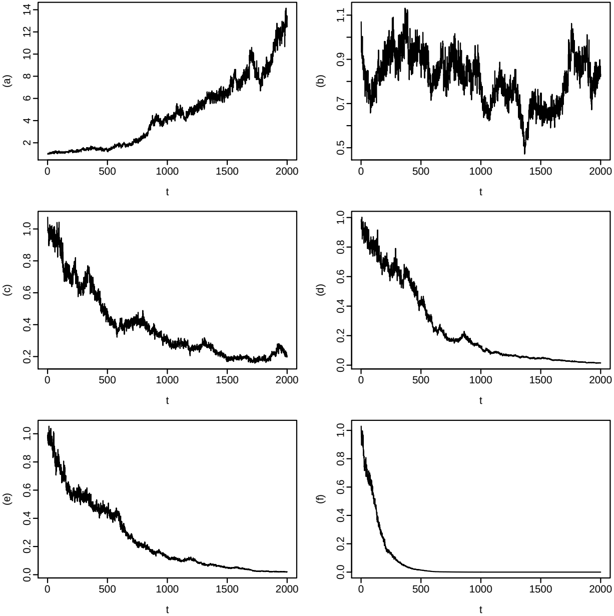
Range	$E(\delta_t^{\alpha})$	$E(\delta_t^{\alpha/2})$	$E(y_h)$	$V(y_h)$
$\mu + \alpha\omega < 0$	< 1	< 1	Decreasing	Decreasing
$\mu + \alpha\omega = 0$	< 1	< 1	Decreasing	Finite
$-\alpha\omega < \mu < -\alpha\omega/2$	< 1	< 1	Decreasing	Increasing
$\mu + \alpha\omega/2 = 0$	$= 1$	< 1	Finite	Increasing
$-\alpha\omega/2 < \mu < -\alpha\omega/4$	> 1	< 1	Increasing	Increasing
$\mu + \alpha\omega/4 = 0$	> 1	$= 1$	Increasing	Increasing
$\mu + \alpha\omega/4 > 0$	> 1	> 1	Increasing	Increasing

Long term forecast behaviour



METS(M,N,N;LN)

$\delta_t \sim \text{logN}(\mu, \omega)$:

(a) $\mu = \alpha\omega/4$

(b) $\mu = 0$

(c) $\mu = -\alpha\omega/4$

(d) $\mu = -3\alpha\omega/8$

(e) $\mu = -\alpha\omega/2$

(f) $\mu = -3\alpha\omega/4$

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Paper: **www.robhyndman.info**

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