Stochastic population forecasts using functional data models

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Outline

1. Functional time series
2. Current state of Australian population forecasting
3. Stochastic population forecasting

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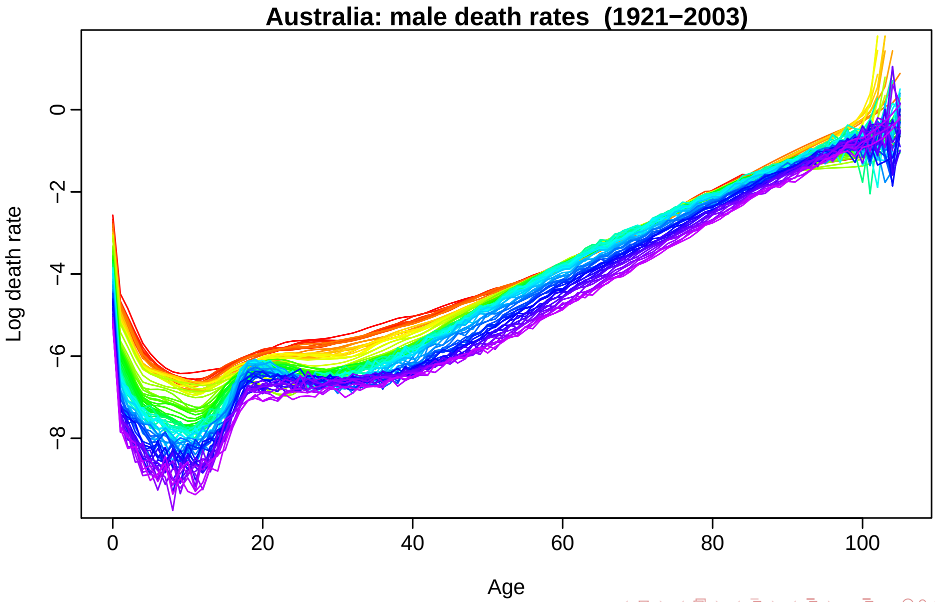
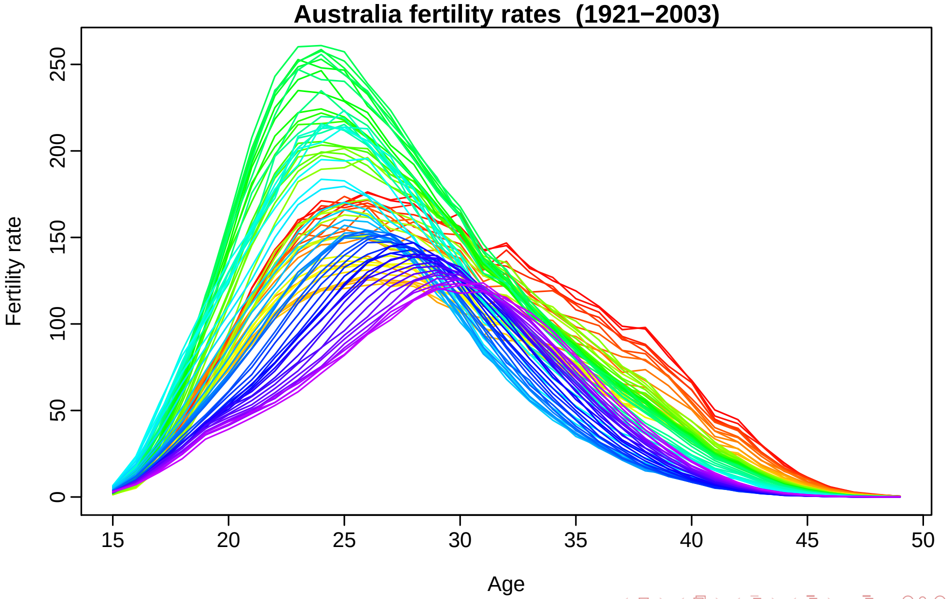
2. Current state of Australian population forecasting
3. Stochastic population forecasting

Mortality rates



Fertility rates



Some notation

Let $y_t(x_i)$ be the observed data in period t at location x_i, $i = 1, \dots, p$, $t = 1, \dots, n$. We assume

$$y_t(x_i) = s_t(x_i) + \sigma_t(x_i)\varepsilon_{t,i}$$

where $\varepsilon_{t,i}$ is an iid standard normal random variable and $\sigma_t(x_i)$ allows the amount of noise to vary with x.

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- We assume $s_t(x)$ is a smooth function of x.
- We need to estimate $s_t(x)$ from the data for $x_1 < x < x_p$.
- We want to forecast **whole curve** $y_t(x)$ for $t = n + 1, \dots, n + h$.

Functional time series model

$$\begin{aligned} y_t(x) &= s_t(x) + \sigma_t(x)\varepsilon_{t,x} \\ s_t(x) &= \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + e_t(x) \end{aligned}$$

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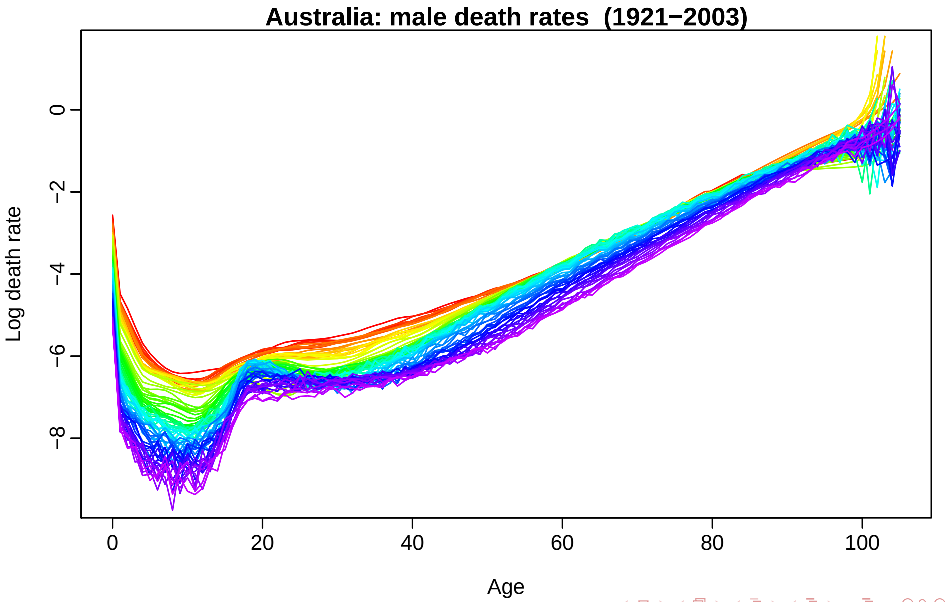

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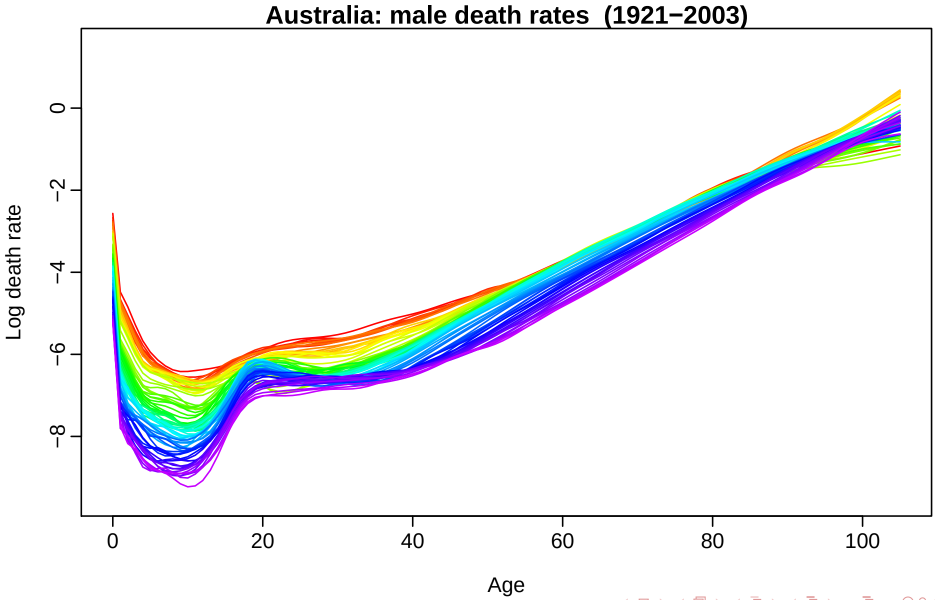

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- This can be done using a modification of the gam function in the mgcv package in **R**.

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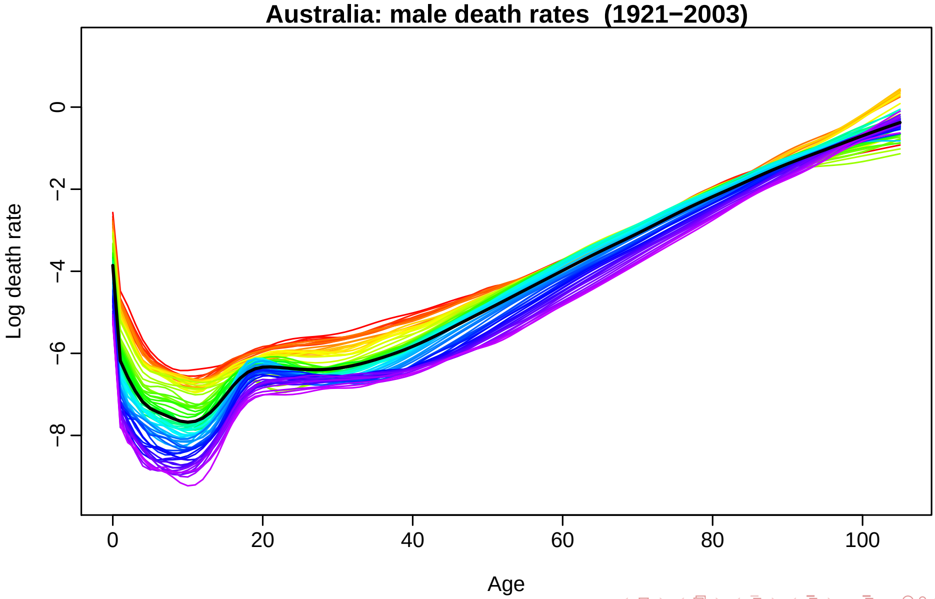
2. Estimate $\mu(x)$



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3. Functional PC

The optimal basis functions

$$s_t(x) = \mu(x) + \sum_{i=0}^{K} \beta_{t,i}\phi_i(x) + e_t(x).$$

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- For a given K, the basis functions $\phi_i(x)$ which minimize

$$\text{MISE} = \frac{1}{n}\sum_{t=1}^{n}\int v_t^2(x)\,dx$$

are the principal components.

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  3. Find the weight function $\phi_3(x)$ that . . .

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Calculating the optimal basis functions

- Define F to be $n \times m$ matrix with (t, j)th element $\hat{s}_t(x_j^*) - \mu(x_j^*)$ where $\{x_1^*, \dots, x_m^*\}$ are a dense grid on x.

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- The basis functions are orthogonal.
- This means the coefficients series are also uncorrelated with each other. i.e., $\text{Corr}(\hat{\beta}_{t,i}, \hat{\beta}_{t,j}) = 0$ for $i \neq j$. However, $\text{Corr}(\hat{\beta}_{t,i}, \hat{\beta}_{s,j}) \neq 0$ in general for $t \neq s$ and $i \neq j$.

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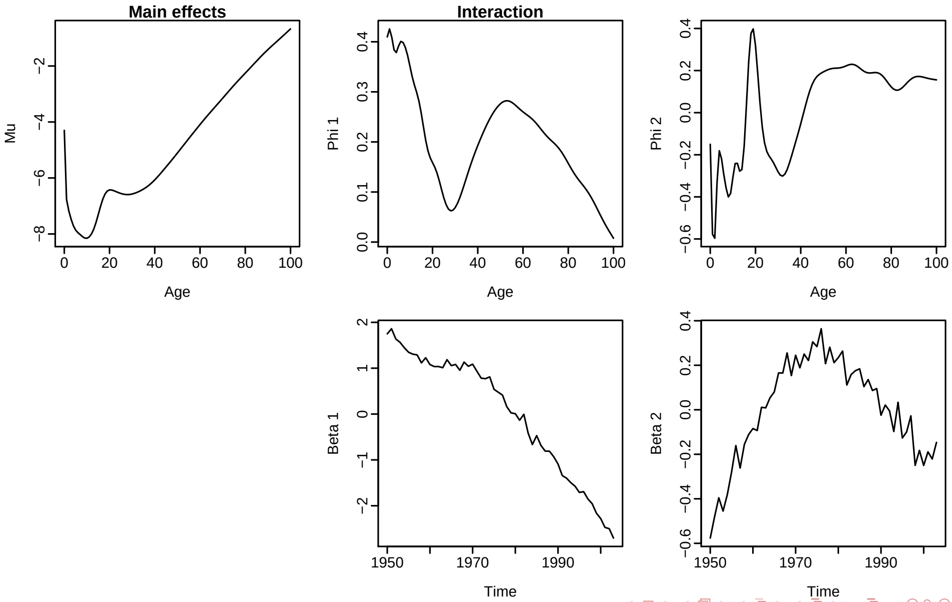

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- We can check if any structure is left in the residuals $\varepsilon_{t,x}$ (smoothing problem) and $e_t(x)$ (modelling problem).

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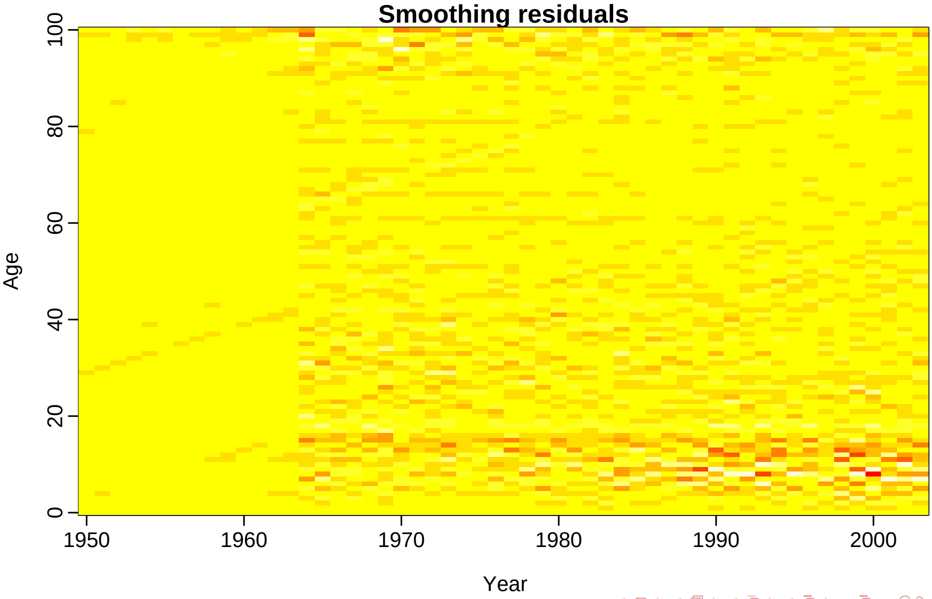

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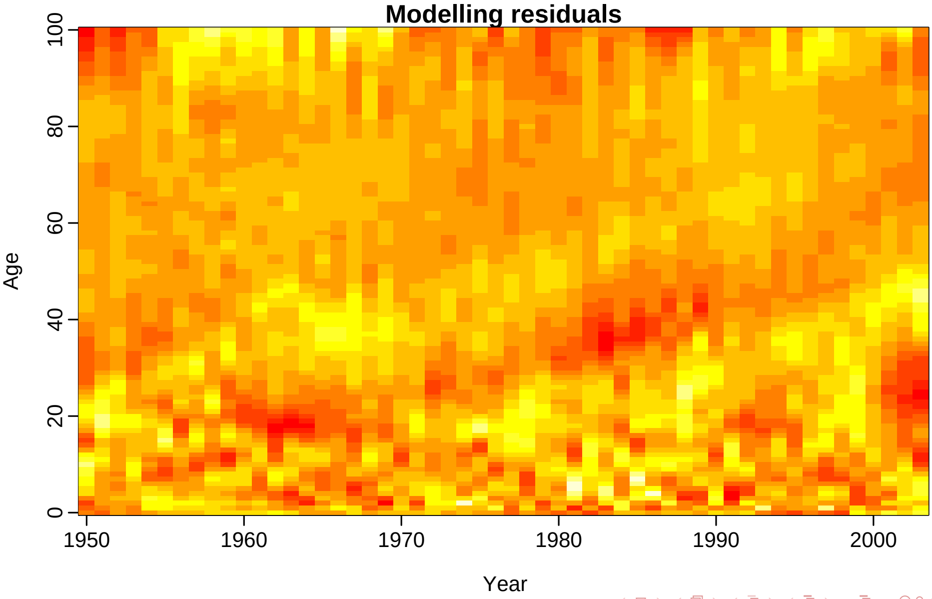

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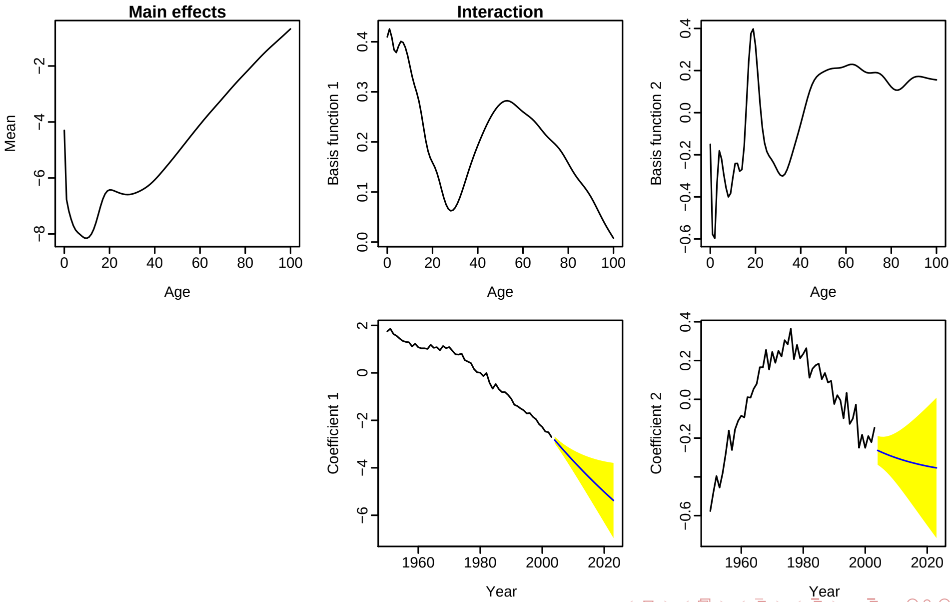

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- Univariate models are ok because the series are uncorrelated.

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Let $\mathcal{I} = \{y_t(x_i); t = 1,\dots,n; i = 1,\dots,p\}$.

- $\mathrm{E}[y_{n+h}(x) \mid \mathcal{I}, \boldsymbol{\Phi}] = \hat{\mu}(x) + \sum_{k=1}^{K} \hat{\beta}_{n+h|n,k}\,\hat{\phi}_k(x)$.

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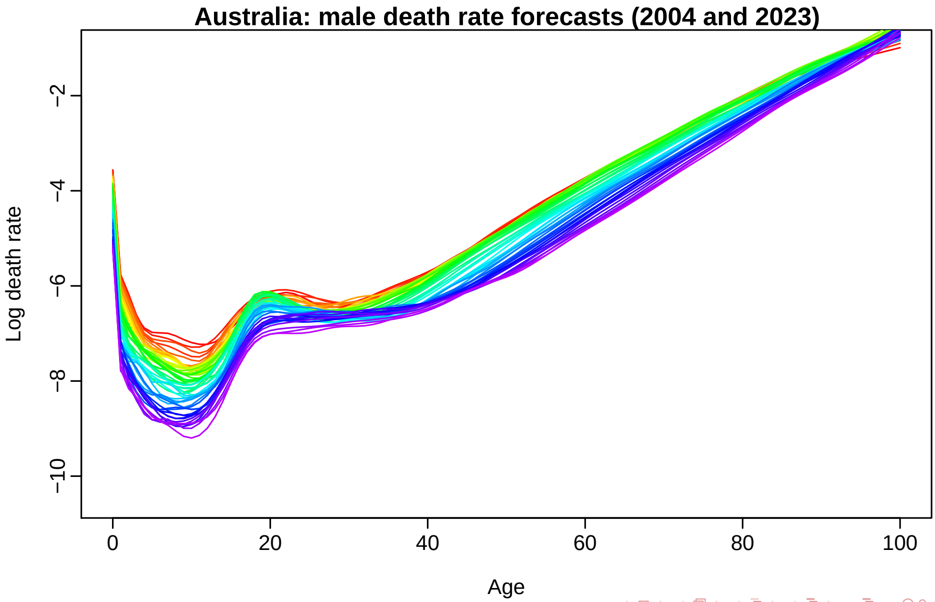
- $\mathrm{E}[y_{n+h}(x) \mid \mathcal{I}, \Phi] = \hat{\mu}(x) + \sum_{k=1}^{K} \hat{\beta}_{n+h|n,k}\,\hat{\phi}_k(x)$.
- $\mathrm{Var}[y_{n+h}(x) \mid \mathcal{I}, \Phi] =$ $\sigma^2_{n+h}(x) + \hat{\sigma}^2_{\mu}(x) + \sum_{k=1}^{K} v_{n+h|n,k}\,\hat{\phi}^2_k(x) + v(x)$ where $v_{n+h|n,k} = \mathrm{Var}(\beta_{n+h,k} \mid \beta_{1,k},\dots,\beta_{n,k})$.

5. Forecasts of $y_t(x)$



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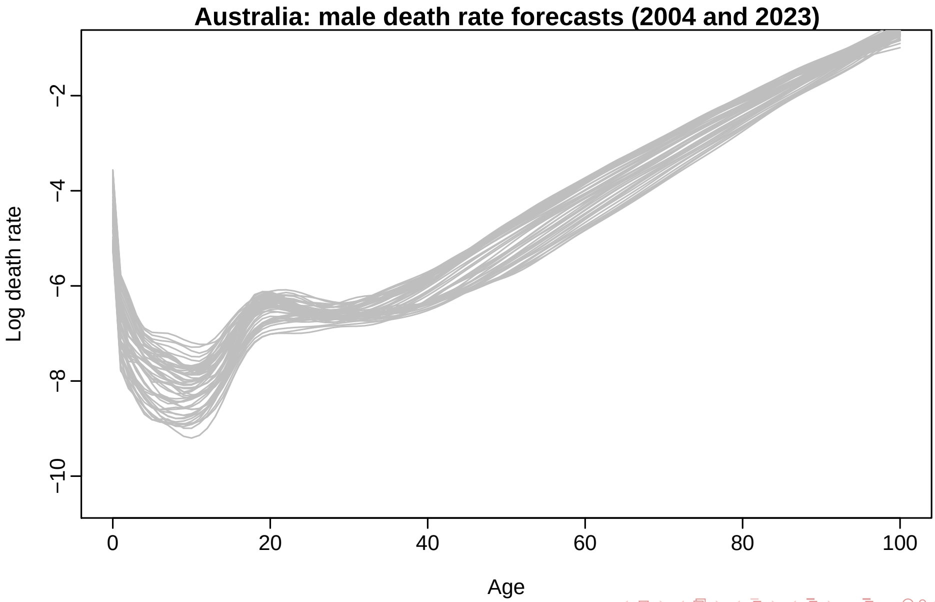

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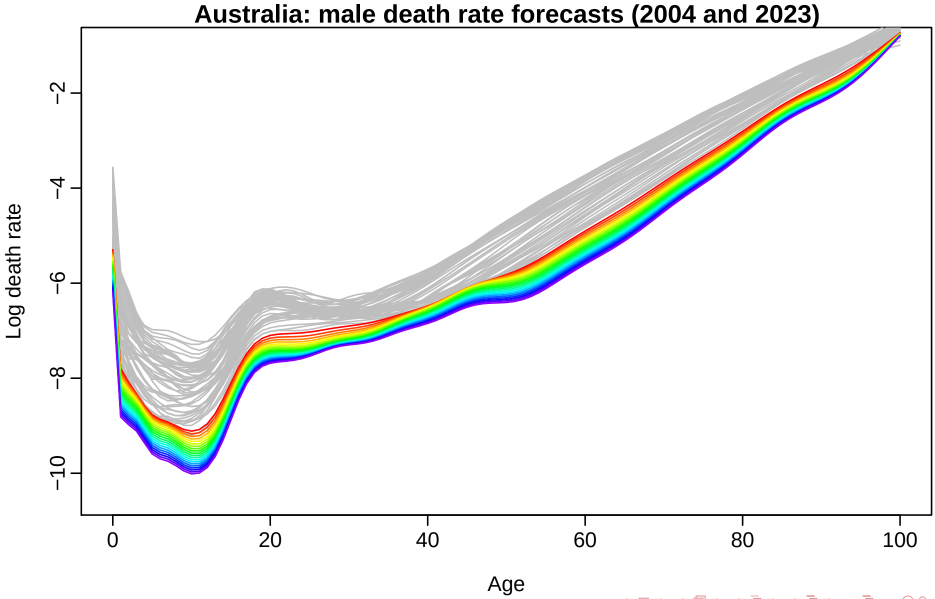

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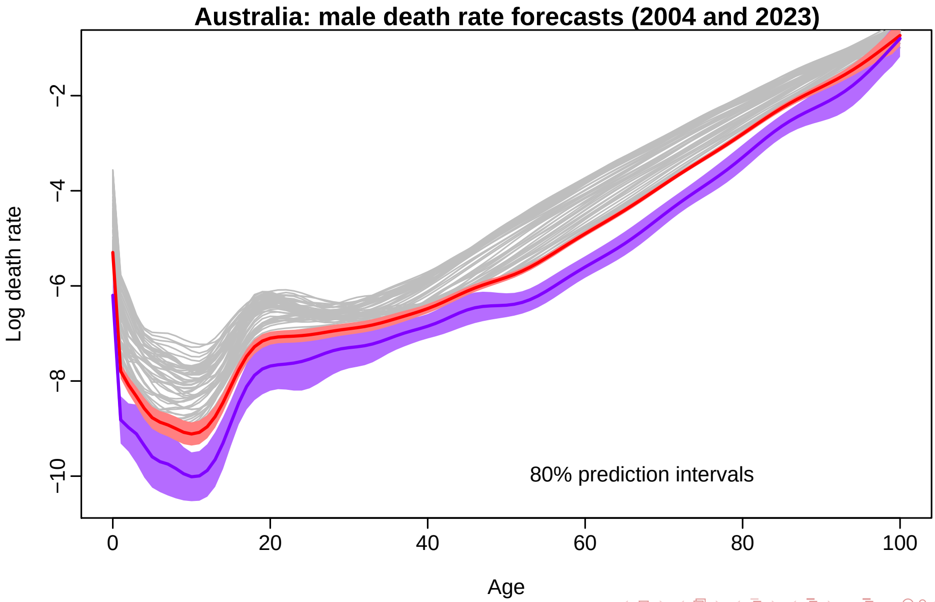

Some references

- Erbas, Hyndman and Gertig (2007) Forecasting age-specific breast cancer mortality using functional data models. *Statistics in Medicine*, **26**(2), 458–470.

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- Hyndman (2006) demography: Forecasting mortality and fertility data. **R package** v0.98. **www.robhyndman.info/Rlibrary/demography**

Outline

1. Functional time series
2. Current state of Australian population forecasting
3. Stochastic population forecasting

ABS population projections

The Australian Bureau of Statistics provide population “projections”.

“The projections are not intended as predictions or forecasts, but are illustrations of growth and change in the population that would occur if assumptions made about future demographic trends were to prevail over the projection period.

While the assumptions are formulated on the basis of an assessment of past demographic trends, both in Australia and overseas, there is no certainty that any of the assumptions will be realised. In addition, no assessment has been made of changes in non-demographic conditions.”

ABS 3222.0 - Population Projections, Australia, 2004 to 2101

ABS population projections

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- Based on assumed mortality, fertility and migration rates

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- Based on assumed mortality, fertility and migration rates
- No objectivity.
- No dynamic changes in rates allowed
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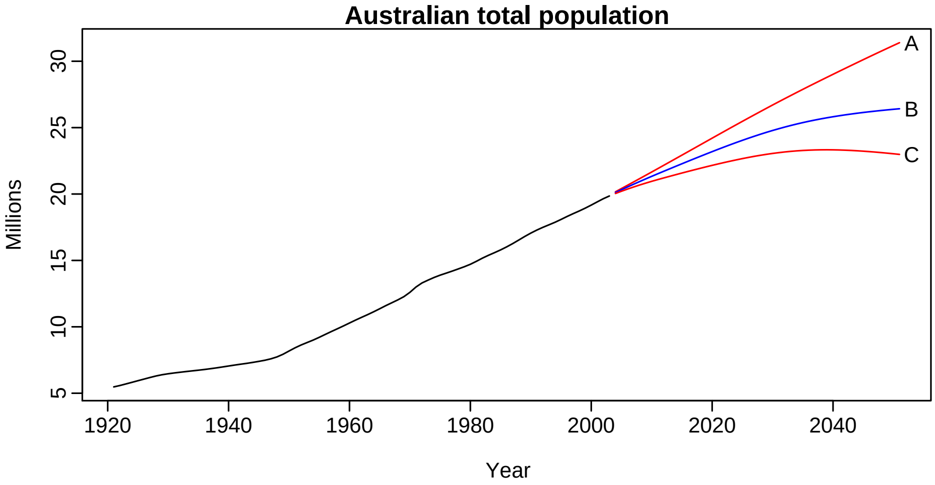
- No probabilistic basis.
- Not prediction intervals.
- Most users use the "Medium" projection, but it is unrelated to the mean, median or mode of the future distribution.

ABS population projections



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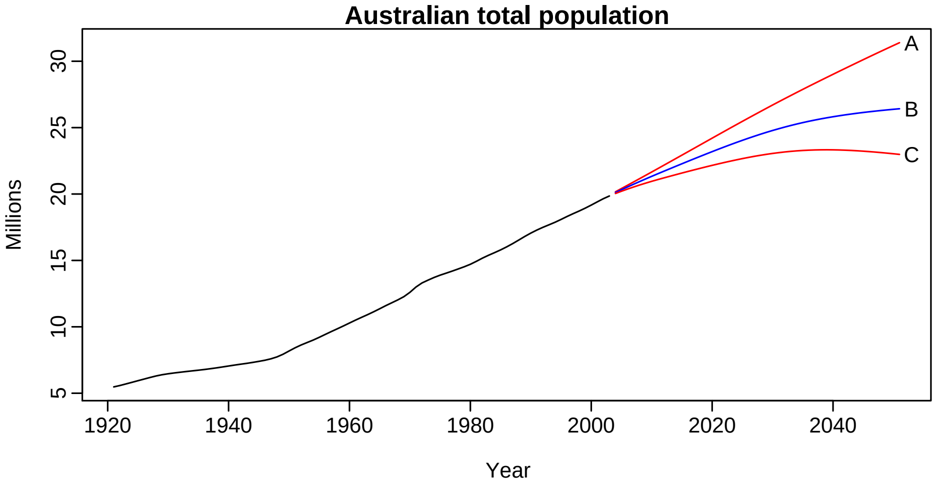

What do these projections mean?

Outline

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Stochastic population forecasts

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- Economic planning is better based on prediction intervals rather than mean or median forecasts.
- Stochastic models allow true policy analysis to be carried out.

Demographic growth-balance equation

Demographic growth-balance equation

$$\begin{aligned} P_{t+1}(x+1) &= P_t(x) - D_t(x,x+1) + G_t(x,x+1) \\ P_{t+1}(0) &= B_t \qquad - D_t(B,0) \qquad + G_t(B,0) \end{aligned}$$

$x = 0, 1, 2, \ldots$.

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$P_t(x) =$ population of age x at 1 January, year t

$B_t =$ births in calendar year t

$D_t(x, x+1) =$ deaths in calendar year t of persons aged x at the beginning of year t

$D_t(B, 0) =$ infant deaths in calendar year t

$G_t(x, x+1) =$ net migrants in calendar year t of persons aged x at the beginning of year t

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Key ideas

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- Combine the results to get *age-specific stochastic population forecasts*.

The available data

In most countries, the following data are available:

$P_t(x) =$ **population** of age x at 1 January, year t
$E_t(x) =$ **population** of age x at 30 June, year t
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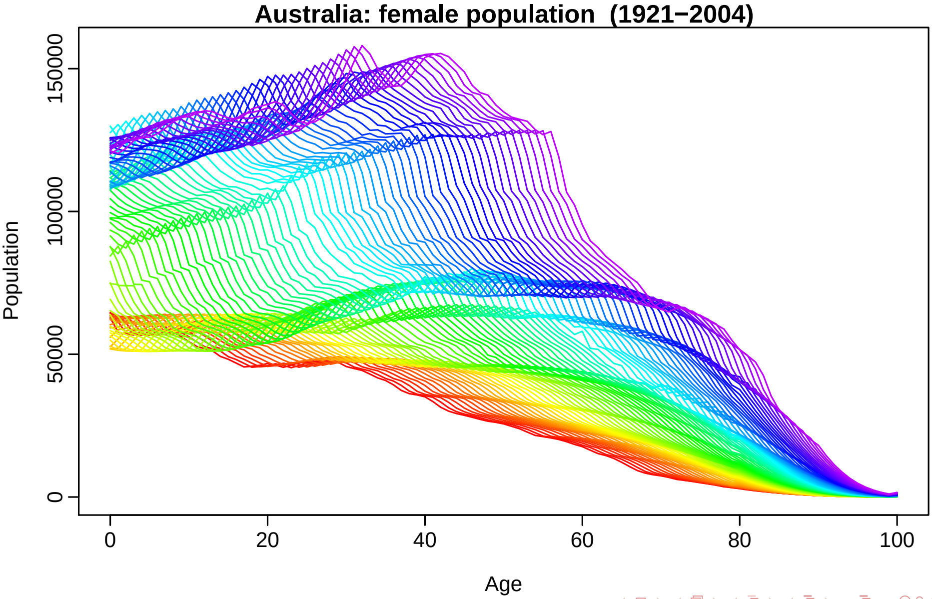
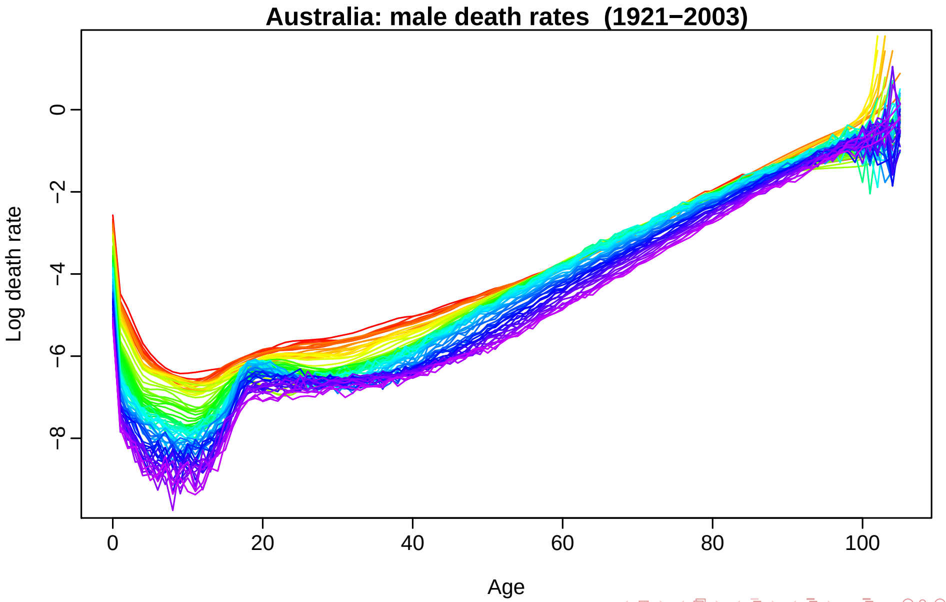
- $m_t(x) = D_t(x)/E_t(x) =$ central death rate in calendar year t;
- $f_t(x) = B_t(x)/E_t^F(x) =$ fertility rate for females of age x in calendar year t.

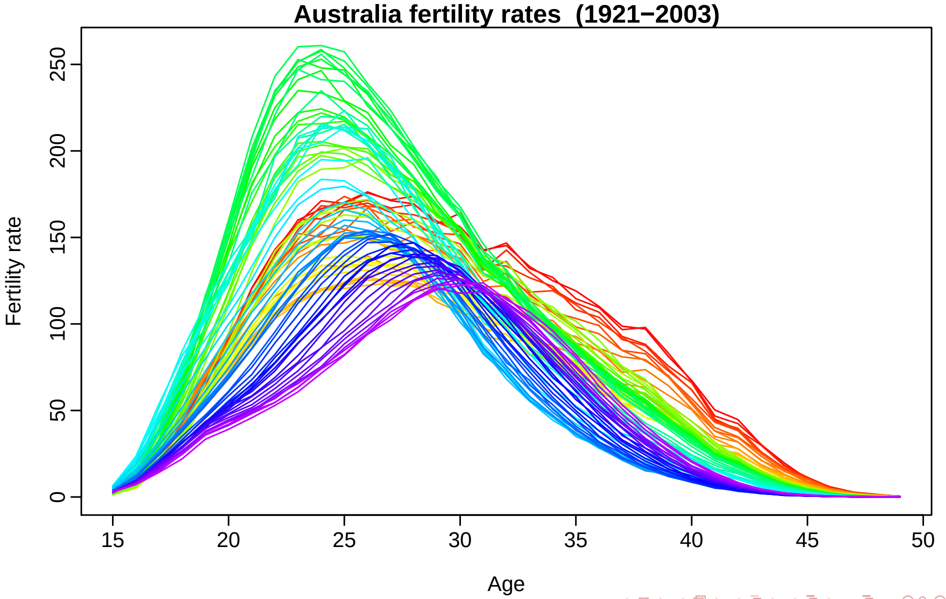
Australia's start-of-year population



Mortality rates



Fertility rates



Net migration

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Demographic growth-balance equation

$$G_t(x, x+1) = P_{t+1}(x+1) - P_t(x) + D_t(x, x+1)$$

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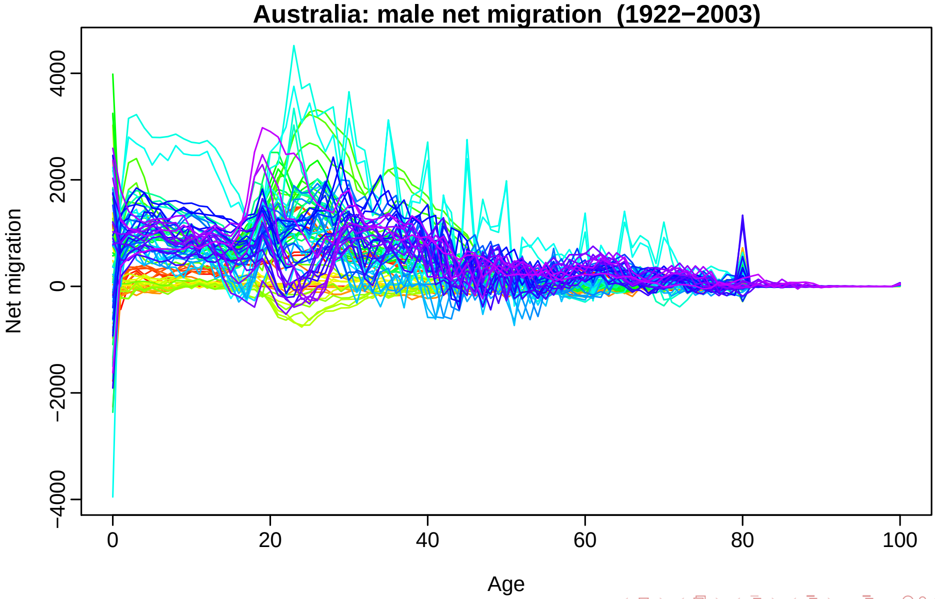
Note: "net migration" numbers also include **errors** associated with all estimates. i.e., a "residual".

Net migration



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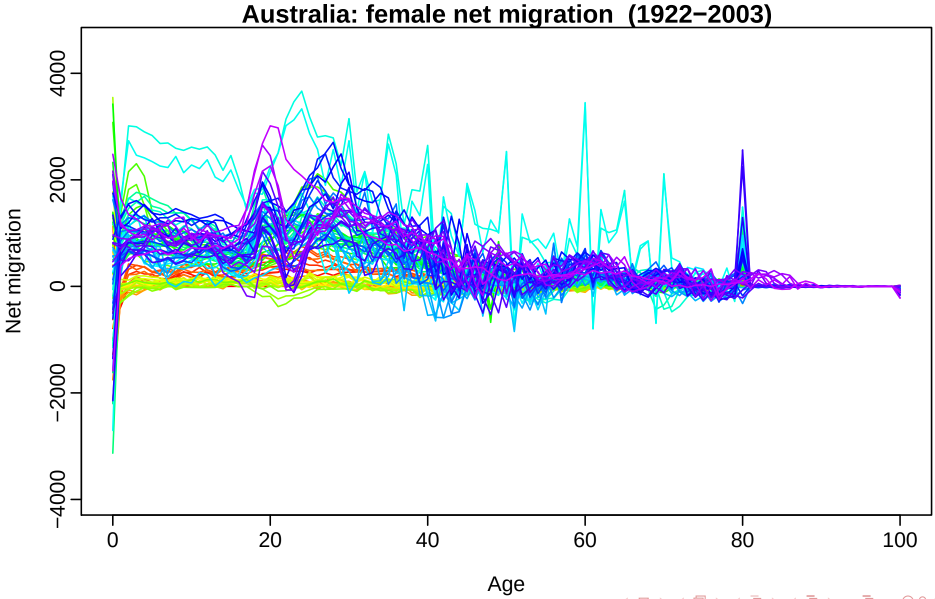

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Component models

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Component models

- Data: age/sex-specific mortality rates, fertility rates and net migration.
- Models: Five functional time series models for mortality (M/F), fertility and net migration (M/F) assuming independence between components.
- For each component:

$$
\begin{aligned}
y_t(x) &= s_t(x) + \sigma_t(x)\varepsilon_{t,x} \\
s_t(x) &= \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + e_t(x)
\end{aligned}
$$

Functional time series

Let $g_\lambda(u) = \begin{cases} \log(u) & \lambda = 0; \\ \frac{x^\lambda - 1}{\lambda} & \lambda > 0. \end{cases}$

- **Mortality rates:**
  $y_t(x_i) = g_0(m_t(x_i))$ where $m_t(x_i) =$ empirical mortality rate at age x_i.

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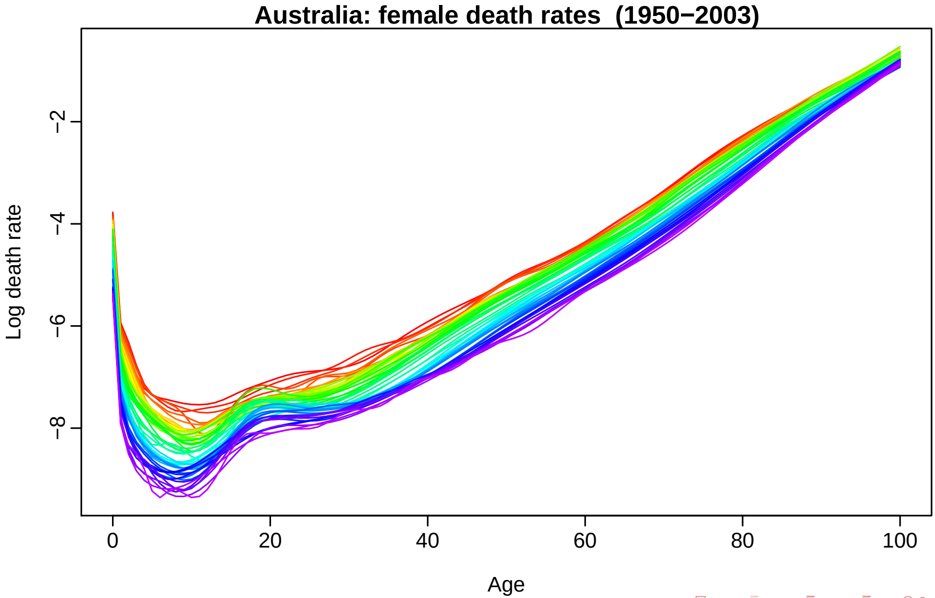
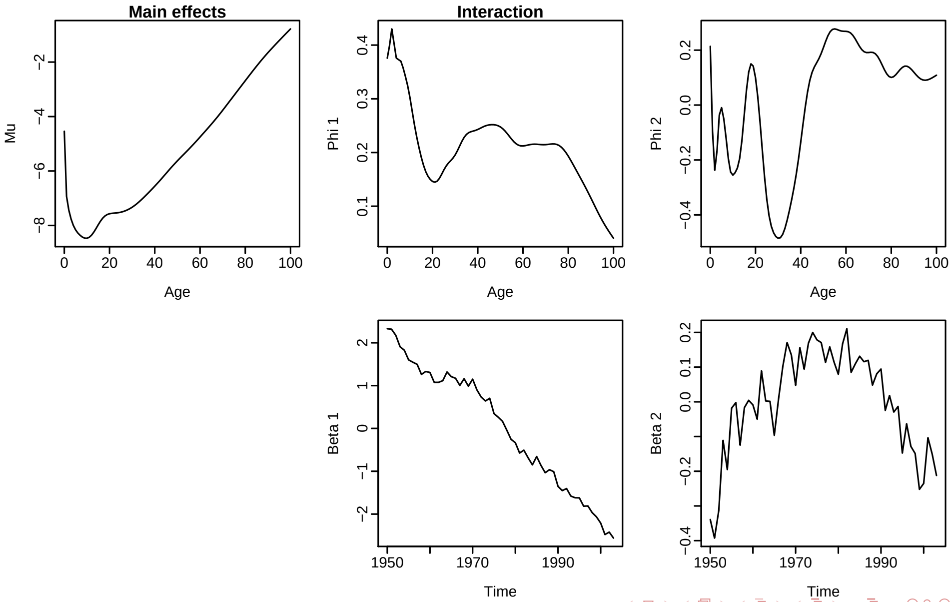
- **Net migration:**
  $y_t(x_i) =$ empirical net migration at age x_i.

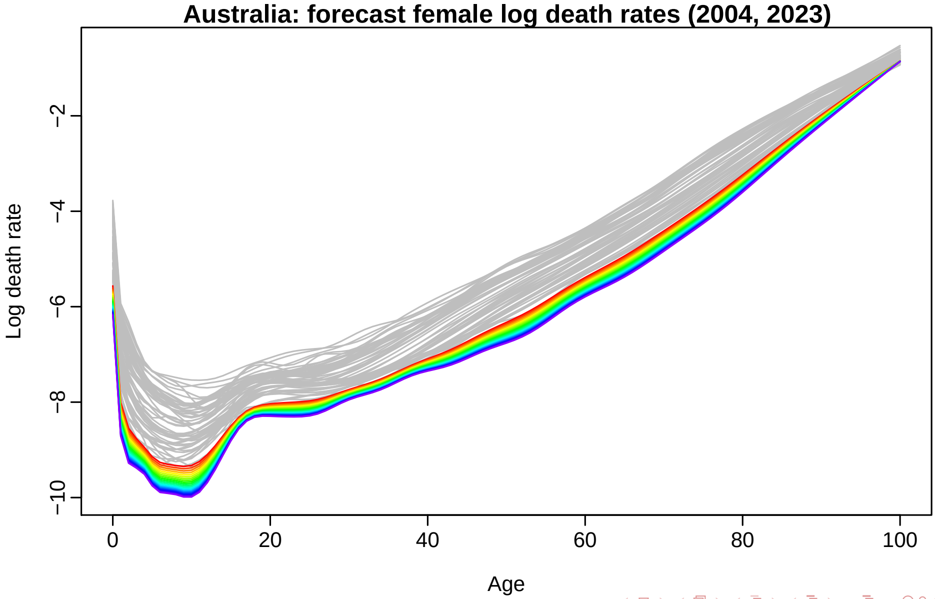
Mortality: female



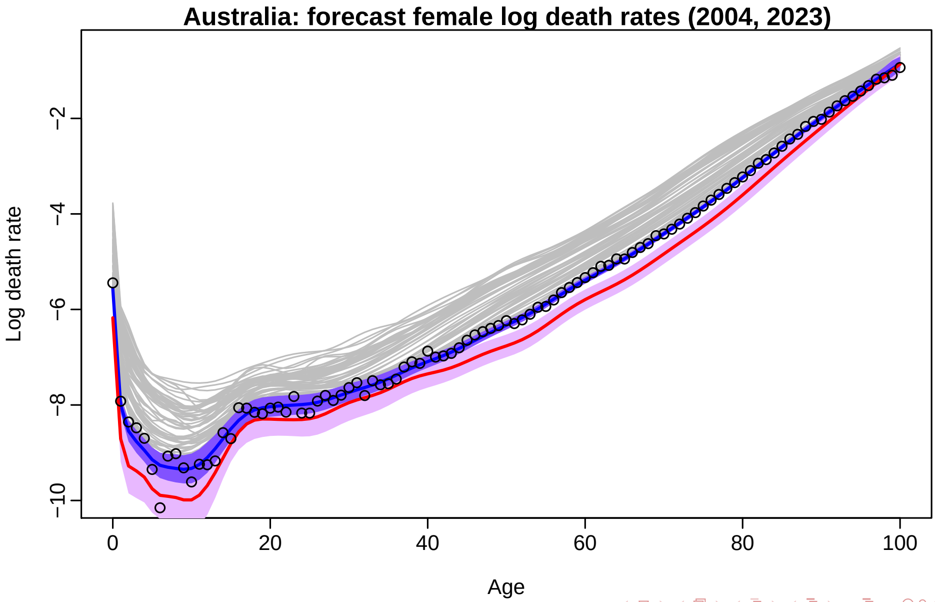
Mortality: female



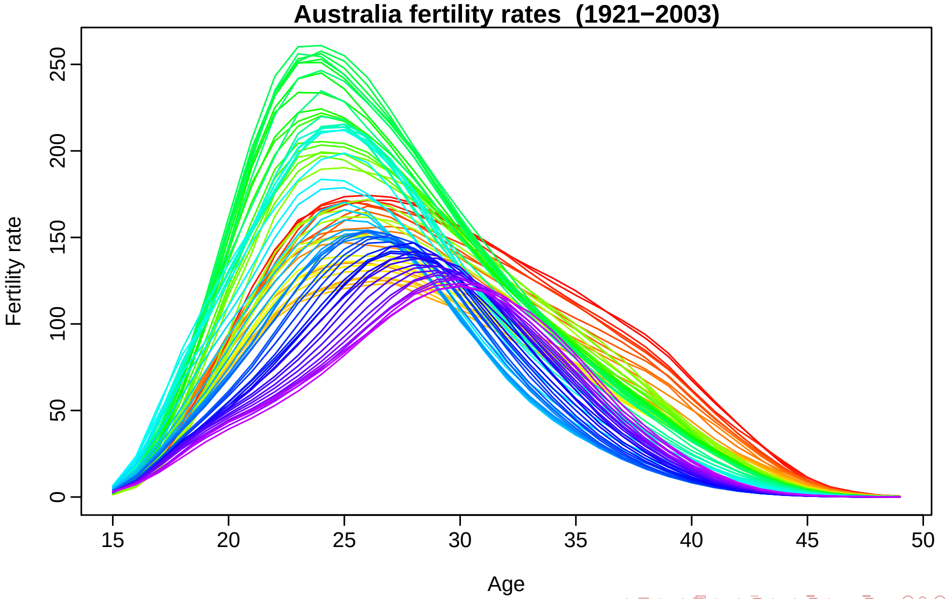
Mortality: female



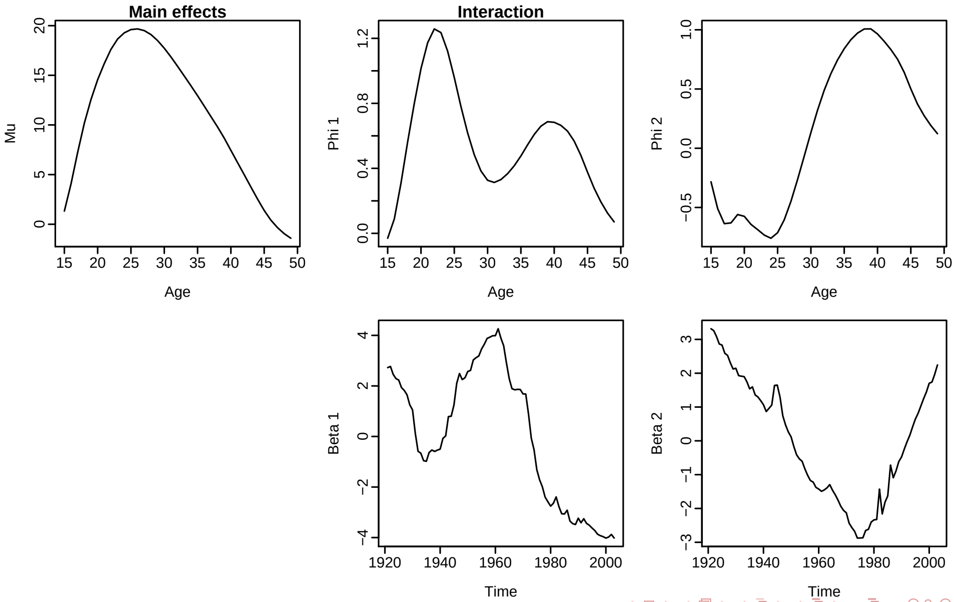
Mortality: female



Fertility


Australia fertility rates (1921−2003)
Fertility rate
0
50
100
150
200
250
15
20
25
30
35
40
45
50
Age

Fertility


Main effects
Mu
0
5
10
15
20
15 20 25 30 35 40 45 50
Age
Interaction
Phi 1
0.0
0.4
0.8
1.2
15 20 25 30 35 40 45 50
Age
Phi 2
−0.5
0.0
0.5
1.0
15 20 25 30 35 40 45 50
Age
Beta 1
−4
−2
0
2
4
1920 1940 1960 1980 2000
Time
Beta 2
−3
−2
−1
0
1
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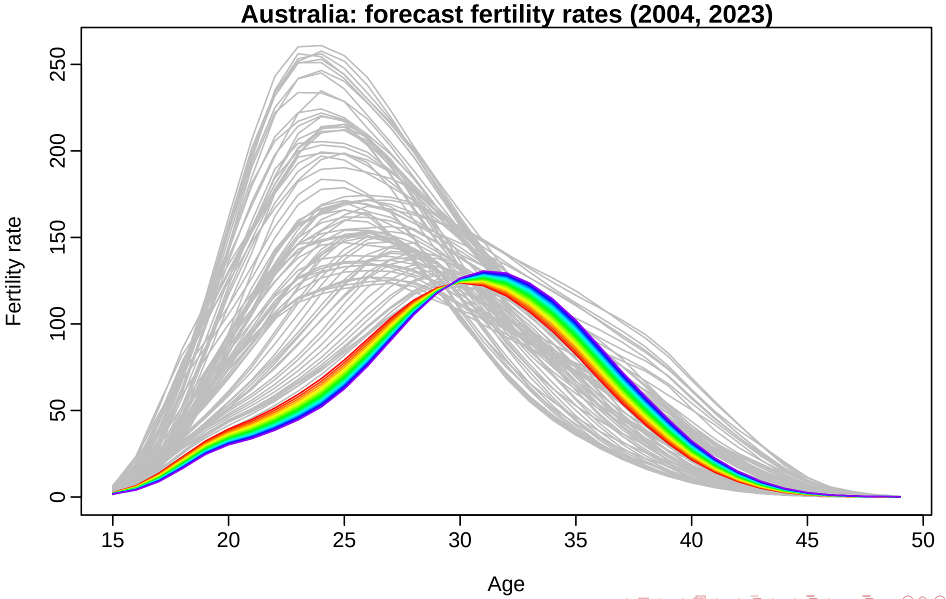
3
1920 1940 1960 1980 2000
Time

Fertility


Australia: forecast fertility rates (2004, 2023)
Fertility rate
Age
0
50
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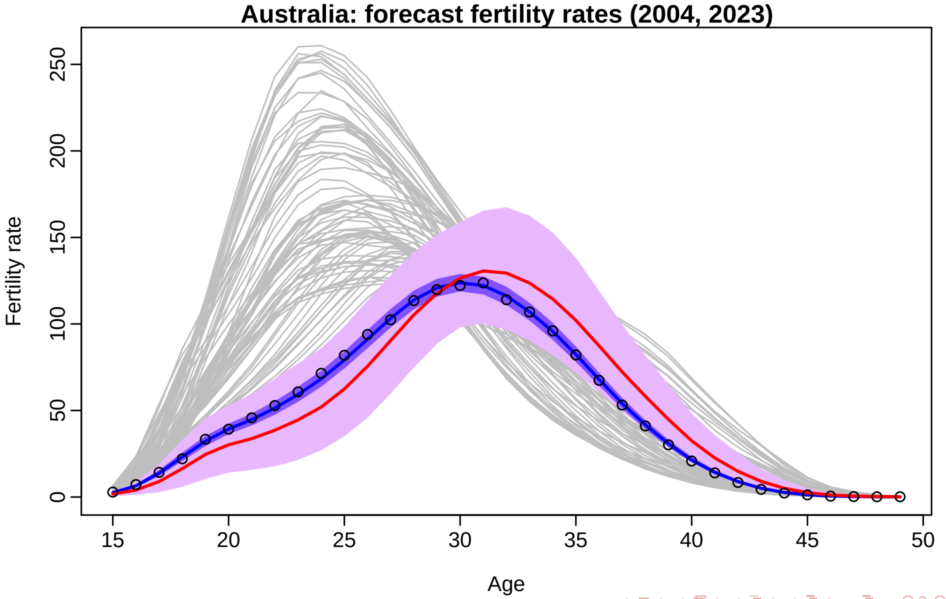
Fertility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Fertility rate
0
50
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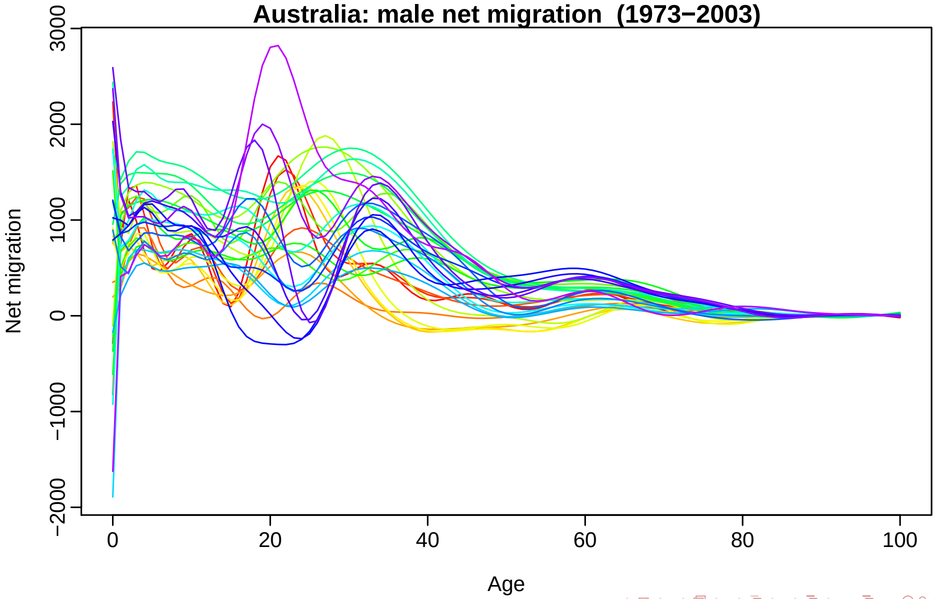
Migration: male



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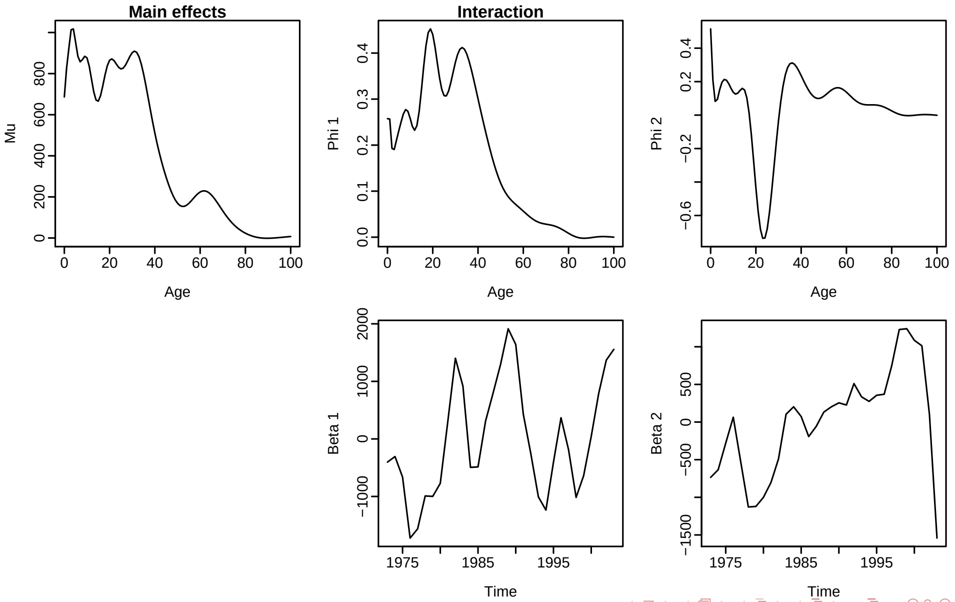
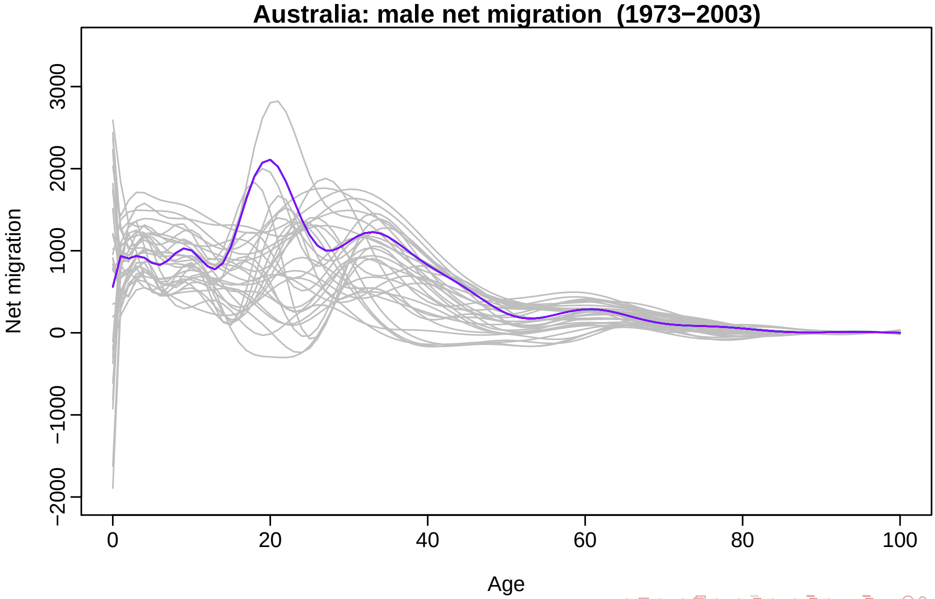
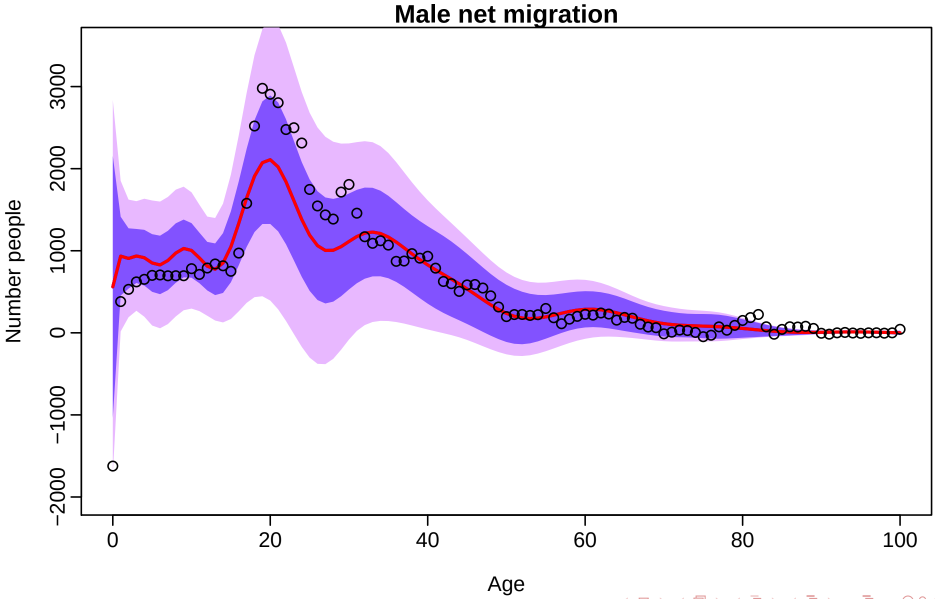


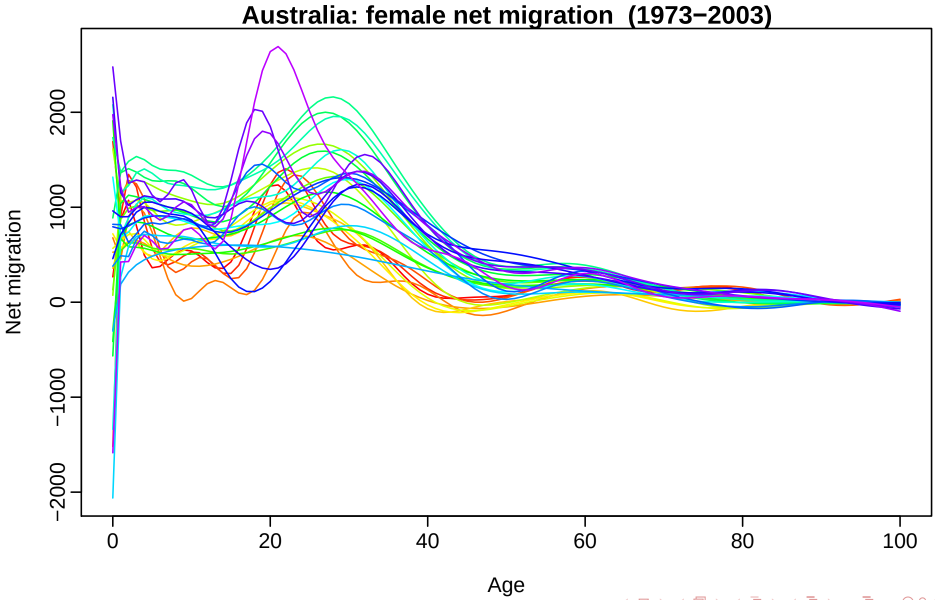
Migration: male



Migration: male



Migration: female



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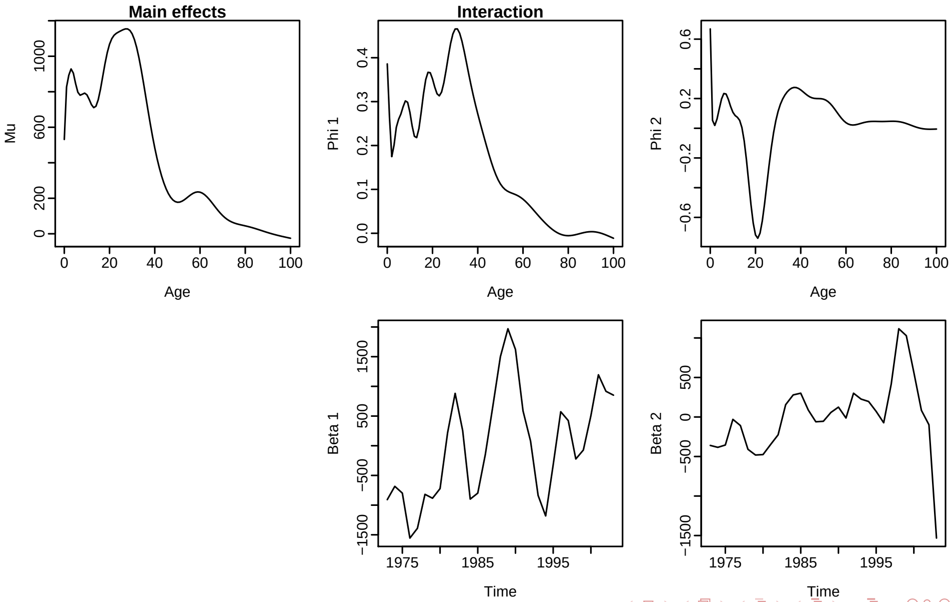

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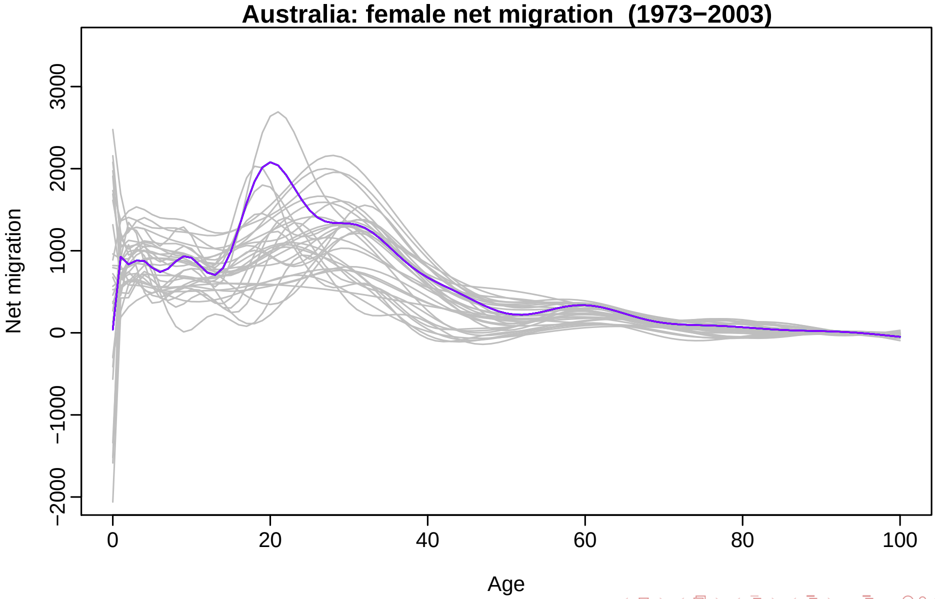

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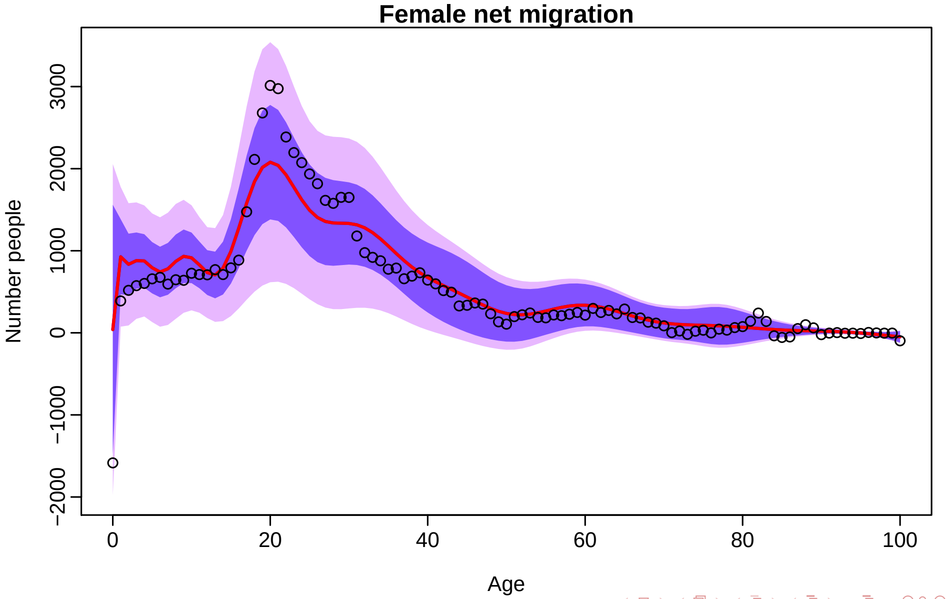

Simulation

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s_t(x) &= \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + e_t(x)
\end{aligned}
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- For each of $m_t^F(x)$, $m_t^M(x)$, $f_t(x)$, $G_t^F(x, x+1)$, and $G_t^M(x, x+1)$:

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- Use simulated rates to generate $B_t(x)$, $D_t^F(x,x+1)$, $D_t^M(x,x+1)$ for $t = n+1,\dots,n+h$, assuming deaths and births are Poisson.

Simulation

Demographic growth-balance equation used to get population sample paths.

Demographic growth-balance equation

$$P_{t+1}(x+1) = P_t(x) - D_t(x, x+1) + G_t(x, x+1)$$
$$P_{t+1}(0) = B_t \quad - D_t(B, 0) \quad + G_t(B, 0)$$

$x = 0, 1, 2, \ldots.$

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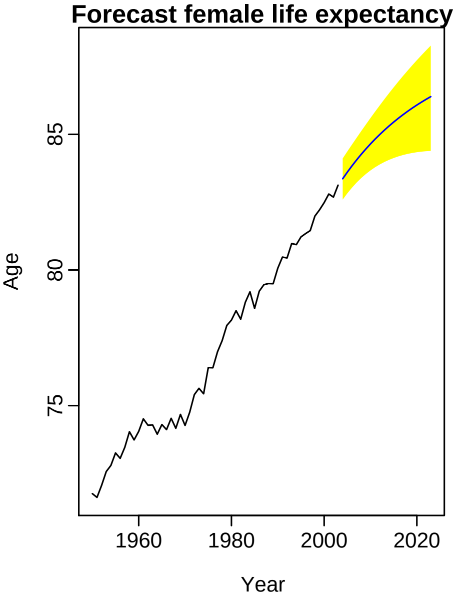
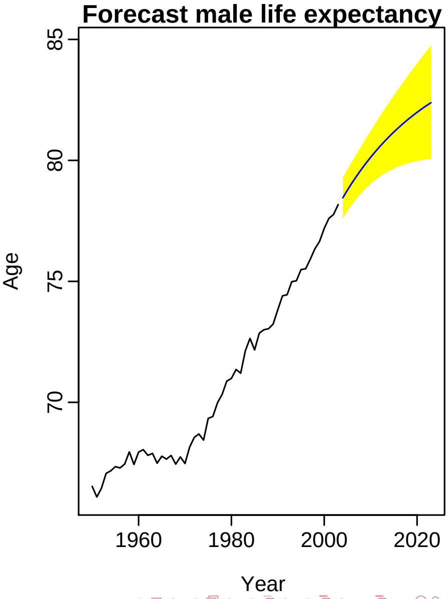
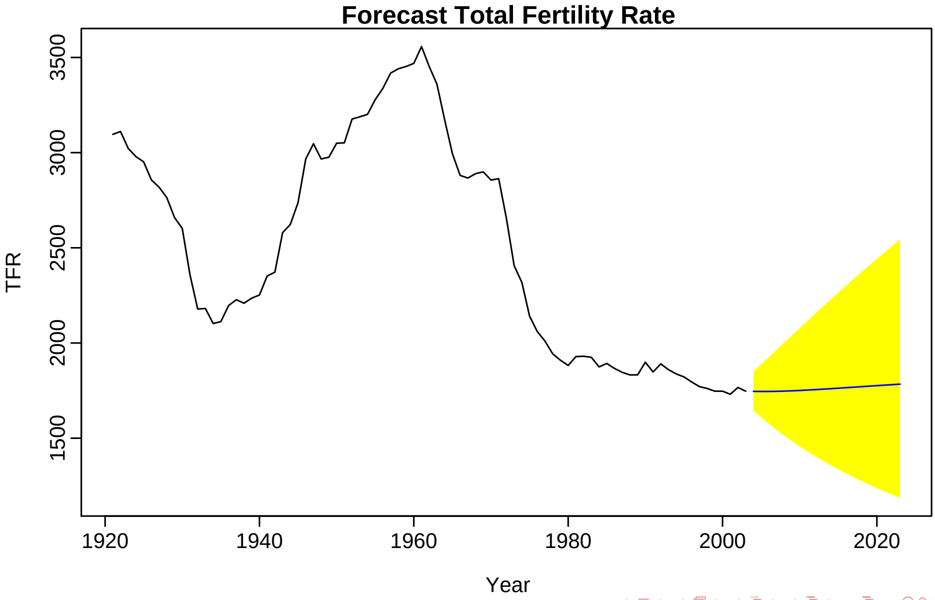
- This allows the computation of the empirical forecast distribution of any demographic quantity that is based on births, deaths and population numbers.

Forecasts of life expectancy at age 0

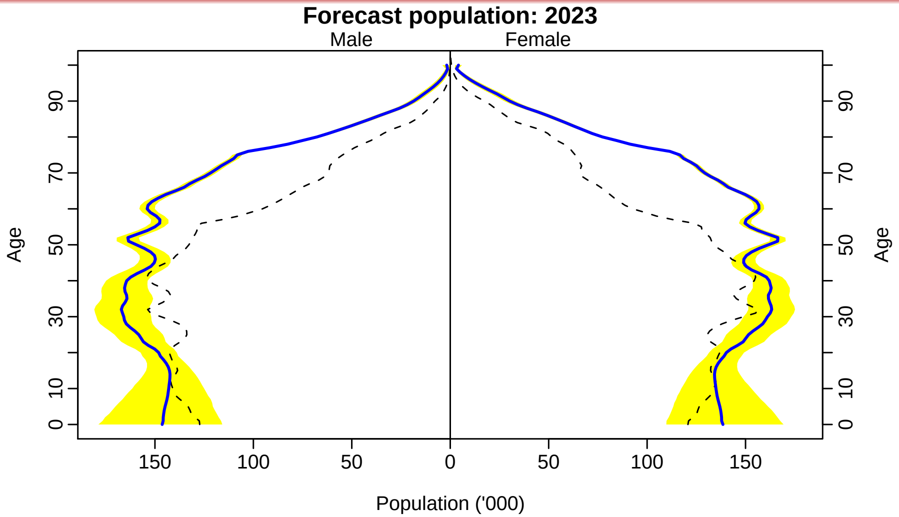




Forecasts of TFR

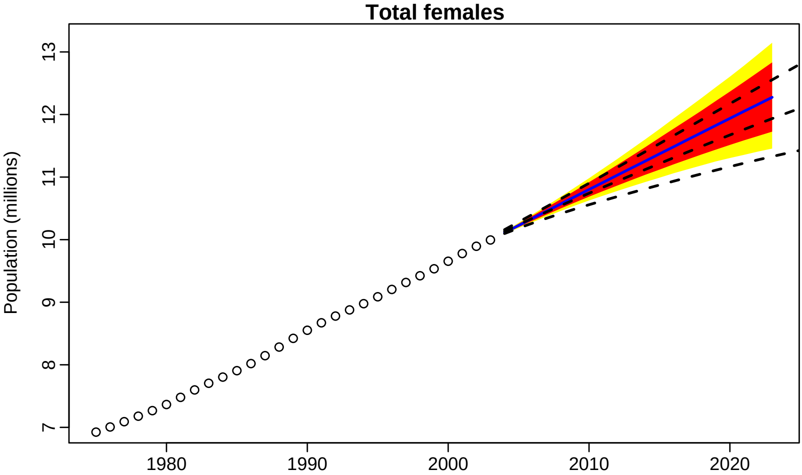


Population forecasts



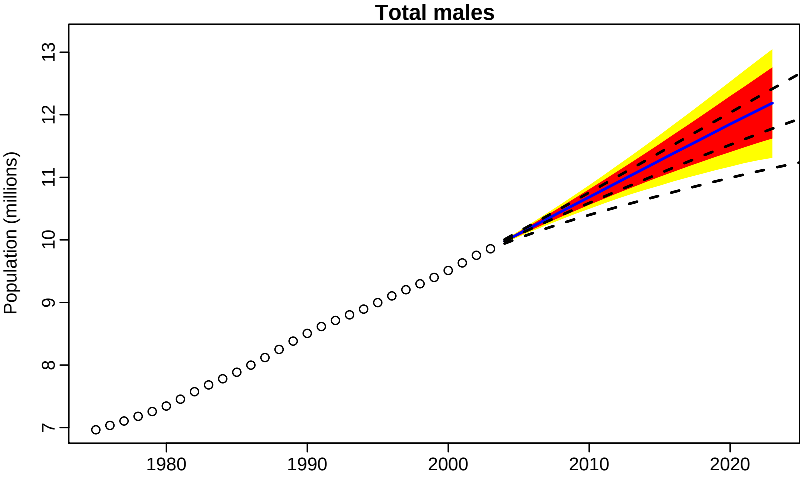
Forecast population pyramid for 2023, along with 80% prediction intervals. Dashed: actual population pyramid for 2003.

Population forecasts



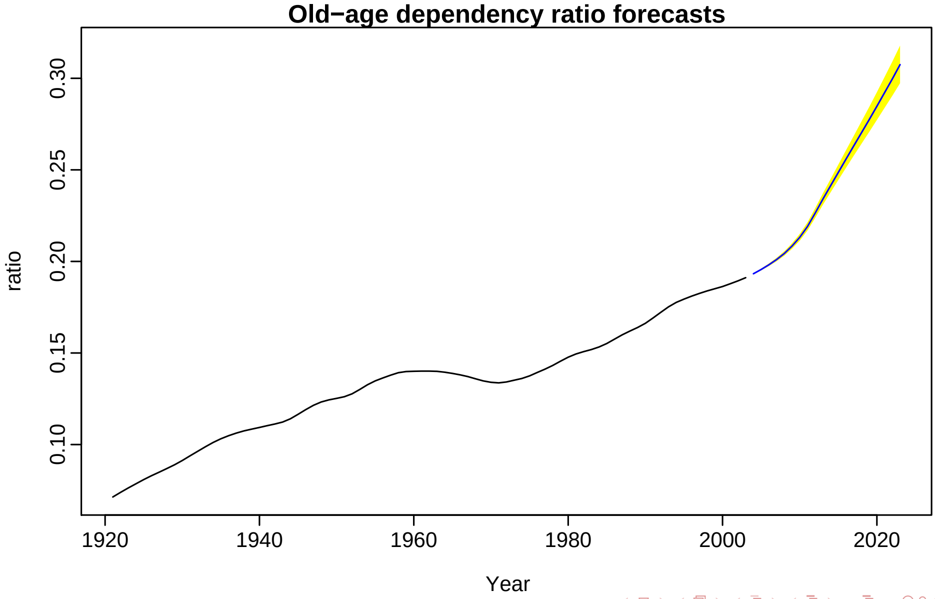
Twenty-year forecasts of total population along with 80% and 95% prediction intervals. Dashed lines show the ABS (2003) projections, series A, B and C.

Population forecasts



Twenty-year forecasts of total population along with 80% and 95% prediction intervals. Dashed lines show the ABS (2003) projections, series A, B and C.

Old-age dependency ratio



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- Stochastic age-specific cohort-component simulation provides a way of forecasting many demographic quantities with prediction intervals.
- No need to select combinations of assumed rates.
- True prediction intervals with specified coverage for population and all derived variables (TFR, life expectancy, old-age dependencies, etc.)

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Software and papers:

Hyndman and Booth (2006). Working paper: “Stochastic population forecasts using functional data models for mortality, fertility and migration”.

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