MONASH University

MONASH BUSINESS SCHOOL

Optimal Forecast Reconciliation

Rob J Hyndman

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Outline

Labour market participation

Australia and New Zealand Standard Classification of Occupations

- 8 major groups
  - 43 sub-major groups
    - 97 minor groups
      - 359 unit groups
        - 1023 occupations

Example: statistician

2 Professionals

22 Business, Human Resource and Marketing Professionals

224 Information and Organisation Professionals

2241 Actuaries, Mathematicians and Statisticians

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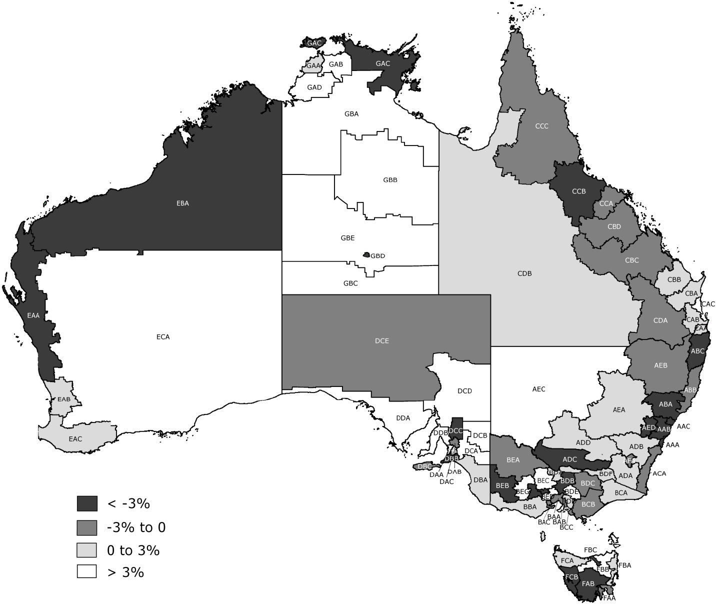
Australian tourism demand



Australian tourism demand

- Quarterly data on visitor night from 1998:Q1 – 2013:Q4
- From: *National Visitor Survey*, based on annual interviews of 120,000 Australians aged 15+, collected by Tourism Research Australia.
- Split by 7 states, 27 zones and 76 regions (a geographical hierarchy)
- Also split by purpose of travel
  - Holiday
  - Visiting friends and relatives (VFR)
  - Business
  - Other
- 304 bottom-level series

> 3%

3. PBS sales

3. PBS sales

ATC drug classification

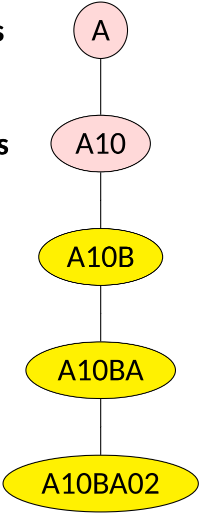
A	Alimentary tract and metabolism
B	Blood and blood forming organs
C	Cardiovascular system
D	Dermatologicals
G	Genito-urinary system and sex hormones
H	Systemic hormonal preparations, excluding sex hormones and insulins
J	Anti-infectives for systemic use
L	Antineoplastic and immunomodulating agents
M	Musculo-skeletal system
N	Nervous system
P	Antiparasitic products, insecticides and repellents
R	Respiratory system
S	Sensory organs
V	Various

3. PBS sales

ATC drug classification

14 classes

84 classes



Alimentary tract and metabolism

Drugs used in diabetes

Blood glucose lowering drugs

Biguanides

Metformin

Spectacle sales

- Monthly UK sales data from 2000 – 2014
- Provided by a large spectacle manufacturer
- Split by brand (26), gender (3), price range (6), materials (4), and stores (600)
- About 1 million bottom-level series

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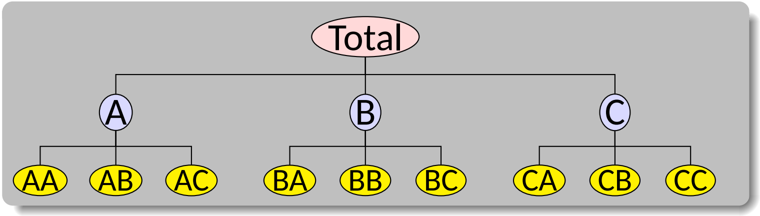

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- Labour turnover by occupation
- Pharmaceutical sales
- Tourism by state and region

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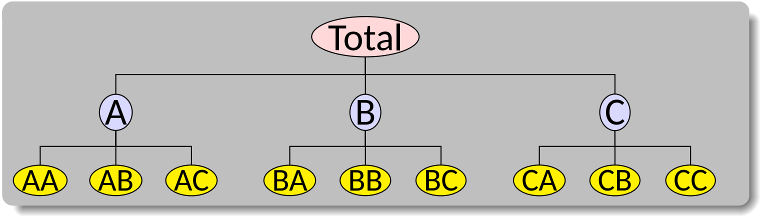

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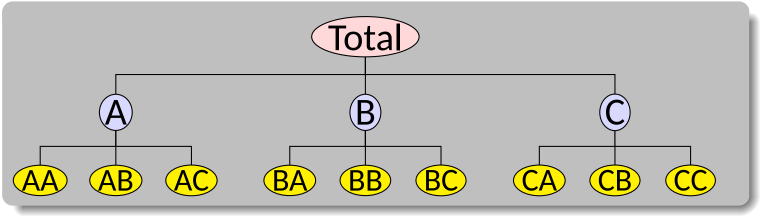

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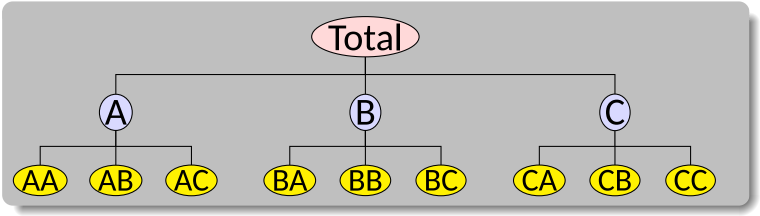

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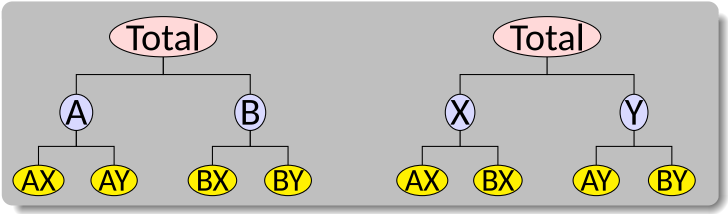

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- Labour turnover by occupation and state
- Spectacle sales by brand, gender, stores, etc.
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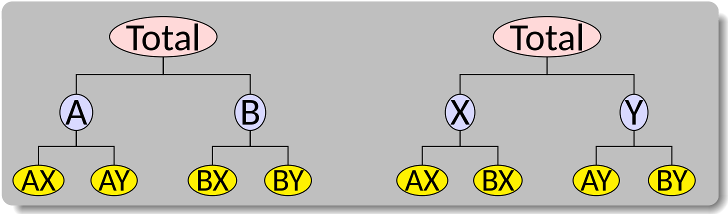

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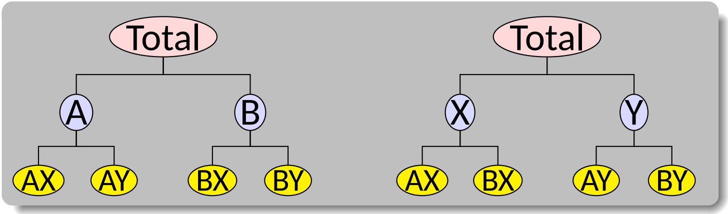

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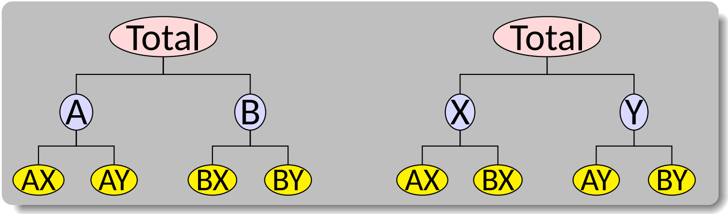

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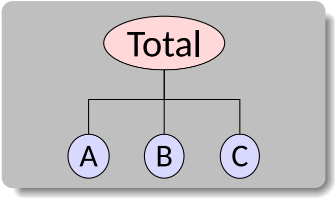

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$\boldsymbol{b}_t$: vector of all series at bottom level in time t.

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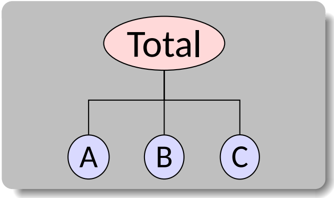

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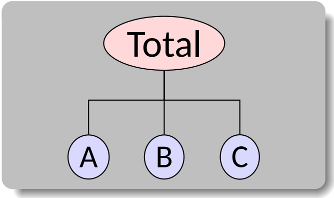

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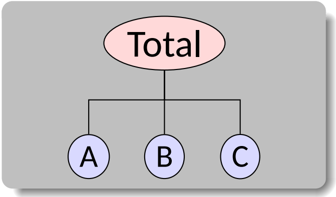

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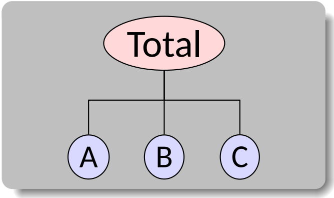

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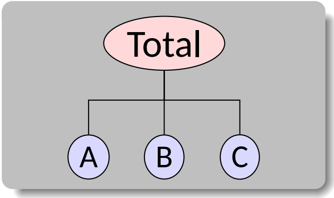

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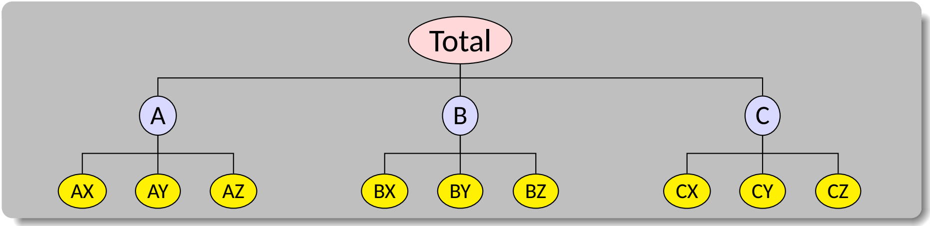

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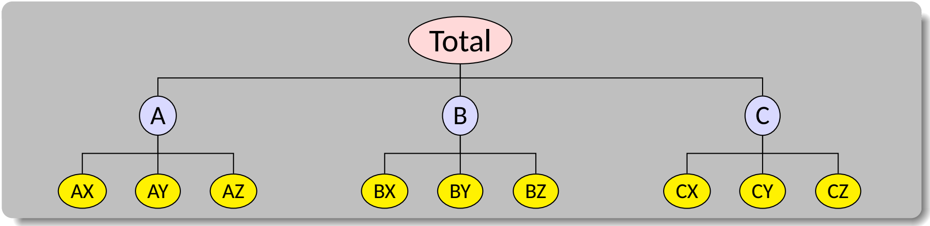

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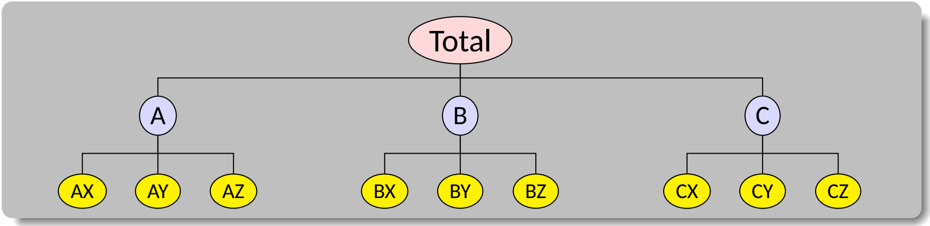

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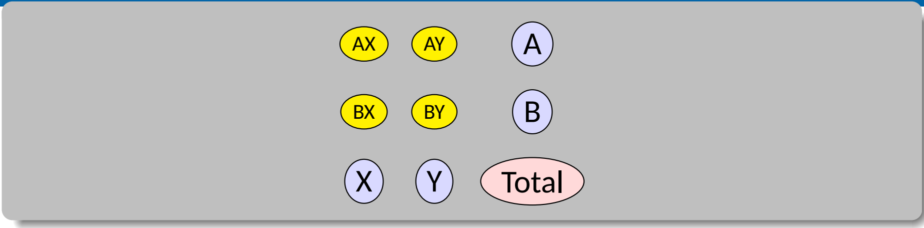

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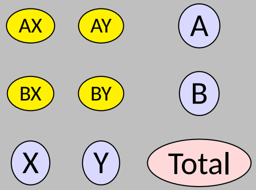

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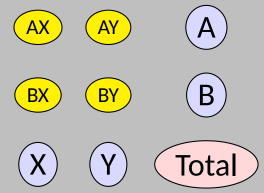

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Hierarchical and grouped time series

Every collection of time series with aggregation constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

where

- $\boldsymbol{y}_t$ is vector of all series at time t
- $\boldsymbol{b}_t$ is vector of the most disaggregated series at time t
- $\boldsymbol{S}$ is "summing matrix" containing the aggregation constraints.

Outline

1. Hierarchical and grouped time series
2. **Forecast reconciliation**
3. Fast computational tricks
4. Temporal hierarchies

Hierarchical/grouped time series

- Forecasts should be “coherent”, unbiased, minimum variance.
- Existing methods:
  - Bottom-up
  - Top-down
  - Middle-out
- How to compute forecast intervals?
- Most research is concerned about relative performance of existing methods.

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Advantages

- Works well in presence of low counts.
- Single forecasting model easy to build
- Provides reliable forecasts for aggregate levels.

Disadvantages

- Loss of information, especially individual series dynamics.
- Distribution of forecasts to lower levels can be difficult
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Bottom-up method

Advantages

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Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.
(In general, they will not "add up".)

Reconciled forecasts must be of the form:
$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$
for some matrix $\boldsymbol{P}$.

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Bottom-up forecasts

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

Bottom-up forecasts are obtained using

$$\boldsymbol{P} = [\mathbf{0} \mid \mathbf{I}] \,,$$

where $\mathbf{0}$ is null matrix and $\mathbf{I}$ is identity matrix.

- $\boldsymbol{P}$ matrix extracts only bottom-level forecasts from $\hat{\boldsymbol{y}}_n(h)$
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Top-down forecasts

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

Top-down forecasts are obtained using

$$\boldsymbol{P} = [\boldsymbol{p} \mid \boldsymbol{0}]$$

where $\boldsymbol{p} = [p_1, p_2, \dots, p_{m_K}]'$ is a vector of proportions that sum to one.

- $\boldsymbol{P}$ distributes forecasts of the aggregate to the lowest level series.
- Different methods of top-down forecasting lead to different proportionality vectors $\boldsymbol{p}$.

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General properties: bias

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

Assume: base forecasts $\hat{\boldsymbol{y}}_n(h)$ are unbiased:

$$\mathrm{E}[\hat{\boldsymbol{y}}_n(h)|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n] = \mathrm{E}[\boldsymbol{y}_{n+h}|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$$

- Let $\hat{\boldsymbol{b}}_n(h)$ be bottom level base forecasts with $\boldsymbol{\beta}_n(h) = \mathrm{E}[\hat{\boldsymbol{b}}_n(h)|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$.
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True for bottom-up, but *not* top-down or middle-out.

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General properties: variance

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

Let error variance of h-step base forecasts $\hat{\boldsymbol{y}}_n(h)$ be

$$\Sigma_h = \text{Var}[\boldsymbol{y}_{n+h} - \hat{\boldsymbol{y}}_n(h) \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

Then the error variance of the corresponding reconciled forecasts is

$$\text{Var}[\boldsymbol{y}_{n+h} - \tilde{\boldsymbol{y}}_n(h) \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{P}\Sigma_h\boldsymbol{P}'\boldsymbol{S}'$$

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This is a general result for all existing methods.

Optimal forecast reconciliation

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

Theorem: MinT Reconciliation

If $\boldsymbol{P}$ satisfies $\boldsymbol{S}\boldsymbol{P}\boldsymbol{S} = \boldsymbol{S}$, then

$$\min_{\boldsymbol{P}} = \text{trace}[\boldsymbol{S}\boldsymbol{P}\Sigma_h\boldsymbol{P}'\boldsymbol{S}']$$

has solution $\boldsymbol{P} = (\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}$.

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}(\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}\hat{\boldsymbol{y}}_n(h)$$

Reconciled forecasts

Base forecasts

- Assume that $\Sigma_h = k_h\Sigma_1$ to simplify computations.

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Solution 1: OLS

- Assume $\Sigma_1 \approx kI$.

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{y}}_n(h)$$

- Reconciliation does not depend on data
- Works surprisingly well.
- Still need to estimate covariance matrix to produce prediction intervals.

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- Approximate Σ_1 by its diagonal.
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Solution 3: GLS

- Estimate Σ_1 using shrinkage to the diagonal.
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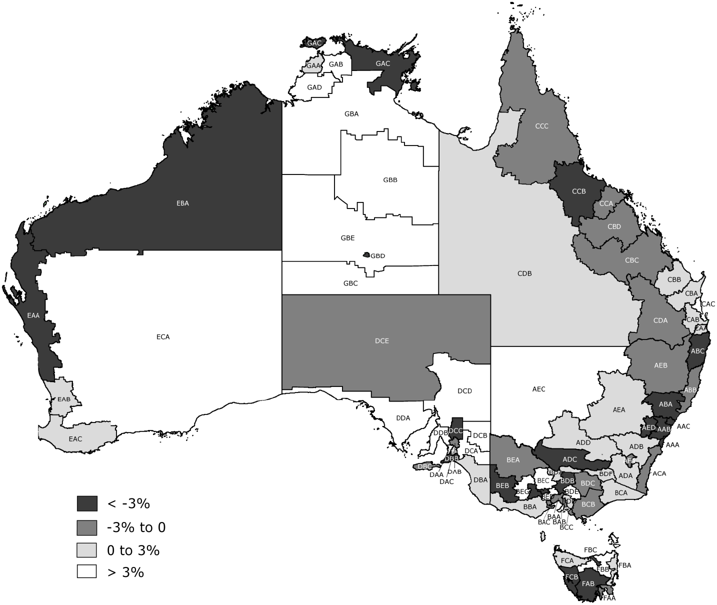
Australian tourism



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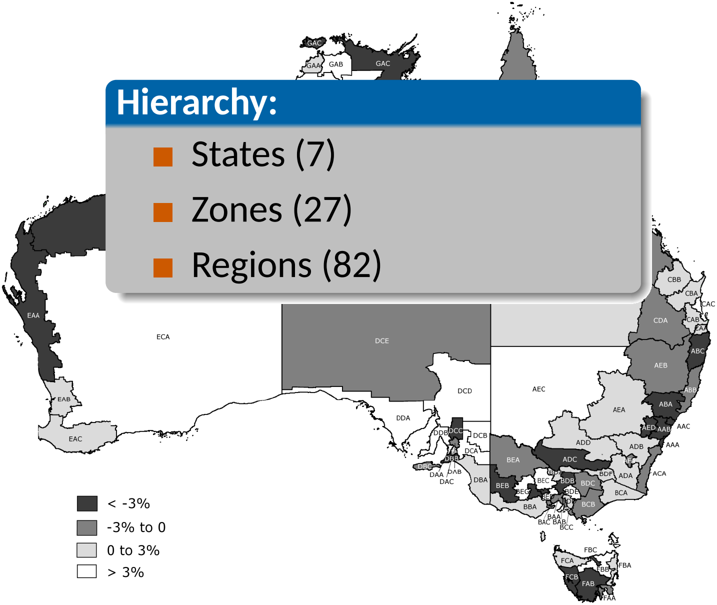
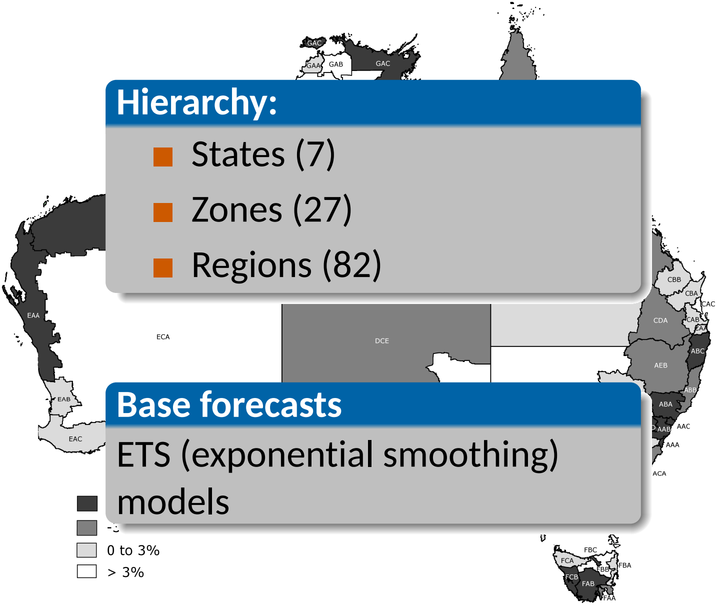


Hierarchy:

- States (7)
- Zones (27)
- Regions (82)

Australian tourism



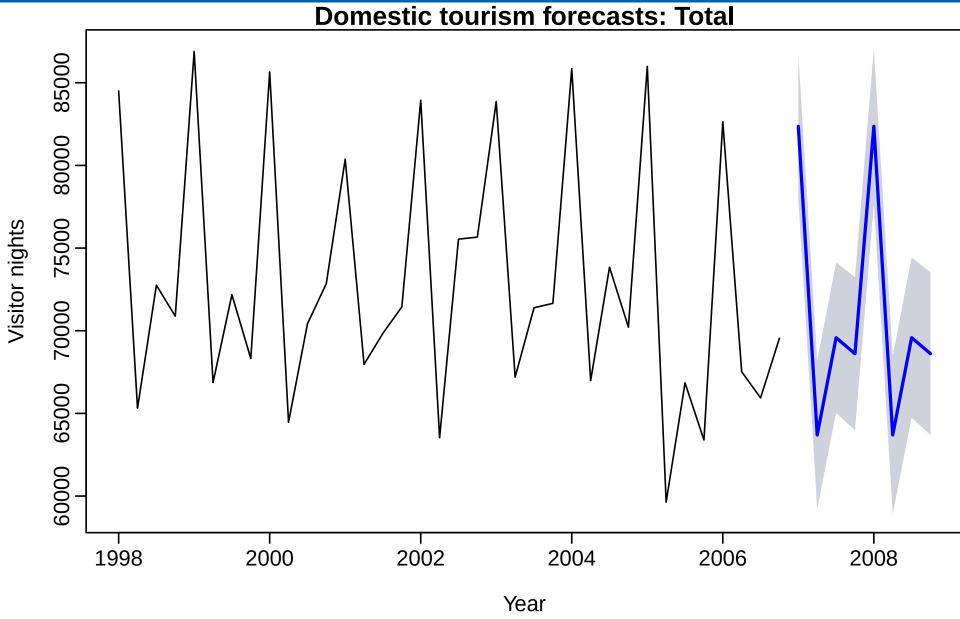
Hierarchy:

- States (7)
- Zones (27)
- Regions (82)

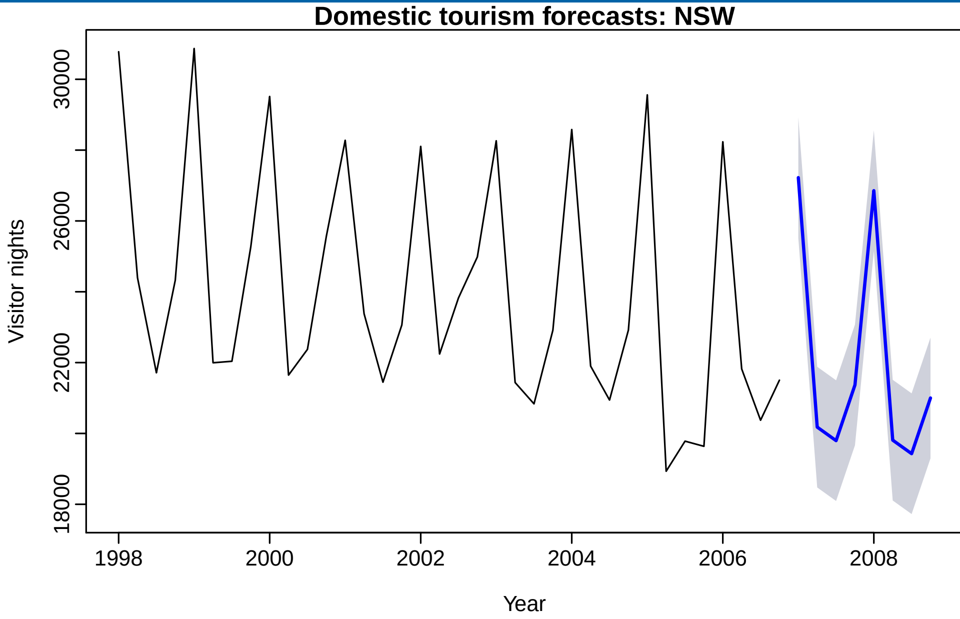
Base forecasts

ETS (exponential smoothing) models

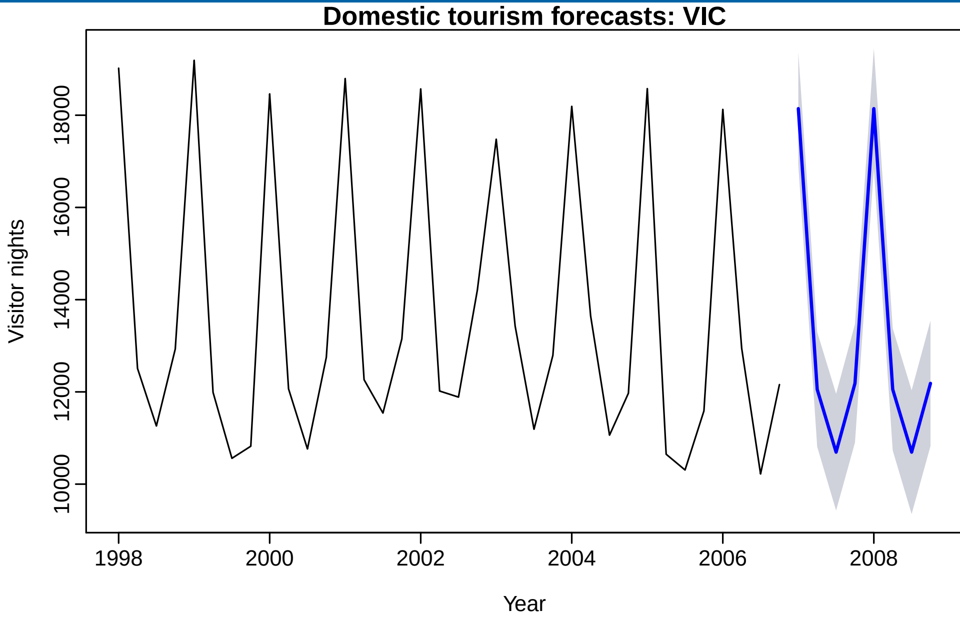
Base forecasts



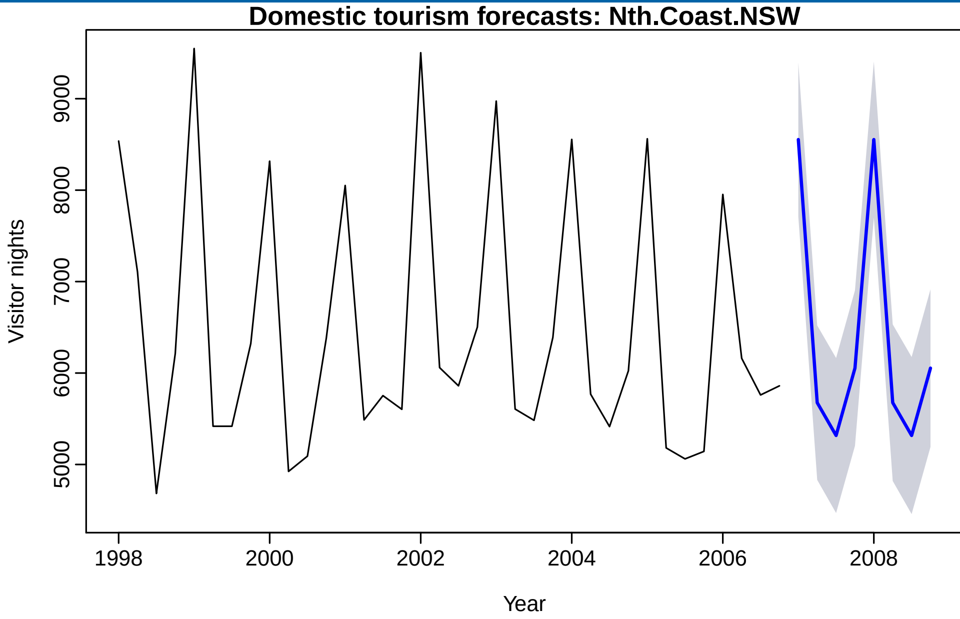
Base forecasts



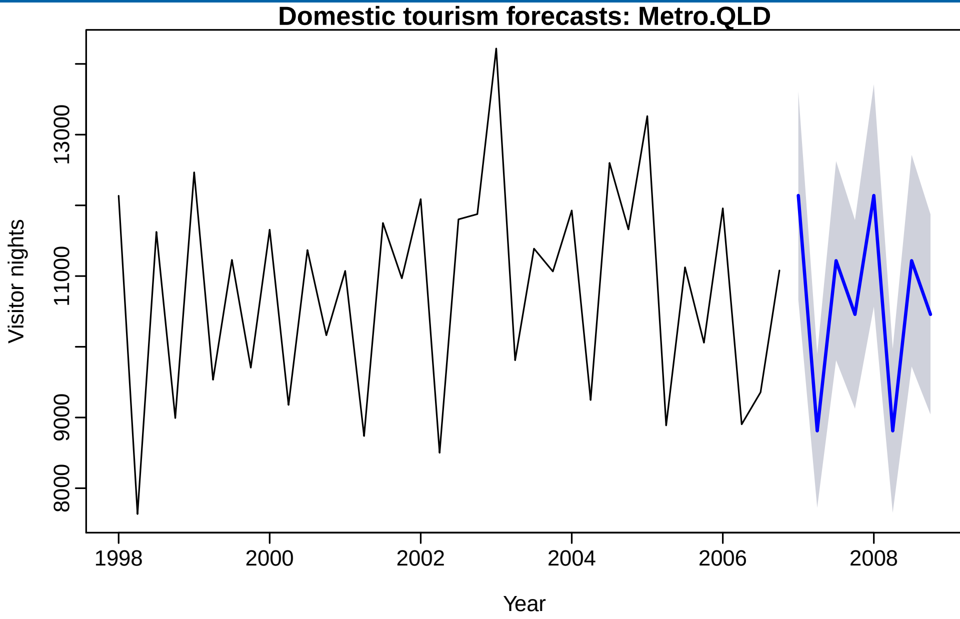
Base forecasts



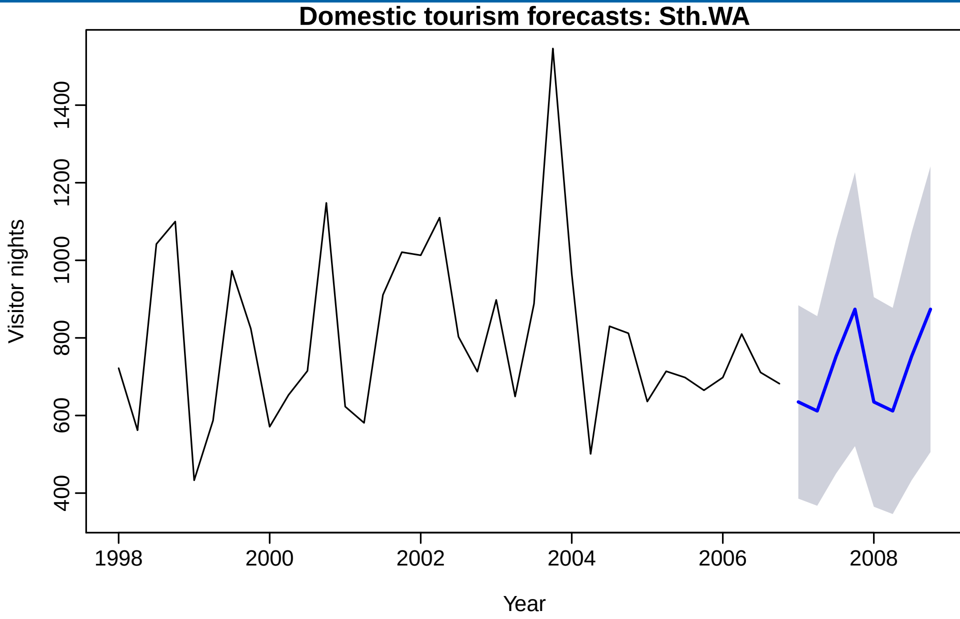
Base forecasts



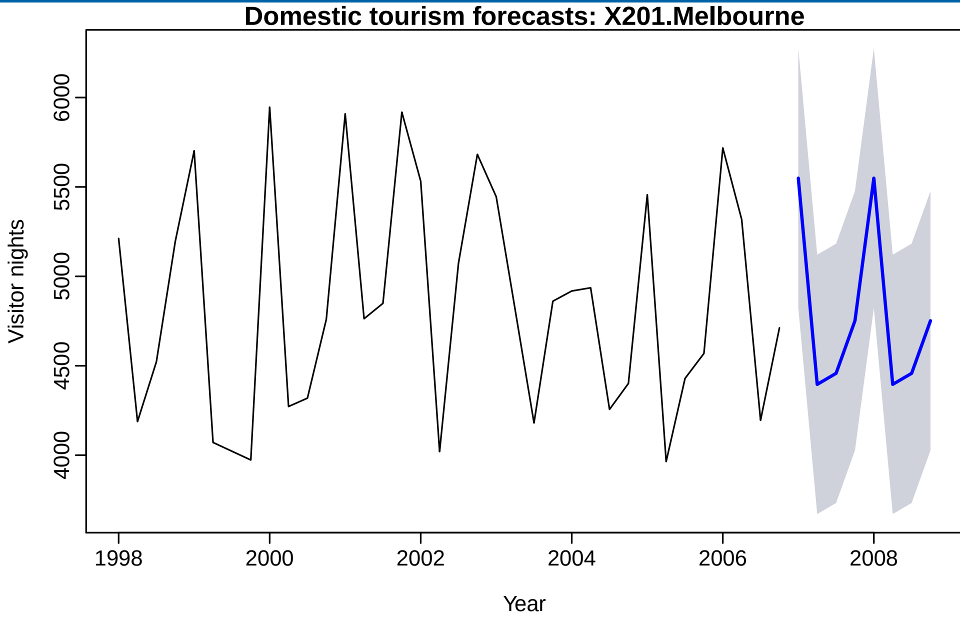
Base forecasts



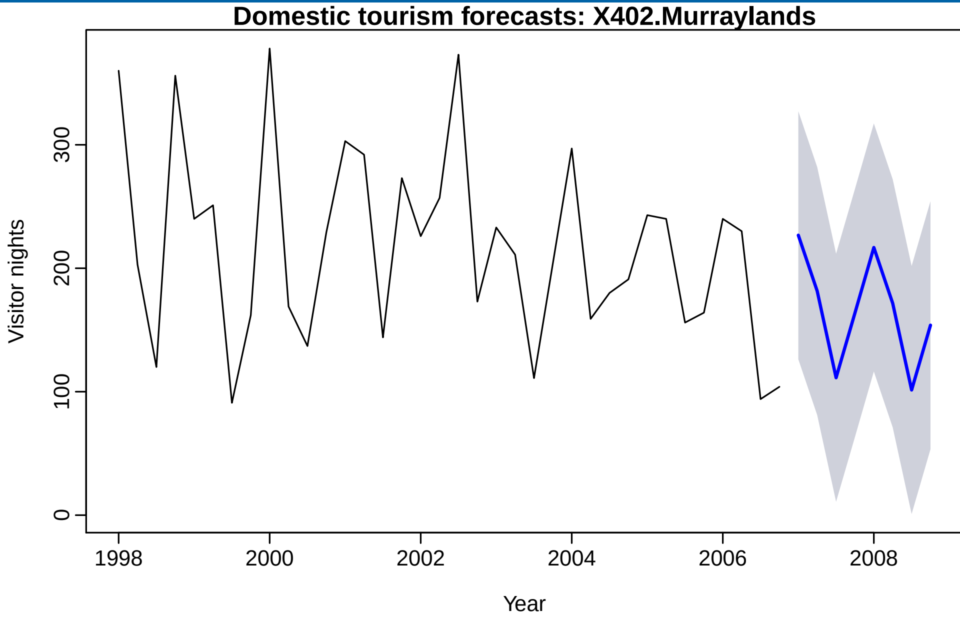
Base forecasts



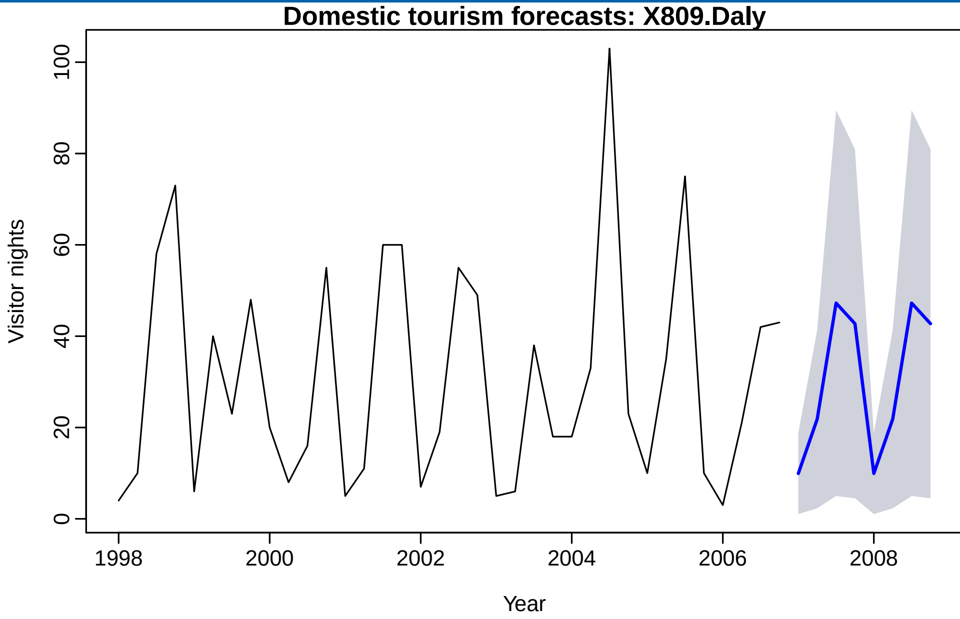
Base forecasts



Base forecasts



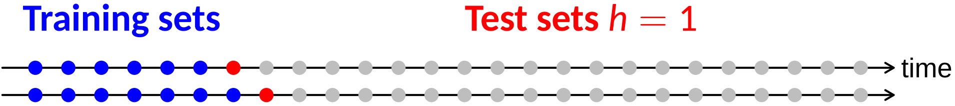
Base forecasts



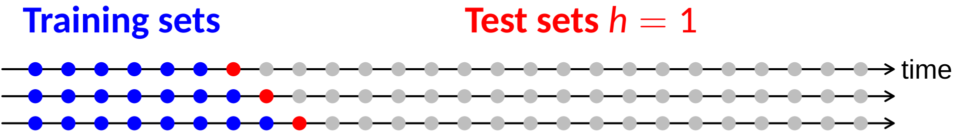
Forecast evaluation

Training sets **Test sets $h = 1$**

Forecast evaluation

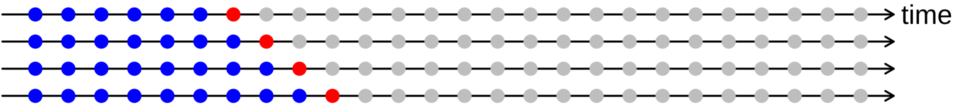


Forecast evaluation



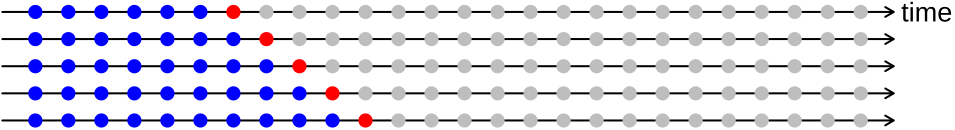
Forecast evaluation

Training sets **Test sets $h = 1$**

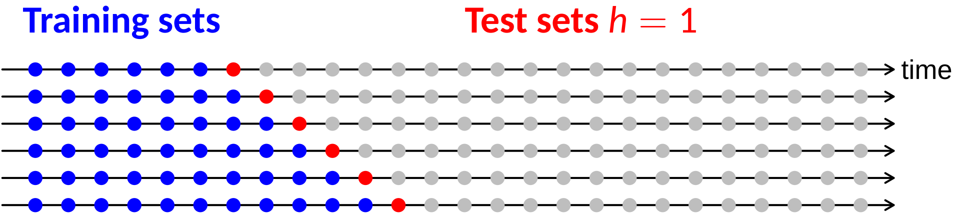


Forecast evaluation

Training sets **Test sets $h = 1$**

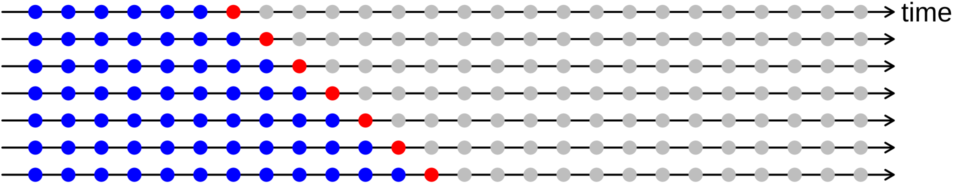


Forecast evaluation



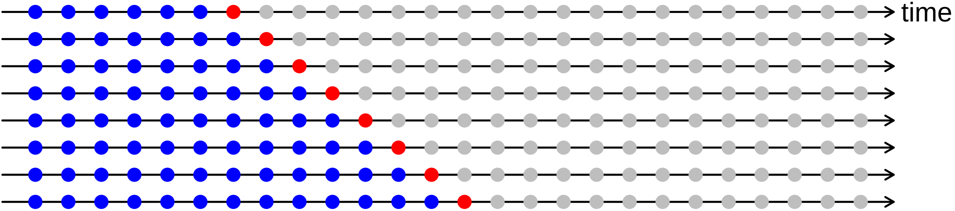
Forecast evaluation

Training sets **Test sets $h = 1$**



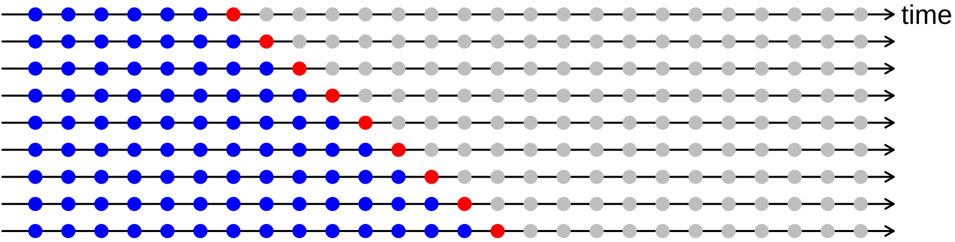
Forecast evaluation

Training sets **Test sets $h = 1$**

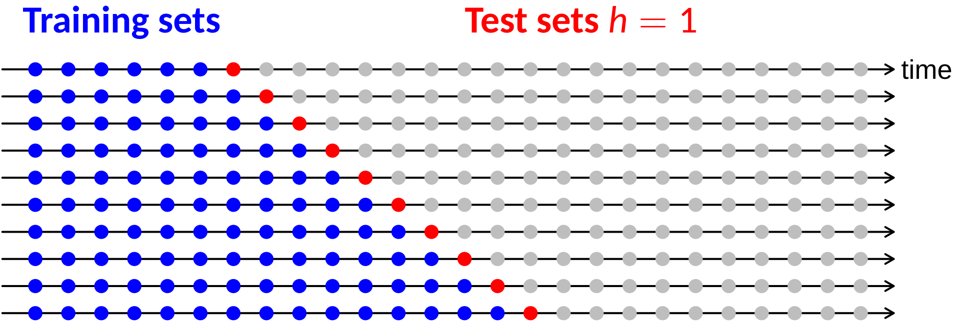


Forecast evaluation

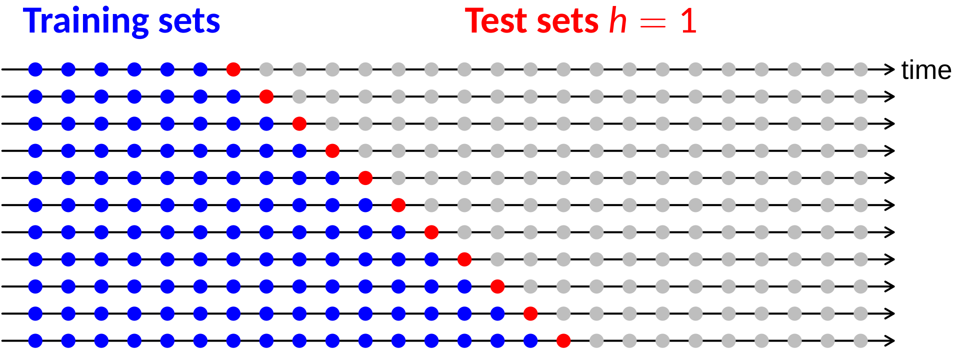
Training sets **Test sets $h = 1$**



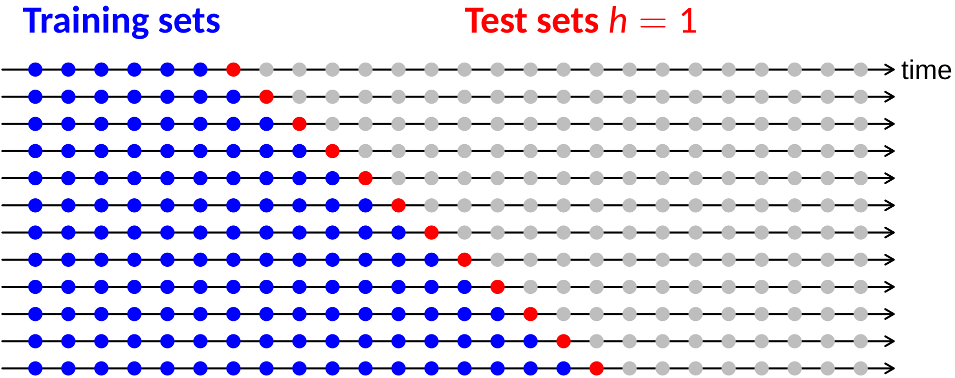
Forecast evaluation



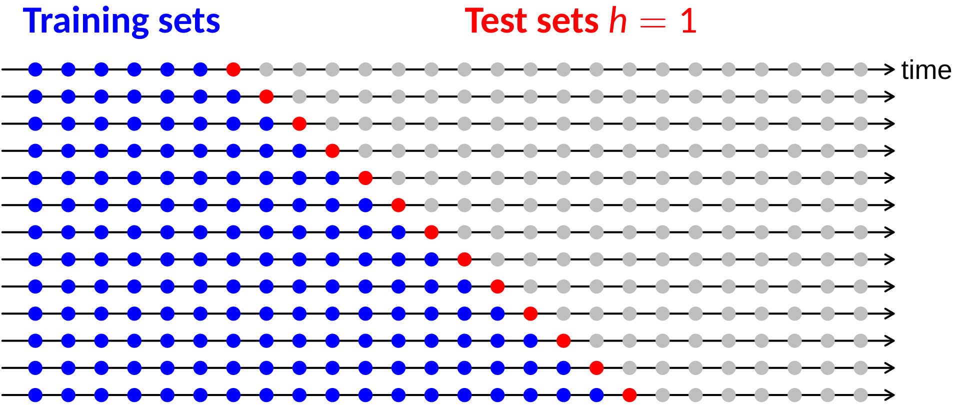
Forecast evaluation



Forecast evaluation

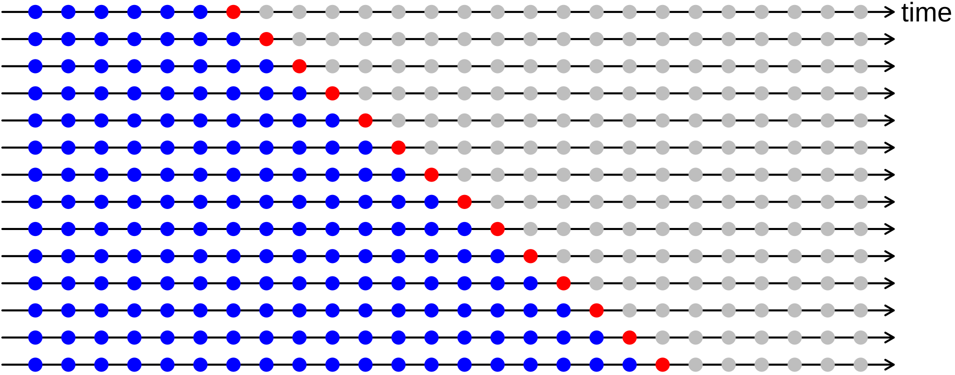


Forecast evaluation

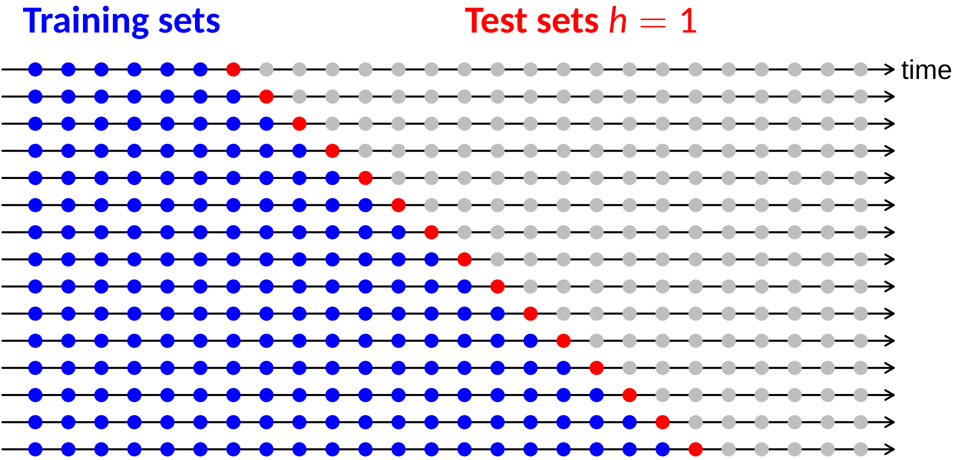


Forecast evaluation

Training sets **Test sets $h = 1$**

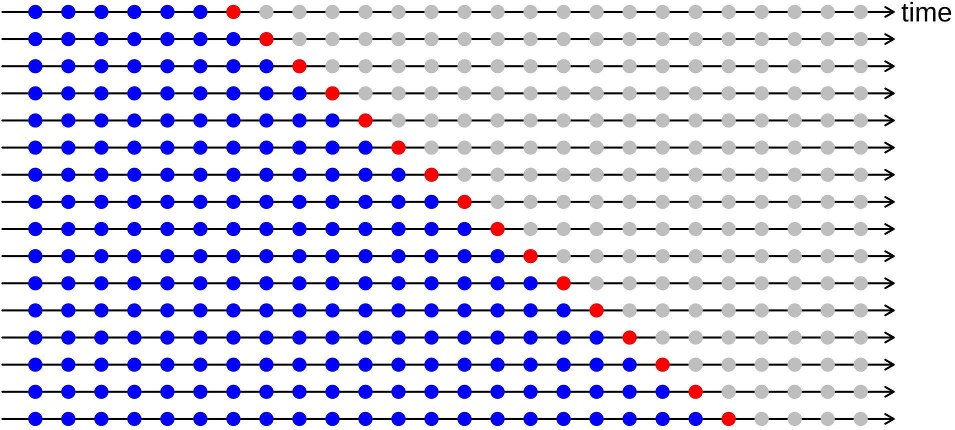


Forecast evaluation

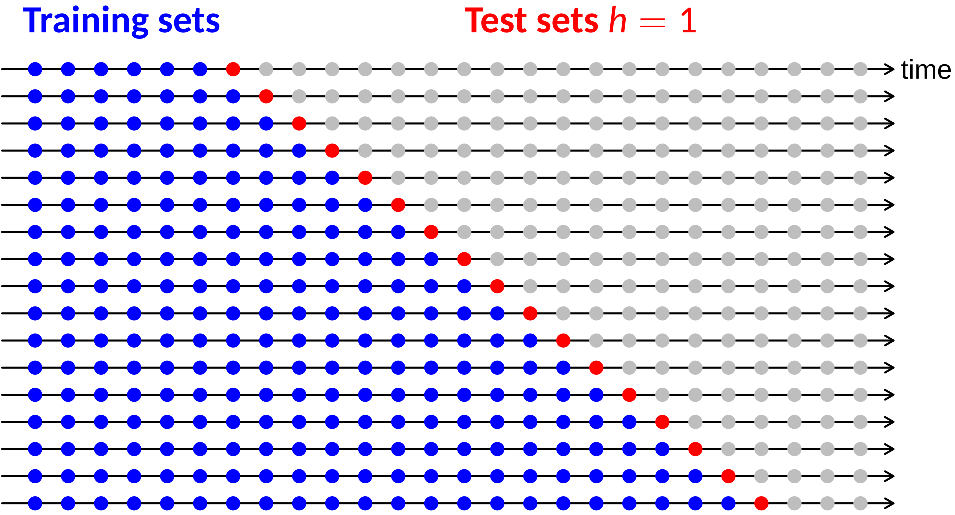


Forecast evaluation

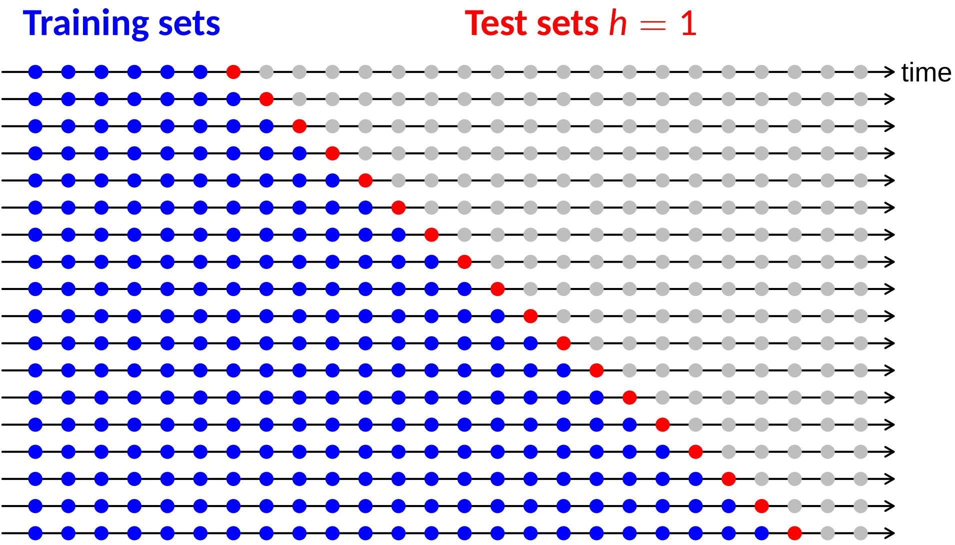
Training sets **Test sets $h = 1$**



Forecast evaluation

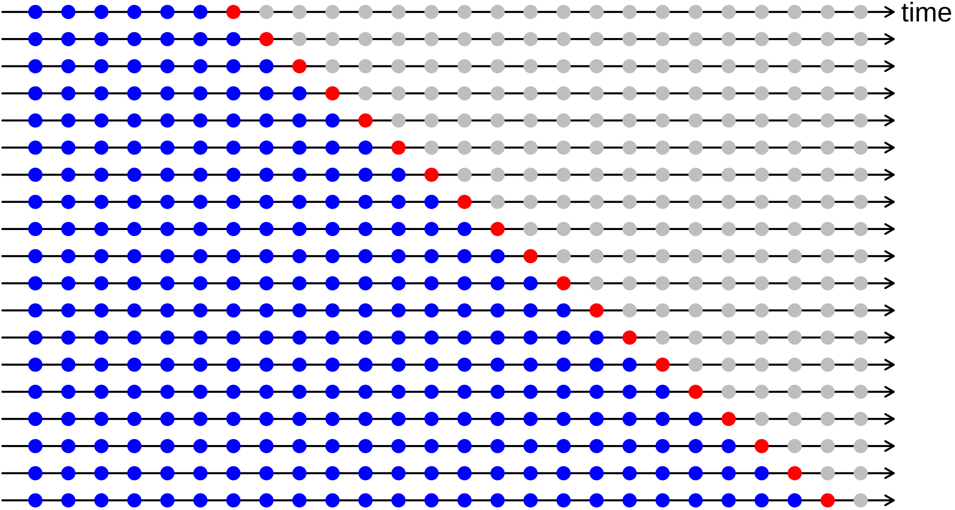


Forecast evaluation

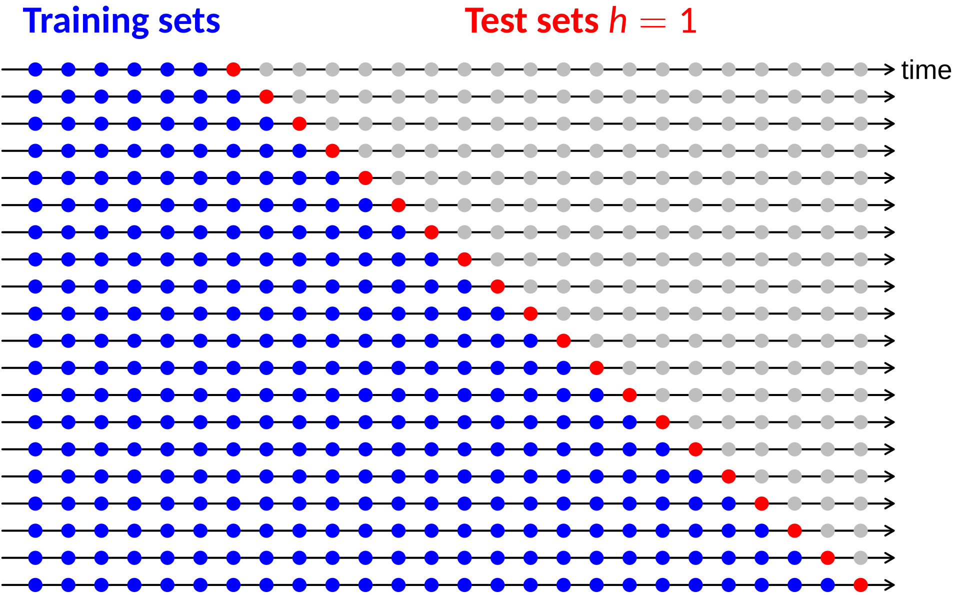


Forecast evaluation

Training sets **Test sets $h = 1$**

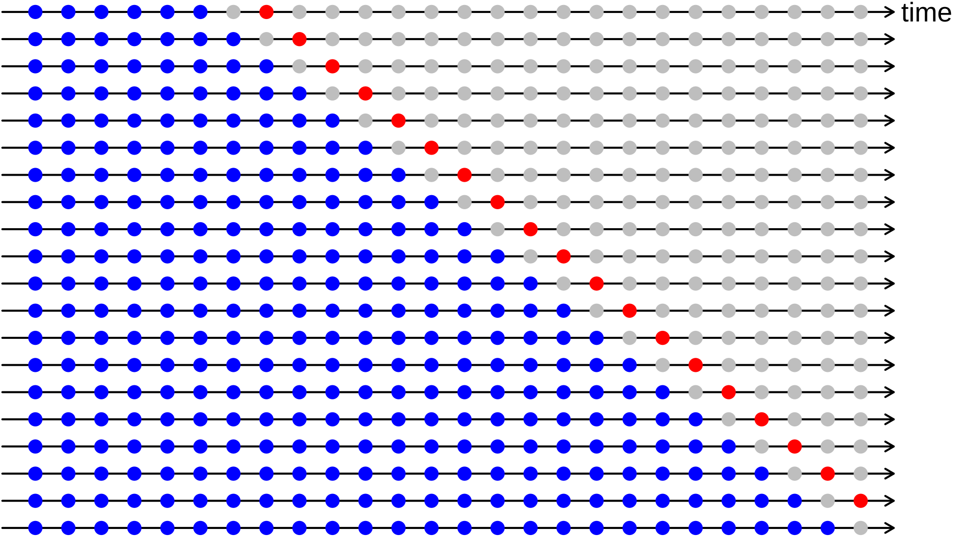


Forecast evaluation

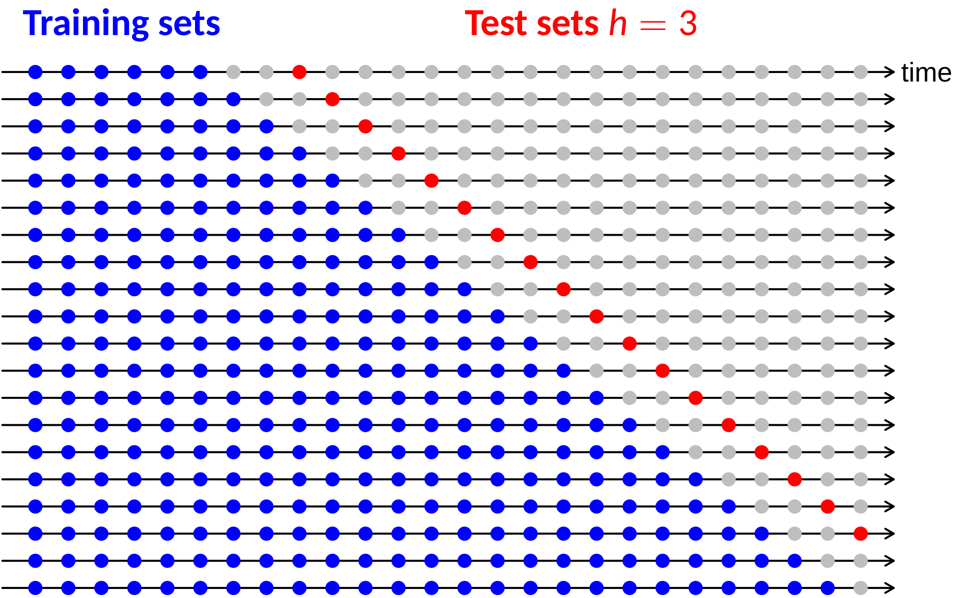


Forecast evaluation

Training sets **Test sets $h = 2$**

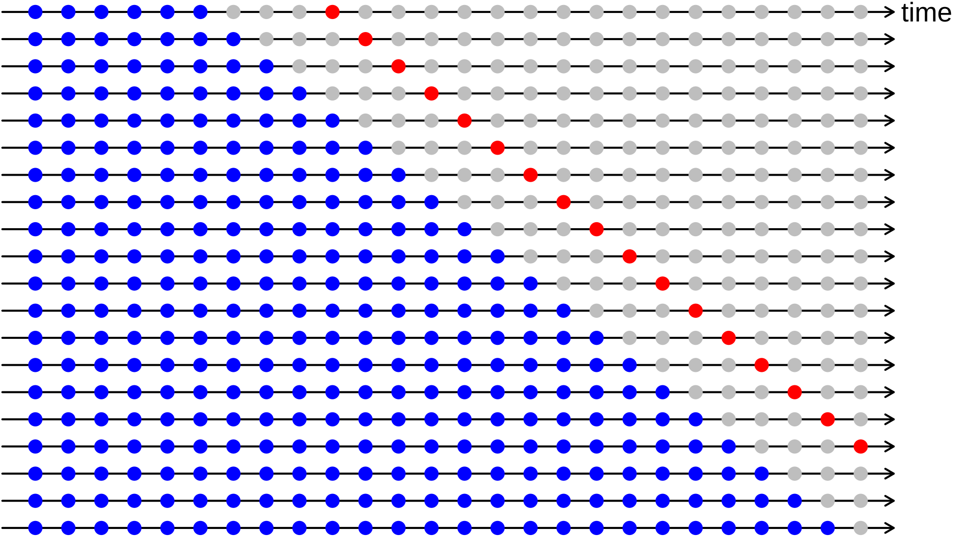


Forecast evaluation

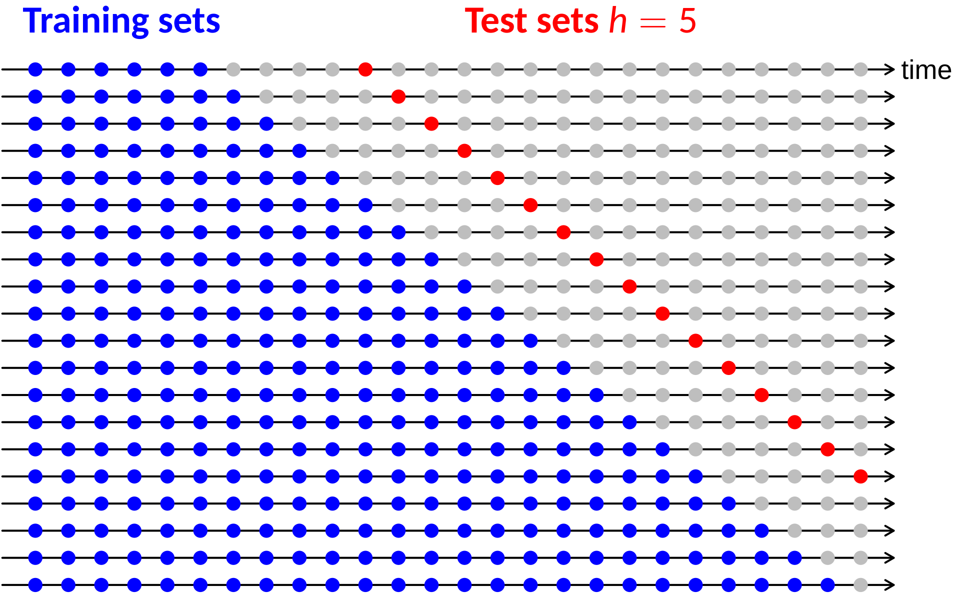


Forecast evaluation

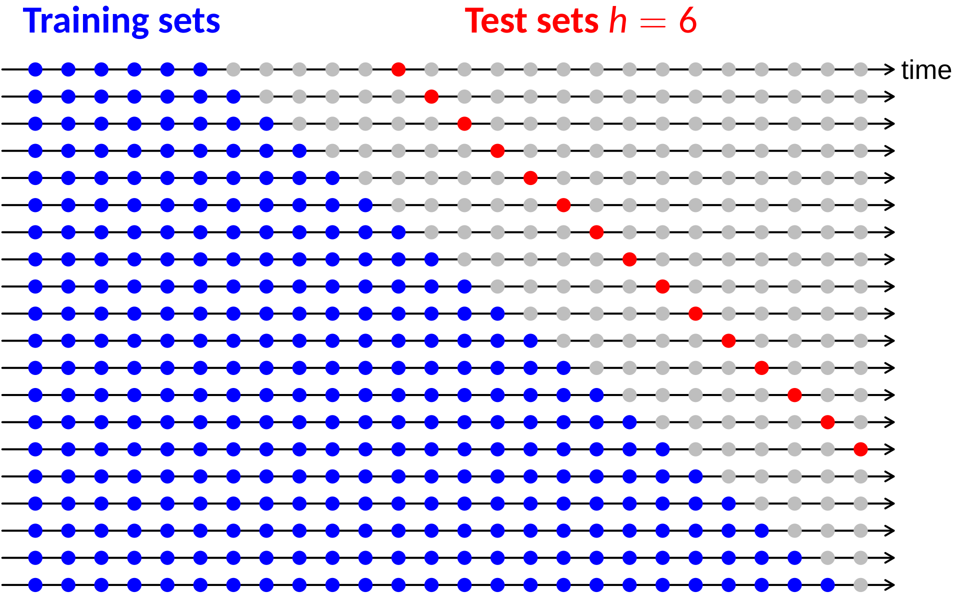
Training sets **Test sets $h = 4$**



Forecast evaluation



Forecast evaluation

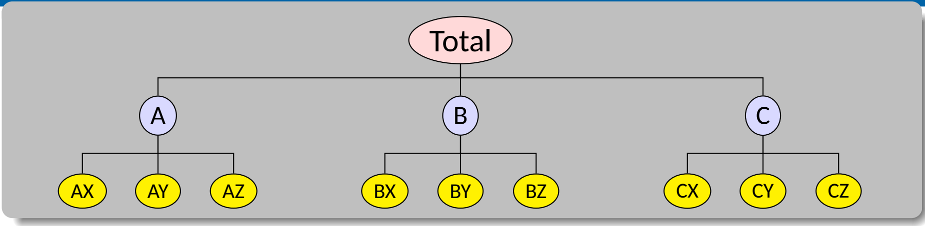


Hierarchy: states, zones, regions

RMSE	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	Ave
				Forecast horizon			
				Australia			
Base	1762.04	1770.29	1766.02	1818.82	1705.35	1721.17	**1757.28**
Bottom	1736.92	1742.69	1722.79	1752.74	1666.73	1687.43	**1718.22**
OLS	1747.60	1757.68	1751.77	1800.67	1686.00	1706.45	**1741.69**
WLS	1705.21	1715.87	1703.75	1729.56	1627.79	1661.24	**1690.57**
GLS	1704.64	1715.60	1705.31	1729.04	1626.36	1661.64	**1690.43**
				States			
Base	399.77	404.16	401.92	407.26	395.38	401.17	**401.61**
Bottom	404.29	406.95	404.96	409.02	399.80	401.55	**404.43**
OLS	404.47	407.62	405.43	413.79	401.10	404.90	**406.22**
WLS	398.84	402.12	400.71	405.03	394.76	398.23	**399.95**
GLS	398.84	402.16	400.86	405.03	394.59	398.22	**399.95**
				Regions			
Base	93.15	93.38	93.45	93.79	93.50	93.56	**93.47**
Bottom	93.15	93.38	93.45	93.79	93.50	93.56	**93.47**
OLS	93.28	93.53	93.64	94.17	93.78	93.88	**93.71**
WLS	93.02	93.32	93.38	93.72	93.39	93.53	**93.39**
GLS	92.98	93.27	93.34	93.66	93.34	93.46	**93.34**

Outline

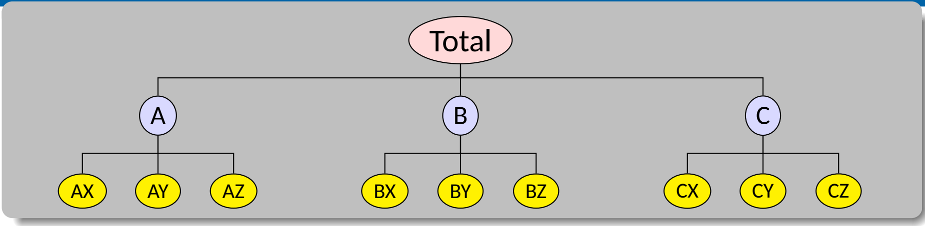
Fast computation: hierarchical data



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

$$\boxed{\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t}$$

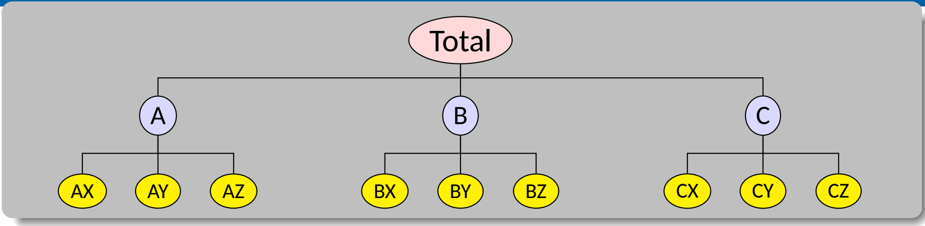
Fast computation: hierarchical data



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{B,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{C,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

$$\boxed{\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t}$$

Fast computation: hierarchical data

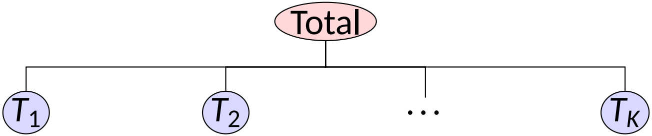


$$
\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{B,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{C,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}}_{\boldsymbol{b}_t}
$$

$$\boxed{\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t}$$

Fast computation: hierarchies

Think of the hierarchy as a tree of trees:



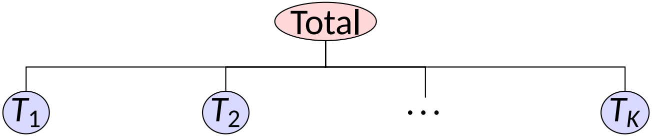
Then the summing matrix contains k smaller summing matrices:

$$S = \begin{bmatrix} \mathbf{1}'_{n_1} & \mathbf{1}'_{n_2} & \cdots & \mathbf{1}'_{n_K} \\ S_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & S_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & S_K \end{bmatrix}$$

where $\mathbf{1}_n$ is an n-vector of ones and tree T_i has n_i terminal nodes.

Fast computation: hierarchies

Think of the hierarchy as a tree of trees:



Then the summing matrix contains k smaller summing matrices:

$$\boldsymbol{S} = \begin{bmatrix} \mathbf{1}'_{n_1} & \mathbf{1}'_{n_2} & \cdots & \mathbf{1}'_{n_K} \\ \boldsymbol{S}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{S}_K \end{bmatrix}$$

where $\mathbf{1}_n$ is an n-vector of ones and tree T_i has n_i terminal nodes.

Fast computation: hierarchies

$$\boldsymbol{S}'\Lambda\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1'\Lambda_1\boldsymbol{S}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S}_2'\Lambda_2\boldsymbol{S}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{S}_K'\Lambda_K\boldsymbol{S}_K \end{bmatrix} + \lambda_0 \boldsymbol{J}_n$$

- λ_0 is the top left element of Λ;
- Λ_k is a block of Λ, corresponding to tree T_k;
- $\boldsymbol{J}_n$ is a matrix of ones;
- $n = \sum_k n_k$.

Now apply the Sherman-Morrison formula …

Fast computation: hierarchies

$$\boldsymbol{S}'\Lambda\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1'\Lambda_1\boldsymbol{S}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S}_2'\Lambda_2\boldsymbol{S}_2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{S}_K'\Lambda_K\boldsymbol{S}_K \end{bmatrix} + \lambda_0 \boldsymbol{J}_n$$

- λ_0 is the top left element of Λ;
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- $\boldsymbol{J}_n$ is a matrix of ones;
- $n = \sum_k n_k$.

Now apply the Sherman-Morrison formula …

Fast computation: hierarchies

$$(\boldsymbol{S}'\Lambda\boldsymbol{S})^{-1} = \begin{bmatrix} (\boldsymbol{S}_1'\Lambda_1\boldsymbol{S}_1)^{-1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{S}_2'\Lambda_2\boldsymbol{S}_2)^{-1} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & (\boldsymbol{S}_K'\Lambda_K\boldsymbol{S}_K)^{-1} \end{bmatrix} - c\boldsymbol{S}_0$$

- $\boldsymbol{S}_0$ can be partitioned into K^2 blocks, with the (k,ℓ) block (of dimension $n_k \times n_\ell$) being

$$(\boldsymbol{S}_k'\Lambda_k\boldsymbol{S}_k)^{-1}\boldsymbol{J}_{n_k,n_\ell}(\boldsymbol{S}_\ell'\Lambda_\ell\boldsymbol{S}_\ell)^{-1}$$

- $\boldsymbol{J}_{n_k,n_\ell}$ is a $n_k \times n_\ell$ matrix of ones.
- $c^{-1} = \lambda_0^{-1} + \sum_k \boldsymbol{1}_{n_k}'(\boldsymbol{S}_k'\Lambda_k\boldsymbol{S}_k)^{-1}\boldsymbol{1}_{n_k}$.
- Each $\boldsymbol{S}_k'\Lambda_k\boldsymbol{S}_k$ can be inverted similarly.
- $\boldsymbol{S}'\Lambda\boldsymbol{y}$ can also be computed recursively.

Fast computation: hierarchies

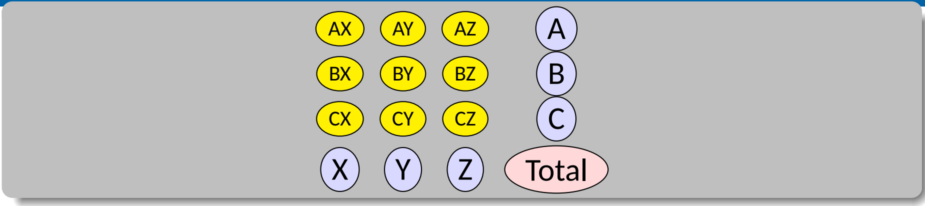
$$(S'\Lambda S)^{-1} = \begin{bmatrix} (S_1'\Lambda_1 S_1)^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (S_2'\Lambda_2 S_2)^{-1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & (S_K'\Lambda_K S_K)^{-1} \end{bmatrix} - cS_0$$

- S_0 can be partitioned into K^2 blocks, with the (k,ℓ) block (of dimension $n_k \times n_\ell$) being

The recursive calculations can be done in such a way that we never store any of the large matrices involved.

- J_{n_k,n_ℓ}
- $c^{-1} = \lambda_0 + \sum_k \ldots 1_{n_k} \ldots 1_{n_k}.$
- Each $S_k'\Lambda_k S_k$ can be inverted similarly.
- $S'\Lambda y$ can also be computed recursively.

Fast computation: grouped data



$$
\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{X,t} \\ y_{Y,t} \\ y_{Z,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}}_{\boldsymbol{b}_t}
$$

$$\boxed{\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t}$$

Fast computation: grouped data

$$S = \begin{bmatrix} \mathbf{1}'_m \otimes \mathbf{1}'_n \\ \mathbf{1}'_m \otimes \mathbf{I}_n \\ \mathbf{I}_m \otimes \mathbf{1}'_n \\ \mathbf{I}_m \otimes \mathbf{I}_n \end{bmatrix}$$

$m =$ number of rows
$n =$ number of columns

$$S'\Lambda S = \lambda_{00} J_{mn} + (\Lambda_R \otimes J_n) + (J_m \otimes \Lambda_C) + \Lambda_U$$

- Λ_R, Λ_C and Λ_U are diagonal matrices corresponding to rows, columns and unaggregated series;
- λ_{00} corresponds to aggregate.

Fast computation: grouped data

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- Λ_R, Λ_C and Λ_U are diagonal matrices corresponding to rows, columns and unaggregated series;
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Fast computation: grouped data

$$(\boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S})^{-1} = \boldsymbol{A} - \frac{\boldsymbol{A}\boldsymbol{1}_{mn}\boldsymbol{1}_{mn}'\boldsymbol{A}}{1/\lambda_{00} + \boldsymbol{1}_{mn}'\boldsymbol{A}\boldsymbol{1}_{mn}}$$

$\boldsymbol{A} = \boldsymbol{\Lambda}_U^{-1} - \boldsymbol{\Lambda}_U^{-1}(\boldsymbol{J}_m \otimes \boldsymbol{D})\boldsymbol{\Lambda}_U^{-1} - \boldsymbol{E}\boldsymbol{M}^{-1}\boldsymbol{E}'$.

$\boldsymbol{D}$ is diagonal with elements $d_j = \lambda_{0j}/(1 + \lambda_{0j}\sum_i \lambda_{ij}^{-1})$.

$\boldsymbol{E}$ has $m \times m$ blocks where $\boldsymbol{e}_{ij}$ has kth element

$$(\boldsymbol{e}_{ij})_k = \begin{cases} \lambda_{i0}^{1/2}\lambda_{ik}^{-1} - \lambda_{i0}^{1/2}\lambda_{ik}^{-2}d_k, & i = j, \\ -\lambda_{j0}^{1/2}\lambda_{ik}^{-1}\lambda_{jk}^{-1}d_k, & i \neq j. \end{cases}$$

$\boldsymbol{M}$ is $m \times m$ with (i, j) element

$$(\boldsymbol{M})_{ij} = \begin{cases} 1 + \lambda_{i0}\sum_k \lambda_{ik}^{-1} - \lambda_{i0}\sum_k \lambda_{ik}^{-2}d_k, & i = j, \\ -\lambda_{i0}^{1/2}\lambda_{j0}^{1/2}\sum_k \lambda_{ik}^{-1}\lambda_{jk}^{-1}d_k, & i \neq j. \end{cases}$$

Fast computation: grouped data

$$(S\Lambda S)^{-1} = A - \frac{A\mathbf{1}_{mn}\mathbf{1}'_{mn}A}{1/\lambda_{00} + \mathbf{1}'_{mn}A\mathbf{1}_{mn}}$$

$A = \Lambda_U^{-1} - \Lambda_U^{-1}(J_m \otimes D)\Lambda_U^{-1} - EM^{-1}E'$.
D is diagonal with elements $d_j = \lambda_{0j}/(1 + \lambda_{0j}\sum_i \lambda_{ij}^{-1})$.
E has $m \times m$ blocks where e_{ij} has kth element

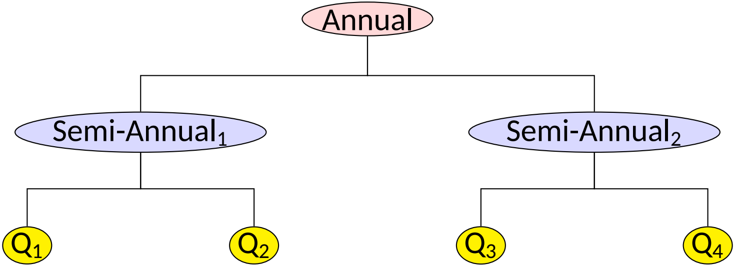
Again, the calculations can be done in such a way that we never store any of the large matrices involved.

M is $m \times$

$$(M)_{ij} = \begin{cases} 1 + \lambda_{i0}\sum_k \lambda_{ik}^{-1} - \lambda_{i0}\sum_k \lambda_{ik}^{-2}d_k, & i = j, \\ -\lambda_{i0}^{1/2}\lambda_{j0}^{1/2}\sum_k \lambda_{ik}^{-1}\lambda_{jk}^{-1}d_k, & i \neq j. \end{cases}$$

Outline

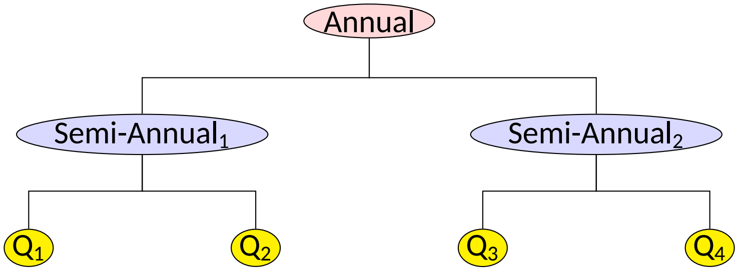
Temporal hierarchies



Basic idea:

- Forecast series at each available frequency.
- Optimally reconcile forecasts within the same year.

Temporal hierarchies



Basic idea:

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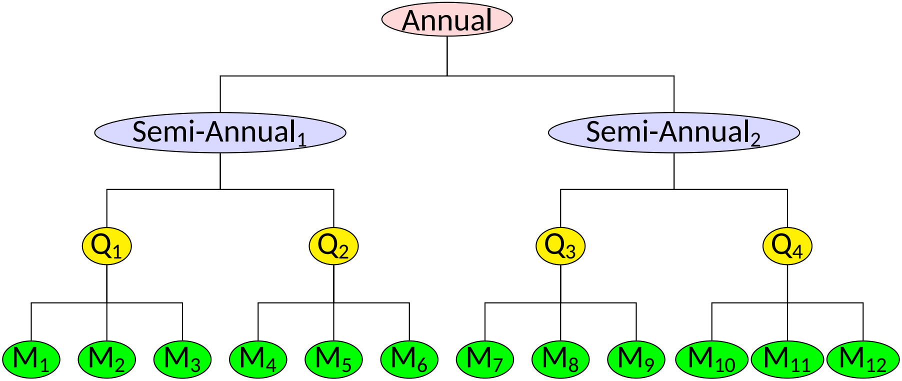
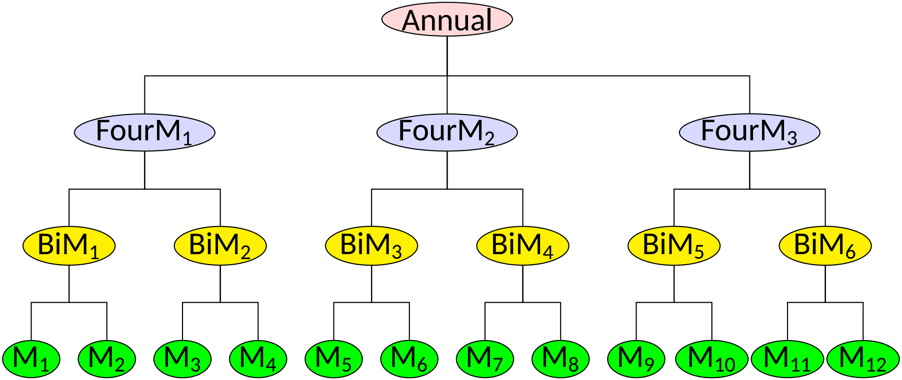
Monthly series



- $k = 2, 4, 12$ nodes
- $k = 3, 6, 12$ nodes
- Why not $k = 2, 3, 4, 6, 12$ nodes?

Monthly series



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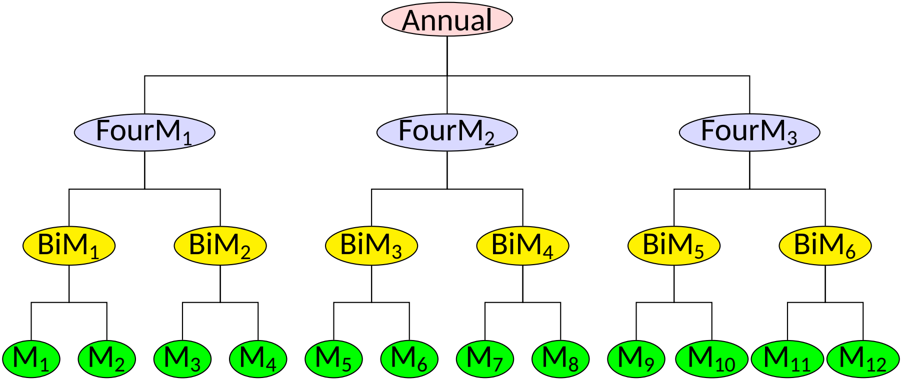

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- Why not $k = 2, 3, 4, 6, 12$ nodes?

Monthly data

$$
\underbrace{\begin{pmatrix} A \\ SemiA_1 \\ SemiA_2 \\ FourM_1 \\ FourM_2 \\ FourM_3 \\ Q_1 \\ \vdots \\ Q_4 \\ BiM_1 \\ \vdots \\ BiM_6 \\ M_1 \\ \vdots \\ M_{12} \end{pmatrix}}_{(28\times 1)}
=
\underbrace{\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & \vdots & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & \vdots & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & & & & \boldsymbol{I}_{12} & & & & &
\end{pmatrix}}_{\boldsymbol{S}}
\underbrace{\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \\ M_9 \\ M_{10} \\ M_{11} \\ M_{12} \end{pmatrix}}_{\boldsymbol{B}_t}
$$

In general

For a time series $y_1, \dots, y_T$, observed at frequency m, we generate aggregate series

$$y_j^{[k]} = \sum_{t=1+(j-1)k}^{jk} y_t, \qquad \text{for } j = 1, \dots, \lfloor T/k \rfloor$$

- $k \in F(m) = \{\text{factors of } m\}$.
- A single unique hierarchy is only possible when there are no coprime pairs in $F(m)$.
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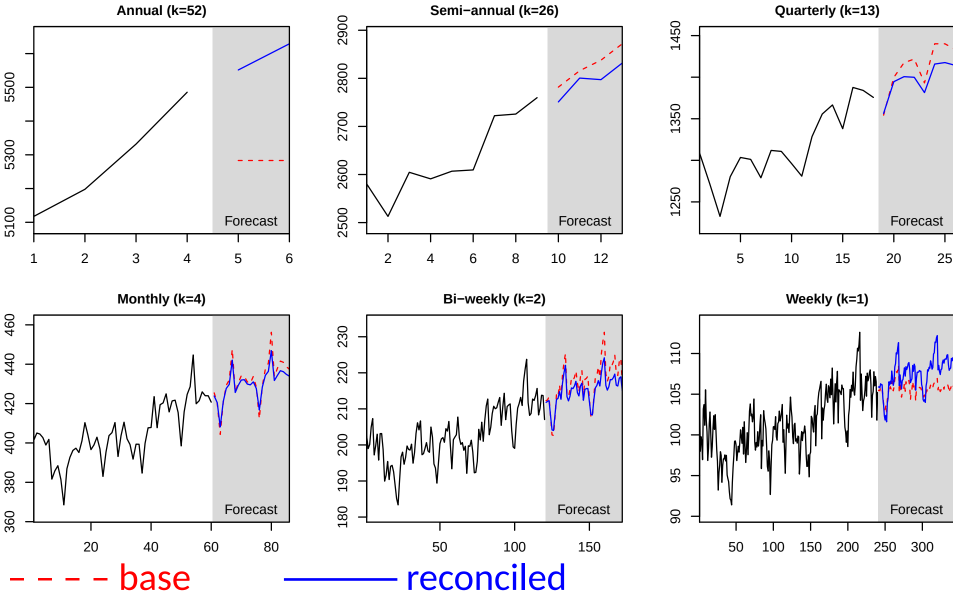
UK Accidents and Emergency Demand



UK Accidents and Emergency Demand

1. Type 1 Departments — Major A&E
2. Type 2 Departments — Single Specialty
3. Type 3 Departments — Other A&E/Minor Injury
4. Total Attendances
5. Type 1 Departments — Major A&E $>$ 4 hrs
6. Type 2 Departments — Single Specialty $>$ 4 hrs
7. Type 3 Departments — Other A&E/Minor Injury $>$ 4 hrs
8. Total Attendances $>$ 4 hrs
9. Emergency Admissions via Type 1 A&E
10. Total Emergency Admissions via A&E
11. Other Emergency Admissions (i.e., not via A&E)
12. Total Emergency Admissions
13. Number of patients spending $>$ 4 hrs from decision to admission

UK Accidents and Emergency Demand

- **Minimum training set**: all data except the last year
- Base forecasts using `auto.arima()`.
- Mean Absolute Scaled Errors for 1, 4 and 13 weeks ahead using a rolling origin.

Aggr. Level	h	Base	Reconciled	Change
Weekly	1	1.6	1.3	−17.2%
Weekly	4	1.9	1.5	−18.6%
Weekly	13	2.3	1.9	−16.2%
Weekly	1–52	2.0	1.9	−5.0%
Annual	1	3.4	1.9	−42.9%

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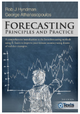
References

RJ Hyndman, RA Ahmed, G Athanasopoulos and HL Shang (2011). Optimal combination forecasts for hierarchical time series. *Computational Statistics & Data Analysis* **55**(9), 2579–2589.

RJ Hyndman, A Lee and E Wang (2016). Fast computation of reconciled forecasts for hierarchical and grouped time series. *Computational Statistics & Data Analysis* **97**, 16–32

SL Wickramasuriya, G Athanasopoulos and RJ Hyndman (2015). *Forecasting hierarchical and grouped time series through trace minimization*. Working paper. Dept Econometrics & Business Statistics, Monash University

G Athanasopoulos, RJ Hyndman, N Kourentzes and F Petropoulos (2017). Forecasting with temporal hierarchies. *European Journal of Operational Research* **262**(1), 60–74



RJ Hyndman and G Athanasopoulos (2018). *Forecasting: principles and practice*. 2nd ed. Melbourne, Australia: OTexts. `OTexts.org/fpp2/`.

R packages

https://github.com/earowang/tsibble

http://pkg.earo.me/sugrrants

http://pkg.robjhyndman.com/forecast

http://pkg.earo.me/hts

http://pkg.robjhyndman.com/thief