ACSE-2

Part B
Continuum mechanics and vector/tensor calculus

Lecture 5

Saskia Goes, s.goes@imperial.ac.uk, room 4.47

Lecture Materials

- Lecture slides (gapped before, complete after lectures)
- Problem sets
- Solutions (after workshop)

On Box:

CourseMaterials-201920/ACSE-2 Lectures and Workshops

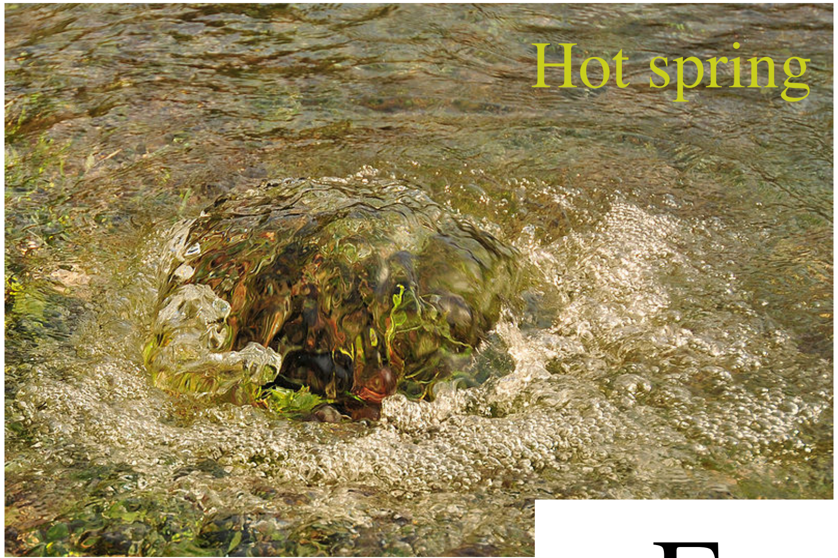
Outline of course

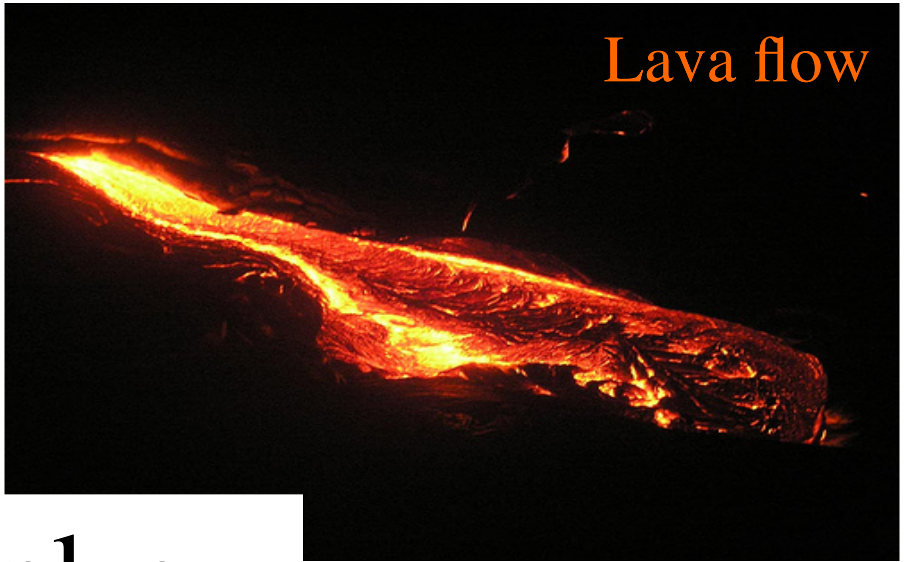
- **1.** Mathematical essentials – *Matthew Piggott*
- **2.** Linear Algebra I – *Matthew Piggott*
- **3.** Linear Algebra II, ODEs– *Matthew Piggott*
- **4.** Verifying models– *Matthew Piggott*
- **5.** Vector and Tensor Calculus - *Saskia Goes*
- **6.** Stress principles - *Saskia Goes*
- **7.** Kinematics and strain - *Saskia Goes*
- **8.** Rheology and conservation equations - *Saskia Goes*
- **9.** Potential flow - *Stephen Neethling*
- **10.** Fluid flow I - *Stephen Neethling*
- **11.** Fluid flow II - *Stephen Neethling*
- **12.** Wave propagation - *Adrian Umpleby*

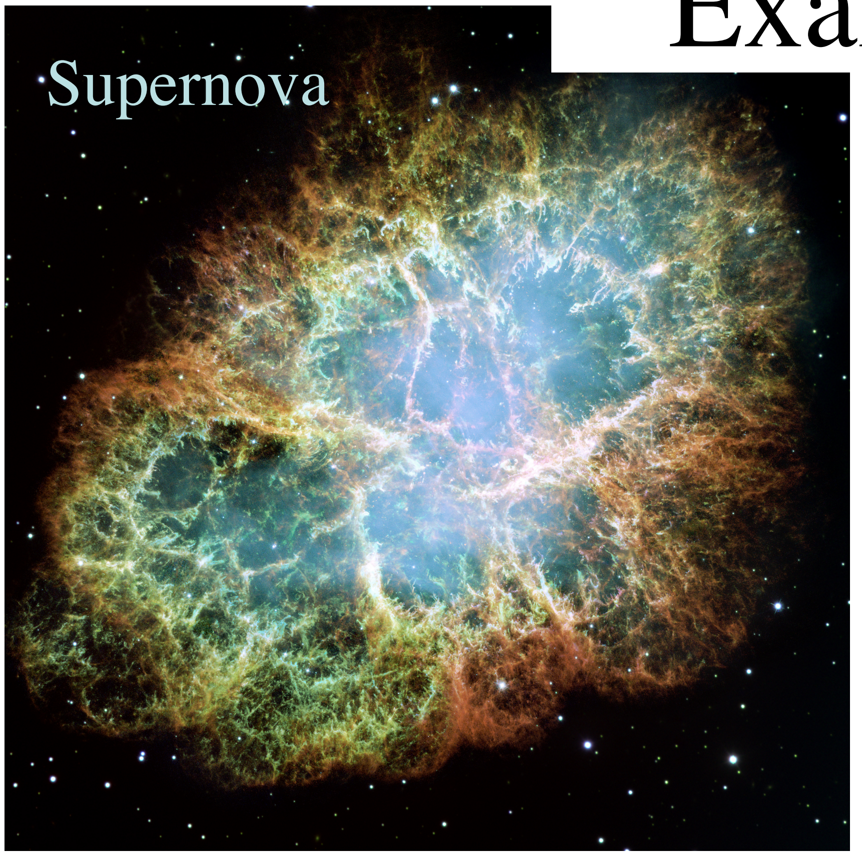
Continuum Mechanics

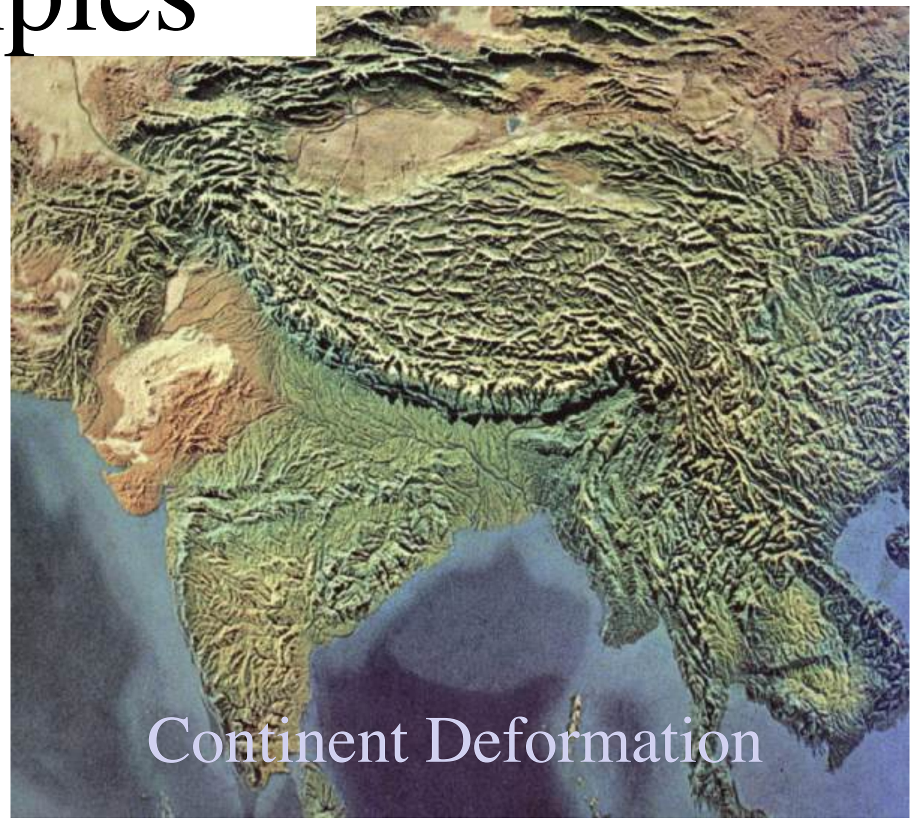
- *Macroscopic description* of the collective behavior of atoms/molecules in the limit where scale >> scale of the individual components
- Treat a material, be it solid, liquid, gas, as hypothetical continuum where all quantities vary continuously so that spatial derivatives exist
- In such a treatment, we can consider infinitesimally small volumes of the material and define point quantities, like mass, velocity, stress
- Such a description has been found to be applicable in a wide range of problems in engineering and physical sciences

Examples

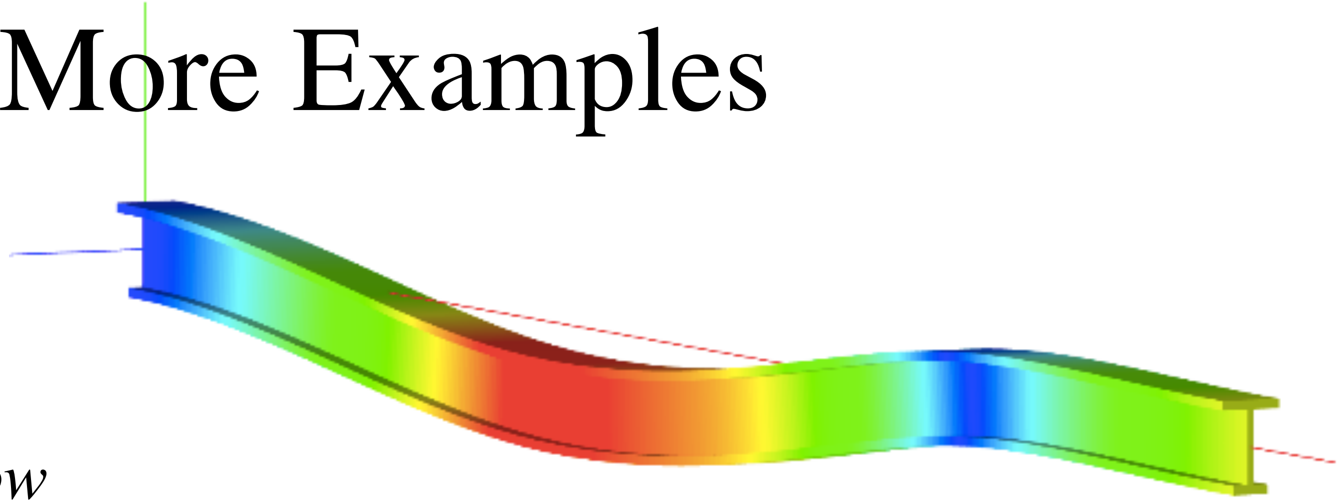






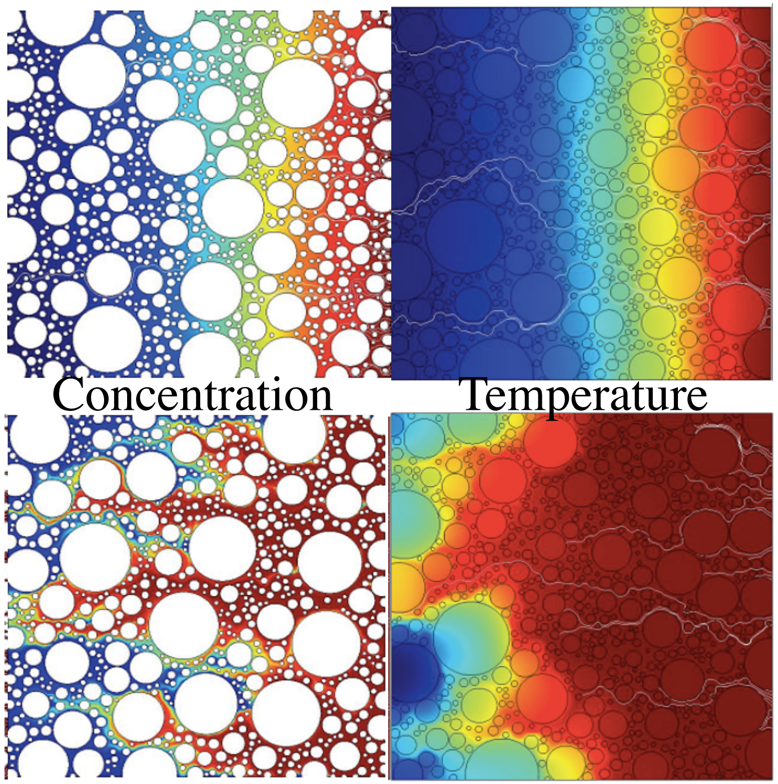


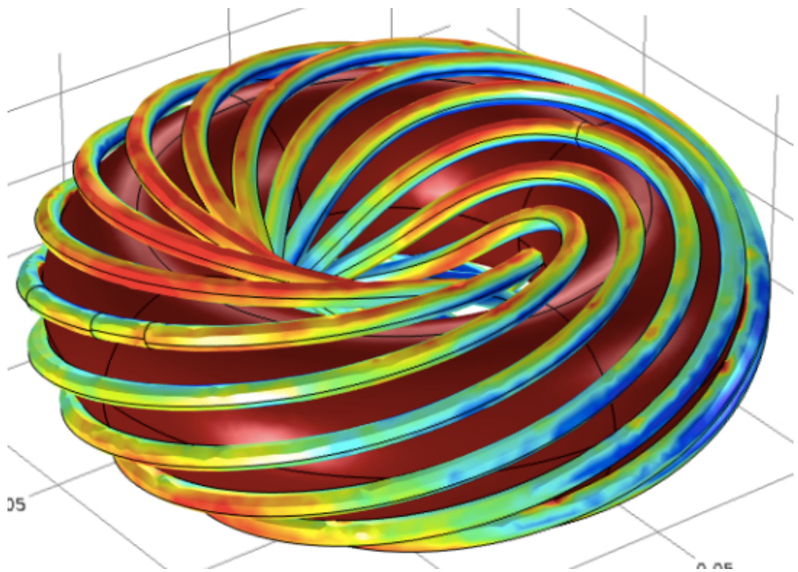
More Examples



Beam bending

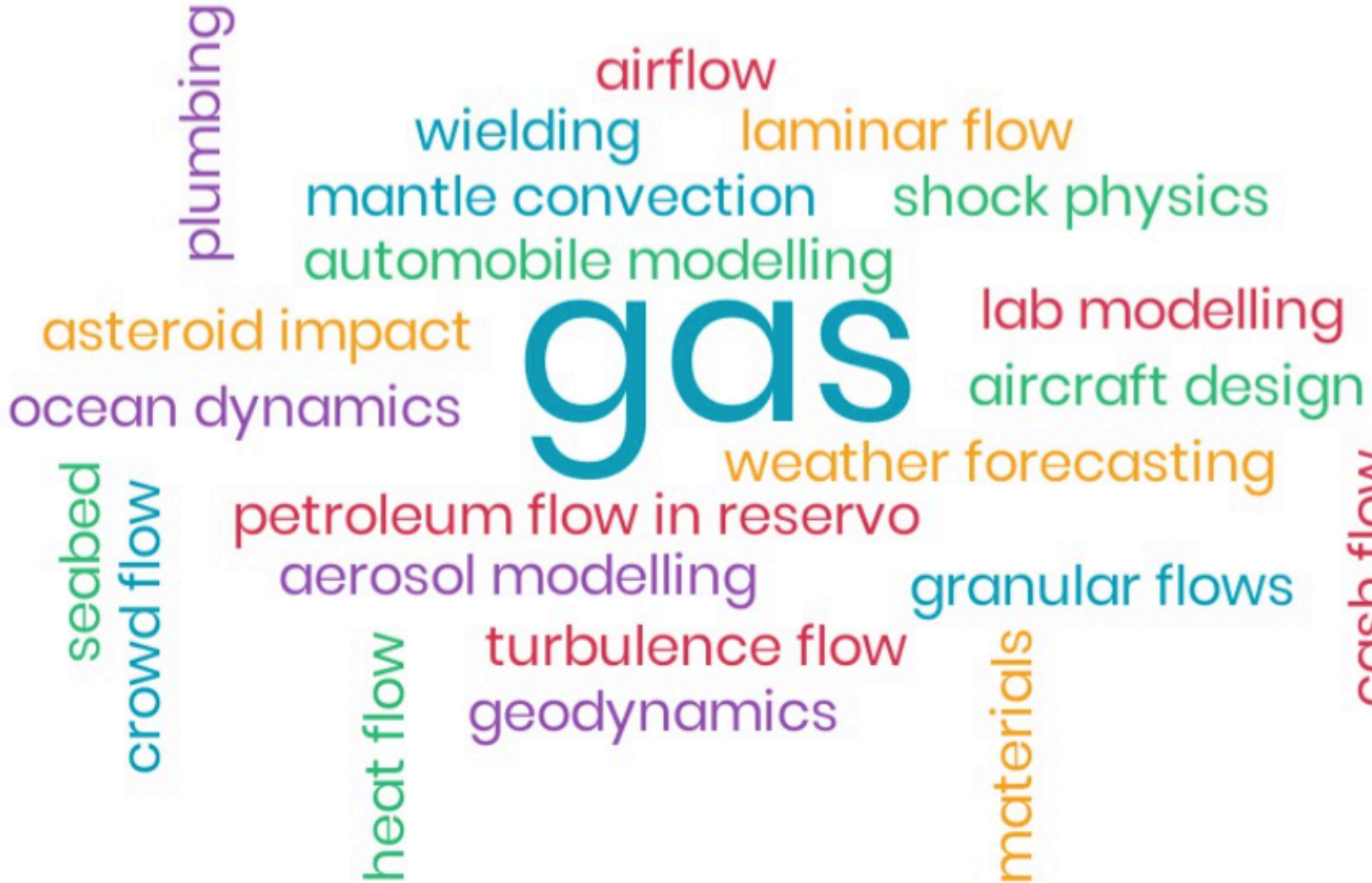
Porous flow





Electromagnetism

Other Examples?



No required text

Possible textbooks for additional background

- Introduction to Continuum Mechanics, W.M. Lai, D. Rubin, E. Krempl, 4th edition, Elsevier – available in electronic form through IC library

- An Introduction to Continuum Mechanics, J.N. Reddy, 2nd edition, Cambridge University Press, 2013

- Khan Academy – online lectures on Maths, Physics

The books use similar notation as this course and cover many of the topics in ACSE-2. However, do note different reading may be suggested for other parts of the course

Continuum Mechanics Equations

General:

1. Kinematics – describing deformation and velocity without considering forces
2. Dynamics – equations that describe force balance, conservation of linear and angular momentum
3. Thermodynamics – relations temperature, heatflux, stress, entropy

Material-specific

4. Constitutive equations – relations describing how material properties vary as a function of T,P, stress,…. Such material properties govern dynamics (e.g., density), response to stress (viscosity, elastic parameters), heat transport (thermal conductivity, diffusivity)

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$\Rightarrow$ *Yields a set of Partial Differential Equations that can be solved for displacement, velocity, temperature,…*

Partial Differential Equations

- **Ordinary Differential Equations** – describe how variables depend on a single independent parameter (e.g., time or distance).

For example: $$m\frac{d^2x}{dt^2} = F$$ *Newton's second law*

- **Partial Differential Equations** – describe how variables depend on several independent parameters (e.g., time, x,y,z coordinates)

For example: $$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$ *Thermal diffusion equation*

∂ - partial derivative

Today
Vectors and Tensors

- Revision vectors
  - Addition, linear independence
  - Orthonormal Cartesian bases, transformation
  - Multiplication
  - Derivatives, del, div, curl
- Revision/introduction tensors
  - Tensors, rank, stress tensor
  - Index notation, summation convention
  - Addition, multiplication
  - Special tensors, δ_{ij} and ε_{ijk}
  - Tensor calculus: gradient, divergence, curl, ..

Learning Objectives

- Be able to perform vector/tensor operations (addition, multiplication) on Cartesian orthonormal bases
- Be able to do basic vector/tensor calculus (time and space derivatives, divergence, curl of a vector field) on these bases.
- Perform transformation of a vector from one to another Cartesian basis.
- Understand differences/commonalities tensor and vector
- Use index notation and Einstein convention
- Be able to use the special tensors δ_{ij} and ε_{ijk}

Key characteristics of a vector?

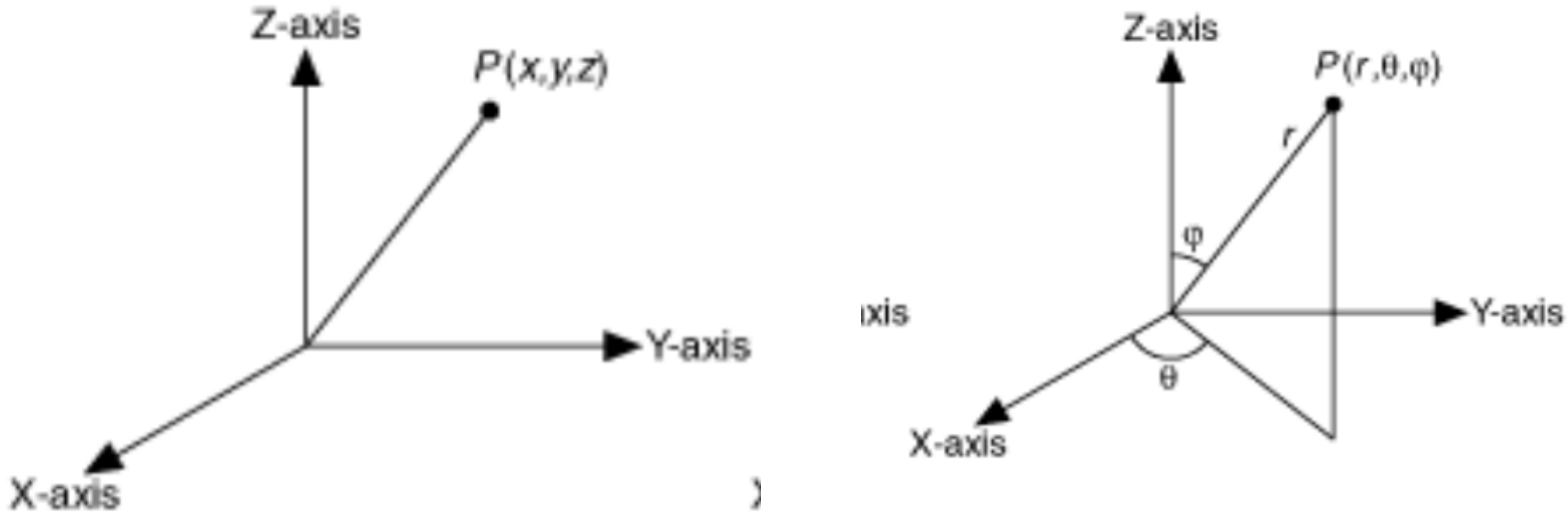
1. Anything that is not a scalar
2. Has three components
3. Has magnitude and direction
4. Depends on coordinate frame
5. Velocity and rotation are two examples
6. Multiplication of two vectors gives another vector

Choose all that are correct
at www.mentimeter.com

Intro Vectors, Tensors

Continuum mechanics equations require vectors and tensors. E.g., velocity is a vector, with magnitude and direction in 3-D, and so are forces like gravity.

The components of a vector depend on the coordinate system chosen to represent them in. However, the actual size and orientation of the vector is not dependent on the choice of coordinate system



Notation

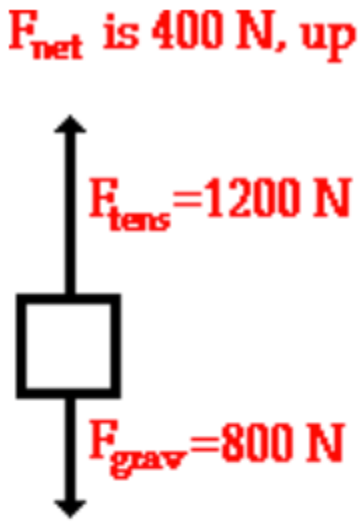
- Vectors as $\mathbf{v}$ or $\vec{v}$ or $\overline{v}$
- Length of vectors v or $|\mathbf{v}|$
- Vector in Cartesian components $v_{\mathrm{x}}, v_{\mathrm{y}}, v_{\mathrm{z}}$
- Index notation v_{i}, $i=x,y,z$ or $i=1,2,3$
- Unit vector along direction of $\mathbf{v}$: $\hat{\mathbf{e}}_{v} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

$$\mathbf{v} = \hat{\mathbf{e}}_{v}\,|\mathbf{v}|$$

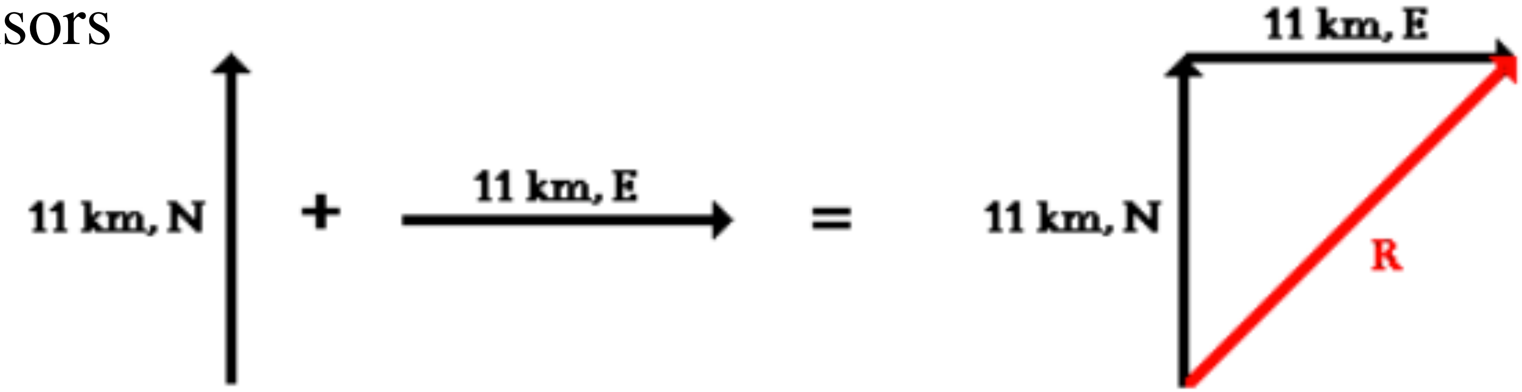
Vectors

Vectors satisfy certain rules of addition and scalar multiplication,

- **a+b=b+a** (commutative)
- (**a+b**)+**c=a**+(**b+c**) (associative)
- α(**a+b**) = α**a**+α**b** (distributive)
- **a+0=a** (zero vector)
- 1·**a=a**·1; 0·**a=0**

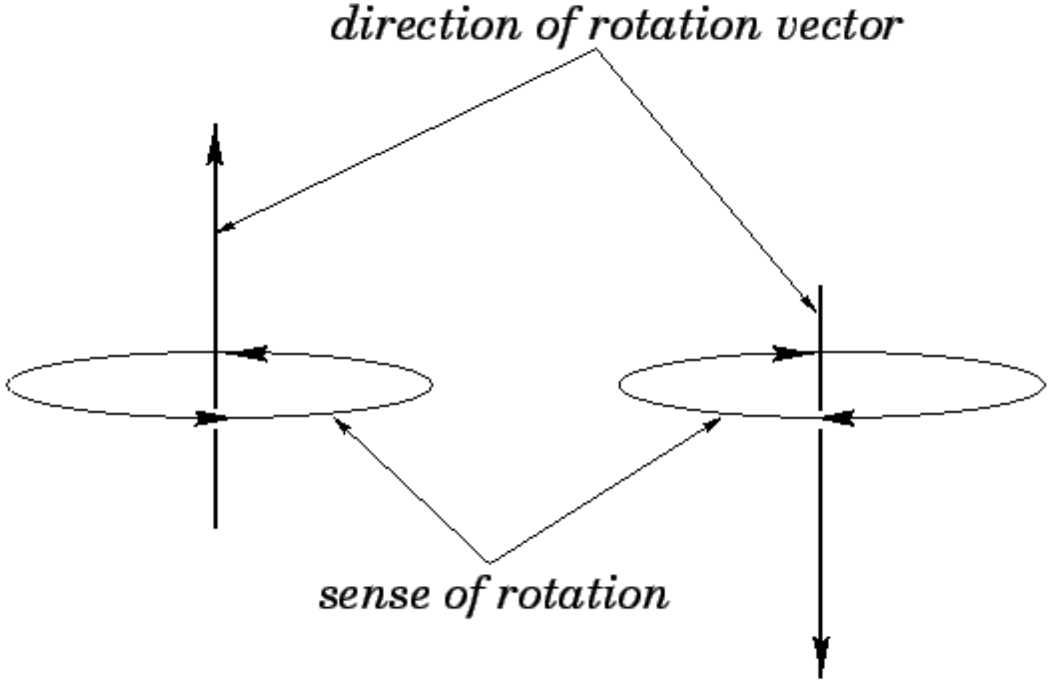


We will see that similar rules apply to tensors



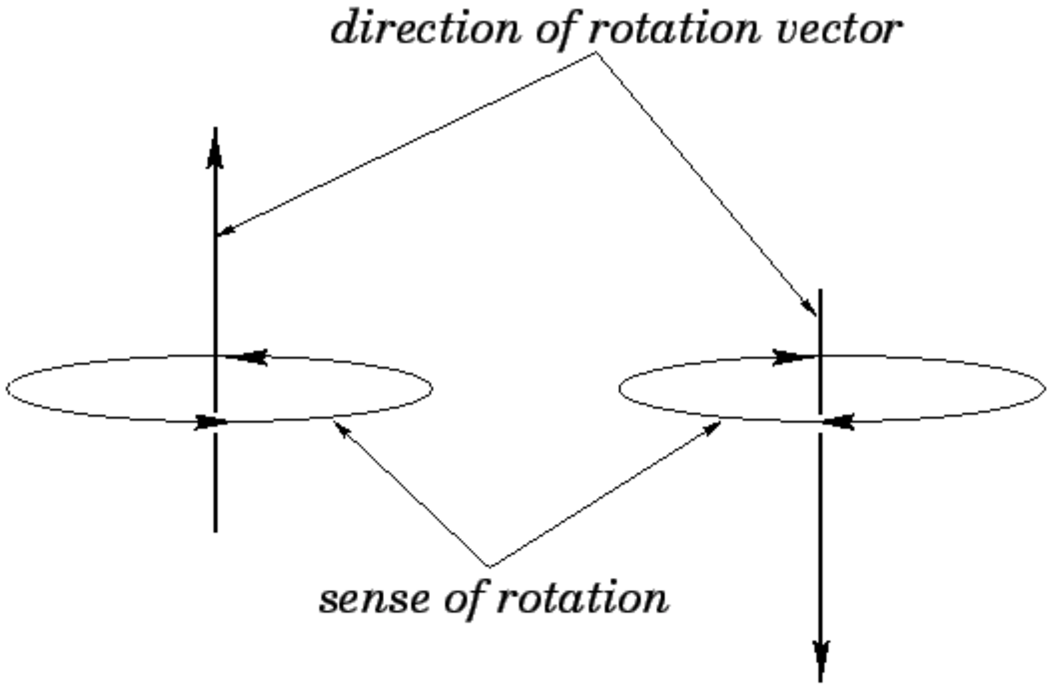
e.g. see http://mathworld.wolfram.com/Vector.html

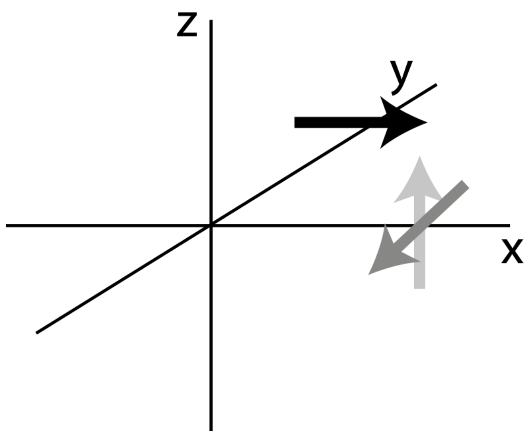
Finite Rotation "Vector"



Right-hand rule

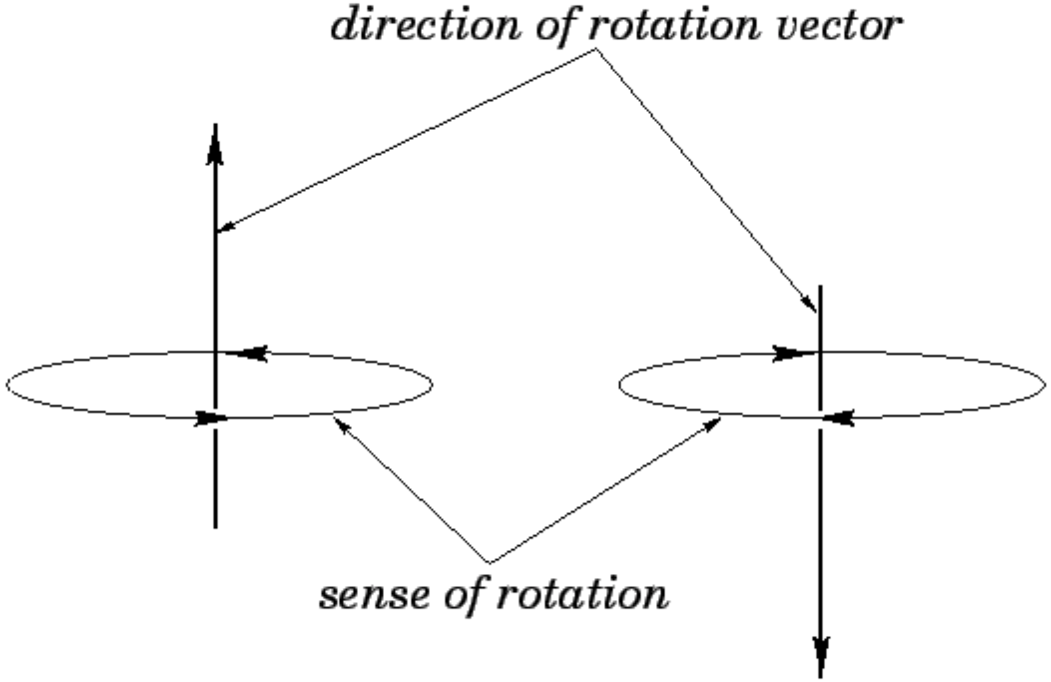
Finite Rotation "Vector"

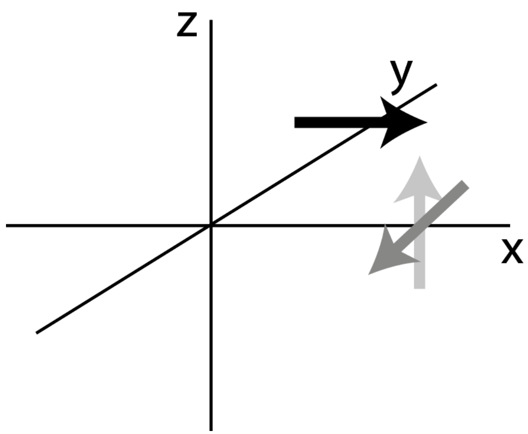




rotate 90° around **x** +
90° around **z**

Finite Rotation "Vector"





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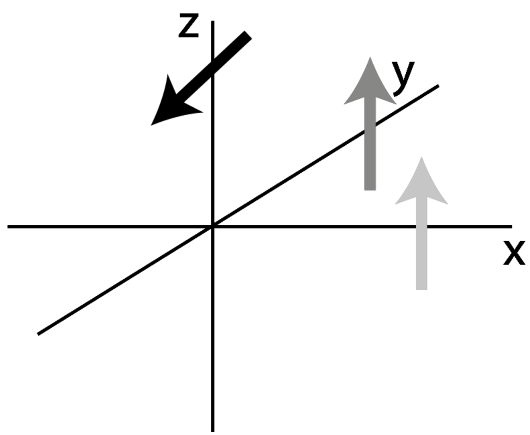


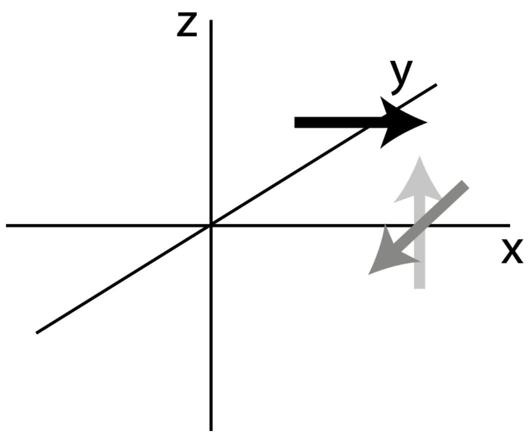
rotate 90° around **z** + 90° around **x**

Finite Rotation "Vector"

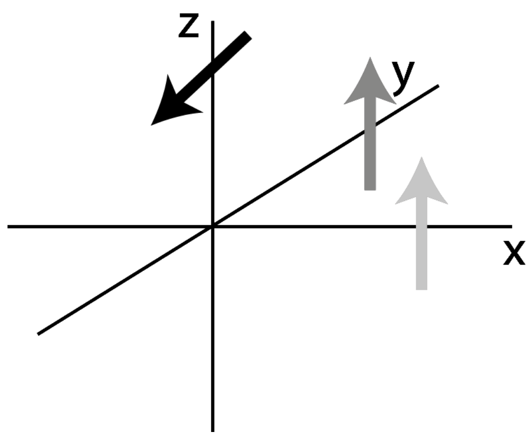
If **a** and **b** are two general vectors, then **a+b=b+a**

However, addition of two finite rotations is not commutative.

Finite rotation is pseudo-vector
Infinitesimal rotation is vector



rotate 90° around **x** + 90° around **z**



rotate 90° around **z** + 90° around **x**

Linear independence

Vectors v_1 through v_n are linearly dependent if coefficients c_i can be found such that:

$$c_1v_1+c_2v_2+c_3v_3+\ldots c_nv_n=0$$

1. if two vectors are linearly dependent, they are?
2. if three vectors are linearly dependent, they are?
3. four or more vectors are always linearly dependent

Important for defining bases, independent solutions to a problem

Try yourself

- *What steps would be required to determine in a code whether two given 2-D vectors* ***a*** *and* ***b*** *are linearly independent or not? Write them out.*

Linear independence two vectors

$\mathbf{a}=(a_1,a_2)$, $\mathbf{b}=(b_1,b_2)$

- $c_1a_1+c_2b_1=0$
- $c_1a_2+c_2b_2=0$

$\Rightarrow c_1=-c_2b_1/a_1$

$\Rightarrow -c_2a_2b_1/a_1+c_2b_2=0$

$\Rightarrow c_2(b_2-a_2b_1/a_1)=0$

$\Rightarrow a_1b_2-a_2b_1=0$

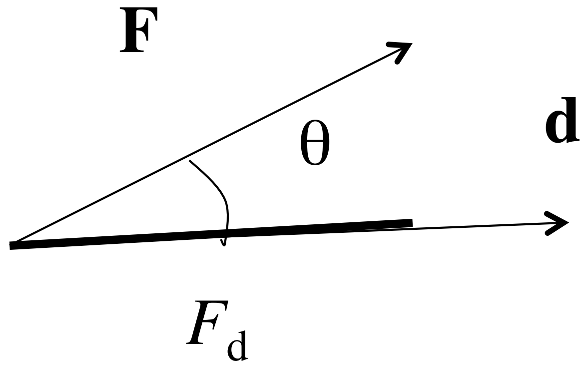
if true, linearly dependent

Products of vectors

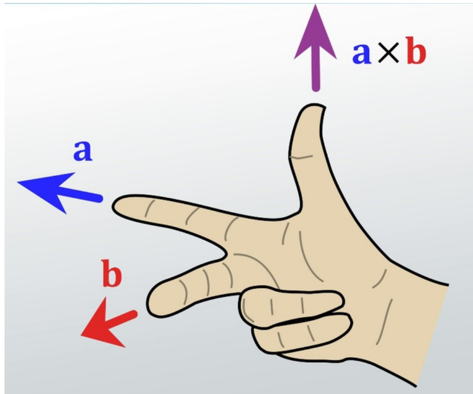
Inner product, dot product, scalar product

Geometric definition

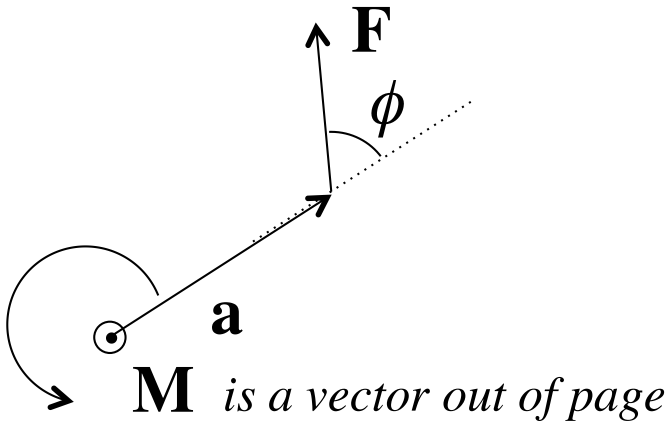
- $\mathbf{F} \cdot \mathbf{d} = |\mathbf{F}||\mathbf{d}|\cos\theta$
  - scalar,
  - projection of **F** on **d** times |**d**|,
  - = 0 if **F** and **d** perpendicular,
  - $\mathbf{F}\cdot\mathbf{d} = \mathbf{d}\cdot\mathbf{F}$



- If **F** is force, **d** is displacement, then **F·d** is the work done by the force **F** for displacement **d**
- $\mathbf{a}\cdot\mathbf{a}=$ length($\mathbf{a}$)^2$= |\mathbf{a}|^2$



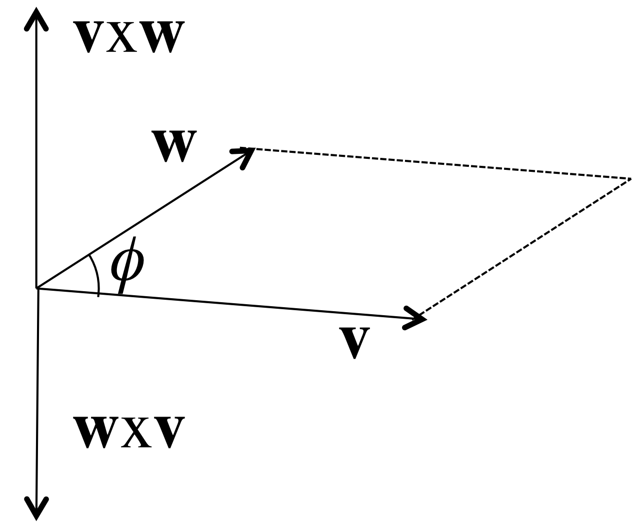
Products of vectors
Cross product, vector product, outer product
Geometric definition



- Example moment:

$$\mathbf{M} = \mathbf{a} \times \mathbf{F} = aF\sin\phi\,\hat{\mathbf{e}}_M$$

- Properties **vxw**
  - vector
  - magnitude = area of parallellogram spanned by **v**, **w**
  - direction is that of plane normal (right-hand rule)
  - =0 if **v** and **w** are parallel
  - **vxw =-wxv**



Products of vectors

Algebraic, in rectangular Cartesian coordinates:

in 2D

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 \qquad \mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1$$

in 3D

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

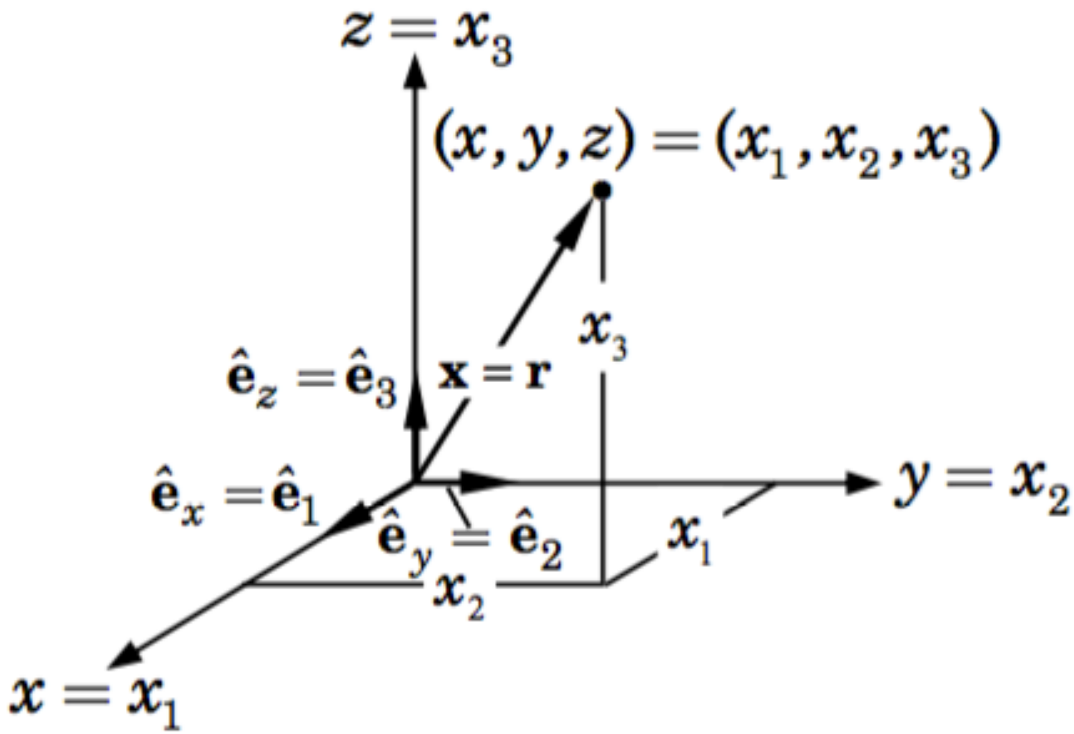
Rectangular Cartesian Coordinate System

Orthonormal basis –
Basis vectors are:
orthogonal

$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0 \qquad \text{if } i \neq j$

and unit length

$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = |\hat{\mathbf{e}}_i|^2 = 1$



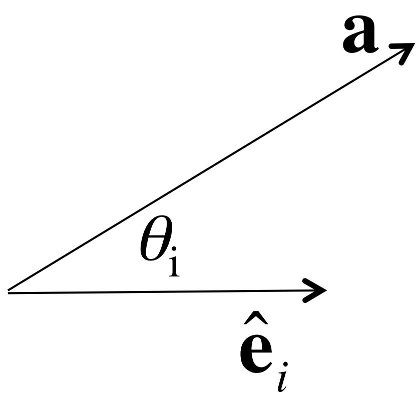
Cartesian – basis vectors with constant length and direction

In following, we will assume Cartesian orthonormal bases

Other orthonormal bases, e.g. polar or spherical, not discussed here

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z$	$\hat{\mathbf{r}} A_r + \hat{\boldsymbol{\phi}} A_\phi + \hat{\mathbf{z}} A_z$	$\hat{\mathbf{R}} A_R + \hat{\boldsymbol{\theta}} A_\theta + \hat{\boldsymbol{\phi}} A_\phi$
Magnitude of $\mathbf{A}$ $\quad |\mathbf{A}| =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\quad \overrightarrow{OP_1} =$	$\hat{\mathbf{x}} x_1 + \hat{\mathbf{y}} y_1 + \hat{\mathbf{z}} z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}} r_1 + \hat{\mathbf{z}} z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}} R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\quad \mathbf{A} \cdot \mathbf{B} =$	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\phi B_\phi + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\quad \mathbf{A} \times \mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $\quad d\mathbf{l} =$	$\hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz$	$\hat{\mathbf{r}}\, dr + \hat{\boldsymbol{\phi}} r\, d\phi + \hat{\mathbf{z}}\, dz$	$\hat{\mathbf{R}}\, dR + \hat{\boldsymbol{\theta}} R\, d\theta + \hat{\boldsymbol{\phi}} R \sin\theta\, d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\, dy\, dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\, dx\, dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\, dx\, dy$	$d\mathbf{s}_r = \hat{\mathbf{r}} r\, d\phi\, dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, dr\, dz$ $d\mathbf{s}_z = \hat{\mathbf{z}} r\, dr\, d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}} R^2 \sin\theta\, d\theta\, d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}} R \sin\theta\, dR\, d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}} R\, dR\, d\theta$
Differential volume $\quad dV =$	$dx\, dy\, dz$	$r\, dr\, d\phi\, dz$	$R^2 \sin\theta\, dR\, d\theta\, d\phi$

Equivalence Cartesian geometric and algebraic dot product



$$\mathbf{a} = \sum_i a_i \hat{\mathbf{e}}_i \qquad \mathbf{b} = \sum_i b_i \hat{\mathbf{e}}_i$$

$$\mathbf{a} \cdot \hat{\mathbf{e}}_i = |\mathbf{a}||\hat{\mathbf{e}}_i| \cos \vartheta_i = |\mathbf{a}| \cos \vartheta_i = a_i$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \sum_i b_i \hat{\mathbf{e}}_i = \sum_i b_i (\mathbf{a} \cdot \hat{\mathbf{e}}_i) = \sum_i b_i a_i = \sum_i a_i b_i$$

Products of vectors

Algebraic, in rectangular Cartesian coordinates:

in 2D

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 \qquad \mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1$$

in 3D

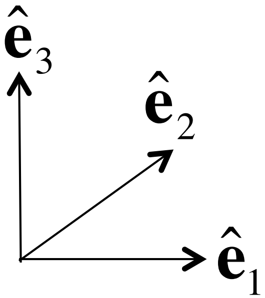
$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

Cartesian algebraic cross product

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_i = 0 \qquad \begin{array}{ll} \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 & \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_2 \end{array}$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2) \times (b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2) \\ &= a_1 b_1 (\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_1) + a_1 b_2 (\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2) \\ &\quad + a_2 b_1 (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1) + a_2 b_2 (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_2) \\ &= (a_1 b_2 - a_2 b_1)\hat{\mathbf{e}}_3 \end{aligned}$$



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$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

Triple products

- $\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$ – vector times scalar
- scalar triple product $\mathbf{a}\cdot(\mathbf{b}\mathrm{x}\mathbf{c})$

  $\mathbf{a}\cdot(\mathbf{b}\mathrm{x}\mathbf{c}) = (\mathbf{a}\mathrm{x}\mathbf{b})\cdot\mathbf{c}$ – *show this*

  $= \mathbf{c}\cdot(\mathbf{a}\mathrm{x}\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\mathrm{x}\mathbf{a})$ (with cyclical permutation) – *show*

  $= -\mathbf{a}\cdot(\mathbf{c}\mathrm{x}\mathbf{b}) = -\mathbf{c}\cdot(\mathbf{b}\mathrm{x}\mathbf{a}) = -\mathbf{b}\cdot(\mathbf{a}\mathrm{x}\mathbf{c})$ (with order changed) – *show*

  $= 0$ if coplanar
- $\mathbf{a}\mathrm{x}(\mathbf{b}\mathrm{x}\mathbf{c})$ – lies in plane formed by $\mathbf{b}\mathrm{x}\mathbf{c}$ and is normal to $\mathbf{a}$

  $\neq (\mathbf{a}\mathrm{x}\mathbf{b})\mathrm{x}\mathbf{c}$

  $= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ *Problem 5-3*

Try now

- *How would you find components of a 3D vector **a** normal and parallel to 3D vector **b**?*

Try later

- *You could program this up and test with special cases*

- *How would you find components of a 3D vector **a** normal and parallel to 3D vector **b**?*

parallel component: $(\mathbf{a} \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|})\hat{\mathbf{e}}_b = (\mathbf{a} \cdot \hat{\mathbf{e}}_b)\hat{\mathbf{e}}_b = |\mathbf{a}|\cos\theta\hat{\mathbf{e}}_b$

perpendicular component: $\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{e}}_b)\hat{\mathbf{e}}_b$

$$= (\hat{\mathbf{e}}_b \cdot \hat{\mathbf{e}}_b)\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{e}}_b)\hat{\mathbf{e}}_b = \hat{\mathbf{e}}_b \times (\mathbf{a} \times \hat{\mathbf{e}}_b)$$

$$= |\mathbf{a}|\sin\theta\left[\hat{\mathbf{e}}_b \times (\hat{\mathbf{e}}_a \times \hat{\mathbf{e}}_b)\right]$$

Vector transformation

- Vector magnitude and direction do not depend on basis
- When defined on orthonormal basis, like rectangular Cartesian, the transformation to other orthonormal bases is simple, with real coefficients.

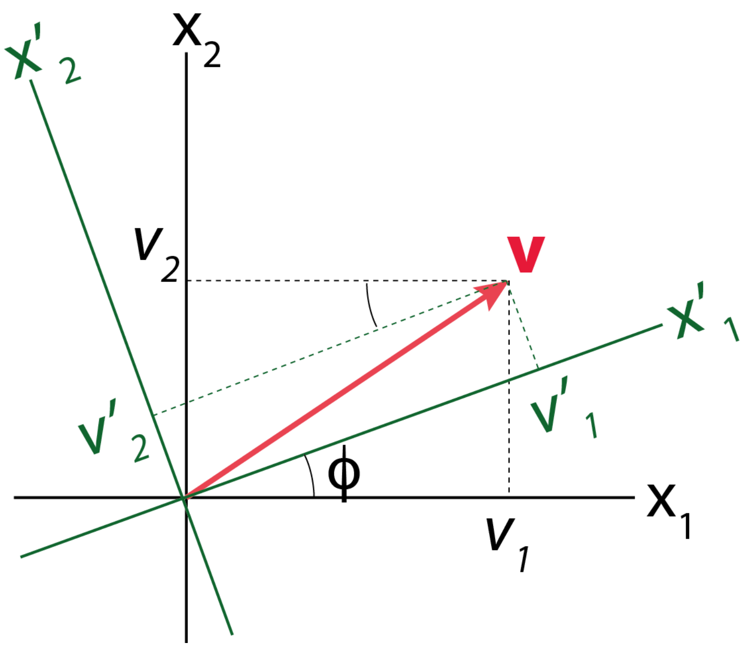
$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0 \qquad \text{if } i \neq j$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = \left|\hat{\mathbf{e}}_i\right|^2 = 1$$

check out Khan Academy lectures on orthonormal bases

Vector transformation

physical parameters should not depend on coordinate frame



for vectors on **orthonormal** basis, transformed vector **v**' depends on **v**:

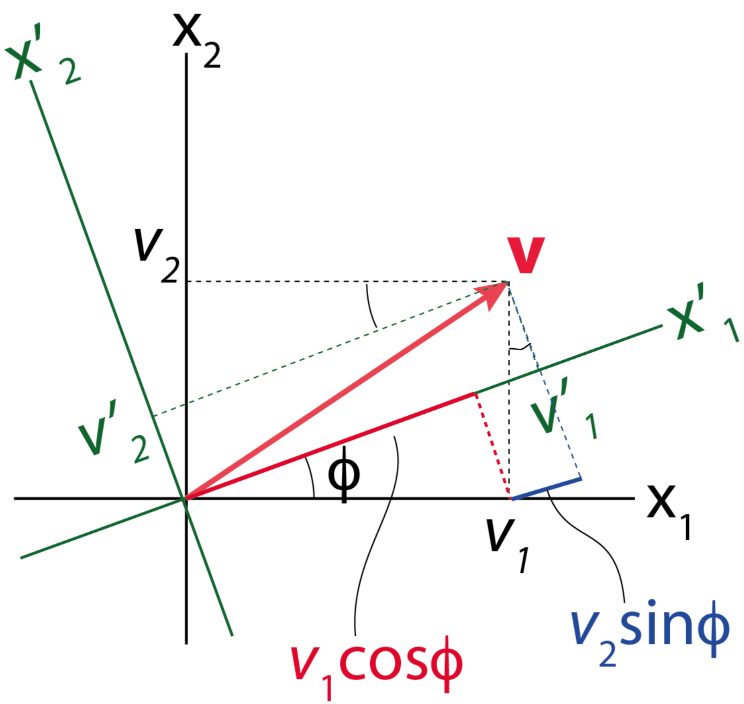
$$v'_1 = \alpha_{11} v_1 + \alpha_{12} v_2$$
$$v'_2 = \alpha_{21} v_1 + \alpha_{22} v_2$$

$$v'_1 = \cos\phi \; v_1 + \sin\phi \; v_2$$
$$v'_2 = -\sin\phi \; v_1 + \cos\phi \; v_2$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

Vector transformation

physical parameters should not depend on coordinate frame



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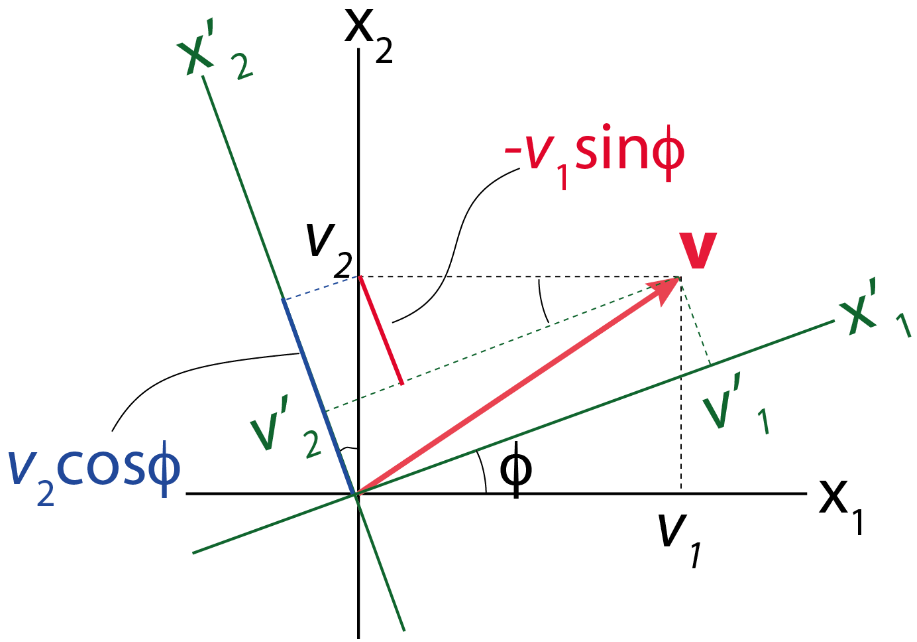
Vector transformation

physical parameters should not depend on coordinate frame



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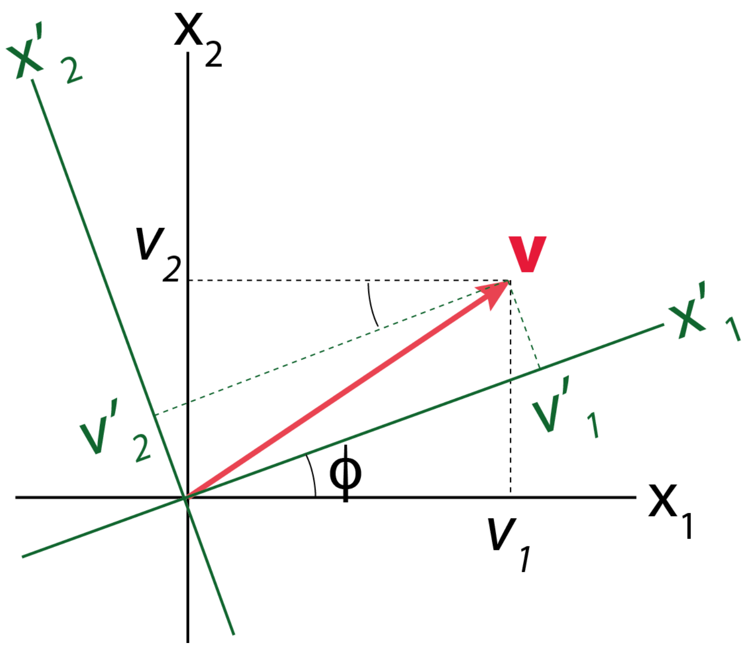
Vector transformation

physical parameters should not depend on coordinate frame



for vectors on **orthonormal** basis, transformed vector **v**' depends on **v**:

$$v'_1 = \alpha_{11} v_1 + \alpha_{12} v_2$$

$$v'_2 = \alpha_{21} v_1 + \alpha_{22} v_2$$

$$\Rightarrow \quad \mathbf{v}' = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mathbf{v}$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v}$$

$$\alpha_{11} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_1 \qquad \alpha_{12} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_2$$

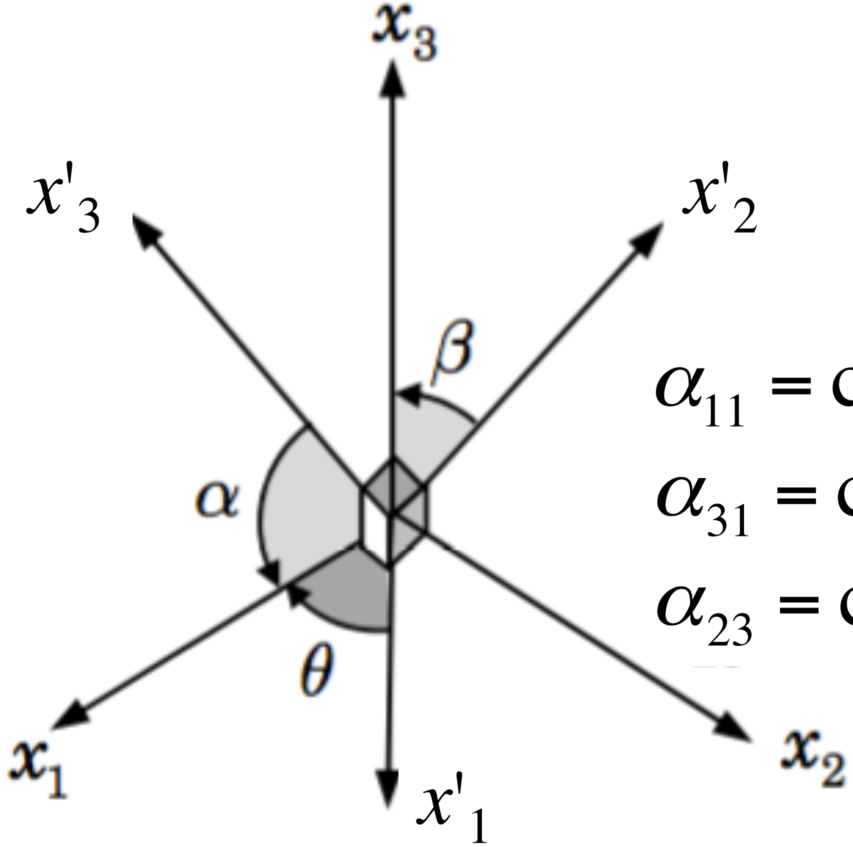
$$\alpha_{21} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_1 \qquad \alpha_{22} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_2$$

Transformation orthonormal bases

$$\hat{\mathbf{e}}'_i = \sum_{j=1,n} \alpha_{ij} \hat{\mathbf{e}}_j$$

In other words:

$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$



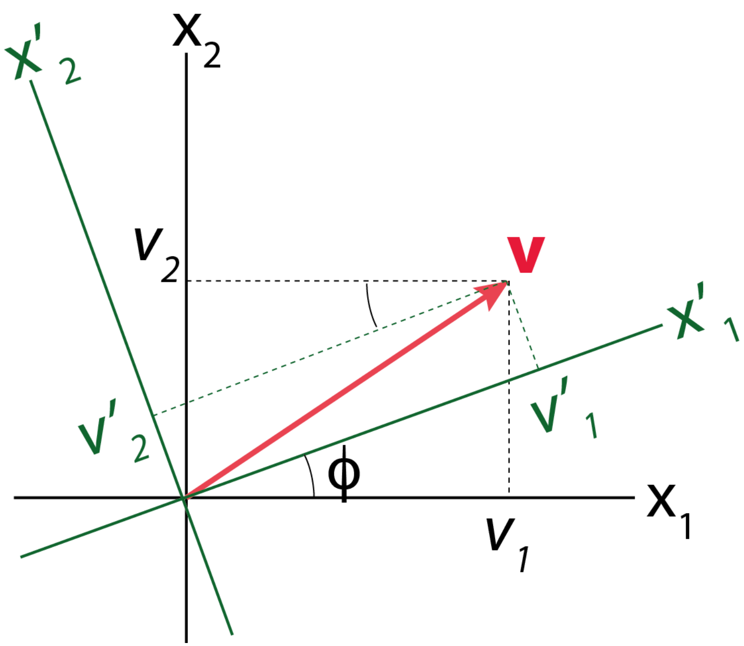
$$\alpha_{11} = \cos\theta$$
$$\alpha_{31} = \cos\alpha$$
$$\alpha_{23} = \cos\beta$$

Transformation orthonormal bases

$$\hat{\mathbf{e}}'_i = \sum_{j=1,n} \alpha_{ij} \hat{\mathbf{e}}_j$$

In other words:

$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$



angle x'_1 and x_2

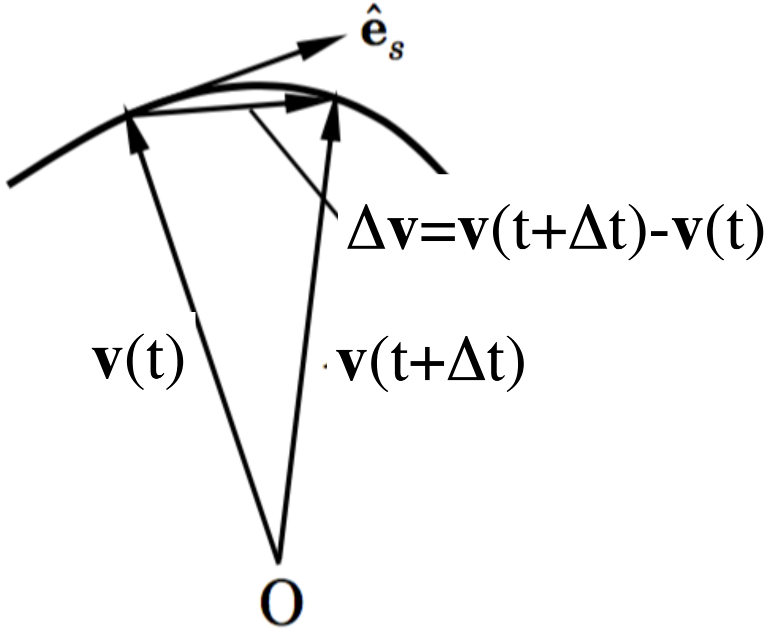
$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v} \qquad \mathbf{v}' = \begin{bmatrix} \cos\phi & \cos(90-\phi) \\ \cos(90+\phi) & \cos\phi \end{bmatrix} \mathbf{v}$$

angle x'_2 and x_1

Vector derivatives
Scalar: e.g. time

$$\boxed{\frac{d\mathbf{v}}{dt} = \begin{pmatrix} \frac{dv_1}{dt} & \frac{dv_2}{dt} & \frac{dv_3}{dt} \end{pmatrix}}$$

$$\frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$$



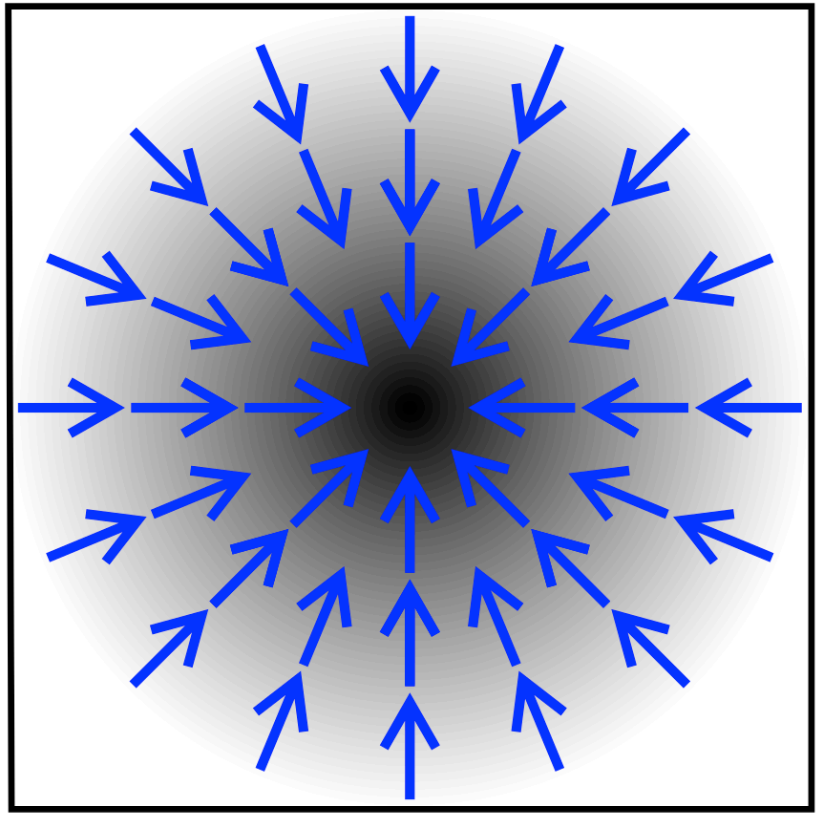
usually has a different direction than **v**

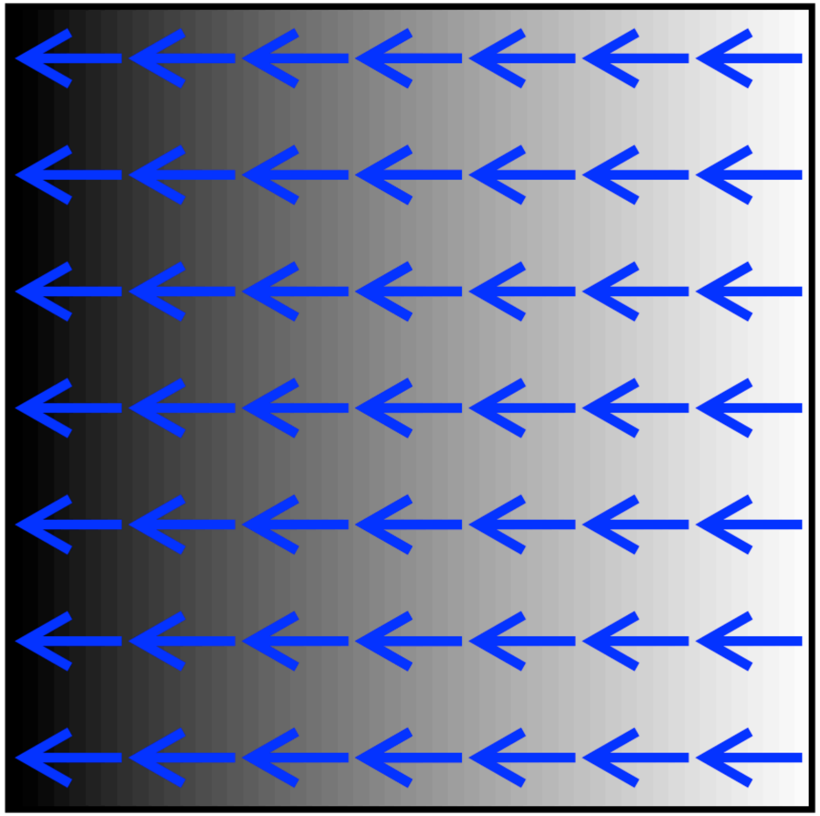
remember: $\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$

$$\frac{d\mathbf{v}}{dt} = \frac{dv_1}{dt}\hat{\mathbf{e}}_1 + \frac{dv_2}{dt}\hat{\mathbf{e}}_2 + \frac{dv_3}{dt}\hat{\mathbf{e}}_3 + \underbrace{v_1\frac{d\hat{\mathbf{e}}_1}{dt} + v_2\frac{d\hat{\mathbf{e}}_2}{dt} + v_3\frac{d\hat{\mathbf{e}}_3}{dt}}_{\text{for Cartesian systems } = 0}$$

Vector derivatives

directional derivative: space





ϕ - high ϕ - low

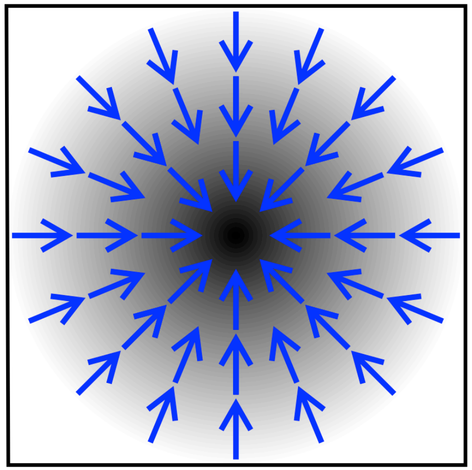
Vector derivatives

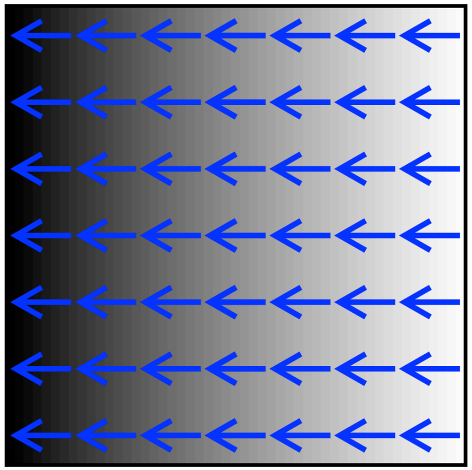
directional derivative: space

∂ - partial derivative

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3$$

$$d\phi = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} & \frac{\partial \phi}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \nabla \phi \cdot \mathbf{dx}$$





Gradient $\nabla \phi$: vector that is a measure of change in scalar field with direction

Try yourself later

- *Define a scalar field, e.g.* $\phi(x,y)=x+2y$
- *Determine values of* ϕ *on a regular grid*
- *Make a contour plot of the field*
- *Numerically determine the gradient* $\nabla\phi$
- *Plot the gradient components as contours, and compare with the analytical solution.*
- *Plot the gradient as vectors*

Del operator

$$\nabla = \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \end{pmatrix}$$

Has some properties of a vector, but not all

$$\mathbf{v} \cdot \nabla \phi \neq \left(\nabla \cdot \mathbf{v} \right) \phi$$

Vector products with derivatives **divergence**, **curl**

- Divergence of a vector: $$\nabla \cdot \mathbf{v} = \sum_{i=1,3} \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

- Curl of a vector: $$\nabla \times \mathbf{v} = \begin{pmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\[2ex] \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\[2ex] \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{pmatrix}$$

Useful calculus theorems

Gauss or divergence theorem: $\int_V \nabla \cdot \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}}\, ds$

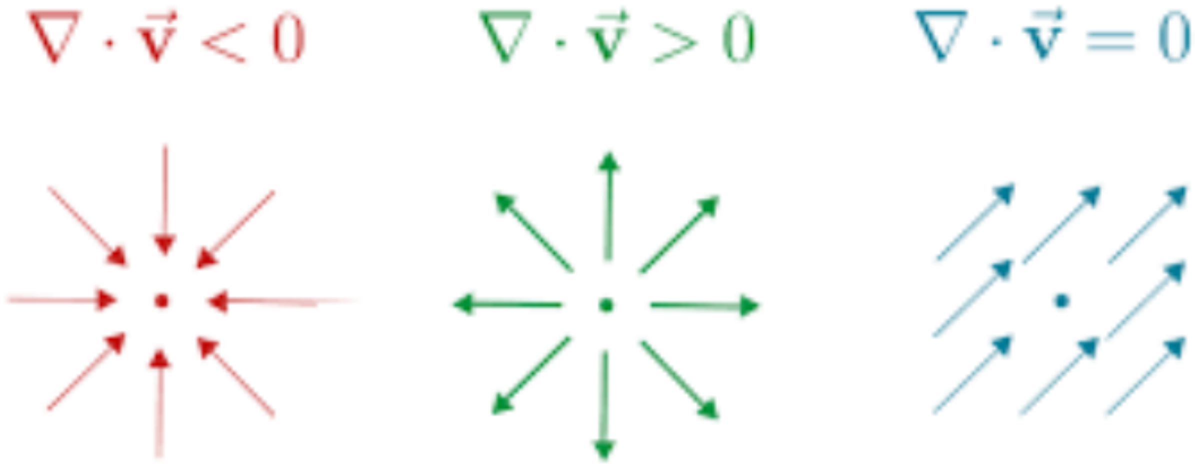
Stokes or curl theorem: $\int_V \nabla \times \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$

Take volume V enclosed by a closed surface S
within a vector field **v** with continuous partial derivatives

Flow perpendicular to the boundary: $\mathbf{v} \cdot \hat{\mathbf{n}}$
Flow parallel to the boundary: $\mathbf{v} \cdot \hat{\mathbf{t}}$

These can be used to simplify integration over volumes or closed surfaces as well as to gain understanding of the meaning of div and curl

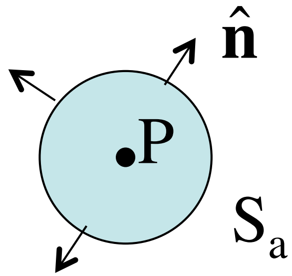
Divergence of a vector field



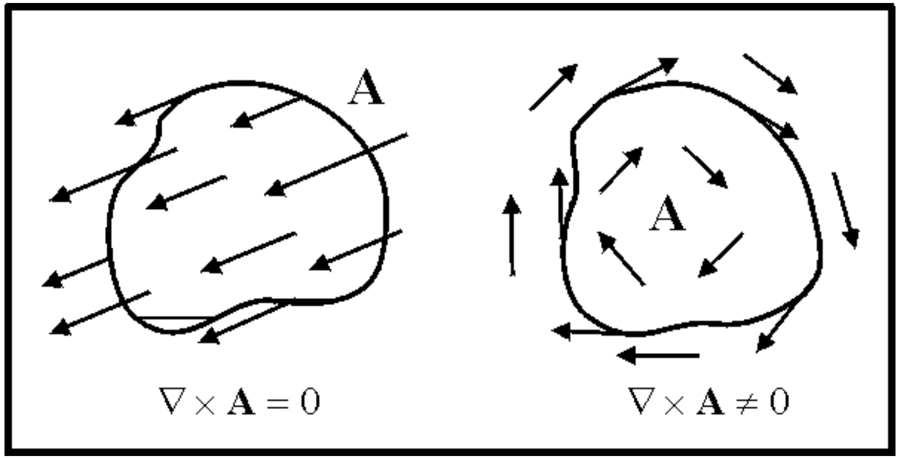
$$\int_V \nabla \cdot \mathbf{v} \, d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}} \, ds$$

Imagine a very small sphere, radius a, with boundary S_a around a point P

$$(\nabla \cdot \mathbf{v})_P \frac{4}{3} \pi a^3 = \oint_{S_a} \mathbf{v} \cdot \hat{\mathbf{n}} \, ds$$



Divergence of a vector field represents the net outward flux per unit volume, i.e. measure of source/sink of flow



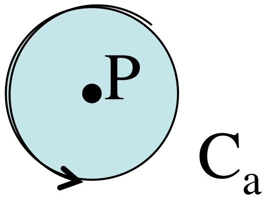
Curl of a vector field

Amount of turn or spin, vorticity, in a vector field

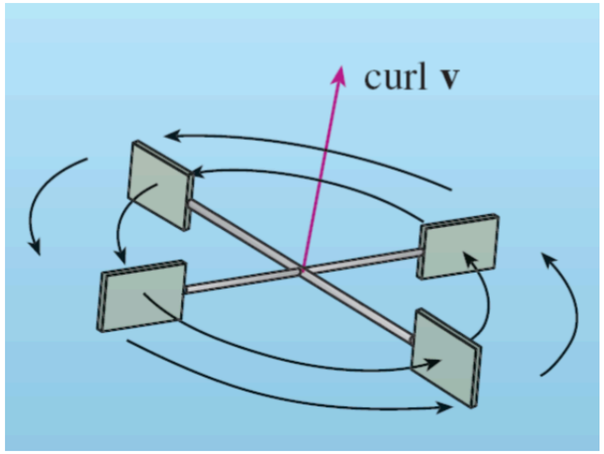
Curl theorem: $\int_V \nabla \times \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$

Right-hand side larger if velocities more parallel to the boundary, spinning in consistent direction, circulation around the boundary

Imagine a very small disk, radius a, with boundary C_a around a point P



$$(\nabla \times \mathbf{v})_P \pi a^2 = \oint_{C_a} \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$$



Try yourself later

e.g. check out Khan Academy lectures on divergence, curl

- *Take a 2D vector field, e.g.* $\mathbf{v} = \begin{pmatrix} xy \\ y^2 - x^2 \end{pmatrix}$
- *Plot the field as vectors on a regular grid*
- *Calculate* $\nabla \cdot \mathbf{v}$ *and plot as a contour plot*
- *Calculate* $\nabla \times \mathbf{v}$ *and plot as contour plot*
- *What do the different values of divergence and curl mean?*

Introduction Tensors

- Tensors, generalisation of vectors to more dimensions
- Use when properties depend on direction in more than one way.
- Stress tensor as example
- Stress is force per area, depends on the direction of the force as well as the chosen cross sectional area (which can be described by its normal) on which the stress is evaluated.

Tensors

Used in

Stress, strain, moment tensors

Electrostatics, electrodynamics, rotation, crystal properties

Tensors describe properties that depend on direction

Tensor rank 0 - scalar - independent of direction

Tensor rank 1 - vector - depends on direction in 1 way

Tensor rank 2 - tensor - depends on direction in 2 ways

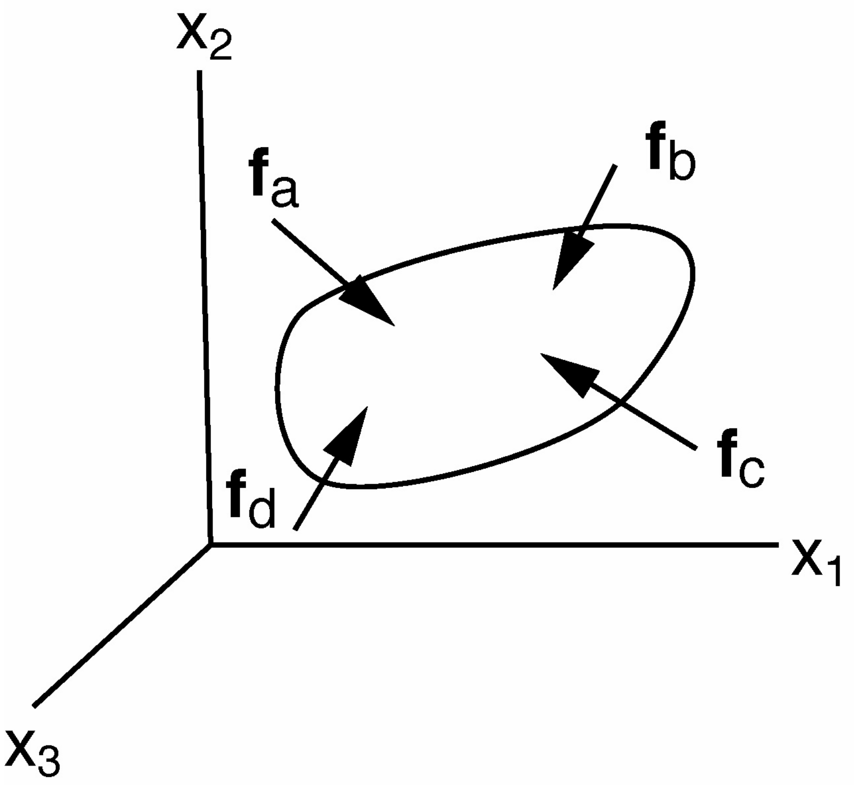
Tensor comes from the word tension (= stress)

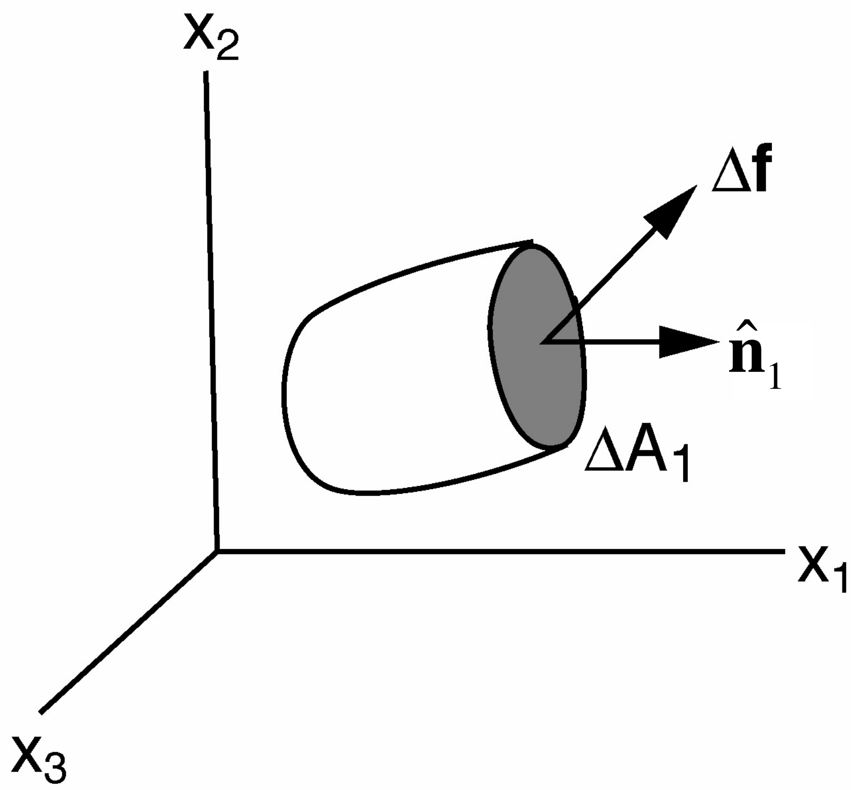
Notation

- Tensors as **T**
- for second order: $\overline{\overline{T}}$ or $\underline{\underline{T}}$
- Index notation T_{ij}, $i,j=x,y,z$ or $i,j=1,2,3$
- But also higher order T_{ijkl}

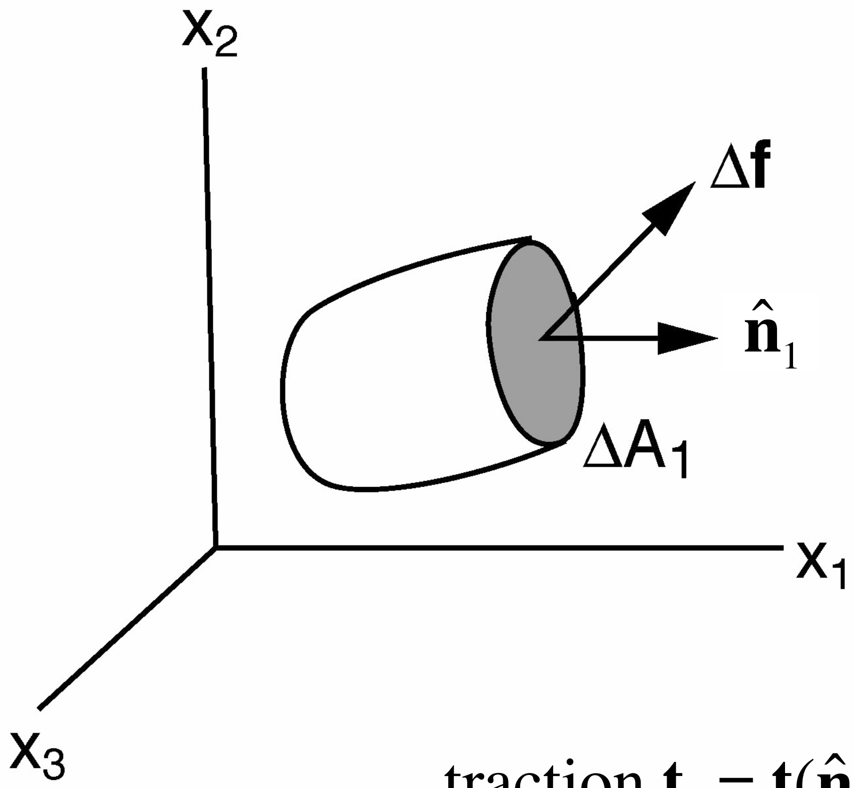
Stress

- *Body forces* - depend on volume, e.g., gravity
- *Surface forces* - depend on surface area, e.g., friction





forces introduce a state of stress in a body

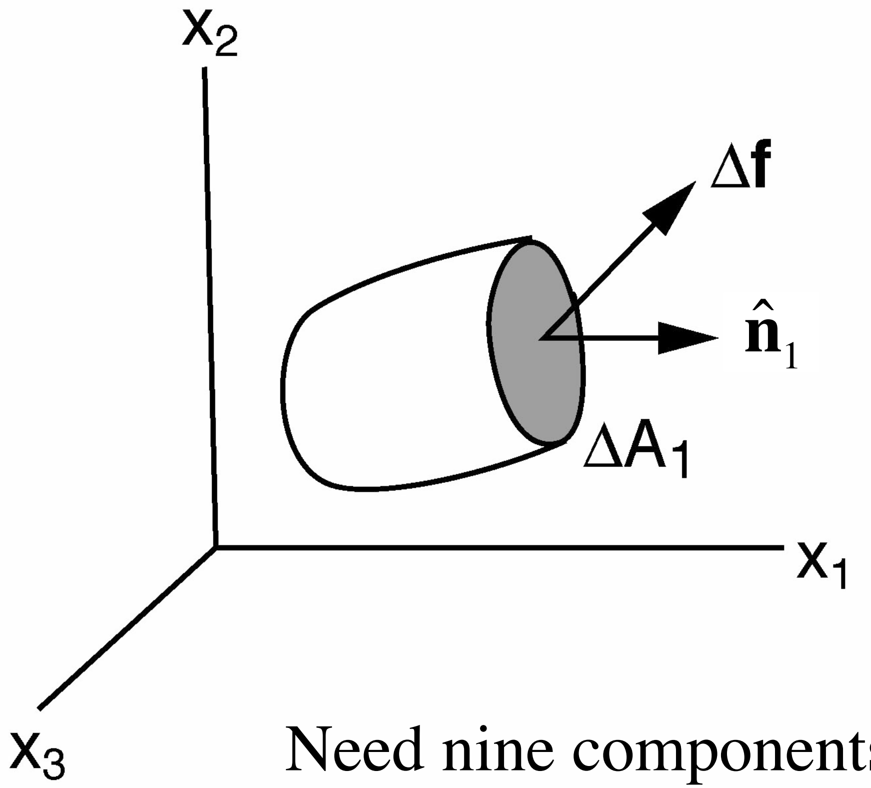


- $\Delta\mathbf{f}$ necessary to maintain equilibrium depends on orientation of the plane, $\hat{\mathbf{n}}_1$

$$\text{traction } \mathbf{t_1} = \mathbf{t}(\hat{\mathbf{n}}_1) = \lim_{\Delta A \to 0} \Delta\mathbf{f}/\Delta A_1$$

$$\mathbf{t_1} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$

$$\sigma_{11} = \lim_{\Delta A1 \to 0} \Delta\mathbf{f}_1/\Delta A_1$$
$$\sigma_{12} = \lim_{\Delta A1 \to 0} \Delta\mathbf{f}_2/\Delta A_1$$
$$\sigma_{13} = \lim_{\Delta A1 \to 0} \Delta\mathbf{f}_3/\Delta A_1$$



$$\mathbf{t_1} = \mathbf{t}(\hat{\mathbf{n}}_1) = \lim_{\Delta A \to 0} \Delta\mathbf{f}/\Delta A_1$$

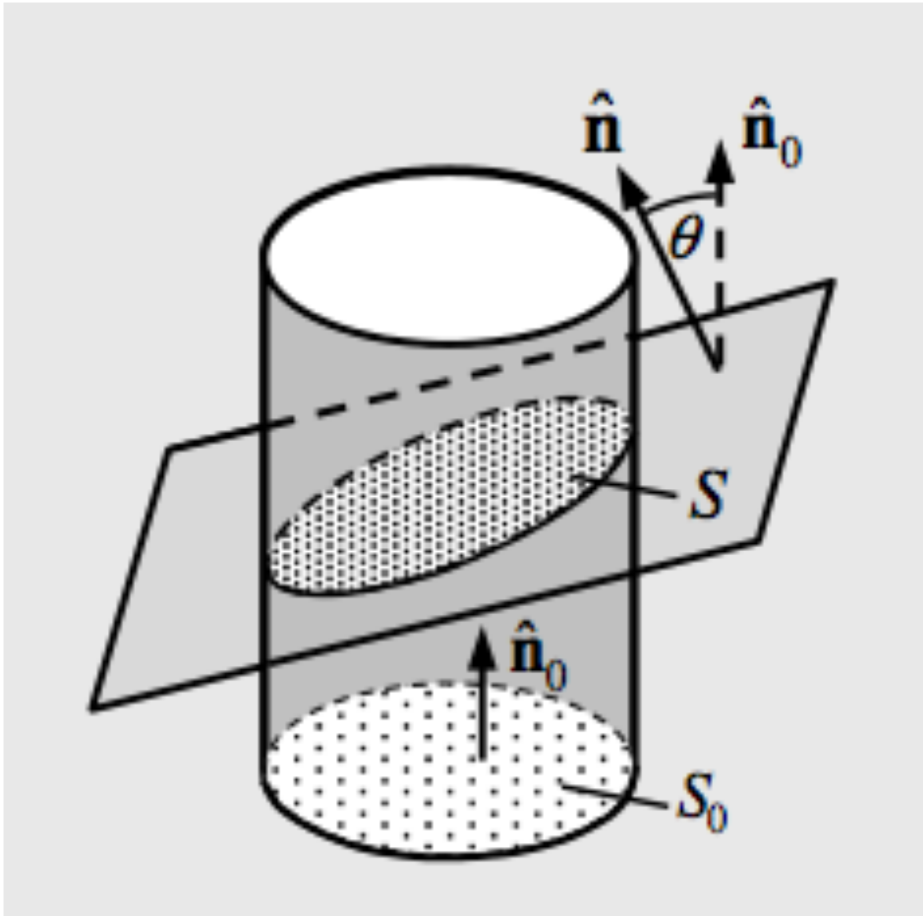
Need nine components to fully describe the stress

$\sigma_{11}, \sigma_{12}, \sigma_{13}$ for ΔA_1
$\sigma_{22}, \sigma_{21}, \sigma_{23}$ for ΔA_2
$\sigma_{33}, \sigma_{31}, \sigma_{32}$ for ΔA_3

first index = orientation of plane
second index = orientation of force

Try now:

Determine the area of plane S assuming S_0 and θ are known. Use vectors to do this.



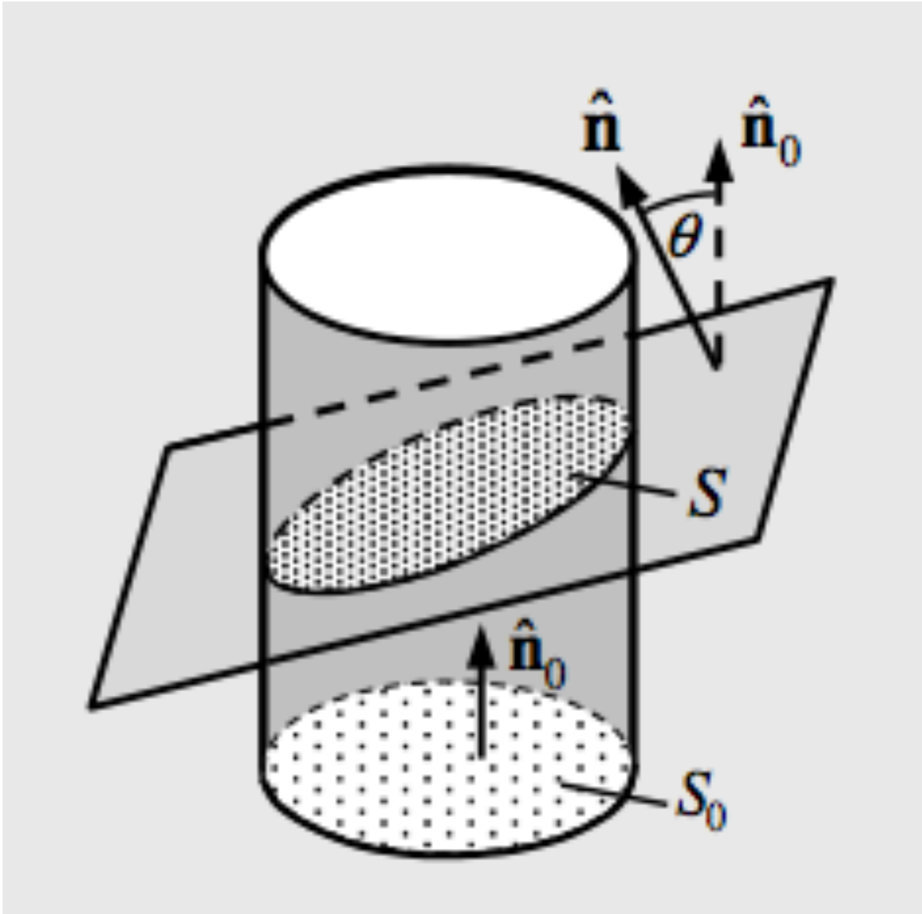
Try now:

Determine the area of plane S assuming S_0 and θ are known. Use vectors to do this.

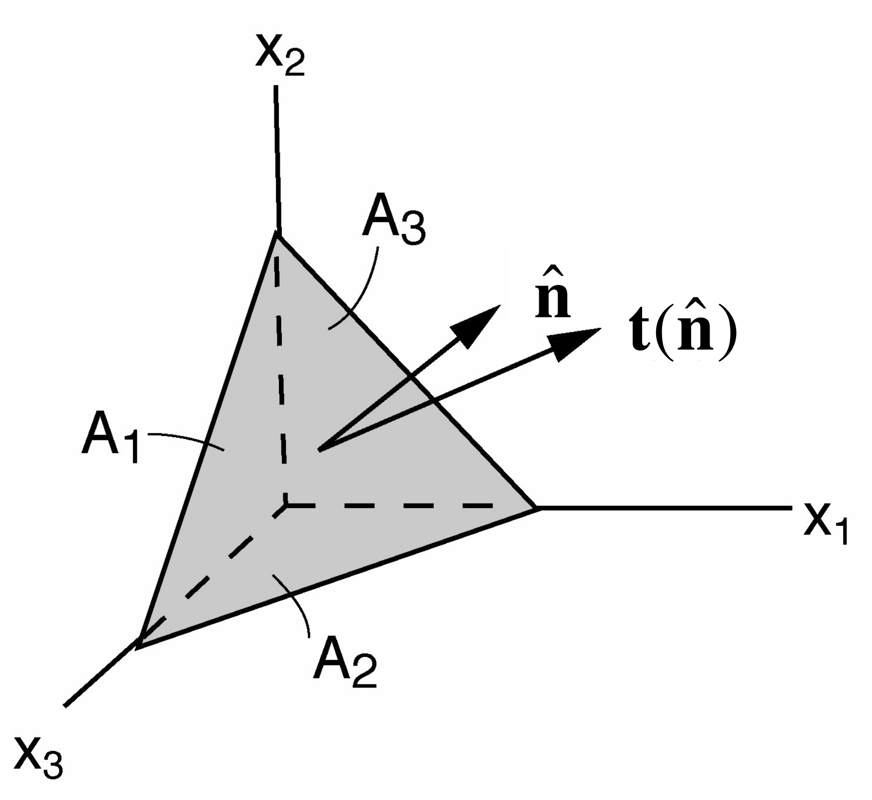
$$S_0 = \mathbf{S} \cdot \hat{\mathbf{n}}_0 =$$

$$S\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 = S\cos\theta$$

$$\Rightarrow S = S_0 / \cos\theta$$

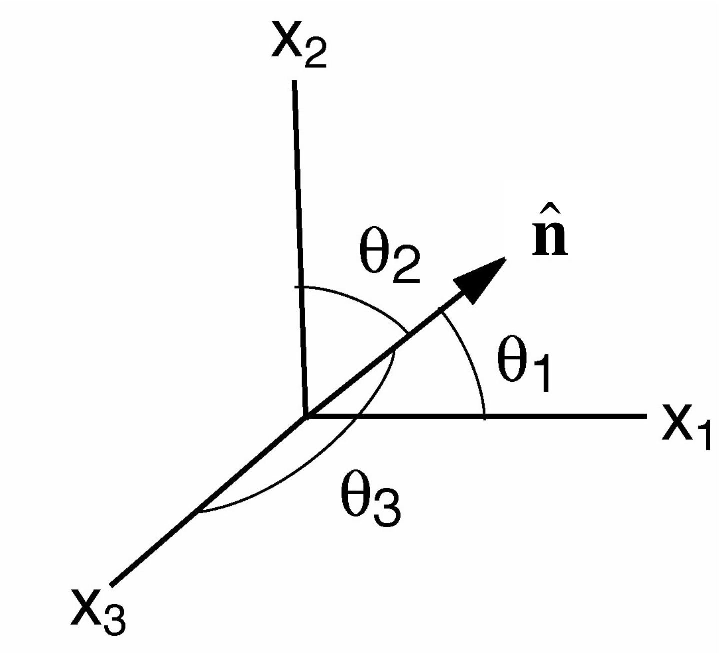


Are nine components sufficient? Demonstrate with equilibrium for a tetrahedron



Given: stress on A_1, A_2, A_3

?: $\mathbf{t}(\hat{\mathbf{n}})$



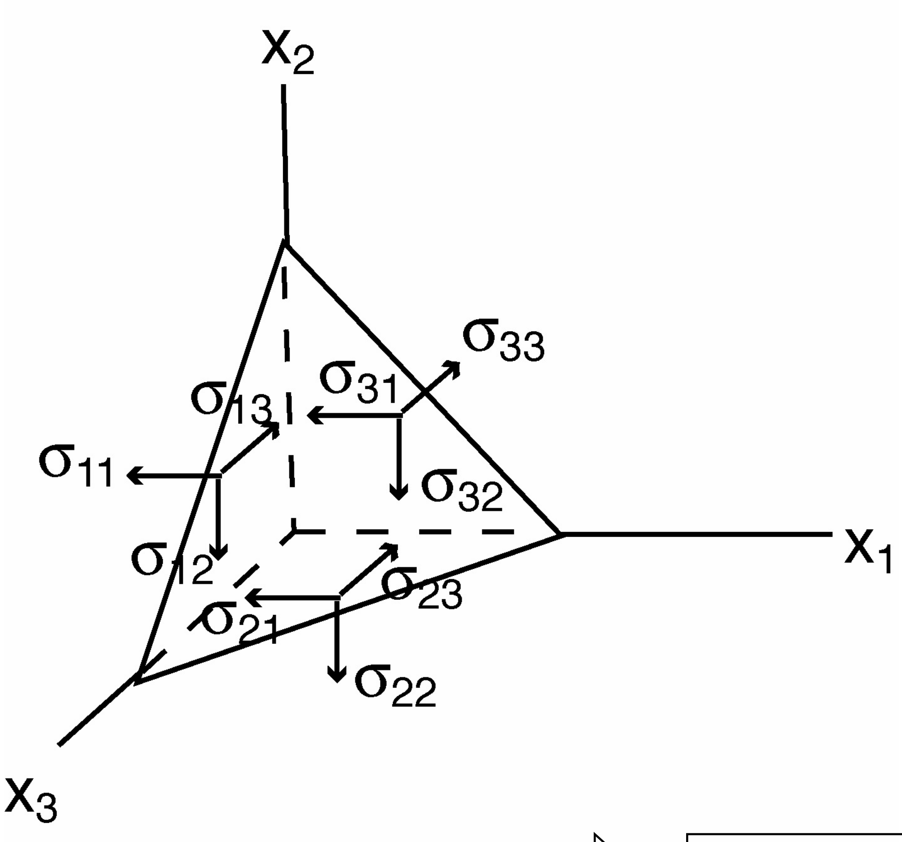
1: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_1$, $\Delta A_1 = \Delta A \cos\theta_1$

2: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_2$, $\Delta A_2 = \Delta A \cos\theta_2$

3: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_3$, $\Delta A_3 = \Delta A \cos\theta_3$

4: $\hat{\mathbf{n}} = (n_1, n_2, n_3)$, $n_i = \cos\theta_i$, $\Delta A_4 = \Delta A$

$$\Sigma f_1 = t_1 \Delta A - \sigma_{11} \Delta A \cos\theta_1 - \sigma_{21} \Delta A \cos\theta_2 - \sigma_{31} \Delta A \cos\theta_3 = 0$$



this gives:

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

similarly:

$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$

$$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

$\Rightarrow$ $\boxed{t_i = \sigma_{ji} n_j}$ (Einstein convention)

How many stress components required in 2D?

Summation (Einstein) convention

When an index in a single term is a duplicate, dummy index, summation implied without writing summation symbol

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \sum_{i=1}^{3} a_i v_i = a_i v_i$$

$$\begin{aligned} \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i y_j = a_{ij} x_i y_j &= a_{11} x_1 y_1 + a_{12} x_1 y_2 + a_{13} x_1 y_3 \\ &+ a_{21} x_2 y_1 + a_{22} x_2 y_2 + a_{23} x_2 y_3 \\ &+ a_{31} x_3 y_1 + a_{32} x_3 y_2 + a_{33} x_3 y_3 \end{aligned}$$

Invalid, indices repeated ~~more than twice~~

$$\sum_{i=1}^{3} a_i b_i v_i \neq a_i b_i v_i$$

Dummy vs free index

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \sum_{i=1}^{3} a_i v_i = \sum_{k=1}^{3} a_k v_k$$

- i,k – dummy index – appears in duplicates and can be substituted without changing equation

$$F_j = A_j \sum_{i=1}^{3} B_i C_i \Rightarrow \begin{aligned} F_1 &= A_1 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right) \\ F_2 &= A_2 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right) \\ F_3 &= A_3 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right) \end{aligned}$$

- j – free index, appears once in each term of the equation

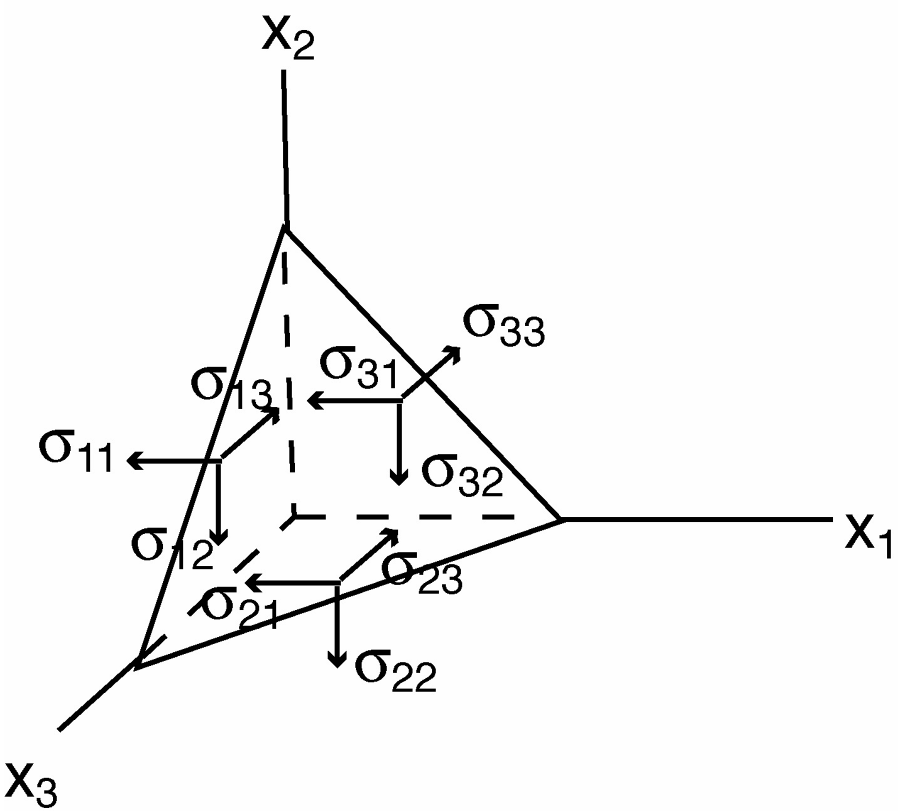
Index notation questions

1. $g_k = h_k(2-3a_ib_i) - p_jq_jf_k$ - *Which dummy, which free indices, how many equations, how many terms in each?*
2. *Are these valid expressions?*
   a) $a_mb_s = c_m(d_r - f_r)$
   b) $x_ix_i = r^2$
   c) $a_ib_jc_j = 3$

Index notation questions

1. $g_k=h_k(2-3a_ib_i)-p_jq_jf_k$ - *Which dummy, which free indices, how many equations, how many terms in each?* *k, i and j, 3, 6*
2. *Are these valid expressions?*
   a) $a_mb_s=c_m(d_r-f_r)$ no
   b) $x_ix_i=r^2$ yes
   c) $a_ib_jc_j=3$ mathematically yes, physically no

$$\Sigma f_1 = t_1 \Delta A - \sigma_{11} \Delta A \cos\theta_1 - \sigma_{21} \Delta A \cos\theta_2 - \sigma_{31} \Delta A \cos\theta_3 = 0$$



this gives:

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

similarly:

$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$

$$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

$$\Rightarrow \boxed{t_i = \sigma_{ji} n_j}$$

(Einstein convention)

How many stress components required in 2D?

$t_i = \sigma_{ji} n_j$ $\quad\quad$ $\mathbf{t} = \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$ $\quad\quad$ Transpose: $\sigma_{ji} = \sigma^T_{ij}$

Note: unusual index order

in matrix notation: $\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$

$\mathbf{t}$ and $\hat{\mathbf{n}}$ - tensors of rank 1 (vectors) in 3-D

$\underline{\underline{\boldsymbol{\sigma}}}$ - tensor of rank 2 in 3-D

compression - negative
tension - positive

σ_{ji} where i=j - normal stresses
σ_{ji} where i≠j - shear stresses

2nd order tensors can be written as square matrices and have algebraic properties similar to some of those of matrices.

Addition and subtraction of tensors

$\mathbf{W} = a\mathbf{T}+b\mathbf{S}$

add each component: $W_{ijkl} = aT_{ijkl} + bS_{ijkl}$

T and **S** must have same rank, dimension and units

W has same rank, dimension and units as **T** and **S**

T and **S** are tensors => **W** is a tensor

commutative, associative

This is same as how vectors and matrices are added.

Multiplication of tensors

Inner product = dot product

$\mathbf{W} = \mathbf{T} \cdot \mathbf{S}$

involves contraction over 1 index: $W_{ik} = T_{ij} S_{jk}$

As normal matrix and matrix-vector multiplication

T and **S** can have different rank, but same dimension

rank$\mathbf{W}$ = rank$\mathbf{T}$+rank$\mathbf{S}$-2, dimension as **T** and **S**,

units as product of units **T** and **S**

T and **S** are tensors => **W** is a tensor

Examples: $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$

$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$ or $\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$ (Hooke's law)

Multiplication of tensors

Tensor product=outer product = dyadic product
≠ cross product

$\mathbf{W} = \mathbf{TS}$ sometimes written as $\mathbf{W} = \mathbf{T} \otimes \mathbf{S}$
no contraction: $W_{ijkl} = T_{ij}S_{kl}$

T and **S** can have different rank, but same dimension
rank**W** = rank**T**+rank**S**, dimension as **T** and **S**,
units as product of units **T** and **S**

T and **S** are tensors => **W** is a tensor

Examples: $\nabla\mathbf{v}$ (gradient of a vector) ≠ $\nabla\cdot\mathbf{v}$ (divergence)

remember gradient is a vector $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$

Multiplication of tensors

For both multiplications

Distributive: $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$

Associative: $\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$

Not commutative: $\mathbf{TS} \neq \mathbf{ST}$, $\mathbf{T}\cdot\mathbf{S} \neq \mathbf{S}\cdot\mathbf{T}$

but: $\mathbf{T}\cdot\mathbf{S}=\mathbf{S}^{T}\cdot\mathbf{T}^{T}$

and: $\mathbf{ab}=(\mathbf{ba})^{\mathbf{T}}$ but only for rank 2

Remember transpose: $\mathbf{a}\cdot\mathbf{T}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{T}^{T}\cdot\mathbf{a} \Rightarrow T_{ji}=T^{T}_{ij}$

Special tensor: Kronecker delta δ_{ij}

$$\delta_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$$

$\delta_{ij} = 1$ for $i=j$, $\delta_{ij} = 0$ for $i \neq j$

In 3-D:

$$\delta = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Isotropic tensors, invariant upon coordinate transformation

- Scalars
- **0** vector
- δ_{ij}

$\mathbf{T}\cdot\boldsymbol{\delta}=\mathbf{T}\cdot\mathbf{I}=\mathbf{T}$ or $T_{ij}\delta_{jk} = T_{ik}$

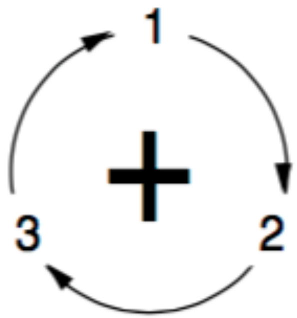
δ is isotropic: $\delta_{ij} = \delta'_{ij}$ upon coordinate transformation
can be used to write dot product: $T_{ij}S_{jl} = T_{ij}S_{kl}\delta_{jk}$
can be used to write trace: $A_{ii} = A_{ij}\delta_{ij}$
orthonormal transformation: $l_{ij}l^T_{jk} = \delta_{ik}$

Special tensor:
Permutation symbol ε_{ijk}

$$\varepsilon_{ijk} = \left(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j\right) \cdot \hat{\mathbf{e}}_k$$

$\varepsilon_{ijk} = 1$ if i,j,k an even permutation of 1,2,3
$\varepsilon_{ijk} = -1$ if i,j,k an odd permutation of 1,2,3
$\varepsilon_{ijk} = 0$ for all other i,j,k

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$
$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$
$\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{222} = \ldots = 0$



Note that $\varepsilon_{ijk} a_i b_j \hat{e}_k$ where $\hat{e}_k$ is the unit vector in k direction is index notation for cross product **a** x **b**

Exercise: useful identity $\varepsilon_{ijm}\, \varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$

Cross product of two vectors:

Try yourself later

Vector Notation	Index Notation
$\vec{a} \times \vec{b} = \vec{c}$	$\epsilon_{ijk} a_j b_k = c_i$

Recall that

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

Now, note that the notation $\epsilon_{ijk} a_j b_k$ represents three terms, the first of which is

$$\begin{aligned} \epsilon_{1jk} a_j b_k &= \\ &= \\ &= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 \\ &= a_2 b_3 - a_3 b_2 \end{aligned}$$

Cross product of two vectors:

Vector Notation	Index Notation
$\vec{a} \times \vec{b} = \vec{c}$	$\epsilon_{ijk} a_j b_k = c_i$

Recall that

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

Now, note that the notation $\epsilon_{ijk} a_j b_k$ represents three terms, the first of which is

$$\begin{aligned}
\epsilon_{1jk} a_j b_k &= \epsilon_{11k} a_1 b_k + \epsilon_{12k} a_2 b_k + \epsilon_{13k} a_3 b_k \\
&= \\
&= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 \\
&= a_2 b_3 - a_3 b_2
\end{aligned}$$

Vector derivatives - curl

Curl of a vector: $$\nabla \times \mathbf{v} = \varepsilon_{ijk} \frac{\partial}{\partial x_i} v_j \hat{\mathbf{e}}_k = \begin{pmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\ \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{pmatrix}$$

In index notation, using special tensor

Some tensor calculus

Gradient of a vector is a tensor: $$\nabla \mathbf{v} = \frac{\partial v_j}{\partial x_i} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Divergence of a vector: $$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

$$\nabla \cdot \mathbf{v} = tr\left(\nabla \mathbf{v}\right)$$

Trace of a tensor is the sum of diagonal elements

Some tensor calculus

Divergence of a tensor:

$$\nabla \cdot T = \frac{\partial T_{ij}}{\partial x_i} = \begin{pmatrix} \dfrac{\partial T_{i1}}{\partial x_i} \\ \dfrac{\partial T_{i2}}{\partial x_i} \\ \dfrac{\partial T_{i2}}{\partial x_i} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial T_{21}}{\partial x_2} + \dfrac{\partial T_{31}}{\partial x_3} \\ \dfrac{\partial T_{12}}{\partial x_1} + \dfrac{\partial T_{22}}{\partial x_2} + \dfrac{\partial T_{32}}{\partial x_3} \\ \dfrac{\partial T_{13}}{\partial x_1} + \dfrac{\partial T_{23}}{\partial x_2} + \dfrac{\partial T_{33}}{\partial x_3} \end{pmatrix}$$

Laplacian = div(grad *f*), where *f* is a scalar function

$$\nabla \cdot \nabla f = \nabla^2 f = \Delta \ f = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial f}{\partial x_1^2} + \frac{\partial f}{\partial x_2^2} + \frac{\partial f}{\partial x_3^2}$$

Summary

- **Vectors**
  - Addition, linear independence
  - Orthonormal Cartesian bases, transformation
  - Multiplication
  - Derivatives, del, div, curl
- **Tensors**
  - Tensors, rank, stress tensor
  - Index notation, summation convention
  - Addition, multiplication
  - Special tensors, δ_{ij} and ε_{ijk}
  - Tensor calculus: gradient, divergence, curl, ..

*Further reading/studying e.g: **Reddy** (2013) 2.2.1-2.2.3, 2.2.5, 2.2.6, 2.4.1, 2.4.4, 2.4.5, 2.4.6, 2.4.8 (not co/contravariant), **Lai, Rubin, Kremple** (2010): 2.1-2.13, 2.16, 2.17, 2.27-2.32, 4.1-4.3, **Khan Academy** – linear algebra, multivariate calculus*