# The interaction between trend and seasonality

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**Comments on "Damping seasonal factors: Shrinkage estimators for the X-12-ARIMA program"**

Miller and Williams have presented an interesting article with a persuasive empirical argument that damping the X-12-ARIMA seasonal component leads, on average, to more accurate seasonal estimates and more accurate forecasts.

One aspect of their results which interests me is the relationship between trend and seasonality that is assumed in the X-12-ARIMA procedure, and the effect this has on the shrinkage estimators of the seasonal component.

In Section 1, I describe the various possible models that could be used for combining trend, seasonality and random error. Then, in Section 2, I explore the effect of model mis-specification on seasonal estimation with damping. Finally, in Section 3, I discuss possible extensions to the X-12-ARIMA methodology to allow a more flexible model formulation.

## 1 Models for trend and seasonality

X-12-ARIMA makes some assumptions about the relationship between trend and seasonality that are not always understood by users. In its simplest form, the observed series $Y_t$ can be considered a combination of three components: the trend-cycle $C_t$, the seasonal component $S_t$, and the irregular or random component $I_t$. Each of these components can enter the equation by multiplication or addition. Thus we have eight possible models:

Model 1: $Y_t = C_t + S_t + I_t$

Model 2: $Y_t = (C_t + S_t)I_t$

Model 3: $Y_t = (C_t + I_t)S_t$

Model 4: $Y_t = (S_t + I_t)C_t$

$$\begin{aligned}
&\text{Model 5:} && Y_t = C_t S_t + I_t \\
&\text{Model 6:} && Y_t = C_t I_t + S_t \\
&\text{Model 7:} && Y_t = S_t I_t + C_t \\
&\text{Model 8:} && Y_t = C_t S_t I_t.
\end{aligned}$$

X-12-ARIMA allows for only two of these eight combinations (models 1 and 8) although model 4 is similar to the pseudo-additive decomposition described by Findley et al. (1998). Sometimes X-12-ARIMA is described as a model-free method of decomposition. But this is clearly not the case as the way trend, seasonality and error are combined is specified. Rather, it is a robust nonparametric estimation procedure designed for models 1 and 8. What Miller and Williams have done is shown how the estimation can be improved using shrinkage. However, as we will see below, the form of the underlying model has an effect on the results from shrinkage.

In the Miller-Williams design A, each non-seasonal series can be expressed using the state space model underlying Holt's method (see Hyndman, et al., 2002):

$$\begin{aligned}
Y_t &= \ell_{t-1} + b_{t-1} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \alpha\beta\varepsilon_t
\end{aligned}$$

where $\ell_t$ denotes the level term, $b_t$ denotes the slope term, and $\{\varepsilon_t\}$ is an uncorrelated error term with mean zero and variance $\sigma^2$. (The series without trend follow the same model but with $\beta = b_0 = 0$.) Once we include seasonality, the data generating process can be written as

$$Y_t = (\ell_{t-1} + b_{t-1} + \varepsilon_t)S_t$$

or, using the X-12-ARIMA notation, $Y_t = (C_t + I_t)S_t$ where $C_t = \ell_{t-1} + b_{t-1}$ and $I_t = \varepsilon_t$. Thus, in simulation design A, Miller and Williams generated data using model 3 but used X-12-ARIMA based on model 8.

In design B, Miller and Williams used the following state space model (also described in Hyndman, et al., 2002):

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\begin{aligned}
Y_t &= (\ell_{t-1} + b_{t-1})s_{t-12}(1+\varepsilon_t) \\
\ell_t &= \ell_{t-1} + b_{t-1} + \alpha(\ell_{t-1} + b_{t-1})\varepsilon_t \\
b_t &= b_{t-1} + \alpha\beta(\ell_{t-1} + b_{t-1})\varepsilon_t \\
s_t &= s_{t-12} + \gamma s_{t-12}\varepsilon_t
\end{aligned}
$$

where $\varepsilon_t$ is as before. The model without trend is obtained with $\beta = b_0 = 0$. Using the X-12-ARIMA notation, this can be written as $Y_t = C_t S_t I_t$. That is, simulation B follows model 8, as does the standard implementation of X-12-ARIMA.

## 2 The interaction between trend and seasonal estimates

It is known (Ittig, 1997) that the estimate of the seasonal component can be biased when there is a trend present. It is also known that the form of the trend (e.g., linear or non-linear, additive or multiplicative) affects the estimate of the seasonal component (Archibald and Koehler, 2000). Therefore it is not surprising to see (Figure 2 in Miller and Williams) the presence of trend affecting the level of overestimation in the seasonal variation.

Because Miller and Williams generate data from model 3 (in simulation design A), the SI ratios are given by

$$SI_t = Y_t/\hat{C}_t = \frac{(C_t + I_t)S_t}{\hat{C}_t} \approx S_t\left[1 + \frac{I_t}{\hat{C}_t}\right] \tag{1}$$

where $\hat{C}_t$ denotes the X-12-ARIMA estimate of $C_t$. The presence of trend will affect the value of $\hat{C}_t$ and so it will affect the values of the SI ratios. Because the estimated seasonal component is a weighted average of the SI ratios, this means that the variance in the estimated seasonal component is also affected by the presence of trend.

This problem does not occur for simulation design B where a true multiplicative model was used, for then

$$SI_t = Y_t\hat{C}_t = \frac{C_t I_t S_t}{\hat{C}_t} \approx S_t I_t \tag{2}$$

and so the variance in the SI ratios does not depend on the trend. Note, however, that when

$C_t$ is much larger than $I_t$, then the variance of the SI ratios is larger for this model than for model 3. This explains why simulation A tends to result in less overestimation of the seasonal variation than simulation B (see Figure 2 of Miller and Williams).

It is interesting that the "accidental" reduction in seasonal variation arising from the model mis-specification is not enough to correct the inflated seasonal variation that Miller and Williams have identified. Consequently, shrinkage is still seen to be a useful procedure despite the model mis-specification.

## 3 A more flexible X-12-ARIMA?

This analysis suggests that there is a need for a more flexible X-12-ARIMA procedure in which a mixture of additive and multiplicative components is possible.

I suggest that it would be useful to extend the X-12-ARIMA algorithm to allow some or all of models 2–7 in addition to models 1 and 8. An automatic choice of the appropriate model could then be employed. Hyndman, et al. (2002) describe a likelihood-based algorithm for distinguishing between analogous models for exponential smoothing (although they do not allow models 3, 4, 6 and 7). Assuming $I_t$ is normally distributed, it is easy to compute the likelihood for each of the models 1–8 and then choose the model with the greatest likelihood. Because the eight models all have the same components, there is no need to use penalized likelihood procedures. Shrinkage methods could be employed for any of these models.

Miller and Williams have shown that shrinkage results in more accurate estimates of the seasonal factors, especially when the data generating process matches the decomposition model (see their Figure 3). By allowing a more flexible combination of trend, seasonal and irregular components, it should be possible to achieve even greater gains in accuracy than those found by Miller and Williams.

## References

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