ACSE-2

Lecture 7

Kinematics of Continua

Description of deformation, motion of a continuum

Outline Lecture 7

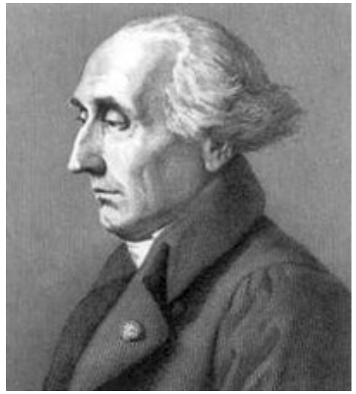
- Material vs. spatial descriptions
- Time derivatives
- Displacement
- Infinitesimal Deformation
- Finite Deformation
- Conservation of Mass

Learning Objectives

- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Be able to find principal strain(rate)s and strain (rate) invariants and know what they represent
- Understand difference between infinitesimal and finite strain
- Be able to use the conservation of mass equation

Two ways to describe motion

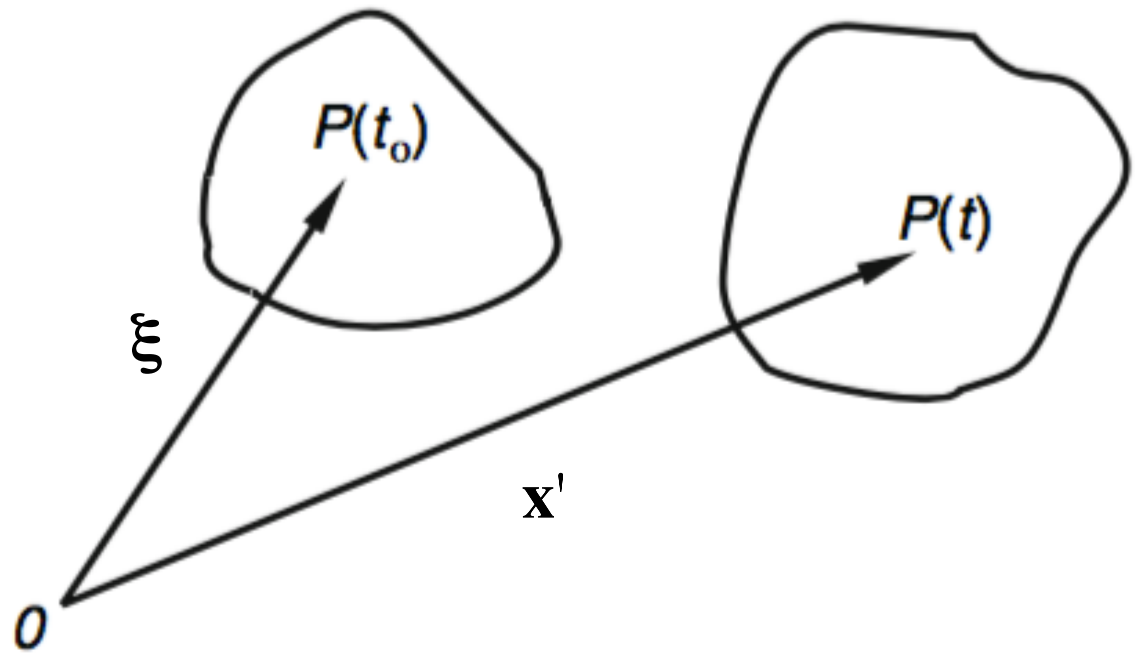
- Material (Lagrangian)
  - following a “particle”



- Spatial (Eulerian)
  - from a fixed observation point

Preferred description depends on application

Material description

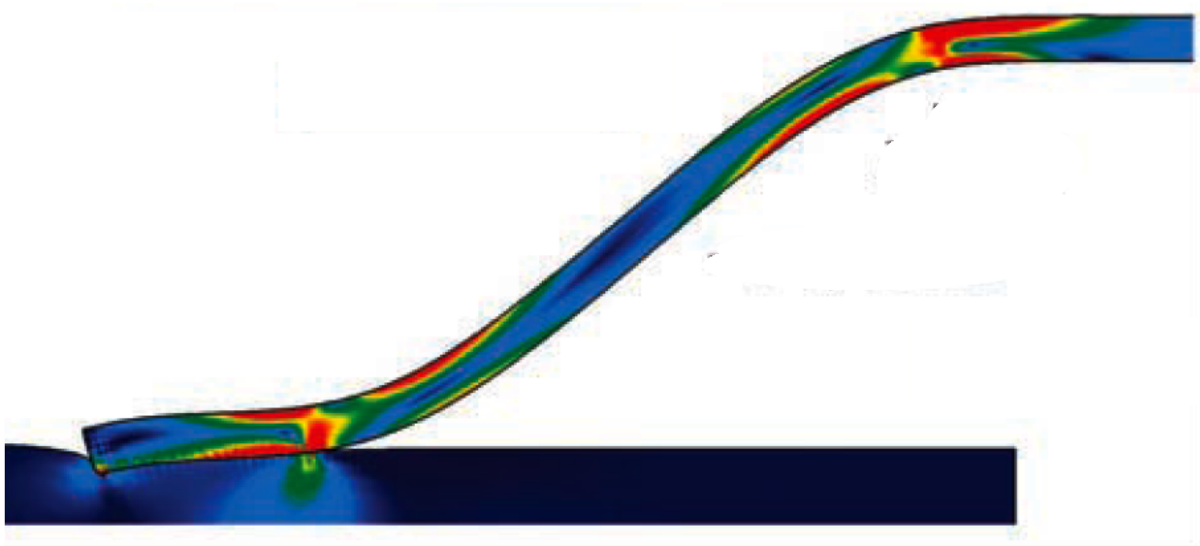


Position vector
$\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$

"Particle" at point $\boldsymbol{\xi}$ at a reference time t_0,
moves to point $\mathbf{x}'$ at a later time t
Field P described as function of $\boldsymbol{\xi}$ and t

Often the preferred description for solids

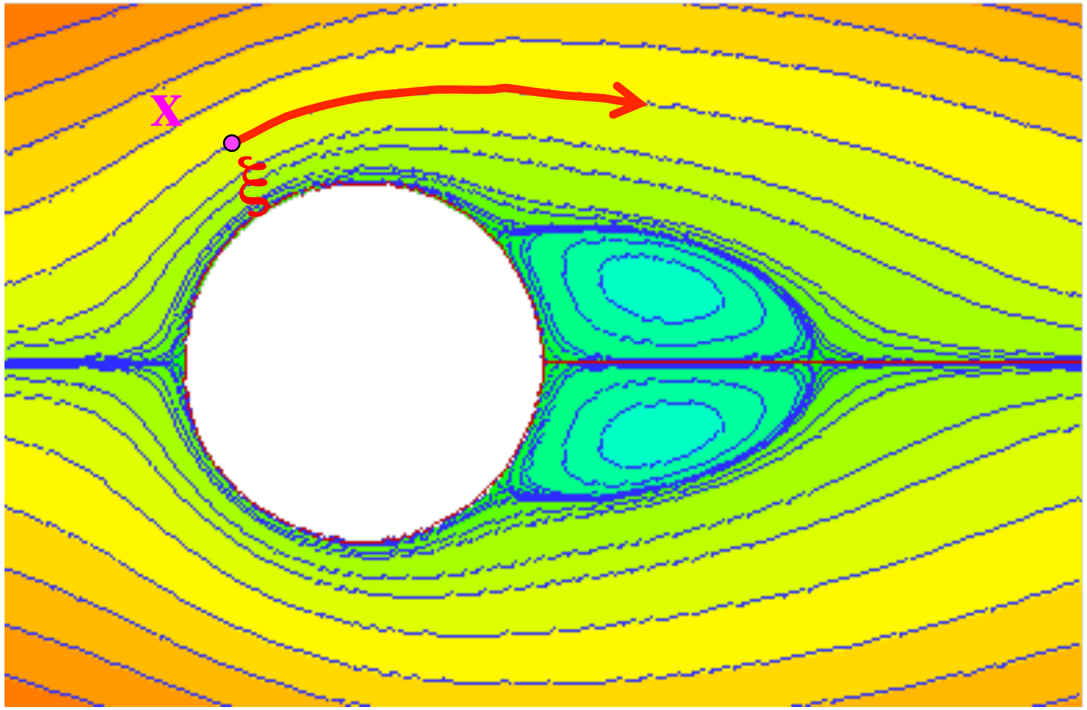
Material description



"Particle" at point ξ at a reference time t_0,
moves to point $\mathbf{x}'$ at a later time t
Field P described as function of ξ and t

Often the preferred description for solids

Spatial description



Field *P* described as function of a given position **x** and *t*

In the example flow, velocity in point **x** *does not change with time, but velocity that a particle originally in same position* **ξ** *experiences with time does change*

Often the preferred description for fluids

Material Derivative

- Rate of change (with time) of a quantity (e.g., T,$\mathbf{v}$,$\boldsymbol{\sigma}$) of a material particle
- In material description, time derivative of P: $$\frac{DP}{Dt} = \left(\frac{\partial P}{\partial t}\right)_{\xi}$$

Note: here $P(\boldsymbol{\xi},t)$

- In spatial description, $$\frac{DP}{Dt} = \left(\frac{\partial P}{\partial t}\right)_{\xi} = \left(\frac{\partial P}{\partial t}\right)_{\mathbf{x}} + \frac{\partial P}{\partial x_i}\left(\frac{\partial x'_i}{\partial t}\right)_{\xi}$$

where $\underbrace{\left(\frac{\partial \mathbf{x}'}{\partial t}\right)_{\xi}}_{material} = \underbrace{\frac{D\mathbf{x}}{Dt}}_{spatial}$ velocity of particle $\boldsymbol{\xi}$

Note: here $P(\mathbf{x},t)$

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \mathbf{v}\cdot\nabla P$$

This definition works in any coordinate frame

Acceleration

- In spatial description: $\mathbf{a} = \dfrac{D\mathbf{v}}{Dt} = \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$

Try yourself:

Determine component a_1 of the acceleration of a particle

in a spatial velocity field: $v_i = \dfrac{kx_i}{1+kt}$

Could start with single component a_1
And then for general case of a_i

Acceleration

- In spatial description: $\mathbf{a} = \dfrac{D\mathbf{v}}{Dt} = \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$

Try yourself:

Determine component a_1 of the acceleration of a particle

in a spatial velocity field: $v_i = \dfrac{kx_i}{1+kt}$

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{k^2 x_i}{(1+kt)^2} + \frac{kx_j}{1+kt}\frac{k\delta_{ij}}{1+kt} = 0$$

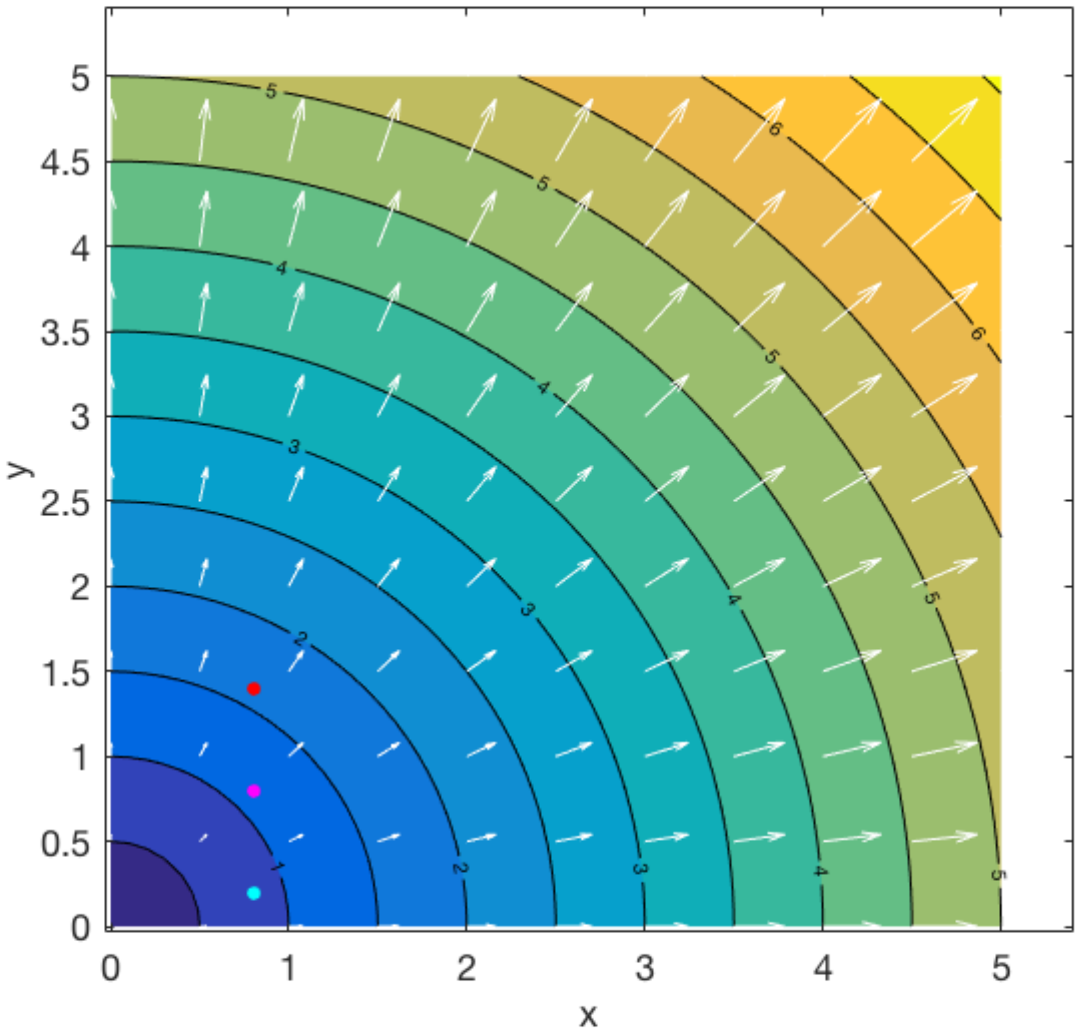
Acceleration

Spatial velocity field:

$$v_i = \frac{kx_i}{1+kt}$$

Acceleration:

velocity field at t=0 (k=1)



contours for magnitude, arrows direction and size

Acceleration

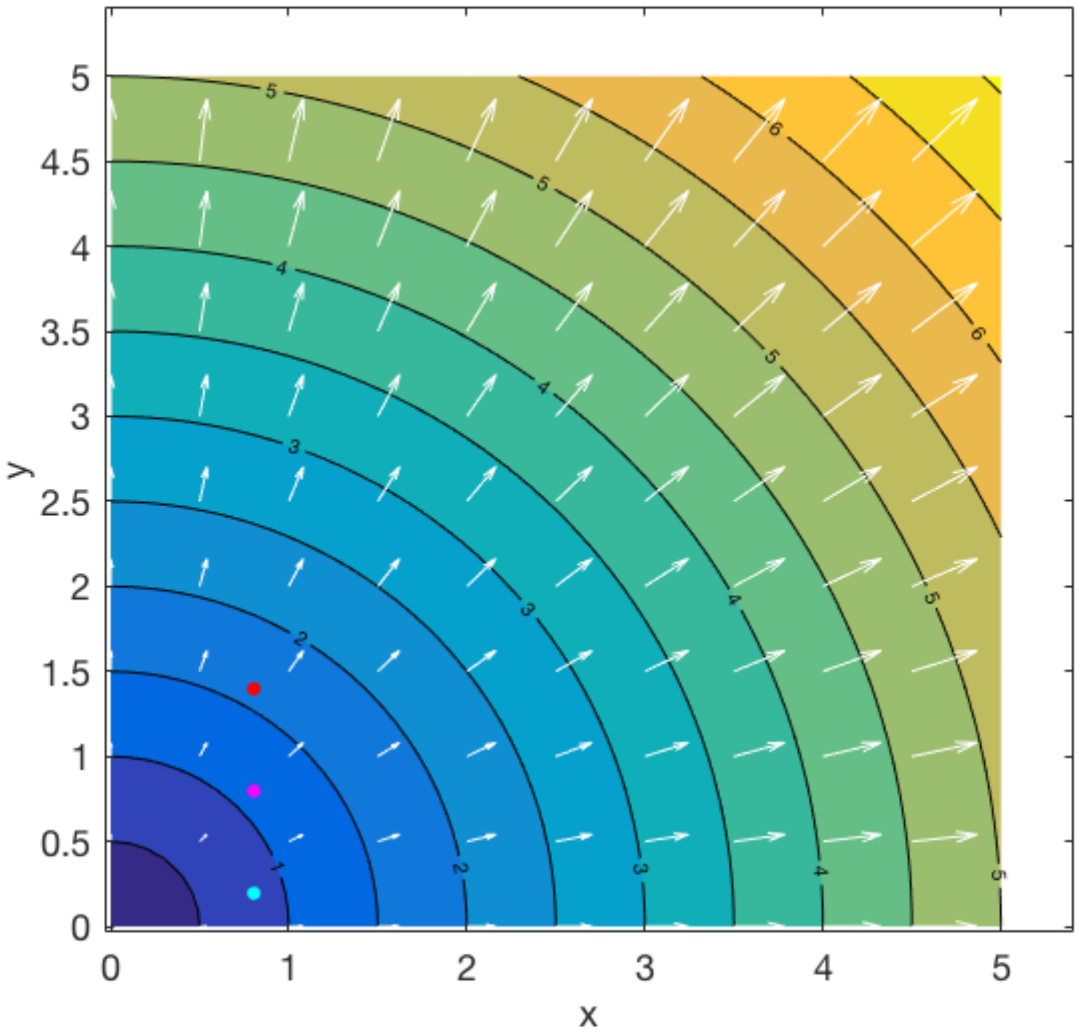
Spatial velocity field:

$$v_i = \frac{kx_i}{1+kt}$$

Acceleration:

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = 0$$

velocity field at t=0 (k=1)



contours for magnitude, arrows direction and size

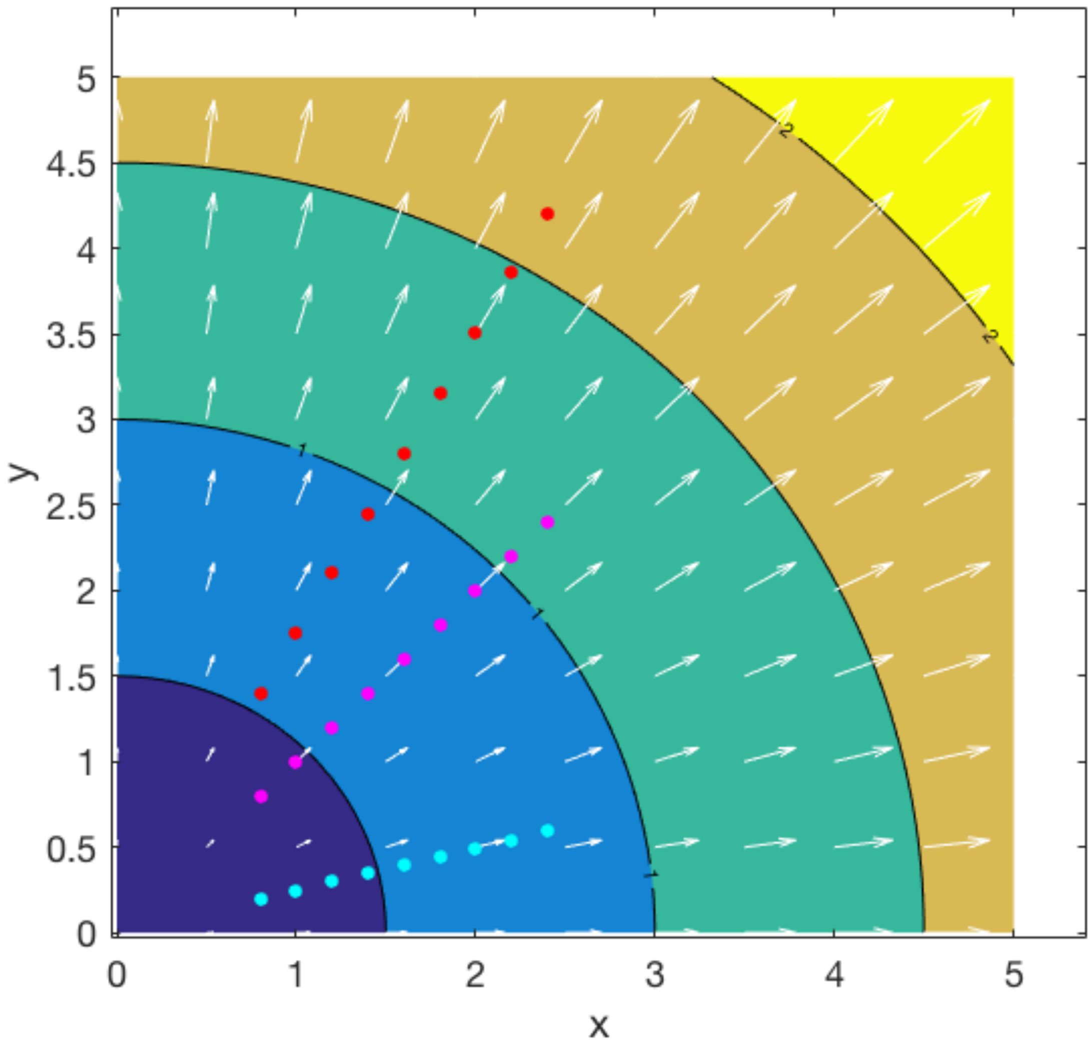
Acceleration

Spatial velocity field:

$$v_i = \frac{kx_i}{1+kt}$$

Acceleration:

velocity field at t=2 (k=1)



marker positions at constant time intervals between [0:2]

Acceleration

Spatial velocity field:

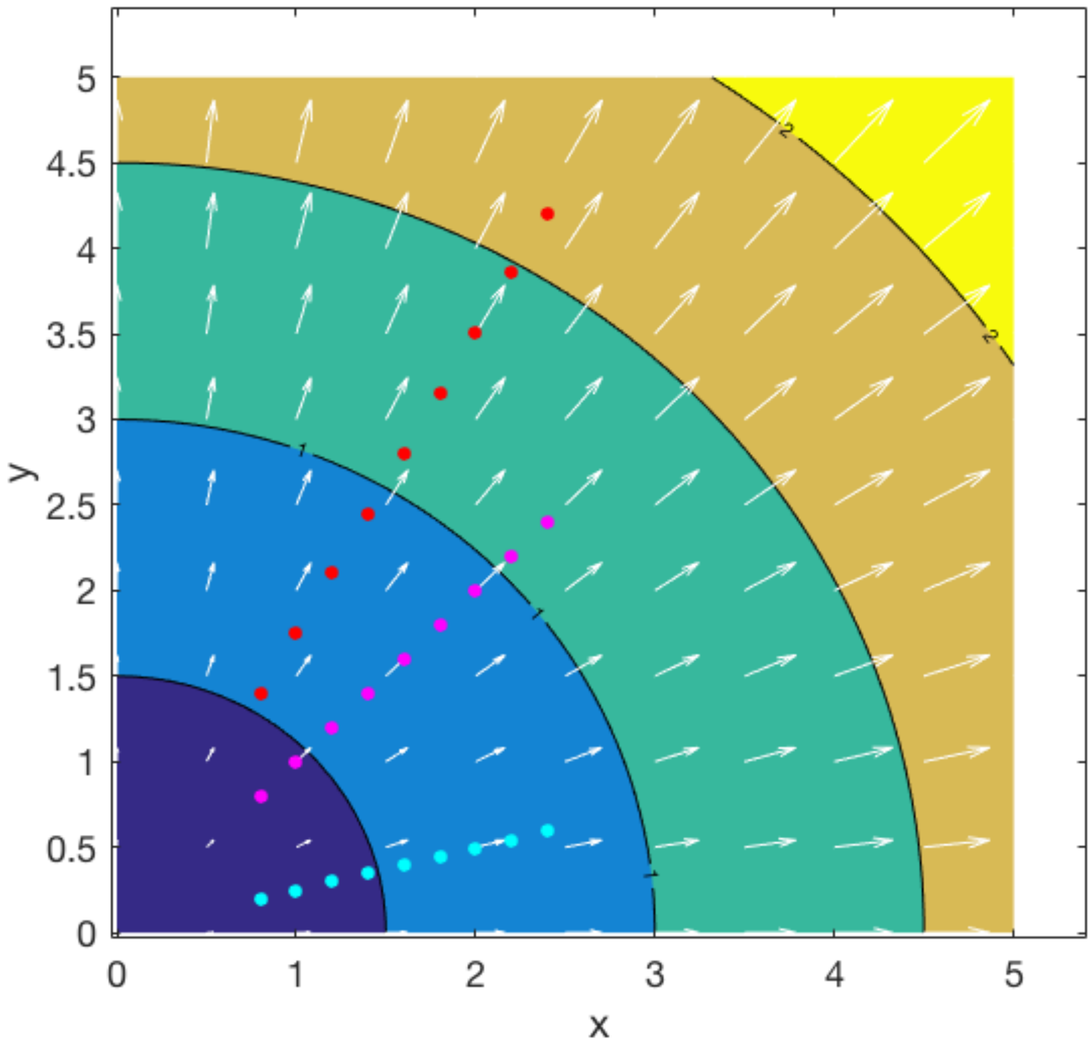
$$v_i = \frac{kx_i}{1+kt}$$

Acceleration:

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = 0$$

*How can you see that **a** = 0?*

velocity field at t=2 (k=1)



marker positions at constant time intervals between [0:2]

Acceleration

- In spatial description: $\mathbf{a} = \dfrac{D\mathbf{v}}{Dt} = \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$

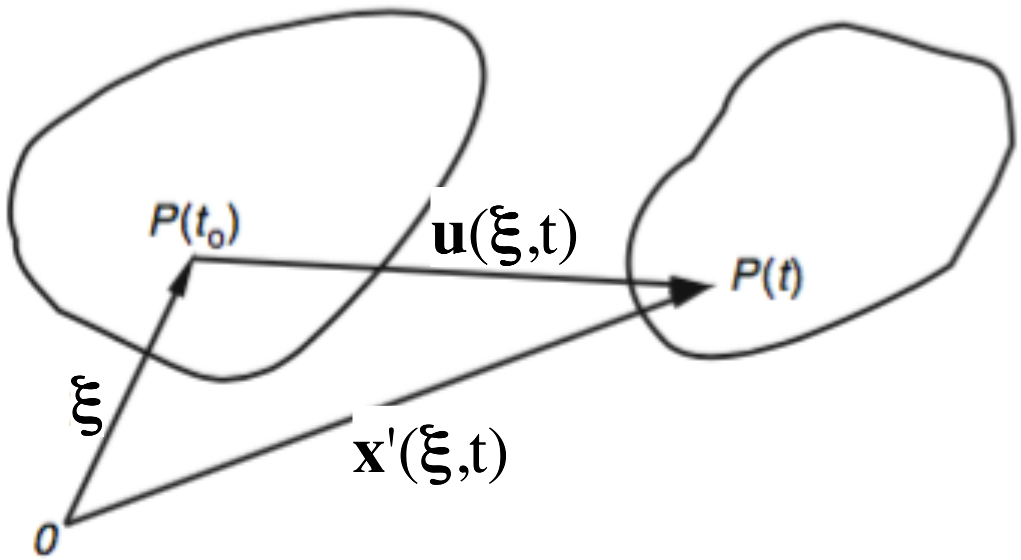
Equation of motion then becomes:

$$\rho \mathbf{a} = \nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right)$$

Displacement

Motion of a continuum can be described by:

- path lines $\mathbf{x}'=\mathbf{x}'(\xi,t)$
- displacement field $\mathbf{u}(\xi,t)=\mathbf{x}'(\xi,t)-\xi$



Pathlines

Try yourself:

Determine the pathline for the x'_1 component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

Realise that
$$v_i = \frac{\partial x'_i}{\partial t} = \frac{kx_i}{1+kt}$$

Pathlines

Try yourself:

Determine the pathline for the x'_1 component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

Realise that $$v_i = \frac{\partial x'_i}{\partial t} = \frac{kx_i}{1+kt}$$

$$\int_{\xi_i}^{x'_i} \frac{dx_i}{kx_i} = \int_0^t \frac{dt}{1+kt}$$

$$\frac{1}{k}\left[\ln x'_i - \ln \xi_i\right] = \frac{1}{k}\left[\ln(1+kt) - \ln(1)\right]$$

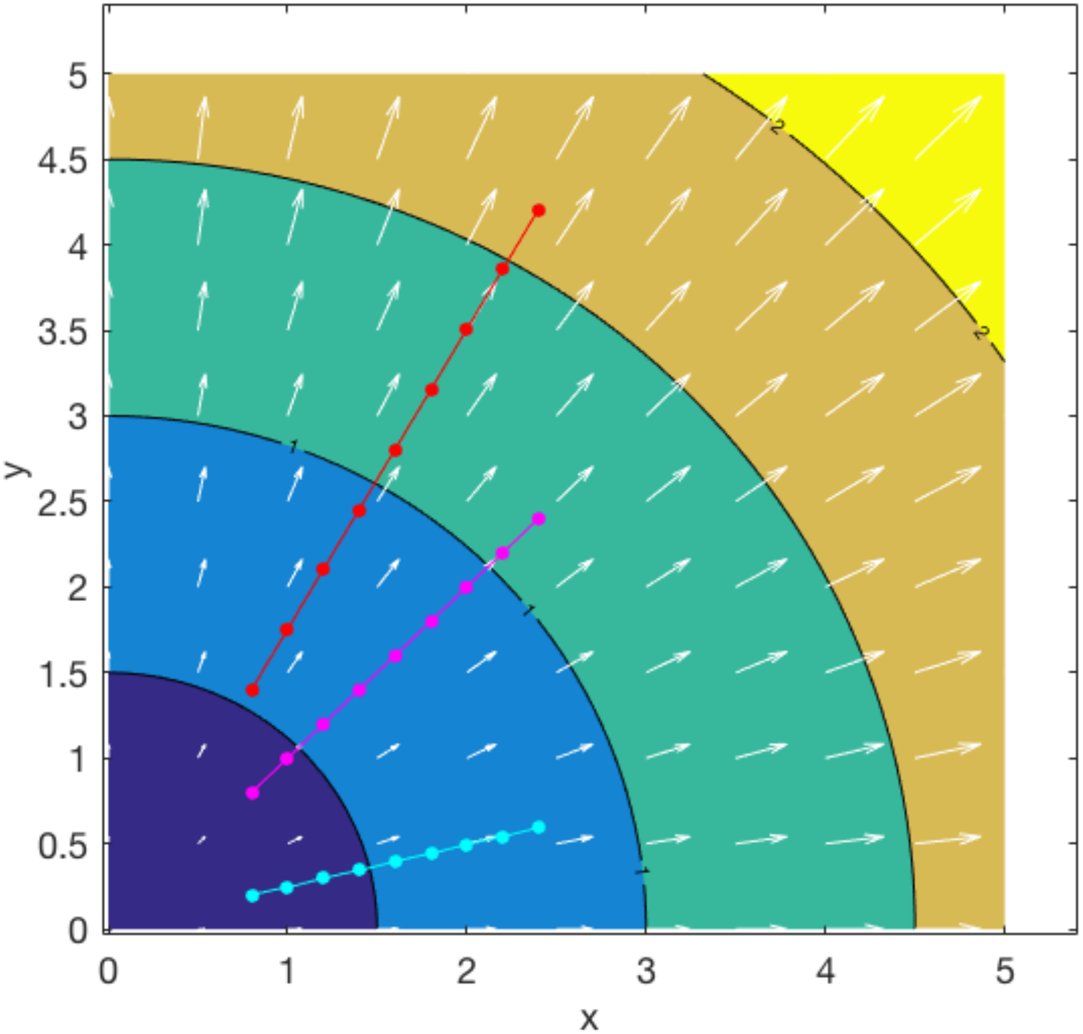
$$x'_i(\xi,t) = (1+kt)\xi_i$$

Pathlines

Determine the pathline for the x'_1 component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

$$x'_i(\xi, t) =$$



Try later: acceleration.ipynb

Pathlines

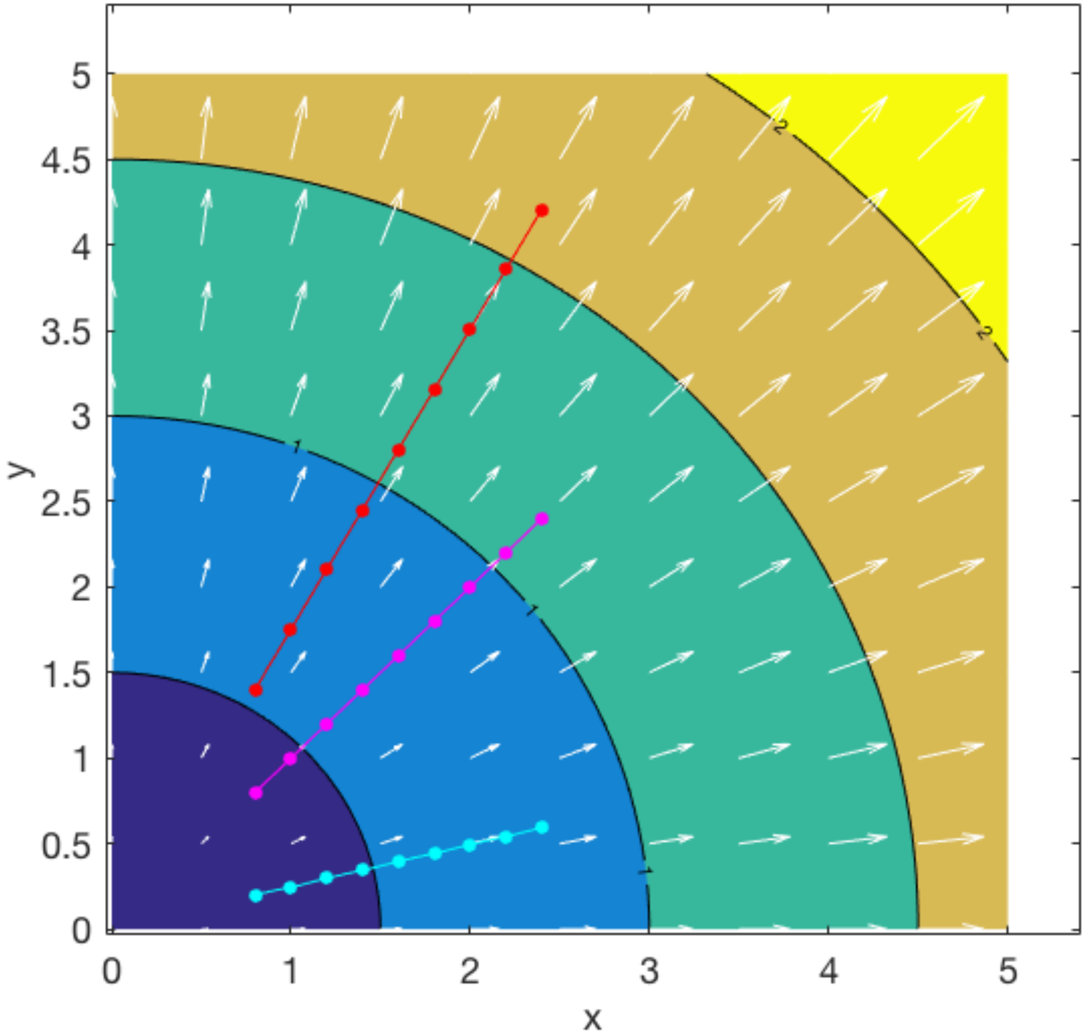
Determine the pathline for the x'_1 component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

$$x'_i(\xi, t) = (1+kt)\xi_i$$

Material velocity field:

$$v'_i = v_i = k\xi_i$$



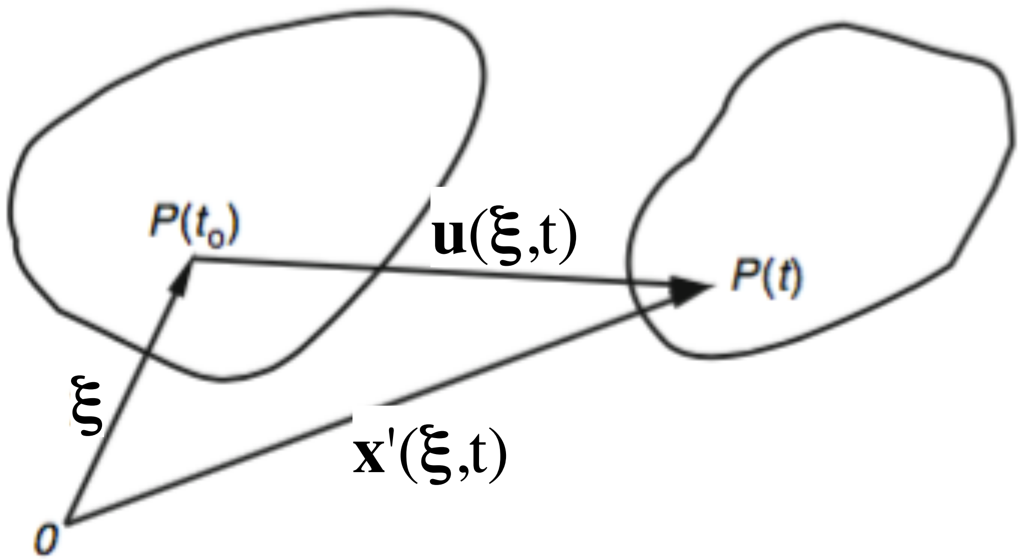
Try later: acceleration.ipynb

Displacement

Can result in

(a) Rigid body motion

(b) Deformation of the body



Rigid body motion

- Translation: $\mathbf{x}'=\boldsymbol{\xi}+\mathbf{c}(t)$, with $\mathbf{c}(0)=\mathbf{0}$

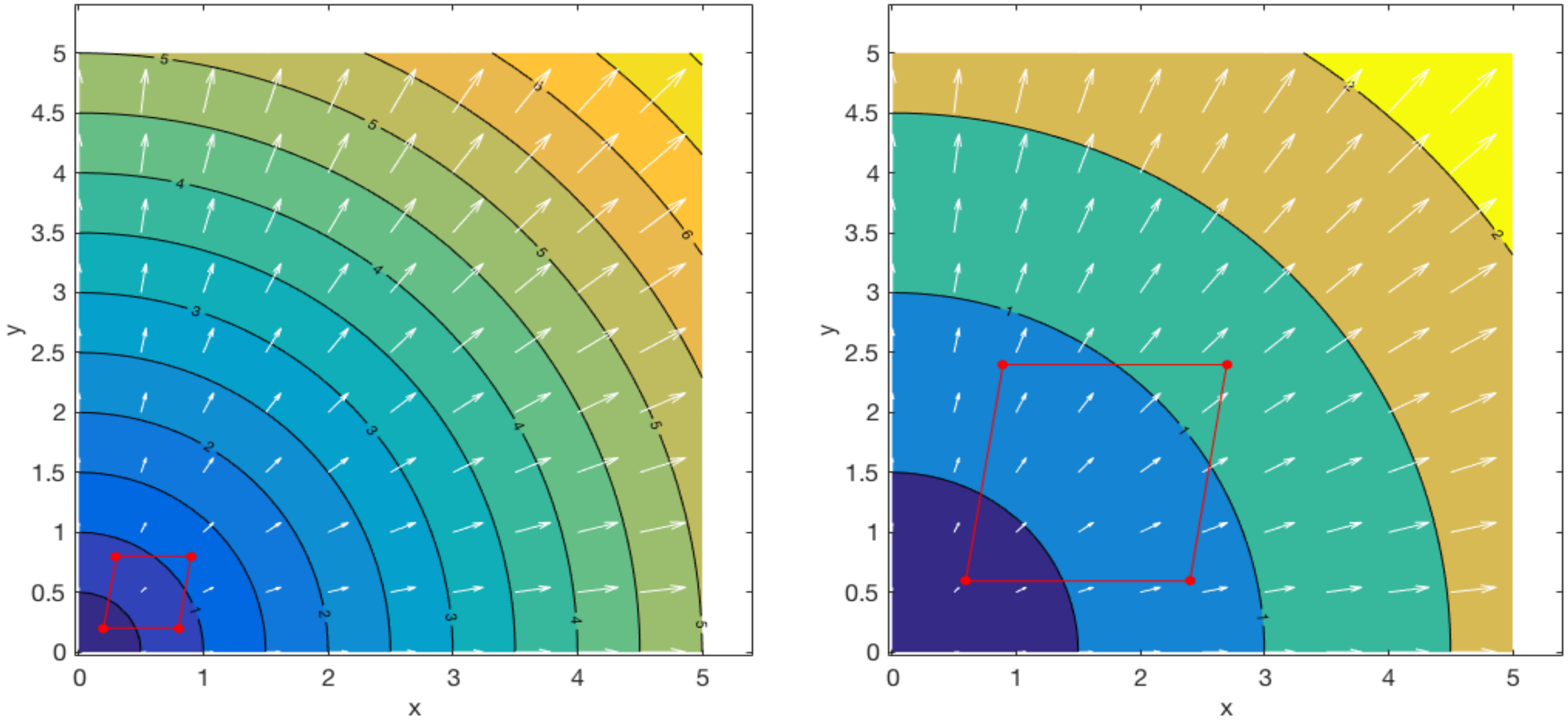
  $\Rightarrow$ $\mathbf{u}=\mathbf{x}'-\boldsymbol{\xi}$, each point same $\mathbf{u}(t)=\mathbf{c}(t)$

- Rotation: $\mathbf{x}'-\mathbf{b}=\mathbf{R}(t)(\boldsymbol{\xi}-\mathbf{b})$, where $\mathbf{R}(t)$ is rotation tensor, with $\mathbf{R}(0)=\mathbf{I}$, $\mathbf{b}$ is the point of rotation. $\mathbf{R}(t)$ is an orthogonal transformation (preserves lengths and angles, $\mathbf{R}^T\mathbf{R}=\mathbf{I}$, $\det(\mathbf{R})=1$)

If $\mathbf{u}$ depends on $\mathbf{x}$ and t, then internal deformation

Displacement

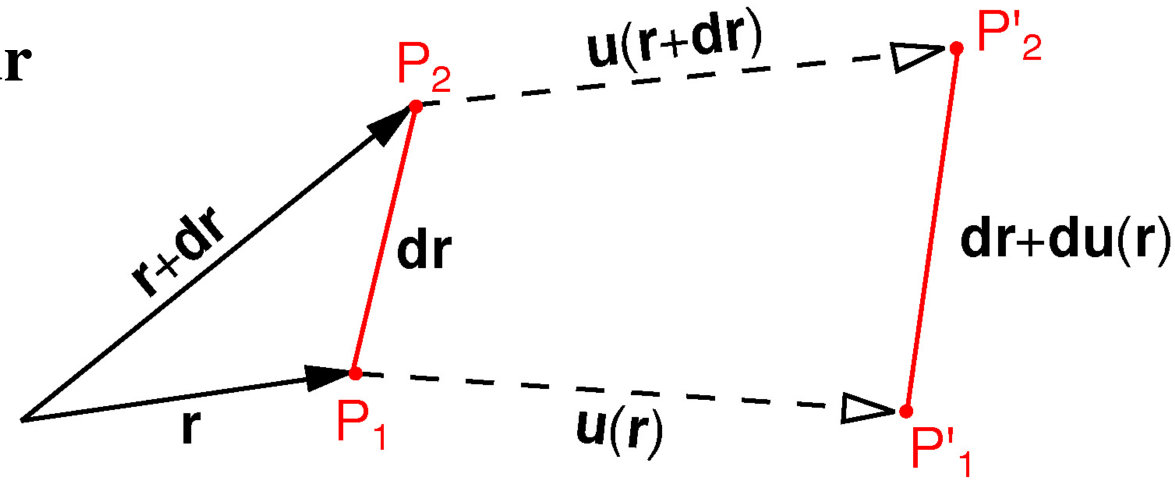
t=0 t=2



translation &
deformation

Deformation tensor

For small **dr**



P_1 at $\mathbf{r}$ -> P'_1 at $\mathbf{r+u(r)}$, P_2 at $\mathbf{r+dr}$ -> P'_2 at $\mathbf{r+dr+u(r+dr)}$.

$\mathbf{dr}' = P'_2 - P'_1 = \mathbf{dr} + [\mathbf{u(r+dr)} - \mathbf{u(r)}] = \mathbf{dr} + \nabla^T\mathbf{u(r)}\cdot\mathbf{dr} = \mathbf{dr} + \mathbf{du(r)}$

deformation of P_2-P_1 described by: $$du_i = \frac{\partial u_i}{\partial x_j} dx_j$$

$\mathbf{du} = \nabla^T\mathbf{u}\cdot\mathbf{dr} = \mathbf{dr}\cdot\nabla\mathbf{u}$

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j \qquad : \qquad \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{E_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij}}$$

Total deformation is:

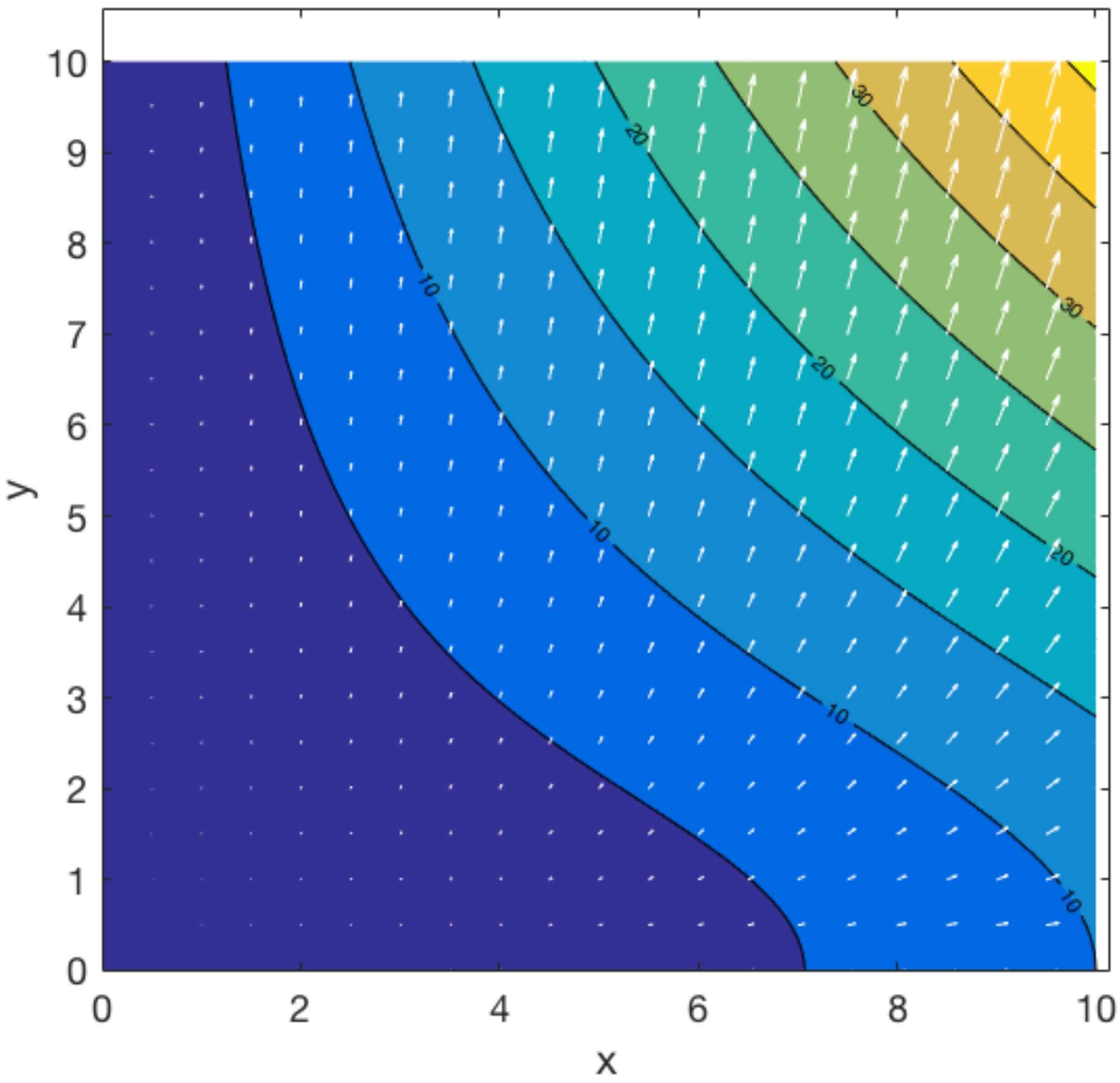
- rigid body translation - $\mathbf{u}(\mathbf{r})$
- rigid body rotation - $\mathbf{\Omega} \cdot \mathbf{dr}$
- internal deformation, strain - $\mathbf{E} \cdot \mathbf{dr}$ - result of stresses

Infinitesimal strain and rotation tensors

$$
E = \begin{bmatrix}
\dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\
\dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \\
\dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) & \dfrac{\partial u_3}{\partial x_3}
\end{bmatrix}
$$

$$
\Omega = \begin{bmatrix}
0 & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1}\right) \\
\dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}\right) & 0 & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} - \dfrac{\partial u_3}{\partial x_2}\right) \\
\dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_1} - \dfrac{\partial u_1}{\partial x_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3}\right) & 0
\end{bmatrix}
$$

Example displacement – infinitesimal strain



displacement in time interval =1

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Example displacement – infinitesimal strain

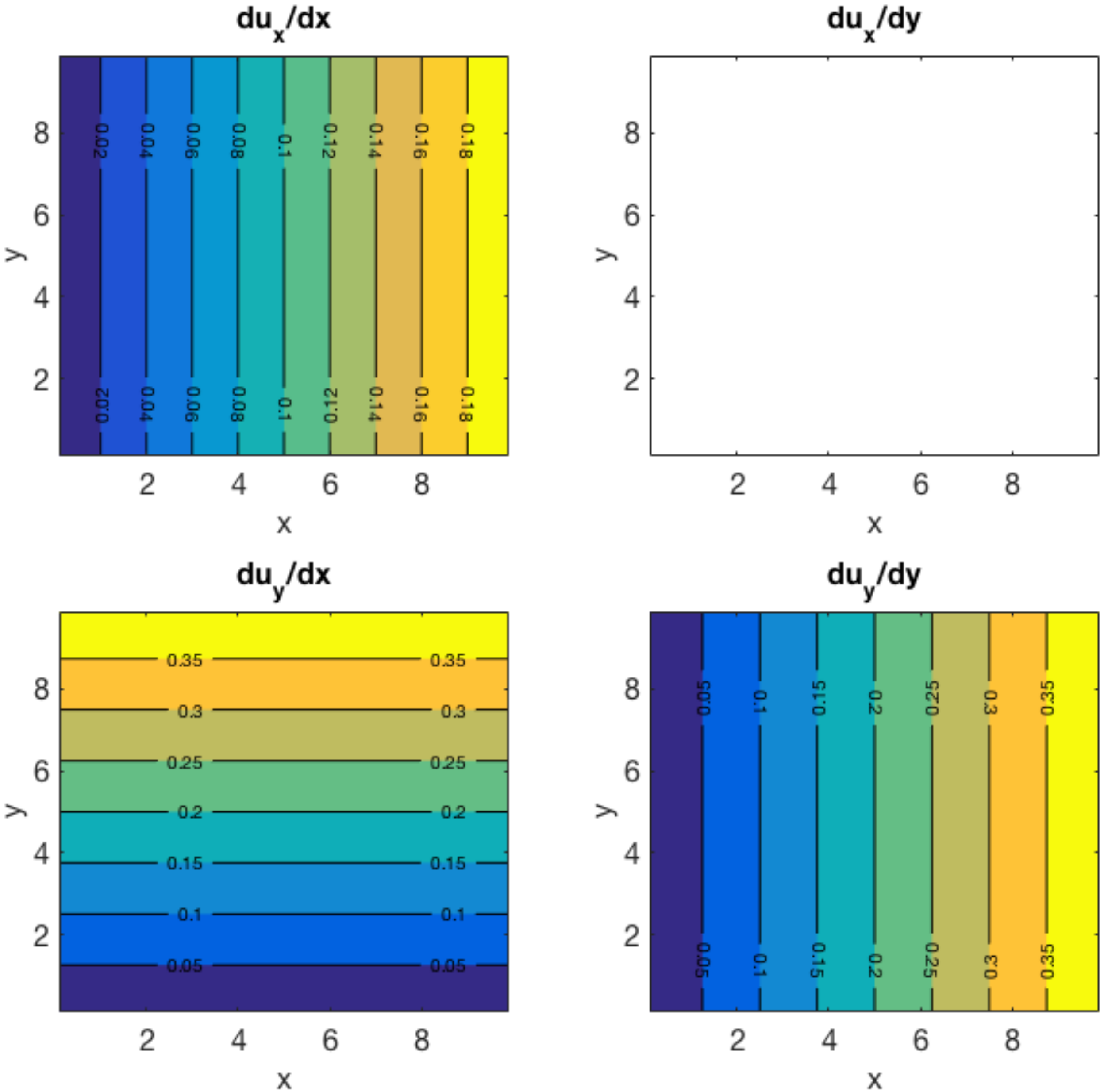
$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\frac{\partial u_x}{\partial y} = 0$$

for small δt
(=0.05)

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$



$$\frac{\partial u_y}{\partial x} = 0.4y$$

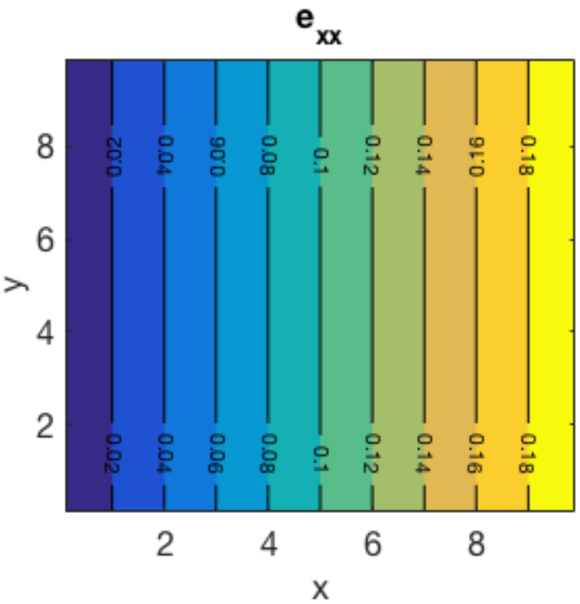
$$\frac{\partial u_y}{\partial y} = 0.4x$$

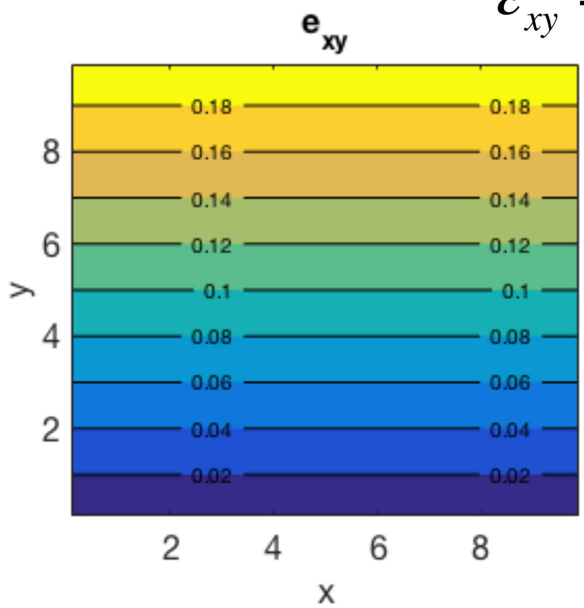
Example displacement – infinitesimal strain

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]$$

$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$





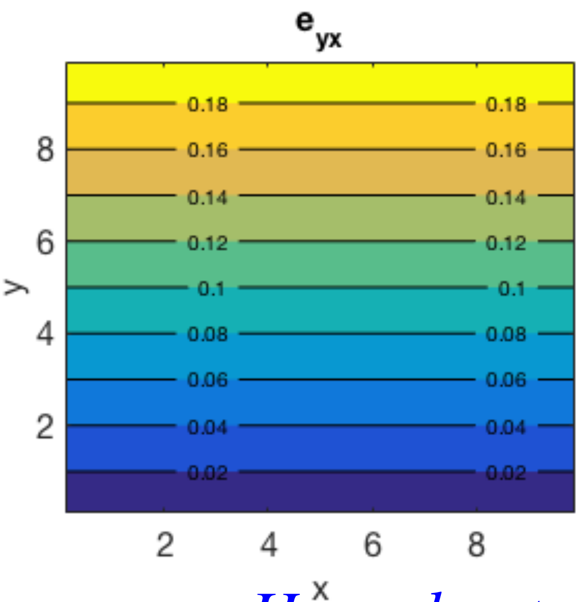
$$\frac{\partial u_x}{\partial y} = 0$$

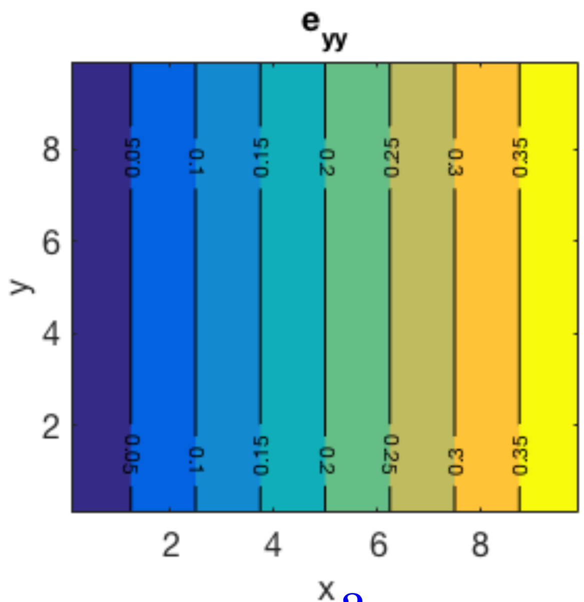
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

$$\frac{\partial u_y}{\partial x} = 0.4y$$

$$\varepsilon_{yx} = \varepsilon_{xy}$$





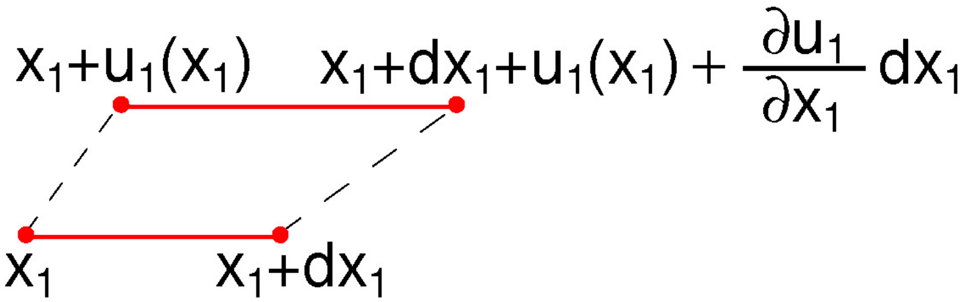
$$\frac{\partial u_y}{\partial y} = 0.4x$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

How about ω_{xx}, ω_{yy}, ω_{xy}, ω_{yx}?

<u>diagonal</u> infinitesimal strain tensor elements

For a line segment
dr = $(dx_1,0,0)$ deforming
in velocity field **u**$=(u_1,0,0)$:



For small dx_1

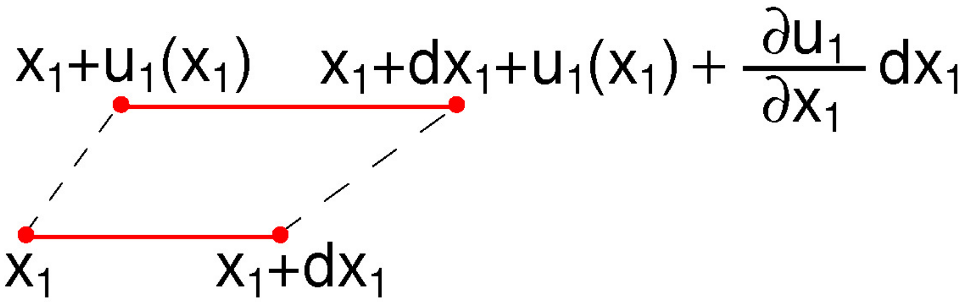
the new length $dx'_1 \approx dx_1 + (\partial u_1/\partial x_1)dx_1 = (1+E_{11})dx_1$

$\Rightarrow$ $E_{11}=[dx'_1 - dx_1]/dx_1$ = the relative change in length of a line element, originally in x_1 direction.

The relative change in volume $(V'-V)/V$ of a cube $V=dx_1dx_2dx_3$ $\approx$?

<u>**diagonal**</u> **infinitesimal strain** tensor elements

For a line segment
dr = (dx_1,0,0) deforming
in velocity field **u**=(u_1,0,0):



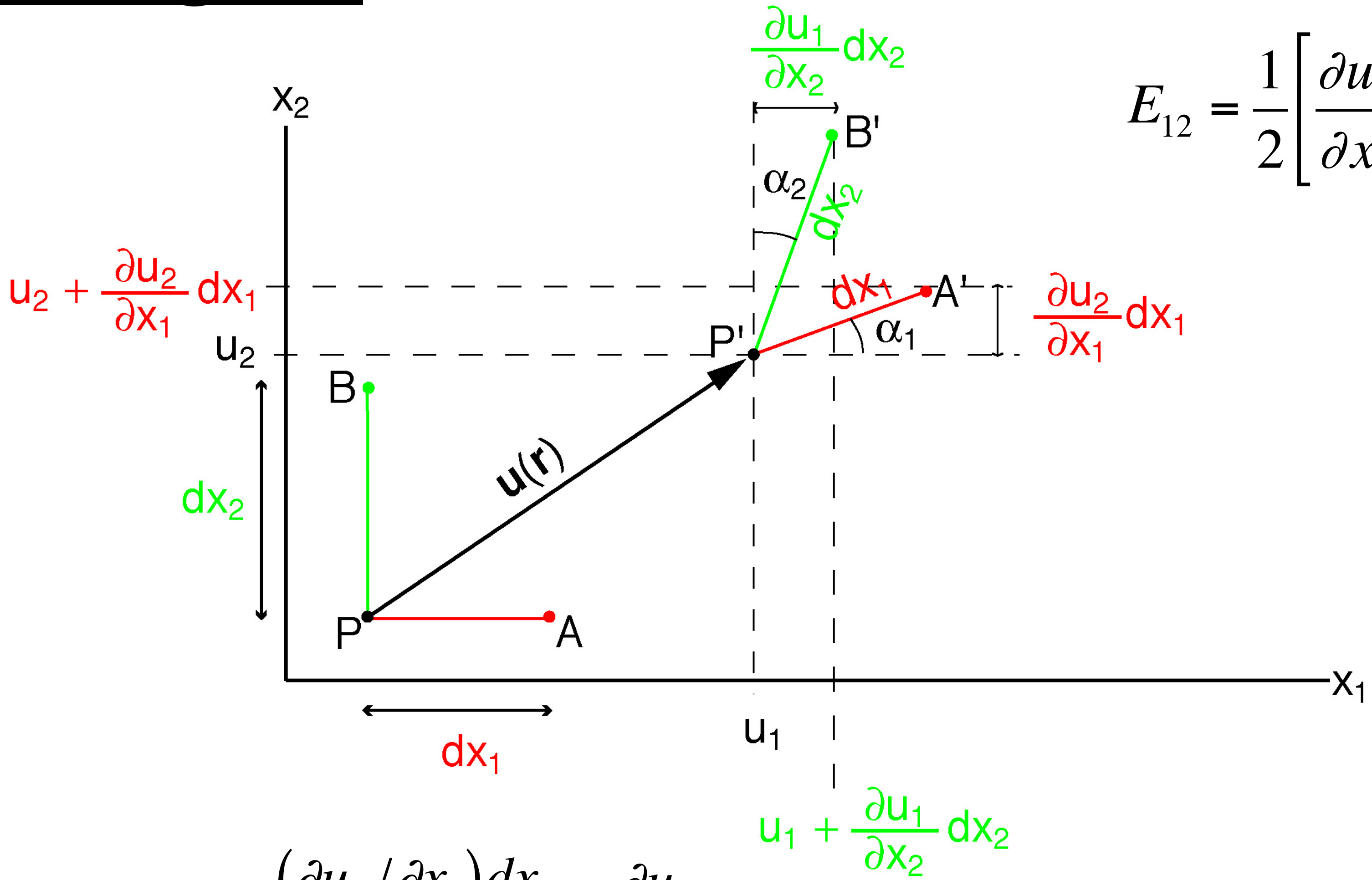
the new length $dx'_1 \approx dx_1 + (\partial u_1/\partial x_1)dx_1 = (1+E_{11})dx_1$

$\Rightarrow E_{11}=[dx'_1 - dx_1]/dx_1$ = the relative change in length of a line element, originally in x_1 direction.

The relative change in volume $(V'-V)/V$ of a cube $V=dx_1dx_2dx_3$
$\approx E_{11}+E_{22}+E_{33} = E_{ii} = tr(\mathbf{E}) = \nabla\cdot\mathbf{u}$.

<u>off-diagonal</u> infinitesimal strain tensor elements

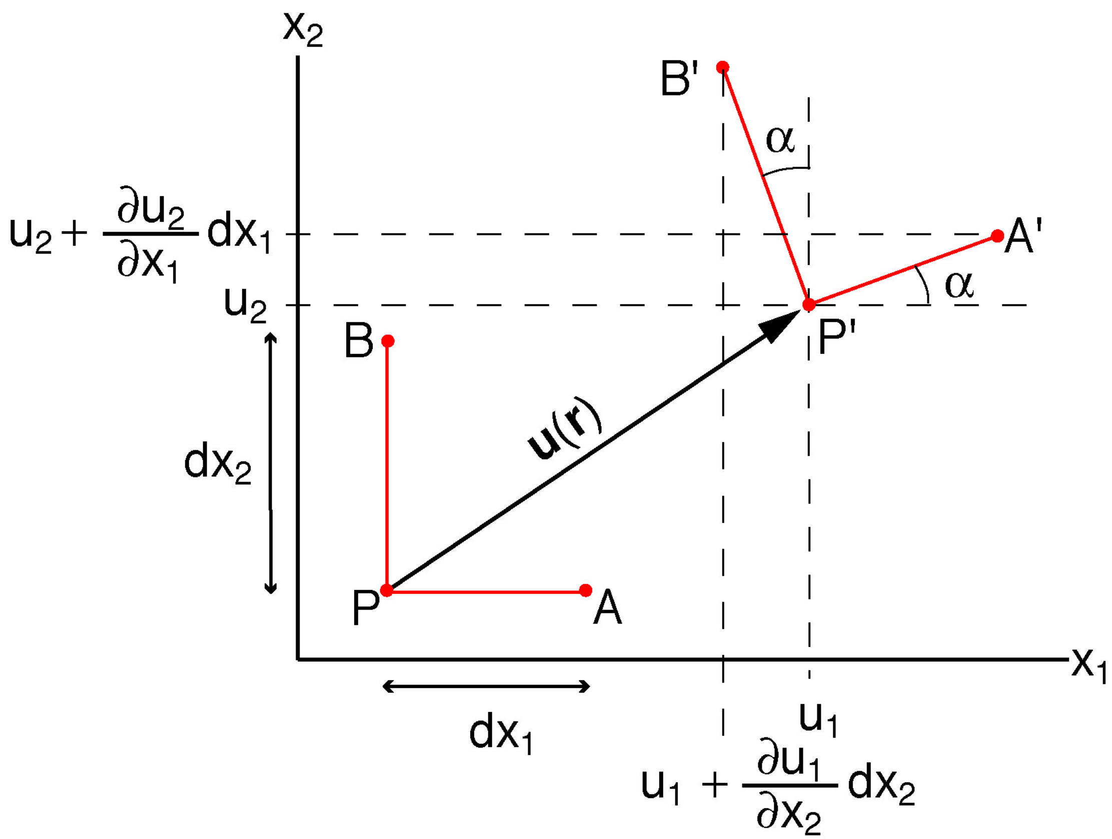
$$E_{12} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right]$$



$$\left.\begin{aligned} \alpha_1 &\approx \sin\alpha_1 = \frac{(\partial u_2 / \partial x_1) dx_1}{dx_1} = \frac{\partial u_2}{\partial x_1} \\ \alpha_2 &\approx \frac{(\partial u_1 / \partial x_2) dx_2}{dx_2} = \frac{\partial u_1}{\partial x_2} \end{aligned}\right\} \quad \boxed{E_{12}=E_{21}=(\alpha_1+\alpha_2)/2}$$

<u>$2E_{12}$ is the change in angle of an originally 90^o angle between dx_1 and dx_2</u>

infinitesimal rotation tensor elements

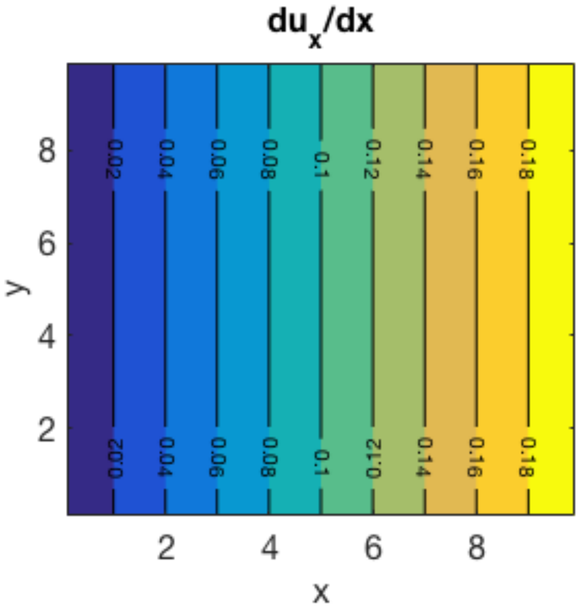


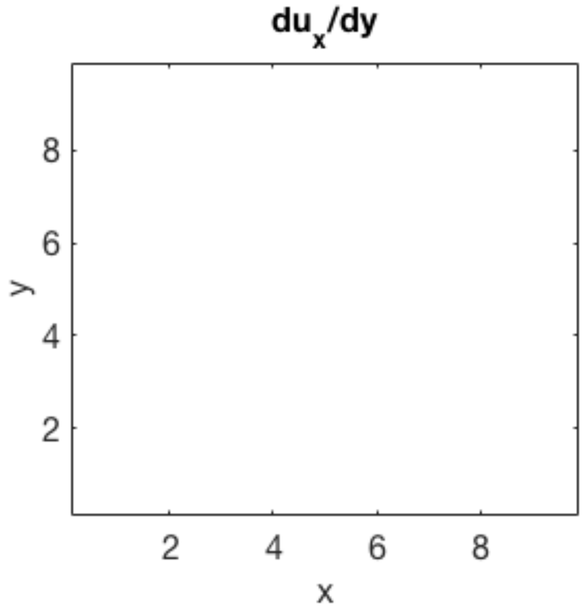
$$\boxed{\Omega_{12}=-\Omega_{21}= [(\partial u_2/\partial x_1)-(\partial u_1/\partial x_2)]/2=(\alpha_1-\alpha_2)/2}$$

Ω_{12} is common rigid rotation angle of vectors in the dx_1 - dx_2 plane (around x_3)

Example displacement – infinitesimal strain

$$\frac{\partial u_x}{\partial x} = 0.2x$$





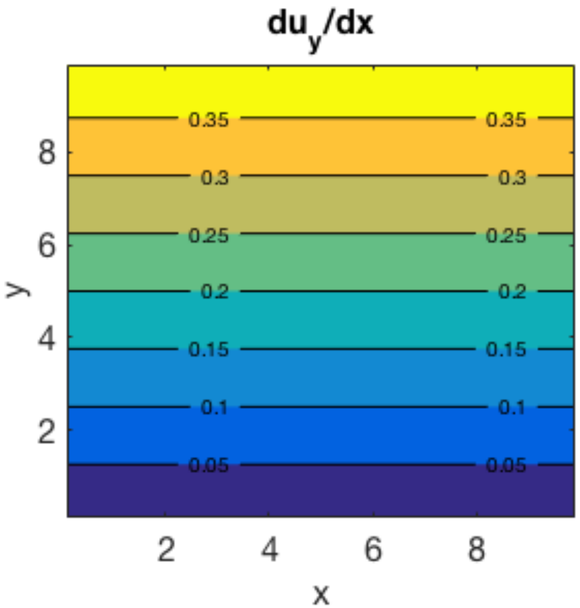
$$\frac{\partial u_x}{\partial y} = 0$$

$$u_x = 0.1x^2$$

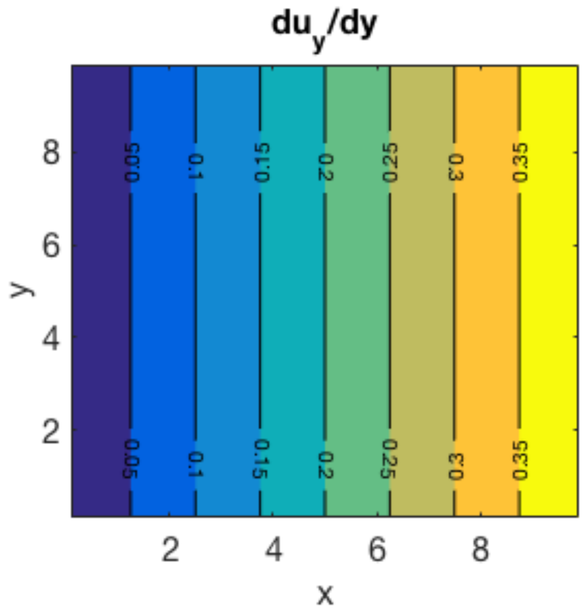
$$u_y = 0.4xy$$

for small δt

(=0.05)



$$\frac{\partial u_y}{\partial x} = 0.4y$$



$$\frac{\partial u_y}{\partial y} = 0.4x$$

Example displacement– infinitesimal strain

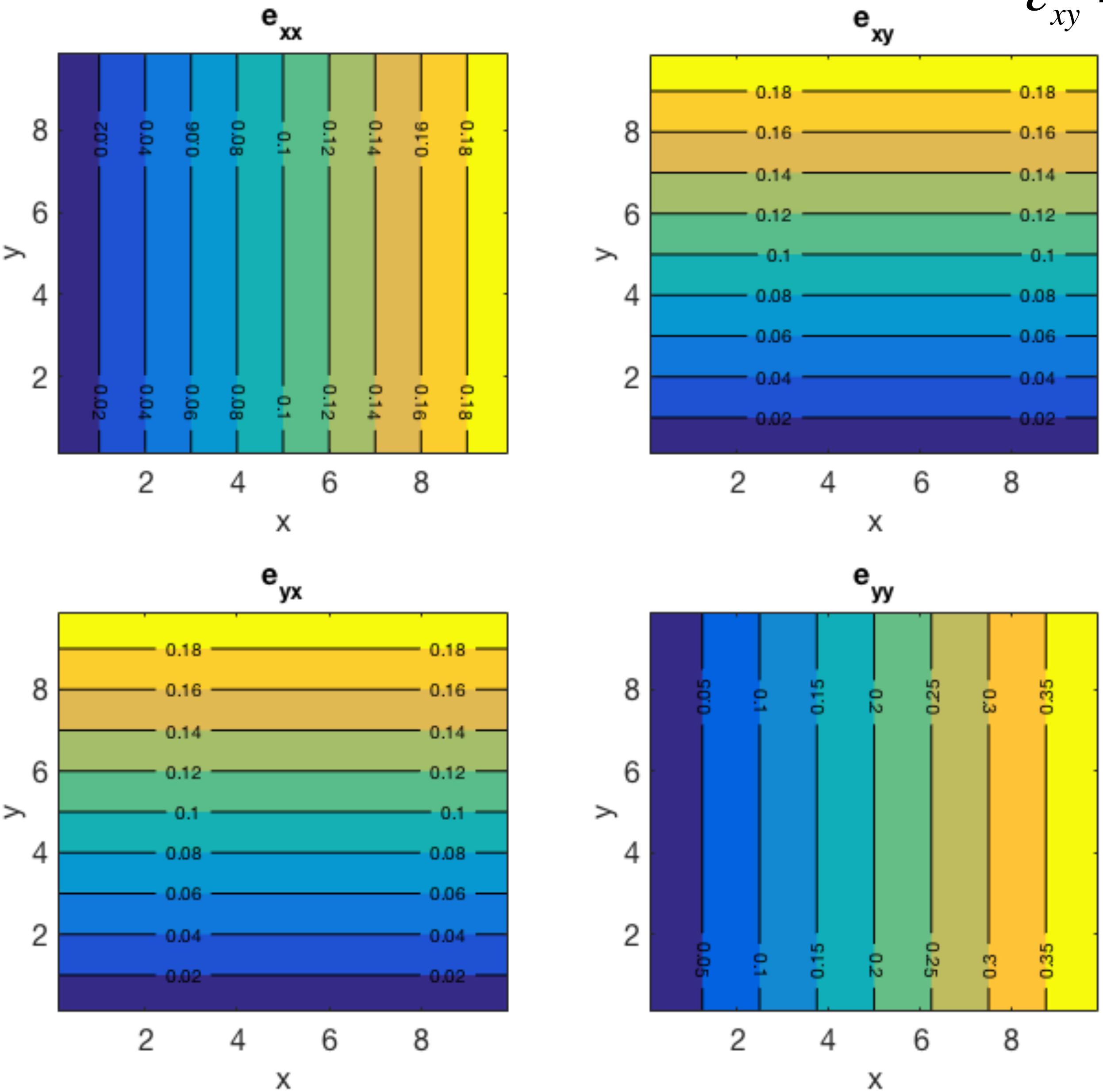
$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]$$

$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\frac{\partial u_y}{\partial x} = 0.4y$$

$$\varepsilon_{yx} = \varepsilon_{xy}$$



$$\frac{\partial u_x}{\partial y} = 0$$

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

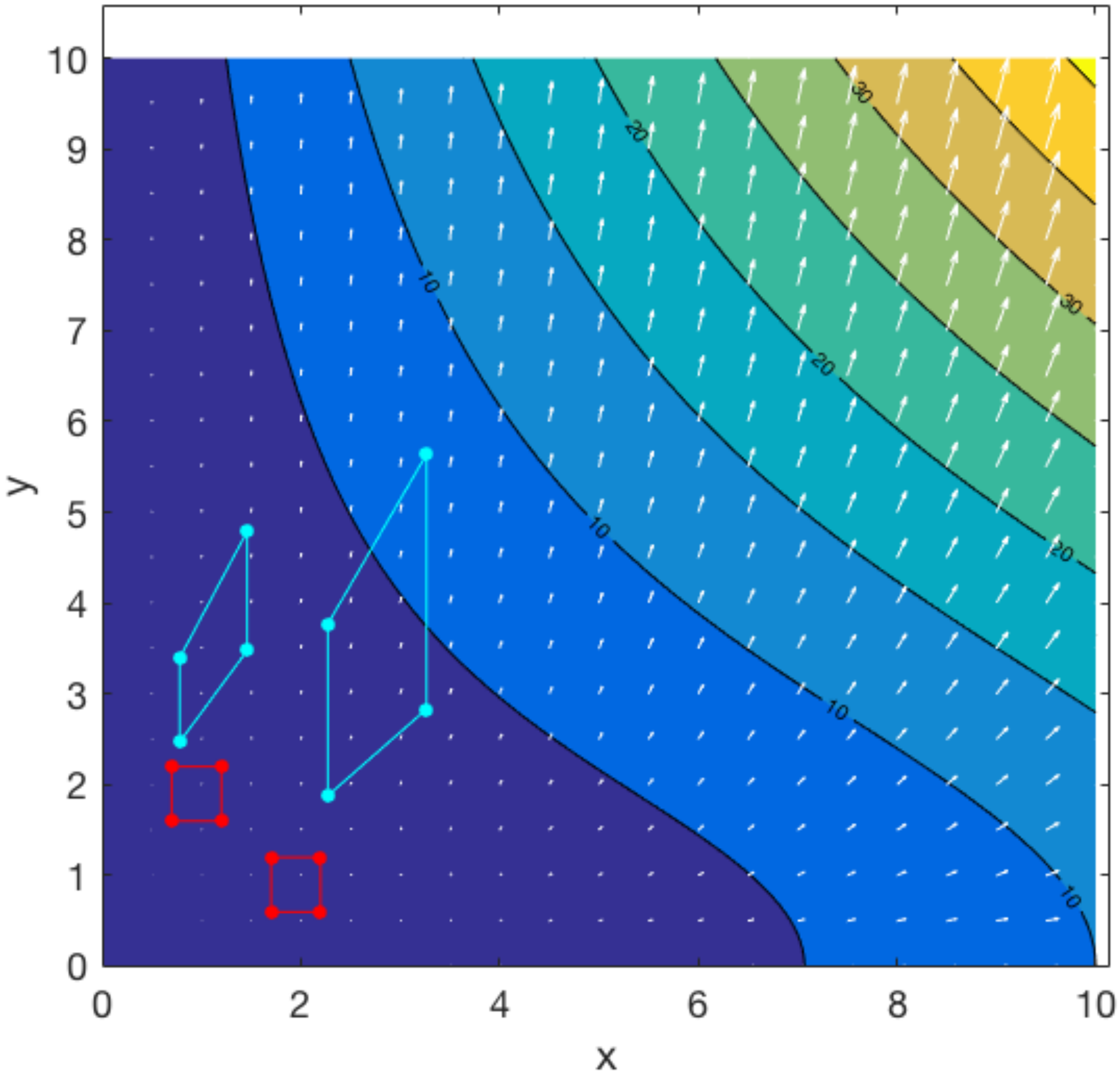
$$\frac{\partial u_y}{\partial y} = 0.4x$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

Deformation after finite strain

original shape

shape at time=1.5

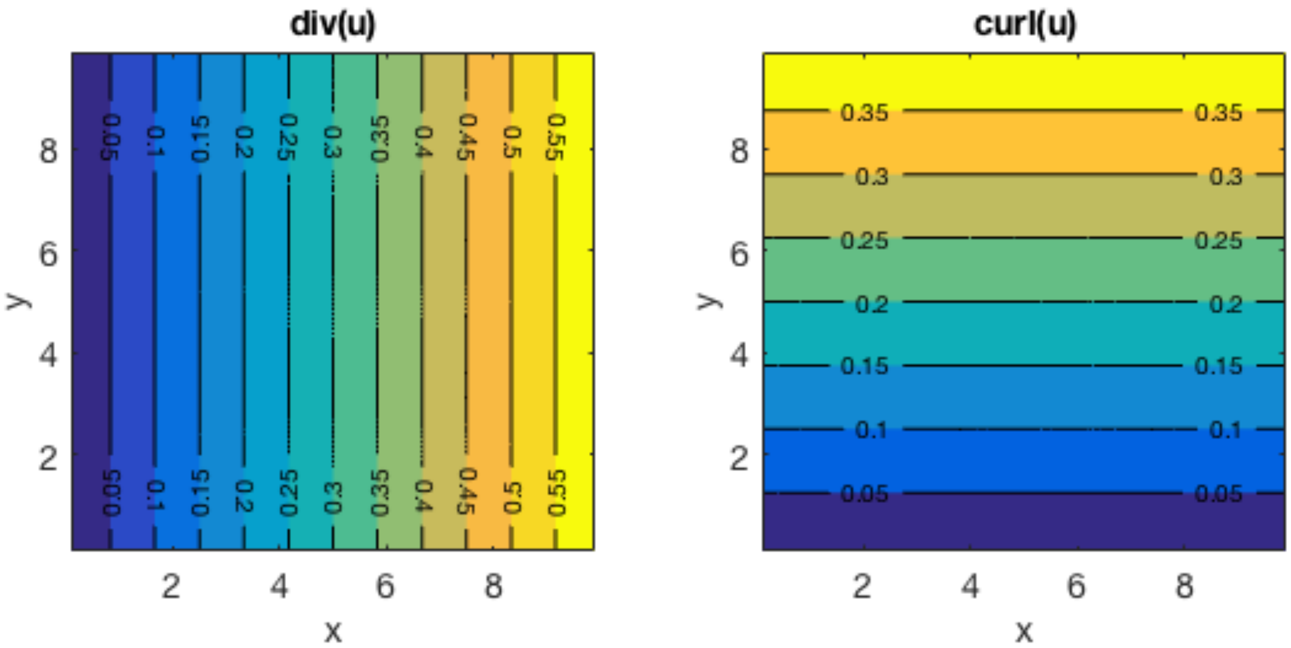


displacement in time interval =1

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Example displacement – infinitesimal strain



$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \qquad \nabla \times \mathbf{u} = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{\mathbf{e}}_z$$

Try later: squarestrain.ipynb

Rotation tensor and rotation vector

For any antisymmetric tensor **W**, a corresponding ***dual*** or ***axial vector*** **w** can be found so that

$$\mathbf{W} \cdot \mathbf{a} = \mathbf{w} \times \mathbf{a}$$

How does vector **w** relate to the components of **W**?

$$\mathbf{w} = \qquad \hat{\mathbf{e}}_1 \qquad \hat{\mathbf{e}}_2 \qquad \hat{\mathbf{e}}_3$$

Rotation tensor and rotation vector

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Vector **w** relates to the components of **W** as:

$$\mathbf{w} = \qquad \hat{\mathbf{e}}_1 \qquad \hat{\mathbf{e}}_2 \qquad \hat{\mathbf{e}}_3$$

For the rotation tensor, an equivalent rotation vector exists:

$$\boldsymbol{\Omega} \cdot \mathbf{dx} = \boldsymbol{\omega} \times \mathbf{dx} \qquad \text{where:} \qquad \boldsymbol{\omega} = \frac{1}{2} \nabla \times \mathbf{u}$$

Note that **ω** only describes the overall rigid body rotation, not the total rotation of each individual segment **dx**, which is also influenced by **E**

Rotation tensor and rotation vector

For any antisymmetric tensor **W**, a corresponding ***dual*** or ***axial vector*** **w** can be found so that

$$\mathbf{W}\cdot\mathbf{a} = \mathbf{w}\times\mathbf{a}$$

Vector **w** relates to the components of **W** as:

$$\mathbf{w} = -W_{23}\hat{\mathbf{e}}_1 + W_{13}\hat{\mathbf{e}}_2 - W_{12}\hat{\mathbf{e}}_3$$

For the rotation tensor, an equivalent rotation vector exists:

$$\boldsymbol{\Omega}\cdot\mathbf{dx} = \boldsymbol{\omega}\times\mathbf{dx} \qquad \text{where:} \qquad \boldsymbol{\omega} = \frac{1}{2}\nabla\times\mathbf{u}$$

Note that **ω** only describes the overall rigid body rotation, not the total rotation of each individual segment **dx**, which is also influenced by **E**

infinitesimal strain tensor properties

transform to fault plane *coordinate frame*:

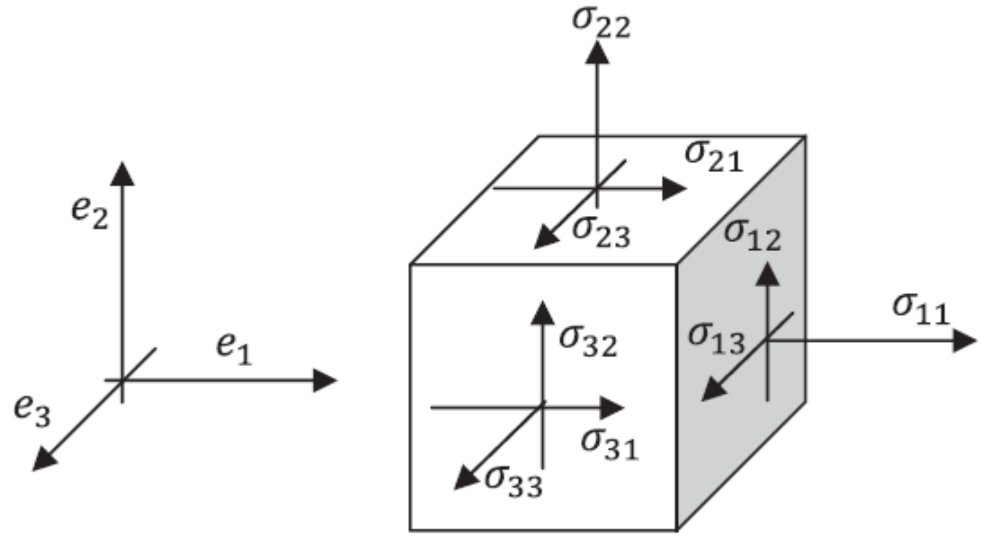
$E_{nn} \quad = E_{11}\cos^2\phi + E_{21}\sin\phi\cos\phi + E_{12}\sin\phi\cos\phi + E_{22}\sin^2\phi$

$E_{ns} \quad = E_{11}\sin\phi\cos\phi + E_{21}\sin^2\phi - E_{12}\cos^2\phi - E_{22}\sin\phi\cos\phi$

E_1, E_2, E_3 - *principal strains* : minimum, maximum and intermediate fractional length changes

isotropic, *deviatoric* strain: $E_{ij} = -(\theta/3)\delta_{ij} + E'_{ij}$

- $tr(\mathbf{E}) = \theta$ = sum of normal strains=volume change
- E'_{ij} is deviatoric strain, change in shape, involves no change in volume
- $tr(\mathbf{E}') = 0$, does not imply $E'_{ij} = 0$ for i=j
- $E_{ij} = 0$ for $i \neq j$ does not ensure no volume change



Stress components

Reminder

traction on a plane $\quad \mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$

what is $\hat{\mathbf{e}}_1 \cdot \mathbf{t} = \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$?

t_1 on plane with normal $\hat{\mathbf{n}}$

what is $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_1$? $\quad \sigma_{11}$

what is $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_2$? $\quad \sigma_{21}$

Strain components

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$$\hat{\mathbf{e}}_1 \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{e}}_1 = \varepsilon_{11}$$

$$\hat{\mathbf{e}}_1 \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{e}}_2 = \varepsilon_{12}$$

$\boldsymbol{\varepsilon} \cdot \hat{\mathbf{p}} = \mathbf{p}'$ the unit vector $\hat{\mathbf{p}}$ after deformation by $\boldsymbol{\varepsilon}$

$\hat{\mathbf{p}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{p}} =$ elongation by $\boldsymbol{\varepsilon}$ of unit vector $\hat{\mathbf{p}}$ in direction $\hat{\mathbf{p}}$

$= \hat{\mathbf{p}} \cdot \mathbf{p}' = |\mathbf{p}'| \cos\alpha$

Strain Rate Tensor

$$\frac{\mathrm{D}}{\mathrm{D}t}\,\mathbf{dr} = \nabla\mathbf{v}$$

In similar way as strain tensor, a tensor that describes the rate of change of deformation can be defined from **velocity gradient**:

$$\nabla\mathbf{v} = \begin{bmatrix} \dfrac{\partial \mathrm{v}_1}{\partial \mathrm{x}_1} & \dfrac{\partial \mathrm{v}_1}{\partial \mathrm{x}_2} & \dfrac{\partial \mathrm{v}_1}{\partial \mathrm{x}_3} \\ \dfrac{\partial \mathrm{v}_2}{\partial \mathrm{x}_1} & \dfrac{\partial \mathrm{v}_2}{\partial \mathrm{x}_2} & \dfrac{\partial \mathrm{v}_2}{\partial \mathrm{x}_3} \\ \dfrac{\partial \mathrm{v}_3}{\partial \mathrm{x}_1} & \dfrac{\partial \mathrm{v}_3}{\partial \mathrm{x}_2} & \dfrac{\partial \mathrm{v}_3}{\partial \mathrm{x}_3} \end{bmatrix}$$

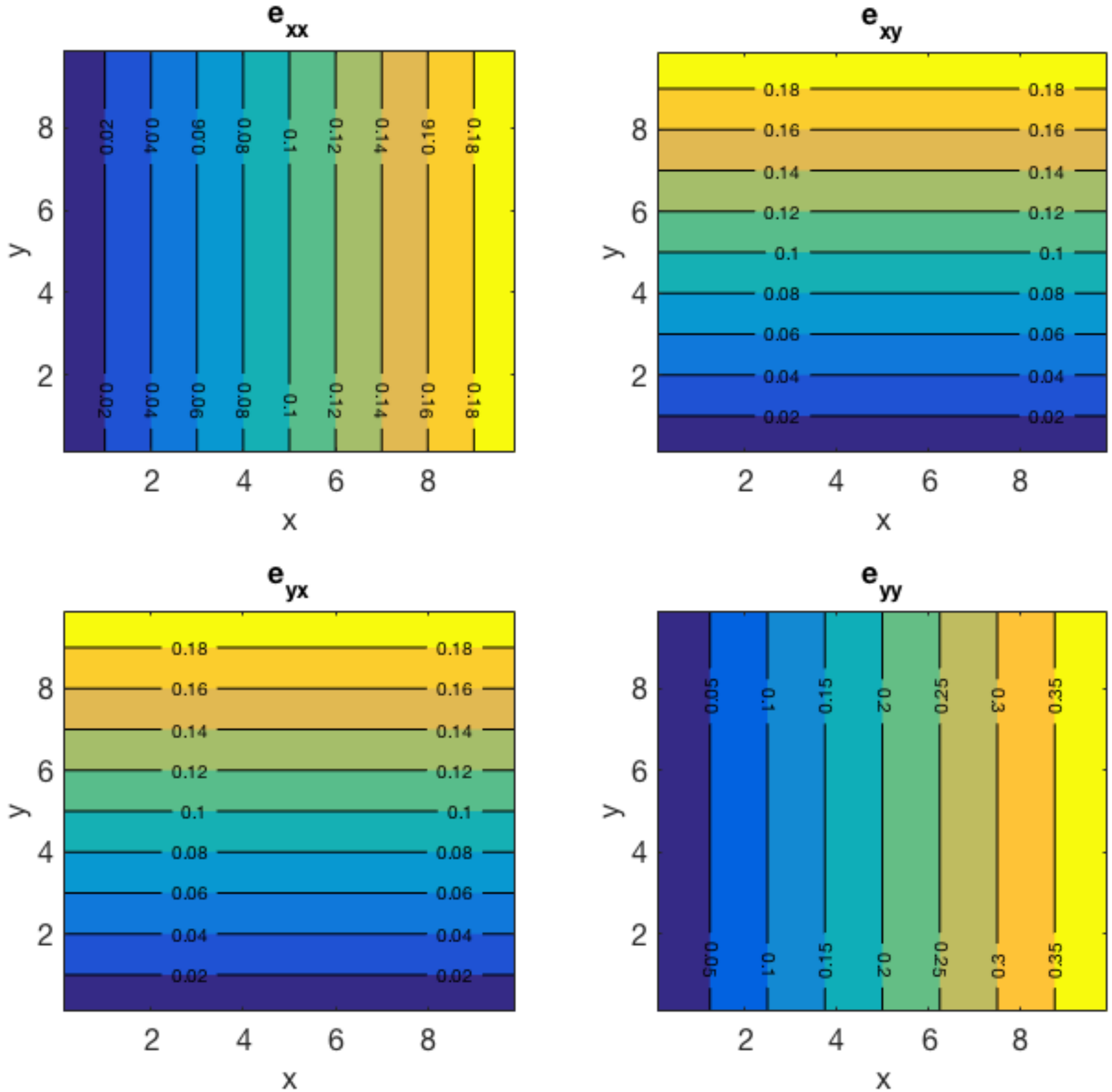
$$\nabla\mathbf{v} = \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right) + \frac{1}{2}\left(\nabla\mathbf{v} - \nabla\mathbf{v}^T\right)$$

$$\nabla\mathbf{v} = \mathbf{D} + \mathbf{W}$$

Velocity gradient tensor is the sum of **strain rate** and **vorticity** tensors

Infinitesimal strain

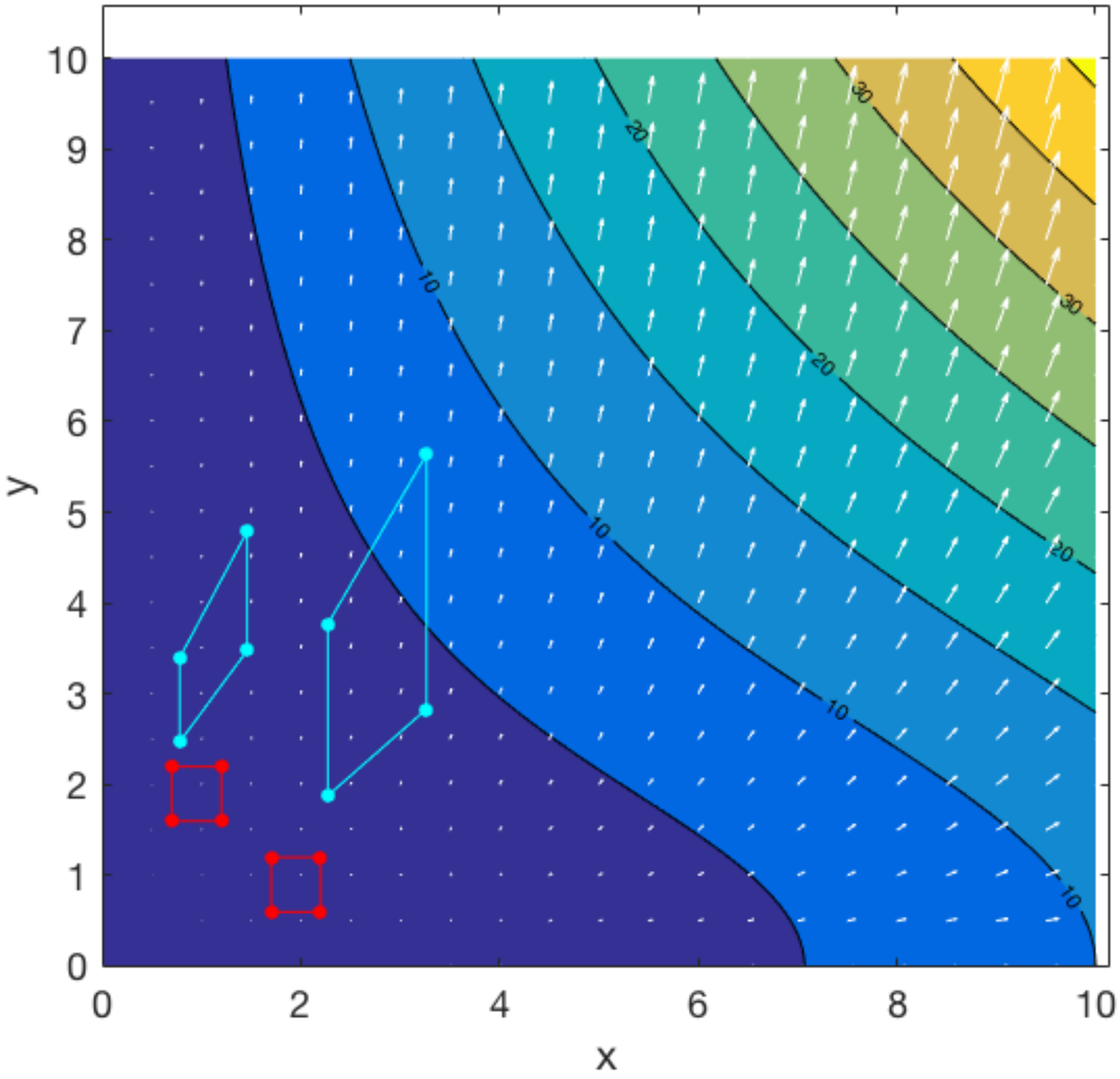
small time step, can assume constant displacement gradient encountered



Deformation after finite strain

original shape

shape at time=1.5



displacement in time interval =1

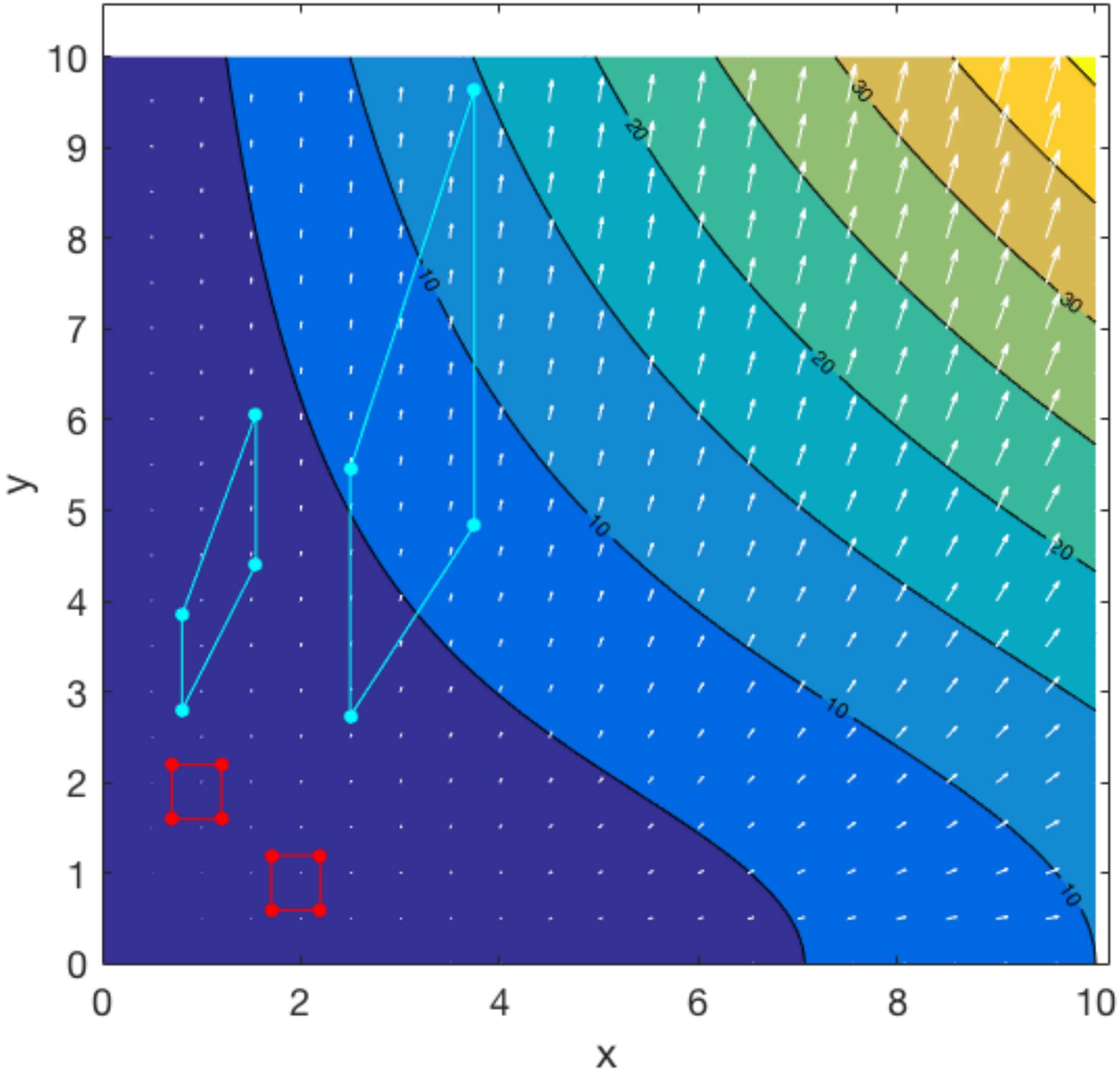
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Deformation after finite strain

original shape

shape at time=1.9



displacement in time interval =1

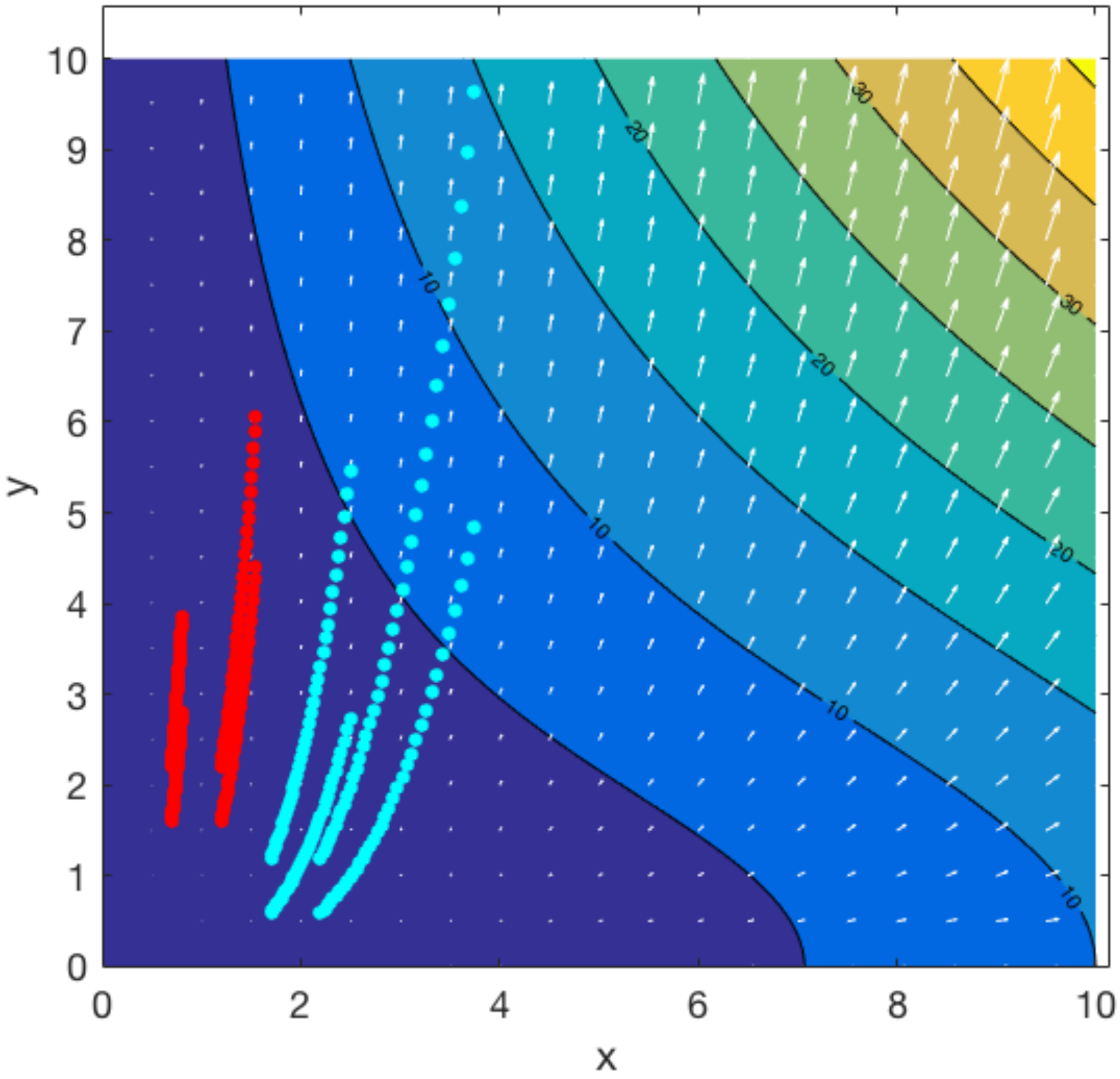
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Deformation after finite strain

points shape 1

points shape 2

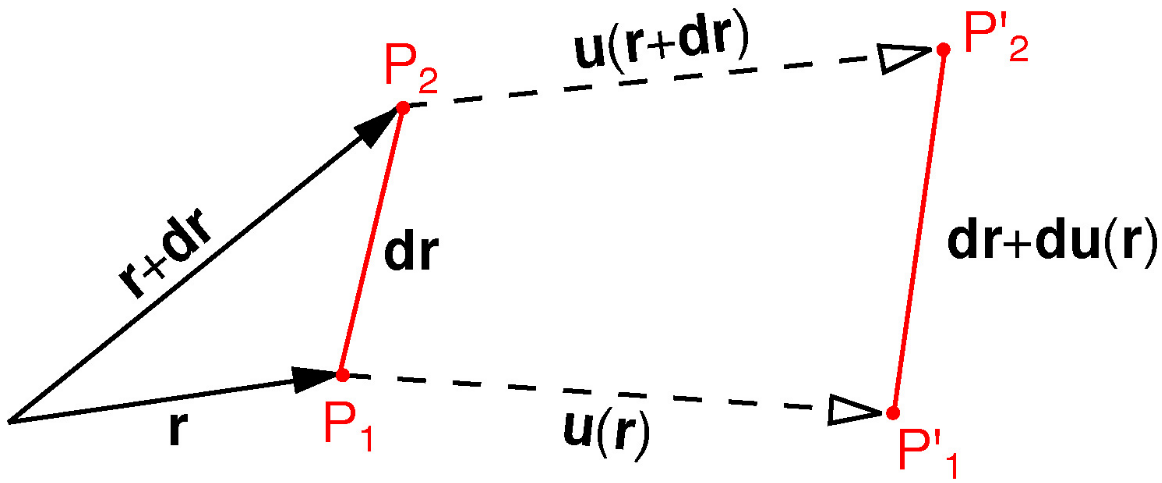


displacement in time interval =1

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Finite Strain



$\mathbf{dr}' = P'_2 - P'_1 = \mathbf{dr} + \nabla\mathbf{u}(\mathbf{r})\cdot\mathbf{dr} = [\mathbf{I} + \nabla\mathbf{u}(\mathbf{r})]\cdot\mathbf{dr} = \mathbf{F}\cdot\mathbf{dr}$

区别于infinitesimal strain deformation

new length of segment P'_2-P'_1:

- $\mathbf{dr}'\cdot\mathbf{dr}' = (\mathbf{F}\cdot\mathbf{dr})\cdot(\mathbf{F}\cdot\mathbf{dr}) = \mathbf{dr}\cdot(\mathbf{F}^T\cdot\mathbf{F})\cdot\mathbf{dr} = \mathbf{dr}\cdot\mathbf{C}\cdot\mathbf{dr}$
- $\mathbf{C} = \mathbf{F}^T\cdot\mathbf{F} = (\mathbf{I}+\nabla\mathbf{u})^T\cdot(\mathbf{I}+\nabla\mathbf{u}) = \mathbf{I} + \nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T\cdot\nabla\mathbf{u}$
- $\mathbf{C} = \mathbf{I} + 2\mathbf{E}^*$
- $\mathbf{E}^* = 1/2\,[\nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T\cdot\nabla\mathbf{u}]$

C - ***right Cauchy-Green deformation tensor***

E* - finite deformation tensor, also called ***Lagrange strain tensor***

Finite Strain

$\mathbf{dr}'\cdot\mathbf{dr}' = \mathbf{dr}\cdot\mathbf{C}\cdot\mathbf{dr}$

$\mathbf{C} = \mathbf{F}^{\mathbf{T}}\cdot\mathbf{F} = \mathbf{I}+2\mathbf{E}^*$ - ***right Cauchy-Green deformation tensor***

$\mathbf{E}^*$ - finite deformation tensor, also called ***Lagrange strain tensor***

$\mathbf{E}^*=1/2\ [\nabla\mathbf{u}+(\nabla\mathbf{u})^{T}+(\nabla\mathbf{u})^{T}\cdot\nabla\mathbf{u}]$

The inverse: $\mathbf{dr}\cdot\mathbf{dr} = \mathbf{dr}'\cdot\mathbf{B}\cdot\mathbf{dr}'$

gives the ***left C-G deformation tensor***: $\mathbf{B}=\mathbf{F}\cdot\mathbf{F}^{\mathbf{T}}=\mathbf{I}+2\mathbf{e}^*$

where $\mathbf{e}^*$ is the ***Euler strain tensor***

$\mathbf{e}^*=1/2\ [\nabla'\mathbf{u}+(\nabla'\mathbf{u})^{T}+(\nabla'\mathbf{u})^{T}\cdot\nabla'\mathbf{u}]$

For small deformation, $\partial/\partial x'\approx\partial/\partial x$ and quadratic term in $\nabla\mathbf{u}$ negligible

$\Rightarrow$ $\mathbf{E}^*=\mathbf{e}^*=$***infinitesimal strain tensor*** E

Meaning Finite Strain Tensor Components

Similar to infinitesimal strain tensor, the components of the finite strain tensor can be related to length and angle changes:

$$\mathbf{dx}^{(1)\prime}\cdot\mathbf{dx}^{(2)\prime} = \mathbf{dx}^{(1)}\cdot(\mathbf{F}^{T}\cdot\mathbf{F})\cdot\mathbf{dx}^{(2)} = \mathbf{dx}^{(1)}\cdot\mathbf{C}\cdot\mathbf{dx}^{(2)} = \mathbf{dx}^{(1)}\cdot(\mathbf{I}+2\mathbf{E}^{*})\cdot\mathbf{dx}^{(2)}$$

$$\mathbf{dx}^{(1)\prime}\cdot\mathbf{dx}^{(2)\prime} - \mathbf{dx}^{(1)}\cdot\mathbf{dx}^{(2)} = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^{*}\cdot\mathbf{dx}^{(2)}$$

Take $\mathbf{dx}^{(1)\prime} = ds_1'\,\hat{\mathbf{s}}$ as the deformed vector of $\mathbf{dx}^{(1)} = ds_1\,\hat{\mathbf{e}}_1$,

So that: $(ds_1')^2 - (ds_1)^2 = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^{*}\cdot\mathbf{dx}^{(1)} = ?$

What is r.h.s
in terms of ds_1?

Then $E^{*}_{11} =$, and similarly for other on-diagonal E_{ij}

For small strain:

Meaning Finite Strain Tensor Components

Similar to infinitesimal strain tensor, the components of the finite strain tensor can be related to length and angle changes:

$$\mathbf{dx}^{(1)'}\cdot\mathbf{dx}^{(2)'} = \mathbf{dx}^{(1)}\cdot(\mathbf{F}^{T}\cdot\mathbf{F})\cdot\mathbf{dx}^{(2)} = \mathbf{dx}^{(1)}\cdot\mathbf{C}\cdot\mathbf{dx}^{(2)} = \mathbf{dx}^{(1)}\cdot(\mathbf{I}+2\mathbf{E}^{*})\cdot\mathbf{dx}^{(2)}$$

$$\mathbf{dx}^{(1)'}\cdot\mathbf{dx}^{(2)'} - \mathbf{dx}^{(1)}\cdot\mathbf{dx}^{(2)} = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^{*}\cdot\mathbf{dx}^{(2)}$$

Take $\mathbf{dx}^{(1)\,'} = ds_1'\,\hat{\mathbf{s}}$ as the deformed vector of $\mathbf{dx}^{(1)} = ds_1\,\hat{\mathbf{e}}_1$,

So that: $(ds_1')^2 - (ds_1)^2 = 2(ds_1)^2\,\hat{\mathbf{e}}_1\cdot\mathbf{E}^{*}\cdot\hat{\mathbf{e}}_1$

dr'.dr' = dr·C·dr

C = I+2E*

Then $E^*_{11} = \dfrac{(ds_1')^2 - (ds_1)^2}{2(ds_1)^2}$, and similarly for other on-diagonal E_{ij}

For small strain:

$$E^*_{11} = \frac{(ds_1'-ds_1)(ds_1'+ds_1)}{2(ds_1)^2} \approx \frac{(ds_1'-ds_1)2ds_1}{2(ds_1)^2} = \frac{(ds_1'-ds_1)}{ds_1} = E_{11}$$

Meaning Finite Strain Tensor Components

Similar to infinitesimal strain tensor, the components of the finite strain tensor can be related to length and angle changes:

$$\mathbf{dx}^{(1)'}\cdot\mathbf{dx}^{(2)'} - \mathbf{dx}^{(1)}\cdot\mathbf{dx}^{(2)} = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^{*}\cdot\mathbf{dx}^{(2)}$$

Take $\mathbf{dx}^{(1)}{}' = ds_1'\,\hat{\mathbf{s}}$ as the deformed vector of $\mathbf{dx}^{(1)} = ds_1\,\hat{\mathbf{e}}_1$,
And $\mathbf{dx}^{(2)}{}' = ds_2'\,\hat{\mathbf{p}}$ as the deformed vector of $\mathbf{dx}^{(2)} = ds_2\,\hat{\mathbf{e}}_2$

So that: $ds_1' ds_2' \cos(\hat{\mathbf{s}},\hat{\mathbf{p}}) - 0 =$

Then $2E^*_{12} =$, and similar for other off-diagonal E_{ij}

For small strain:

$$\cos(\hat{\mathbf{s}},\hat{\mathbf{p}}) = \sin(90^{o} - (\hat{\mathbf{s}},\hat{\mathbf{p}})) \approx 90^{o} - (\hat{\mathbf{s}},\hat{\mathbf{p}}) \qquad \frac{ds_i'}{ds_i} \approx 1$$

$$E^*_{12} \approx E_{12}$$

Meaning Finite Strain Tensor Components

Similar to infinitesimal strain tensor, the components of the finite strain tensor can be related to length and angle changes:

$$\mathbf{dx}^{(1)\prime}\cdot\mathbf{dx}^{(2)\prime} - \mathbf{dx}^{(1)}\cdot\mathbf{dx}^{(2)} = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^*\cdot\mathbf{dx}^{(2)}$$

Take $\mathbf{dx}^{(1)}{}' = ds_1'\,\hat{\mathbf{s}}$ as the deformed vector of $\mathbf{dx}^{(1)} = ds_1\,\hat{\mathbf{e}}_1$,
And $\mathbf{dx}^{(2)}{}' = ds_2'\,\hat{\mathbf{p}}$ as the deformed vector of $\mathbf{dx}^{(2)} = ds_2\,\hat{\mathbf{e}}_2$

sin(90 –(s,p))

So that: $ds_1' ds_2' \cos(\hat{\mathbf{s}},\hat{\mathbf{p}}) - 0 = 2 ds_1 ds_2 \hat{\mathbf{e}}_1 \cdot \mathbf{E}^* \cdot \hat{\mathbf{e}}_2$

Then $2\mathrm{E}^*_{12} = \dfrac{ds_1'}{ds_1}\,\dfrac{ds_2'}{ds_2}\cos(\hat{\mathbf{s}},\hat{\mathbf{p}})$, and similar for other off-diagonal E_{ij}

For small strain:

$$\cos(\hat{\mathbf{s}},\hat{\mathbf{p}}) = \sin(90^\circ - (\hat{\mathbf{s}},\hat{\mathbf{p}})) \approx 90^\circ - (\hat{\mathbf{s}},\hat{\mathbf{p}}) \qquad \frac{ds_i'}{ds_i} \approx 1$$

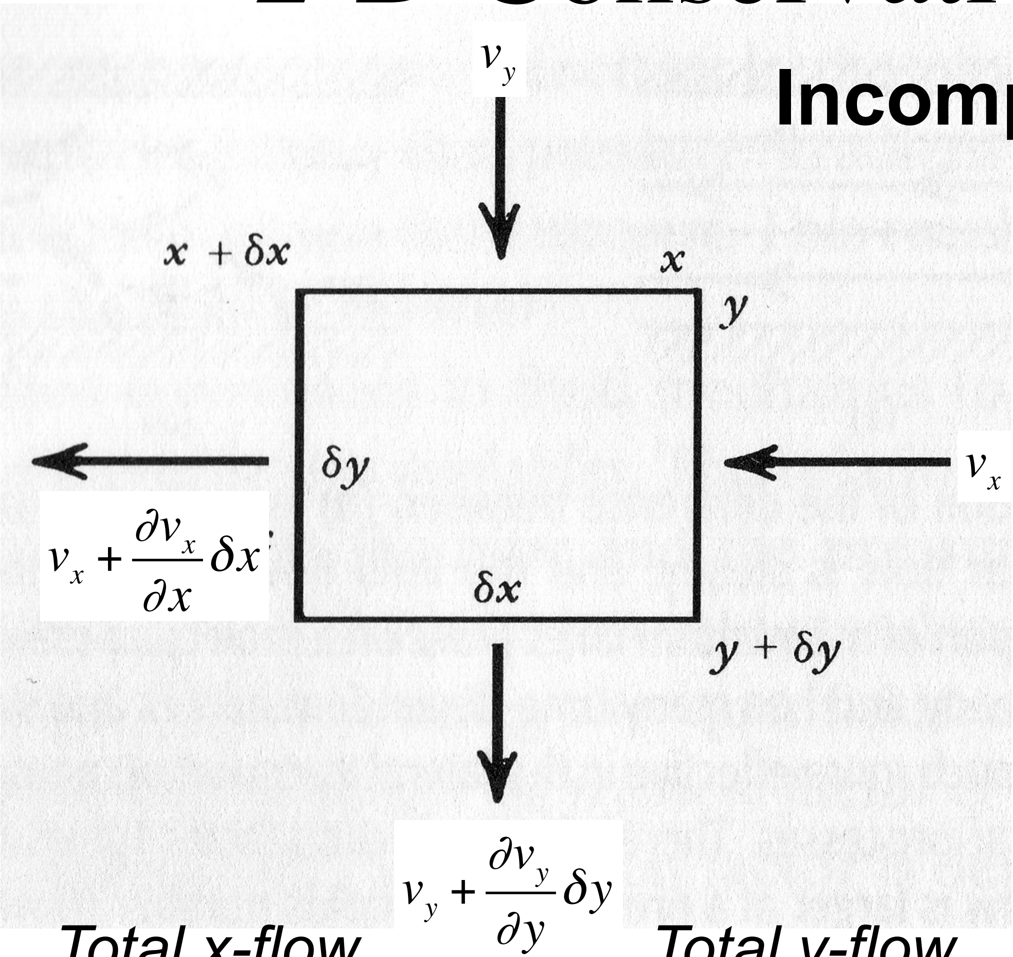
$$\mathrm{E}^*_{12} \approx \mathrm{E}_{12}$$

Compatibility equations

Computing strain (rate) field from a displacement (velocity) field is straightforward.

The inverse is only defined if the strain rate field satisfies a set of ***compatibility equations*** to ensure that the 6 strain components uniquely relate to a continuous field of 3 displacement components.

2-D Conservation of Mass



Incompressible

Continuity Equation

x-flow in: $v_x \delta y$

x-flow out: $(v_x + \frac{\partial v_x}{\partial x}\delta x)\delta y$

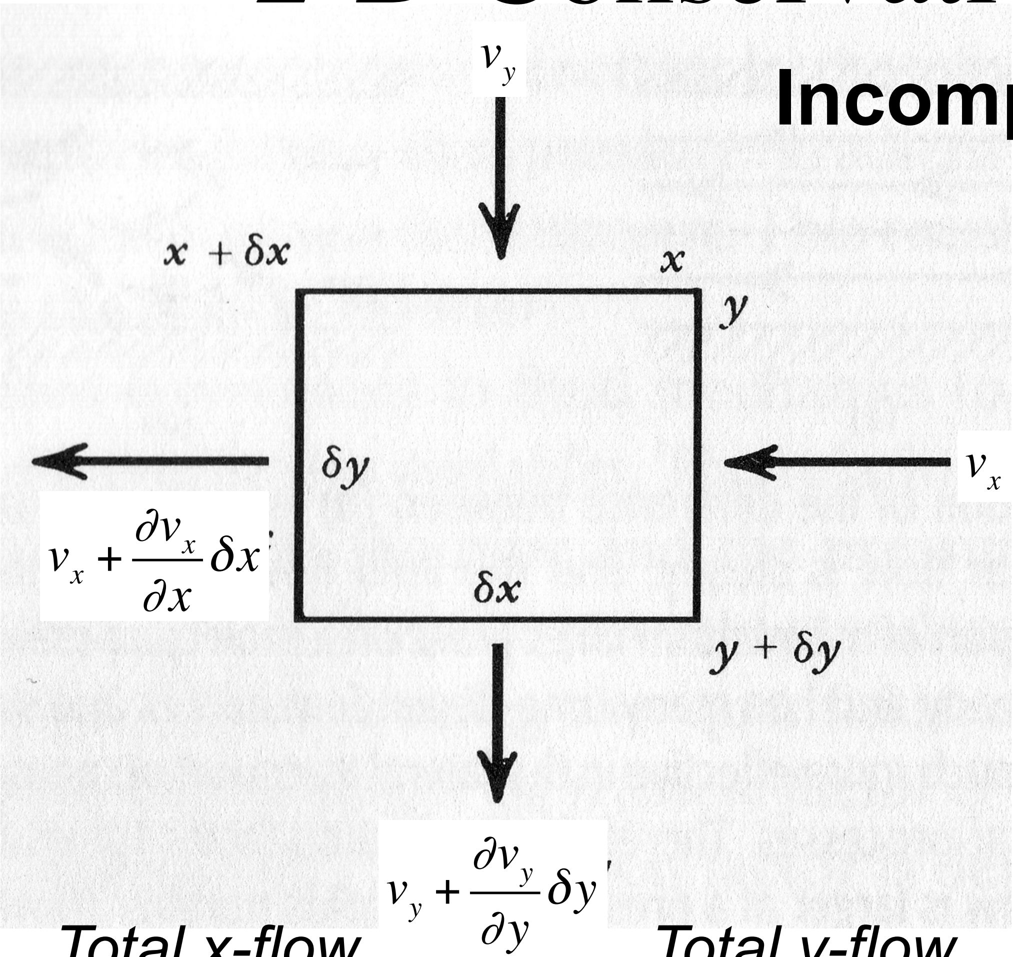
Total x-flow | *Total y-flow* | *Per unit area*

$$\left\{ \qquad \right\}\delta y + \left\{ \qquad \right\}\delta x = 0 \Rightarrow$$

i.e., which also applies in 3-D

No volume changes!

2-D Conservation of Mass



Incompressible

Continuity Equation

x-flow in: $v_x \delta y$

x-flow out: $(v_x + \frac{\partial v_x}{\partial x}\delta x)\delta y$

Total x-flow *Total y-flow* *Per unit area*

$$\left\{(v_x + \frac{\partial v_x}{\partial x}\delta x) - v_x\right\}\delta y + \left\{(v_y + \frac{\partial v_y}{\partial y}\delta y) - v_y\right\}\delta x = 0 \Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

i.e., $\nabla \cdot \mathbf{v} = 0$ which also applies in 3-D

No volume changes!

Conservation of Mass

Full expression: compressible

$$\frac{D\rho dV}{Dt} = 0$$

ρ − density
dV − infinitesimal volume

density changes

$$\frac{D\rho}{Dt} dV + \rho \frac{DdV}{Dt} = 0$$

volume changes

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

In spatial description: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$, where $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$

ρ(time) advected

Outline Lecture 7

- Material vs. spatial descriptions
- Time derivatives
- Displacement
- Infinitesimal Deformation
- Finite Deformation
- Conservation of Mass

Further reading on the topics in the lecture can be done in for example: Lai, Rubin, Kremple (2010): Ch. 3-1 through 3-15 and we covered some of the basics discussed in 3-20 to 3-26