Evaluating extreme quantile forecasts

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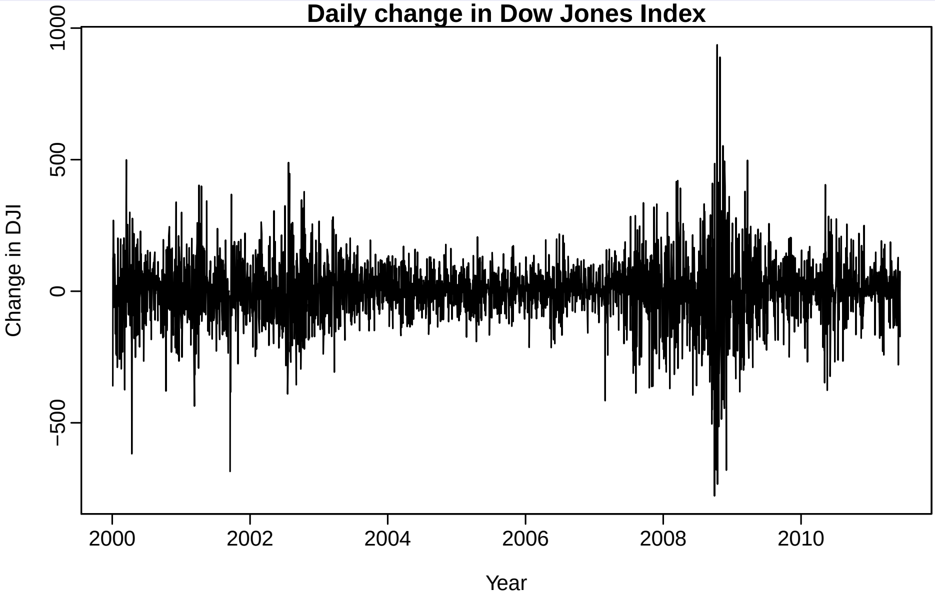
Outline

Extreme quantile forecasting



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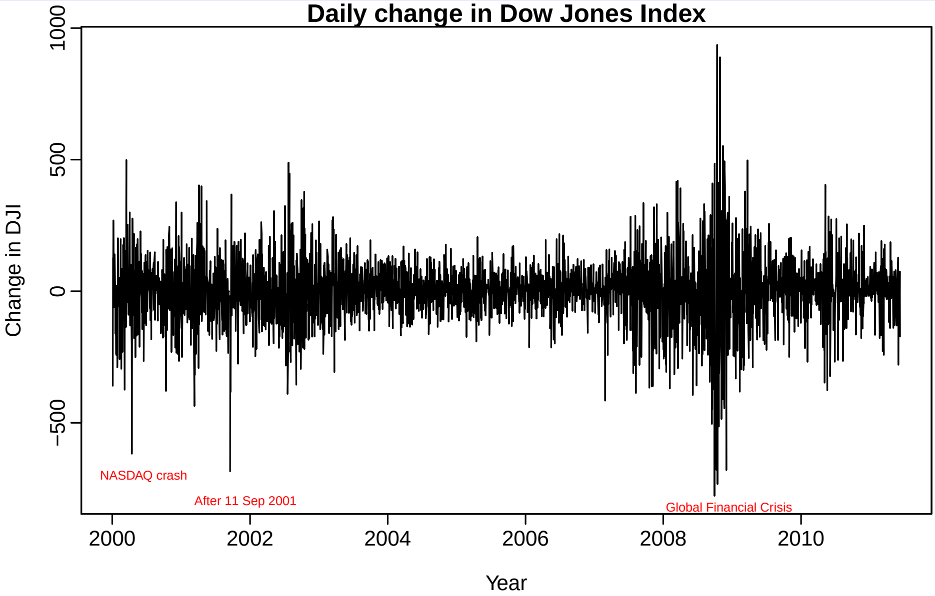

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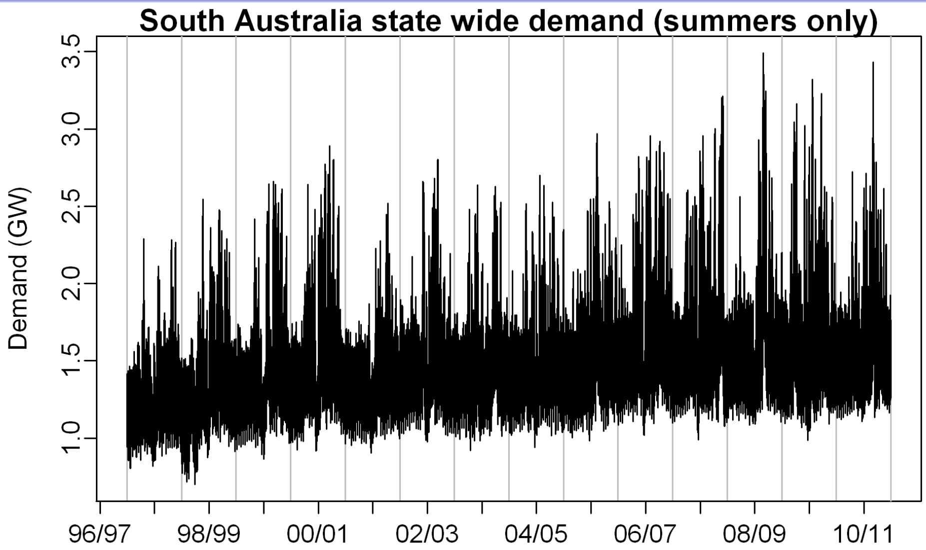

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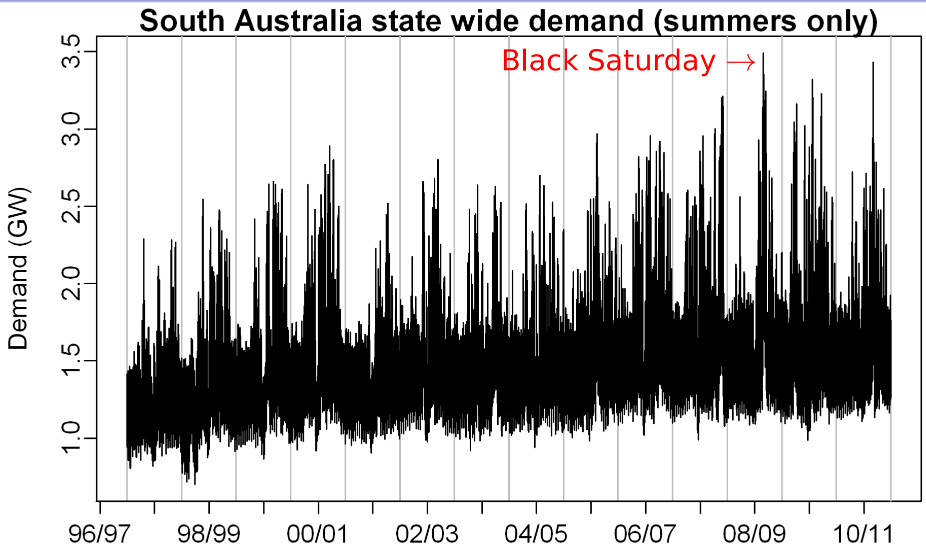

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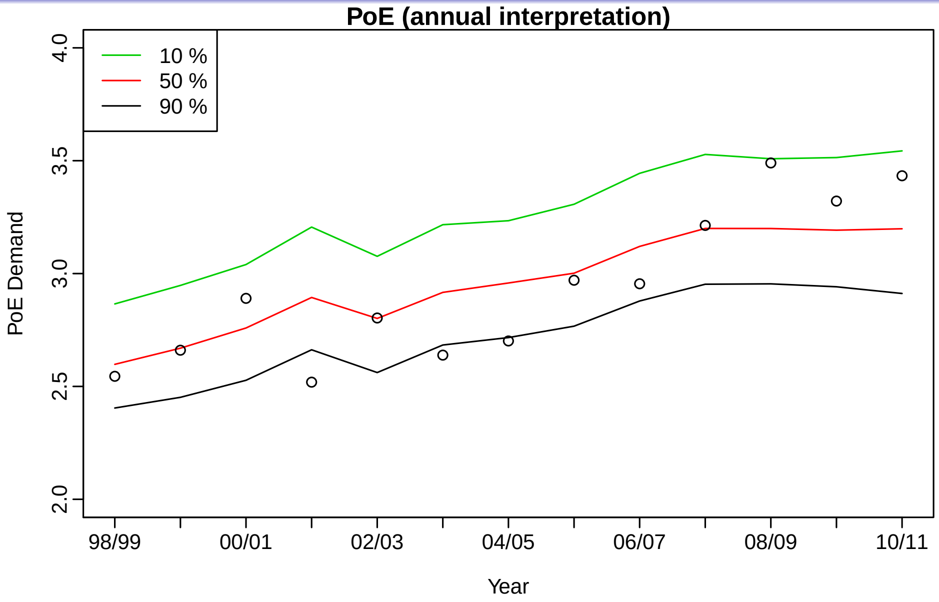

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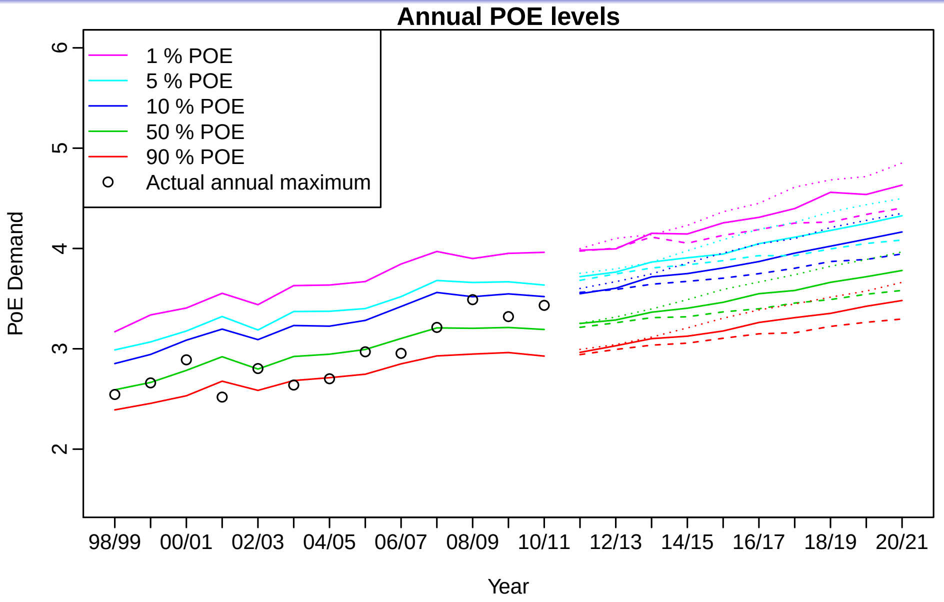

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Density evaluation

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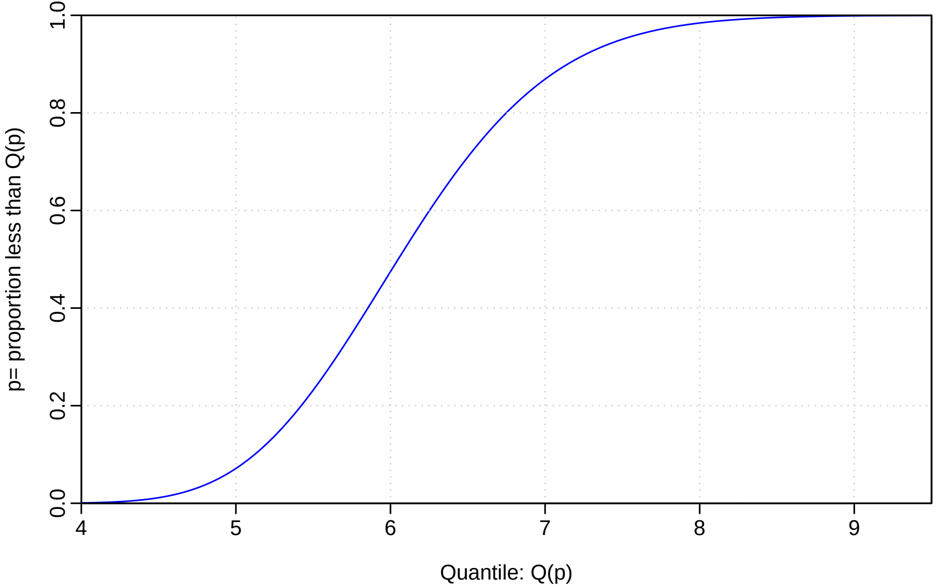

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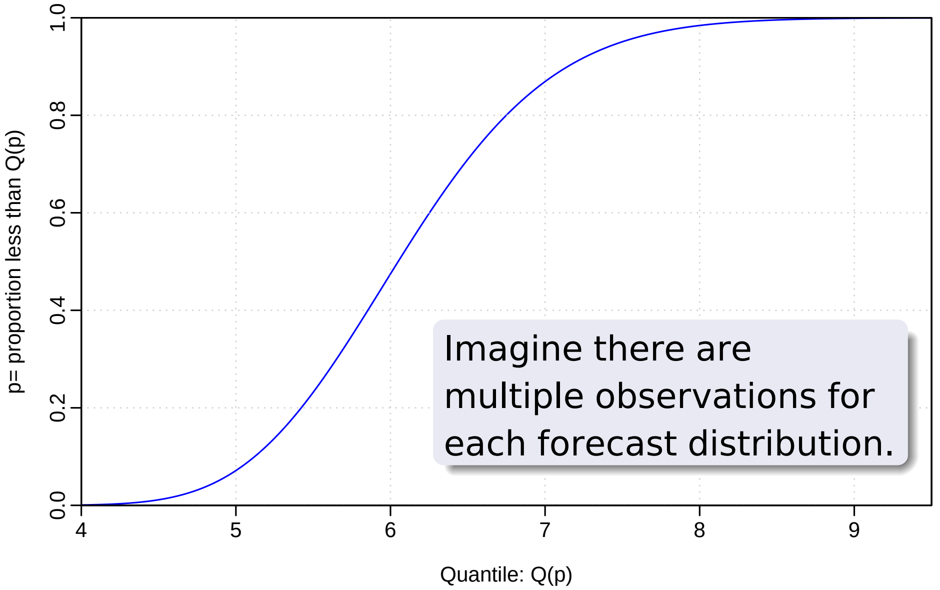

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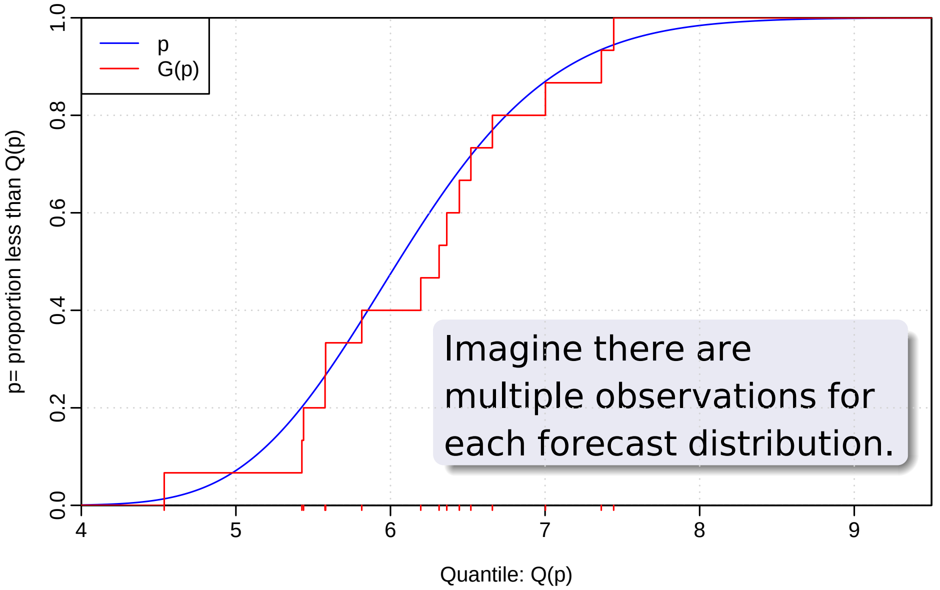

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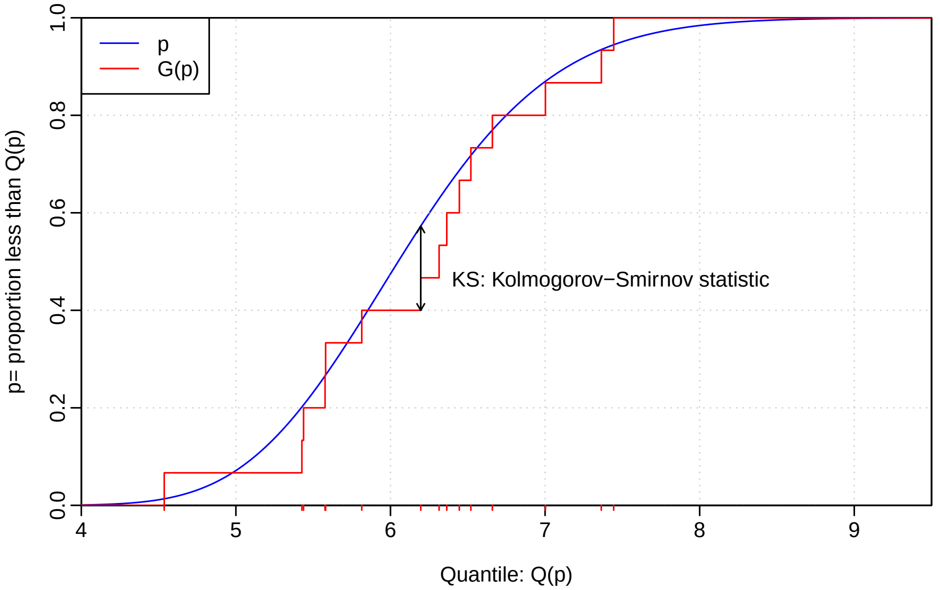

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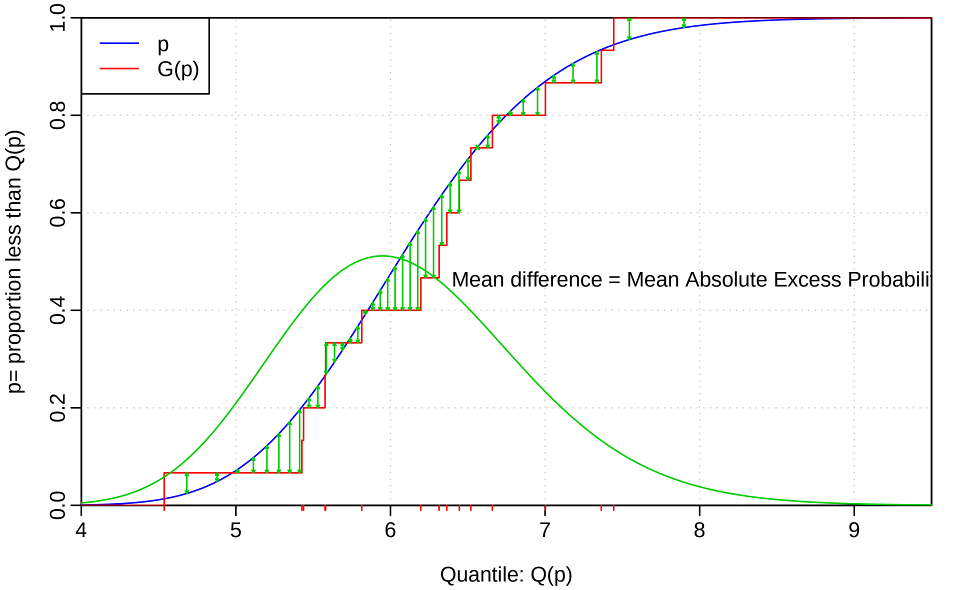

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- Cramer-von-Mises $= \int_0^1 E^2(p)\, dp$

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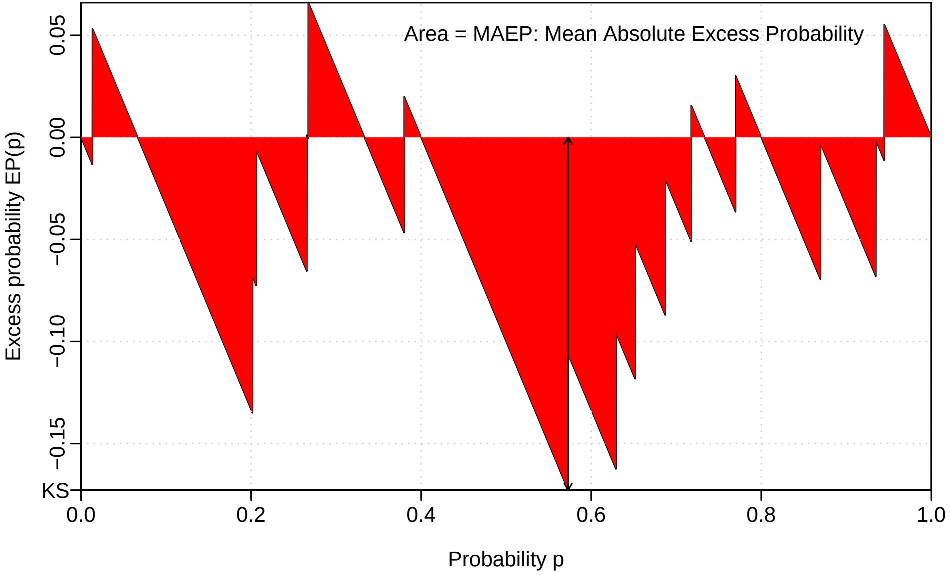

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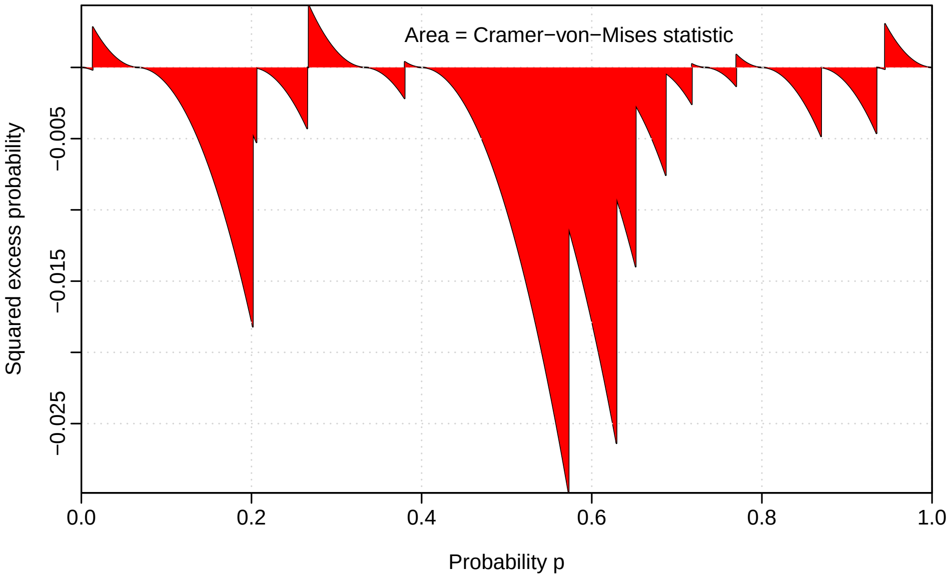

Probability integral transform

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- If $F_t(y)$ is correct, then Z_t will follow a $U(0, 1)$ distribution.

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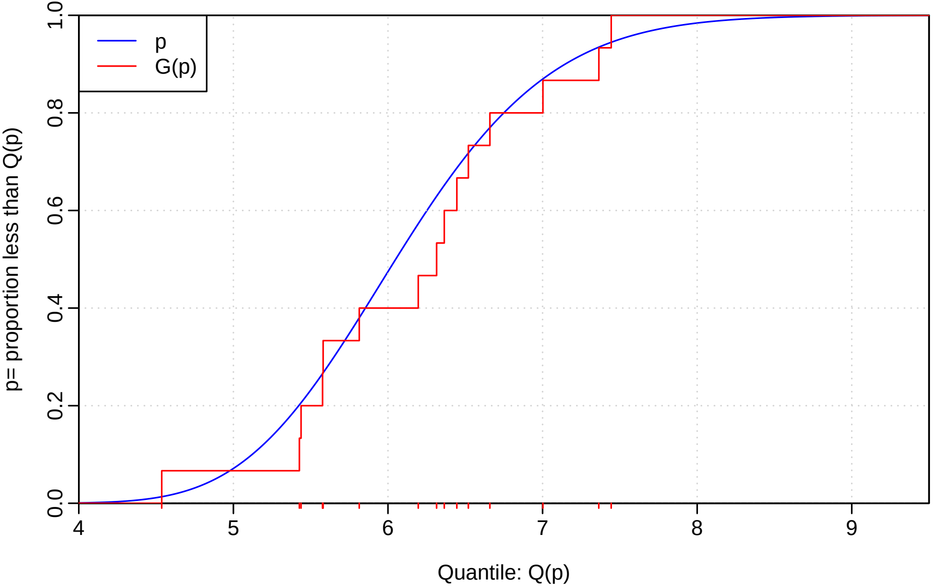

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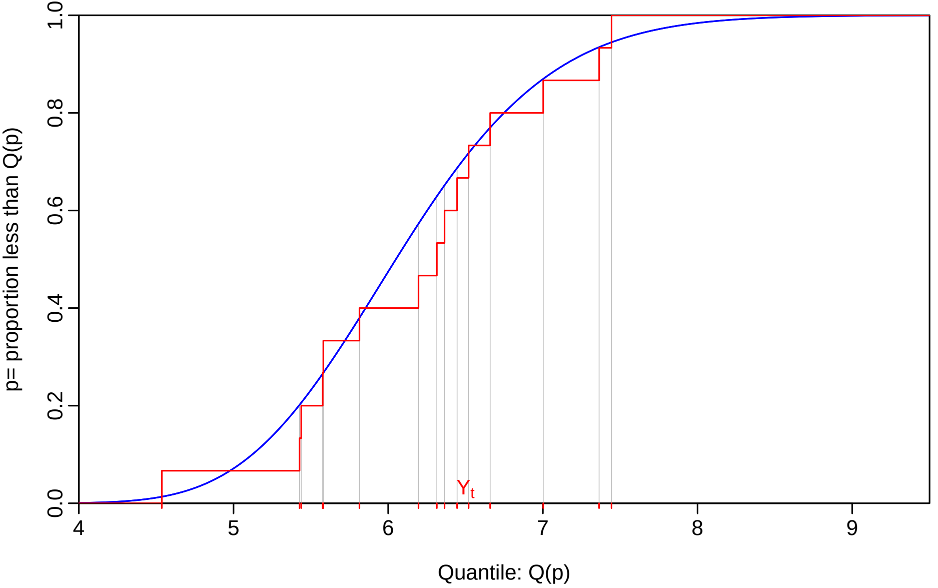

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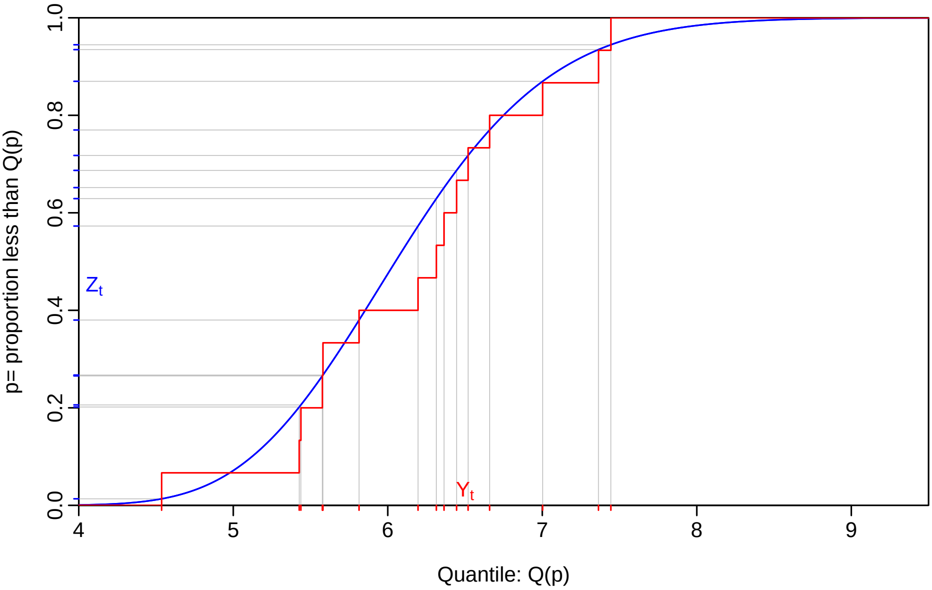

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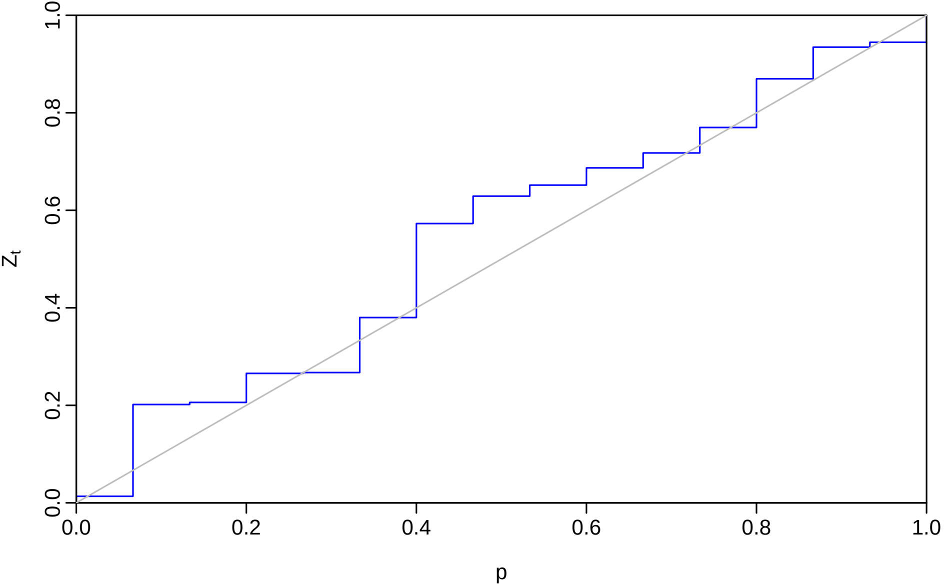

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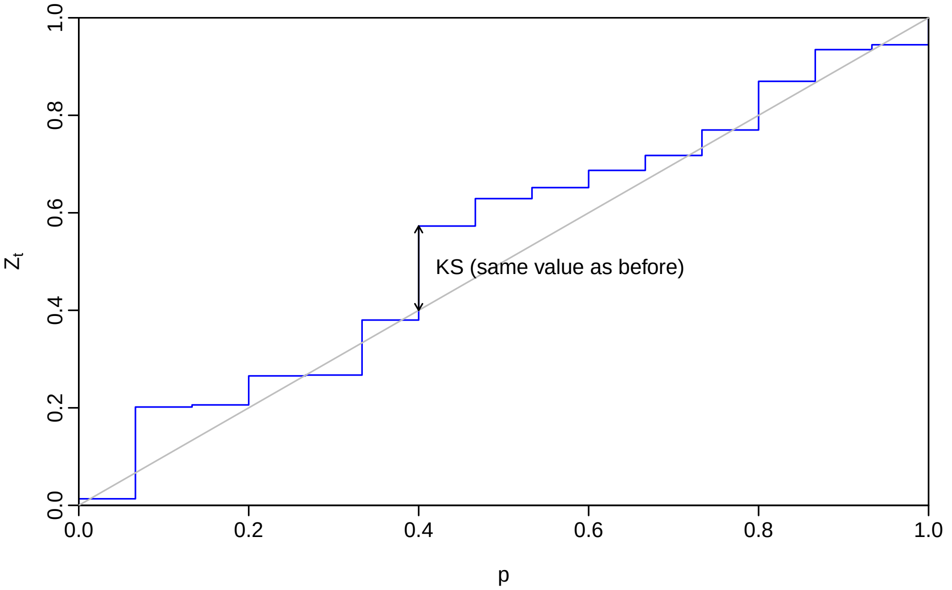

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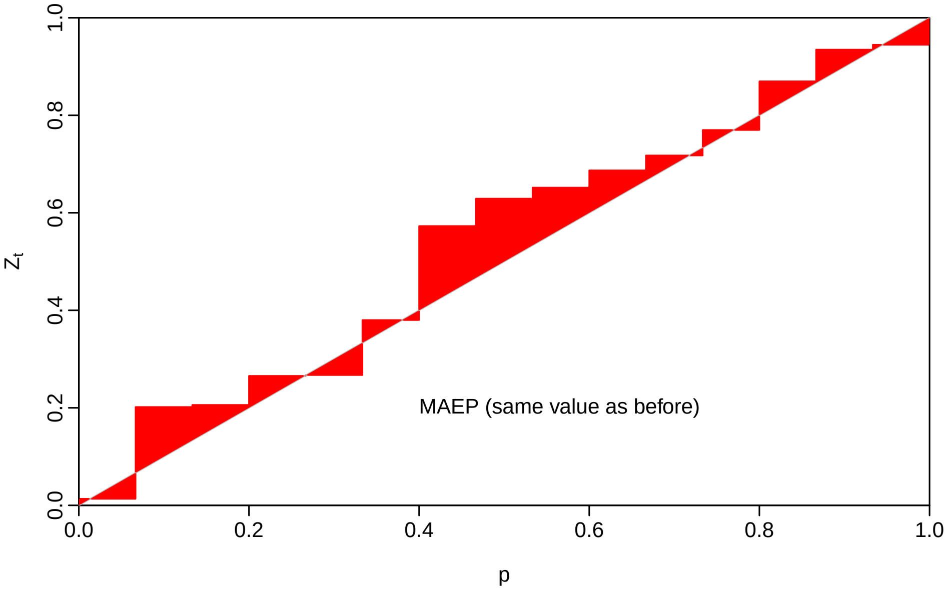
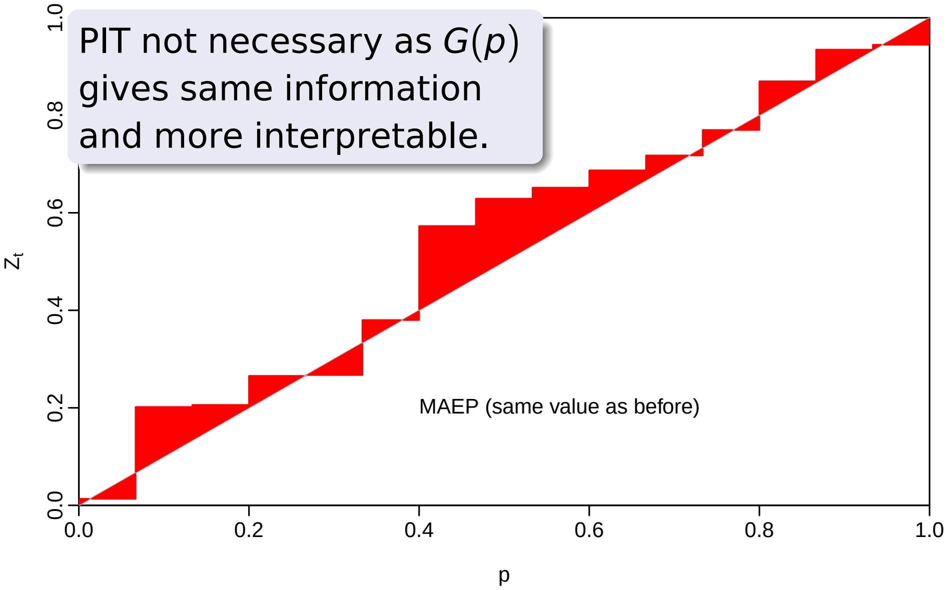


Probability integral transform

PIT not necessary as $G(p)$ gives same information and more interpretable.



Distribution of MAEP

$$Z_i = F_i(y_i)$$

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- $\text{E}(\text{MAEP}) = \frac{1}{\sqrt{10n}}$
- $\text{V}(\text{MAEP}) = \frac{1}{54n}$
- Get p-values by simulation.

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Apply density evaluation measures to tail of distribution only.

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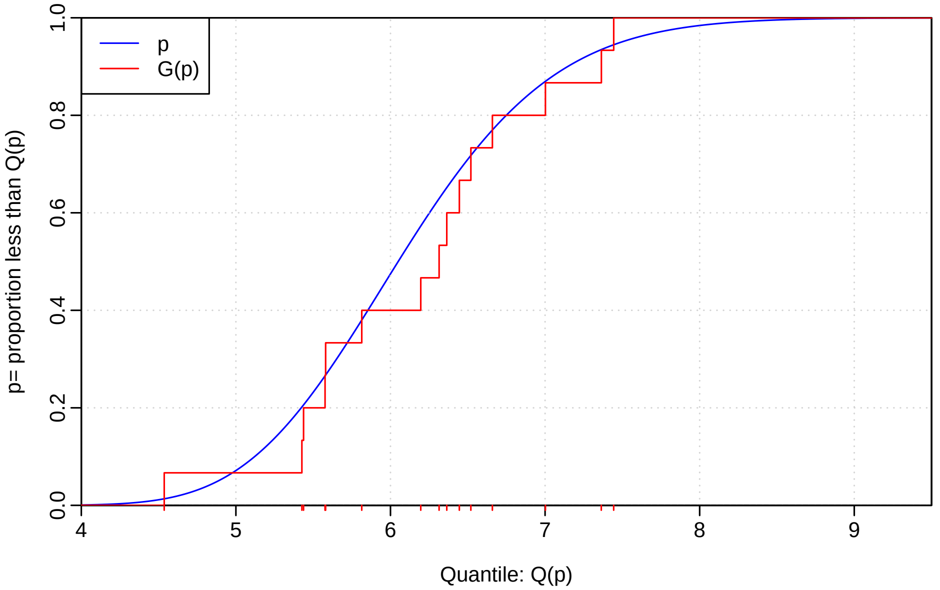

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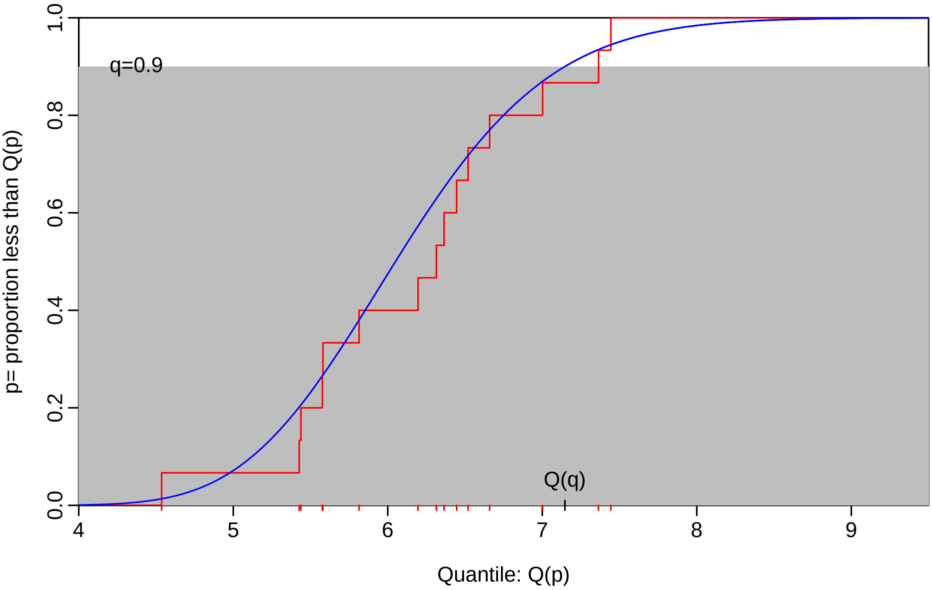

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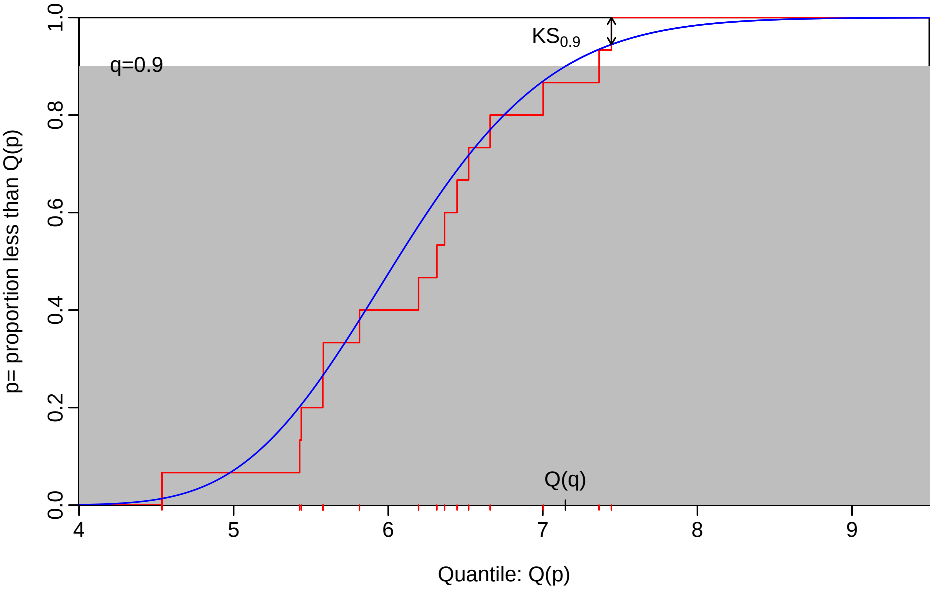

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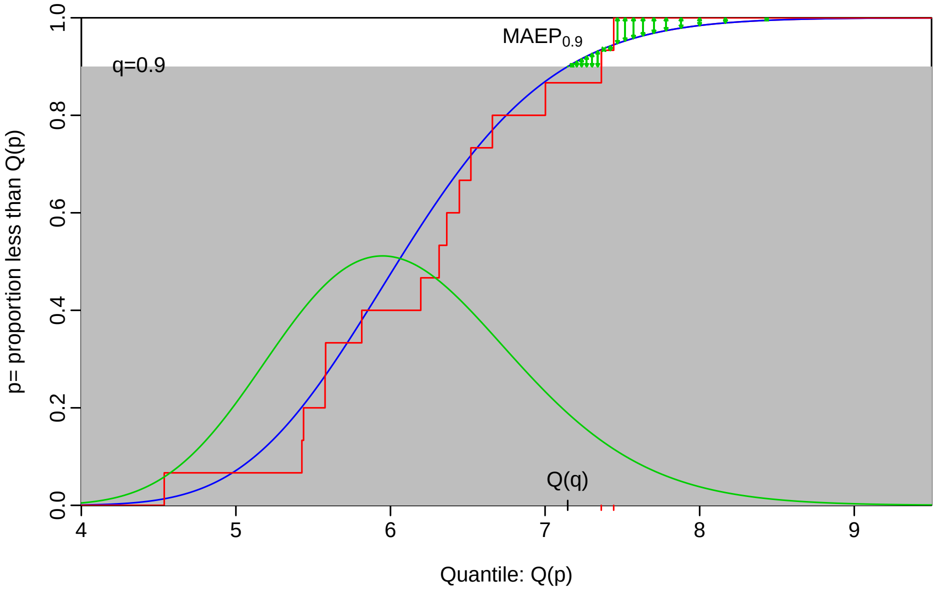

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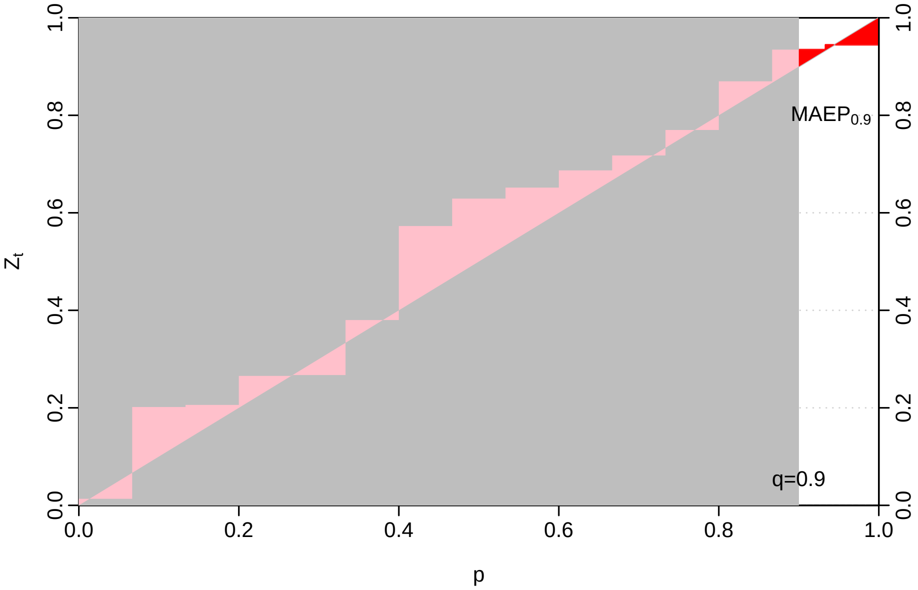
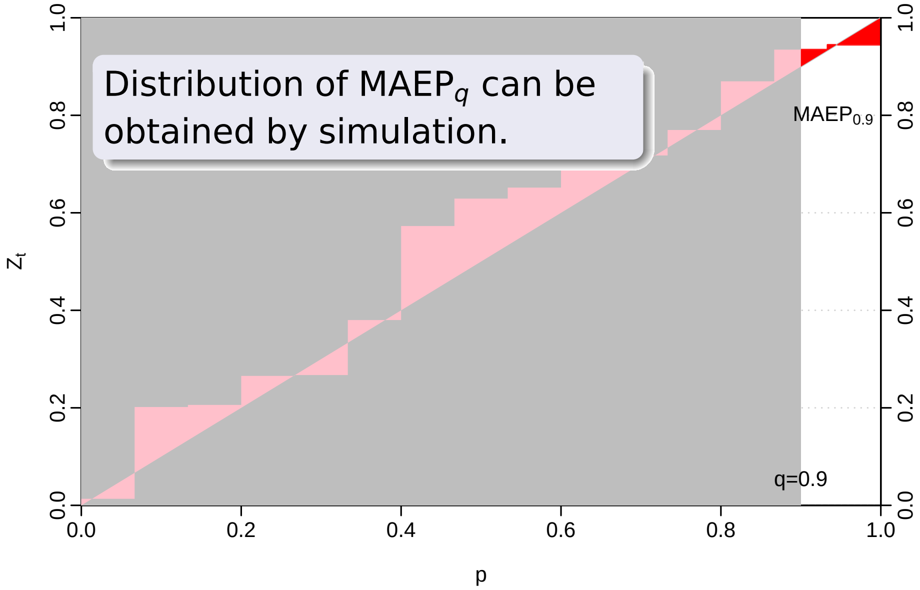


Quantile evaluation measures

Distribution of MAEP_q can be obtained by simulation.



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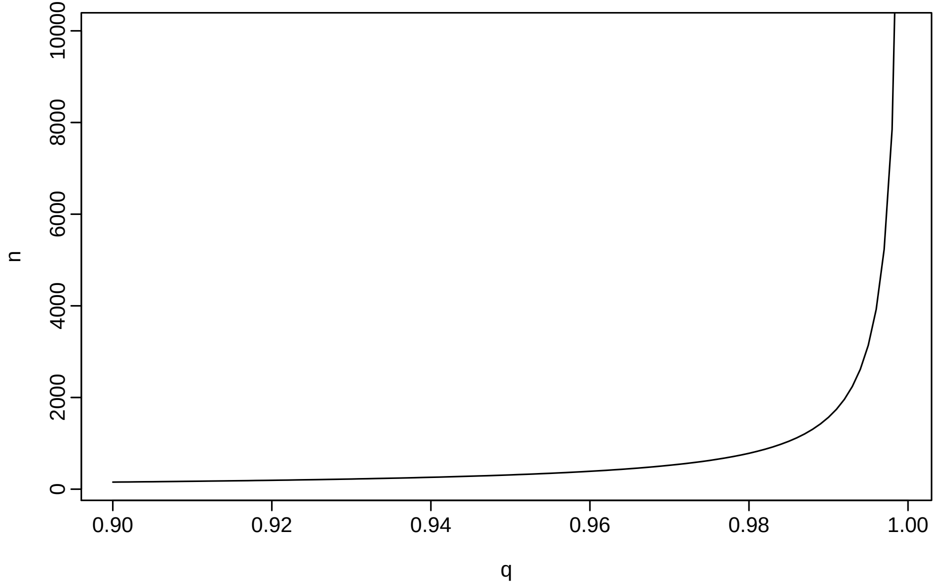
- $q = 0.95 \Rightarrow n > 181$.
- $q = 0.99 \Rightarrow n > 913$.

Sample size needed



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- For these weekly forecasts, $q = (1-\alpha)^{1/21}$.
- For 15 years of data, $n = 315$.
- Therefore $q \le 0.971$ and $\alpha \ge 0.46$.

Model evaluation for electricity demand

	$q = 0.95$	$q = 0.90$	$q = 0.50$	$q = 0.10$	$q = 0.0$
Ex ante	4.35%	5.59%	9.25%	10.73%	10.31%
Ex post	3.79%	4.28%	5.24%	7.95%	8.24%