MONASH University

Rob J Hyndman

Forecasting:

Principles and Practice

3. Exponential smoothing I

OTexts.com/fpp/7/

Outline

State space perspective

- Observed data: $y_1, \dots, y_T$.
- Unobserved state: $\boldsymbol{x}_1, \dots, \boldsymbol{x}_T$.

- Forecast $\hat{y}_{T+h|T} = \mathrm{E}(y_{T+h} | \boldsymbol{x}_T)$.
- The "forecast variance" is $\mathrm{Var}(y_{T+h} | \boldsymbol{x}_T)$.
- A prediction interval or "interval forecast" is a range of values of y_{T+h} with high probability.

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Simple Exponential Smoothing

Component form

Forecast equation $\hat{y}_{t+h|t} = \ell_t$

Smoothing equation $\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$

$\ell_1 = \alpha y_1 + (1-\alpha)\ell_0$

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$$\vdots$$

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Simple Exponential Smoothing

Forecast equation

$$\hat{y}_{t+h|t} = \sum_{j=1}^{t} \alpha(1-\alpha)^{t-j} y_j + (1-\alpha)^t \ell_0, \qquad (0 \le \alpha \le 1)$$

	Weights assigned to observations for:			
Observation	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
y_t	0.2	0.4	0.6	0.8
y_{t-1}	0.16	0.24	0.24	0.16
y_{t-2}	0.128	0.144	0.096	0.032
y_{t-3}	0.1024	0.0864	0.0384	0.0064
y_{t-4}	$(0.2)(0.8)^4$	$(0.4)(0.6)^4$	$(0.6)(0.4)^4$	$(0.8)(0.2)^4$
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- Limiting cases: $\alpha \to 1, \quad \alpha \to 0.$

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- $e_t = y_t - \ell_{t-1} = y_t - \hat{y}_{t|t-1}$ for $t = 1, \dots, T$, the one-step within-sample forecast error at time t.
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- Need to estimate α and ℓ_0.

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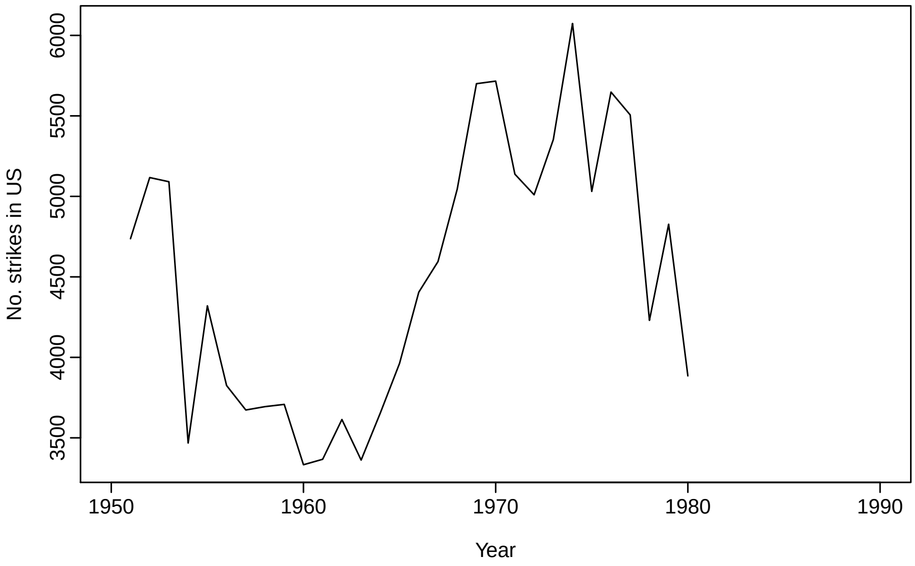

Simple exponential smoothing

Optimisation

- Need to choose value for α and ℓ_0
- Similarly to regression — we choose α and ℓ_0 by minimising MSE:

$$\text{MSE} = \frac{1}{T}\sum_{t=1}^{T}(y_t - \hat{y}_{t|t-1})^2 = \frac{1}{T}\sum_{t=1}^{T} e_t^2.$$

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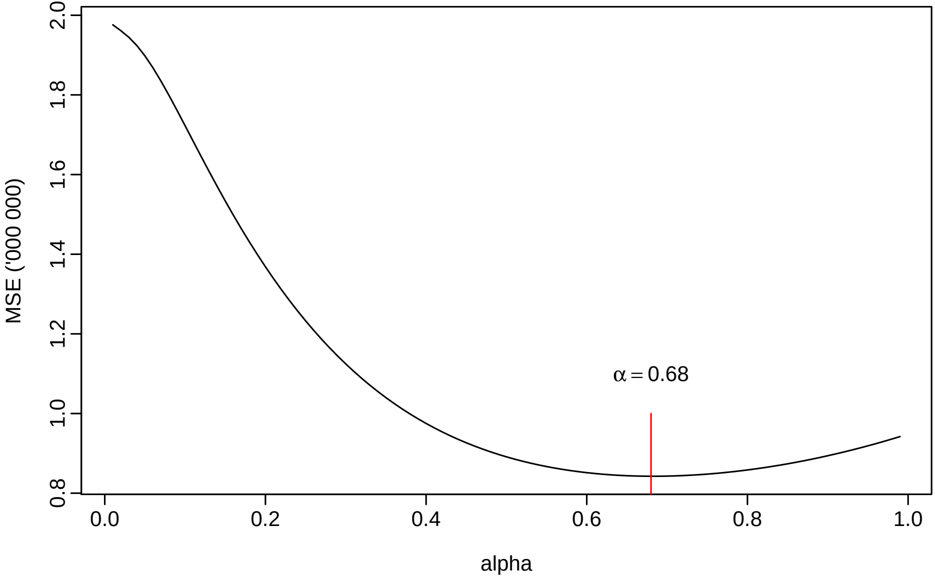

Simple exponential smoothing

Multi-step forecasts

$$\hat{y}_{T+h|T} = \hat{y}_{T+1|T}, \qquad h = 2, 3, \ldots$$

- A "flat" forecast function.
- Remember, a forecast is an estimated mean of a future value.
- So with no trend, no seasonality, and no other patterns, the forecasts are constant.

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SES in R

```
library(fpp)

fit <- ses(oil, h=3)

plot(fit)

summary(fit)
```

Outline

Holt's local trend method

- Holt (1957) extended SES to allow forecasting of data with trends.
- Two smoothing parameters: α and β^* (with values between 0 and 1).

$$\hat{y}_{t+h|t} = \ell_t + hb_t$$
$$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$$
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Holt's linear trend

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Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},$

State space form

Observation equation	$y_t = \ell_{t-1} + b_{t-1} + e_t$
State equations	$\ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t$
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- $\beta = \alpha\beta^*$
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Holt’s method in R

```
fit2 <- holt(ausair, h=5)

plot(fit2)

summary(fit2)
```

Holt's method in R

```r
fit1 <- holt(strikes)
plot(fit1$model)
plot(fit1, plot.conf=FALSE)
lines(fitted(fit1), col="red")
fit1$model

fit2 <- ses(strikes)
plot(fit2$model)
plot(fit2, plot.conf=FALSE)
lines(fit1$mean, col="red")

accuracy(fit1)
accuracy(fit2)
```

Comparing Holt and SES

- Holt's method will almost always have better in-sample RMSE because it is optimized over one additional parameter.
- It may not be better on other measures.
- You need to compare out-of-sample RMSE (using a test set) for the comparison to be useful.
- But we don't have enough data.
- A better method for comparison will be in the next session!

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Exponential trend method

Multiplicative version of Holt's method

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	$b_t = b_{t-1} + \beta e_t / \ell_{t-1}$

- ℓ_t denotes an estimate of the level of the series at time t
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- In R: holt(x, exponential=TRUE)

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Additive damped trend

- Gardner and McKenzie (1985) suggested that the trends should be "damped" to be more conservative for longer forecast horizons.
- Damping parameter $0 < \phi < 1$.

State space form

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t$
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Forecast equation	$\hat{y}_{t+h\|t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t$
Observation equation	$y_t = \ell_{t-1} + \phi b_{t-1} + e_t$
State equations	$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha e_t$
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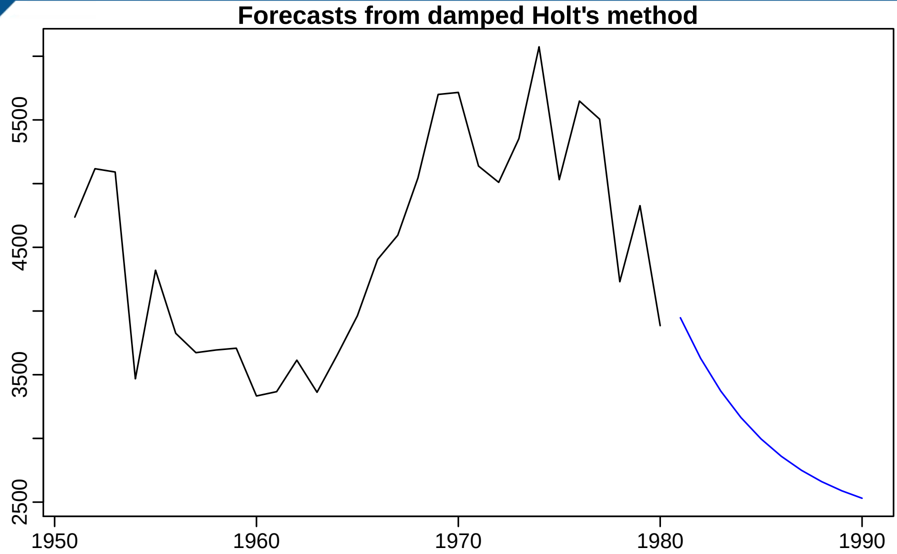
Damped trend method



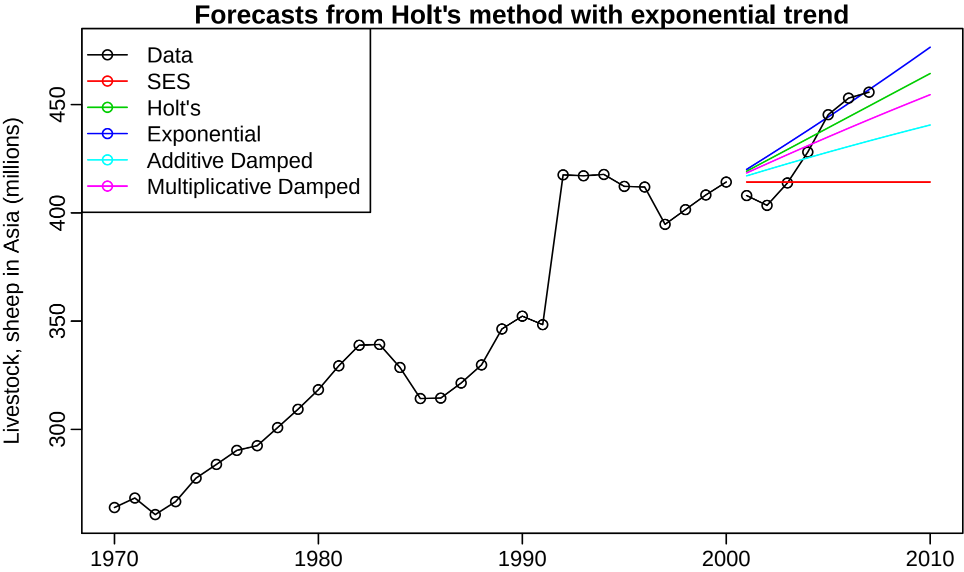
Trend methods in R

```
fit4 <- holt(air, h=5, damped=TRUE)

plot(fit4)

summary(fit4)
```

Example: Sheep in Asia



Multiplicative damped trend method

Taylor (2003) introduced multiplicative damping.

$$
\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t b_t^{(\phi+\phi^2+\dots+\phi^h)} \\
\ell_t &= \alpha y_t + (1-\alpha)(\ell_{t-1} b_{t-1}^{\phi}) \\
b_t &= \beta^*(\ell_t/\ell_{t-1}) + (1-\beta^*) b_{t-1}^{\phi}
\end{aligned}
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- Forecasts converge to $\ell_T + b_T^{\phi/(1-\phi)}$ as $h \to \infty$.

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Outline

Holt-Winters additive method

- Holt and Winters extended Holt's method to capture seasonality.
- Three smoothing equations—one for the level, one for trend, and one for seasonality.
- Parameters: $0 \le \alpha \le 1$, $0 \le \beta^* \le 1$, $0 \le \gamma \le 1-\alpha$ and $m =$ period of seasonality.

State space form

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t-m+h_m^+}$$

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + e_t$$

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$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t-m+h_m^+} \qquad h_m^+ = \lfloor (h-1) \mod m \rfloor + 1$$

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- Most textbooks use $s_t = \gamma(y_t/\ell_t) + (1-\gamma)s_{t-m}$
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Seasonal methods in R

```r
aus1 <- hw(austourists)
aus2 <- hw(austourists, seasonal="mult")

plot(aus1)
plot(aus2)

summary(aus1)
summary(aus2)
```

Holt-Winters damped method

Often the single most accurate forecasting method for seasonal data:

State space form

$$
\begin{aligned}
y_t &= (\ell_{t-1} + \phi b_{t-1}) s_{t-m} + e_t \\
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\end{aligned}
$$

Seasonal methods in R

```
aus3 <- hw(austourists, seasonal="mult",
    damped=TRUE)

summary(aus3)

plot(aus3)
```

Outline

Exponential smoothing methods

- Simple exponential smoothing: no trend.
  `ses(x)`
- Holt's method: linear trend.
  `holt(x)`
- Exponential trend method.
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  `hw(x, damped=TRUE, exponential=TRUE, seasonal="additive")`

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