Solutions
Problem Set 7
ACSE-2 2019

①

(a) $T(\vec{\xi},t) = \alpha\left[\xi_1 + kt\xi_2 + (1+kt)\xi_2\right]$

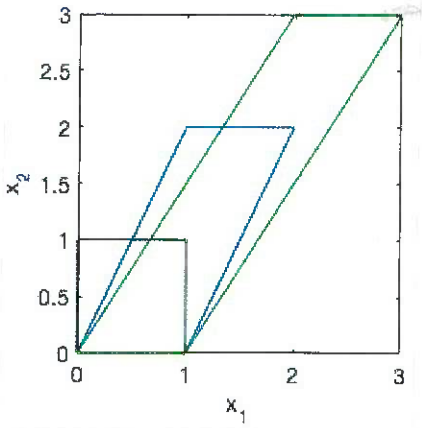
(b) $\vec{v}(\vec{\xi},t) = (k\xi_2, k\xi_2, 0)$

$$\vec{v}(\vec{x},t) = \left(\frac{kx_2}{1+kt}, \frac{kx_2}{1+kt}, 0\right)$$

(c) $$\frac{DT}{dt}(\vec{\xi},t) = 2\alpha k\xi_2$$

$$\frac{DT}{dt}(\vec{x},t) = \frac{2\alpha k x_2}{1+kt}$$

(d)



black at $t=0$
blue at $t=1$
green at $t=2$

②

(a) unit elongation in direction $\hat{s} = \frac{1}{3}(2,2,1)$

$$\hat{s}\cdot\underline{\underline{E}}\cdot\hat{s} = \frac{58}{9}\cdot 10^{-4}$$

(b) original lines in direction $\hat{s} = \frac{1}{3}(2,2,1)$

$$\hat{p} = \frac{1}{\sqrt{45}}(3,0,-6)$$

change in angle $2\times \hat{p}\cdot E\cdot\hat{s} = 2\times\hat{s}\cdot E\cdot\hat{p}$

$= \frac{32}{\sqrt{45}}\cdot 10^{-4}$ in radians

③ maximum unit elongation

$\rightarrow$ equals maximum eigenvalue of strain tensor

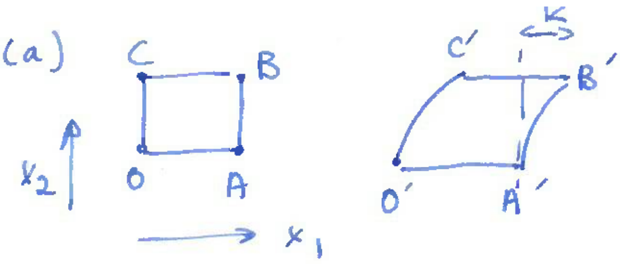
$$E = \frac{1}{2}(\nabla\bar{u} + \nabla\bar{u}^T) = \begin{bmatrix} 2k & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 2k \end{bmatrix} \text{ in point } (1,0,0)$$

eigenvalues $\lambda = 3k, k, 0$

$\uparrow$ max. elongation

in direction $\hat{x} = \frac{1}{2}\sqrt{2}(1,0,1)$

(4) (a)



original deformed

$\vec{x}' = \vec{\xi} + \vec{u}(\vec{\xi})$

(exaggerated, as k is only 10^{-4})

$0' = 0$; $A' = A$

lines O'-C' and A'-B' parabolic in shape

(b) $d\vec{x}'^{(i)} = d\vec{\xi}^{(i)} + \underline{\underline{\nabla u}} \cdot d\vec{\xi}^{(i)}$

$$\underline{\underline{\nabla u}} = \begin{bmatrix} 0 & 2k\xi_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \xi_2 = 1$$

at C

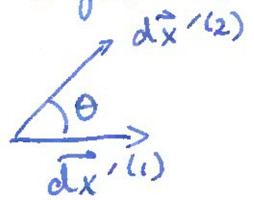
$d\vec{x}'^{(1)} = (d\xi_1, 0, 0)$

$d\vec{x}'^{(2)} = d\xi_2\,(2k, 1, 0)$

(c) $|d\vec{x}'^{(1)}| - |d\vec{\xi}^{(1)}| = 0$

$|d\vec{x}'^{(2)}| - |d\vec{\xi}^{(2)}| = d\xi_2\left(\sqrt{4k^2+1} - 1\right)$

(d) angle $d\vec{\xi}^{(1)}$ and $d\vec{\xi}^{(2)}$ was $90°$



$$\cos\theta = \frac{2k\,d\xi_2}{\sqrt{4k^2+1}\,d\xi_2} = \frac{2k}{\sqrt{4k^2+1}} = \sin(90° - \theta)$$

= sine of change in angle

(e) $$\underline{\underline{E}} = \begin{bmatrix} 0 & k\xi_2 & 0 \\ k\xi_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(f) change in length $d\vec{\xi}^{(1)}$ according to $E_{11} = 0$

" " " $d\vec{\xi}^{(2)}$ " " $E_{22} = 0$

(g) change in angle $d\vec{\xi}^{(1)}$ and $d\vec{\xi}^{(2)}$ from $2E_{12} = 2k$ (radians)

(h) compare result (f) with (c)

→ same if k small enough

(g) with (d)

→ $\sin(d\varphi) = d\varphi$ if φ small enough

small $k \rightarrow \frac{2k}{\sqrt{4k^2+1}} \approx 2k$

(5) $\rho(t)$ - not function of x

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} = -\frac{3\rho k}{1+kt}$$

$\rightarrow$ integrate

$$\int_{\rho_0}^{\rho} \frac{d\rho'}{\rho'} = \int_0^t \frac{-3k\,dt'}{1+kt'}$$

$$\Rightarrow \quad \rho(t) = \frac{\rho_0}{(1+kt)^3}$$