The 3D Wave Equation

Elastic & Acoustic

November 2019

Starting points

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- Particle position (displacement): $\mathbf{u}(\mathbf{r},t)$
- Equation of motion (i.e: $\mathbf{F} = m\mathbf{a}$)
- Hooke's Law: ($\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{e}$)

Assumptions:

- isotropic (direction isn't important)
- small perturbations in $\mathbf{u}$

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Assumptions:

- form of $\boldsymbol{\sigma}$: $_{ij} = e_{kk}\delta_{ij} + 2\mu e_{ij}$
- form of $\boldsymbol{e}$: $e_{ij} = \tfrac{1}{2}(\nabla_i u_j + \nabla_j u_i)$

3d elastic isotropic wave equation

After some working, and use of identity:

$$\nabla^2 \mathbf{a} \equiv \nabla(\nabla\cdot\mathbf{a}) - \nabla\times\nabla\times\mathbf{a}$$

$$\boxed{\begin{aligned}\rho\,\frac{\partial^2\mathbf{u}}{\partial t^2} = {} & (\nabla\cdot\mathbf{u})\,(\nabla\lambda) + \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)(\nabla\mu) \\ & + (\lambda+2\mu)\,\nabla(\nabla\cdot\mathbf{u}) - \mu\,\nabla\times\nabla\times\mathbf{u}\end{aligned}}$$

Note: if Lamé parameters, λ and μ, vary slowly in space then top two terms on RHS are small

Also: above supports both P-waves and S-waves

3d acoustic isotropic wave equation

In the acoustic case we have $\lambda = \rho c^2$ and $\mu = 0$

Also, force is negative gradient of pressure, p

Eliminating $\mathbf{u}$ from 3d elastic equation gives:

$$\boxed{\frac{1}{\rho c^2}\frac{\partial^2 p}{\partial t^2} = \nabla \cdot \left(\frac{1}{\rho}\nabla p\right)}$$

Note: if density varies slowly enough in space then ρ cancels from both sides, and RHS is just $\nabla^2 p$