ACSE-2

Lecture 6

Stress and Tensors

Outline

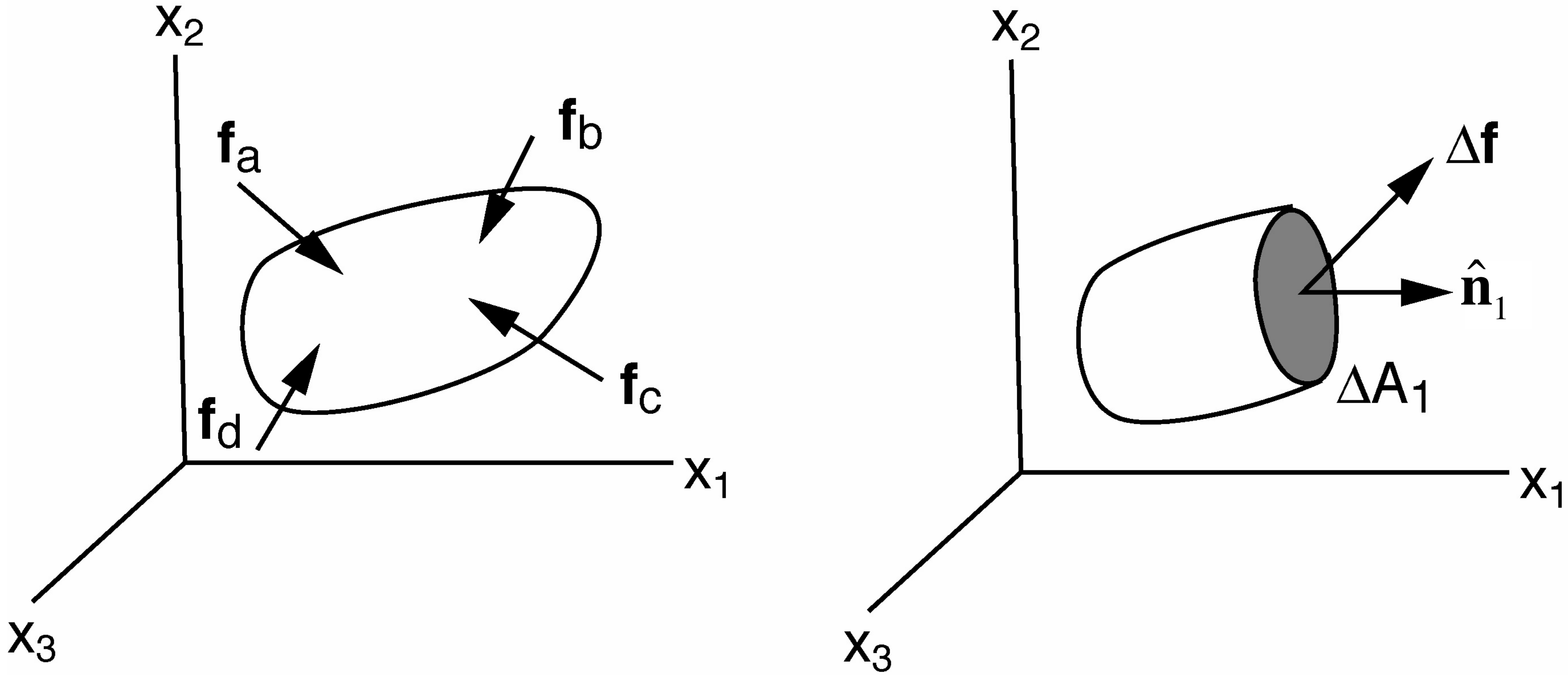
- Cauchy stress tensor recap
- Coordinate transformation (stress) tensors
- (Stress) tensor symmetry
- Tensor invariants
- Diagonalising, eigenvalues, eigenvectors
- Special stress states
- Equation of motion

Learning Objectives

- Understand meaning of different components of 3D Cauchy stress tensor, and know how to determine state of stress on given plane
- Be able to transform rank 2 tensor to a new basis.
- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components
- Be able to find principal stresses and stress invariants and know what they represent
- Be able to balance body forces and stresses

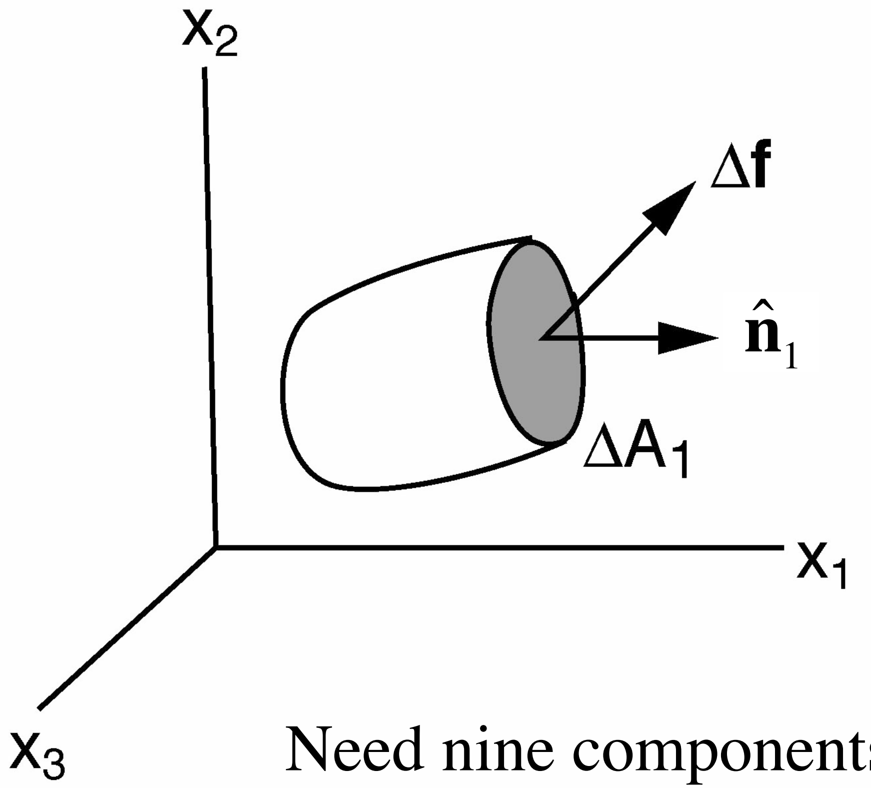
Cauchy Stress

Stress in a point, measured in medium as deformed by the stress experienced.



forces introduce a state of stress in a body

(Other stress measures, e.g., Piola-Kirchhoff tensor, used in Lagrangian formulations)



traction, stress vector

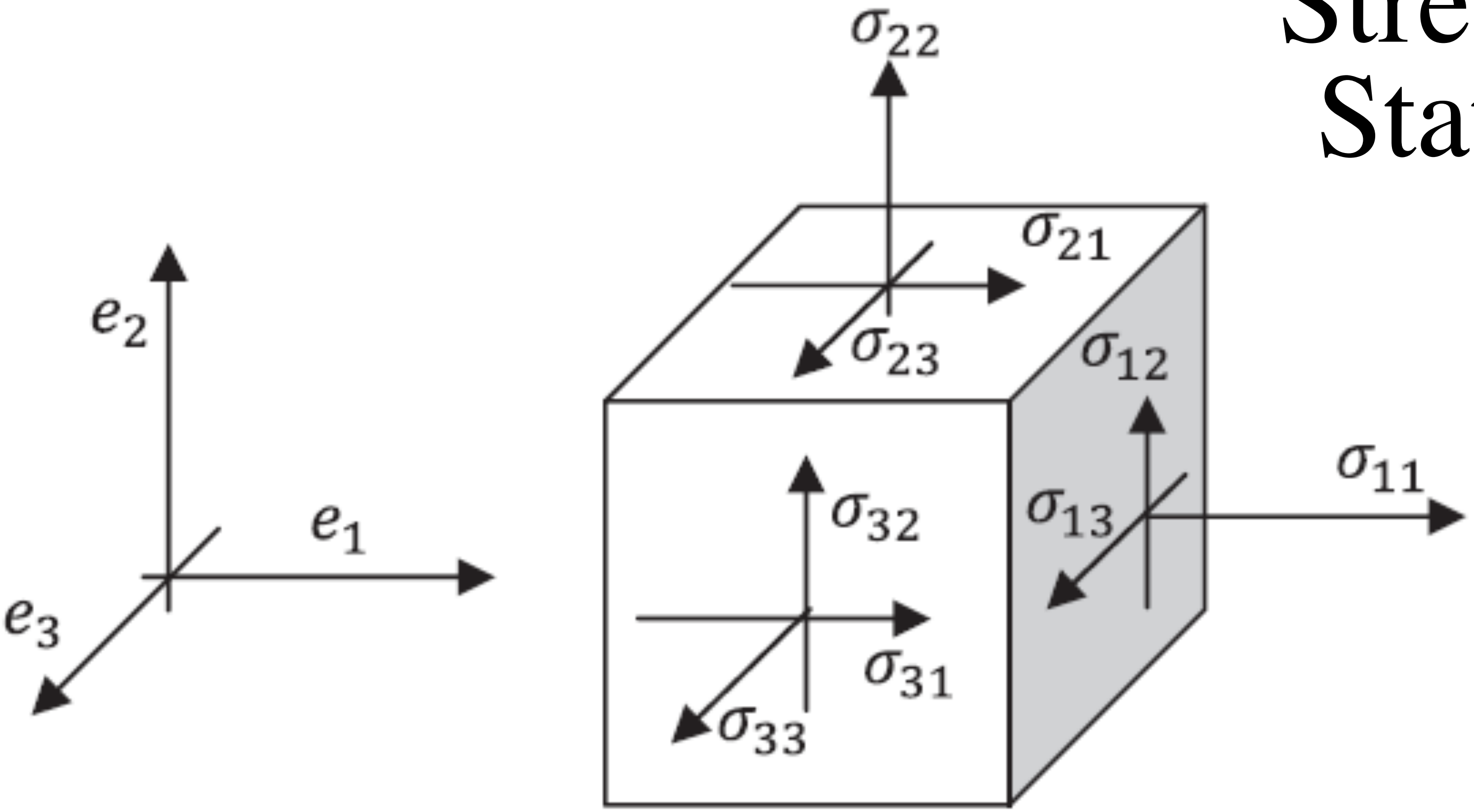
$$\mathbf{t}_{\hat{\mathbf{n}}_1} = \mathbf{t}(\hat{\mathbf{n}}_1) = \lim_{\Delta A \to 0} \Delta \mathbf{f}/\Delta A_1$$

Need nine components to fully describe the stress

$\sigma_{11}, \sigma_{12}, \sigma_{13}$ for ΔA_1
$\sigma_{22}, \sigma_{21}, \sigma_{23}$ for ΔA_2
$\sigma_{33}, \sigma_{31}, \sigma_{32}$ for ΔA_3

first index = orientation of plane
second index = orientation of force

3-D Stress State



first index = orientation of plane
second index = orientation of force

Positive if force in direction of normal (as shown)

$t_i = \sigma_{ji} n_j$ $\mathbf{t} = \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$ Transpose: $\sigma_{ji} = \sigma^T_{ij}$

Note: unusual index order

in matrix notation: $$\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$$

$\mathbf{t}$ and $\hat{\mathbf{n}}$ - tensors of rank 1 (vectors) in 3-D

$\underline{\underline{\boldsymbol{\sigma}}}$ - tensor of rank 2 in 3-D

compression - negative

tension - positive

σ_{ji} where i=j - normal stresses

σ_{ji} where i≠j - shear stresses

2nd order tensors can be written as square matrices and have algebraic properties similar to some of those of matrices.

Example to try

Assume state of stress in a point described by stress tensor

$$\boldsymbol{\sigma} = -p\mathbf{I}$$

How could you show that there is no shearing stress on any plane containing this point?

Example to try

Assume state of stress in a point described by stress tensor

$$\boldsymbol{\sigma} = -p\mathbf{I}$$

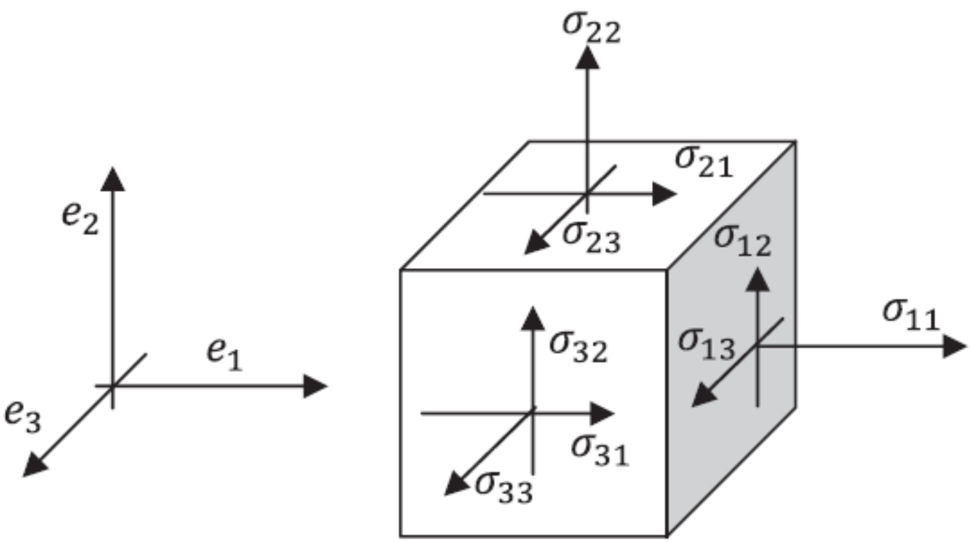
How could you show that there is no shearing stress on any plane containing this point?

By showing that traction vector on any plane with normal $\hat{\mathbf{n}}$

$$\mathbf{t} = \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}} = -p\hat{\mathbf{n}}$$

i.e., normal stress, no matter which orientation of a plane

Stress components



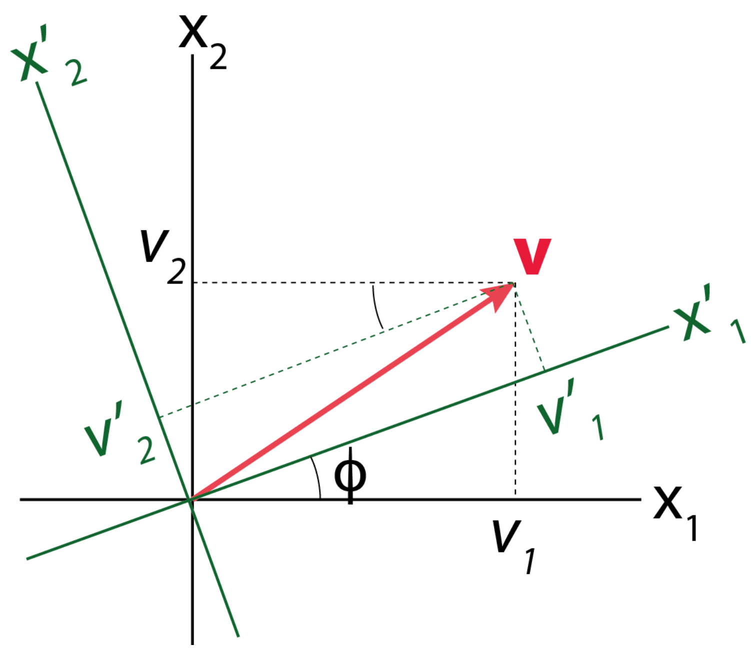
traction on a plane $\quad \mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$

what is $\hat{\mathbf{e}}_1 \cdot \mathbf{t} = \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$?

what is $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_1$? what is $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_2$?

physical parameters should not depend on coordinate frame

$\Rightarrow$ **tensors follow linear transformation laws**



for vectors on orthonormal basis:

$$v'_1 = \alpha_{11}v_1 + \alpha_{12}v_2$$

$$v'_2 = \alpha_{21}v_1 + \alpha_{22}v_2$$

$$\Rightarrow \quad \mathbf{v}' = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mathbf{v}$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v} = \begin{bmatrix} \cos\phi & \cos(90-\phi) \\ \cos(90+\phi) & \cos\phi \end{bmatrix} \mathbf{v}$$

$$\boxed{\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j}$$

In a new coordinate system:

traction $t'_i = \alpha_{ik} t_k$

normal $n'_j = \alpha_{jl} n_l$

$t_k = \sigma^T_{kl} n_l$

$t'_i = \sigma'^T_{ij} n'_j$

Relation $\boldsymbol{\sigma}'$ to $\boldsymbol{\sigma}$?

$\Rightarrow$ *transformation for stress tensor*

$$t'_i = \alpha_{ik} \sigma^T_{kl} n_l$$
$$= \alpha_{ik} \sigma^T_{kl} \alpha^{-1}_{lj} n'_j$$
$$= \alpha_{ik} \sigma^T_{kl} \alpha_{jl} n'_j$$

- transformation matrices are orthogonal
  $\alpha^{-1}_{jl} = \alpha_{lj}$ ($\mathbf{A}^{-1} = \mathbf{A}^T$)
- *remember* $\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$
  $\alpha^{-1}_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}'_j = \alpha_{ji} = \alpha^T_{ij}$

$\Rightarrow$ $\sigma'^T_{ij} = \alpha_{ik} \sigma^T_{kl} \alpha_{jl} = \alpha_{ik} \alpha_{jl} \sigma^T_{kl}$

$\boldsymbol{\sigma}'^T = \mathbf{A} \boldsymbol{\sigma}^T \mathbf{A}^T$

$\Rightarrow$ each dependence on direction transforms as a vector, requiring two transformations

An *n-dimensional* tensor of *rank r* consists of n^r components

This tensor $T_{i1,i2,\dots,in}$ is defined relative to a basis of the real, linear n-dimensional space S_n

and under a coordinate transformation **T** transforms as:

$T'_{ij\dots n} = \alpha_{ip}\alpha_{jq}\dots\alpha_{nt}\, T_{pq\dots t}$

For *orthonormal* bases the matrices α_{ik} are *orthogonal* transformations, i.e. $\alpha_{ik}^{-1} = \alpha_{ki}$. (columns and rows are orthogonal and have length =1, i.e., perpendicular unit vectors are transformed to unit vectors)
If the basis is *Cartesian*, α_{ik} are *real*.

Difference tensor and its matrix

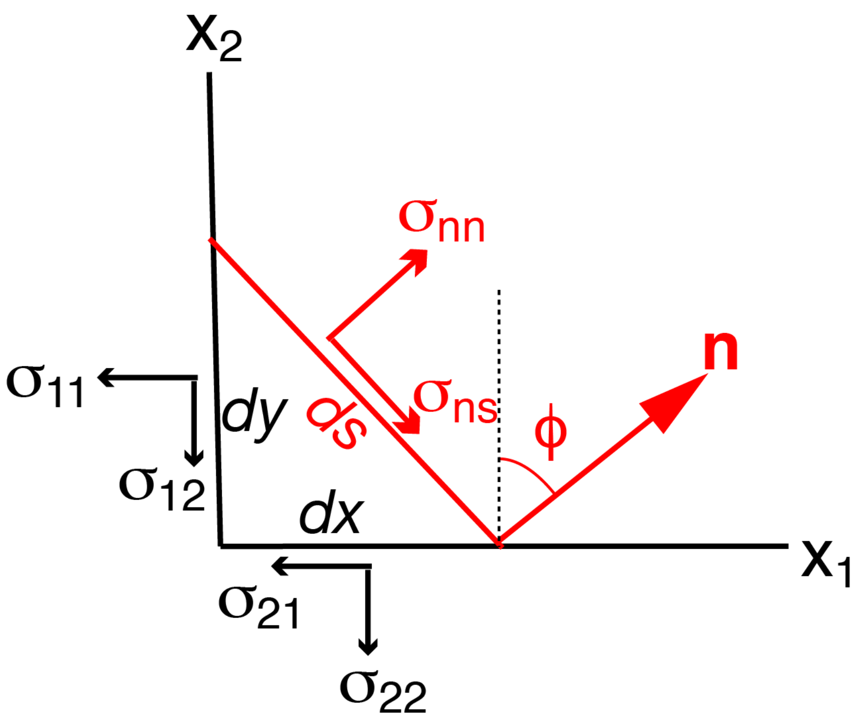
Tensor – physical quantity which is independent of coordinate system used

Matrix of a tensor – contains components of that tensor in a particular coordinate frame

Could test that indeed tensor addition and multiplication satisfy transformation laws

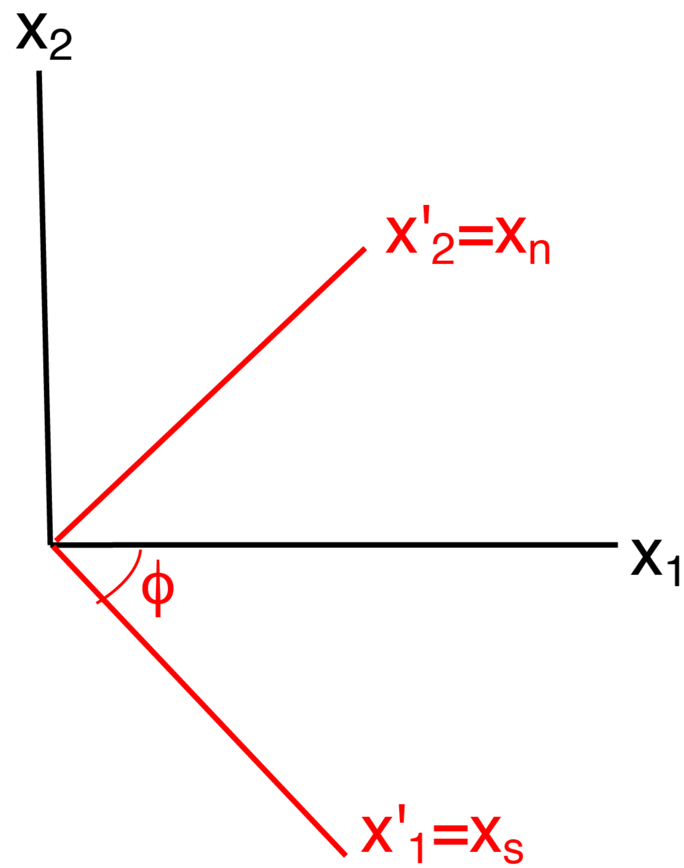
Transforming the 2-D stress tensor

(determining normal and shear stress on a plane)



$dx = \cos\phi \, ds$

$dy = \sin\phi \, ds$



Try writing force balance in x_1 direction

Force balance

in x_1 direction: (1) *Try writing force balance in x_1 direction*

in x_2 direction: (2)

$$\sigma_{12}dy + \sigma_{22}dx = \sigma_{nn}\cos\phi ds - \sigma_{ns}\sin\phi ds$$

$$\sigma_{12}\sin\phi + \sigma_{22}\cos\phi = \sigma_{nn}\cos\phi - \sigma_{ns}\sin\phi$$

(1)·sinϕ + (2)·cosϕ: *verify yourself*

$$\sigma_{nn} = \sigma_{11}\sin^2\phi + \sigma_{21}\cos\phi\sin\phi + \sigma_{12}\cos\phi\sin\phi + \sigma_{22}\cos^2\phi$$

(1)·cosϕ − (2)·sinϕ:

$$\sigma_{ns} = \sigma_{11}\cos\phi\sin\phi + \sigma_{21}\cos^2\phi - \sigma_{12}\sin^2\phi - \sigma_{22}\cos\phi\sin\phi$$

This is equivalent to the **tensor transformation** $\sigma'_{qp}=\alpha_{pi}\alpha_{qj}\sigma_{ji}$

$$\sigma'_{nn}=\alpha_{ni}\alpha_{nj}\sigma_{ji}$$

$$\sigma'_{ns}=\alpha_{si}\alpha_{nj}\sigma_{ji}$$

With $\alpha_{n1} = \sin\phi$, $\alpha_{n2} = \cos\phi$, $\alpha_{s1} = \cos\phi$, $\alpha_{s2} = -\sin\phi$

Force balance

in x_1 direction: (1) $$\sigma_{11}dy + \sigma_{21}dx = \sigma_{nn}\sin\phi ds + \sigma_{ns}\cos\phi ds$$

$$\sigma_{11}\sin\phi + \sigma_{21}\cos\phi = \sigma_{nn}\sin\phi + \sigma_{ns}\cos\phi$$

in x_2 direction: (2) $$\sigma_{12}dy + \sigma_{22}dx = \sigma_{nn}\cos\phi ds - \sigma_{ns}\sin\phi ds$$

$$\sigma_{12}\sin\phi + \sigma_{22}\cos\phi = \sigma_{nn}\cos\phi - \sigma_{ns}\sin\phi$$

(1)·sinϕ + (2)·cosϕ: *verify yourself*

$$\sigma_{nn} = \sigma_{11}\sin^2\phi + \sigma_{21}\cos\phi\sin\phi + \sigma_{12}\cos\phi\sin\phi + \sigma_{22}\cos^2\phi$$

(1)·cosϕ − (2)·sinϕ:

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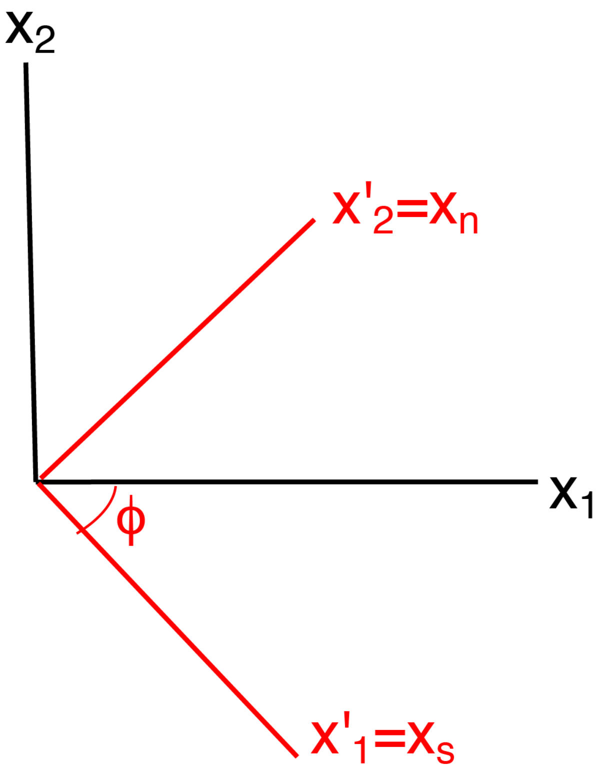
With $\alpha_{n1} = \sin\phi$, $\alpha_{n2} = \cos\phi$, $\alpha_{s1} = \cos\phi$, $\alpha_{s2} = -\sin\phi$

Write out transformation

$\sigma'_{qp}=\alpha_{pi}\alpha_{qj}\sigma_{ji}$



$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$

$$\alpha_{s1} = \hat{\mathbf{e}}_s \cdot \hat{\mathbf{e}}_1 = \cos\phi$$

$$\alpha_{s2} = \hat{\mathbf{e}}_s \cdot \hat{\mathbf{e}}_2 = -\sin\phi$$

$$\alpha_{n2} = \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2 = \cos\phi$$

$$\alpha_{n1} = \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1 = \sin\phi$$

In tensor notation:

$\boldsymbol{\sigma}'^{T} = \mathbf{A}\cdot\boldsymbol{\sigma}^{T}\cdot\mathbf{A}^{T}$

In matrix notation:

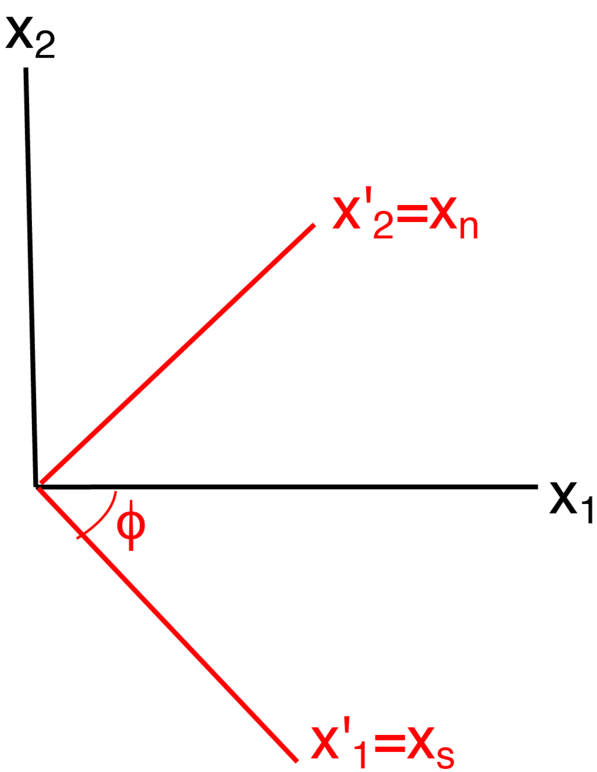
$$\begin{bmatrix} \sigma_{ss} & \sigma_{ns} \\ \sigma_{sn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Write out matrices A and A^T

Check that the expressions for σ_{nn}, σ_{ns} of previous slide obtained

Write out transformation

$\sigma'_{qp}=\alpha_{pi}\alpha_{qj}\sigma_{ji}$



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In tensor notation:

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In matrix notation:

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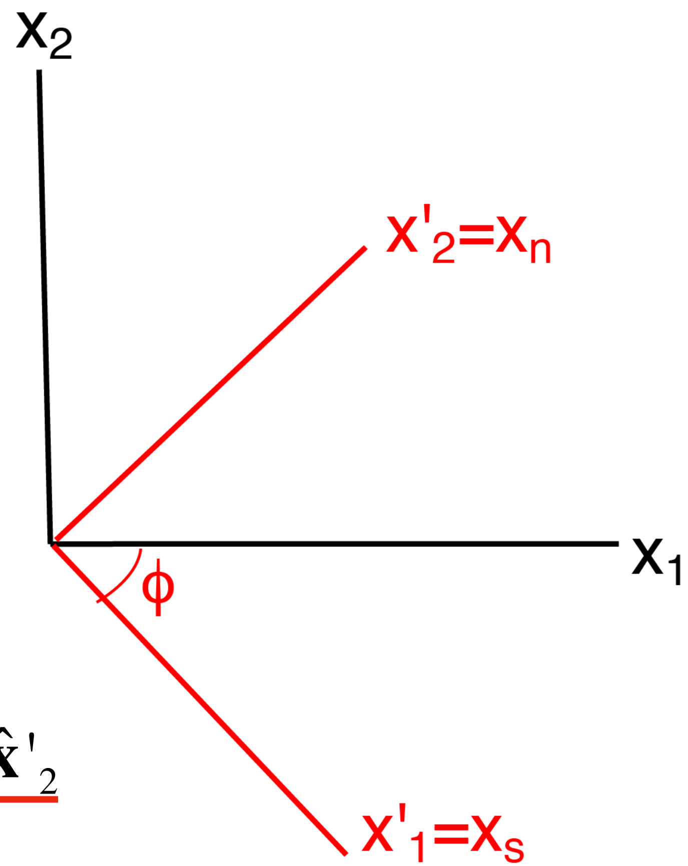
$$\begin{bmatrix} \sigma_{ss} & \sigma_{ns} \\ \sigma_{sn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

For $\hat{\mathbf{x}}_1=(1,0)$, $\hat{\mathbf{x}}_2=(0,1)$,
first row of $\mathbf{A}$ consists of $\hat{\mathbf{x}}'_1$, second of $\hat{\mathbf{x}}'_2$

$$\hat{\mathbf{x}}'_1 = (\cos\phi, -\sin\phi)$$

$$\hat{\mathbf{x}}'_2 = (\sin\phi, \cos\phi)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{x}'_1 \cdot \mathbf{x}_1 & \mathbf{x}'_1 \cdot \mathbf{x}_2 \\ \mathbf{x}'_2 \cdot \mathbf{x}_1 & \mathbf{x}'_2 \cdot \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$



You may recognise $\mathbf{A}$ *as a matrix that describes a rigid-body rotation over and angle* $-\phi$

A^T *describes a rotation over angle* ϕ

First column of $\mathbf{A}^{\mathrm{T}}$ consists of $\hat{\mathbf{x}}'_1$, second of $\hat{\mathbf{x}}'_2$

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{x}'_1 & \mathbf{x}_1 \cdot \mathbf{x}'_2 \\ \mathbf{x}_2 \cdot \mathbf{x}'_1 & \mathbf{x}_2 \cdot \mathbf{x}'_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

Tensor symmetry

A tensor can be symmetric in 1 or more indices

In 2-D:

$S_{ij}= S_{ji}$ => $\mathbf{S}=\mathbf{S}^T$ symmetric

$S_{ij}= -S_{ji}$ => $\mathbf{S}=-\mathbf{S}^T$ antisymmetric

Higher rank:

e.g., $S_{ijk}= S_{jik}$ for all i,j,k => symmetric in i,j

Write out general antisymmetric **T** *rank 2, n=3 => how many independent components?*

antisymmetric **T** of rank 2

symmetric **T** of rank 2

has n(n+1)/2 independent components

Any **T** of rank 2 can be decomposed in symm. and antisymm. part:

$$T_{ij}= (T_{ij}+T_{ji})/2 + (T_{ij}-T_{ji})/2$$

Tensor symmetry

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Higher rank:

e.g., $S_{ijk}= S_{jik}$ for all i,j,k => symmetric in i,j

antisymmetric **T** of rank 2

=> $T_{ij}=0$ for i=j, trace(**T**)=0

has n(n-1)/2 independent components

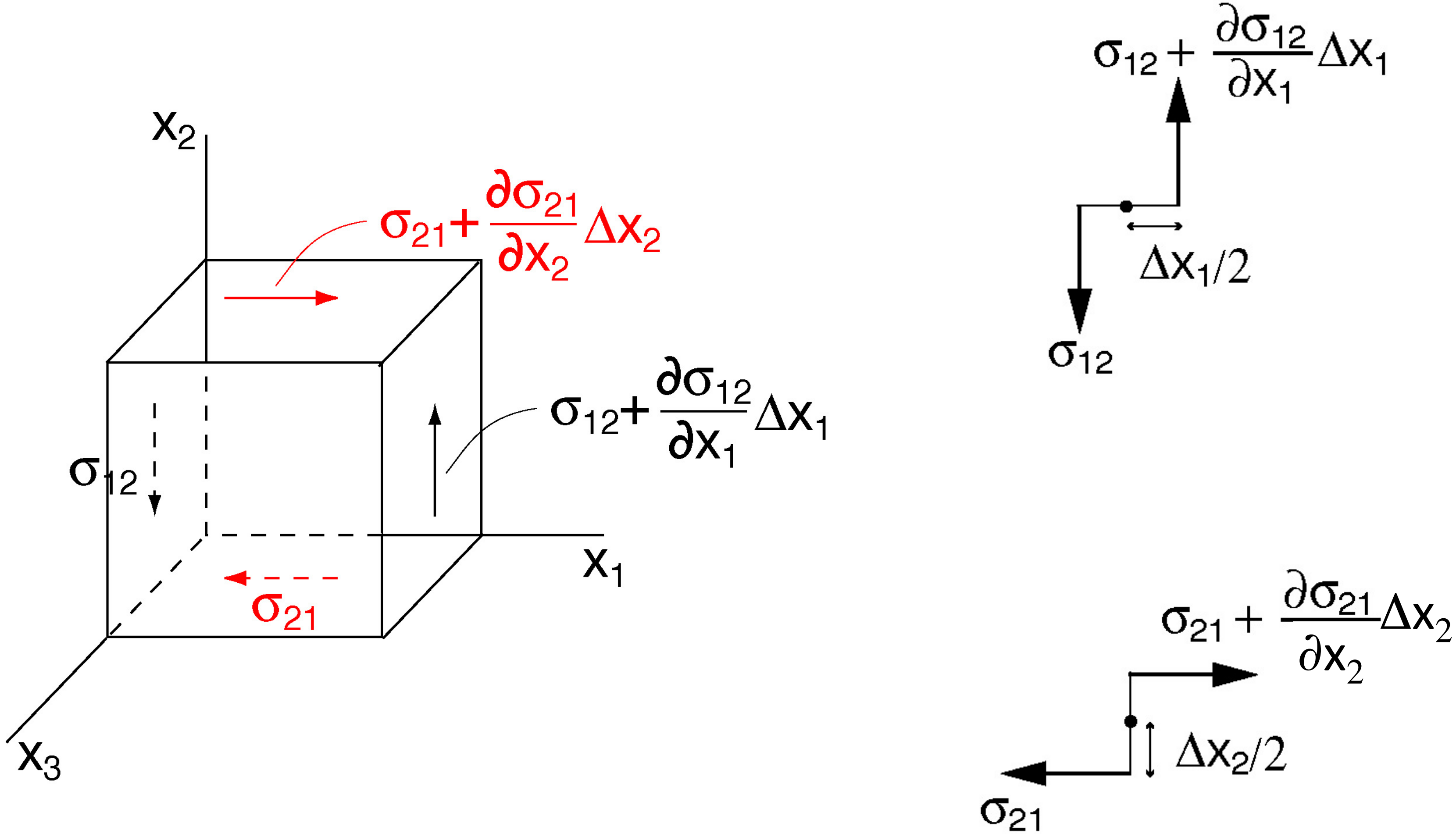
symmetric **T** of rank 2

has n(n+1)/2 independent components

Any **T** of rank 2 can be decomposed in symm. and antisymm. part:

$$T_{ij}= (T_{ij}+T_{ji})/2 + (T_{ij}-T_{ji})/2$$

Symmetry of the stress tensor



Try writing out the balance of moments in x_3 direction,

assuming static equilibrium

A balance of moments in x_3 direction:

$$m_3 = [\qquad\qquad\qquad\qquad\qquad] \qquad \cdot \Delta x_1 / 2$$

$$-[\qquad\qquad\qquad\qquad\qquad] \qquad \cdot \Delta x_2 / 2 = 0$$

$$\Rightarrow [2\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})] - [2\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})] = 0$$

$$\lim_{\Delta x_1, \Delta x_2} \to 0 \Rightarrow \boxed{\sigma_{12} = \sigma_{21}}$$

Note: if body force induced rotation:

$$I_{33} \frac{\partial \omega}{\partial t} = O(\Delta x^2)$$

Balancing m_1 and m_2: $\boxed{\sigma_{23}=\sigma_{32}}$ *and* $\boxed{\sigma_{13}=\sigma_{31}}$

*thus, the stress tensor is **symmetric***

$$\mathbf{t} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \hat{\mathbf{n}} \Rightarrow \mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$

A balance of moments in x_3 direction:

$$m_3 = [\sigma_{12} + (\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})]\Delta x_2 \Delta x_3 \cdot \Delta x_1 / 2$$
$$-[\sigma_{21} + (\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})]\Delta x_1 \Delta x_3 \cdot \Delta x_2 / 2 = 0$$

$$\Rightarrow [2\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})] - [2\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})] = 0$$

$$\lim_{\Delta x_1, \Delta x_2} \to 0 \Rightarrow \boxed{\sigma_{12} = \sigma_{21}}$$

Note: if body force induced rotation:

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thus, the stress tensor is ***symmetric***

$$\mathbf{t} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \hat{\mathbf{n}} \Rightarrow \mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$

Diagonalizing

Real-valued, symmetric rank 2 tensors (square, symmetric matrices) can be diagonalized, i.e. a coordinate frame can be found, such that only the diagonal elements (normal stresses) remain.

For stress tensor, these elements, $\sigma_1, \sigma_2, \sigma_3$ are called the principal stresses

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Such a transformation can be cast as:

$\mathbf{T} \cdot \mathbf{x} = \lambda \mathbf{x}$

where $\mathbf{x}_i$ are eigenvectors or characteristic vectors

and λ_i are the eigenvalues, characteristic or principal values

$\Rightarrow (\mathbf{T} - \lambda \boldsymbol{\delta}) \cdot \mathbf{x} = \mathbf{0}$

Non-trivial solution only if $\det(\mathbf{T} - \lambda \boldsymbol{\delta}) = 0$

Determinant

For 2-dimensional rank 2 tensor

$$\det(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} = T_{11}T_{22} - T_{12}T_{21}$$

$\det(\mathbf{T}) \neq 0$
columns of **T**
are linearly
independent,
and $\mathbf{T}^{-1}$ exists

$$\det(\mathbf{a},\mathbf{b}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \mathbf{a} \times \mathbf{b}$$

signed area

For 3-dimensional rank 2 tensor $\mathbf{T}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}$

$$\mathbf{T} \cdot \hat{\mathbf{e}}_1 = \mathbf{a}$$
$$\mathbf{T} \cdot \hat{\mathbf{e}}_2 = \mathbf{b}$$
$$\mathbf{T} \cdot \hat{\mathbf{e}}_3 = \mathbf{c}$$

$$\underline{\det(\mathrm{T})} = \det(\mathbf{a},\mathbf{b},\mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

$$= \underline{\varepsilon_{ijk} a_i b_j c_k = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}}$$

signed volume

Determinant and cross product

Can write cross product as a determinant

$$\mathbf{a}\times\mathbf{b} = \varepsilon_{ijk} a_i b_j \hat{\mathbf{e}}_k = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\hat{\mathbf{e}}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\mathbf{e}}_2 \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Diagonalising

Write out characteristic equation for n=2

$\det(\mathbf{T}-\lambda\boldsymbol{\delta}) = 0$ => eigenvalues λ_i $\quad$ i=1,n

$\det(\mathbf{T}-\lambda\boldsymbol{\delta}) = -\lambda^3+\mathrm{tr}(\mathbf{T})\lambda^2-\mathrm{minor}(\mathbf{T})\lambda+\det(\mathbf{T})=0$ $\quad$ for n=3

characteristic equation + coefficients are *tensor invariants*

$$I_1 = \mathrm{tr}(\mathbf{T}) = T_{11} + T_{22} + T_{33}$$

$$I_2 = \mathrm{minor}(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{31} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{32} \\ T_{32} & T_{33} \end{vmatrix}$$

$$=$$

$$I_3 = \det(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} & T_{31} \\ T_{21} & T_{22} & T_{32} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} =$$

Diagonalising

Write out characteristic equation for n=2

$\det(\mathbf{T}-\lambda\boldsymbol{\delta}) = 0 \Rightarrow$ eigenvalues λ_i $i=1,n$

$\det(\mathbf{T}-\lambda\boldsymbol{\delta}) = -\lambda^3 + \mathrm{tr}(\mathbf{T})\lambda^2 - \mathrm{minor}(\mathbf{T})\lambda + \det(\mathbf{T}) = 0$ for n=3

characteristic equation + coefficients are *tensor invariants*

$$I_1 = \mathrm{tr}(\mathbf{T}) = T_{11} + T_{22} + T_{33}$$

$$I_2 = \mathrm{minor}(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{31} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{32} \\ T_{32} & T_{33} \end{vmatrix}$$

$$= T_{11}T_{22} + T_{22}T_{33} + T_{11}T_{33} - T_{21}^2 - T_{32}^2 - T_{31}^2$$

$$I_3 = \det(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} & T_{31} \\ T_{21} & T_{22} & T_{32} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = T_{11}T_{22}T_{33} + 2T_{21}T_{32}T_{31} - T_{11}T_{32}^2 - T_{22}T_{31}^2 - T_{33}T_{21}^2$$

Eigenvalues, eigenvectors

For real-valued, symmetric rank 2 order *n* tensors

- All eigenvalues are real
- If S is positive definite, then eigenvalues are positive
- Eigenvectors for two distinct λ are orthogonal.
- There are *n* linearly independent eigenvectors

$$\mathbf{T}\cdot\mathbf{x}_1=\lambda_1\mathbf{x}_1$$
$$\mathbf{T}\cdot\mathbf{x}_2=\lambda_2\mathbf{x}_2$$

where $\lambda_1\neq\lambda_2$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1=\lambda_1\mathbf{x}_2\cdot\mathbf{x}_1 \qquad \mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2=\lambda_2\mathbf{x}_1\cdot\mathbf{x}_2=\lambda_2\mathbf{x}_2\cdot\mathbf{x}_1$$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1=\mathbf{x}_1\cdot\mathbf{T}^{\mathrm{T}}\cdot\mathbf{x}_2 \text{ with symmetry } =\mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2$$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1-\mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2=(\lambda_1-\lambda_2)\mathbf{x}_2\cdot\mathbf{x}_1=0$$

$$\Rightarrow \mathbf{x}_2\cdot\mathbf{x}_1=0$$

Eigenvectors

- If x is an eigenvector with eigenvalue λ, then any multiple $\alpha\mathbf{x}$ is also an eigenvector: $\mathbf{T}\cdot\alpha\mathbf{x} = \alpha\lambda\mathbf{x}$

  $\Rightarrow$ Eigenvectors often scaled to unit vectors

- For repeated λ, infinite range of possible $\mathbf{x}$, usually set of orthonormal vectors chosen

Example: $\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Write out the characteristic equation. What are λ_i?

Try finding eigenvectors so that $\mathbf{T}\cdot\mathbf{x}_i=\lambda_i\mathbf{x}_i$

Eigenvectors

- If x is an eigenvector with eigenvalue λ, then any multiple $\alpha\mathbf{x}$ is also an eigenvector: $\mathbf{T}\cdot\alpha\mathbf{x} = \alpha\lambda\mathbf{x}$

  $\Rightarrow$ Eigenvectors often scaled to unit vectors

- For repeated λ, infinite range of possible $\mathbf{x}$, usually set of orthonormal vectors chosen

Example: $$\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Characteristic equation: $(2-\lambda)^2(3-\lambda)=0$

$\Rightarrow \lambda=2$ (twice), $\lambda=3$

Easy to verify that: $\mathbf{T}\cdot\hat{\mathbf{e}}_1=2\hat{\mathbf{e}}_1, \mathbf{T}\cdot\hat{\mathbf{e}}_2=2\hat{\mathbf{e}}_2, \mathbf{T}\cdot\hat{\mathbf{e}}_3=3\hat{\mathbf{e}}_3$

$\Rightarrow$ $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ eigenvectors, but so are any $a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2$

Try yourself

- Program finding eigenvalues for a 2-dimensional, rank 2 tensor with components input by the user. *What would you like to check for before starting calculations? What needs to be done to find eigenvalues?*
- Find eigenvectors for the eigenvalues. Bear in mind that because $|\sigma-\lambda I|=0$, the two linear equations for a single λ will be multiples of each other. *What additional requirement do you need to impose to obtain unique vectors? What different cases are there?*
- What would you need to find the eigenvalues for a 3-dimensional, rank-2 tensor?
- *How would you deal with finding eigenvectors for repeated eigenvalues?*

Try yourself

- Program finding eigenvalues for a 2-dimensional, rank 2 tensor with components input by the user. *What would you like to check for before starting calculations? Symmetry. What needs to be done to find eigenvalues? Solve quadratic equation (quadratic formula).*
- Find eigenvectors for the eigenvalues. Bear in mind that because $|\sigma-\lambda I|=0$, the two linear equations for a single λ will be multiples of each other. *What additional requirement do you need to impose to obtain unique vectors? What different cases are there? If* $T_{21} = 0$, *then already diagonal. Otherwise, choose value for* x_1, *solve for* x_2, *then normalise to unit length.*
- What would you need to find the eigenvalues for a 3-dimensional, rank-2 tensor? *Root finder to solve cubic equation*
- *How would you deal with finding eigenvectors for repeated eigenvalues? Find eigenvector for unique* λ, *others perpendicular. In 2-D, repeated eigenvalues => isotropic stress*

Invariants

$$I_1 = \mathrm{tr}(\mathbf{T}) \qquad = T_{ii}$$
$$I_2 = \mathrm{minor}(\mathbf{T}) = T_{ii}T_{jj} + T_{ij}T_{ji}$$
$$I_3 = \det(\mathbf{T}) \qquad = \varepsilon_{ijk}T_{i1}T_{j2}T_{k3}$$

In terms of eigenvalues, invariants simplify to:

$$I_1 = \mathrm{tr}(\mathbf{T}) \qquad = \lambda_1 + \lambda_2 + \lambda_3$$
$$I_2 = \mathrm{minor}(\mathbf{T}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$
$$I_3 = \det(\mathbf{T}) \qquad = \lambda_1\lambda_2\lambda_3$$

Check yourself

Stress components

Diagonalizing
=> principal stress coordinate frame

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

(σ_1 to σ_3 usually ordered from largest to smallest)

$\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$

$tr(\boldsymbol{\sigma})$ = sum of normal stresses
$tr(\boldsymbol{\sigma})/3$ = - pressure = average normal stress = ***hydrostatic stress***
$\Rightarrow$ volume change

$\boldsymbol{\sigma}'_{ij}$ is ***deviatoric stress*** $= \sigma_{ij}+p\delta_{ij}$
$\Rightarrow$ shape change

$tr(\boldsymbol{\sigma}')$ = ?

Second invariant deviatoric stress

σ'_{ij} is deviatoric stress = $\sigma_{ij}+p\delta_{ij}$

$$\text{minor}(\boldsymbol{\sigma}') = \sigma'_{11}\sigma'_{22} + \sigma'_{22}\sigma'_{33} + \sigma'_{11}\sigma'_{33} - \sigma'^2_{21} - \sigma'^2_{32} - \sigma'^2_{31} \quad (1)$$

$$=$$

$$- \sigma'^2_{21} - \sigma'^2_{32} - \sigma'^2_{31} \quad (2)$$

Rewrite first three terms using expression for tr(σ') i.e.,

$\sigma'_{22} = -\sigma'_{11} - \sigma'_{33}$

$$= \tfrac{1}{2}\,[(1)+(2)]$$

$$= -\tfrac{1}{2}\,[\sigma'^2_{11} + \sigma'^2_{22} + \sigma'^2_{33} + \sigma'^2_{21} + \sigma'^2_{32} + \sigma'^2_{31}]$$

$\text{minor}(\boldsymbol{\sigma})=\tfrac{1}{2}[\text{tr}(\boldsymbol{\sigma}^2)-(\text{tr}\boldsymbol{\sigma})^2]$, $\text{minor}(\boldsymbol{\sigma}')=\tfrac{1}{2}\text{tr}(\boldsymbol{\sigma}'^2)$

measure of stress magnitude, important in flow and plastic yielding

Second invariant deviatoric stress

σ'_{ij} is deviatoric stress = $\sigma_{ij}+p\delta_{ij}$

$$\text{minor}(\boldsymbol{\sigma}') = \sigma'_{11}\sigma'_{22} + \sigma'_{22}\sigma'_{33} + \sigma'_{11}\sigma'_{33} - \sigma'^{2}_{21} - \sigma'^{2}_{32} - \sigma'^{2}_{31} \quad (1)$$

$$= - \sigma'^{2}_{11} - \sigma'^{2}_{22} - \sigma'^{2}_{33}$$
$$- \sigma'_{11}\sigma'_{33} - \sigma'_{11}\sigma'_{22} - \sigma'_{22}\sigma'_{33} \quad (2)$$
$$- \sigma'^{2}_{21} - \sigma'^{2}_{32} - \sigma'^{2}_{31}$$

Using that:
$\mathrm{tr}(\sigma')=\sigma'_{11}+\sigma'_{22}+\sigma'_{33}$
$=0$

$$= \tfrac{1}{2}\,[(1)+(2)]$$

$$= - \tfrac{1}{2}\,[\sigma'^{2}_{11} + \sigma'^{2}_{22} + \sigma'^{2}_{33} + \sigma'^{2}_{21} + \sigma'^{2}_{32} + \sigma'^{2}_{31}]$$

$\text{minor}(\boldsymbol{\sigma})=\tfrac{1}{2}[\mathrm{tr}(\boldsymbol{\sigma}^2)-(\mathrm{tr}\boldsymbol{\sigma})^2]$, $\text{minor}(\boldsymbol{\sigma}')=\tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}'^2)$

measure of stress magnitude, important in flow and plastic yielding

Maximum shear stress

Principal stresses include largest and smallest normal stresses in given stress system (*see proof in Lai et al*.)

If σ_1 is largest and σ_3 smallest principal stress, then maximum shear stress

$$\left|\sigma_s^{\max}\right| = \frac{\sigma_1 - \sigma_3}{2}$$

- Show this using case of 2-D stress in σ_1, σ_3 coordinate frame,
- Determine the orientation of the corresponding direction relative to the σ_1, σ_3 coordinate frame

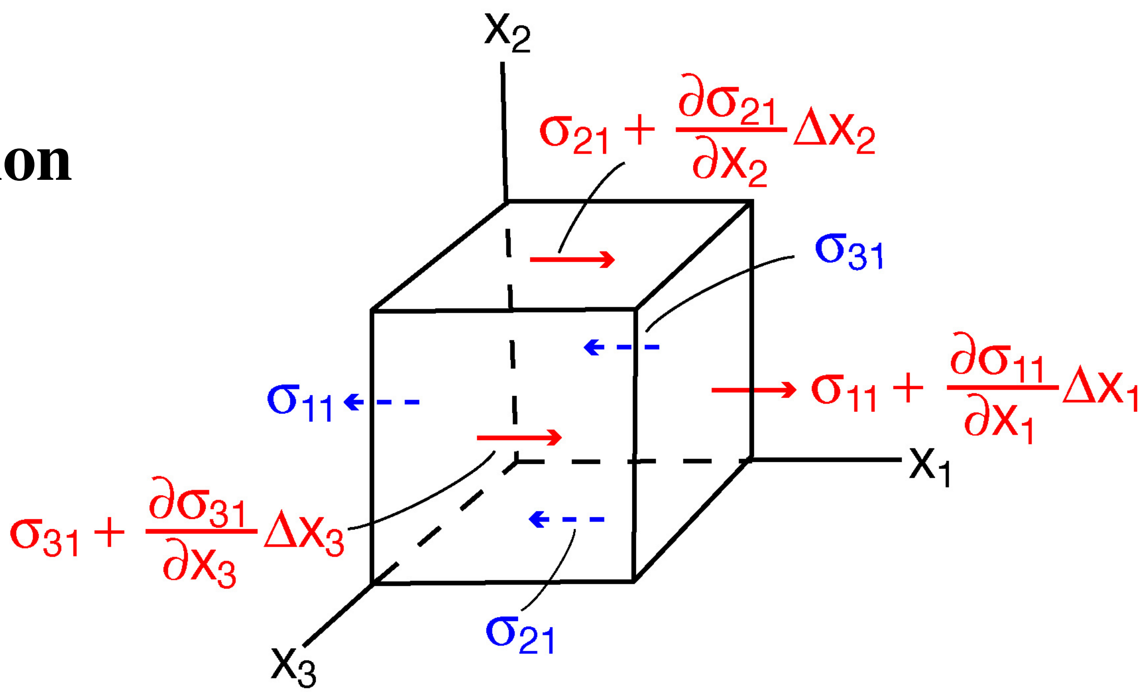
Maximum shear stress important for yield criteria

Equation of motion

Force balance:

$\mathbf{F}_{body}+\mathbf{F}_{stress}=m\mathbf{a}$

In x_1- direction:



$+$

$+$

$+$

$= \rho\Delta x_1\Delta x_2\Delta x_3\partial^2 u_1/\partial t^2$

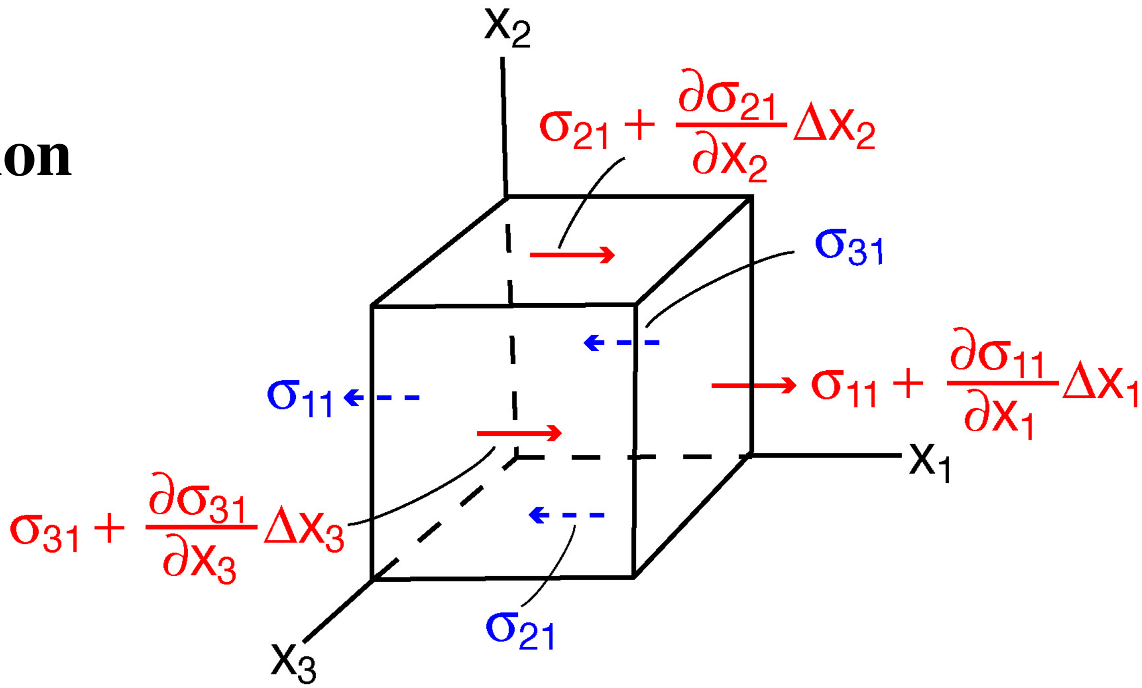
$\Rightarrow\ f_1+\partial\sigma_{11}/\partial x_1 + \partial\sigma_{21}/\partial x_2 + \partial\sigma_{31}/\partial x_3 = \rho\partial^2 u_1/\partial t^2$

$\Rightarrow\ f_i + \partial\sigma_{ji}/\partial x_j\ = \rho\partial^2 u_i/\partial t^2$

$\Rightarrow\ \mathbf{f} + \nabla\cdot\underline{\underline{\boldsymbol{\sigma}}} = \rho\partial^2\mathbf{u}/\partial t^2$

Equation of motion

Force balance:



$\mathbf{F}_{body}+\mathbf{F}_{stress}=\mathbf{ma}$

In x_1- direction:

$f_1\Delta x_1\Delta x_2\Delta x_3+$
$(\sigma_{11}+\Delta x_1\partial\sigma_{11}/\partial x_1 - \sigma_{11})\Delta x_2\Delta x_3 +$
$(\sigma_{21}+\Delta x_2\partial\sigma_{21}/\partial x_2 - \sigma_{21})\Delta x_1\Delta x_3 +$
$(\sigma_{31}+\Delta x_3\partial\sigma_{31}/\partial x_3 - \sigma_{31})\Delta x_1\Delta x_2 = \rho\Delta x_1\Delta x_2\Delta x_3\partial^2 u_1/\partial t^2$

$\Rightarrow\ f_1+\partial\sigma_{11}/\partial x_1 + \partial\sigma_{21}/\partial x_2 + \partial\sigma_{31}/\partial x_3 = \rho\partial^2 u_1/\partial t^2$

$\Rightarrow\ f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

$\Rightarrow\ \underline{\mathbf{f} + \nabla\cdot\underline{\boldsymbol{\sigma}} = \rho\partial^2\mathbf{u}/\partial t^2}$

Summary
Stress Tensors

- Cauchy stress tensor
- Tensor coordinate transformation
- (Stress) tensor symmetry
- Tensor invariants
- Diagonalizing, eigenvalues, eigenvectors
- Special stress states
- Equation of motion

Further reading on the topics in the lecture can be done in for example: Lai, Rubin, Kremple (2010): Ch. 2.18 through 2.25, 4.4 through 4.7