Adjoint Models for Data Assimilation

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Model under discussion is (not quite the one from the lecture notes)

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}$$

Take the continuous case first (always recommended as a sanity check. First, substitute $\psi \to \psi + \delta\psi$ and collect all the terms involving *one* copy of $\delta\psi$ to build our tangent linear model

$$\frac{\partial\left(\psi+\delta\psi\right)}{\partial t} = \frac{\partial\left(\psi+\delta\psi\right)}{\partial y}\frac{\partial\left(\psi+\delta\psi\right)}{\partial x}$$

$$\frac{\partial\psi}{\partial t} + \frac{\partial\delta\psi}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} + \frac{\partial\delta\psi}{\partial y}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\delta\psi}{\partial x} + \frac{\partial\delta\psi}{\partial y}\frac{\partial\delta\psi}{\partial x}$$

i.e.

$$\frac{\partial\delta\psi}{\partial t} = \frac{\partial\delta\psi}{\partial y}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\delta\psi}{\partial x}$$

This turns up in the continuous cost function in an integral, so can integrate by parts (watch minus signs!)

$$\int\lambda\left(\frac{\partial\delta\psi}{\partial t} - \frac{\partial\delta\psi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y}\frac{\partial\delta\psi}{\partial x}\right)d\Omega dt$$

$$\int\delta\psi\left(-\frac{\partial\lambda}{\partial t} + \frac{\partial}{\partial y}\left(\lambda\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\lambda\frac{\partial\psi}{\partial y}\right)\right)d\Omega dt$$

This gives us our continuous adjoint equation

$$\frac{\partial\lambda}{\partial t} = \frac{\partial}{\partial y}\left(\lambda\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\lambda\frac{\partial\psi}{\partial y}\right)$$

Remember, we are going to integrate this *backwards* in time.

If we discretize in time (not space) then we pick up a whole load of terms like

$$\lambda_0 \left(\psi_1 - \psi_0 - \Delta t \frac{\partial \psi_0}{\partial y} \frac{\partial \psi_0}{\partial x} \right) + \lambda_1 \left(\psi_2 - \psi_1 - \Delta t \frac{\partial \psi_1}{\partial y} \frac{\partial \psi_1}{\partial x} \right)$$

Or (playing the same trick).

$$\lambda_0 - \lambda_1 + \Delta t \left(\frac{\partial}{\partial y} \lambda_1 \frac{\partial \psi_1}{\partial x} + \frac{\partial}{\partial x} \lambda_1 \frac{\partial \psi_1}{\partial y} \right)$$

Now for space (the nastiest part).

$$\psi_1^{(i,j)} = \psi_0^{(i,j)} + \Delta t \left[\frac{\psi_0^{(i+1,j)} - \psi_0^{(i-1,j)}}{2\Delta x} \frac{\psi_0^{(i,j+1)} - \psi_0^{(i,j-1)}}{2\Delta x} \right]$$

Or in vector form (pick a numbering, eg $I = ny \cdot i + j$ or $I = nx \cdot j + i$)

$$\boldsymbol{\psi}_1 = \boldsymbol{m}\left(\boldsymbol{\psi}_0\right)$$

Play the linearization trick again for each of the individual $\psi_k^{(i,j)}$:

$$\delta\psi_1^{(i,j)} = \delta\psi_0^{(i,j)} + \Delta t \left[\frac{\delta\psi_0^{(i+1,j)} - \delta\psi_0^{(i-1,j)}}{2\Delta x} \frac{\psi_0^{(i,j+1)} - \psi_0^{(i,j-1)}}{2\Delta x} + \frac{\psi_0^{(i+1,j)} - \psi_0^{(i-1,j)}}{2\Delta x} \frac{\delta\psi_0^{(i,j+1)} - \delta\psi_0^{(i,j-1)}}{2\Delta x} \right]$$

Now having picked your numbering, you can actually write down the form of your matrix

$$M = \begin{pmatrix} 1 & \Delta t \frac{\psi_0^{(0,j+1)} - \psi_0^{(0,j-1)}}{4\Delta x^2} & 0 \\ -\Delta t \frac{\psi_0^{(1,j+1)} - \psi_0^{(1,j-1)}}{4\Delta x^2} & 1 & \Delta t \frac{\psi_0^{(1,j+1)} - \psi_0^{(1,j-1)}}{4\Delta x^2} \\ 0 & -\Delta t \frac{\psi_0^{(2,j+1)} - \psi_0^{(2,j-1)}}{4\Delta x^2} & 1 \\ & & \end{pmatrix}$$

Adjoint model is the transpose (remember our terms look like $\boldsymbol{\lambda_1} \cdot (\boldsymbol{\delta\psi_2} - \boldsymbol{M(\psi_1)} \cdot \delta\boldsymbol{\psi_1}) + \boldsymbol{\lambda_0} \cdot (\boldsymbol{\delta\psi_1} - \boldsymbol{M}(\psi_0) \cdot \delta\boldsymbol{\psi_0})$ etc)

$$M^T = \begin{pmatrix} 1 & -\Delta t \frac{\psi_0^{(1,j+1)} - \psi_0^{(1,j-1)}}{4\Delta x^2} & 0 \\ \Delta t \frac{\psi_0^{(0,j+1)} - \psi_0^{(0,j-1)}}{4\Delta x^2} & 1 & -\Delta t \frac{\psi_0^{(2,j+1)} - \psi_0^{(2,j-1)}}{4\Delta x^2} \\ 0 & \Delta t \frac{\psi_0^{(1,j+1)} - \psi_0^{(1,j-1)}}{4\Delta x^2} & 1 \\ & & \end{pmatrix}$$

i.e

$$\boldsymbol{\lambda}_0 = -\boldsymbol{M}^T \boldsymbol{\lambda_1} + \boldsymbol{f}$$

where $\boldsymbol{f}$ is the forcing term which is non-zero when there are new observations.