MONASH University

MONASH BUSINESS SCHOOL

Forecasting using R

Rob J Hyndman

3.2 Dynamic regression

Outline

Regression with ARIMA errors

Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + e_t,$$

- y_t modeled as function of k explanatory variables $x_{1,t}, \ldots, x_{k,t}$.
- In regression, we assume that e_t was WN.
- Now we want to allow e_t to be autocorrelated.

Example: ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + n_t,$$
$$(1 - \phi_1 B)(1 - B)n_t = (1 + \theta_1 B)e_t,$$

where e_t is white noise .

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Residuals and errors

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- Be careful in distinguishing n_t from e_t.
- Only the errors n_t are assumed to be white noise.
- In ordinary regression, n_t is assumed to be white noise and so $n_t = e_t$.

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Estimation

If we minimize $\sum n_t^2$ (by using ordinary regression):

1. Estimated coefficients $\hat{\beta}_0, \dots, \hat{\beta}_k$ are no longer optimal as some information ignored;
2. Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
3. *p*-values for coefficients usually too small ("spurious regression").
4. AIC of fitted models misleading.

- Minimizing $\sum e_t^2$ avoids these problems.
- Maximizing likelihood is similar to minimizing $\sum e_t^2$.

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Stationarity

Regression with ARMA errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + n_t,$$

where n_t is an ARMA process.

- All variables in the model must be stationary.
- If we estimate the model while any of these are non-stationary, the estimated coefficients can be incorrect.
- Difference variables until all stationary.
- If necessary, apply same differencing to all variables.

Stationarity

Model with ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + n_t,$$
$$(1 - \phi_1 B)(1 - B)n_t = (1 + \theta_1 B)e_t,$$

Equivalent to model with ARIMA(1,0,1) errors

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + n'_t,$$
$$(1 - \phi_1 B)n'_t = (1 + \theta_1 B)e_t,$$

where $y'_t = y_t - y_{t-1}$, $x'_{t,i} = x_{t,i} - x_{t-1,i}$ and $n'_t = n_t - n_{t-1}$.

Stationarity

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where $y'_t = y_t - y_{t-1}$, $x'_{t,i} = x_{t,i} - x_{t-1,i}$ and $n'_t = n_t - n_{t-1}$.

Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

Original data

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + n_t$$

$$\text{where} \quad \phi(B)(1-B)^d N_t = \theta(B) e_t$$

After differencing all variables

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + n'_t.$$

$$\text{where} \quad \phi(B) N_t = \theta(B) e_t$$

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After differencing all variables

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$$\text{where} \quad \phi(B) N_t = \theta(B) e_t$$

$$\text{and} \quad y'_t = (1-B)^d y_t$$

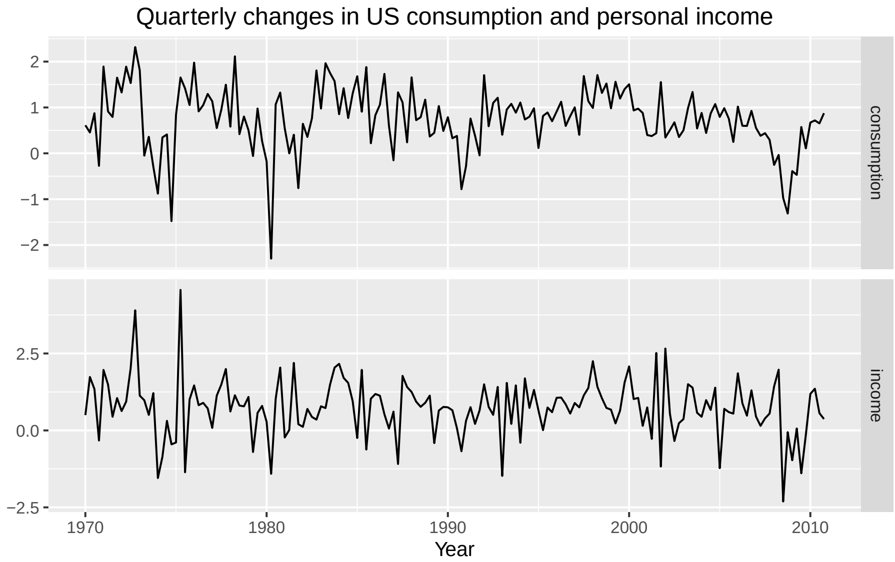
Model selection

- Check that all variables are stationary. If not, apply differencing. Where appropriate, use the same differencing for all variables to preserve interpretability.
- Fit regression model with automatically selected ARIMA errors.
- Check that e_t series looks like white noise.

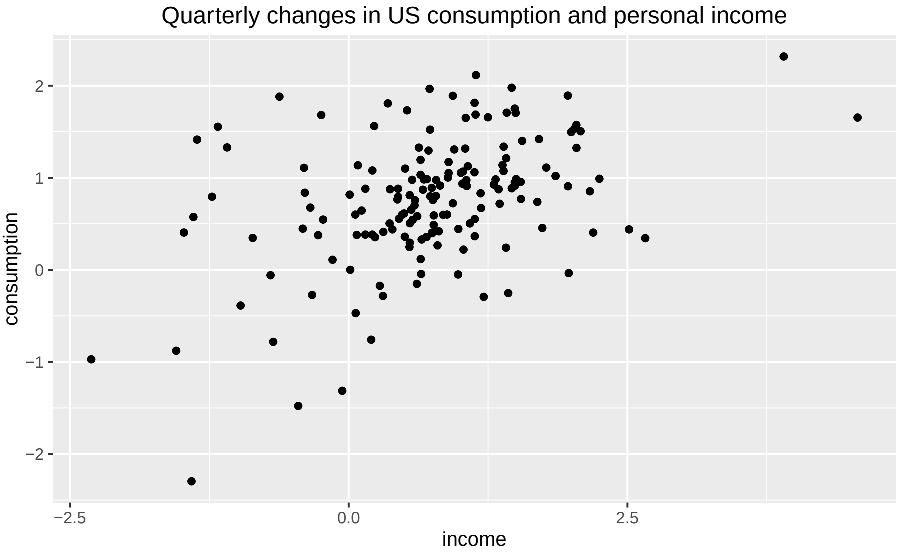
Selecting predictors

- AICc can be calculated for final model.
- Repeat procedure for all subsets of predictors to be considered, and select model with lowest AIC value.

US personal consumption & income



US personal consumption & income



US Personal Consumption and income

- No need for transformations or further differencing.
- Increase in income does not necessarily translate into instant increase in consumption (e.g., after the loss of a job, it may take a few months for expenses to be reduced to allow for the new circumstances). We will ignore this for now.

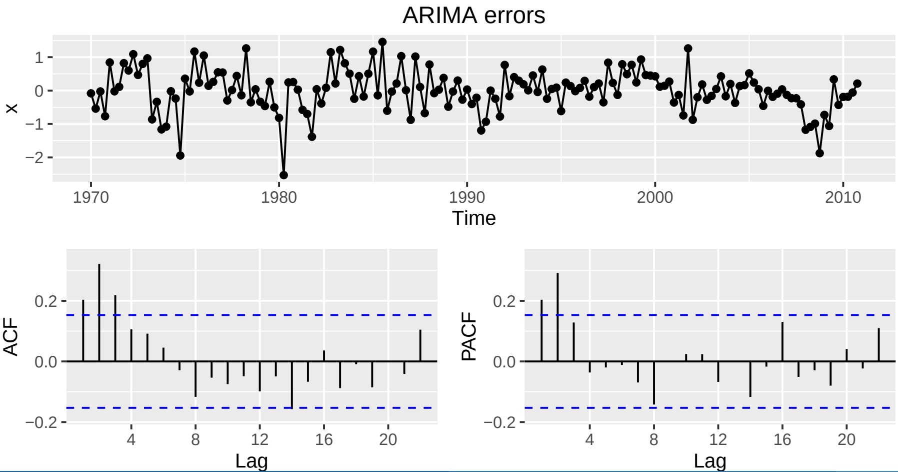
US personal consumption & income

```r
(fit <- auto.arima(usconsumption[,1],
    xreg=usconsumption[,2]))
```

```
## Series: usconsumption[, 1]
## ARIMA(1,0,2) with non-zero mean
##
## Coefficients:
##          ar1      ma1     ma2  intercept  usconsumption[, 2]
##       0.6516  -0.5440  0.2187     0.5750              0.2420
## s.e.  0.1468   0.1576  0.0790     0.0951              0.0513
##
## sigma^2 estimated as 0.3502:  log likelihood=-144.27
## AIC=300.54   AICc=301.08   BIC=319.14
```

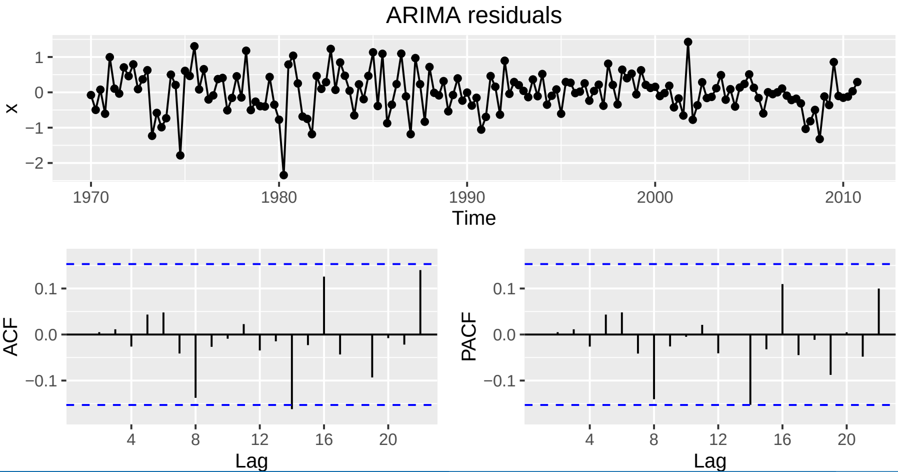
US personal consumption & income

```r
ggtsdisplay(arima.errors(fit),
  main="ARIMA errors")
```



US personal consumption & income

```r
ggtsdisplay(residuals(fit),
  main="ARIMA residuals")
```



US Personal Consumption and Income

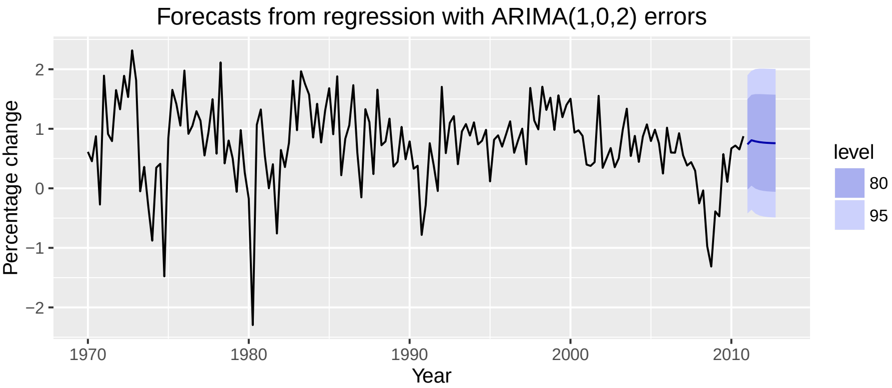
A Ljung-Box test shows the residuals are uncorrelated.

```
Box.test(residuals(fit),
  fitdf=5, lag=10, type="Ljung")
```

```
##
##  Box-Ljung test
##
## data:  residuals(fit)
## X-squared = 4.5948, df = 5, p-value = 0.4673
```

US Personal Consumption and Income

```
fcast <- forecast(fit,
  xreg=rep(mean(usconsumption[,2]),8), h=8)
autoplot(fcast) + xlab("Year") +
  ylab("Percentage change") +
  ggtitle("Forecasts from regression with ARIMA(1,0,2) errors"
```



Forecasting

- To forecast a regression model with ARIMA errors, we need to forecast the regression part of the model and the ARIMA part of the model and combine the results.
  - Some explanatory variable are known into the future (e.g., time, dummies).
  - Separate forecasting models may be needed for other explanatory variables.

Outline

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + n_t$$

where n_t is ARMA process.

Stochastic trend

$$y_t = \beta_0 + \beta_1 t + n_t$$

where n_t is ARIMA process with $d \geq 1$.

Difference both sides until n_t is stationary:

$$y'_t = \beta_1 + n'_t$$

where n'_t is ARMA process.

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Difference both sides until n_t is stationary:

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where n'_t is ARMA process.

International visitors

International visitors

Deterministic trend

```
(fit1 <- auto.arima(austa, d=0, xreg=1:length(austa)))
```

```
## Series: austa
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2  intercept  1:length(austa)
##       1.0371  -0.3379     0.4173           0.1715
## s.e.  0.1675   0.1797     0.1866           0.0102
##
## sigma^2 estimated as 0.02854:  log likelihood=12.7
## AIC=-15.4   AICc=-13   BIC=-8.23
```

$$
\begin{aligned}
y_t &= 0.4173 + 0.1715t + n_t \\
n_t &= 1.0371n_{t-1} - 0.3379n_{t-2} + e_t \\
e_t &\sim \text{NID}(0, 0.02854).
\end{aligned}
$$

International visitors

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\end{aligned}
$$

International visitors

Stochastic trend

```r
(fit2 <- auto.arima(austa,d=1))
```

```
## Series: austa
## ARIMA(0,1,0) with drift
##
## Coefficients:
##        drift
##       0.1537
## s.e.  0.0323
##
## sigma^2 estimated as 0.03241:  log likelihood=9.38
## AIC=-14.76   AICc=-14.32   BIC=-11.96
```

$$
\begin{aligned}
y_t - y_{t-1} &= 0.1538 + e_t \\
y_t &= y_0 + 0.1538t + n_t \\
n_t &= n_{t-1} + e_t
\end{aligned}
$$

International visitors

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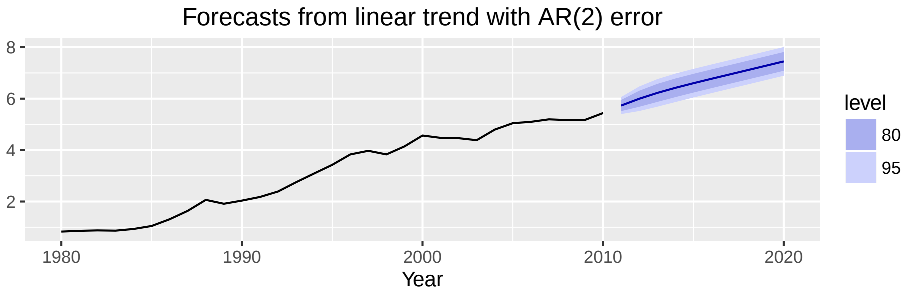
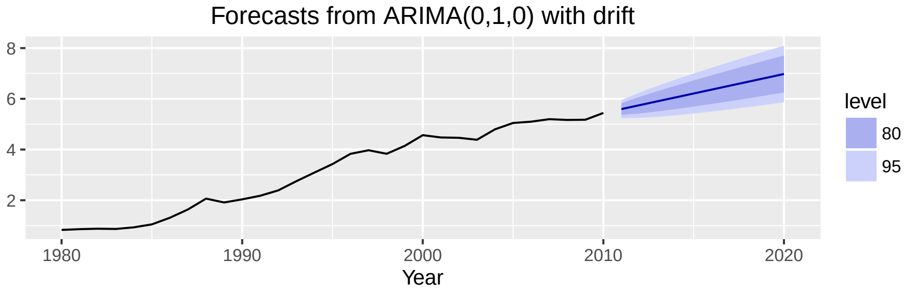
```

$$y_t - y_{t-1} = 0.1538 + e_t$$
$$y_t = y_0 + 0.1538t + n_t$$
$$n_t = n_{t-1} + e_t$$

International visitors





Forecasting with trend

- Point forecasts are almost identical, but prediction intervals differ.
- Stochastic trends have much wider prediction intervals because the errors are non-stationary.
- Be careful of forecasting with deterministic trends too far ahead.

Outline

Fourier terms for seasonality

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = \sum_{k=1}^{K} [\alpha_k s_k(t) + \beta_k c_k(t)] + n_t$$

- n_t is non-seasonal ARIMA process.
- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc.

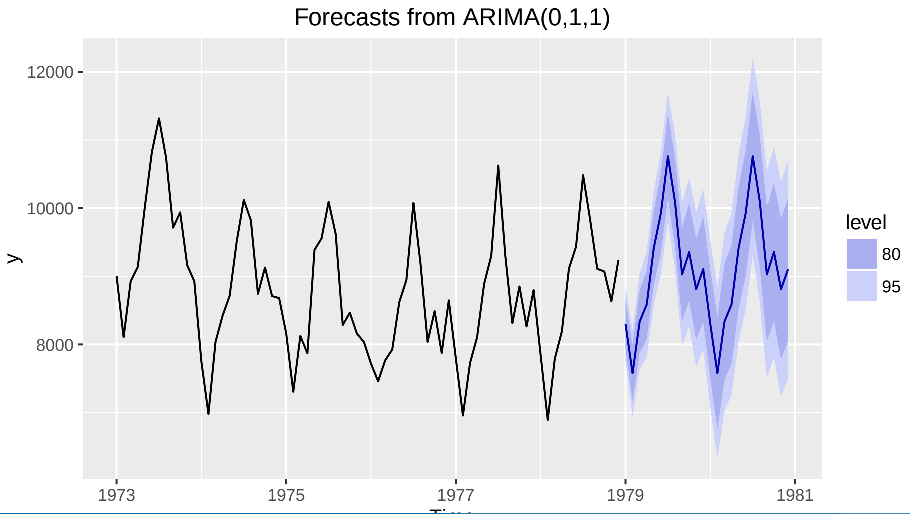
US Accidental Deaths

```
fit <- auto.arima(USAccDeaths,
         xreg=fourier(USAccDeaths, 5),
         seasonal=FALSE)

fc <- forecast(fit,
         xreg=fourier(USAccDeaths, 5, 24))
```

US Accidental Deaths

```
autoplot(fc)
```



Outline

Lab Session 14

Outline

Dynamic regression models

Sometimes a change in x_t does not affect y_t instantaneously

- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

- These are dynamic systems with input (x_t) and output (y_t).
- x_t is often a leading indicator.
- There can be multiple predictors.

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Lagged explanatory variables

The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \ldots.$

$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \cdots + \nu_k x_{t-k} + n_t$$

where n_t is an ARIMA process.

Rewrite model as

$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots + \nu_k B^k) x_t + n_t \\ &= a + \nu(B) x_t + n_t. \end{aligned}$$

- $\nu(B)$ is called a *transfer function* since it describes how change in x_t is transferred to y_t.
- x can influence y, but y is not allowed to influence x.

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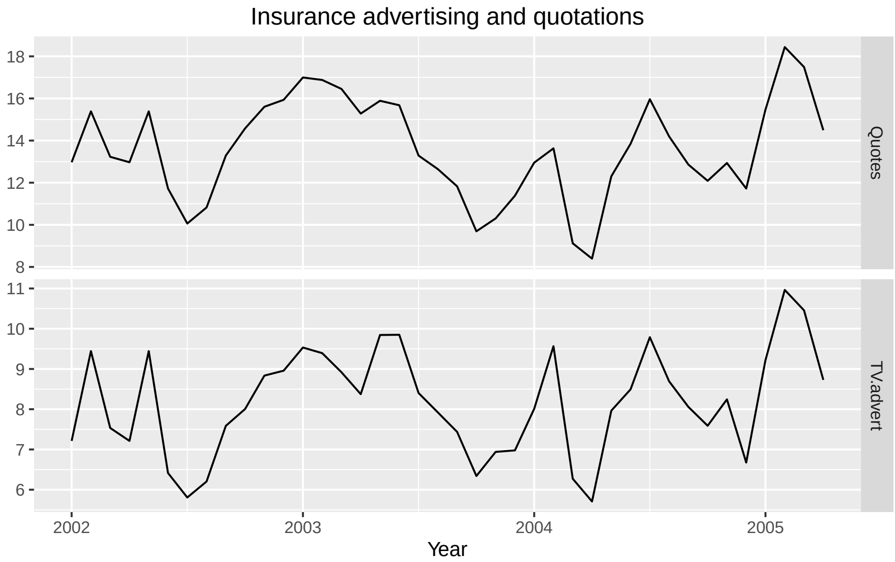
Example: Insurance quotes and TV adverts



Example: Insurance quotes and TV adverts

```r
Advert <- cbind(insurance[,2], c(NA,insurance[1:39,2]))
colnames(Advert) <- paste("AdLag",0:1,sep="")
(fit <- auto.arima(insurance[,1], xreg=Advert, d=0))
```

```
## Series: insurance[, 1]
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2     ar3  intercept  AdLag0  AdLag1
##       1.4117  -0.9317  0.3591     2.0393  1.2564  0.1625
## s.e.  0.1698   0.2545  0.1592     0.9931  0.0667  0.0591
##
## sigma^2 estimated as 0.2165:  log likelihood=-23.89
## AIC=61.78   AICc=65.28   BIC=73.6
```

$$y_t = 2.05 + 1.26x_t + 0.16x_{t-1} + n_t$$
$$n_t = 1.41n_{t-1} - 093n_{t-2} + 0.36n_{t-3}$$

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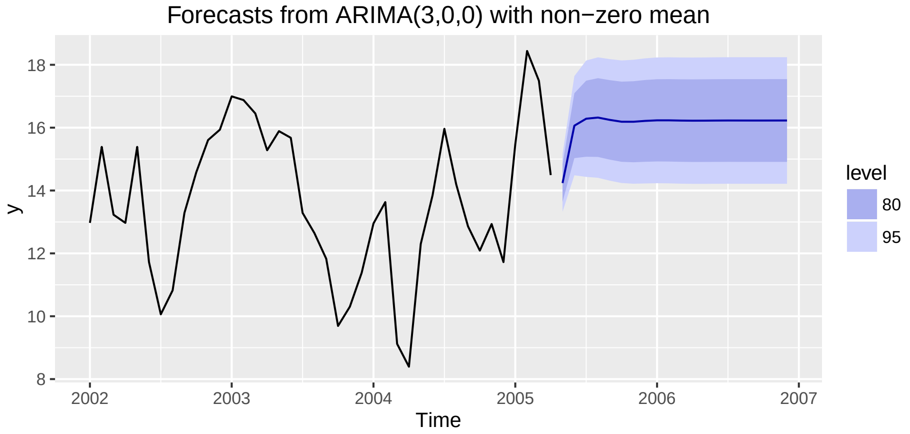
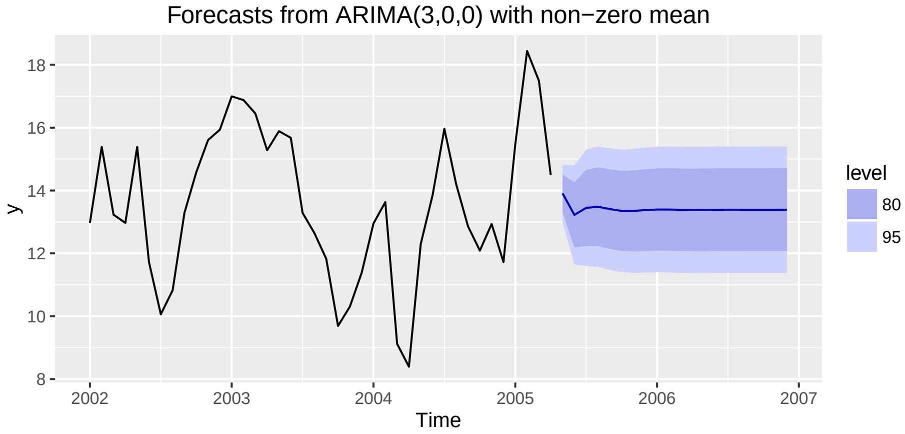
fc <- forecast(fit, h=20,
  xreg=cbind(c(Advert[40,1],rep(10,19)), rep(10,20)))
autoplot(fc)
```



Example: Insurance quotes and TV adverts

```
fc <- forecast(fit, h=20,
  xreg=cbind(c(Advert[40,1],rep(8,19)), rep(8,20)))
autoplot(fc)
```



Example: Insurance quotes and TV adverts

```
fc <- forecast(fit, h=20,
  xreg=cbind(c(Advert[40,1],rep(6,19)), rep(6,20)))
autoplot(fc)
```

Dynamic regression models

$$y_t = a + \nu(B)x_t + n_t$$

where n_t is an ARMA process. So

$$\phi(B)n_t = \theta(B)e_t \qquad \text{or} \qquad n_t = \frac{\theta(B)}{\phi(B)}e_t = \psi(B)e_t.$$

$$y_t = a + \nu(B)x_t + \psi(B)e_t$$

- ARMA models are rational approximations to general transfer functions of e_t.
- We can also replace $\nu(B)$ by a rational approximation.
- There is no R package for forecasting using a general transfer function approach.

Dynamic regression models

$$y_t = a + \nu(B)x_t + n_t$$

where n_t is an ARMA process. So

$$\phi(B)n_t = \theta(B)e_t \qquad \text{or} \qquad n_t = \frac{\theta(B)}{\phi(B)}e_t = \psi(B)e_t.$$

$$y_t = a + \nu(B)x_t + \psi(B)e_t$$

- ARMA models are rational approximations to general transfer functions of e_t.
- We can also replace $\nu(B)$ by a rational approximation.
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Dynamic regression models

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