Lee-Carter mortality forecasting: a multi-country comparison of variants and extensions

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Abstract: We compare the short- to medium- term accuracy of five variants or extensions of the Lee-Carter method for mortality forecasting. These include the original Lee-Carter, the Lee-Miller and Booth-Maindonald-Smith variants, and the more flexible Hyndman-Ullah and De Jong-Tickle extensions. These methods are compared by applying them to sex-specific populations of 10 developed countries using data for 1986–2000 for evaluation. All variants and extensions are more accurate than the original Lee-Carter method for forecasting log death rates, by up to 61%. However, accuracy in log death rates does not necessarily translate into accuracy in life expectancy. There are no significant differences among the five methods in forecast accuracy for life expectancy.

Key words: Functional data, Lee-Carter method, mortality forecasting, nonparametric smoothing, principal components, state space.

1 Introduction

The future of human survival has attracted renewed interest in recent decades. The historic rise in life expectancy shows little sign of slowing, and increased survival is a significant contributor to population ageing. In this context, forecasting mortality has gained prominence. The future of mortality is of interest not only in its own right, but also in the context of population forecasting, on which economic, social and health planning is based. The future provision of health and social security for ageing populations is now a central concern of countries throughout the developed world.

This renewed interest in mortality forecasting has been accompanied by the development of new and more sophisticated methods; for a review, see Booth (2006). A significant milestone was the publication of the Lee-Carter method (Lee and Carter, 1992), although a principal components approach had previously been employed by Bell and Monsell (1991); see also Bell (1997). The Lee-Carter method is regarded as among the best currently available and has been widely applied (e.g., Lee and Tuljapurkar, 1994; Wilmoth, 1996; Tuljapurkar et al., 2000; Li et al., 2004; Lundström and Qvist, 2004; Buettner and Zlotnik, 2005). The Lee-Carter method was a significant departure from previous approaches: in particular it involves a two-factor (age and time) model and uses matrix decomposition to extract a single time-varying index of the level of mortality, which is then forecast using a time series model. The strengths of the method are its simplicity and robustness in the context of linear trends in age-specific death rates. While other methods have subsequently been developed (e.g., Brouhns et al., 2002; Renshaw and Haberman, 2003a,b; Currie et al., 2004; Bongaarts, 2005; Girosi and King, 2006), the Lee-Carter method is often taken as the point of reference.

The underlying principle of the Lee-Carter method is the extrapolation of past trends. The method was designed for long-term forecasting based on a lengthy time series of historic data. However, significant structural changes have occurred in mortality patterns over the twentieth century, reducing the validity of experience in the more distant past for present forecasts. Thus, judgement is inevitably involved in determining the appropriate fitting period. If a longer fitting period is not advantageous, the heavy data demands of the Lee-Carter method can be somewhat relaxed. Whether length of fitting period significantly affects forecast accuracy has not been systematically evaluated.

Indeed, evaluation is limited by the lengthy forecast horizon. However, the forecast can be evaluated in the shorter term using historical data to evaluate out-of-sample forecasts. Shorter term evaluation is relevant to the increasing number of applications that adopt the Lee-Carter method for short- to medium-term forecasting. Shorter term evaluation also informs the longer term prospects of the forecast because errors in forecasting trends can be identified.

Two modifications of the original Lee-Carter method have been proposed: the first by Lee and Miller (2001) and the second by Booth et al. (2002). These three variants of the Lee-Carter method were first evaluated by Booth et al. (2005). In addition, there have been several extensions of the Lee-Carter method, retaining some of its flavour but adding additional statistical features such as non-parametric smoothing, Kalman filtering and multiple principal components. Two such extensions are by Hyndman and Ullah (2007) and De Jong and Tickle (2006). It is not known how these extensions perform compared with the Lee-Carter method and its variants.

This paper presents the results of an evaluation of these five mortality forecasting methods: Lee-Carter, Lee-Miller, Booth-Maindonald-Smith, Hyndman-Ullah and De Jong-Tickle. Each method is applied to data by sex for ten countries. The evaluation involves fitting the different methods to data up to 1985, forecasting for the period 1986–2000, and comparing the forecasts with actual mortality in that period. This paper does not address forecast uncertainty, which has been a recent research focus particularly in relation to long-term forecasting (see Lutz and Goldstein, 2004; Booth, 2006). Rather, it focuses on short- to medium-term forecast accuracy.

2 The five methods

2.1 *The Lee-Carter method*

The Lee-Carter method of mortality forecasting combines a demographic model of mortality with time-series methods of forecasting. The method is generally interpreted as making the use of the longest available time series of data. The Lee-Carter model of mortality is

$$\ln m_{x,t} = a_x + b_x k_t + \varepsilon_{x,t} \tag{1}$$

where $m_{x,t}$ is the central death rate at age x in year t, k_t is an index of the level of mortality at time t, a_x is the average pattern of mortality by age across years, b_x is the relative speed of change at each age, and $\varepsilon_{x,t}$ is the residual at age x and time t. The a_x are calculated as the average of $\ln m_{x,t}$ over time, and the b_x and k_t are estimated by singular value decomposition (Trefethen and Bau, 1997). Constraints are imposed to obtain a unique solution: the a_x are set equal to the means over time of ln $m_{x,t}$ and the b_x sum to 1; the k_t sum to zero.

The Lee-Carter method adjusts k_t by refitting to total observed deaths. This adjustment gives greater weight to ages at which deaths are high, thereby partly counterbalancing the effect of using the logarithm of rates in the Lee-Carter model. The adjusted k_t is extrapolated using ARIMA time series models (e.g., Makridakis et al., 1998). Lee and Carter used a random walk with drift model. The model is

$$k_t = k_{t-1} + d + e_t \tag{2}$$

where d is the average annual change in k_t, and e_t are uncorrelated errors. Lee and Carter used a dummy variable to take account of the outlier resulting from the 1918 influenza epidemic. Forecast age-specific death rates are obtained using extrapolated k_t and fixed a_x and b_x. In this case, the jump-off rates (i.e., the rates in the last year of the fitting period or jump-off year) are fitted rates.

It should be noted that the Lee-Carter method does not prescribe the linear time series model of a random walk with drift for all situations. However, this model has been judged to be appropriate in almost all cases; even where a different model was indicated, the more complex model was found to give results which were only marginally different to the random walk with drift (Lee and Miller, 2001). Further, Tuljapurkar et al. (2000) found that the rate of decline in mortality was constant (i.e., k_t was linear) for the G7 countries, reinforcing the use of a random walk with drift as an integral part of the Lee-Carter method.

2.2 *The Lee-Miller variant*

The Lee-Miller variant differs from this basic Lee-Carter method in three ways:

1. the fitting period is reduced to commence in 1950;
2. the adjustment of k_t involves fitting to $e(0)$ in year t;
3. the jump-off rates are taken to be the actual rates in the jump-off year.

In their evaluation of the Lee-Carter method, Lee and Miller (2001) noted that for US data the forecast was biased when using the fitting period 1900–1989 to forecast the period 1990–1997. The main source of error was the mismatch between fitted rates for the last year of the fitting period (1989) and actual rates in that year; this jump-off error or bias amounted to 0.6 years in life expectancy for males and females combined (Lee and Miller, 2001, p.539). Jump-off bias was avoided by constraining the model such that k_t passes through zero in the jump-off year.

It was also noted that the pattern of change in mortality was not fixed over time, as the Lee-Carter model assumes. Based on different age patterns of change (or b_x patterns) for 1900–1950 and 1950–1995, Lee and Miller (2001) adopted 1950 as the first year of the fitting period. This solution to evolving age patterns of change had been adopted by Tuljapurkar et al. (2000).

The adjustment of k_t by fitting to $e(0)$ was adopted to avoid the use of population data as required for fitting to D_t (Lee and Miller, 2001).

2.3 *The Booth-Maindonald-Smith variant*

The Booth-Maindonald-Smith variant also differs from the Lee-Carter method in three ways:

1. the fitting period is chosen based on statistical goodness-of-fit criteria under the assumption of linear k_t;
2. the adjustment of k_t involves fitting to the age distribution of deaths;
3. the jump-off rates are taken to be the fitted rates based on this fitting methodology.

Booth et al. (2002) fitted the Lee-Carter model to Australian data for 1907–1999 and found that the 'universal pattern' (Tuljapurkar et al., 2000) of constant mortality decline as represented by linear

k_t did not hold over that fitting period. In addition, problems were encountered in meeting the assumption of constant b_x in the underlying Lee-Carter model. Taking assumption of linearity in k_t as a starting point, the Booth-Maindonald-Smith variant seeks to maximize the fit of the overall model by restricting the fitting period to maximize fit to the linearity assumption, which also results in the assumption of constant b_x being better met. The choice of fitting period is based on the ratio of the mean deviances of the fit of the underlying Lee-Carter model to the overall linear fit: this ratio is computed for all possible fitting periods (i.e., varying the starting year but holding the jump-off year fixed) and the chosen fitting period is that for which this ratio is substantially smaller than for periods starting in previous years.

The procedure for the adjustment of k_t was modified. Rather than fit to total deaths, D_t, the Booth-Maindonald-Smith variant fits to the age distribution of deaths, $D_{x,t}$, using the Poisson distribution to model the death process and the deviance statistic to measure goodness of fit (Booth et al., 2002). The jump-off rates are taken to be the fitted rates under this adjustment.

2.4 *The Hyndman-Ullah functional data method*

The approach of Hyndman and Ullah (2007) uses the functional data paradigm (Ramsay and Silverman, 2005) for modelling log death rates. It extends the Lee-Carter method in the following ways:

1 mortality is assumed to be a smooth function of age that is observed with error; smooth death rates are estimated using nonparametric smoothing methods;

2 more than one set of (k_t, b_x) components is used;

3 more general time series methods than random walk with drift are used for forecasting the coefficients; state space models for exponential smoothing are used;

4 robust estimation can be used to allow for unusual years due to wars or epidemics;

5 it does not adjust k_t.

The Hyndman-Ullah approach can be expressed using the equation

$$\ln m_{x,t} = a(x) + \sum_{j=1}^{J} k_{t,j} b_j(x) + e_t(x) + \sigma_t(x)\varepsilon_{x,t} \tag{3}$$

where $a(x)$ is the average pattern of mortality by age across years, $b_j(x)$ is a "basis function" and $k_{t,j}$ is a time series coefficient. The use of $a(x)$ rather than a_x is intended to show that $a(x)$ is a smooth function of age where age is a continuous quantity. It is estimated by applying penalized regression splines (Wood 2000) to each year of data and averaging the results. The pairs $(k_{t,j}, b_j(x))$ for $j = 1, \ldots, J$ are estimated using principal component decomposition. The error term $\sigma_t(x)\varepsilon_{x,t}$ accounts for observational error that varies with age; i.e., it is the difference between the observed rates and the spline curves. The error term $e_t(x)$ is modelling error; i.e., it is the difference between the spline curves and the fitted curves from the model.

In our implementation of the Hyndman-Ullah method, we do not use robust estimation. Rather, the fitting period is restricted to 1950 on, thus avoiding outliers. This was found to give slightly more accurate forecasts than using all the data with robust estimation. We use $J = 6$ for all data sets. The results seem relatively insensitive to the choice of J provided J is large enough. We forecast the time series coefficients $k_{t,j}$ for each j using damped Holt's method based on the state space formulation of Hyndman et al. (2002).

2.5 *The De Jong-Tickle LC(smooth) model*

The approach of De Jong and Tickle (2006) uses the state space framework (Harvey, 1989) for modelling log death rates. State space models encompass a wide range of flexible multivariate time series models of which the Lee-Carter model is a special case. The general framework admits a host of specialisations and generalisations, and includes estimation of unknown parameters, inference, diagnostic checking and forecasting including forecast error calculations.

The Lee-Carter model (1) may be written in the form

$$\boldsymbol{y}_t = \boldsymbol{a} + \boldsymbol{b} k_t + \boldsymbol{\varepsilon}_t \tag{4}$$

where $\boldsymbol{y}_t$ is the vector of the log-central death rates at each age in year t, $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of the corresponding Lee-Carter parameters for each age, k_t is an index of the level of mortality in year t as in the Lee-Carter model, and ε_t is a vector of error terms at each age in year t.

De Jong and Tickle (2006) developed the more general specification

$$\boldsymbol{y}_t = X\boldsymbol{a} + X\boldsymbol{b}k_t + \varepsilon_t \tag{5}$$

where X is a known "design" matrix with more rows than columns, unless $X = I$ in which case the model reduces to (4). Model (5) addresses an issue with LC model (4) where there is an a and a b parameter for each age, which means that the k_t time series has an independent impact at each age. In model (5), X having fewer columns than rows means that there are fewer a and b parameters than there are age groups. The effects of the k_t time series are not independent across age but are constrained by the structure of X, imposing across-age smoothness. The authors thus termed the model LC(smooth).

It is possible to include several time series components in which case k_t is a vector and b is a matrix with one column for each component of k_t. Various forms of the matrix X and the time series k_t are possible. In the current analysis, the matrix X is based on B-splines (Hastie and Tibshirani, 1990) which impose a quadratic form on log-mortality between knots at various ages. A single random walk with drift time series has been used. Maximum likelihood estimates of the model are derived using Kalman filtering and smoothing (Harvey, 1989). The $\boldsymbol{a}$ parameters are derived from the average of the rates in the jump-off year and the previous year, with the effect that the jump-off rates are smoothed average actual rates. As for Hyndman-Ullah, the fitting period is restricted to 1950 on to avoid outliers.

3 Data and accuracy measures

The data for this study are taken from the Human Mortality Database (2006). Ten countries were selected giving 20 sex-specific populations for analysis. The ten countries selected are those with reliable data series commencing in 1921 or earlier. It was desirable to use only countries for which the available time series of data commenced somewhat earlier than 1950 in order to maintain the full and consistent comparison of the three variants. Lee and Carter (1992) used US data for the full period available, 1900–1989. Therefore this multi-country analysis uses data for the period commencing in 1900 where possible. Though for some countries the data extend back to the nineteenth century, these were truncated at 1900: the use of pre-1900 data would both reduce comparability of methods across countries and necessitate a time series model with a non-linear trend which falls outside the scope of both applications to date and the current analysis. The selected countries are shown in Table 1 along with the dates used to define the fitting periods.

Table 1: *Start year for different countries and methods.*

Country	LC	LM	BMS [m]	BMS [f]	HU	DJT
Australia	1921	1950	1968	1970	1950	1950
Canada	1921	1950	1974	1976	1950	1950
Denmark	1900	1950	1968	1967	1950	1950
England and Wales	1900	1950	1968	1972	1950	1950
Finland	1900	1950	1971	1971	1950	1950
France	1900	1950	1971	1969	1950	1950
Italy	1900	1950	1968	1968	1950	1950
Norway	1900	1950	1969	1963	1950	1950
Sweden	1900	1950	1976	1969	1950	1950
Switzerland	1900	1950	1962	1962	1950	1950

Note: The fitting period is defined by start year to 1985; the forecasting period is defined by 1986 to 2000.

The data consist of central death rates and mid-year populations by sex and single years of age to 110 years. In the evaluation, data at older ages (age 95 and above) were grouped in order to avoid problems associated with erratic rates at these ages. The evaluation seeks to focus on the performance of methods in the context of reasonably regular data rather than on their ability to cope with irregularities. The data for Australia differ from those used in previous work in that overseas World War II deaths have been excluded.

The five methods were fitted to periods ending in 1985 and used to forecast death rates from 1986 to 2000. The methods are evaluated by comparing forecast log death rates with actual log death rates.

Forecasting error in log death rates (forecast − actual) is averaged over forecast years, countries or ages to give different views of the relative bias of the five methods. The absolute errors are also averaged to provide measures of forecast accuracy. In addition to these errors in log death rates, the error in life expectancy (forecast – actual) is examined. Again, these (and the absolute errors) are averaged over countries or years to give different summary measures.

We investigate forecast bias in the methods using t-tests of zero mean applied to the errors in log death rates averaged across forecast horizon and age. The sexes are treated separately. Similarly, we test for zero mean in the errors in life expectancy averaged across forecast horizon.

4 Forecast evaluation of the five methods

We refer to the three Lee-Carter variants as LC, LM and BMS, and the two extensions as HU and DJT. The overall mean errors for the 20 populations are shown in Table 2. The p-values in the bottom row are based on t-tests of zero mean applied to the mean errors given in each column. These results confirm earlier findings (Lee and Miller, 2001; Booth et al., 2005) that the original Lee-Carter method consistently and substantially under-estimates mortality especially for females, as indicated by the relatively large negative average errors. Results for the remaining four methods are fairly similar, but only BMS and HU show no evidence of bias in either female or male mortality. Sex differences in this measure are related to the cancellation of positive and negative errors (compare Table 3).

Table 3 provides a summary of forecast accuracy based on mean absolute error. Again, LC performs least well and there are only minor differences among the other four methods. It is notable that the simple variations on the LC method used in LM and BMS provide substantial improvements in forecast accuracy which are only marginally improved by the more sophisticated HU and DJT methods. It is also notable that for this absolute measure, female and male mortality are equally difficult to forecast. Some countries (notably the Nordic countries) proved more difficult to forecast than others.

We used a 2-way ANOVA model (with method and country as factors) on the mean absolute errors to test whether the methods are significantly different. A test for differences between methods was

Table 2: *Overall mean error by sex, method and country. Mean taken over age and year of the error in log death rates. The p-value is a test of bias (a t-test for the average mean error to be zero).*

	Male					Female				
	LC	LM	BMS	HU	DJT	LC	LM	BMS	HU	DJT
Australia	−0.23	0.10	0.04	0.00	0.08	−0.16	0.05	0.01	0.06	0.02
Canada	−0.13	0.04	−0.06	−0.07	0.04	−0.24	−0.03	−0.07	−0.08	−0.05
Denmark	0.04	0.12	0.11	0.13	0.10	−0.36	0.04	0.03	0.03	0.02
England	−0.28	0.03	0.03	0.02	0.03	−0.20	0.00	0.02	0.00	−0.02
Finland	−0.24	0.01	−0.05	−0.02	−0.02	−0.68	−0.16	−0.17	−0.13	−0.17
France	−0.19	0.08	0.07	0.06	0.08	−0.27	0.02	0.03	0.02	0.02
Italy	−0.06	0.00	−0.03	0.02	0.01	−0.24	−0.06	−0.08	−0.05	−0.06
Norway	0.17	0.10	0.11	0.07	0.09	−0.57	0.00	−0.04	−0.01	−0.05
Sweden	−0.09	0.06	−0.01	0.04	0.07	−0.61	−0.01	−0.04	−0.05	−0.03
Switzerland	−0.12	0.02	0.02	0.06	0.02	−0.44	−0.02	−0.03	0.02	−0.03
Average	−0.11	0.06	0.02	0.03	0.05	−0.38	−0.02	−0.03	−0.02	−0.03
p-value	0.03	0.00	0.27	0.09	0.00	0.00	0.43	0.12	0.34	0.08

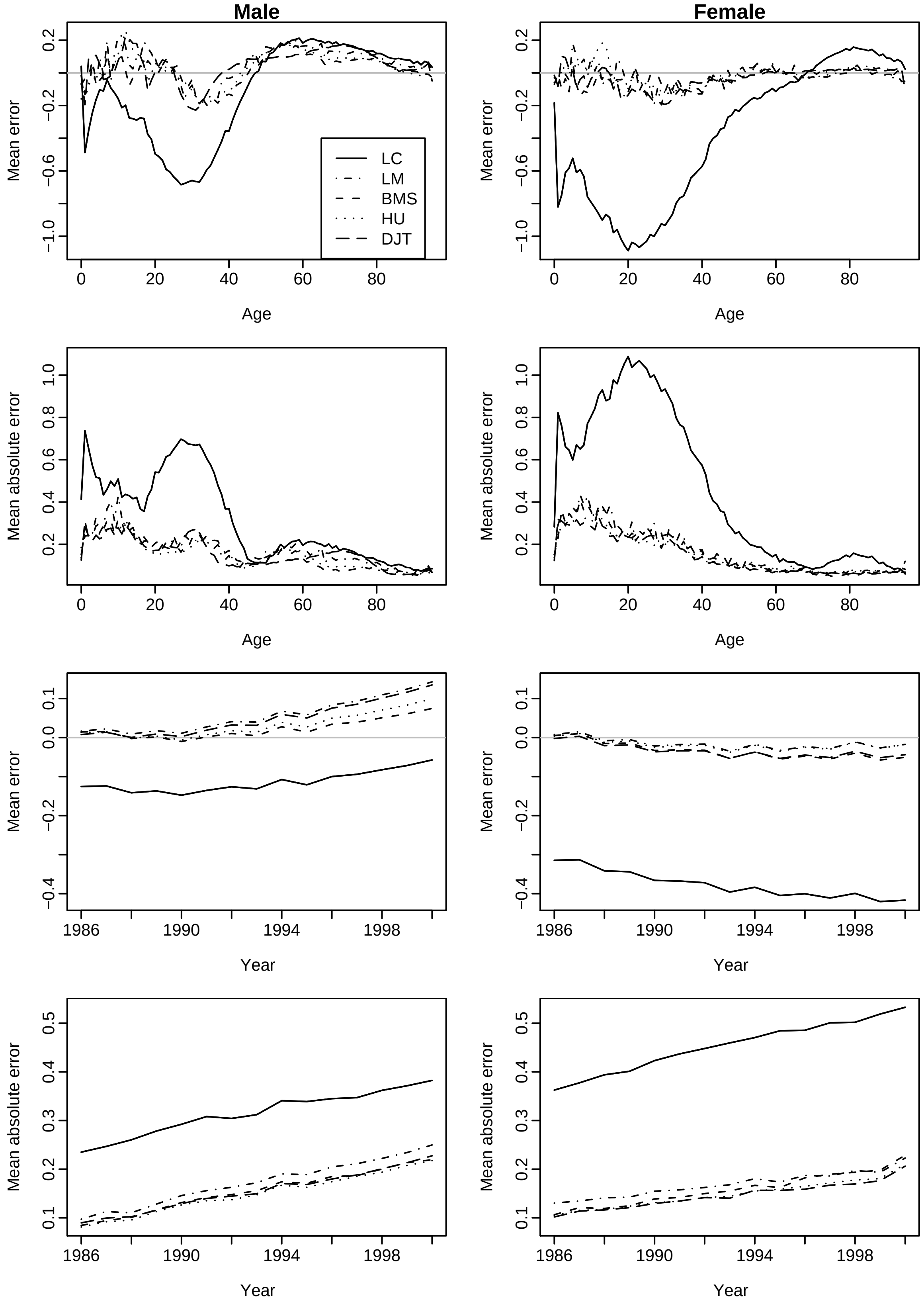
Table 3: *Overall mean absolute error by sex, method and country. Mean taken over age and year of the absolute error in log death rates.*

	Male					Female				
	LC	LM	BMS	HU	DJT	LC	LM	BMS	HU	DJT
Australia	0.46	0.18	0.13	0.12	0.15	0.30	0.15	0.12	0.12	0.11
Canada	0.30	0.11	0.12	0.12	0.11	0.26	0.10	0.12	0.11	0.09
Denmark	0.19	0.21	0.18	0.19	0.17	0.42	0.23	0.21	0.20	0.18
England	0.44	0.14	0.12	0.12	0.14	0.26	0.12	0.10	0.11	0.11
Finland	0.44	0.20	0.19	0.18	0.19	0.76	0.27	0.26	0.22	0.25
France	0.35	0.13	0.12	0.12	0.12	0.36	0.11	0.10	0.09	0.09
Italy	0.25	0.18	0.18	0.15	0.17	0.34	0.15	0.15	0.15	0.15
Norway	0.23	0.20	0.18	0.17	0.18	0.65	0.19	0.18	0.18	0.18
Sweden	0.24	0.20	0.16	0.17	0.17	0.67	0.18	0.18	0.16	0.14
Switzerland	0.25	0.18	0.16	0.15	0.15	0.50	0.20	0.18	0.15	0.15
Average	0.31	0.17	0.15	0.15	0.15	0.45	0.17	0.16	0.15	0.15

highly significant ($p < 0.001$). However, using Tukey's Honest Significant Differences to see which pairs of methods were different showed that the original LC method was significantly different from all other methods ($p < 0.001$), but the other four methods were not significantly different from each other (all p-values greater than 0.86).

Age patterns of error in the log death rates are similar across countries; the average of all countries is shown in Figure 1. There is a tendency for all methods to underestimate mortality for males aged 30–40 and overestimate mortality for males aged 45+. Similarly, all methods underestimate female

Figure 1: *Mean error and mean absolute error in log death rates by sex and method. Top two rows: averaged over countries and years. Bottom two rows: averaged over countries and ages.*



mortality at ages 20–45. The LC method produces large negative mean errors at the younger ages, particularly for females, and small positive mean errors at the older ages. This is due to the fact that the longer LC fitting period produces estimates of b_x that do not reflect the age pattern of change in the forecasting period. The dominance of the large negative errors at the younger ages accounts for the overall underestimation observed for LC in Table 2, and for males the greater cancellation of errors accounts for their less-biased forecasts.

Averages across age are shown over time in the lower half of Figure 1. All methods show similar trends in mean errors, though LC starts from a different level (in line with the overall underestimation of this variant). However, it is clear that divergence is occurring in mean errors; this reflects differences in the estimates of the average annual change in k_t.

Errors in life expectancy are shown in Table 4. In general an underestimate of overall mortality (when measuring error in log death rates — Table 2) does not necessarily translate into an overestimate of life expectancy (and vice versa), because of the implicit weights applied to the age pattern of errors over age (Figure 1). Statistical significance is also affected by this transformation. For males, all methods underestimate life expectancy, whereas for females no method significantly over- or underestimates life expectancy despite, in the case of LC, significant underestimation of log death rates. For this measure, LC does not always produce larger errors than the other methods.

Table 4: *Overall mean error in life expectancy by sex, method and country. Mean taken over age and year of the error in life expectancy.*

	Male					**Female**				
	LC	LM	BMS	HU	DJT	LC	LM	BMS	HU	DJT
Australia	−1.09	−1.56	−0.64	−0.29	−1.35	−0.80	−0.87	−0.22	−0.68	−0.56
Canada	−0.76	−0.74	0.17	0.27	−0.76	0.42	0.42	0.40	0.83	0.50
Denmark	−0.53	−1.10	−1.18	−1.20	−0.90	1.45	0.48	0.40	0.99	0.66
England	−0.57	−1.07	−0.84	−0.80	−1.04	0.03	−0.44	−0.43	−0.30	−0.34
Finland	−0.66	−0.60	−0.11	−0.46	−0.40	0.52	0.47	0.81	0.66	0.53
France	−0.56	−1.01	−0.85	−0.86	−1.06	−0.35	−0.41	−0.23	−0.29	−0.47
Italy	−1.33	−1.13	−0.80	−0.92	−1.24	−0.65	−0.50	−0.23	−0.53	−0.55
Norway	−1.59	−1.50	−1.12	−0.91	−1.23	0.73	0.02	0.34	−0.06	0.18
Sweden	−0.63	−1.24	−0.59	−1.00	−1.12	0.65	0.10	0.13	0.63	0.26
Switzerland	0.04	−0.39	−0.28	−0.66	−0.45	0.76	0.28	0.51	0.01	0.26
Average	−0.77	−1.03	−0.62	−0.68	−0.96	0.28	−0.04	0.15	0.12	0.05
p-value	0.00	0.00	0.00	0.00	0.00	0.25	0.78	0.28	0.53	0.76

Table 5: *Overall mean absolute error in life expectancy by sex, method and country. Mean taken over age and year of the absolute error in life expectancy.*

	Male					Female				
	LC	LM	BMS	HU	DJT	LC	LM	BMS	HU	DJT
Australia	1.19	1.56	0.64	0.39	1.35	0.80	0.87	0.24	0.69	0.57
Canada	0.80	0.74	0.19	0.28	0.76	0.42	0.42	0.40	0.83	0.50
Denmark	0.53	1.10	1.18	1.20	0.90	1.45	0.49	0.40	0.99	0.66
England	0.70	1.07	0.84	0.80	1.04	0.19	0.44	0.43	0.30	0.34
Finland	0.84	0.62	0.27	0.53	0.53	0.55	0.48	0.81	0.66	0.53
France	0.63	1.01	0.85	0.86	1.06	0.40	0.41	0.23	0.30	0.47
Italy	1.33	1.13	0.80	0.92	1.24	0.66	0.50	0.23	0.53	0.55
Norway	1.59	1.51	1.15	1.10	1.32	0.73	0.21	0.34	0.32	0.22
Sweden	0.79	1.24	0.61	1.00	1.12	0.65	0.16	0.17	0.63	0.26
Switzerland	0.49	0.49	0.40	0.66	0.52	0.76	0.29	0.51	0.14	0.27
Average	0.89	1.05	0.69	0.78	0.98	0.66	0.43	0.38	0.54	0.44

Figure 2: *Mean error and mean absolute error in life expectancy by sex and method, averaged over countries.*

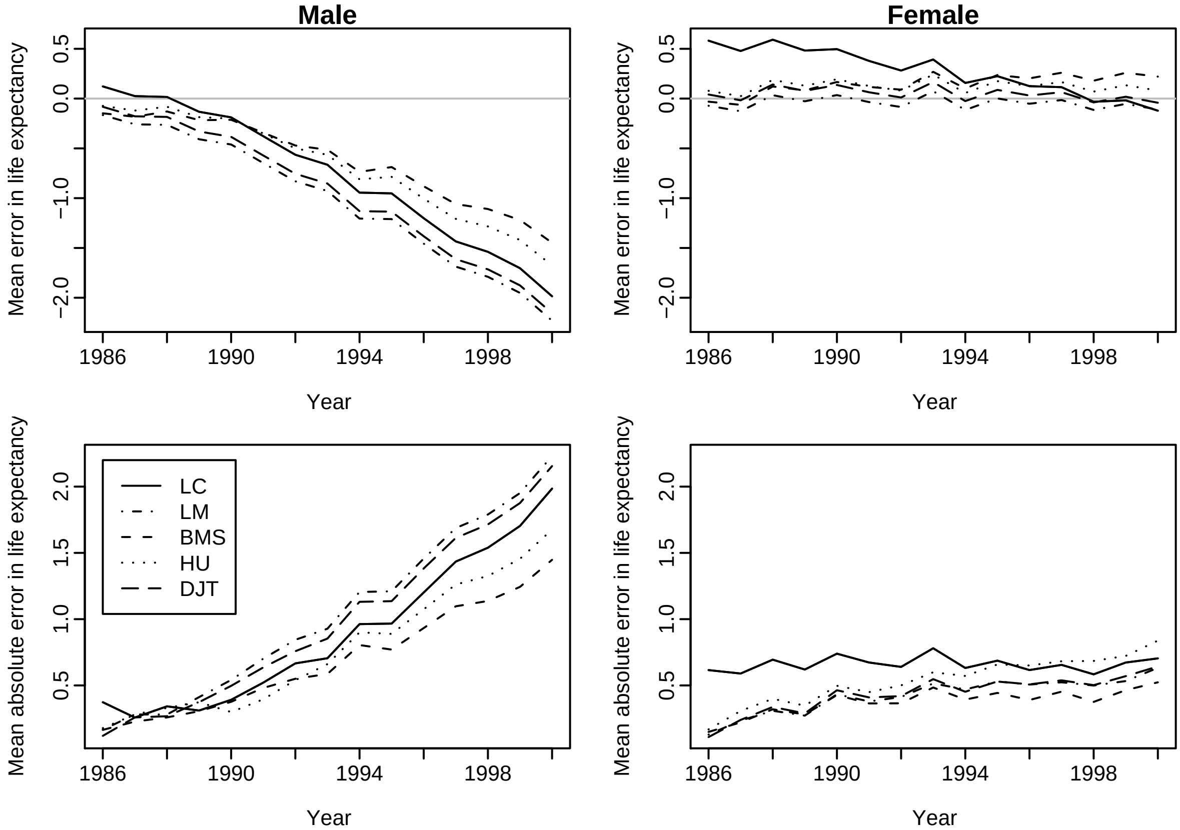


Table 5 shows mean absolute errors in life expectancy. Again, we used a 2-way ANOVA model (with method and country as factors) on the mean absolute errors in life expectancy to test whether the methods are significantly different. In fact, there is no significant difference between the five methods ($p = 0.21$) in the accuracy of life expectancy forecasts.

The results are further summarized in Figure 2 showing the mean error and mean absolute error in life expectancy by year, averaged across countries. The rate of improvement in male life expectancy is underestimated by all five methods: the shorter fitting period for BMS gives the best results except in the very early years. For females, the rate of improvement is underestimated by LC, and slightly overestimated by BMS.

5 Decomposition of differences among the three LC variants

The LC variants evaluated in the previous section are just three of many possible combinations of the different adjustment methods, fitting periods and jump-off choices. In this section, we investigate the effect of each of these factors by comparing all combinations.

The three fitting periods are denoted "long", "1950" and "short", corresponding to the periods used in the LC, LM and BMS variants respectively (Table 1). Similarly, the adjustment methods used are denoted D_t, $e(0)$ and $D_{x,t}$. We also include no adjustment. The two jump-off choices are fitted rates (as in LC and BMS) or actual rates (as in LM) for jump-off. Thus we have $3 \times 4 \times 2 = 24$ Lee-Carter variations.

The three factors (fitting period, method of adjustment and jump-off rates) are independent for LC and LM. For BMS, choice of fitting period is dependent on the shape of the fitted k_t, which in turn is influenced to some extent by the method of adjustment, particularly where deviations from linearity occur (see Figure 3).

Figure 3: *k_t and adjusted k_t for Australia, both sexes combined, 1921–2000. The $e(0)$ method of Lee and Miller (2001) and the $D_{x,t}$ method of Booth et al. (2002) give almost identical results.*

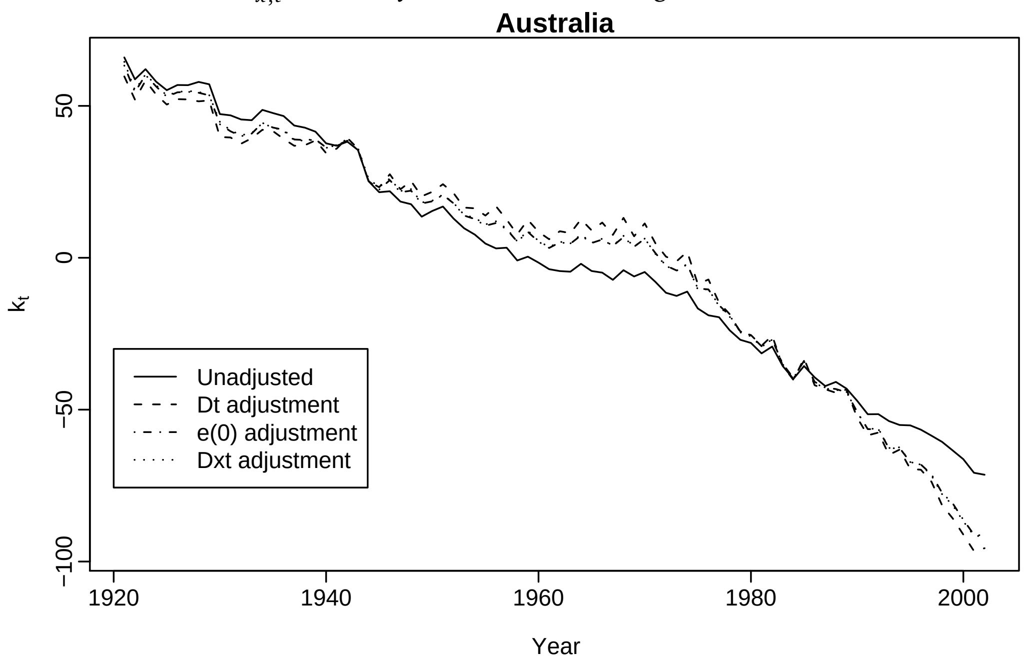


Table 6: *Mean absolute error in log death rates for different Lee-Carter variations, averaged over country, sex, forecast year and age. The LC, LM and BMS variants are marked in bold.*

	Jump-off:	Fitted				Actual			
	Adjustment:	None	D_t	$e(0)$	$D_{x,t}$	None	D_t	$e(0)$	$D_{x,t}$
Fitting period									
long		0.236	**0.384**	0.309	0.300	0.177	0.184	0.181	0.181
1950		0.175	0.187	0.179	0.178	0.171	0.171	**0.171**	0.171
short		0.154	0.171	0.157	**0.157**	0.166	0.172	0.169	0.169

Table 7: *Mean absolute error in life expectancy for different Lee-Carter variations, averaged over country, sex and forecast year. The LC, LM and BMS variants are marked in bold.*

	Jump-off:	Fitted				Actual			
	Adjustment:	None	D_t	$e(0)$	$D_{x,t}$	None	D_t	$e(0)$	$D_{x,t}$
Fitting period									
long		1.809	**0.775**	0.802	0.983	0.826	0.718	0.744	0.764
1950		0.956	0.850	0.758	0.878	0.749	0.756	**0.735**	0.757
short		0.492	0.535	0.498	**0.534**	0.484	0.498	0.494	0.502

The mean absolute error in log death rates from each of the combinations is given in Table 6, averaged over country, sex, forecast year and age. The mean absolute error in life expectancy is similarly given in Table 7. In both tables, the LC, LM and BMS variants are marked in bold. Tables 6 and 7 show that amongst the three variants, BMS is best for both accuracy measures (log death rates and life expectancy). However, the tables suggest that a better method would use the short fitting period of BMS, but with no adjustment. In fact, for log death rates, the use of no adjustment is most accurate in all cases.

The largest improvement in forecast accuracy of log death rates compared with the LC method is from 0.384 to 0.154 or 60%. The maximum improvement in forecast accuracy of life expectancy rates is from 0.775 to 0.484 or 38%, but poorer accuracy also occurs (despite not occurring for log death rates).

By way of comparison, the mean absolute error in log death rates for HU is 0.149 and for DJT is 0.150 (an improvement of 61% over LC in both cases). The mean absolute error in life expectancy for HU is 0.657 and for DJT it is 0.711. This is consistent with the earlier findings, that HU and DJT are more accurate than the other methods in forecasting log death rates, but this doesn't translate into greater accuracy for life expectancy forecasts. An indication of the gain in accuracy attributable to the greater statistical sophistication of HU and DJT can be obtained by comparing them with the four results for 1950/fitted rates. The maximum gain in accuracy for HU is 20% for log death rates and 31% for life expectancy. DJT achieves gains of up to 20% and 26% respectively.

The effect of different fitting periods is essentially measuring the effect of different trends in k_t. It is seen that mean absolute error in log death rates is consistently greatest for the long fitting period, while mean error in life expectancy is consistently smallest for the short fitting period. The use of 1950 to define the fitting period produces less consistent results: for log death rates some errors are smallest while for life expectancy some errors are largest. These results refer, of course, to the 15-year forecasting period under consideration; a different pattern may emerge for longer forecasting periods.

The effect of adjustment is small compared with the effect of fitting period and jump-off bias, and in some cases is extremely marginal. When fitted jump-off rates are used, any adjustment worsens the forecasts of log death rates; this is partly because the fit to the base model is no longer statistically optimal. Adjustment to D_t consistently produces the largest errors in log death rates. For life expectancy, any adjustment tends to improve the forecast, except with a short fitting period, but the optimal adjustment varies. The effect of the different adjustments on life expectancy is complex and depends on the cancellation of errors.

Comparison of fitted and actual jump-off rates gives an indication of the contribution of jump-off error to forecast error. Using actual jump-off rates is generally advantageous. The gain in accuracy is largest when the fitting period is long and when adjustment to D_t is used. This explains why jump-off error is particularly large for LC (as indicated by Figures 1 and 2). When forecast error is small, jump-off error is marginal. When actual rates are used there are only marginal differences in errors in log death rates between fitting periods or adjustment methods. Given the potentially significant error associated with the use of fitted jump-off rates, actual jump-off rates would seem preferable.

6 Discussion and conclusions

The results of this comparative evaluation of forecasts for the period 1986–2000 show that while each of the four variants and extensions is more accurate in forecasting log death rates than the original Lee-Carter method, none is consistently more accurate than the others. It was found that on average HU and DJT provided the most accurate forecasts of log death rates; however, the differences among the four methods are small and are not significant. BMS provided marginally more accurate forecasts of life expectancy but there were no significant differences between the five methods for this measure.

The changed ranking of methods depending on the measure of interest highlights the conceptual problem in defining forecast accuracy. Demographers have traditionally focussed on life expectancy but, as has been seen, there is little relation between the relative accuracy of this measure and that of the underlying log death rates which are actually modelled. The two transformations, namely exponentiation and the life table (involving the cancellation of errors and implicit weights), are highly complex in combination such that the finer degree of accuracy in forecasting life expectancy is largely a matter of luck. Even if forecast life expectancy is accurate, compensating age-specific errors can be relatively substantial (see Figure 1) and in the long-term lead to unrealistic forecasts of the age pattern of mortality, with flow-on effects on forecasts of population structure. While accuracy in forecasting life expectancy may be important, it is not sufficient. To gain an understanding of forecast error, the evaluation of error in log death rates is essential.

Among the factors defining the three Lee-Carter variants, it has been possible to identify those that are generally advantageous. The shorter fitting periods of LM and BMS result in greater accuracy on average than the longer fitting period, though earlier results show that the ranking of LM and BMS in this respect differs by sex (Booth et al., 2005, Table 7). Actual jump-off rates generally do better than fitted jump-off rates, particularly when the model is not a good fit to the data. However, there is no compelling evidence in favour of any of the adjustment methods. Further, among the possible combinations of factors, the combination of short fitting period, no adjustment of k_t and fitted jump-off rates produced the smallest errors in log death rates (0.154) while actual jump-off rates were more advantageous for life expectancy. Either of these combinations might thus be adopted at least for the short forecast horizons considered here.

There is some evidence that the absolute error in the log death rates increases as the fitting period increases in length. This suggests that model misspecification may be present, probably due to the assumed linearity in modelling k_t and the assumed fixed age pattern of change, b_x. Given a changing pattern of mortality decline, such as occurred over the twentieth century, a shorter fitting period often results in more appropriate k_t and b_x for the forecasting period. This highlights the limitations of the model for longer fitting periods. The random walk with drift model is in general a poor model for k_t because it does not allow for dynamic changes in slope. Shorter fitting periods tend to work better with this model (at least for shorter forecast horizons) because they capture the most recent trend. Adaptive time series models such as those used in HU, which place more weight on recent than distant experience, tend to perform better for the same reason; our empirical results support this for the fifteen-year period in question. Similarly, the assumption of fixed b_x is less of a limitation for shorter fitting periods because the recent pattern of change is most relevant. HU overcomes this assumption to some extent by the use of multiple functions, thus allowing for more flexible mortality changes.

It is noted that Tuljapurkar et al. (2000) did not adjust k_t; they combined this with the 1950 fitting period and fitted jump-off rates. The results of this evaluation show that, for 1950/fitted rates, the choice of no adjustment is advantageous for the accuracy of forecast log death rates but disadvantageous for the accuracy of life expectancy. While these effects are moderate for the 1950 fitting period, they are substantial when the long fitting period is used. It is seen in Figure 3 that adjustment makes a noticeable difference to the trend in k_t: specifically, when no adjustment is used the decline is less rapid leading to a lower fitted life expectancy in the jump-off year. This general pattern is observed for all ten populations included in this evaluation. (When the fitting period begins in 1950, adjustment makes little difference to the trend.) For life expectancy, the slower rate of increase from a lower jump-off point produces significant underestimation especially in the longer term. Thus caution should be exercised in using no adjustment with longer fitting periods, especially when combined with fitted jump-off rates.

The results confirm the findings of Lee and Miller (2001) for a smaller group of populations. The LM use of actual jump-off rates in order to avoid jump-off bias is generally endorsed. This is particularly important for very short horizons. In the longer term, jump-off bias becomes less important because it diminishes in size over time due to entropy of the life table. In contrast, error in the trend accumulates over time and quickly comes to dominate total error (Figure 2). The indication that actual jump-off rates give greater forecast accuracy than fitted rates might be regarded as undermining the model. However, in all three Lee-Carter variants the model is already less than statistically optimal by virtue of the adjustment of k_t. BMS and HU aim to reduce jump-off bias by achieving a better fit to the underlying model; for HU this also involves the use of several basis functions. It is noted that the drift term of a random walk with drift is defined by the first and last points of the fitting period. Thus the better the fit of the underlying model (or its first basis function) to the last point in particular, the smaller the jump-off bias and the more accurate the drift.

The LM variant is, in fact, widely referred to by Lee and others as the Lee-Carter method and it is this variant that is now widely applied. However, the original Lee-Carter method (specifically adjustment of k_t to match total deaths) is still used as a point of reference (e.g. Renshaw and Haberman, 2003c; Brouhns et al., 2002). This analysis suggests that not only is the original LC method a rather poor point of reference when the evaluation is focused on log death rates, but also that the LM variant is not the optimal point of reference (at least on the basis of these averaged results). Actual jump-off rates and no adjustment of k_t appears to be a better point of reference for all but the short fitting period where fitted rates are advantageous. Bongaarts (2005) uses as a reference the Lee-Carter method without adjustment. Actual rates may be replaced by the average observed rates over the last two or three years of the fitting period (Renshaw and Haberman, 2003a).

There has been no attempt in this paper to compare the five forecasting methods on any basis other than forecast accuracy. Further research is needed to compare forecast uncertainty among the five methods; a comparison of LC and BMS standard errors and prediction intervals appears in Booth et al. (2002). HU and DJT provide a general framework that is readily adapted to deal with more complex forecasting problems including forecasting several populations with related dynamics such as a common trend. They also produce forecast rates that are smooth across age, which may be an advantage in some applications.

While the results are limited to the forecasting period and countries adopted, it is likely that they may be more widely generalised to other developed countries. The extent to which they may be generalised to other forecasting periods, including longer periods, is less clear. Other research comparing different forecasting methods has shown that forecast accuracy is highly dependent on the particular period or population (e.g. Keyfitz, 1991; Murphy, 1995). In this comparison, however, the methods do not differ substantially, and it remains to be examined whether the details of the basic Lee-Carter method have a different effect in different forecasting periods. It is expected that the more flexible methods of HU and DJT will be better able to forecast less regular mortality patterns (e.g. where the time index does not show a linear trend). For the forecasting period adopted and the countries included, however, these methods do not deliver a marked increase in forecast accuracy.

A final consideration is the ease with which the methods can be implemented. To this end, Hyndman (2006) is an R package which implements the HU, LC, LM and BMS methods, as well as other variants of the Lee-Carter method.

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