Example ACSE-2 Autumn 2019

Imperial College London 2019

Please answer all questions

Total time for the exam: 90 minutes

Calculators will be provided

Total marks= 80

(1) Give a mathematical definition of the Taylor series. Using an expansion about the point x=0 use Taylor series to estimate the value of exp(x) when x=0.1 accurate to 6 significant figures. [*6 marks*]

(2)

(a) Consider the ordinary differential equation system

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}, \qquad where\ A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$$

The solution takes the form: $\boldsymbol{x}(t) = \boldsymbol{C}exp(tA)$, where $\boldsymbol{C}$ is a constant (vector) of integration. Using matrix diagonalization to compute this matrix exponential, show that the solution can be written as

$$\boldsymbol{x}(t) = \frac{1}{2}(C_1 + C_2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} exp(2t) + \frac{1}{2}(C_2 - C_1)\begin{pmatrix} -1 \\ 1 \end{pmatrix} exp(-4t)$$

[*10 marks*]

(b) What choice of initial condition for this problem yields a solution that tends to zero as *t* tends to infinity? [*4 marks*]

(3) Define the vector one, two and max norms mathematically. Consider a vector with two components, plot the shapes mapped out in 2D by all vectors with unit norm, i.e. the "unit circle", using each of these three norms. [*6 marks*]

(4) Consider a plane of reflection that passes through the origin. Let $\hat{\mathbf{n}}$ be the unit normal vector to the plane and let $\mathbf{r}$ be the position vector for a point in space.

(a) Show that the reflected vector for $\mathbf{r}$ is given by $\mathbf{T} \cdot \mathbf{r} = \mathbf{r} - 2(\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$, where $\mathbf{T}$ is the transformation that corresponds to the reflection. [*6 marks*]

(b) Let $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$. Find the matrix of $\mathbf{T}$. [*5 marks*]

(5) Given a vector field $\mathbf{v} = x_1^2\hat{\mathbf{e}}_1 + x_3^2\hat{\mathbf{e}}_2 + x_2^2\hat{\mathbf{e}}_3$. For the point $\mathbf{x}$=(1,1,0), find the following: [*7 marks*]

i. $\nabla\mathbf{v}$

ii. $\nabla \cdot \mathbf{v}$

iii. $\nabla\times\mathbf{v}$

iv. The differential $\mathbf{dv}$ for $\mathbf{dx} = ds(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)/\sqrt{3}$

(6)

(a) Given the following strain tensor $\boldsymbol{\varepsilon} = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times 10^{-5}$ in a point

i. Sketch and describe how two lines originally in $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ direction, respectively, would be deformed by this strain field. [*5 marks*]

ii. Find the principal strains [*6 marks*]

iii. Find the principal strain directions and use these to sketch how a sphere would deform in this strain field [*9 marks*]

(7)

Consider a case with material flowing in a channel of a width $2b$, driven by a pressure drop along the channel of ΔP over every distance L. Take x_1 to be the direction of flow and x_2 the direction across the width of the channel, i.e. ranging from $-b$ on the bottom to b on the top. The channel can be considered infinite in x_3 direction.

For this case, the Navier-Stokes equation simplifies to:

$$\nabla \cdot \boldsymbol{\tau} = \nabla p$$

where $\boldsymbol{\tau}$ is the deviatoric stress tensor and p is pressure. Assume that there is linear relation between deviatoric stress $\boldsymbol{\tau}$ tensor and strain rate $\mathbf{D}$: $\boldsymbol{\tau} = 2\eta\mathbf{D}$, where η is viscosity.

(a) Write out the relevant components of the force balance for this case, i.e. for the non-zero components of the stress divergence and pressure gradient. [*4 marks*]

(b) Show that the following is the velocity profile:

$$v_1(y) = \frac{1}{2\eta}\frac{\Delta P}{L}(b^2 - x_2^2)$$

[*5 marks*]

(c) Derive equations to describe the pathlines of the flow, i.e. relating position of a fluid particle $\mathbf{x}(t)$ to its initial position $\boldsymbol{\xi}$ and time t. Sketch the pathlines of 3 particles with positions chosen to illustrate the character of the flow. [*7 marks*]