

# Fluid Flow

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## Outline of Lecture

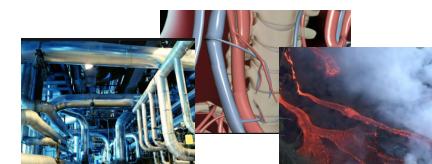
- Fluid flow in simple geometries
- Reynolds number and the onset of turbulence
- Modelling turbulence
- Strain-rate dependent rheologies

## Learning Objectives

- Be able to set up Navier-Stokes equations and find analytical solutions for simple steady flow problems:
  - channel flow, pipe flow and flow down a slope,
  - with Newtonian or non-Newtonian viscosities,
  - gain understanding in physical behaviour of such flows.
- Understanding different common methods to model turbulence

## Why study flow in simple geometries?

- Channel flow
- Pipe flow
- Flow down a slope



Analytical solutions - first estimates of flow characteristics, benchmarks

Numerical solutions => complex geometries, complex rheology, non-steady problems

## The Navier-Stokes Equation

- Momentum balance:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g}$$

- Mass balance (continuity equation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Where:  
 $\mathbf{u}$  – velocity  
 $P$  – pressure  
 $\boldsymbol{\tau}$  – deviatoric stress tensor  
 $\rho$  – density  
 $\mathbf{g}$  – gravitational acceleration

Material derivative:  
 $\frac{Du}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$   
or equivalently

$$\frac{Du}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u} \mathbf{u}$$

*Use index notation to check  
that these are equivalent*

## Assumptions in this lecture

Where:  
 $\mathbf{S}$  – strain rate tensor  
 $\mu$  – (shear) viscosity

- We will only be considering incompressible flow – Constant density:

$$\nabla \cdot \mathbf{u} = 0$$

- Start by considering Newtonian Flow:

- Shear stress is proportional to the strain rate

$$\boldsymbol{\tau} = 2\mu \mathbf{S}$$

$$\mathbf{S} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

- This assumes viscosity is isotropic and also ignores the bulk viscosity, which is associated with changes in the volume/density of the fluid (we are making an incompressible assumption here). Bulk viscosity can be important in, for instance shocks

## Incompressible Newtonian Fluid

- In an incompressible Newtonian fluid the gradient of viscous stress can be further simplified:

$$\nabla \cdot \boldsymbol{\tau} = \mu \nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T) = \mu \nabla^2 \mathbf{u}$$

- This requires the implication of the incompressible assumption:

$$\nabla \cdot \mathbf{u} = 0$$

*Check, e.g. using index  
notation*

## Side note: Dynamic and Kinematic Viscosity

Be careful when talking about viscosity as to what is meant

Dynamic viscosity  $\mu$  (sometimes called absolute viscosity)

- (units: Pa·s)
- Proportionality between shear stress and strain rate

Kinematic viscosity  $\nu = \mu/\rho$

- (units: m<sup>2</sup>/s)
- Equivalent to a diffusivity for momentum

### Pressure Driven Steady Flow between Parallel Plates

- Can be derived easily either from the Navier-Stokes equation or by considering a force balance on a unit of fluid:

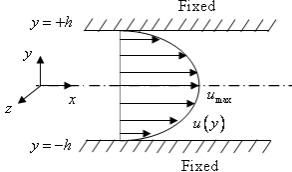
- No time dependency
- Only non-zero velocity components are in the  $x$  direction
- Ignore body forces
- Flow in direction of pressure drop

$$\frac{dP}{dx} = -\frac{\Delta P}{L} = \mu \frac{d^2 u_x}{dy^2}$$

- Integrate and note that  $u_x$  at  $y=h$  and  $y=-h$ :

$$u_x = \frac{\Delta P}{2\mu L} (h^2 - y^2)$$

*Check this*



### Pressure Driven Steady Flow between Parallel Plates

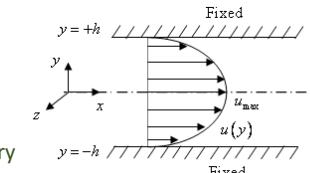
- For boundary condition  $u_x$  at  $y=h$  and  $y=-h$ :

$$u_x = \frac{\Delta P}{2\mu L} (h^2 - y^2)$$

*What is the volumetric flow rate (per meter of width) ?*

- What is the solution for the boundary condition  $u_x=U$  at  $y=h$  and  $u_x=0$  at  $y=-h$ ?

This type of flow is called Couette flow.



### Pressure Driven Steady Flow in a pipe

- Similar derivation as before, but done in cylindrical coordinates:

- Navier Stokes momentum equation in cylindrical coordinates:

$r$  direction:  $\rho \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (ru_r) \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2} \right] + \rho g_r$

$\theta$  direction:  $\rho \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta u_r}{r} + u_z \frac{\partial u_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2} \right] + \rho g_\theta$

$z$  direction:  $\rho \left( \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right] + \rho g_z$

### Pressure Driven Steady Flow in a pipe

- Need only consider momentum balance in  $z$  direction

- $z$  direction only non-zero velocity and no time dependency:

$$\frac{dP}{dz} = -\frac{\Delta P}{L} = \mu \frac{1}{r} \frac{d}{dr} \left( r \frac{du_z}{dr} \right)$$

- Integrate once:

$$\frac{du_z}{dr} = -\frac{\Delta P}{2\mu L} r \quad \text{Constant of integration is zero as symmetry means that there is a zero velocity gradient at the centre of the pipe}$$

- Integrate again and note that velocity is zero at the wall ():

$$u_z = \frac{\Delta P}{4\mu L} (R^2 - r^2) \quad \text{What is the volumetric flow rate in this case?}$$

## Calculating flow and pressure

- For simple geometries either simplify the Navier-Stokes equation or do your own momentum balance
- Free surface flow on an inclined plane using a momentum balance
  - For a given slope angle, volumetric fluid flow-rate, fluid density and fluid viscosity, how thick will the fluid layer be (assuming a large enough plane that the thickness is constant)?

## Calculations:

- Assume  $x$  is distance from free surface and  $z$  is distance down slope
- Force balance on control in direction parallel to slope (for unit width)
 
$$p_0\Delta x - p_L\Delta x + \rho g \cos(\beta)\Delta x L + \tau_{xz}(x + \Delta x)L - \tau_{xz}(x)L = 0$$

Pressure force on ends of control volume      Gravity force down the slope  
 Shear force on sides of control volume (note that  $\tau_{xz}$  refers to the stress on the  $x$  direction face acting in the  $z$  direction)  
 Momentum in by viscous transfer      Momentum Out by viscous transfer  
 Gravity Force      Direction of gravity
- As the system is at steady state, there are no viscous stresses exerting a force in the direction normal to the slope
  - Pressure with depth below free surface purely hydrostatic  
– no pressure gradient down the slope ( $p_0 = p_L$ )
- Rearranging the equation:

$$\frac{\tau_{xz}(x + \Delta x) - \tau_{xz}(x)}{\Delta x} = -\rho g \cos(\beta) \text{ Taking the limit as } \Delta x \text{ goes to zero: } \frac{d\tau_{xz}}{dx} = -\rho g \cos(\beta)$$

## Calculations continued:

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos(\beta)$$

- Integrate to get shear stress as a function of the depth below the surface:  

$$\tau_{xz} = -\rho g \cos(\beta)x + A$$
- As the air above the free surface is much lower viscosity than the liquid we can assume that it exerts negligible viscous stress:  

$$\tau_{xz} = 0 \text{ at } x = 0 \text{ therefore } A = 0$$
- If we assume a Newtonian fluid then the shear stress is proportional to the strain rate, with viscosity being the proportionality:
  - Note that the sign depends on whether you consider the stress to be the force exerted by the fluid or on the fluid – Either can be used, but it must be used consistently
$$\tau_{xz} = \mu \frac{dy_z}{dx}$$
- Substituting and integrating again yields:  

$$v_z = -\frac{\rho g \cos(\beta)}{2\mu} x^2 + B$$

## Calculations continued:

$$v_z = -\frac{\rho g \cos(\beta)}{2\mu} x^2 + B$$

- Constant of integration, obtain from bottom boundary condition:
- If  $h$  is the thickness of the slide, then  $v_z = 0$  at  $x = h$  (noting that we have defined  $x$  as being the distance below the free surface)
 
$$B = \frac{\rho g \cos(\beta)}{2\mu} h^2$$
- The velocity profile is therefore parabolic with the following shape
 
$$v_z = \frac{\rho g \cos(\beta)}{2\mu} (h^2 - x^2)$$
- Note that this is the same velocity profile as would be obtained for similar flow between 2 parallel plates  $2h$  apart
  - Planes of symmetry and free surfaces are physically very similar as they involve no shear stress

### Question 1 - worksheet

- A fluid is flowing between two large parallel plates (density  $\rho$  and viscosity  $\mu$ ). The plates are a distance  $h$  apart and the pressure drop is  $\Delta PL$ . The plates are inclined at an angle  $\vartheta$ . Assuming that the flow is laminar, calculate the volumetric flowrate through the system per meter of width as a function of the inclination angle. Simplify the Navier-Stokes equation in order to carry out this calculation.
- At what angle does the flow stop in terms of the other variables (you can also check this based on a force balance over the entire system)?

### Not the end of the story – Why?

- An explicit assumption in these derivations is that the flow is steady (no time dependency)
- What happens if there is a perturbation in the flow?
  - If this perturbation grows, it means that the flow can not be steady and that this assumption is invalid
- Perturbation will be damped if viscous forces are stronger than the inertial force associated with the perturbation

### Reynolds Number

- Reynolds number represents the balance between inertial and viscous forces

$$\frac{\text{Inertial Force}}{\text{Viscous Force}} \propto \frac{\rho v \frac{dv}{dx}}{\mu \frac{d^2 v}{dx^2}} \propto \frac{\rho v \frac{v}{d}}{\mu \frac{d^2}{d^2}} \quad Re = \frac{\rho d v}{\mu}$$

- If Reynolds number is small perturbations are suppressed – Laminar Flow
- If Reynolds number is large perturbations grow – Turbulent Flow

### Reynolds Number – Flow in Pipe

- Pipe diameter as characteristic length scale
- Average velocity as characteristic velocity

Laminar	$Re < 2100$
Transition	$2100 < Re < 4000$
Turbulent	$Re > 4000$

$$Re = \frac{\rho d v}{\mu}$$

## Flow in Pipe

- As Reynolds number increases beyond transition average flow profile goes from parabolic to closer to plug flow
  - Laminar boundary layer near the wall – Gets thinner as the Reynolds number increases
  - Thickness of boundary layer also influenced by surface roughness

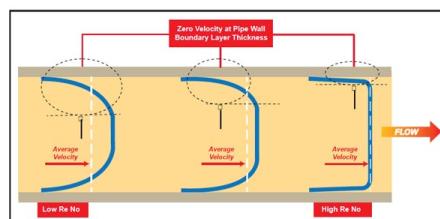


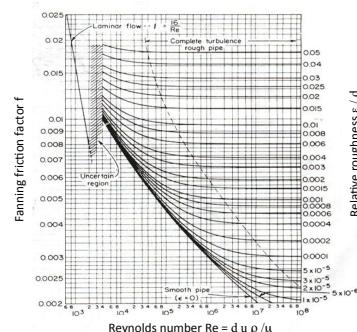
Figure 1. Boundary layer influence on flow profile and Reynolds Number

## Flow in Pipes – The Engineering Approach

- Friction factor is a function of Reynolds Number and pipe roughness:
  - For laminar flow  $f=16/Re$
  - For turbulent flow empirical relationships or charts are used

$$f = \frac{\Delta P R}{L \rho v^2}$$

Note that that Moody friction factor is often used instead of the Fanning friction factor, but it is simply 4 times larger



## Flow in Pipes – The Engineering Approach

- Fanning Friction factor:

- Ratio of shear stress exerted on the fluid to its kinetic energy:

$$f = \frac{\tau}{\rho v^2} \frac{2}{R}$$

- For fully developed flow the shear stress on the wall must balance the pressure drop:

$$\Delta P \pi R^2 = \tau 2 \pi R L$$

- Combine to give Fanning friction factor:  $f = \frac{\Delta P R}{L \rho v^2}$

## Non-dimensionalisation

Use characteristic length and velocity:

$$r' = \frac{r}{R} \quad u' = \frac{u}{\bar{u}}$$

$$\text{Equation of Motion} \quad \frac{du}{dr} = \frac{r}{2\mu} \frac{dp}{dx} \Rightarrow \frac{du'}{dr'} = \frac{R^2}{\bar{u}} \frac{r'}{2\mu} \frac{dp}{dx} = -\frac{f Re}{4} r'$$

dimensionless pressure gradient	Friction factor
$f = \frac{-R}{\bar{u}^2} \frac{dp}{dx}$	$Re = \frac{\bar{u} D}{\mu}$

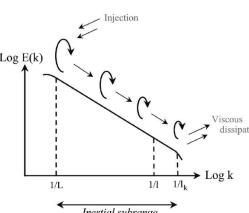
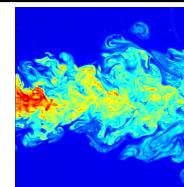
Reynolds number	inertial forces / viscous forces
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$$\text{Solution laminar flow} \quad u'(r') = \frac{f Re}{8} (1 - r'^2) \quad u'_{\max} = 2\bar{u}' = \frac{f Re}{8}$$

$$\bar{u}' = 1 = \frac{f Re}{16} \Rightarrow f = 16/Re$$

## What happens in Turbulent Flow?

- At high Reynolds numbers, large scale flow structures and instabilities will develop smaller scale perturbations
- This cascade of larger perturbations generating smaller flow structures will continue down until they are small enough that viscosity can “win”
  - The Kolmogorov length scale
- At this finest level the energy in the flow is dissipated as heat via the viscosity



## Direct Numerical Simulation (DNS)

- Solve Navier-Stokes equation with enough resolution to resolve all the eddies down to their laminar cores
  - Very computationally expensive – prohibitively so for all but the lowest Reynolds numbers and smallest systems
- Can get a rough estimate of the resolution required based on Kolmogorov length scales and dimensional arguments

$$\Delta x \sim \left( L \left( \frac{\mu}{\rho |\mathbf{v}|} \right)^3 \right)^{\frac{1}{4}}$$

Water ( $\mu=1 \cdot 10^{-3}$  Pa s,  $\rho = 1 \cdot 10^3$  kg/m<sup>3</sup> at 1 m/s in a 1 m channel would require a resolution of about 10  $\mu\text{m}$  (about  $10^{15}$  elements in cubic box)!

*Check dimensions*

## Turbulence Modelling

- As DNS is only useable on a small range of problems, we need to model the effect of turbulence
- Turbulence dissipates more energy than the macroscopic strain rate would suggest
  - Need to calculate this extra dissipation

3 main approaches

- Simply increase the viscosity!
  - Trivial to do – often used in large scale simulations such as ocean modelling
- Assume that you can separate the turbulent and mean components of the flow
  - Reynold Average Navier Stokes (RANS) type models
  - We will look at the k- $\epsilon$  model as it is the most commonly used of this type
- Resolve as much as you can and then use a model for the bits you can't resolve
  - Large Eddy Simulations (LES)

Both RANS and LES approaches involve the calculation of an additional turbulent viscosity that needs to be added to the underlying fluid viscosity

## Mean and Fluctuating Flow Components

- Turbulent structures are typically embedded within macroscopic flow structures
- In analysing and modelling turbulence it is thus useful to distinguish between these flow components
- Assume that the flow at any point can be decomposed into a mean component and a fluctuating component:

where  $\mathbf{v}$  is the instantaneous velocity,  $\bar{\mathbf{v}}$  is the average velocity and  $\mathbf{v}'$  is the fluctuating component of the velocity

## RANS version of the Navier Stokes Equation

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}$$

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u} \mathbf{u}$$

- Assume that there is a mean and fluctuating component of the velocity and pressure
  - Fluctuating component of pressure has no impact on final equation form
- $\mathbf{u} = \mathbf{U} + \mathbf{u}' \quad p = P + p'$
- Substituting and applying the RANS averaging rules:

$$\rho \frac{D\mathbf{U}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{U} + \rho \mathbf{g} - \rho \nabla \cdot \overline{\mathbf{u}' \mathbf{u}'}$$

Why only a single term with  $\mathbf{u}'$ ?

## Reynolds Stress Tensor

- The averaged equation is similar to the original except for the inclusion of the divergence of a rank 2 tensor:

$$\rho \nabla \cdot \overline{\mathbf{u}' \mathbf{u}'}$$

- Known as the Reynolds Stress Tensor

$$\rho \overline{\mathbf{u}' \mathbf{u}'} = \rho \begin{bmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'v'} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'w'} \end{bmatrix} \quad \mathbf{u}' = \{u', v', w'\}$$

## Reynolds Stress Tensor

- Need to relate the Reynolds Stress Tensor to the mean flow
  - Know as the turbulence closure problem
  - No right way to do this, but some approximations are better than others
- The Reynolds Stress Tensor represents the transfer of momentum due to the fluctuations
- In the standard Navier-Stokes equation the viscous stress term represents the “diffusion” of momentum
  - Can think of Reynolds Stress Tensor in terms of a diffusion of momentum due to velocity fluctuations – a turbulent eddy viscosity

## Turbulent Eddy Viscosity

- Write the Reynolds Stress Tensor in terms of the macroscopic strain rate:  $\overline{\mathbf{u}' \mathbf{u}'} = -\nu_T (\nabla \mathbf{U} + \nabla \mathbf{U}^T)$ 
  - Where  $\nu_T$  is a kinematic viscosity
- This means that you can combine the liquid viscosity and the turbulent losses into a single effective viscosity:
  - In turbulence modelling it is often more convenient to use the kinematic viscosity
- $\nu_{\text{effective}} = \nu + \nu_T$
- We now need to calculate this turbulent eddy viscosity

## Zero Equation Models

- The simplest type of turbulence model
  - Assume no explicit transport of turbulence
- Mainly based on dimensional arguments
  - $l_0$  is a turbulent length scale,  $t_0$  is a turbulent time scale
- Zero equation models are not very good in steady state RANS type models where the turbulent length scale is not well defined
  - Often adequate in LES models where the turbulent structures are resolved down to the grid resolution
    - Unresolved turbulent length scale similar to mesh resolution
    - Less transport of small scale turbulence before it is dissipated

$$v_T = \frac{l_0^2}{t_0}$$

## Zero Equation Models

- If we assume that we know the turbulent length scale, we still need the turbulent time scale from some macroscopic flow property
  - Two obvious candidates
    - Magnitude of the strain rate or magnitude of the vorticity
    - Both have units of inverse time
- Based on strain rate – Smagorinsky model  $v_T = (C_S l_0)^2 |\mathbf{S}|$
- $\mathbf{S} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$
- $C_S$  is a tuneable constant. Typical values are 0.1-0.2
- Based on vorticity – Baldwin-Lomaz model  $v_T = (k l_0)^2 |\boldsymbol{\omega}|$
- $\boldsymbol{\omega} = \nabla \times \mathbf{U}$
- Less widely used than Smagorinsky – Can be used for boundary layers
- In boundary layers is a function of distance from the wall

## Two Equation Model

- We will revisit zero equation models in the context of LES
- For RANS modelling zero equation models are inadequate
- Need to consider the generation, transport and dissipation of turbulence
  - Two quantities can be used to model this
    - Turbulent kinetic energy -  $k$
    - Turbulent dissipation rate -  $\epsilon$
- So called turbulence model
  - Other models exist, but this is one of the most widely used and illustrates the idea

$$v_T = C_\mu \frac{k^2}{\epsilon}$$

## Turbulent Kinetic Energy - $k$

- The fluctuating components of the turbulence stores energy
  - This is known as the turbulent kinetic energy, which, per mass of fluid, can be written as follows:
- $k = \frac{1}{2} (\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$   $\mathbf{u}' = \{u', v', w'\}$
- Note that the turbulent kinetic energy is therefore closely related to the Root Mean Square (RMS) of the velocity distribution
- $u_{rms} = \sqrt{|\mathbf{u}'|^2} = \sqrt{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}$
- This turbulent kinetic energy is generated at the large scale and dissipated at the small scale
  - It can also be transported around the system by the mean flow

## Turbulent Dissipation Rate - $\varepsilon$

- This is equal to the viscous losses associated with the fluctuating component of the velocity
  - I.e. the rate at which the turbulent kinetic energy is converted into thermal energy

$$\varepsilon = \nu \overline{\mathbf{S}' \cdot \mathbf{S}'}$$

$$\mathbf{S}' = \frac{1}{2}(\nabla \mathbf{u}' + \nabla \mathbf{u}'^T)$$

$\nu$  – kinematic viscosity

- While this is how the energy is dissipated, this equation is not directly useful as we do not know the fluctuation components (or their gradients)

## Equations for $k$ and $\varepsilon$

- Both the turbulent kinetic energy ( $k$ ) and the dissipation rate ( $\varepsilon$ ) can be transported, generated and destroyed and are represented by a pair of coupled ODEs

- Don't worry too much about how these are derived – note, though, the additional complexity associated with modelling turbulence

$$\frac{\partial k}{\partial t} + \mathbf{U} \cdot \nabla k = \nabla \cdot \left( \frac{\nu_t}{\sigma_k} \nabla k \right) + 2 \nu_t \mathbf{S} \cdot \mathbf{S} - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \mathbf{U} \cdot \nabla \varepsilon = \nabla \cdot \left( \frac{\nu_t}{\sigma_\varepsilon} \nabla \varepsilon \right) + C_{1\varepsilon} \frac{\varepsilon}{k} \mathbf{S} \cdot \mathbf{S} - C_{2\varepsilon} \frac{\varepsilon^2}{k}$$

$$\nu_T = C_\mu \frac{k^2}{\varepsilon}$$

The semi-empirical constants are typically assigned the following values:

$$C_\mu = 0.09, \sigma_k = 1.00, \sigma_\varepsilon = 1.30, C_{1\varepsilon} = 1.44, C_{2\varepsilon} = 1.92$$

## Boundaries in Turbulence modelling

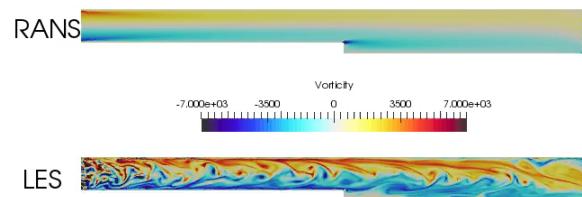
- At inlets both  $k$  and  $\varepsilon$  need to be assigned values
  - If appropriate values are not assigned, long inlet regions will be required
- At walls  $k$  is zero (no slip), but the transition typically occurs over a very narrow boundary layer
  - Typically need to model this boundary layer as it is usually computationally very expensive to explicitly resolve

## Large Eddy Simulations - LES

- Transient simulations that resolve as much of the dynamics as possible
- Zero equation models are usually appropriate
  - No time averaging or smoothing of equations above the grid resolution
  - If unresolved resolution is small then the assumption of no turbulent transport between generation and dissipation is appropriate
- Can use Smagorinsky model with length scale as the grid resolution

$$\nu_T = (C_S \Delta x)^2 |\mathbf{S}|$$

## RANS vs LES



## Question 4 - worksheet

- Using an approximation for the relation between friction factor and Reynolds number for laminar and turbulent flow, write a program that returns the volumetric flow rate for given pressure drop, pipe length, pipe radius, pipe roughness, fluid density and fluid viscosity.
- See detailed instructions on the worksheet

## Other Rheologies

- Thus far we have only considered Newtonian rheology
  - Shear stress proportional to the strain rate
  - A good approximation for many fluids, importantly including water and virtually all gases
- Other Rheologies are possible
  - Shear stress a more complex function of strain rate
  - Can also be a function of strain history
    - Thixotropic fluids

## Shear thinning

- Apparent viscosity decreases with strain rate
  - Quite common
    - Dense suspensions, emulsions, foams etc...
  - Sometimes called pseudo-plastic fluids
- Some physical origins:
  - Structures in the fluid can break down at high shears
    - E.g. some suspensions where particles can form flocs
  - Molecules align with or are extended in the shear direction
    - E.g. molten plastics

## Shear-thinning viscosity



"Frog saliva switches between being thin and watery as the whip-like tongue hits its target, to thick and sticky as the insect is reeled in"

## Yield Stress

- An extreme example of shear thinning are materials with a yield stress
- Below a yield stress a material does not flow and acts like a solid
- Above the yield stress the material flows and acts like a liquid
- Quite common
  - It is really a matter of the size of the yield stress
  - Logical extension of plastic deformation in a solid

## Shear thickening

- Apparent viscosity increases with strain rate
  - Much less common
    - Classic example is corn starch in water
- Some physical origins:
  - Structures in the fluid jam and can't pass one another as easily at high strain rates
- Lots of current interest in shear-thickening fluids
  - Flexible protection and armouring

## Modelling non-Newtonian Rheologies

- Need to incorporate these rheologies into the Navier-Stokes equation
  - Require a relationship between shear stress and strain rate tensors
- Often easier to incorporate rheology as an apparent viscosity that depends on strain rate
  - Note that in most solvers it is the velocity (and thus strain rate) that is assumed to be known

## Power law fluid

- The simplest model for both shear thinning and shear thickening
  - The separation of the strain rate into two terms is so that stress tensor points in the correct direction

$$\tau = 2k|2\mathbf{S}|^{n-1} \mathbf{S} \quad \mathbf{S} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

- If  $n=1$  this is Newtonian
- If  $n>1$  this is shear thickening
- If  $n<1$  this is shear thinning
- If the velocity only changes in one direction

$$\tau_{xy} = k \frac{du_y}{dx} \left| \frac{du_y}{dx} \right|^{n-1}$$

$$\mu_{eff} = k|2\mathbf{S}|^{n-1}$$

## Question 2 - worksheet

- A power law fluid is flowing in a cylindrical pipe driven by a pressure gradient of  $\Delta P/L$ . The flow is occurring in the  $z$  direction, with the shear stress thus being the result of a velocity gradient in the  $r$  direction:

$$\tau = k \frac{\partial u_y}{\partial x} \left| \frac{\partial u_y}{\partial x} \right|^{n-1}$$

- Use a force/moment balance to calculate the velocity profile as a function of the power law exponent  $n$ .
- Plot these results in order to see how the velocity profile is influenced by changes in the rheology.

## Bingham Plastic

$$\mathbf{S} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

- The simplest type of yield behaviour is to assume that there is a yield stress followed by a linearly increasing stress with increasing strain rate

$$\text{if } |\tau| > \tau_0: \quad \tau = (2\mu_\infty|\mathbf{S}| + \tau_0) \frac{\mathbf{S}}{|\mathbf{S}|}$$

$$\text{if } |\tau| < \tau_0: \quad \mathbf{S} = 0$$

$$\text{if } |\tau| > \tau_0: \quad \mu_{eff} = 2\mu_\infty + \frac{\tau_0}{|\mathbf{S}|}$$

$$\text{if } |\tau| < \tau_0: \quad \mu_{eff} = \infty$$

- If the velocity only changes in one direction

$$\text{if } |\tau| > \tau_0: \quad \tau = sign\left(\frac{\partial u_y}{\partial x}\right) \left( \mu_\infty \left| \frac{\partial u_y}{\partial x} \right| + \tau_0 \right)$$

$$\text{if } |\tau| < \tau_0: \quad \frac{\partial u_y}{\partial x} = 0$$

## Herschel-Bulkley Fluid

- Combines yield stress with power law post-yield behaviour:

$$\text{if } |\tau| > \tau_0: \quad \tau = (k(2|\mathbf{S}|)^n + \tau_0) \frac{\mathbf{S}}{|\mathbf{S}|}$$

$$\text{if } |\tau| < \tau_0: \quad \mathbf{S} = 0$$

$$\text{if } |\tau| > \tau_0: \quad \mu_{eff} = k(2|\mathbf{S}|)^{n-1} + \frac{\tau_0}{|\mathbf{S}|}$$

$$\text{if } |\tau| < \tau_0: \quad \mu_{eff} = \infty$$

- If the velocity only changes in one direction

$$\text{if } |\tau| > \tau_0: \quad \tau = sign\left(\frac{\partial u_y}{\partial x}\right) \left( k \left( \left| \frac{\partial u_y}{\partial x} \right| \right)^n + \tau_0 \right)$$

$$\text{if } |\tau| < \tau_0: \quad \frac{\partial u_y}{\partial x} = 0$$

## Free surface flow on an inclined plane

- This is the problem that we solved earlier for a Newtonian fluid
- Let us consider a Bingham plastic with a yield stress  $\tau_0$

### Calculation

- The shear stress distribution in this problem is the same as that for the Newtonian fluid and can be obtained in the same manner:

$$\tau_{xz} = -\rho g \cos(\beta)x$$

Note that again  $x$  is defined as the distance below the free surface

$$\tau_{xz} = -\rho g \cos(\beta)x$$

## Calculation continued:

- As the  $x$  component is the only non-zero shear stress:

$$\text{if } |\tau| > \tau_0: \quad \tau = \text{sign}\left(\frac{\partial u_z}{\partial x}\right) \left( \mu_\infty \left| \frac{\partial u_z}{\partial x} \right| + \tau_0 \right)$$

$$\text{if } |\tau| < \tau_0: \quad \frac{\partial u_z}{\partial x} = 0$$

- If we assume that the geometry is defined such that  $\tau_{xz}$  is negative, so  $|\tau_{xz}| = -\tau_{xz}$  and  $|du_z/dx| = -du_z/dx$  (note that if  $\tau_{xz}$  was positive, which would have been the case if we had defined  $x$  as being the distance above the bottom of the flow,  $|\tau_{xz}| = \tau_{xz}$  and  $|du_z/dx| = du_z/dx$ )

$$\text{if } |\tau| > \tau_0: \quad \tau_{xz} = -\left( \mu_\infty \frac{-du_z}{dx} + \tau_0 \right) = \mu_\infty \frac{du_z}{dx} - \tau_0$$

$$\text{if } |\tau| < \tau_0: \quad \frac{du_z}{dx} = 0$$

## Calculation continued:

- We need to do this in two regions
  - A deep region when  $\tau_{xz} > \tau_0$  and a shallow region where  $\tau_{xz} < \tau_0$
  - Let  $x_0$  be the depth below the liquid surface of this transition:

$$x_0 = \frac{\tau_0}{\rho g \cos(\beta)}$$

- In the deep region ( $x > x_0$ ):

$$\mu_\infty \frac{du_z}{dx} - \tau_0 = -\rho g \cos(\beta)x$$

- Which can be integrated to give:

$$u_z = -\frac{\rho g \cos(\beta)}{2\mu_\infty} x^2 + \frac{\tau_0}{\mu_\infty} x + B$$

- $B$  can be obtained from no slip at the bottom ( $u_z=0$  at  $x=h$ )

$$u_z = \frac{\rho g \cos(\beta)}{2\mu_\infty} (h^2 - x^2) - \frac{\tau_0}{\mu_\infty} (h - x)$$

$$\text{if } |\tau| > \tau_0: \quad \tau_{xz} = \mu_\infty \frac{du_z}{dx} - \tau_0$$

$$\text{if } |\tau| < \tau_0: \quad \frac{du_z}{dx} = 0$$

$$\tau_{xz} = -\rho g \cos(\beta)x$$

## Calculation continued:

- From previous slide:

$$\text{if } x > x_0: \quad u_z = \frac{\rho g \cos(\beta)}{2\mu_\infty} (h^2 - x^2) - \frac{\tau_0}{\mu_\infty} (h - x)$$

- The velocity in the top portion is constant and can be obtained by setting

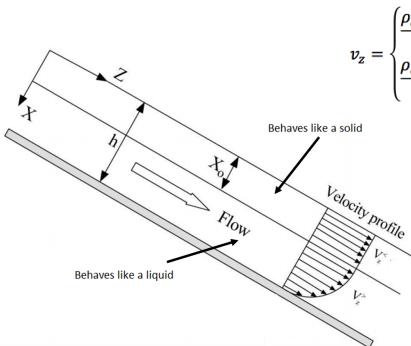
$$\text{if } x > x_0: \quad u_z = \frac{\rho g \cos(\beta)}{2\mu_\infty} (h^2 - x_0^2) - \frac{\tau_0}{\mu_\infty} (h - x_0)$$

- Yielding following overall velocity profile:

$$u_z = \begin{cases} \frac{\rho g \cos(\beta)}{2\mu_\infty} (h^2 - x_0^2) - \frac{\tau_0}{\mu_\infty} (h - x_0) & \text{if } x < x_0 \\ \frac{\rho g \cos(\beta)}{2\mu_\infty} (h^2 - x^2) - \frac{\tau_0}{\mu_\infty} (h - x) & \text{if } x > x_0 \end{cases}$$

$$x_0 = \frac{\tau_0}{\rho g \cos(\beta)}$$

### Bingham plastic velocity profile



$$v_z = \begin{cases} \frac{\rho g \cos(\beta)}{2\mu_\infty} (h^2 - x_0^2) - \frac{\tau_0}{\mu_\infty} (h - x_0) & \text{if } x < x_0 \\ \frac{\rho g \cos(\beta)}{2\mu_\infty} (h^2 - x^2) - \frac{\tau_0}{\mu_\infty} (h - x) & \text{if } x > x_0 \end{cases}$$

### Question 3 - worksheet

- Assume that a soil flow can be approximated as a Bingham plastic. Consider a layer of soil with density 2500 kg/m<sup>3</sup> on a slope that can be assumed to be infinitely wide and long. The layer is 1 m thick and is at an angle of 30° from the horizontal.
- Assume that it is initially at its angle of repose, which implies that the yield stress is just able to balance the gravitational down the slope. force Gravitational acceleration = 9.8 m/s<sup>2</sup>. After heavy rain, the yield stress of the material decreases by 20%.
- Assuming that the  $\mu_\infty=100 \text{ Pa.s}$ , what is the maximum velocity that the slide achieves down the slope?

### Outline of Lecture

- Fluid flow in simple geometries
- Reynolds number and the onset of turbulence
- Modelling turbulence
- Strain-rate dependent rheologies