MONASH University

MONASH BUSINESS SCHOOL

ETC3550: Applied forecasting for business and economics

Ch8. ARIMA models

OTexts.org/fpp2/

Outline

Stationarity

Definition

If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \dots, y_{t+s})$ does not depend on t.

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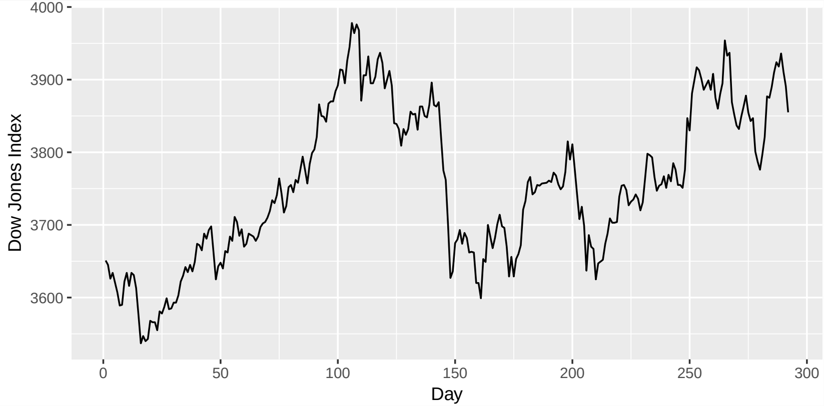
A **stationary series** is:

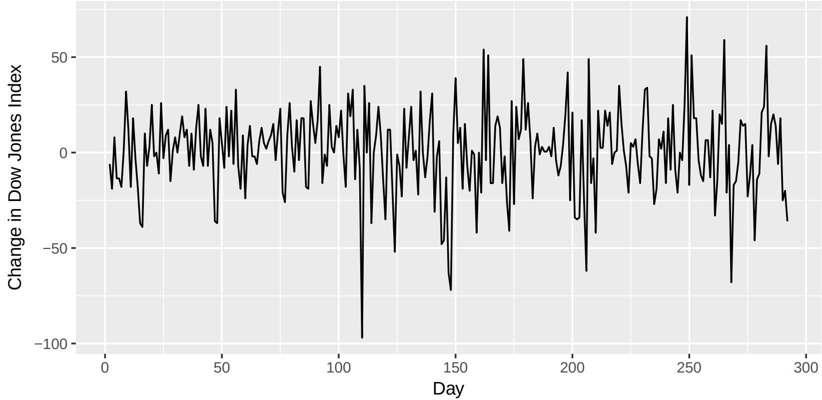
- roughly horizontal
- constant variance
- no patterns predictable in the long-term

Stationary?



4000
3900
3800
3700
3600
Dow Jones Index
0
50
100
150
200
250
300
Day

Stationary?

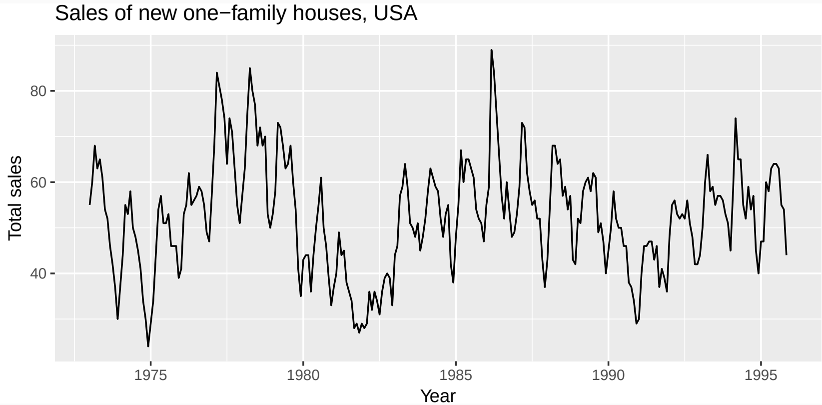


Change in Dow Jones Index
50
0
−50
−100
0
50
100
150
200
250
300
Day

Stationary?

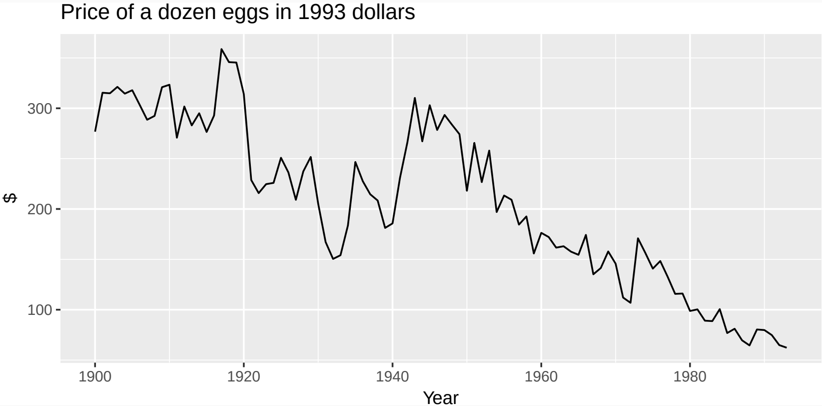
6000
5000
4000
Number of strikes
1950
1955
1960
1965
1970
1975
1980
Year

Stationary?



Sales of new one−family houses, USA
Total sales
80
60
40
1975
1980
1985
1990
1995
Year

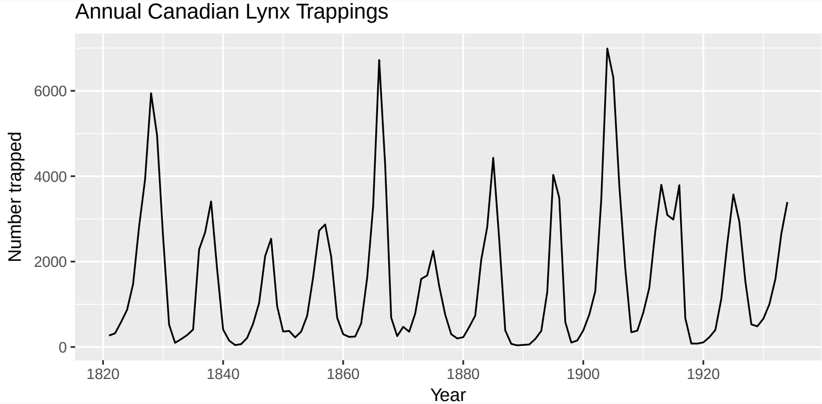
Stationary?


Price of a dozen eggs in 1993 dollars
$
300
200
100
1900
1920
1940
1960
1980
Year

Stationary?


Number of pigs slaughtered in Victoria
thousands
110
100
90
80
1990
1991
1992
1993
1994
1995
Year

Stationary?


Annual Canadian Lynx Trappings
Number trapped
6000
4000
2000
0
1820
1840
1860
1880
1900
1920
Year

Stationary?


Australian quarterly beer production
megalitres
500
450
400
1995
2000
2005
2010
Year

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Transformations help to **stabilize the variance**.

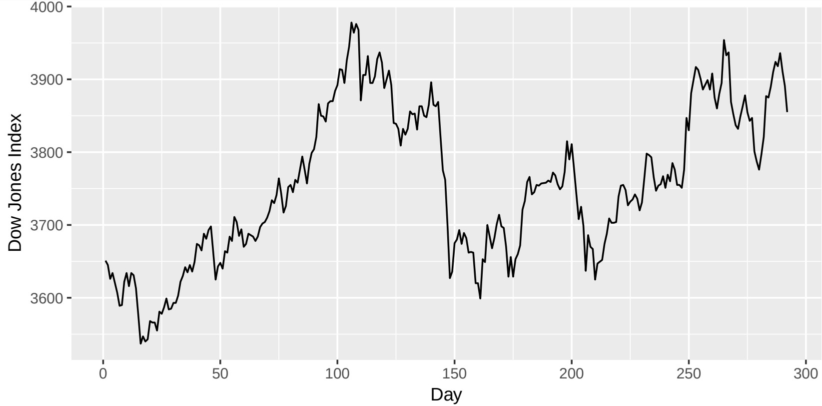
For ARIMA modelling, we also need to **stabilize the mean**.

Non-stationarity in the mean

Identifying non-stationary series

- time plot.
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly.
- For non-stationary data, the value of r_1 is often large and positive.

Example: Dow-Jones index



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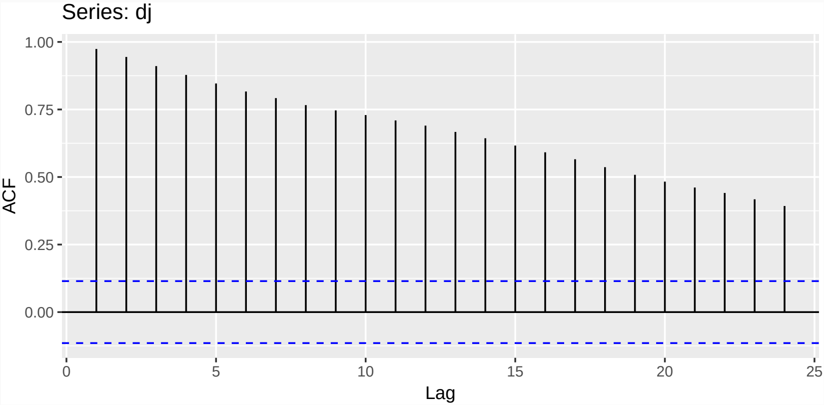

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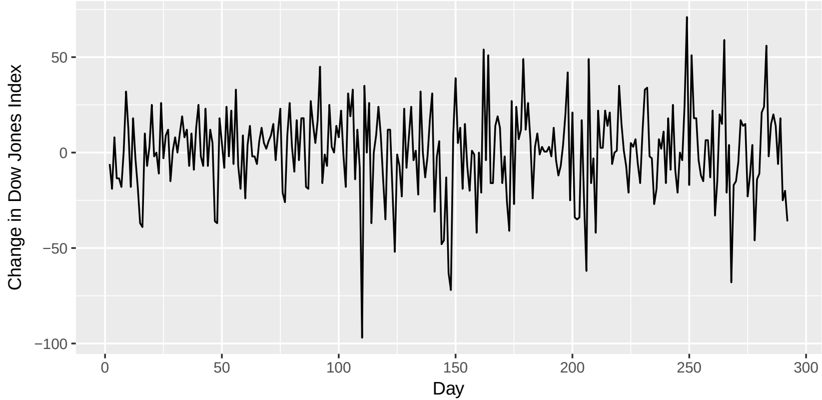

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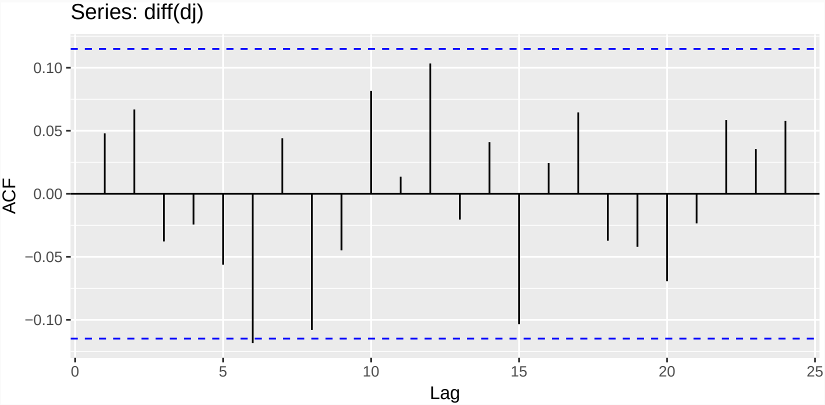

Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series: $y'_t = y_t - y_{t-1}$.
- The differenced series will have only $T - 1$ values since it is not possible to calculate a difference y'_1 for the first observation.

Second-order differencing

Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time:

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$$\begin{aligned} y''_t &= y'_t - y'_{t-1} \\ &= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\ &= y_t - 2y_{t-1} + y_{t-2}. \end{aligned}$$

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- y''_t will have $T - 2$ values.
- In practice, it is almost never necessary to go beyond second-order differences.

Seasonal differencing

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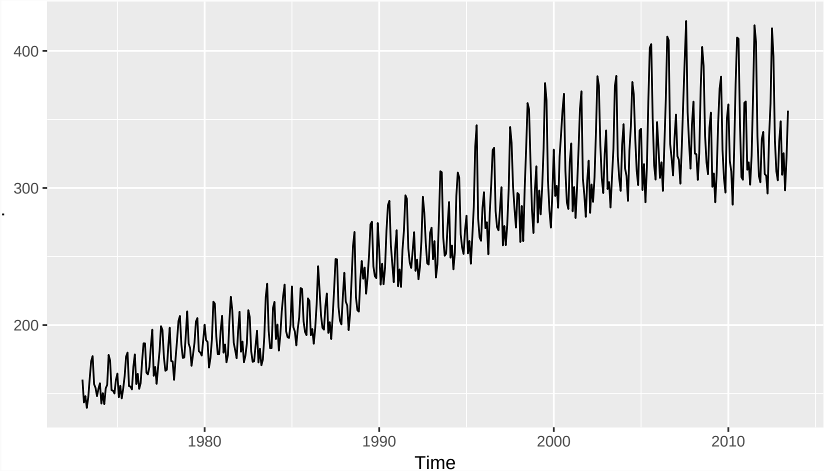
- For monthly data $m = 12$.
- For quarterly data $m = 4$.

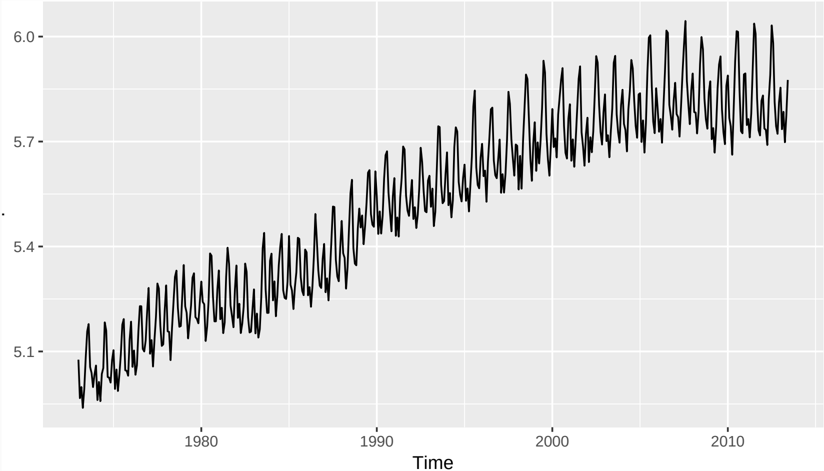
Electricity production

```r
usmelec %>% autoplot()
```



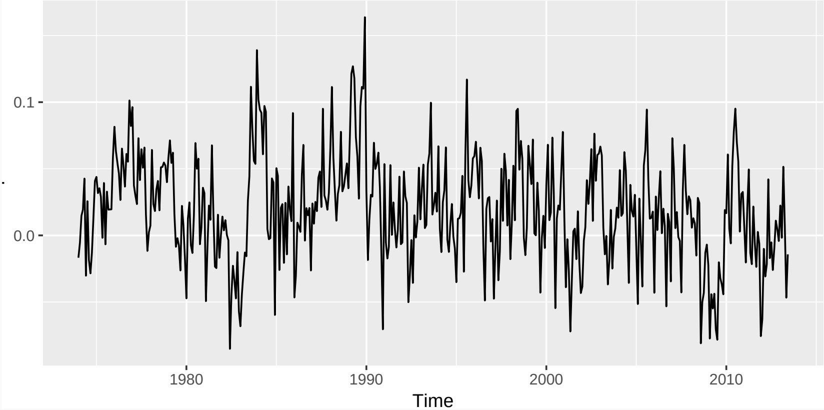
Electricity production

```
usmelec %>% log() %>% autoplot()
```



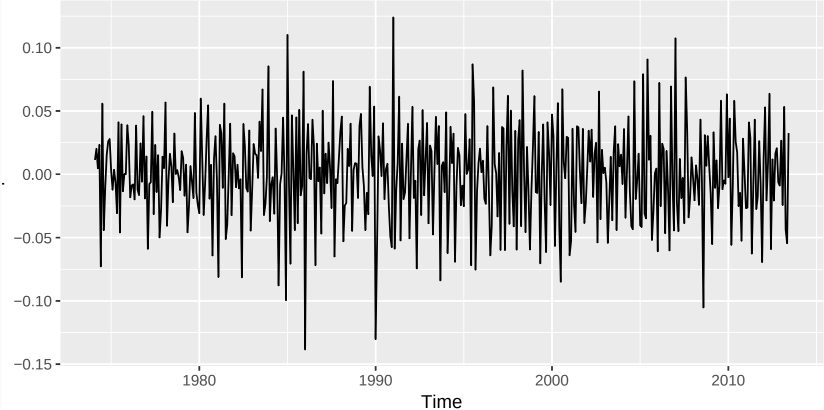
Electricity production

```
usmelec %>% log() %>% diff(lag=12) %>%
  autoplot()
```



Electricity production

```
usmelec %>% log() %>% diff(lag=12) %>%
  diff(lag=1) %>% autoplot()
```



Electricity production

- Seasonally differenced series is closer to being stationary.
- Remaining non-stationarity can be removed with further first difference.

If $y'_t = y_t - y_{t-12}$ denotes seasonally differenced series, then twice-differenced series i

$$\begin{aligned} y^*_t &= y'_t - y'_{t-1} \\ &= (y_t - y_{t-12}) - (y_{t-1} - y_{t-13}) \\ &= y_t - y_{t-1} - y_{t-12} + y_{t-13}\,. \end{aligned}$$

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When both seasonal and first differences are applied…

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- it makes no difference which is done first—the result will be the same.
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It is important that if differencing is used, the differences are interpretable.

Interpretation of differencing

- first differences are the change between **one observation and the next**;
- seasonal differences are the change between **one year to the next**.

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But taking lag 3 differences for yearly data, for example, results in a model which cannot be sensibly interpreted.

Unit root tests

Statistical tests to determine the required order of differencing.

1. Augmented Dickey Fuller test: null hypothesis is that the data are non-stationary and non-seasonal.
2. Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: null hypothesis is that the data are stationary and non-seasonal.
3. Other tests available for seasonal data.

Dickey-Fuller test

Test for "unit root"

- Estimate regression model
$$y'_t = \phi y_{t-1} + b_1 y'_{t-1} + b_2 y'_{t-2} + \cdots + b_k y'_{t-k}$$
where y'_t denotes differenced series $y_t - y_{t-1}$.
- Number of lagged terms, k, is usually set to be about 3.
- If original series, y_t, needs differencing, $\hat{\phi} \approx 0$.
- If y_t is already stationary, $\hat{\phi} < 0$.
- In R: Use `tseries::adf.test()`.

Dickey-Fuller test in R

```r
tseries::adf.test(x,
 alternative = c("stationary", "explosive"),
 k = trunc((length(x)-1)^(1/3)))
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- Set alternative = `stationary`.

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```

- $k = \lfloor T - 1 \rfloor^{1/3}$
- Set alternative = `stationary`.

```r
tseries::adf.test(dj)
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  dj
## Dickey-Fuller = -1.9872, Lag order = 6, p-value = 0.5816
```

How many differences?

```r
ndiffs(dj)
```

```
## [1] 1
```

```r
nsdiffs(USAccDeaths)
```

```
## [1] 1
```

Your turn

For the `visitors` series, find an appropriate differencing (after transformation if necessary) to obtain stationary data.

Backshift notation

A very useful notational device is the backward shift operator, B, which is used as follows:

$$By_t = y_{t-1} \ .$$

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$$B(By_t) = B^2 y_t = y_{t-2} \ .$$

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For monthly data, if we wish to shift attention to “the same month last year,” then B^{12} is used, and the notation is $B^{12} y_t = y_{t-12}$.

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The backward shift operator is convenient for describing the process of *differencing*.

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$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B)y_t\ .$$

Note that a first difference is represented by $(1 - B)$. Similarly, if second-order differences (i.e., first differences of first differences) have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1 - B)^2 y_t\ .$$

Backshift notation

- Second-order difference is denoted $(1 - B)^2$.
- *Second-order difference* is not the same as a *second difference*, which would be denoted $1 - B^2$;
- In general, a dth-order difference can be written as
$$(1 - B)^d y_t.$$
- A seasonal difference followed by a first difference can be written as
$$(1 - B)(1 - B^m)y_t\ .$$

Backshift notation

The "backshift" notation is convenient because the terms can be multiplied together to see the combined effect.

$$\begin{aligned}(1-B)(1-B^m)y_t &= (1-B-B^m+B^{m+1})y_t\\ &= y_t-y_{t-1}-y_{t-m}+y_{t-m-1}.\end{aligned}$$

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For monthly data, $m = 12$ and we obtain the same result as earlier.

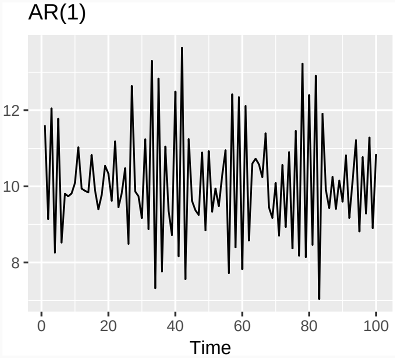
Outline

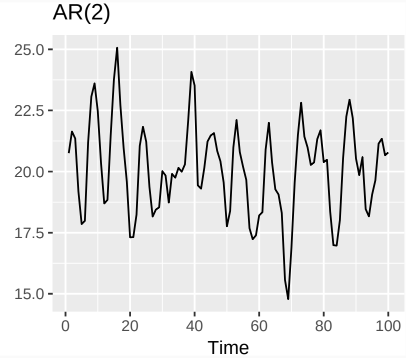
Autoregressive models

Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + e_t,$$

where e_t is white noise. This is a multiple regression with **lagged values** of y_t as predictors.

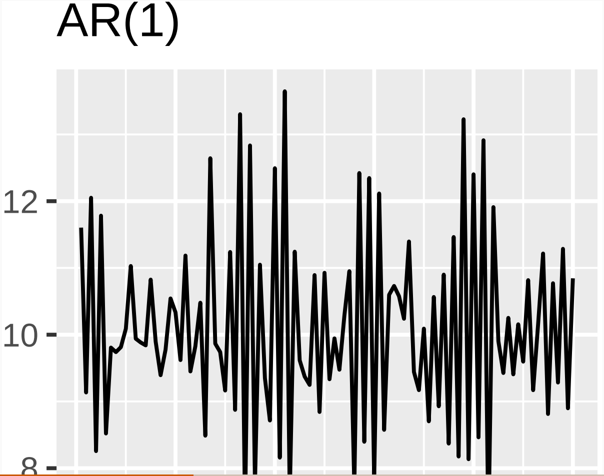




AR(1) model

$$y_t = 2 - 0.8y_{t-1} + e_t$$

$e_t \sim N(0, 1), \quad T = 100.$



AR(1) model

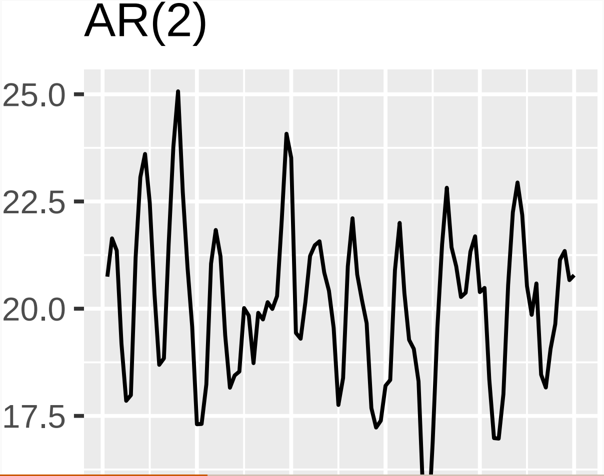
$$y_t = c + \phi_1 y_{t-1} + e_t$$

- When $\phi_1 = 0$, y_t is **equivalent to WN**
- When $\phi_1 = 1$ and $c = 0$, y_t is **equivalent to a RW**
- When $\phi_1 = 1$ and $c \neq 0$, y_t is **equivalent to a RW with drift**
- When $\phi_1 < 0$, y_t tends to **oscillate between positive and negative values**.

AR(2) model

$$y_t = 8 + 1.3y_{t-1} - 0.7y_{t-2} + e_t$$

$e_t \sim N(0, 1), \qquad T = 100.$



Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ lie outside the unit circle on the complex plane.

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Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ lie outside the unit circle on the complex plane.

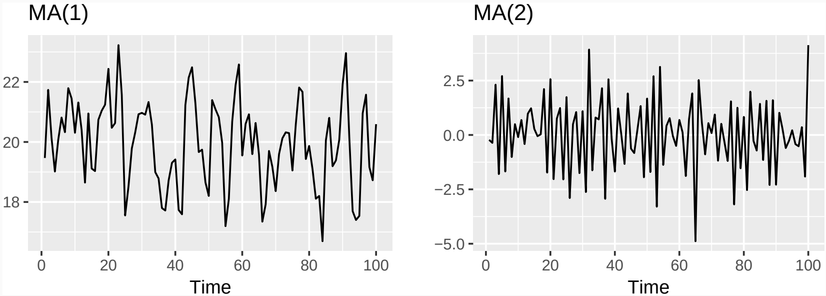
- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$:\
  $-1 < \phi_2 < 1 \qquad \phi_2 + \phi_1 < 1 \qquad \phi_2 - \phi_1 < 1.$
- More complicated conditions hold for $p \geq 3$.
- Estimation software takes care of this.

Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q},$$

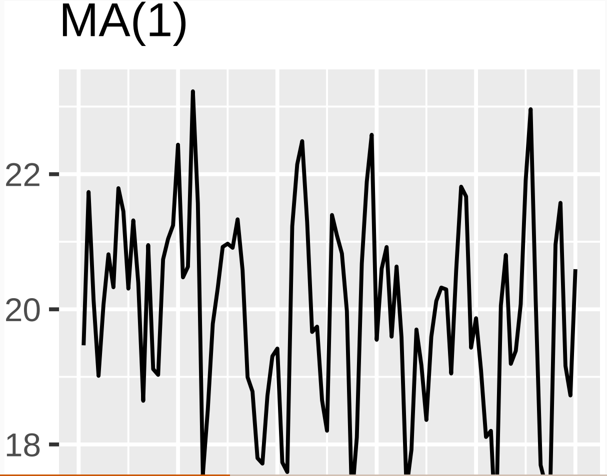
where e_t is white noise. This is a multiple regression with ***past errors*** as predictors. *Don't confuse this with moving average smoothing!*



MA(1) model

$$y_t = 20 + e_t + 0.8e_{t-1}$$

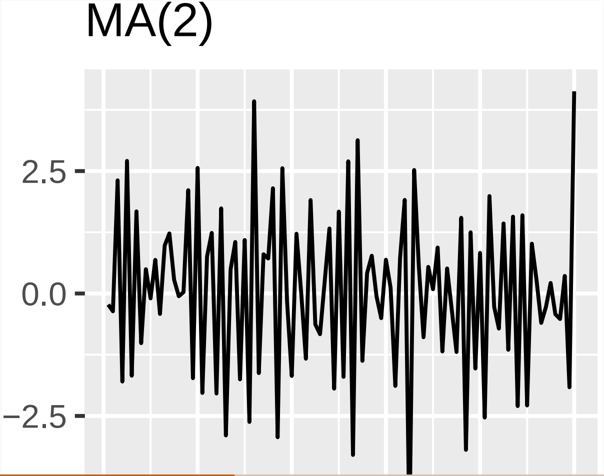
$e_t \sim N(0, 1), \quad T = 100.$



MA(2) model

$$y_t = e_t - e_{t-1} + 0.8e_{t-2}$$

$e_t \sim N(0, 1), \quad T = 100.$



MA(∞) models

It is possible to write any stationary AR(p) process as an MA(∞) process.

Example: AR(1)

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + e_t \\
&= \phi_1(\phi_1 y_{t-2} + e_{t-1}) + e_t \\
&= \phi_1^2 y_{t-2} + \phi_1 e_{t-1} + e_t \\
&= \phi_1^3 y_{t-3} + \phi_1^2 e_{t-2} + \phi_1 e_{t-1} + e_t \\
&\dots
\end{aligned}
$$

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&\dots
\end{aligned}
$$

Provided $-1 < \phi_1 < 1$:

$$y_t = e_t + \phi_1 e_{t-1} + \phi_1^2 e_{t-2} + \phi_1^3 e_{t-3} + \cdots$$

Invertibility

- Any MA(q) process can be written as an AR(∞) process if we impose some constraints on the MA parameters.
- Then the MA model is called "invertible".
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

Invertibility

General condition for invertibility

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ lie outside the unit circle on the complex plane.

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- For $q = 2$:

  $-1 < \theta_2 < 1 \qquad \theta_2 + \theta_1 > -1 \qquad \theta_1 - \theta_2 < 1.$
- More complicated conditions hold for $\{q \geq 3.\}$
- Estimation software takes care of this.

ARIMA models

Autoregressive Moving Average models:

$$\begin{aligned} y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} + e_t. \end{aligned}$$

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$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} + e_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

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Autoregressive Moving Average models:

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- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing**.
- $(1 - B)^d y_t$ follows an ARMA model.

ARIMA models

Autoregressive Integrated Moving Average models

ARIMA(p, d, q) model

AR: p = order of the autoregressive part
I: d = degree of first differencing involved
MA: q = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA(p,0,0)
- MA(q): ARIMA(0,0,q)

Backshift notation for ARIMA

- ARMA model:

$$y_t = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t$$

$$\text{or} \quad (1 - \phi_1 B - \cdots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \cdots + \theta_q B^q) e_t$$

- ARIMA(1,1,1) model:

$$\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1 - \phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1 - B) y_t} \quad = \quad \underset{\substack{\uparrow \\ \text{MA(1)}}}{c + (1 + \theta_1 B) e_t}$$

Backshift notation for ARIMA

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$$y_t = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t$$

$$\text{or} \quad (1 - \phi_1 B - \cdots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \cdots + \theta_q B^q) e_t$$

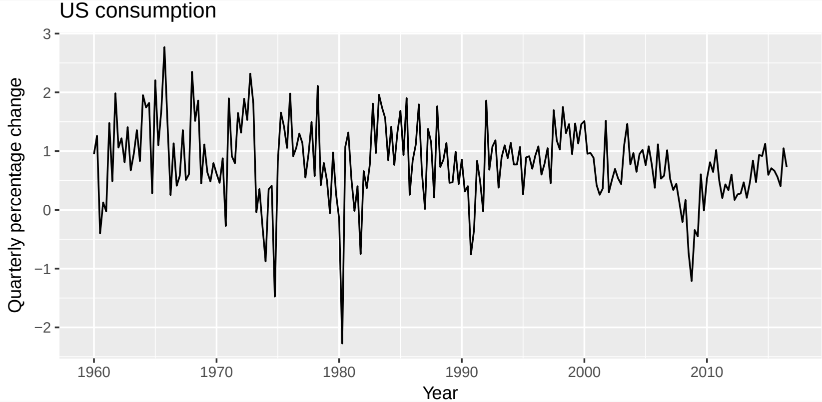
- ARIMA(1,1,1) model:

$$\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1 - \phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1 - B) y_t} \quad = \quad c + \underset{\substack{\uparrow \\ \text{MA(1)}}}{(1 + \theta_1 B) e_t}$$

Written out:

$$y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 e_{t-1} + e_t$$

US personal consumption



US personal consumption

```r
(fit <- auto.arima(uschange[,"Consumption"],
    seasonal=FALSE))
```

```
## Series: uschange[, "Consumption"]
## ARIMA(0,1,1)
##
## Coefficients:
##           ma1
##       -0.7080
## s.e.   0.0637
##
## sigma^2 estimated as 0.4095:  log likelihood=-219.64
## AIC=443.28   AICc=443.33   BIC=450.12
```

US personal consumption

```r
(fit <- auto.arima(uschange[,"Consumption"],
    seasonal=FALSE))
```

```
## Series: uschange[, "Consumption"]
## ARIMA(0,1,1)
##
## Coefficients:
##           ma1
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## s.e.   0.0637
##
## sigma^2 estimated as 0.4095:  log likelihood=-219.64
## AIC=443.28   AICc=443.33   BIC=450.12
```
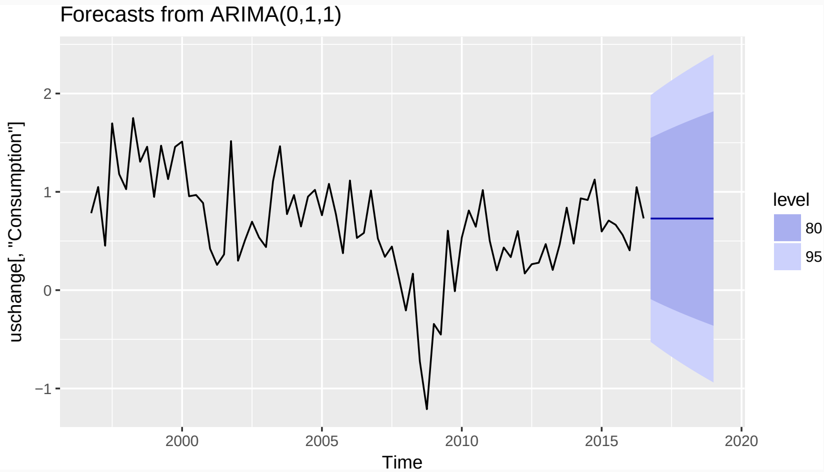

ARIMA(0,1,1) model:

US personal consumption

```
fit %>% forecast(h=10) %>% autoplot(include=8
```



Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 2$, the long-term forecasts will follow a quadratic trend.

Understanding ARIMA models

Forecast variance and d

- The higher the value of d, the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Cyclic behaviour

- For cyclic forecasts, $p > 2$ and some restrictions on coefficients are required.
- If $p = 2$, we need $\phi_1^2 + 4\phi_2 < 0$. Then average cycle of length

Outline

Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$.

Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters c, $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$.

- MLE is very similar to least squares estimation obtained by minimizing
$$\sum_{t-1}^{T} e_t^2.$$
- The `Arima()` command allows CLS or MLE estimation.
- Non-linear optimization must be used in either case.
- Different software will give different estimates.

Partial autocorrelations

Partial autocorrelations measure relationship between y_t and y_{t-k}, when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

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$$\begin{aligned} \alpha_k &= k\text{th partial autocorrelation coefficient} \\ &= \text{equal to the estimate of } b_k \text{ in regression:} \\ & \qquad y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k}. \end{aligned}$$

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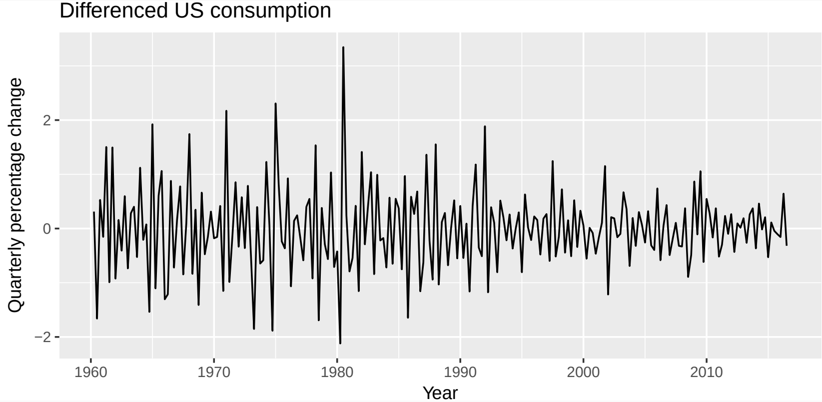
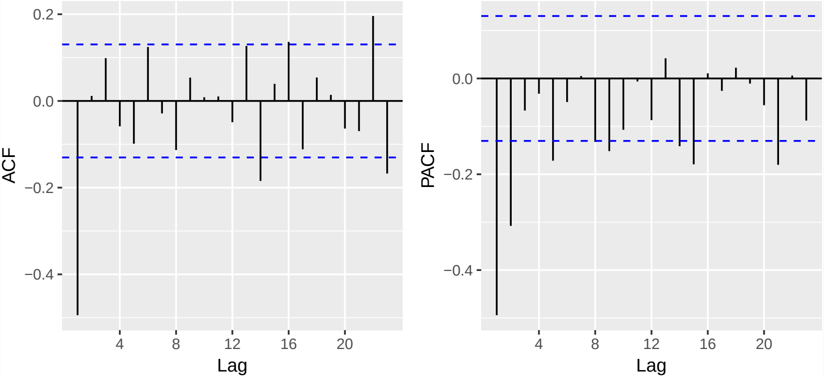
- Varying number of terms on RHS gives α_k for different values of k.
- There are more efficient ways of calculating α_k.
- $\alpha_1 = \rho_1$
- same critical values of $\pm 1.96/\sqrt{T}$ as for ACF

Example: US consumption



Example: US consumption



ACF and PACF interpretation

AR(1)

$$\rho_k = \phi_1^k \qquad \text{for } k = 1, 2, \dots;$$

$$\alpha_1 = \phi_1 \qquad \alpha_k = 0 \qquad \text{for } k = 2, 3, \dots.$$

So we have an AR(1) model when

- autocorrelations exponentially decay
- there is a single significant partial autocorrelation.

ACF and PACF interpretation

AR(p)

- ACF dies out in an exponential or damped sine-wave manner
- PACF has all zero spikes beyond the pth spike

So we have an AR(p) model when

- the ACF is exponentially decaying or sinusoidal
- there is a significant spike at lag p in PACF, but none beyond p

ACF and PACF interpretation

MA(1)

$$\rho_1 = \theta_1 \qquad \rho_k = 0 \qquad \text{for } k = 2, 3, \ldots;$$

$$\alpha_k = -(-\theta_1)^k$$

So we have an MA(1) model when

- the PACF is exponentially decaying and
- there is a single significant spike in ACF

ACF and PACF interpretation

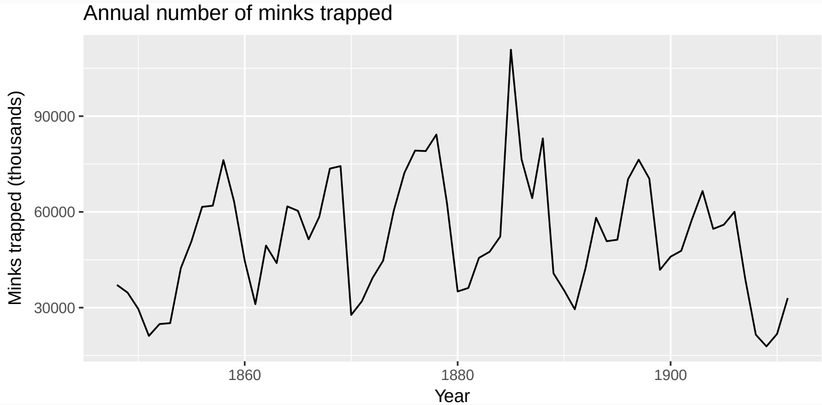
MA(q)

- PACF dies out in an exponential or damped sine-wave manner
- ACF has all zero spikes beyond the qth spike

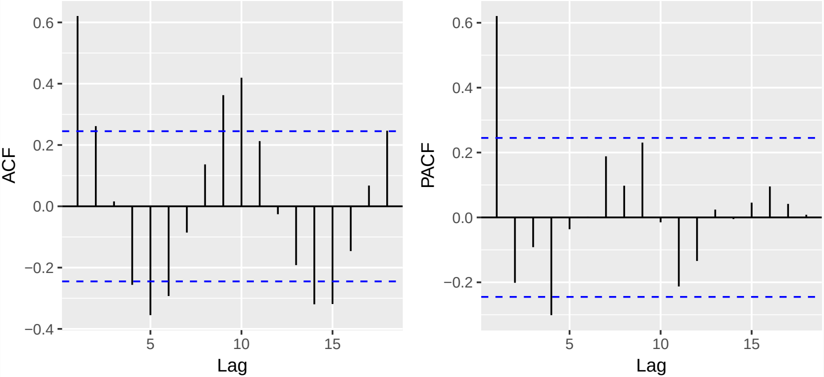
So we have an MA(q) model when

- the PACF is exponentially decaying or sinusoidal
- there is a significant spike at lag q in ACF, but none beyond q

Example: Mink trapping



Example: Mink trapping



Information criteria

Akaike's Information Criterion (AIC):

$$\text{AIC} = -2\log(L) + 2(p + q + k + 1),$$

where L is the likelihood of the data,

$k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p+q+k+1)(p+q+k+2)}{T-p-q-k-2}.$$

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Bayesian Information Criterion:

$$\text{BIC} = \text{AIC} + \log(T)(p+q+k-1).$$

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Bayesian Information Criterion:

$$\text{BIC} = \text{AIC} + \log(T)(p+q+k-1).$$

Good models are obtained by minimizing either the AIC, AICc or BIC. Our preference is to use the AICc.

Outline

How does auto.arima() work?

A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders: p, q, d

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d and D via unit root tests.
- Select p, q by minimising AICc.
- Use stepwise search to traverse model space.

How does auto.arima() work?

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \tfrac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k=1$ if $c\neq 0$ and $k=0$ otherwise.

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where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

Step1: Select current model (with smallest AICc) from:
ARIMA($2, d, 2$)
ARIMA($0, d, 0$)
ARIMA($1, d, 0$)
ARIMA($0, d, 1$)

How does auto.arima() work?

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

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Step1: Select current model (with smallest AICc) from:
ARIMA$(2, d, 2)$
ARIMA$(0, d, 0)$
ARIMA$(1, d, 0)$
ARIMA$(0, d, 1)$

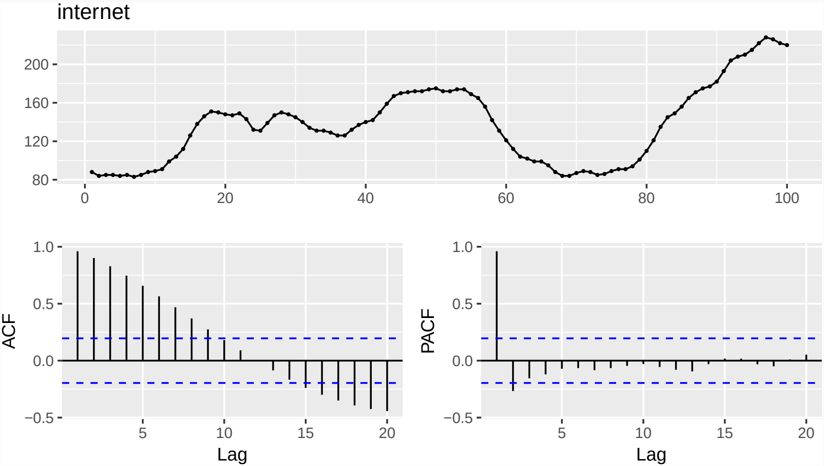
Step 2: Consider variations of current model:

- vary one of $p, q,$ from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

Choosing your own model

```
ggtsdisplay(internet)
```



Choosing your own model

```
tseries::adf.test(internet)
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  internet
## Dickey-Fuller = -2.6421, Lag order = 4, p-value = 0.3107
## alternative hypothesis: stationary
```

```
tseries::kpss.test(internet)
```

```
##
##  KPSS Test for Level Stationarity
##
## data:  internet
## KPSS Level = 0.72197, Truncation lag parameter = 2, p-value =
```

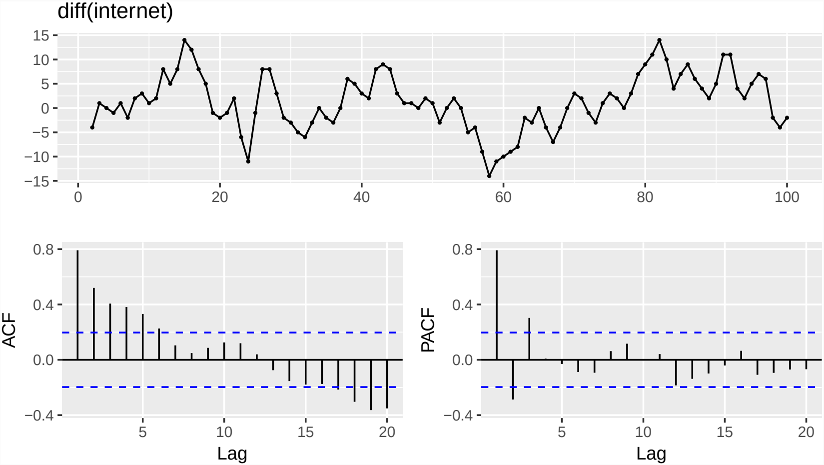
Choosing your own model

```
tseries::kpss.test(diff(internet))
```

```
##
##  KPSS Test for Level Stationarity
##
## data:  diff(internet)
## KPSS Level = 0.26352, Truncation lag parameter = 2, p-value = (
```

Choosing your own model

```
ggtsdisplay(diff(internet))
```



Choosing your own model

```r
(fit <- Arima(internet,order=c(3,1,0)))
```

```
## Series: internet
## ARIMA(3,1,0)
##
## Coefficients:
##          ar1      ar2     ar3
##       1.1513  -0.6612  0.3407
## s.e.  0.0950   0.1353  0.0941
##
## sigma^2 estimated as 9.656:  log likelihood=-252
## AIC=511.99   AICc=512.42   BIC=522.37
```

Choosing your own model

```
auto.arima(internet)
```

```
## Series: internet
## ARIMA(1,1,1)
##
## Coefficients:
##          ar1     ma1
##       0.6504  0.5256
## s.e.  0.0842  0.0896
##
## sigma^2 estimated as 9.995:  log likelihood=-254
## AIC=514.3   AICc=514.55   BIC=522.08
```

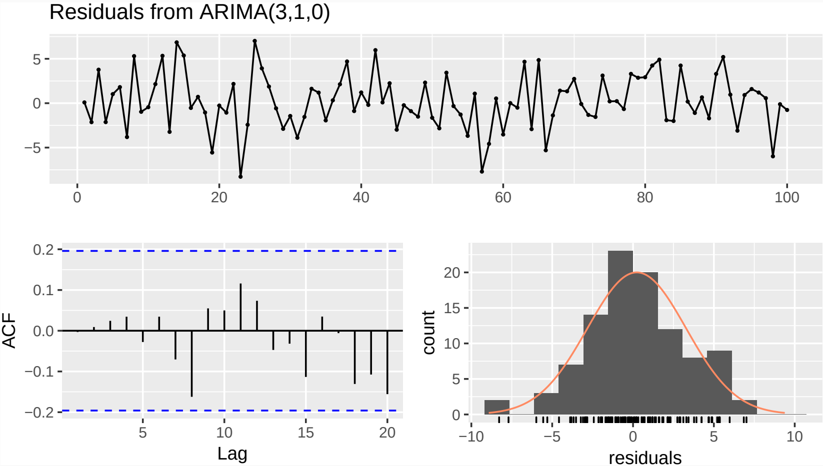
Choosing your own model

```r
auto.arima(internet, stepwise=FALSE,
  approximation=FALSE)
```

```
## Series: internet
## ARIMA(3,1,0)
##
## Coefficients:
##          ar1      ar2     ar3
##       1.1513  -0.6612  0.3407
## s.e.  0.0950   0.1353  0.0941
##
## sigma^2 estimated as 9.656:  log likelihood=-252
## AIC=511.99   AICc=512.42   BIC=522.37
```

Choosing your own model

```
checkresiduals(fit)
```

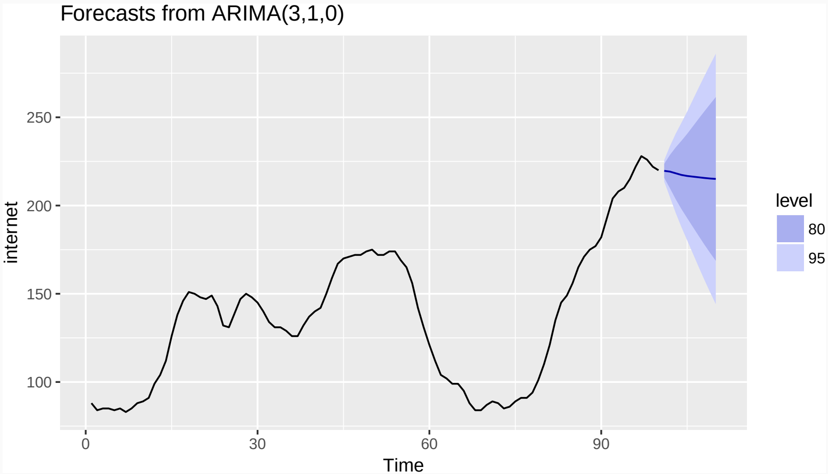


Choosing your own model

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(3,1,0)
## Q* = 4.4913, df = 7, p-value = 0.7218
##
## Model df: 3.   Total lags used: 10
```

Choosing your own model

```
fit %>% forecast %>% autoplot
```



Modelling procedure with `Arima`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
3. If the data are non-stationary: take first differences of the data until the data are stationary.
4. Examine the ACF/PACF: Is an AR(p) or MA(q) model appropriate?
5. Try your chosen model(s), and use the AICc to search for a better model.
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.

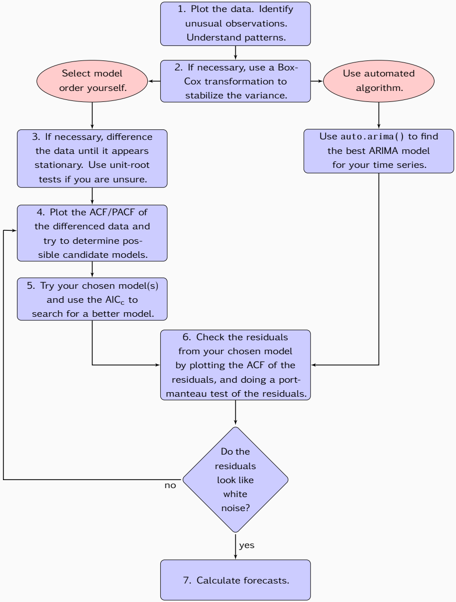
Modelling procedure with `auto.arima`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.

3. Use `auto.arima` to select a model.

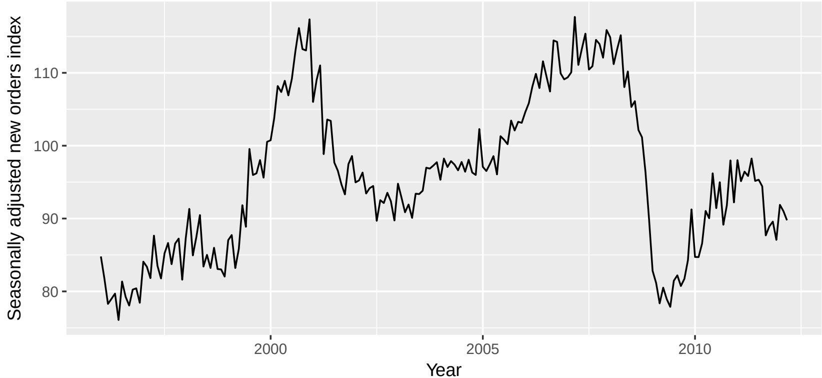
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

Modelling procedure



Seasonally adjusted electrical equipment

```r
eeadj <- seasadj(stl(elecequip, s.window="periodic
autoplot(eeadj) + xlab("Year") +
  ylab("Seasonally adjusted new orders index")
```

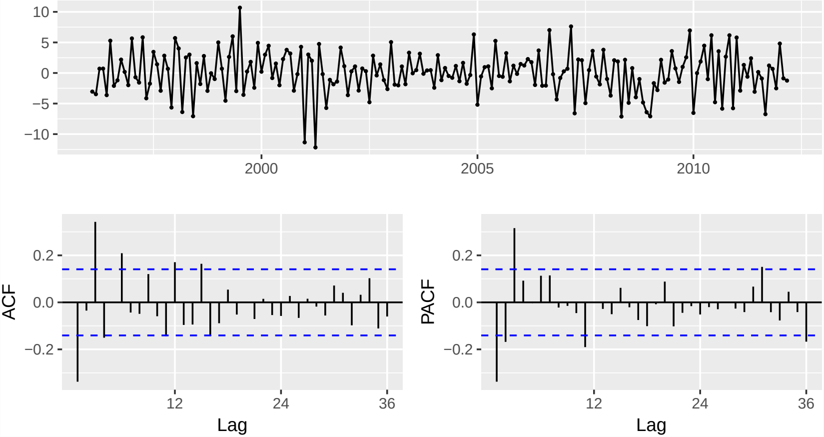


Seasonally adjusted electrical equipment

1. Time plot shows sudden changes, particularly big drop in 2008/2009 due to global economic environment. Otherwise nothing unusual and no need for data adjustments.
2. No evidence of changing variance, so no Box-Cox transformation.
3. Data are clearly non-stationary, so we take first differences.

Seasonally adjusted electrical equipment

```r
ggtsdisplay(diff(eeadj), main="")
```



Seasonally adjusted electrical equipment

4. PACF is suggestive of AR(3). So initial candidate model is ARIMA(3,1,0). No other obvious candidates.
5. Fit ARIMA(3,1,0) model along with variations: ARIMA(4,1,0), ARIMA(2,1,0), ARIMA(3,1,1), etc. ARIMA(3,1,1) has smallest AICc value.

Seasonally adjusted electrical equipment

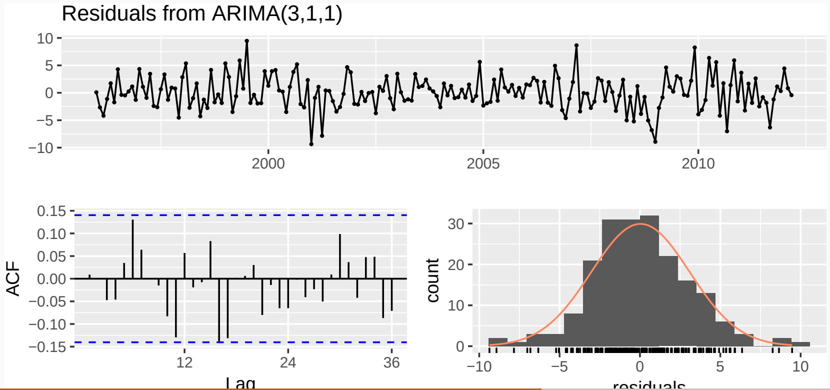
```
fit <- Arima(eeadj, order=c(3,1,1))
summary(fit)
```

```
## Series: eeadj
## ARIMA(3,1,1)
##
## Coefficients:
##          ar1     ar2     ar3      ma1
##       0.0044  0.0916  0.3698  -0.3921
## s.e.  0.2201  0.0984  0.0669   0.2426
##
## sigma^2 estimated as 9.577:  log likelihood=-492.69
## AIC=995.38   AICc=995.7   BIC=1011.72
##
## Training set error measures:
##                     ME     RMSE      MAE          MPE     MAP
## Training set 0.0328818 3.054718 2.357169 -0.006470086 2.48160
```

Seasonally adjusted electrical equipment

6. ACF plot of residuals from ARIMA(3,1,1) model look like white noise.

```
checkresiduals(fit)
```

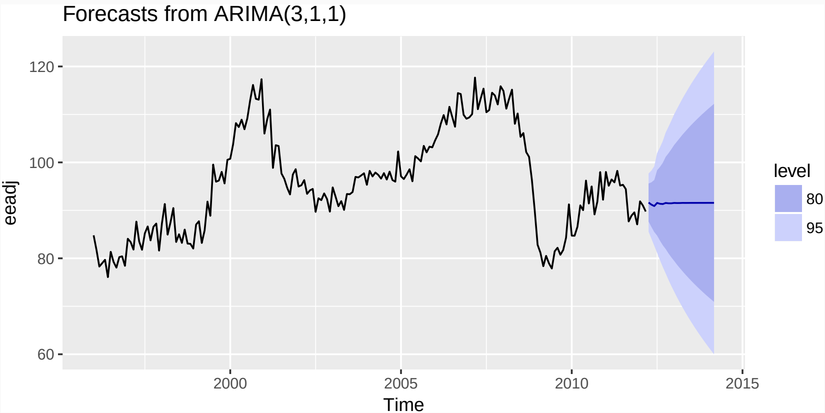


Seasonally adjusted electrical equipment

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(3,1,1)
## Q* = 24.034, df = 20, p-value = 0.2409
##
## Model df: 4.   Total lags used: 24
```

Seasonally adjusted electrical equipment

```
fit %>% forecast %>% autoplot
```



Outline

Point forecasts

1. Rearrange ARIMA equation so y_t is on LHS.
2. Rewrite equation by replacing t by $T + h$.
3. On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \ldots$.

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B) y_t = (1 + \theta_1 B) e_t,$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B) y_t = (1 + \theta_1 B) e_t,$$

$$\begin{aligned}\left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4\right] y_t \\ = (1 + \theta_1 B) e_t,\end{aligned}$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B) y_t = (1 + \theta_1 B) e_t,$$

$$\begin{aligned}\left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4\right] y_t \\ = (1 + \theta_1 B) e_t,\end{aligned}$$

$$\begin{aligned}y_t - (1 + \phi_1) y_{t-1} + (\phi_1 - \phi_2) y_{t-2} + (\phi_2 - \phi_3) y_{t-3} \\ + \phi_3 y_{t-4} = e_t + \theta_1 e_{t-1}.\end{aligned}$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B) y_t = (1 + \theta_1 B) e_t,$$

$$\left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4\right] y_t$$
$$= (1 + \theta_1 B) e_t,$$

$$y_t - (1 + \phi_1) y_{t-1} + (\phi_1 - \phi_2) y_{t-2} + (\phi_2 - \phi_3) y_{t-3}$$
$$+ \phi_3 y_{t-4} = e_t + \theta_1 e_{t-1}.$$

$$y_t = (1 + \phi_1) y_{t-1} - (\phi_1 - \phi_2) y_{t-2} - (\phi_2 - \phi_3) y_{t-3}$$
$$- \phi_3 y_{t-4} + e_t + \theta_1 e_{t-1}.$$

Point forecasts (h=1)

$$
\begin{aligned}
y_t = (1+\phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\
- \phi_3 y_{t-4} + e_t + \theta_1 e_{t-1}.
\end{aligned}
$$

Point forecasts (h=1)

$$\begin{aligned} y_t = (1+\phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + e_t + \theta_1 e_{t-1}. \end{aligned}$$

ARIMA(3,1,1) forecasts: Step 2

$$\begin{aligned} y_{T+1} = (1+\phi_1)y_{T} - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3 y_{T-3} + e_{T+1} + \theta_1 e_{T}. \end{aligned}$$

Point forecasts (h=1)

$$y_t = (1+\phi_1)y_{t-1} - (\phi_1-\phi_2)y_{t-2} - (\phi_2-\phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + e_t + \theta_1 e_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1+\phi_1)y_T - (\phi_1-\phi_2)y_{T-1} - (\phi_2-\phi_3)y_{T-2} \\ - \phi_3 y_{T-3} + e_{T+1} + \theta_1 e_T.$$

ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+1|T} = (1+\phi_1)y_T - (\phi_1-\phi_2)y_{T-1} - (\phi_2-\phi_3)y_{T-2} \\ - \phi_3 y_{T-3} + \theta_1 \hat{e}_T.$$

Point forecasts (h=2)

$$\begin{aligned} y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + e_t + \theta_1 e_{t-1}. \end{aligned}$$

Point forecasts (h=2)

$$
\begin{aligned}
y_t = (1+\phi_1)y_{t-1} - (\phi_1-\phi_2)y_{t-2} - (\phi_2-\phi_3)y_{t-3} \\
- \phi_3 y_{t-4} + e_t + \theta_1 e_{t-1}.
\end{aligned}
$$

ARIMA(3,1,1) forecasts: Step 2

$$
\begin{aligned}
y_{T+2} = (1+\phi_1)y_{T+1} - (\phi_1-\phi_2)y_{T} - (\phi_2-\phi_3)y_{T-1} \\
- \phi_3 y_{T-2} + e_{T+2} + \theta_1 e_{T+1}.
\end{aligned}
$$

Point forecasts (h=2)

$$\begin{aligned} y_t = (1+\phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + e_t + \theta_1 e_{t-1}. \end{aligned}$$

ARIMA(3,1,1) forecasts: Step 2

$$\begin{aligned} y_{T+2} = (1+\phi_1)y_{T+1} - (\phi_1 - \phi_2)y_{T} - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3 y_{T-2} + e_{T+2} + \theta_1 e_{T+1}. \end{aligned}$$

ARIMA(3,1,1) forecasts: Step 3

$$\begin{aligned} \hat{y}_{T+2|T} = (1+\phi_1)\hat{y}_{T+1|T} - (\phi_1 - \phi_2)y_{T} - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3 y_{T-2}. \end{aligned}$$

Prediction intervals

95% Prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

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where $v_{T+h|T}$ is estimated forecast variance.

- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = e_t + \sum_{i=1}^{q} \theta_i e_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \dots.$$

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- AR(1): Rewrite as MA(∞) and use above result.
- Other models beyond scope of this workshop.

Prediction intervals

- Prediction intervals **increase in size with forecast horizon**.
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed**.
- Prediction intervals tend to be too narrow.
  - the uncertainty in the parameter estimates has not been accounted for.
  - the ARIMA model assumes historical patterns will not change during the forecast period.
  - the ARIMA model assumes uncorrelated future

Your turn

For the `usgdp` data:

- if necessary, find a suitable Box-Cox transformation for the data;
- fit a suitable ARIMA model to the transformed data using `auto.arima()`;
- check the residual diagnostics;
- produce forecasts of your fitted model. Do the forecasts look reasonable?

Outline

Backshift notation

A very useful notational device is the backward shift operator, B, which is used as follows:

$$By_t = y_{t-1} \; .$$

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$$B(By_t) = B^2 y_t = y_{t-2} \,.$$

For monthly data, if we wish to shift attention to "the same month last year," then B^{12} is used, and the notation is $B^{12}y_t = y_{t-12}$.

Backshift notation

- First difference: $1 - B$.
- Double difference: $(1 - B)^2$.
- dth-order difference: $(1 - B)^d y_t$.
- Seasonal difference: $1 - B^m$.
- Seasonal difference followed by a first difference: $(1 - B)(1 - B^m)$.
- Multiply terms together together to see the combined effect:

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

Backshift notation for ARIMA

- ARMA model:

$$\begin{aligned} y_t &= c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \\ &= c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t \end{aligned}$$

$$\phi(B) y_t = c + \theta(B) e_t$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$

and $\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$.

Backshift notation for ARIMA

- ARMA model:

$$
\begin{aligned}
y_t &= c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \\
&= c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t
\end{aligned}
$$

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where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$

and $\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$.

- ARIMA(1,1,1) model:

$$
\begin{array}{ccccc}
(1 - \phi_1 B) & (1 - B) y_t & = & c + (1 + \theta_1 B) e_t \\
\uparrow & \uparrow & & \uparrow \\
\text{AR(1)} & \text{First} & & \text{MA(1)} \\
 & \text{difference} & &
\end{array}
$$

Backshift notation for ARIMA

- ARIMA(p, d, q) model:

$$\underset{\substack{\uparrow \\ \text{AR}(p)}}{(1 - \phi_1 B - \cdots - \phi_p B^p)} \quad \underset{\substack{\uparrow \\ d \text{ differences}}}{(1 - B)^d y_t} \;=\; c + \underset{\substack{\uparrow \\ \text{MA}(q)}}{(1 + \theta_1 B + \cdots + \theta_q B^q) e_t}$$

Outline

Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$	$\uparrow$
	Non-seasonal part of the model	Seasonal part of of the model

where m = number of observations per year.

Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

Seasonal ARIMA models

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$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$$

Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)$_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$$

$\left(\begin{array}{c}\text{Non-seasonal}\\ \text{AR(1)}\end{array}\right)$ $\left(\begin{array}{c}\text{Seasonal}\\ \text{AR(1)}\end{array}\right)$ $\left(\begin{array}{c}\text{Non-seasonal}\\ \text{difference}\end{array}\right)$ $\left(\begin{array}{c}\text{Seasonal}\\ \text{difference}\end{array}\right)$ $\left(\begin{array}{c}\text{Non-seasonal}\\ \text{MA(1)}\end{array}\right)$ $\left(\begin{array}{c}\text{Seasonal}\\ \text{MA(1)}\end{array}\right)$

Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t \;=\; (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$

All the factors can be multiplied out and the general model written as follows:

$$
\begin{aligned}
y_t = {} & (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\
& - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\
& - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \\
& + e_t + \theta_1 e_{t-1} + \Theta_1 e_{t-4} + \theta_1\Theta_1 e_{t-5}.
\end{aligned}
$$

Common ARIMA models

In the US Census Bureau uses the following models most often:

ARIMA$(0,1,1)(0,1,1)_m$	with log transformation
ARIMA$(0,1,2)(0,1,1)_m$	with log transformation
ARIMA$(2,1,0)(0,1,1)_m$	with log transformation
ARIMA$(0,2,2)(0,1,1)_m$	with log transformation
ARIMA$(2,1,2)(0,1,1)_m$	with no transformation

Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)$_{12}$ will show:

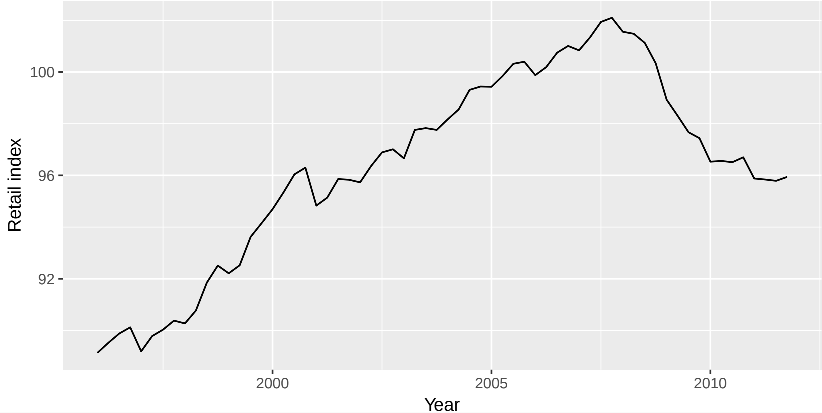
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ….

ARIMA(0,0,0)(1,0,0)$_{12}$ will show:

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

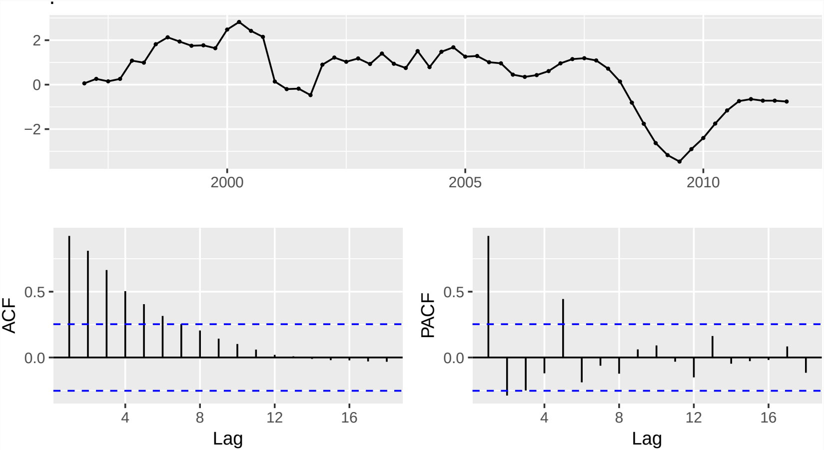
European quarterly retail trade

```r
autoplot(euretail) +
  xlab("Year") + ylab("Retail index")
```



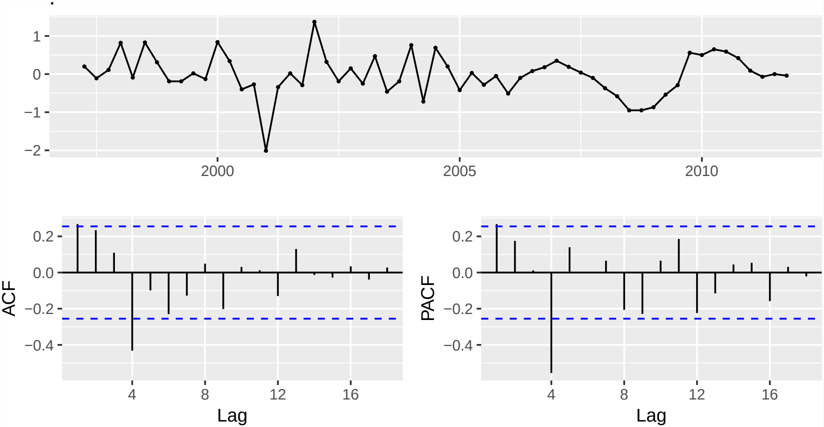
European quarterly retail trade

```r
euretail %>% diff(lag=4) %>% ggtsdisplay()
```



European quarterly retail trade

```r
euretail %>% diff(lag=4) %>% diff() %>%
  ggtsdisplay()
```

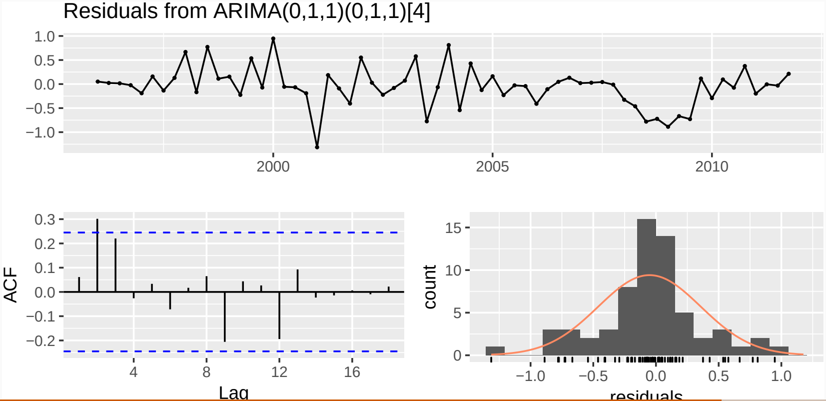


European quarterly retail trade

- $d = 1$ and $D = 1$ seems necessary.
- Significant spike at lag 1 in ACF suggests non-seasonal MA(1) component.
- Significant spike at lag 4 in ACF suggests seasonal MA(1) component.
- Initial candidate model: ARIMA(0,1,1)(0,1,1)$_4$.
- We could also have started with ARIMA(1,1,0)(1,1,0)$_4$.

European quarterly retail trade

```
fit <- Arima(euretail, order=c(0,1,1),
  seasonal=c(0,1,1))
checkresiduals(fit)
```



European quarterly retail trade

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(0,1,1)(0,1,1)[4
## Q* = 10.654, df = 6, p-value = 0.09968
##
## Model df: 2.   Total lags used: 8
```

European quarterly retail trade

- ACF and PACF of residuals show significant spikes at lag 2, and maybe lag 3.
- AICc of ARIMA$(0,1,2)(0,1,1)_4$ model is 74.36.
- AICc of ARIMA$(0,1,3)(0,1,1)_4$ model is 68.53.

European quarterly retail trade

- ACF and PACF of residuals show significant spikes at lag 2, and maybe lag 3.
- AICc of ARIMA(0,1,2)$(0,1,1)_4$ model is 74.36.
- AICc of ARIMA(0,1,3)$(0,1,1)_4$ model is 68.53.

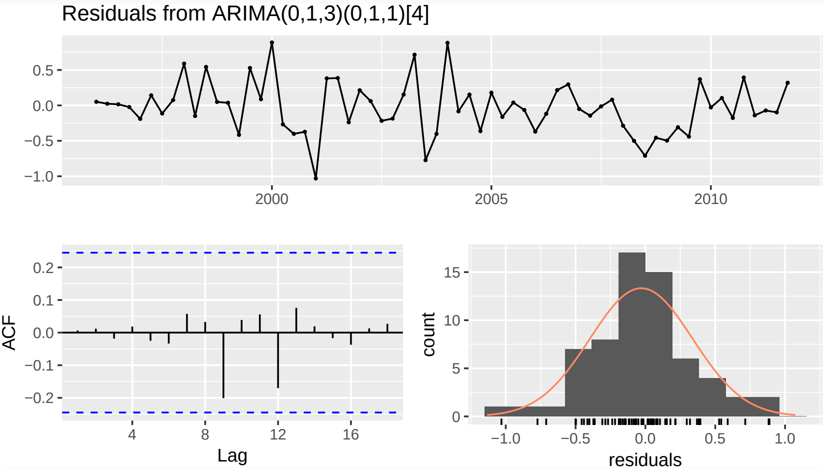
```
fit <- Arima(euretail, order=c(0,1,3),
  seasonal=c(0,1,1))
checkresiduals(fit)
```

European quarterly retail trade

```
## Series: euretail
## ARIMA(0,1,3)(0,1,1)[4]
##
## Coefficients:
##          ma1     ma2     ma3     sma1
##       0.2630  0.3694  0.4200  -0.6636
## s.e.  0.1237  0.1255  0.1294   0.1545
##
## sigma^2 estimated as 0.156:  log likelihood=-28.63
## AIC=67.26   AICc=68.39   BIC=77.65
```

European quarterly retail trade

```
checkresiduals(fit)
```

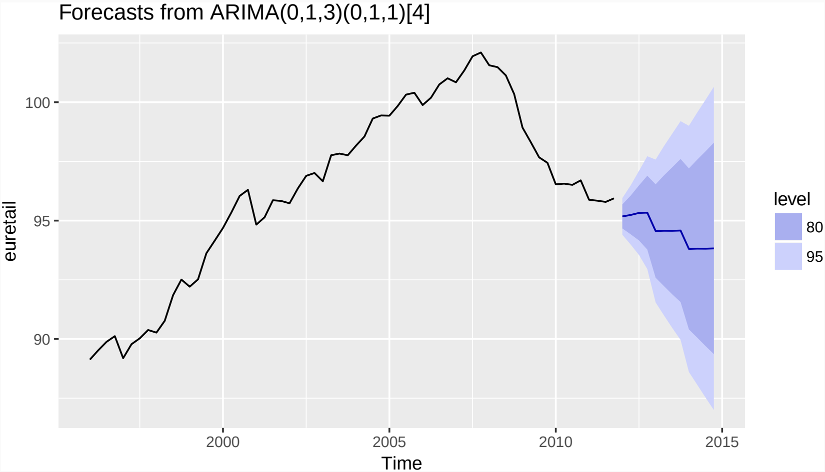


European quarterly retail trade

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(0,1,3)(0,1,1)[4]
## Q* = 0.51128, df = 4, p-value = 0.9724
##
## Model df: 4.   Total lags used: 8
```

European quarterly retail trade

```
autoplot(forecast(fit, h=12))
```



European quarterly retail trade

```r
auto.arima(euretail)
```

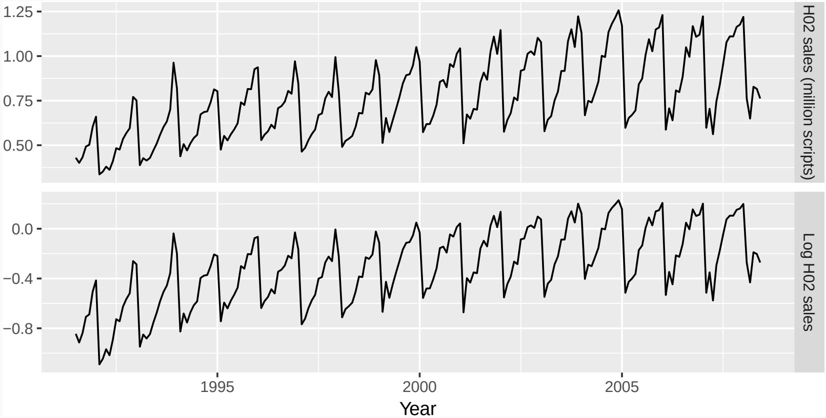
```
## Series: euretail
## ARIMA(1,1,2)(0,1,1)[4]
##
## Coefficients:
##          ar1      ma1     ma2     sma1
##       0.7362  -0.4663  0.2163  -0.8433
## s.e.  0.2243   0.1990  0.2101   0.1876
##
## sigma^2 estimated as 0.1587:  log likelihood=-29.62
## AIC=69.24   AICc=70.38   BIC=79.63
```

European quarterly retail trade

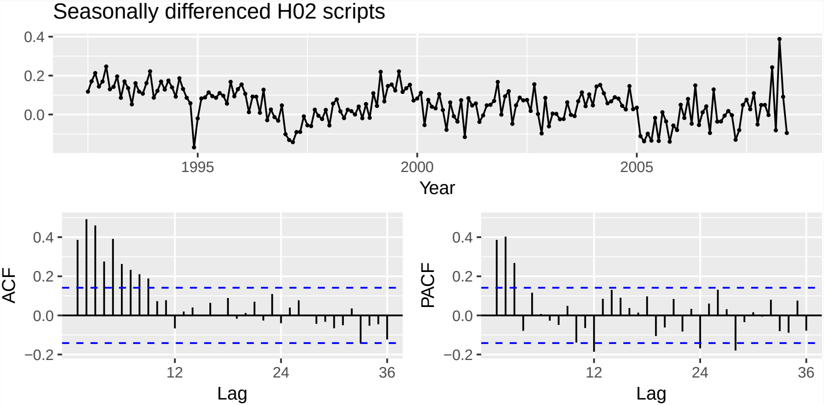
```
auto.arima(euretail, stepwise=FALSE, approximation=FALS
```

```
## Series: euretail
## ARIMA(0,1,3)(0,1,1)[4]
##
## Coefficients:
##          ma1     ma2     ma3     sma1
##       0.2630  0.3694  0.4200  -0.6636
## s.e.  0.1237  0.1255  0.1294   0.1545
##
## sigma^2 estimated as 0.156:  log likelihood=-28.63
## AIC=67.26   AICc=68.39   BIC=77.65
```

Cortecosteroid drug sales



Cortecosteroid drug sales



Cortecosteroid drug sales

- Choose $D = 1$ and $d = 0$.
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model: ARIMA(3,0,0)(2,1,0)$_{12}$.

Cortecosteroid drug sales

Model	AICc
ARIMA$(3,0,0)(2,1,0)_{12}$	-475.12
ARIMA$(3,0,1)(2,1,0)_{12}$	-476.31
ARIMA$(3,0,2)(2,1,0)_{12}$	-474.88
ARIMA$(3,0,1)(1,1,0)_{12}$	-463.40
ARIMA$(3,0,1)(0,1,1)_{12}$	-483.67
ARIMA$(3,0,1)(0,1,2)_{12}$	-485.48
ARIMA$(3,0,1)(1,1,1)_{12}$	-484.25

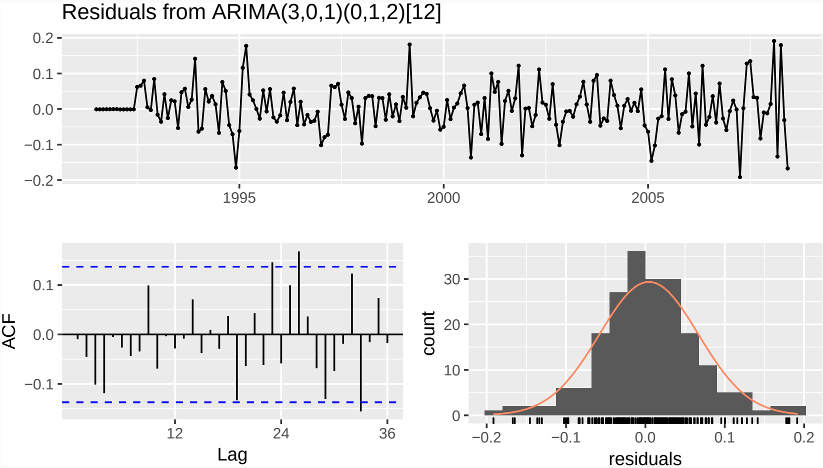
Cortecosteroid drug sales

```
(fit <- Arima(h02, order=c(3,0,1), seasonal=c(0,1,2),
   lambda=0))
```

```
## Series: h02
## ARIMA(3,0,1)(0,1,2)[12]
## Box Cox transformation: lambda= 0
##
## Coefficients:
##           ar1     ar2     ar3     ma1     sma1     sma2
##       -0.1603  0.5481  0.5678  0.3827  -0.5222  -0.1768
## s.e.   0.1636  0.0878  0.0942  0.1895   0.0861   0.0872
##
## sigma^2 estimated as 0.004278:  log likelihood=250.04
## AIC=-486.08   AICc=-485.48   BIC=-463.28
```

Cortecosteroid drug sales

```r
checkresiduals(fit)
```



Cortecosteroid drug sales

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(3,0,1)(0,1,2)[1
## Q* = 23.663, df = 18, p-value = 0.1664
##
## Model df: 6.   Total lags used: 24
```

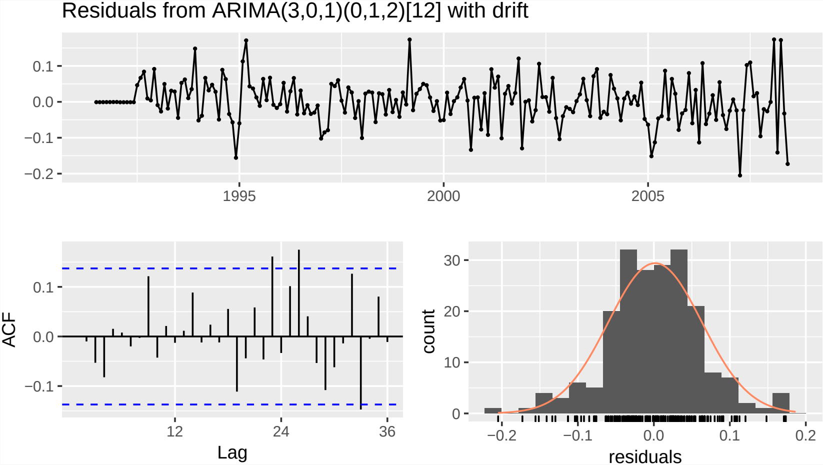
Cortecosteroid drug sales

```
(fit <- auto.arima(h02, lambda=0, d=0, D=1, max.order=9,
  stepwise=FALSE, approximation=FALSE))
```

```
## Series: h02
## ARIMA(3,0,1)(0,1,2)[12] with drift
## Box Cox transformation: lambda= 0
##
## Coefficients:
##           ar1     ar2     ar3     ma1     sma1     sma2   dri
##       -0.2653  0.5011  0.5394  0.4572  -0.5031  -0.2030  0.00
## s.e.   0.1691  0.0813  0.0848  0.1904   0.0847   0.0871  0.00
##
## sigma^2 estimated as 0.004176:  log likelihood=252.99
## AIC=-489.99   AICc=-489.2   BIC=-463.93
```

Cortecosteroid drug sales

```
checkresiduals(fit)
```



Cortecosteroid drug sales

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(3,0,1)(0,1,2)[12] wi
## Q* = 19.369, df = 17, p-value = 0.3078
##
## Model df: 7.   Total lags used: 24
```

Cortecosteroid drug sales

Training data: July 1991 to June 2006
Test data: July 2006–June 2008

```
getrmse <- function(x,h,...)
{
  train.end <- time(x)[length(x)-h]
  test.start <- time(x)[length(x)-h+1]
  train <- window(x,end=train.end)
  test <- window(x,start=test.start)
  fit <- Arima(train,...)
  fc <- forecast(fit,h=h)
  return(accuracy(fc,test)[2,"RMSE"])
}
getrmse(h02,h=24,order=c(3,0,0),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,2),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(1,1,0),lambda=0)
```

Cortecosteroid drug sales

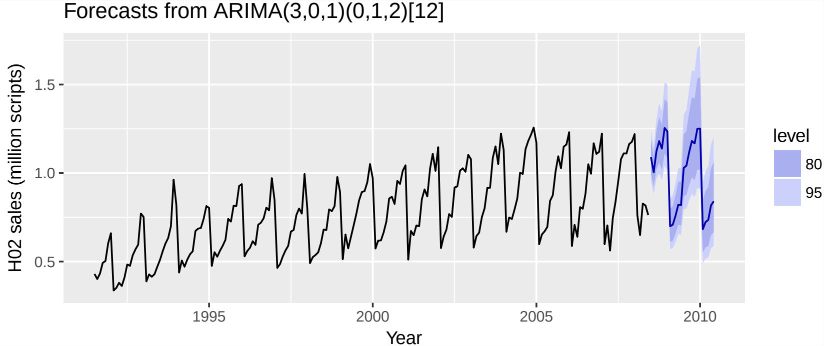
Model	RMSE
ARIMA(3,0,0)(2,1,0)[12]	0.0661
ARIMA(3,0,1)(2,1,0)[12]	0.0646
ARIMA(3,0,2)(2,1,0)[12]	0.0645
ARIMA(3,0,1)(1,1,0)[12]	0.0679
ARIMA(3,0,1)(0,1,1)[12]	0.0644
ARIMA(3,0,1)(0,1,2)[12]	0.0622
ARIMA(3,0,1)(1,1,1)[12]	0.0630
ARIMA(4,0,3)(0,1,1)[12]	0.0648
ARIMA(3,0,3)(0,1,1)[12]	0.0639
ARIMA(4,0,2)(0,1,1)[12]	0.0648
ARIMA(3,0,2)(0,1,1)[12]	0.0644
ARIMA(2,1,3)(0,1,1)[12]	0.0634

Cortecosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- No model passes all the residual tests.
- Use the best model available, even if it does not pass all tests.
- In this case, the ARIMA(3,0,1)(0,1,2)$_{12}$ has the lowest RMSE value and the best AICc value for models with fewer than 6 parameters.

Cortecosteroid drug sales

```
fit <- Arima(h02, order=c(3,0,1), seasonal=c(0,1,2),
  lambda=0)
autoplot(forecast(fit)) +
  ylab("H02 sales (million scripts)") + xlab("Year")
```



Outline

ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$
		$\theta_2 = 1 - \alpha$
ETS(A,A,N)	ARIMA(1,1,2)	$\phi_1 = \phi$
		$\theta_1 = \alpha + \phi\beta - 1 - \phi$
		$\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA(0,0,m)(0,1,0)$_m$	
ETS(A,A,A)	ARIMA(0,1,$m+1$)(0,1,0)$_m$	
ETS(A,A,A)	ARIMA(1,0,$m+1$)(0,1,0)$_m$	

Your turn

For the `condmilk` series:

- Do the data need transforming? If so, find a suitable transformation.
- Are the data stationary? If not, find an appropriate differencing which yields stationary data.
- Identify a couple of ARIMA models that might be useful in describing the time series.
- Which of your models is the best according to their AIC values?
- Estimate the parameters of your best model and do diagnostic testing on the residuals. Do the residuals resemble white noise? If not, try to find another ARIMA model which fits better.
- Forecast the next 24 months of data using your