(1) (a) Energy equation

1-D, steady state, no flow, no strain

$\frac{D}{Dt} = 0$ no advection $\underline{\underline{\sigma}} : \underline{\underline{D}} = 0$

$$\Rightarrow \quad 0 = \nabla \cdot k \nabla T + A$$

in 1-D $\quad 0 = \frac{d}{dy}\left(k \frac{dT}{dy}\right) + A$

(b) k constant

$y = h \quad \frac{dT}{dy} = 0$

$y = 0 \quad T = T_0$

$$\frac{d^2T}{dy^2} = -\frac{A}{k}$$

integrate once $\quad \frac{dT}{dy} = \frac{-Ay}{k} + B$

use $\frac{dT}{dy}(y=h) = 0 \Rightarrow B = \frac{Ah}{k}$

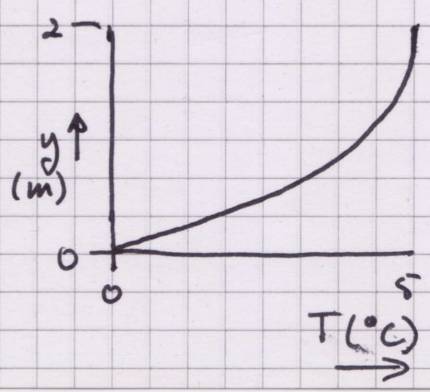
integrate again $\quad T(y) = -\frac{A}{2k} y^2 + By + C$

use $T(y=0) = T_0 \Rightarrow C = T_0$

solution $\quad T(y) = -\frac{A}{2k} y^2 + \frac{Ah}{k} y + T_0$

(c) $k = 80$ W/m/K ; $A = 200$ W/m^3, $h = 2$ m, $T_0 = 0$

$$T(y) = \frac{A}{2k}(2h - y)\, y$$



(2) (a) principal stress and strain axes same for Hookean solid?

Stress $\underline{\underline{\sigma}}$, Strain $\underline{\underline{\varepsilon}}$

Take $\vec{x}_1$ as eigenvector of $\underline{\underline{\varepsilon}}$ with eigenvalue c_1

$$\underline{\underline{\varepsilon}} \cdot \vec{x}_1 = c_1 \vec{x}_1$$

$$\Rightarrow \underline{\underline{\sigma}} \cdot \vec{x}_1 = \lambda \Theta \underline{\underline{I}} \cdot \vec{x}_1 + 2\mu \underline{\underline{\varepsilon}} \cdot \vec{x}_1$$

where $\Theta = tr(\underline{\underline{\varepsilon}})$, i.e. constant

$$\underline{\underline{\sigma}} \cdot \vec{x}_1 = \lambda \Theta \vec{x}_1 + 2\mu c_1 \vec{x}_1$$
$$= (\lambda \Theta + 2\mu c_1) \vec{x}_1$$

$\Rightarrow \vec{x}_1$ is also an eigenvector of $\underline{\underline{\sigma}}$ with as eigenvalue $\lambda\Theta + 2\mu c_1$

(b) Relation between principal stress and strain components

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

$$\left.\begin{aligned} \sigma_1 &= \lambda\Theta + 2\mu\varepsilon_1 \\ \sigma_2 &= \lambda\Theta + 2\mu\varepsilon_2 \\ \sigma_3 &= \lambda\Theta + 2\mu\varepsilon_3 \end{aligned}\right\} \sigma_i = \lambda\Theta + 2\mu\varepsilon_i$$

(3) Young's modulus E and Poisson's ratio ν in terms of Lamé parameters λ, μ

Defined for uniaxial stress

$\sigma_1 = \sigma_1$, $\sigma_2 = \sigma_3 = 0$

Strain for this case:

$\varepsilon_1 = \varepsilon_1$, $\varepsilon_2 = \varepsilon_3$

Young's modulus $\quad E = \dfrac{\sigma_1}{\varepsilon_1}$

Poisson's ratio $\quad \nu = \dfrac{-\varepsilon_3}{\varepsilon_1}$

$$\sigma_1 = \lambda(\varepsilon_1 + 2\varepsilon_3) + 2\mu\varepsilon_1$$

$$\sigma_3 = \lambda(\varepsilon_1 + 2\varepsilon_3) + 2\mu\varepsilon_3 = 0$$

$$\Rightarrow 2(\lambda + \mu)\varepsilon_3 = -\lambda\varepsilon_1 \quad \Rightarrow \quad 2\varepsilon_3 = \frac{-\lambda}{\lambda+\mu}\varepsilon_1$$

$$\Rightarrow \nu = \boxed{\frac{\lambda}{2\lambda+\mu}}$$

$$\Rightarrow \theta = \left(\frac{\lambda+\mu}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}\right)\varepsilon_1 = \frac{\mu}{\lambda+\mu}\varepsilon_1$$

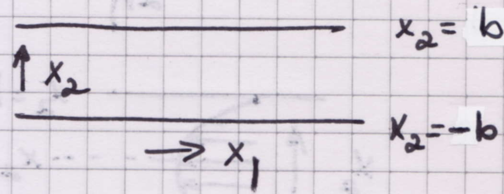
$$\sigma_1 = \left(\frac{\lambda\mu}{\lambda+\mu} + 2\mu\right)\varepsilon_1$$

$$= \left(\frac{\lambda\mu + 2\mu\lambda + 2\mu^2}{\lambda+\mu}\right)\varepsilon_1$$

$$= (3\lambda + 2\mu)\frac{\mu}{\lambda+\mu}\varepsilon_1$$

$$\boxed{E = \frac{\sigma}{\varepsilon_1} = \frac{(3\lambda + 2\mu)\mu}{\lambda+\mu}}$$

(4) Steady unidirectional flow
constant viscosity η
incompressible



$v_1 = v(x_2)$, $v_2 = v_3 = 0$

$v(-b) = v(b) = 0$

Steady $\Rightarrow$ $\frac{D}{Dt} = 0$

no external force $\Rightarrow$ $\vec{f} = 0$

(a) $\Rightarrow$ $-\nabla p + \eta \nabla^2 \vec{v} = 0$ only $\frac{\partial^2 v_1}{\partial x_2^2} \neq 0$

$$-\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2}{\partial x_j^2} v_i = 0$$

(b) $-\frac{\partial p}{\partial x_1} + \eta \frac{d^2 v}{d x_2^2} = 0$, $-\frac{\partial p}{\partial x_2} = -\frac{\partial p}{\partial x_3} = 0$

$\Rightarrow$ pressure does not vary in x_2 and x_3 direction

$$\frac{\partial p}{\partial x_1} = \eta \frac{d^2 v}{\partial x_2^2}$$

$\frac{\partial^2 p}{\partial x_1^2} = 0$ because v only depends on x_2 so $\frac{d^2 v}{d x_2^2}$ also does not vary with x_1

$\Rightarrow$ $\frac{\partial p}{\partial x_1} = \text{constant} = -c$ $\rightarrow$ set to $-c$ realising flow will move in the direction of decreasing p

(c) $\eta \frac{d^2 v}{d x_2^2} = -c$

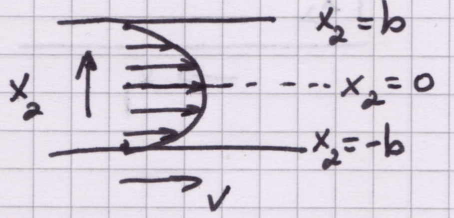
Integrate twice as for T solution of problem (1)

$$\Rightarrow v(x_2) = -\frac{c}{2\eta} x_2^2 + B_1 x + B_2$$

$v(b) = v(-b) = 0$

$$\left.\begin{aligned} -\frac{c}{2\eta} b^2 + B_1 b + B_2 &= 0 \\ -\frac{c}{2\eta} b^2 - B_1 b + B_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} B_1 &= 0 \\ B_2 &= +\frac{c}{2\eta} b^2 \end{aligned}$$

$$\Rightarrow \quad v(x_2) = \frac{c}{2\eta}\left(b^2 - x_2^2\right)$$



parabolic flow profile