Improved Interval Estimation of Long Run Response from a Dynamic Linear Model: A Highest Density Region Approach

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Abstract

This paper proposes a new method of interval estimation for the long run response (or elasticity) parameter from a general linear dynamic model. We employ the bias-corrected bootstrap, in which small sample biases associated with the parameter estimators are adjusted in two stages of the bootstrap. As a means of bias-correction, we use alternative analytic and bootstrap methods. To take atypical properties of the long run elasticity estimator into account, the highest density region (HDR) method is adopted for the construction of confidence intervals. From an extensive Monte Carlo experiment, we found that the HDR confidence interval based on indirect analytic bias-correction performs better than other alternatives, providing tighter intervals with excellent coverage properties. Two case studies (demand for oil and demand for beef) illustrate the results of the Monte Carlo experiment with respect to the superior performance of the confidence interval based on indirect analytic bias-correction.

Keywords: ARDL model, Bias-correction, Bootstrapping, Highest density region, Long run elasticity

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1. Introduction

The estimation of long run response parameters, such as the own price elasticity, is an important issue in many areas of applied economics. The long run response is often estimated from a dynamic linear model as a non-linear function of the unknown parameters. The partial adjustment model (Nerlove, 1979) serves as a classical example, while the autoregressive distributed lag (ARDL) model is widely used to allow for a more general dynamic structure (see, for example, Hendry and Ericsson, 1991). Despite the popularity of long run response estimation in applied economics, the extant literature has identified three main problems with it. First, point estimates of the long run response are often extremely disparate (see Marquez and McNeilly, 1988; and Askari and Cummings, 1977). Second, the sampling variability of the long run response estimator is difficult to estimate accurately (see Li and Maddala, 1999). Third, the long run response estimator can be severely biased, especially in small samples. Within the context of the research presented in this paper, Pesaran and Zhao (1999) and Focarelli (2005) examined various bias-correction methods for the long run elasticity estimator.

In this paper, we propose a new method of interval estimation for the long run response from an ARDL model. Following Li and Maddala (1999), we use the bias-corrected bootstrap of Kilian (1998a, 1998b) to approximate the sampling distribution of the long run response estimator. It involves bias-correction in two stages of the bootstrap procedure. In the first stage, the biases in the parameter estimates of the model from the original data are adjusted; and in the second stage, those from the bootstrap samples are corrected and then the bias-corrected long run response estimate is obtained. The bootstrap distribution of the long run response estimator is obtained

by repeating the second stage a sufficient number of times. As we will later demonstrate, this bootstrap distribution can be heavily skewed, with extreme values on the longer tail of the distribution. This is mainly because the estimator typically takes a ratio form and does not possess finite sample moments, as has also been noted by Bewley and Fiebig (1990) and Diebold and Lamb (1997). In this case, the usual percentile interval (Efron and Tibshirani, 1993) can often be excessively wide and uninformative. Indeed, this feature is evident from the Monte Carlo results presented in Li and Maddala (1999; Table 6), where the bias-corrected bootstrap confidence intervals are too wide when the model is close to unit root non-stationarity, although their coverage rates are reasonably close to the nominal level[2].

In order to obtain a more sensible confidence interval from the bootstrap distribution of the long run response estimator, this paper adopts the highest density region (HDR) method proposed by Hyndman (1996). It provides tighter intervals than the percentile method or the normal approximation when the underlying distribution is asymmetric. It can also produce disjoint confidence intervals when the distribution is multi-modal. Thus, it is expected that the HDR method will yield a tighter confidence interval than the conventional percentile method, in the context of long run response estimation. In a recent study of half-life estimation, Kim *et al.* (2007) found that the HDR interval estimators are far superior (i.e., tighter confidence intervals and better coverage rates) to those of the conventional methods.

The bias-corrected bootstrap used in this paper requires the choice of a bias-correction method. Li and Maddala (1999) used the bootstrap as a means of bias-correction,

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[2] Li and Maddala (1999) also reported that the confidence intervals based on the conventional methods, such as the delta method and Fieller's method, can be too short and optimistic, with the coverage rates substantially lower than the nominal level.

following what is called the bootstrap-after-bootstrap method proposed by Kilian (1998a). In this paper, analytic bias-correction based on asymptotic approximation is also adopted, as in Kilian (1998b). We consider indirect analytic bias-correction based on the asymptotic formula given by Kiviet and Phillips (1994), as well as the direct analytic bias-correction of the long run response estimator as a generalization of the asymptotic method proposed by Pesaran and Zhao (1999).

Our extensive Monte Carlo experiment reveals that the bias-corrected bootstrap confidence interval based on the HDR method provides much tighter intervals than those based on the conventional percentile method. Among the HDR intervals, the confidence interval based on indirect analytic bias-correction shows the best overall performance, with the shortest length and the most accurate coverage probabilities. To illustrate the utility of the bias-corrected bootstrap HDR confidence interval for the long run response developed in this paper, we present two case studies: demand for oil and the demand for beef. These case studies contribute to a large body of applied literature that has employed bootstrap methods to construct confidence intervals for elasticities (e.g., Dorfman *et al.,* 1990, Vinod and McCullough, 1994, Letson and McCullough, 1998). These authors noted the importance of providing a variability measure associated with a point estimate, and argued that a point estimate should always be reported with a confidence interval. Our case studies extend the existing literature because our new bootstrap approach provides tighter confidence intervals with accurate coverage properties.

The structure of our paper is as follows. In the next section, we present the model and alternative bias-correction methods for the long run response estimator. In Section 3,

we outline the bias-corrected bootstrap procedures for interval estimation of the long run response parameter. In Section 4, the HDR method is described in the context of long run response estimation. Section 5 presents the results of the Monte Carlo experiment, which compares the small sample properties of alternative confidence intervals. Two empirical case studies are given in Section 6, and conclusions are drawn in Section 7.

2. Long run response estimation and bias-correction

2.1. Model and parameter estimation

Following the notation of Kiviet and Phillips (1994), we consider a general linear autoregressive (ARDL) distributive lag model

$$Y_t = \sum_{i=1}^{p} \gamma_i Y_{t-i} + \sum_{j=1}^{m} \sum_{i=0}^{p(j)} \beta_{ji} X_{t-i}^{j} + u_t \, , \tag{1}$$

where X_t^j is an exogenous variable and u_t is an iid error term. X_t^j may include a deterministic component such as a constant, a time trend or dummy variables. The above model can be written in matrix form as

$$Y = Z\alpha + u, \tag{2}$$

where $Y = (Y_1, \ldots, Y_n)'$, $u = (u_1, \ldots, u_n)'$, and $Z = [W{:}X]$ is an $(n \times (p+k))$ matrix with $k = \sum_{j=1}^{m} p(j)$, while W is an $(n \times p)$ matrix of lagged dependent variables and X is an $(n \times k)$ matrix of exogenous variables. The vector of unknown coefficients is written as $\alpha = (\gamma : \beta)' = (\gamma_1, \ldots, \gamma_p : \beta_{10}, \ldots, \beta_{mp(m)})'$. This model can be expressed in an equivalent form as

$$Y_t = \lambda Y_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta Y_{t-i} + \sum_{j=1}^{m} \sum_{i=0}^{p(j)} \beta_{ji} X_{t-i}^{j} + u_t \, , \tag{3}$$

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where $\Delta = 1 - B$, B is the lag operator, $\lambda = \gamma_1 + \ldots + \gamma_p$ and $\phi_j = -\sum_{k=j+1}^{p} \gamma_k$.

It is assumed that the model satisfies the following conditions given by Kiviet and Phillips (1994): (i) all elements of the X matrix are fixed and finite; (ii) $\lim_{n\to\infty}(X'X)/n$ is finite and non-singular; (iii) the model is stationary; (iv) the initial values (Y_{1-p}, …, Y_0) are non-stochastic; and (v) u_t is NID($0,\sigma^2$). The least-squares (LS) estimators for α and σ^2 are $\hat{\alpha} \equiv (\hat{\gamma}:\hat{\beta}) \equiv (\hat{\gamma}_1,...,\hat{\gamma}_p,\hat{\beta}_{10},...,\hat{\beta}_{mp(m)}) = (Z'Z)^{-1}Z'Y$ and $s^2 = e'e/(n-p-k)$ respectively, while $e = (e_1, …, e_n)'$ denotes the vector of residuals.

The long run response of Y with respect to X^j ($j = 1, …, m$) is written as

$$\theta_j = \frac{\beta_{j0} + \beta_{j1} + ... + \beta_{jp(j)}}{1 - \lambda},$$

where $\lambda = \gamma_1 + \ldots + \gamma_p$. It can be estimated by replacing the unknowns with their LS estimates as

$$\hat{\theta}_j = \frac{\hat{\beta}_{j0} + \hat{\beta}_{j1} + ... + \hat{\beta}_{jp(j)}}{1 - \hat{\lambda}},$$

where $\hat{\lambda} = \hat{\gamma}_1 + ... + \hat{\gamma}_p$. It is well known that $\hat{\theta}_j$ is a biased estimator for θ_j in small samples. The bias is generated from (i) the biases associated with $\hat{\alpha}$; and (ii) the non-linearity of θ_j. Pesaran and Zhao (1999) discussed alternative methods of bias-correction for $\hat{\theta}$ in a simple AR(1) model with an exogenous variable. In this section, we present alternative bias-correction methods for the long run response estimated from the ARDL model given in (1), based on asymptotic and bootstrap approximations.

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2.2. Analytic bias-correction based on asymptotic approximation

2.2.1. Indirect Bias-Correction

Kiviet and Phillips (1994) derived analytic formulae for the bias of $\hat{\alpha}$ to $O(n^{-1})$. That is,

$$E(\hat{\alpha} - \alpha) = B_{\hat{\alpha}}(\alpha, \sigma^2, Z) + o(n^{-1})$$

where $B_{\hat{\alpha}}(\alpha, \sigma^2, Z) \equiv (B_{\hat{\gamma}} : B_{\hat{\beta}})$ denotes the bias as a function of the unknown parameters and the data matrix, the explicit form of which is given in Theorem 1B of Kiviet and Phillips (1994). The bias-corrected estimator for α can be obtained as

$$\hat{\alpha}^c = [\hat{\gamma}^c : \hat{\beta}^c] = \hat{\alpha} - B_{\hat{\alpha}}(\hat{\alpha}, s^2, Z). \tag{4}$$

The bias-corrected estimator for θ, based on what is called the indirect method, is written as

$$\hat{\theta}_j^{IC} = \frac{\hat{\beta}_{j0}^c + \hat{\beta}_{j1}^c + ... + \hat{\beta}_{jp(j)}^c}{1 - \hat{\lambda}^c},$$

where $\hat{\lambda}^c = \hat{\gamma}_1^c + ... + \hat{\gamma}_p^c$.

It is possible that the bias-correction given in (4) pushes $\hat{\gamma}^c$ to the non-stationary part of the parameter space, especially when the sample size is small and the model is close to unit root non-stationarity. In this case, the stationarity-correction proposed by Kilian (1998a, 1998b) is implemented. This procedure is conducted as follows: if $\hat{\gamma}^c$ implies non-stationarity, then let $\delta_1 = 1$, $\Delta_1 = B_{\hat{\gamma}}$ and $\hat{\gamma}^c = \hat{\gamma} - \Delta_i$. Set $\Delta_{i+1} = \delta_i \Delta_i$,

and $\delta_{i+1} = \delta_i - 0.01$ for $i = 1, 2, 3, \ldots$. Iterate until $\hat{\gamma}^c$ satisfies the condition of stationarity.

2.2.2. Direct Bias-Correction

In a similar fashion to Pesaran and Zhao (1999; Theorem 1), we derive an expression for the bias to $O(n^{-1})$ for the long run response estimator $\hat{\theta}_j$. Based on a Nagar (1959) type expansion (see Appendix A for details), it can be shown that[3]

$$E\left(\frac{\hat{\beta}_{ji}}{1-\hat{\lambda}} - \frac{\beta_{ji}}{1-\lambda}\right) = \psi_{ji}(\alpha, \sigma^2, Z) + o(n^{-1}),$$

where $\psi_{ji}(\alpha, \sigma^2, Z) \equiv \left[\dfrac{B_{\hat{\beta}_{ji}}}{1-\lambda} + \dfrac{\beta_{ji}B_{\hat{\lambda}} + C\,\mathrm{ov}(\hat{\lambda}, \hat{\beta}_{ji})}{(1-\lambda)^2} - \dfrac{\beta_{ji}MSE(\hat{\lambda})}{(1-\lambda)^3}\right]$, $B_{\hat{\lambda}} = \sum_{i=1}^{p} B_{\hat{\gamma}_i}$, and

$MSE(\hat{\lambda}) = Var(\hat{\lambda}) + B_{\hat{\lambda}}^2$. This leads to

$$E(\hat{\theta}_j - \theta_j) = \sum_{i=0}^{p(j)} \psi_{ji}(\alpha, \sigma^2, Z) + o(n^{-1}).$$

The bias term can be estimated by replacing the unknowns with their LS estimates,

i.e., $\sum_{i=0}^{p(j)} \psi_{ji}(\hat{\alpha}, s^2, Z)$, while $Cov(\hat{\lambda}, \hat{\beta}_{ji})$ and $Var(\hat{\lambda})$ are obtained from the covariance matrix of the LS estimation of model (3). The bias-corrected estimator for θ_j based on what is called direct bias-correction can be obtained as

$$\hat{\theta}_j^{DC} = \hat{\theta}_j - \sum_{i=0}^{p(j)} \psi_{ji}(\hat{\alpha}, s^2, Z). \tag{5}$$

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[3] Jan Kiviet provided us with a simpler bias-expression and proof than those given in Pesaran and Zhao (1999) for the AR(1) model with an exogenous variable, which was used here to derive the bias-expression for the case of a general ARDL model.

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In calculating (5), we have attempted the use of $\hat{\alpha}^c$ in place of $\hat{\alpha}$, but this resulted in highly volatile values of $\hat{\theta}_j^{DC}$ in repeated sampling. Based on this, the use of $\hat{\alpha}$ is maintained for the calculation (5) throughout the paper.

2.3. Bootstrap (indirect) bias-correction

As an alternative to the analytic bias-correction methods presented above, the non-parametric bootstrap based on residual resampling can be applied to model (1) to estimate the bias of $\hat{\alpha}$ to $O(n^{-1})$, as in Pesaran and Zhao (1999). First, a pseudo-data set is generated as

$$Y_t^* = \sum_{i=1}^{p} \hat{\gamma}_i Y_{t-i}^* + \sum_{j=1}^{m} \sum_{i=0}^{p(j)} \hat{\beta}_{ji} X_{t-i}^j + e_t^*; \text{ or } Y^* = Z^* \hat{\alpha} + e^*$$

using $\{Y_t\}_{t=1}^{p}$ as starting values, where e_t^* is a random draw with replacement from $\{e_t\}_{t=1}^{n}$, $Z^* = [W^*:X]$, and W^* is a matrix of lagged dependent variables constructed from $\{Y_t^*\}_{t=1}^{n}$. Using (Y^*, Z^*), the unknown parameters are estimated to obtain $\hat{\alpha}^*$. This procedure is then repeated B_1 times to obtain $\{\hat{\alpha}^*(j)\}_{j=1}^{B_1}$.

The bias of $\hat{\alpha}$ is estimated in the usual way as $Bias(\hat{\alpha}) = \bar{\alpha}^* - \hat{\alpha}$, where $\bar{\alpha}^*$ is the sample mean of $\{\hat{\alpha}^*(j)\}_{j=1}^{B_1}$. The bootstrap bias-corrected estimator $\hat{\alpha}_B^c$ is obtained as

$$\hat{\alpha}_B^c = \hat{\alpha} - Bias(\hat{\alpha}) \equiv [\hat{\gamma}_B^c : \hat{\beta}_B^c].$$ As with the analytic indirect bias-correction, the stationarity-correction is applied to $\hat{\gamma}_B^c$ if necessary. The bootstrap (indirect) bias-corrected estimator for θ_j can be obtained as

$$\hat{\theta}_j^{BC} = \frac{\hat{\beta}_{B,j0}^c + \hat{\beta}_{B,j1}^c + ... + \hat{\beta}_{B,jp(j)}^c}{1 - \hat{\lambda}_B^c}.$$

3. Bias-corrected Bootstrap for the Long Run Response

We now describe the bias-corrected bootstrap procedure for interval estimation of the long run response parameter. As mentioned before, the bias-corrected bootstrap of Kilian (1998a, 1998b) is used to approximate the sampling distribution of the long run response estimator. As a means of bias-correction, we use the alternative direct and indirect methods discussed in the previous section. The bootstrap procedures are described in three stages as follows.

3.1. Bias-corrected bootstrap with indirect bias-correction

3.1.1. Analytic Bias-Correction

Stage 1

Calculate the LS estimators $\hat{\alpha}$ and s^2 for model (1), as well as their bias-corrected versions $\hat{\alpha}^c$ following (4), and s^{2c}, which is the estimator for σ^2 calculated from $\hat{\alpha}^c$.

Stage 2

Generate pseudo-data sets recursively as

$$Y_t^* = \sum_{i=1}^{p} \hat{\gamma}_i^c Y_{t-i}^* + \sum_{j=1}^{m} \sum_{i=0}^{p(j)} \hat{\beta}_{ji}^c X_{t-i}^j + e_t^*; \text{ or } Y^* = Z^* \hat{\alpha}^c + e^*$$

using $\{Y_t\}_{t=1}^{p}$ as starting values, where e_t^* is a random draw from $N(0, s^{2c})$. Using (Y^*, Z^*), calculate $\hat{\alpha}^*$ and s^{2*}; the bootstrap versions of $\hat{\alpha}$ and s^2; and the bias-corrected estimator $\hat{\alpha}^{c*} = [\hat{\gamma}^{c*} : \hat{\beta}^{c*}] = \hat{\alpha}^* - B_{\hat{\alpha}}(\hat{\alpha}^*, s^{2*}, Z^*)$ following (4). Then calculate

$$\hat{\theta}_j^{IC*} = \frac{\hat{\beta}_{j0}^{c*} + \hat{\beta}_{j1}^{c*} + ... + \hat{\beta}_{jp(j)}^{c*}}{1 - \hat{\lambda}^{c*}}, \text{ where } \hat{\lambda}^{c*} = \hat{\gamma}_1^{c*} + ... + \hat{\gamma}_p^{c*}.$$

Stage 3

Repeat Stage 2 B times to obtain $\{\hat{\theta}_j^{IC*}(l)\}_{l=1}^{B}$, which is the bootstrap distribution for the long-run response estimator.

3.1.2. Bootstrap Bias-Correction

As an alternative to the analytic bias-correction, one may use the bootstrap bias-corrected estimator $\hat{\alpha}_B^c$, following Kilian (1998a) and Li and Maddala (1999). The procedure is similar to that of the analytic bias-correction, except that (i) e_t^* is a random draw from the residuals with replacement and (ii) the bias estimate obtained in Stage 1 is used as an approximation to the bias in Stage 2 (for a more detailed description, see Kilian, 1998a). The bootstrap distribution, based on the bootstrap bias-correction, is denoted by $\{\hat{\theta}_j^{BC*}(l)\}_{l=1}^B$.

3.2. Bias-corrected bootstrap with direct analytic bias-correction

The procedure detailed in Section 3.1 is based on indirect bias-correction. As an alternative, the direct bias-correction detailed in Section 2.2.2 can be used in Stage 2 as follows:

Stage 1

Identical to Stage 1 of the indirect analytic bias-correction case

Stage 2

Identical to Stage 2 of the indirect analytic bias-correction case, except that, from (Y^*, Z^*), calculate $\hat{\theta}_j^{DC*} = \hat{\theta}_j^* - \sum_{i=1}^k \psi_{ji}(\hat{\alpha}^*, s^{2*}, Z^*)$ following (5).

Stage 3

Repeat Stage 2 B times to obtain $\{\hat{\theta}_j^{DC*}(l)\}_{l=1}^B$, which gives the bootstrap distribution for the long run response estimator based on direct estimation.

As a further note, it is often the case in practice that model (1) is estimated with linear restrictions imposed. In this case, $\hat{\theta}_j$ can be obtained using a restricted version of $\hat{\alpha}$, denoted as $\hat{\alpha}_R$. Bias-correction for $\hat{\alpha}_R$ and $\hat{\theta}_j$ can be conducted using the bias formula for $\hat{\alpha}_R$ given in Theorem 2 of Kiviet and Phillips (1994). The bias-corrected bootstrap can also be implemented as a special case of the unrestricted case detailed above.

3.3. Percentile intervals

Note that $\{\hat{\theta}_j^{IC*}(l)\}_{l=1}^B$, $\{\hat{\theta}_j^{DC*}(l)\}_{l=1}^B$ and $\{\hat{\theta}_j^{BC*}(l)\}_{l=1}^B$ provide alternative small-sample approximations to the unknown sampling distribution of $\hat{\theta}_j$. The $100(1-\alpha)\%$ confidence interval for θ_j can be constructed as $[\hat{\theta}_\tau, \hat{\theta}_{1-\tau}]$ where $\hat{\theta}_\tau$ is the τth percentile of $\{\hat{\theta}_j^{\bullet C*}(l)\}_{l=1}^B$ obtained in Stage 3 above while $\tau = 0.5\alpha$. This is the percentile interval of Efron and Tibshirani (1993), which is widely used for bootstrap confidence intervals. However, when the bootstrap distribution is extremely skewed, this interval can be excessively wide and uninformative, although it may have reasonably good coverage properties (see, for example, Li and Maddala, 1999). As an alternative, we now consider the highest density region (HDR) method of Hyndman (1996) for constructing the confidence intervals.

4. HDR Method for Long Run Response Estimation

Let $f(z)$ be the density function for a random variable Z. The $100(1-\alpha)\%$ HDR is defined (see Hyndman, 1996) as the subset $R(f_\alpha)$ of the sample space of Z such that $R(f_\alpha) = \{z: f(z) \geq f_\alpha\}$, where f_α is the largest constant such that $\Pr[Z \in R(f_\alpha)] \geq 1 - \alpha$.

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Thus, $R(f_\alpha)$ represents the smallest region with a given probability content. It can take disjoint intervals when the underlying distribution is multi-modal, and it consists of the intervals associated with the modes of the distribution, whereas the percentile interval is centred on the median. In the present context, Z is the long run response estimator and its density can be estimated from the bootstrap distributions detailed in Section 3.

4.1. Kernel Density Estimation

We estimate the density $f(z)$ using a kernel estimator with the Gaussian kernel, with the bandwidth selected using the Sheather-Jones rule (Sheather and Jones, 1991). We observe that the bootstrap distribution is heavily skewed, especially when the value of λ is close to 1, in which case kernel density estimation of $f(z)$ can be problematic due to the uneven amount of smoothing required. That is, a long tail will be under-smoothed and modes will be over-smoothed. One way of overcoming these problems is to use the transformation kernel density estimator proposed by Wand *et al*. (1991). First, let $V = h(Z)$, where h is an increasing, differentiable function on the support of $f(z)$. Second, the transformation h is chosen so that the density function of V, denoted $g(v)$, can easily be estimated using the standard kernel density estimation method. From the kernel density of $g(v)$, that of $f(z)$ can readily be obtained using the relationship $f(v)=g(h(z))h'(z)$.

Wand *et al*. (1991) proposed a general class of convex transformations called the shifted power family, for when f is an extremely skewed distribution. In this paper, we use a special case $h(Z) = Z^{0.1}$ which we have found to be the most suitable in the present context. In the preliminary analysis, we tried various other transformations

such as $h(Z) = Z^{0.3}$ and $h(Z) = Z^{0.5}$. They gave the kernel density estimates which were often rough in the tails and over-smoothed in the peaks. This tendency of kernel density estimates gets stronger as the value of the transformation exponent increases. We established that the transformation $h(Z) = Z^{0.1}$ gave the best balance in allowing sharp resolution in the peaks without undue roughness in the tails.

4.2. HDR Confidence Intervals

To illustrate the HDR method, we present an example with a set of simulated data to demonstrate how the HDR intervals can be constructed from $\{\hat{\theta}_j^{IC*}(l)\}_{l=1}^B$. Similar illustrations can be drawn from other bootstrap distributions $\{\hat{\theta}_j^{DC*}(l)\}_{l=1}^B$ and $\{\hat{\theta}_j^{BC*}(l)\}_{l=1}^B$, although, for simplicity, they are not included. We consider a simple model $Y_t = 0.9Y_{t-1} + 0.5X_t + u_t$, $(\theta = 5)$, where $u_t \sim \text{NID}(0, \sigma^2)$ and $X_t = 0.3X_{t-1} + v_t$, where $v_t \sim \text{NID}(0,1)$. We set $\sigma^2 = 0.96$ so that the signal-to-noise ratio is 2 with the value of population R^2 of 0.67. The sample size n is set to 50. The number of bootstrap iterations is set at 1,000. A set of realized data gives parameter estimates $\hat{\alpha}$ = (0.80, 0.67) that yield $\hat{\theta}$ = 3.39, while analytic bias-correction gives $\hat{\alpha}^c$ = (0.83, 0.67).

{Approximate Position of Figure 1}

The kernel density estimate of $\{\hat{\theta}_j^{IC*}(l)\}_{l=1}^B$ is given in Figure 1. Notice that the distribution is heavily skewed with a substantially longer tail on the right hand side. This reflects the property that a small sampling error in the estimation of λ can result in an extremely large value in the estimation of the long run response parameter. Vinod and McCullough (1994), who considered a bootstrap without bias-correction,

also noted this skewness property. The degree of skewness, however, is further accentuated by bias-correction. The upper and lower horizontal lines in Figure 1 correspond to 75% and 90% HDR intervals of [1.95, 9.20] and [1.44, 14.67], respectively. Although they are not indicated in the figure, the 75% and 90% percentile intervals are [2.18, 10.66] and [1.68, 19.23]. This example clearly demonstrates that the HDR confidence intervals are much tighter than the percentile intervals.

Remark

It is possible that a small number of negative outliers occur in $\{\hat{\theta}_j^{\cdot C^*}(l)\}_{l=1}^B$ even when $\hat{\theta}_j > 0$, due to variability in the bootstrap procedure, especially when the model is close to unit root non-stationarity. In this case, the density estimation may fail. To circumvent this, the outliers in the negative tail of the $\{\hat{\theta}_j^{\cdot C^*}(l)\}_{l=1}^B$ distribution are removed. An element of $\{\hat{\theta}_j^{\cdot C^*}(l)\}_{l=1}^B$ is classified as an outlier if it is less than $Q_1 - 1.5(Q_3 - Q_1)$, where Q_1 and Q_3 are the 1st and 3rd quartiles of $\{\hat{\theta}_j^{\cdot C^*}(l)\}_{l=1}^B$. Note that, when $\hat{\theta}_j < 0$, outliers in the positive tail of $\{\hat{\theta}_j^{\cdot C^*}(l)\}_{l=1}^B$ are removed from the density estimation in the same way.

5. Monte Carlo Experiment

To examine the small sample properties of the alternative confidence interval methods described, we undertake a Monte Carlo experiment. The experimental design loosely follows those of Bewley and Fiebig (1990) and Li and Maddala (1999). We considered a model of the form

$$Y_t = \lambda Y_{t-1} + \beta X_t + u_t;$$

$$X_t = \phi X_{t-1} + v_t,$$

where $\lambda \in \{0.75, 0.9\}$ and $\phi \in \{0.3, 0.8\}$, $v_t \sim \text{NID}(0, 1)$ and $u_t \sim \text{NID}(0, \sigma^2)$. We set $\beta = 0.5$, without loss of generality, which yields $\theta \in \{2, 5\}$. The sample sizes considered are 25 and 50. We chose the values of σ^2 so that the values of the signal-to-noise ratio (denoted g) are set to 2 and 8 (see Bewley and Fiebig, 1990). The nominal coverage $(1-\alpha)$ of the confidence interval is set to 0.75 and 0.90. The number of bootstrap iterations B is set to 1,000, and so is the number of Monte Carlo trials. The number of bootstrap iterations B_1 for bootstrap bias-correction is set to 500. All calculations were carried out using R (R Development Core Team, 2006) and its hdrcde package (Hyndman and Einbeck, 2006).

To facilitate a comparison of alternative confidence intervals, we use the mean coverage rate over Monte Carlo trials. To compare their length properties, the median and inter-quartile range (IQR) for the length of the confidence intervals are reported.

5.1. Point Estimation

We have compared the small sample properties of alternative bias-corrected point estimators for the long run response parameter presented in Section 2. Although the details are not reported, we obtained similar results to those reported in Pesaran and Zhao (1999). Although bias-correction may improve the accuracy when the value of λ is small, this is not necessarily the case when it is relatively close to one. That is, for the range of λ values considered in this paper, none of the bias-corrected point estimators outperform $\hat{\theta}$. This suggests that, for point estimation, bias-correction adds variability, which outweighs the possible gain in accuracy, under the parameter space of interest in this paper.

5.2. Interval Estimation

We now compare the small sample properties of three HDR-based confidence intervals with the percentile interval based on bootstrap bias-correction. The latter was examined by Li and Maddala (1999), who found that this interval performs better than other conventional confidence intervals. Although they are included in the simulation, we do not report the percentile intervals based on analytic bias-correction for the sake of simplicity, because these percentile intervals are almost always inferior to the HDR intervals. Typically, the percentile intervals show wider and more volatile length properties, although the coverage rates are close to those of the HDR intervals. For ease of exposition, the HDR interval obtained from $\{\hat{\theta}_j^{IC*}(l)\}_{l=1}^B$ is denoted as HDR$_{IC}$, while those from $\{\hat{\theta}_j^{DC*}(l)\}_{l=1}^B$ and $\{\hat{\theta}_j^{BC*}(l)\}_{l=1}^B$ are denoted as HDR$_{DC}$ and HDR$_{BC}$ respectively. The percentile interval from $\{\hat{\theta}_j^{BC*}(l)\}_{l=1}^B$ is denoted as PER.

{Approximate Position of Table 1}

Table 1 reports the coverage rates of alternative confidence intervals for the nominal coverages 0.75 and 0.90. For all cases, the coverage rates are reasonably close to the nominal level. There is a tendency for the coverage rate to be higher than the nominal level when $\lambda = 0.9$. However, the coverage rates are closer to the nominal level when $\lambda = 0.75$ and when the sample size is larger. Overall, it is evident that all confidence intervals show desirable coverage properties.

{Approximate Position of Table 2}

The length properties of alternative confidence intervals are reported in Table 2 for the nominal coverages of 0.75 and 0.90. To begin with, we pay attention to the HDR_{BC} and PER intervals. As we can see in Table 2, the former almost always have much shorter lengths than the latter, and their IQR values are also substantially lower. The PER intervals can be too wide and are highly volatile. This feature is also evident for the case of analytic bias-correction, as mentioned above.

We now compare HDR intervals based on analytic and bootstrap bias-correction. In general, the HDR_{BC} intervals are inferior to those based on HDR_{IC} and HDR_{DC}. Overall, the former are wider and more volatile than the latter, especially when $\lambda = 0.9$. Comparing the HDR intervals based on analytic bias-correction, it is almost always the case that HDR_{DC} intervals are inferior to HDR_{IC}, as they show wider and more volatile length properties. The former can often be inferior to the HDR_{BC} intervals. When $\lambda = 0.75$, the HDR_{IC} intervals perform better than the HDR_{BC} intervals overall, except when $\{\phi = 0.3, g=2\}$ and $\{\phi = 0.8, g=8\}$. However, when the sample size is small and $\lambda = 0.9$, the HDR_{IC} interval is the clear winner.

These results strongly suggest that the HDR_{IC} intervals provide tighter and more informative intervals with excellent coverage properties. Although the HDR_{BC} intervals also perform reasonably well, they can occasionally yield very wide and highly volatile intervals when the model is close to unit root non-stationarity. When the model is close to unit root non-stationarity and the sample size is small, the HDR_{IC} intervals provide the most reliable and accurate interval estimation.

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6. Case Studies

6.1. The Demand for Oil

We present an application where the long run own price elasticity of demand for oil is estimated. The use of a dynamic linear model is common in this body of literature, see Pesaran *et al.* (1998), Gately and Huntington (2002), Cooper (2003) and Griffin and Schulman (2005). In terms of the antecedent literature, the short and long run own price elasticity of demand for oil is typically found to be inelastic. For the OECD countries, the estimates reported in these papers vary from almost zero to -0.6. However, in all of these papers, the long run elasticity estimates are reported without confidence intervals. As a result, it is difficult to assess the degree of uncertainty associated with the elasticity estimates reported.

We consider an ARDL model of the form

$$D_t = \beta_0 + \sum_{i=1}^{p} \gamma_i D_{t-i} + \sum_{i=0}^{p(1)} \beta_{1i} P_{t-i} + \sum_{i=0}^{p(2)} \beta_{2i} Y_{t-i} + u_t, \qquad (6)$$

where D_t is the quantity demand for oil as at year t, P_t is the real price for oil in year t, Y_t is the real GDP per capita in year t and u is a random error. The long run own price elasticity parameter to be estimated is $\theta = \dfrac{\beta_{10} + \beta_{11} + ... + \beta_{1p(1)}}{1 - \lambda}$, where $\lambda = \gamma_1 + ... + \gamma_p$.

We have used annual data from 1970 to 2004 for a group of 10 OECD countries: Australia, Canada, France, Germany, Italy, Japan, Spain, Sweden, the UK and the USA. Our data are drawn from two sources, the OECD Fact Book and various issues of the BP Statistical Review of World Energy. All data are in natural logs.

Table 3 presents the model selection and basic statistics results. The lag orders are selected following a simple-to-general strategy. First, setting $p(1) = p(2) = 0$, the value of p is determined as the smallest number such that the residuals mimic a white noise. Higher values of $p(1)$ and $p(2)$ than zero are included if they provide statistically significant coefficients at the 10% level of significance. The number of bootstrap iterations is set to 5,000.

{Approximate Position of Table 3}

The results presented in Table 3 show that there is some variation in the model specification. For example, the US equation has an autoregressive order of two whereas all others have one. There are also some differences in the estimated lag orders of income and own price. In general, the long run own price elasticity estimates are all inelastic and within the range typically reported in the literature. We also note that when the own price coefficient is statistically insignificant, the resulting long run own price elasticity is close to zero.

{Approximate Position of Table 4}

Table 4 reports three 90% HDR-based confidence intervals. The results obtained in the Monte Carlo experiment are neatly illustrated in these estimates. When the sum of AR coefficient estimates ($\hat{\lambda}$) are close to one, the HDR_{BC} and HDR_{DC} intervals are substantially wide, while the HDR_{IC} intervals are much tighter. For example, for Canada where $\hat{\lambda} = 0.93$, the 90% HDR_{IC} confidence interval is nearly twice and six times shorter than HDR_{DC} and HDR_{BC}, respectively. When the value of $\hat{\lambda}$ is smaller, all three HDR-based intervals are similar (see, for example, the case for the UK). According to the HDR_{IC} confidence intervals reported in Table 3, the own price

elasticity is statistically significant for Canada, France, Italy, Japan, UK and US, at 10% level of significance. The demand is found to be inelastic only for UK, while the results also suggest that the long run elasticity for certain countries (e.g., Canada, France and Japan) could be far more elastic than the point estimates would suggest.

6.2. The Demand for Beef

In this case study, we present the results for the long run own price elasticity of demand for beef in the UK. In the applied literature, most analyses of meat demand have typically focussed on short run elasticity estimates (e.g., Fraser and Moosa, 2002). In the few papers that estimated the long run elasticity of the demand for beef, researchers have employed a dynamic specification such as an error correction model e.g., Burton and Young (1996) and Mazzocchi (2006). From the results reported in the literature, the long run elasticity ranges from -1.5 to less than -1. As with the long run elasticity estimates in the oil demand literature, few of the point estimates in this body of literature are supported by confidence intervals.

To estimate long run elasticity of the demand for beef with respect to its on price, we have employed monthly data from 1975:01 to 2000:12 (312 observations, seasonally unadjusted). The data are drawn from the National Food Survey which was collected by the Ministry of Agriculture, Fisheries and Food (MAFF) (now DEFRA). We estimate the following model:

$$Q_t = \beta_0 + \sum_{i=1}^{p} \gamma_i Q_{t-i} + \sum_{i=0}^{p(1)} \beta_{1i} P_{bt-i} + \sum_{i=0}^{p(2)} \beta_{2i} I_{t-i} + \beta_3 P_{lt} + \beta_4 P_{pt} + \beta_5 P_{ct} + u_t , \qquad (7)$$

where Q_t is the quantity of beef consumed, P_{bt} is the price of beef, P_{lt} is the price of lamb, P_{pt} is the price of pork, P_{ct} is the price of chicken, and I_t is a measure of total

expenditure on meat products as a proxy for income assuming two stage budgeting (i.e., weakly separable demand). Although not included in (7), the model also contains a linear time trend, monthly seasonal dummy variables, and a dummy variable for the BSE (Bovine Spongiform Encephalopathy; i.e., mad cow decease) crisis following Leeming and Turner (2004). The BSE dummy variable takes a value of 1 for the period from 1996:01 to 2000:12, and 0 elsewhere[4]. The long run own price elasticity parameter to be estimated is $\theta = \dfrac{\beta_{10} + \beta_{11} + ... + \beta_{1p(1)}}{1 - \lambda}$, where $\lambda = \gamma_1 + ... + \gamma_p$.

To demonstrate the robustness of the methods we developed in this paper, we have used a rolling window as follows. We take the first window of 120 observations starting from 1975:1 to 1984:12. This window of 120 observations moves six months forward to cover the period of 1975:7 and 1985:6, and this continues until the last window for the period of 1991:1 to 2000:12. We set the lag orders to $p = p(1) = p(2) = 1$, which are found to provide the residuals that mimic a white noise, according to the residual statistics. Higher order lag values such as 3 and 6 were attempted, but qualitative similar results were obtained. Figure 2 reports the 75% and 90% HDR_{IC} intervals. Other HDR intervals are not reported because they are similar. The number of bootstrap iterations is set to 1,000.

{Approximate Position of Figure 2}

From Figure 2, it is evident that all point and interval estimates exhibit the correct sign according to economic theory. We can also observe that the confidence intervals are reasonably tight. For the 90% intervals, the average upper and lower bounds across all windows are -1.22 and -0.63, with an average point estimate of -0.86. These

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[4] Although the existence of the BSE can be traced back to 1984, it was in May 1995 that we observed the first recorded death from the human variant Creutzfeldt-Jakob Disease (v-CJD), and it was only in March 1996 that the UK government officially recognised the link between the consumption of infected beef and the development of CJD.

results are is in keeping with the estimates previously reported in the literature. It is also interesting to note that elasticity has become progressively lower across the sample, although this trend has been sharply reversed in the last couple of years. Finally, although we do not report the results here, we did vary the size of the rolling window, and, as might be expected, the width of the confidence intervals increased (decreased) as we reduced (increased) the window size.

7. Conclusion

This paper has proposed new methods for interval estimation for the long run response parameter from a general linear dynamic model. Our proposed method is based on the bias-corrected bootstrap first proposed by Kilian (1998a, 1998b), and later adopted by Li and Maddala (1999) for long run elasticity estimation. It is distinct from past studies in the following ways. First, we have considered analytic bias-correction in addition to the bootstrap bias-correction used by Li and Maddala (1999). For analytic bias-correction, we have considered both indirect bias-correction using the asymptotic formula derived by Kiviet and Phillips (1994), and direct bias-correction as a generalisation of the method proposed by Pesaran and Zhao (1999). Secondly, we have adopted the highest density region (HDR) method of Hyndman (1996), instead of the conventional percentile method, for constructing confidence intervals from the bootstrap distribution of the long run response estimator.

We have conducted an extensive Monte Carlo experiment to compare the small sample properties of alternative confidence intervals. It is found that the HDR method provides much tighter and more stable confidence intervals than the percentile method, with accurate coverage properties. In particular, the HDR interval based on indirect

analytic bias-correction overall outperforms all other alternatives in small samples, especially when the model is close to unit root non-stationarity.

In order to illustrate our new method in an applied context, we have presented two empirical case studies (long run elasticity estimation for demand for oil and demand for beef). For both case studies, the empirical results we obtained are consistent with those previously reported in the literature in terms of point estimates. However, unlike much of the existing literature, we also report confidence intervals using the methods developed in this paper. Both case studies provide results that further support the Monte Carlo experiment.

Appendix A

$$\frac{\hat{\beta}_{ji}}{1-\hat{\lambda}} = \hat{\beta}_{ji}\left\{(1-\lambda)\left[1-\frac{\hat{\lambda}-\lambda}{1-\lambda}\right]\right\}^{-1} = \frac{\hat{\beta}_{ji}}{1-\lambda}\left[1+\frac{\hat{\lambda}-\lambda}{1-\lambda}-\frac{(\hat{\lambda}-\lambda)^2}{(1-\lambda)^2}+...\right]$$

$$= \frac{\hat{\beta}_{ji}}{1-\lambda}+\frac{\hat{\beta}_{ji}(\hat{\lambda}-\lambda)}{(1-\lambda)^2}-\frac{\hat{\beta}_{ji}(\hat{\lambda}-\lambda)^2}{(1-\lambda)^3}+o_p(n^{-1})$$

$$= \frac{\hat{\beta}_{ji}}{1-\lambda}+\frac{\hat{\beta}_{ji}(\hat{\lambda}-\lambda)}{(1-\lambda)^2}-\frac{\beta_{ji}(\hat{\lambda}-\lambda)^2}{(1-\lambda)^3}+o_p(n^{-1})$$

since $\hat{\beta}_{ji} = (\hat{\beta}_{ji} - \beta_{ji}) + \beta_{ji}$ and $(\hat{\beta}_{ji} - \beta_{ji})(\hat{\lambda}-\lambda)^2 = O_p(n^{-3/2})$.

$$E\left(\frac{\hat{\beta}_{ji}}{1-\hat{\lambda}} - \frac{\beta_{ji}}{1-\lambda}\right) = \frac{E(\hat{\beta}_{ji}-\beta_{ji})}{1-\lambda}+\frac{\beta_{ji}E(\hat{\lambda}-\lambda)}{(1-\lambda)^2}+\frac{E(\hat{\beta}_{ji}-\beta_{ji})(\hat{\lambda}-\lambda)}{(1-\lambda)^2}-\frac{\beta_{ji}E(\hat{\lambda}-\lambda)^2}{(1-\lambda)^3}+o(n^{-1})$$

$$= \frac{B_{\hat{\beta}_{ji}}}{1-\lambda}+\frac{\beta_{ji}B_{\hat{\lambda}}+Cov(\hat{\beta}_{ji},\hat{\lambda})}{(1-\lambda)^2}-\frac{\beta_{ji}MSE(\hat{\lambda})}{(1-\lambda)^3}+o(n^{-1})$$

Figure 1. An example of the 75% and 90% HDR intervals for the long run response based on simulated data
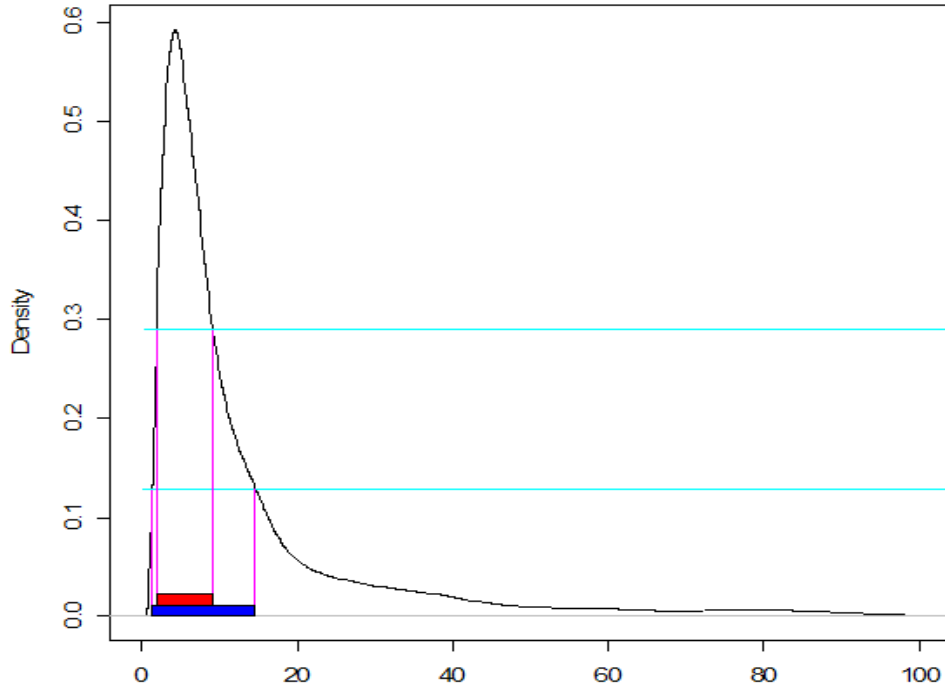

Figure 2. 75% and 90% confidence intervals for the long run own price elasticity for beef demand. (HDR intervals based on indirect analytic bias-correction)*
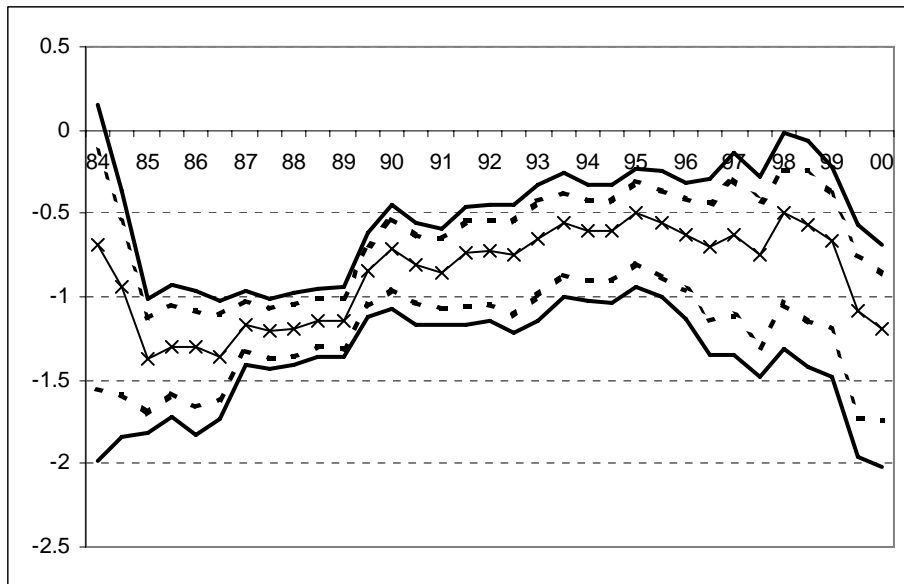
*Solid line: 90% confidence intervals; Dotted line: 75% confidence intervals; Crossed line: Point estimates of long run elasticity

Table 1. Coverage Rate of Confidence Intervals

λ = 0.75

	1-α =0.75				1-α =0.90			
	Analytic		Bootstrap		Analytic		Bootstrap	
	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER
n=25								
ϕ = 0.3								
g = 2	0.78	0.78	0.82	0.75	0.92	0.91	0.94	0.90
g = 8	0.72	0.70	0.75	0.72	0.88	0.89	0.90	0.86
ϕ = 0.8								
g = 2	0.77	0.71	0.77	0.75	0.89	0.87	0.87	0.88
g = 8	0.80	0.79	0.76	0.74	0.92	0.93	0.89	0.88
n=50								
ϕ = 0.3								
g = 2	0.76	0.75	0.77	0.75	0.93	0.92	0.92	0.90
g = 8	0.76	0.75	0.75	0.74	0.91	0.90	0.90	0.89
ϕ = 0.8								
g = 2	0.73	0.66	0.78	0.75	0.91	0.85	0.90	0.89
g = 8	0.76	0.69	0.76	0.76	0.89	0.87	0.90	0.89

λ = 0.9

	1-α =0.75				1-α =0.90			
	Analytic		Bootstrap		Analytic		Bootstrap	
	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER
n=25								
ϕ = 0.3								
g = 2	0.84	0.84	0.82	0.81	0.93	0.94	0.89	0.91
g = 8	0.79	0.80	0.84	0.78	0.90	0.94	0.91	0.93
ϕ = 0.8								
g = 2	0.83	0.84	0.82	0.78	0.90	0.91	0.89	0.88
g = 8	0.77	0.74	0.82	0.81	0.91	0.91	0.91	0.91
n=50								
ϕ = 0.3								
g = 2	0.82	0.80	0.85	0.77	0.94	0.95	0.93	0.92
g = 8	0.82	0.82	0.84	0.74	0.92	0.95	0.92	0.90
ϕ = 0.8								
g = 2	0.77	0.76	0.83	0.77	0.91	0.90	0.92	0.90
g = 8	0.82	0.81	0.81	0.76	0.92	0.94	0.93	0.91

Analytic: analytic bias-correction; Bootstrap: bootstrap bias-correction;
HDR_{IC}: bias-corrected bootstrap HDR interval based on indirect bias-correction; HDR_{DC}: bias-corrected bootstrap HDR interval based on direct bias-correction; HDR_{BC}: bias-corrected bootstrap HDR interval based on bootstrap bias-correction; PER: bias-corrected bootstrap percentile interval.
λ: AR(1) coefficient in the ARDL model; ϕ: the coefficient in the AR(1) model for the exogenous variable; g: signal-to-noise ration, n: sample size

Table 2. Comparison of Length Properties

$\lambda = 0.75$, *n*=25

	1-α =0.75				1-α =0.90			
	Analytic		Bootstrap		Analytic		Bootstrap	
	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER
$\phi = 0.3$								
g = 2								
MED	3.68	4.45	2.95	3.14	6.34	7.97	5.03	5.52
IQR	4.56	5.90	3.69	5.05	10.60	13.15	8.32	13.69
g = 8								
MED	1.75	1.98	1.98	2.01	2.60	2.98	3.07	3.13
IQR	1.36	1.70	1.96	2.15	2.13	2.71	3.38	4.01
$\phi = 0.8$								
g = 2								
MED	2.51	3.57	5.97	6.00	3.94	5.92	10.08	10.81
IQR	3.19	4.97	11.95	19.47	7.29	10.52	32.20	196.84
g = 8								
MED	4.08	5.23	1.23	1.19	7.00	9.36	1.81	1.74
IQR	5.95	8.37	0.82	0.77	17.25	21.50	1.26	1.20

$\lambda = 0.75$, *n*=50

	1-α =0.75				1-α =0.90			
	Analytic		Bootstrap		Analytic		Bootstrap	
	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER
$\phi = 0.3$								
g = 2								
MED	2.20	2.55	2.12	2.10	3.36	3.94	3.21	3.14
IQR	1.82	2.36	1.64	1.69	3.15	4.15	2.60	2.74
g = 8								
MED	1.06	1.15	1.15	1.15	1.55	1.70	1.69	1.70
IQR	0.54	0.62	0.64	0.66	0.82	1.00	0.98	1.01
$\phi = 0.8$								
g = 2								
MED	2.11	2.48	2.87	2.74	3.09	3.66	4.38	4.12
IQR	1.40	1.88	1.96	1.99	2.14	2.94	3.35	3.27
g = 8								
MED	1.14	1.23	1.08	1.06	1.66	1.80	1.57	1.54
IQR	0.58	0.68	0.57	0.56	0.87	1.05	0.90	0.85

Analytic: analytic bias-correction; Bootstrap: bootstrap bias-correction;
HDR_{IC}: bias-corrected bootstrap HDR interval based on indirect bias-correction; HDR_{DC}: bias-corrected bootstrap HDR interval based on direct bias-correction; HDR_{BC}: bias-corrected bootstrap HDR interval based on bootstrap bias-correction; PER: bias-corrected bootstrap percentile interval.
λ: AR(1) coefficient in the ARDL model; ϕ: the coefficient in the AR(1) model for the exogenous variable; g: signal-to-noise ration, n: sample size

Table 2. Continued

λ = 0.9, *n*=25

| | 1-α =0.75 | | | | 1-α =0.90 | | | |
| | Analytic | | Bootstrap | | Analytic | | Bootstrap | |
	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER
φ = 0.3								
g = 2								
MED	13.51	16.65	24.05	175.77	27.12	30.93	50.31	708.90
IQR	32.48	31.57	39.63	661.58	73.93	93.51	71.17	1964.44
g = 8								
MED	5.16	6.63	13.52	22.73	8.32	11.49	35.02	136.37
IQR	6.72	9.97	29.44	513.77	13.19	22.21	68.06	1598.47
φ = 0.8								
g = 2								
MED	29.04	32.12	16.45	30.75	60.69	80.80	41.63	233.93
IQR	59.26	60.02	61.01	320.72	111.62	177.16	177.53	974.73
g = 8								
MED	6.56	8.38	19.79	50.93	10.49	13.70	45.49	509.20
IQR	7.99	11.47	40.60	536.72	16.80	24.20	75.56	1722.40

λ = 0.9, *n*=50

| | 1-α =0.75 | | | | 1-α =0.90 | | | |
| | Analytic | | Bootstrap | | Analytic | | Bootstrap | |
	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER	HDR_{IC}	HDR_{DC}	HDR_{BC}	PER
φ = 0.3								
g = 2								
MED	11.01	13.10	10.08	11.15	18.60	22.28	17.77	20.70
IQR	17.69	19.18	21.59	50.79	44.61	44.64	64.15	534.35
g = 8								
MED	7.14	9.10	6.49	7.10	12.18	16.54	10.72	12.28
IQR	9.30	13.80	9.41	13.44	21.84	34.61	20.99	36.57
φ = 0.8								
g = 2								
MED	8.95	11.65	15.63	18.06	15.55	20.56	31.15	49.82
IQR	12.89	17.12	45.78	186.50	34.05	41.31	99.68	734.64
g = 8								
MED	9.91	12.46	6.24	6.18	17.93	22.53	9.71	9.89
IQR	16.20	19.94	9.28	12.38	47.95	55.67	21.66	42.69

Analytic: analytic bias-correction; Bootstrap: bootstrap bias-correction;
HDR_{IC}: bias-corrected bootstrap HDR interval based on indirect bias-correction; HDR_{DC}: bias-corrected bootstrap HDR interval based on direct bias-correction; HDR_{BC}: bias-corrected bootstrap HDR interval based on bootstrap bias-correction; PER: bias-corrected bootstrap percentile interval.
λ: AR(1) coefficient in the ARDL model; φ: the coefficient in the AR(1) model for the exogenous variable; g: signal-to-noise ration, n: sample size

Table 3. Model specification and basic statistics for demand for oil

	$\hat{\lambda}$	p	$p(1)$	$p(2)$	R^2	$\hat{\theta}$
Australia	0.92	1	0*	0	0.80	-0.29
Canada	0.93	1	0*	1	0.92	-0.63
France	0.91	1	0*	0	0.94	-0.64
Germany	0.65	1	0	0	0.87	-0.07
Italy	0.66	1	0*	1	0.80	-0.17
Japan	0.86	1	0*	1	0.87	-0.50
Spain	0.86	1	0	0	0.96	-0.05
Sweden	0.69	1	0	0	0.97	-0.02
UK	0.49	1	0*	0	0.80	-0.12
US	0.85	2	0*	0	0.89	-0.23

For all cases, the intercept in included. The linear time trend is included only for Spain and Germany.
p: the value of the autoregressive order
p(1): the value of the maximum lag order of price, the zero indicates that only the contemporaneous values are included. The starred value indicates that the corresponding coefficient is statistically significant at the 5% level.
p(2): the value of the maximum lag order of income

Table 4. 90% Confidence Intervals of the Long Run Price Elasticity

	Analytic				Bootstrap	
	HDR$_{IC}$		HDR$_{DC}$		HDR$_{BC}$	
Australia	-2.04	0.08	-2.33	0.07	-10.24	0.58
Canada	-4.80	-0.01	-12.15	0.03	-29.87	0.34
France	-6.66	-0.08	-14.13	-0.07	-7.21	-0.10
Germany	-0.56	0.08	-0.41	0.07	-1.79	0.21
Italy	-0.38	-0.10	-0.41	-0.11	-041	-0.10
Japan	-5.30	-0.13	-4.82	-0.16	-6.61	-0.15
Spain	-1.14	0.26	-0.37	0.26	-10.95	0.60
Sweden	-0.39	0.11	-0.35	0.13	-1.82	0.25
UK	-0.23	-0.07	-0.26	-0.07	-0.24	-0.06
US	-1.77	-0.01	-1.19	0.11	-2.78	0.10

Analytic: analytic bias-correction; Bootstrap: bootstrap bias-correction;
HDR$_{IC}$: bias-corrected bootstrap HDR interval based on indirect bias-correction; HDR$_{DC}$: bias-corrected bootstrap HDR interval based on direct bias-correction; HDR$_{BC}$: bias-corrected bootstrap HDR interval based on bootstrap bias-correction.

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