Potential Flow and Related Problems

Stephen Neethling

Potential Flow

- Often used for the specific case of inviscid (zero viscosity), incompressible fluid flow
- In this lecture I will be using potential flow to describe a wider number of problems which can be modelled using Laplace's equation
- Will look at both analytical and numerical solutions of Laplace's equation
- Will also examine some of the deviations away from pure potential flow

What is potential flow?

- Flow is proportional to the gradient of a potential:

$$\mathbf{v} = -k\nabla\varphi$$

- Flow continuity holds:

$$\nabla \cdot \mathbf{v} = 0$$

- If the proportionality is constant this results in Laplace’s Equation:

$$\nabla^2\varphi = 0$$

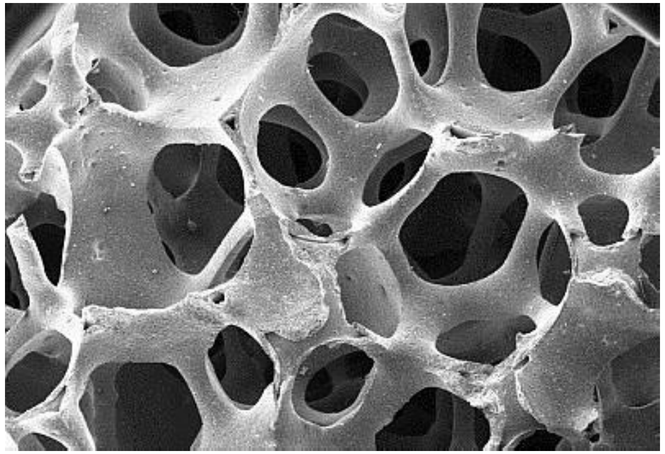
Where potential flow is a good approximation

One type of situation where potential flow is a good approximation

- The potential exerts a “force” on the quantity being conserved
  - Electrical current, fluid volume…
- The resistance to the flow is proportional to the flow-rate of the conserved quantity

- Good approximation for a number systems
  - Heat flow
  - Electrical current
  - Saturated flow in porous media

Flow in Porous Media



- Wide range of systems
  - Packed bed catalytic reactors
  - Water flow in aquifers
  - Petroleum reservoirs
  - Porous electrodes in flow batteries and fuel cells
  - …

Flow in Porous Media

- Often characterised by a permeability, k, with viscosity as a separate term in order to separate the porous media and the fluid effect:

$$\boldsymbol{v} = -\frac{k}{\mu}\nabla P$$

- Note that this velocity is the superficial velocity (volumetric flowrate per area of porous media) rather then the actual average velocity of the fluid
- More complicated for multi-phase flow in porous media
  - Introduction of relative permeabities that are a function of volume fraction of that phase in the pore space
  - Capillarity (forces due to surface tension) can further complicate the situation

Where potential flow is a good approximation

- Random motion can also lead to potential flow
- Diffusion is a good example
  - Thermal motion of molecules in solvent leads to molecules of the solute being randomly nudged – **Brownian motion**
  - At the microscopic scale heat flow follows a similar process as the kinetic energy of the particles is randomly transferred to neighbouring particles
- If particle motion is random, the number of particles leaving a small volume in a given time will be proportional to the number of particles in the volume

Diffusion through a random walk

- If we assume that Brownian motion causes a particle of the solute to take a random step of size Δx in time Δt, this will result in the following macroscopic diffusion coefficient:
  - Δx is the Root Mean Square (RMS) of the step size if the step size varies

$$D = \frac{\Delta x^2}{2\Delta t}$$

- The macroscopic flux can then be expressed as a function of the diffusion coefficient and the gradient of the concentration:

$$\boldsymbol{F} = -D\nabla C$$

Implications of Potential Flow

- Flow is irrotational (curl of the velocity vector is zero):

$$\nabla \times \mathbf{v} = 0$$

参考textbook有证明 p48

$$\nabla \times \mathbf{v} = \begin{pmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{pmatrix}$$

- Prove?

- Reverse is also true: Irrotational flows can be written as potential flows

What is vorticity?

- Potential flow implies irrotationality, which means that it has a vorticity of zero

$$\omega = \nabla \times \mathbf{v}$$

- Vorticity, ω, is a measure of the rotation of the flow
  - This is true, but can be misleading – fluid that has a curving flow path can have a zero vorticity, while fluid going in straight line can have a finite vorticity
  - Vorticity is a measure of the rotation of a small parcel of fluid – if you put a small twig in the flow will it rotate (the average path it follows is immaterial to the vorticity)?
  - In a 3D flow, the axis of the rotation is in the direction that the vorticity vector points and the length of the vector is proportional to how fast it rotates.

Incompressible Inviscid Flow

- Why is incompressible inviscid flow a potential flow problem?

Navier-Stokes Equations for Incompressible Newtonian Flow:

- You have already derived the NS equation (more detail on it later in the course)
- Need to know what each term represents

Flow of momentum (Inertial force) Viscous force

Momentum balance:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho\, \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \mu \nabla^2 \mathbf{u} + \rho\, \mathbf{g}$$

Rate of change of momentum Pressure force Body force

Mass balance (continuity):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\mathbf{u} \cdot \nabla \mathbf{u} = (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \mathbf{u} \cdot (\nabla \mathbf{u}) = \begin{pmatrix} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \\ u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \\ u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \end{pmatrix}$$

Incompressible Inviscid Flow

We can obtain the vorticity equation by taking the curl ($\nabla \times$) of the Navier-Stokes equation. Using $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ as the variable for the vorticity:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \frac{\mu}{\rho} \nabla^2 \boldsymbol{\omega} + \nabla \times \mathbf{g}$$

Note that the pressure does not appear in this equation if the density is constant (incompressible flow) as the curl of the gradient of a scalar is always zero.

Try and derive this equation for yourself

- You will also need to use the incompressible assumption: $\nabla \cdot \mathbf{u} = \mathbf{0}$.

Useful identities for deriving vorticity equation:

$$\nabla \times (\mathbf{a} + \mathbf{b}) = \nabla \times \mathbf{a} + \nabla \times \mathbf{b}$$

$$\nabla \times (\nabla^2 \mathbf{a}) = \nabla^2 (\nabla \times \mathbf{a})$$

$$\nabla \times (\nabla a) = \mathbf{0}$$

$$\nabla \times \frac{\partial \mathbf{a}}{\partial t} = \frac{\partial}{\partial t} (\nabla \times \mathbf{a})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = -\mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

$$(\mathbf{u} \cdot \nabla)\, \mathbf{u} = \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) - \mathbf{u} \times \boldsymbol{\omega}$$

Show that this implies: $$\nabla \times \big((\mathbf{u} \cdot \nabla)\mathbf{u} \big) = (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} + (\nabla \cdot \mathbf{u})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$$

Incompressible Inviscid Flow

If we further assume that the flow is inviscid and that there is a constant (or conservative, $\nabla \times \mathbf{g} = 0$) body force:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$$

The LHS of this equation represents the evolution of the vorticity of a parcel of fluid. This implies that if the vorticity starts with a zero value it will remain zero for all positions and all times.

If we therefore assume that our flow is initially irrotational (zero vorticity), it will remain irrotational if it is incompressible and inviscid.

Incompressible Inviscid Flow

- Side note:

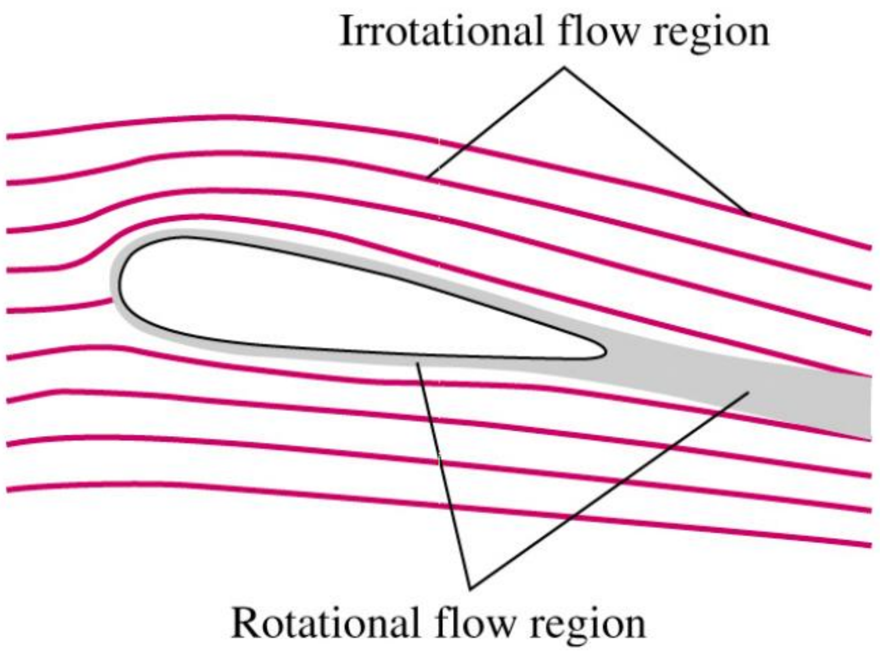
Zero vorticity is also a solution to the equation with viscosity

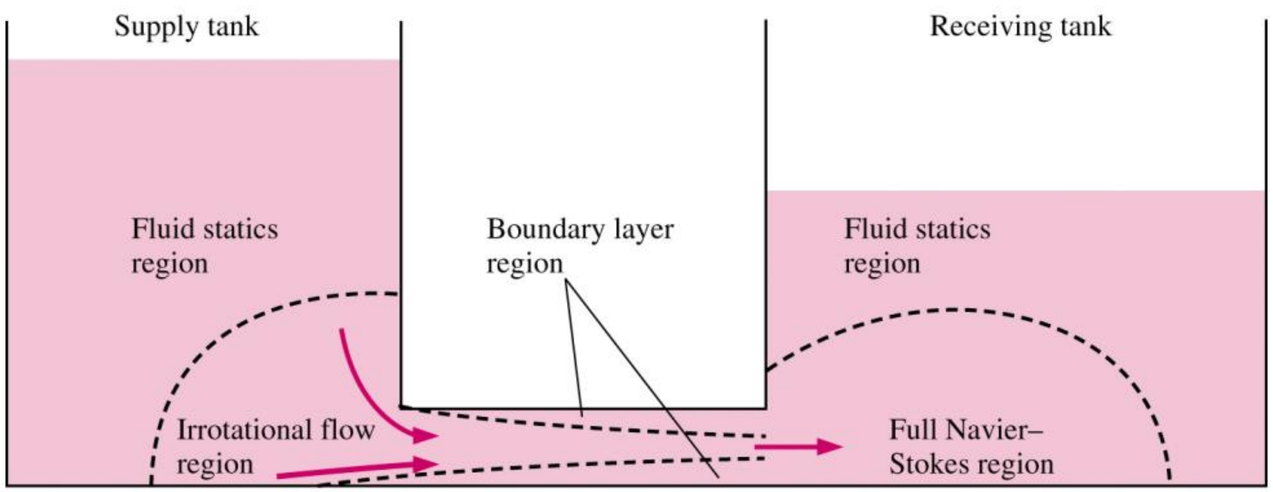
$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \frac{\mu}{\rho} \nabla^2 \boldsymbol{\omega}$$

…,but it is incompatible with shear stress at the boundaries. A shear stress at the boundary will generate vorticity at the boundary. The viscous term will then cause the vorticity to “diffuse” into the interior.

Incompressible Inviscid Flow

- Because vorticity diffuses in from boundaries, systems can have regions where potential flow is a good approximation and regions where it is not:





Incompressible Inviscid Flow

From our earlier discussion, irrotational flows can be described using a potential:

$$\mathbf{u} = -\nabla\varphi$$

As $\nabla \cdot \mathbf{u} = \mathbf{0}$, this further implies that

$$\nabla^2\varphi = 0$$

Incompressible Inviscid Flow

The potential described previously is NOT the same as the fluid pressure. To convert between them we need to use the Navier-Stokes Equation.

Navier-Stokes Equations for Incompressible Newtonian Flow:

$$\cancel{\frac{\partial \mathbf{u}}{\partial t}} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho} \nabla^2 \mathbf{u}} + \mathbf{g}$$

$$\nabla \cdot \mathbf{u} = 0$$

Assuming steady-state and inviscid

Incompressible Inviscid Flow

Left with

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\nabla P}{\rho} + \mathbf{g}$$

$$\nabla\cdot\mathbf{u} = 0$$

We can also use the identity from earlier:

$$(\mathbf{u}\cdot\nabla)\mathbf{u}=\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times\boldsymbol{\omega}$$ Textbook p48

Since $\boldsymbol{\omega}$ is zero, this means that for an incompressible fluid (constant ρ):

$$\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = \nabla\left(-\frac{P}{\rho} + \mathbf{g}\cdot\mathbf{x}\right)$$ Where $\mathbf{x}$ is the location vector

Incompressible Inviscid Flow

Since $\mathbf{u} = -\nabla\varphi$ based on our previous derivation:

$$\nabla\left(\frac{|\nabla\varphi|^2}{2}\right) = \nabla\left(\frac{P}{\rho} - \mathbf{g}\cdot\mathbf{x}\right)$$

This means that (less a constant of integration):

$$P = \rho\left(\frac{|\nabla\varphi|^2}{2} + \mathbf{g}\cdot\mathbf{x}\right)$$

This means that the potential can be solved for first and the pressure subsequently calculated even though they are not the same as one another

Solving Potential Flow Problems

- While potential flow problems have a variety of physics that can be represented, the underlying equation that is being solved is always Laplace's Equation

$$\nabla^2 \varphi = 0$$

- There is no time dependency in this equation and therefore its solution takes the form of a Boundary Value Problem

- We therefore need to specify all boundaries

Boundaries in Potential Flow Problems

- Two linear boundary types are the specification of the potential at the boundary or the flux through the boundary

- Potential specification is a Dirichlet Boundary condition in potential:

$$\varphi = f(\mathbf{x})$$

- Flux specification is a Neumann boundary condition in potential:

$$\mathbf{v} \cdot \mathbf{n} = -k\nabla\varphi \cdot \mathbf{n} = F(\mathbf{x})$$

Where $\mathbf{n}$ is the unit normal to the boundary and F is the scalar flux through the boundary (in the direction of $\mathbf{n}$)

Stream Function Formulation

- Dirichlet boundary conditions are easier to implement than Neumann boundaries and typically result in quicker convergence when using a numerical solution scheme
- Stream function formulation allows flux boundary conditions to be specified using Dirichlet boundary conditions
  - Still doesn’t allow a mixture of potential and flux boundaries, which would still require a combination of Dirichlet and Neumann boundaries

Stream Functions

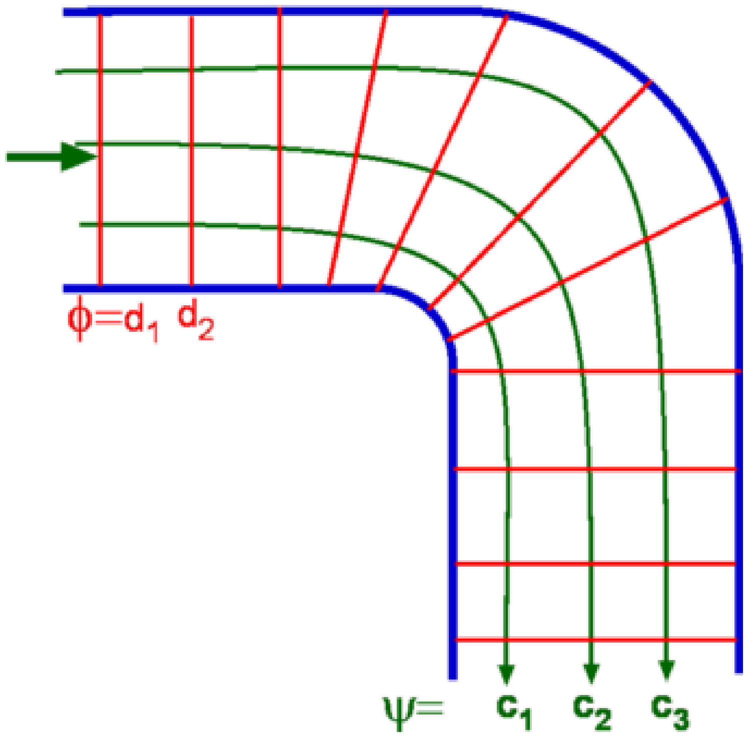
- Can be used to represent 2D steady state systems that exhibit continuity
- A stream function is a scalar field where flow follows a constant value of the function
  - In other words a constant value of the stream function represents a streamline
- Stream functions are just one way to solve potential flow problems and they only work in 2D
  - Can also be used in the analysis or solution of other flow problems where continuity of flux holds
- Make specifying flux boundary conditions very easy

Stream Functions

Other properties of stream functions

- Differences in value of the stream function are equal to the flowrate of the conserved quantity between the streamlines represented by those stream function values

- In potential flow problems lines of constant value of the stream function are orthogonal to lines of constant value of the potential



Stream Functions (cont.)

- The stream function ψ is related to flux (volume flux is a velocity):

$$F_x = \frac{\partial \psi}{\partial y} \qquad F_y = -\frac{\partial \psi}{\partial x}$$

- If this is substituted into the steady state continuity equation you can see that it is always satisfied

$$\nabla \cdot \hat{F} = 0$$ in 2D $$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = 0$$

Stream Functions (cont.)

- We also know that potential flow implies that there is zero vorticity

$$\nabla \times \hat{F} = 0$$

- Substituting the definitions of the fluxes into this equation results in Laplace's equation (strictly speaking the curl is a vector, but in 2D there is only one non-zero component):

$$\nabla^2 \psi = 0$$

Boundary Conditions

- Stream functions are useful if the boundaries are flux boundary conditions

$$F_x = \frac{\partial \psi}{\partial y} \qquad F_y = -\frac{\partial \psi}{\partial x}$$

- These definitions imply that the stream function is a constant value along a boundary for a zero flux condition and changes linearly for a constant flux
  - Other flux boundaries are readily solvable

Solving the Problem Numerically

- Using the stream function formulation we can calculate the values at the boundaries
- We then need to use the values at the boundaries to calculate the internal values based on Laplace's equation

- We can either do this numerically or analytically (though in the form of an infinite series)

- We will do both, starting with a numerical solution

Finite difference approximation

- Finite difference is the easiest way to approximate differentials on a grid
- There are a number of ways in which first derivatives can be approximated on a grid, 3 of which are:
  - Central difference: $\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x}$ (2nd order accurate)
  - Forward difference: $\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1} - u_i}{\Delta x}$ (1st order accurate)
  - Backward difference: $\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_i - u_{i-1}}{\Delta x}$ (1st order accurate)
- 2nd derivatives can also be approximated

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

Finite Difference Approximation of a Linear 2nd Order PDEs

- Using finite difference approximations, any 2D steady state linear 2nd order PDE can be approximated as follows:

$$a_{i,j}u_{i+1,j} + b_{i,j}u_{i-1,j} + c_{i,j}u_{i,j+1} + d_{i,j}u_{i,j-1} + e_{i,j}u_{i,j} = f_{i,j}$$

- The prefactors *a*, *b*, *c*, *d*, *e* and *f* are matrices (i.e. functions of position) in the general case, since the linear coefficients in the PDE can also be functions of position

Approximating Laplaces Equation

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

- Using our earlier approximations for the 2nd derivatives and with $x = x_0 + i\,\Delta x$ and y $= y_0 + j\,\Delta y$ as the definition for the grid:

$$\nabla^2\psi \approx \frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{\Delta x^2} + \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{\Delta y^2} = 0$$

- If we further assume that the grid is square ($\Delta x = \Delta y$), then we need to solve the following approximation:

$$\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{i,j} = 0$$

- Using the form on the previous page: $a = b = c = d = 1,\ e = -4, f = 0$

Iterative and Inversion Methods

$$a_{i,j}u_{i+1,j} + b_{i,j}u_{i-1,j} + c_{i,j}u_{i,j+1} + d_{i,j}u_{i,j-1} + e_{i,j}u_{i,j} = f_{i,j}$$

- Since there is one of the above equations for each point in the solution grid, the problem takes the form of a set of coupled linear equations

There are therefore two main approaches that can be used:

- Matrix inversion
  - Can reach the solutions in one step, though iterative matrix solvers more common for large sparse systems
  - "Stiff" problems (usually problems with strong convective terms) can result in near singular matrices and thus issues with numerical precision
  - Can requires a lot of memory unless sparse matrix techniques are used (which increases the complexity of the implementation)
- Iterative methods
  - Iteratively approaching the solution requires a large number of steps, each (hopefully) closer to the solution than the previous one
  - Easier to implement
  - Generally has lower memory requirements than inversion techniques
  - Usually slower

Iterative Methods

- Iterative methods work by successively reducing the error in the solution
- The error we will use is the residual, which is the error in the approximation of the differential equation
  - As we don't know the final solution a priori, we can't use the error in the dependent variable
- The residual at point *i, j* is:

$$\xi_{i,j} = a_{i,j}u_{i+1,j} + b_{i,j}u_{i-1,j} + c_{i,j}u_{i,j+1} + d_{i,j}u_{i,j-1} + e_{i,j}u_{i,j} - f_{i,j}$$

- …with iterations continuing until the average magnitude of the residual drops below a certain value
  - You might also wish to ensure that the maximum residual is also below a set value

$$\text{Continue iterations until } \frac{\sum\sum\left|\xi_{i,j}\right|}{i_{\max} j_{\max}} < \xi_{Critical}$$

Simultaneous Over Relaxation (SOR)

- From the finite difference approximation:

$$a_{i,j}u_{i+1,j}^{correct} + b_{i,j}u_{i-1,j}^{correct} + c_{i,j}u_{i,j+1}^{correct} + d_{i,j}u_{i,j-1}^{correct} + e_{i,j}u_{i,j}^{correct} - f_{i,j} = 0$$

- …and the definition of the residual:

$$a_{i,j}u_{i+1,j}^{guess} + b_{i,j}u_{i-1,j}^{guess} + c_{i,j}u_{i,j+1}^{guess} + d_{i,j}u_{i,j-1}^{guess} + e_{i,j}u_{i,j}^{guess} - f_{i,j} = \xi_{i,j}$$

- …we get:

$$a_{i,j}\left(u_{i+1,j}^{guess} - u_{i+1,j}^{correct}\right) + b_{i,j}\left(u_{i-1,j}^{guess} - u_{i-1,j}^{correct}\right) + c_{i,j}\left(u_{i,j+1}^{guess} - u_{i,j+1}^{correct}\right)$$
$$+ d_{i,j}\left(u_{i,j-1}^{guess} - u_{i,j-1}^{correct}\right) + e_{i,j}\left(u_{i,j}^{guess} - u_{i,j}^{correct}\right) = \xi_{i,j}$$

- If we assume that the error at the central point is more important than the other errors
    - A good assumption for strongly diffusive problems, but less good for advective problems

$$u_{i,j}^{correct} \approx u_{i,j}^{guess} - \frac{\xi_{i,j}}{e_{i,j}}$$

Simultaneous Over Relaxation (SOR) (cont.)

- We can use this approximation to update our estimates:

$$u_{i,j}^{new} = u_{i,j}^{old} - \omega \frac{\xi_{i,j}}{e_{i,j}}$$

- ω is know as the relaxation factor
  - Solutions are generally stable for values between 0 and 2
  - Below 1 is known as under-relaxation and convergence is always slower (can be used to improve stability when using this method for, for instance, non-linear systems)
  - Above 1 the convergence can be quicker, sometimes dramatically so
  - For some systems the optimum value of can ω be obtained theoretically, though generally we find the best value by trial and error

Checker-boarding

- SOR is potentially unstable if new values are used in the updating as they become available
- One way to ensure that this doesn't happen is to have two arrays, one for new values and one for the old values
  - This uses twice as much memory
- The other alternative is to do checker-boarding
  - Imagine that your solution grid is like a chess board.
  - Solve for black squares on one iteration and the white squares on another iteration
  - ...therefore solve for a grid point if ((i+j)%2==count%2) where count is the number of iterations that have been done

You will be applying this method in Python this afternoon and so will have a chance to ask for help in doing so

It will also form the basis for part of you coursework – If you can get this working this afternoon your coursework next week will be much easier to do!

Analytical Solutions to Laplace's Equation

- While this course is about computational solutions, where they exist, analytical solutions are useful for testing numerical algorithms

- Laplace's equation, while quite widely applicable, is also linear and it is therefore possible to find analytical solutions even in quite complex geometries
    - Usually in the form of infinite series

A digression – Fourier Series

- To find the analytical solution to Laplace's equation we need to use Fourier Series
- Any function can be converted into an infinite series of sines and/or cosines
    - Tractable if function is either periodic or if we require a solution only over a finite extent
- If we only require the function to be approximated between 0 and *L*, then the following version of the Fourier Series can be used:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \qquad a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Fourier Series

$$f(x) = \sum_{n=1}^{\infty} a_n \, sin\left(\frac{n\pi x}{L}\right)$$

- Simple Example – Straight Line

$f(x) = mx + c$

$$a_n = \frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right) dx$$

- Solve:

$$a_n = \frac{2}{L}\int_0^L (mx + c) sin\left(\frac{n\pi x}{L}\right) dx = \frac{2m}{L}\int_0^L x sin\left(\frac{n\pi x}{L}\right) dx + \frac{2c}{L}\int_0^L sin\left(\frac{n\pi x}{L}\right) dx$$

$$a_n = \begin{cases} -\dfrac{2m}{\pi n L} & if\ n\ is\ even \\ \dfrac{2m}{\pi n L} + \dfrac{4c}{\pi n} & if\ n\ is\ odd \end{cases}$$

Try and obtain this yourself

Fourier Series

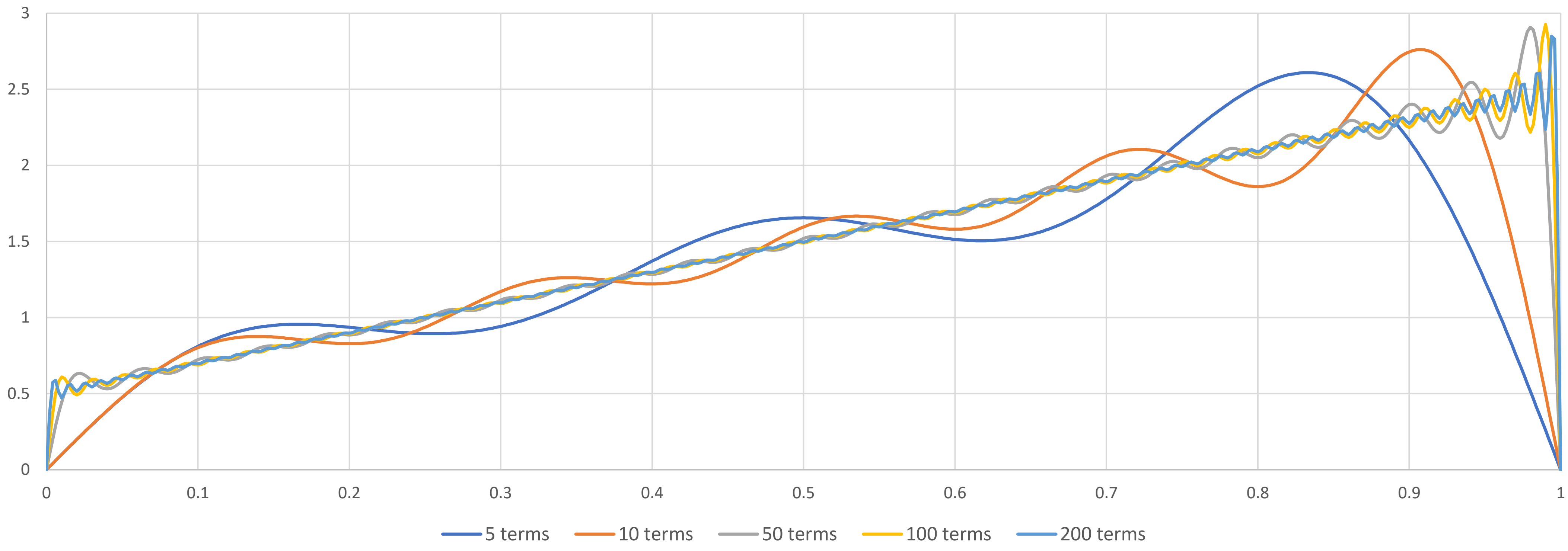
$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \begin{cases} -\frac{2m}{\pi n L} & \text{if } n \text{ is even} \\ \frac{2m}{\pi n L} + \frac{4c}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

$f(x) = mx + c$

- How quickly does this converge?

(c = 0.5, m = 2)



Analytical Solutions to Laplace's Equation

How to find analytical solutions

- Laplace's equation is linear and therefore solutions to simpler problems can be combined:

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \cdots$$

$$0 = \nabla^2\varphi = \nabla^2(\varphi_1 + \varphi_2 + \varphi_3 + \cdots) = \nabla^2\varphi_1 + \nabla^2\varphi_2 + \nabla^2\varphi_3 + \cdots$$

- As long as the sum of the potentials satisfy the boundary conditions, the sum of the solutions to the individual Laplace's equation will satisfy the overall system

Analytical Solutions to Laplace’s Equation

- We will do 2D solutions as they are more tractable than a 3D analytical solution
- A useful trick for trying to find analytical solutions to PDEs is to convert them into ODEs (note that this is only typically possible for some very simple linear PDEs):
- Separation of variables

Lets write our potential out as the product of a function of x and a function of y (i.e. assume that the function is separable):

$$\varphi = X(x)Y(y)$$

Analytical Solutions to Laplace’s Equation

- How does this help: $\varphi = X(x)Y(y)$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{d^2 X(x)}{dx^2} Y(y) \qquad \frac{\partial^2 \varphi}{\partial^2 y} = \frac{d^2 Y(y)}{dy^2} X(x)$$

Note the change from partial to ordinary differential equations

- Therefore $\nabla^2 \varphi = 0$ implies that (assuming separability)

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = 0$$

Analytical Solutions to Laplace's Equation

- The equation now consists of the sum of two independent ODEs:

$$0 = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$$

- We can therefore solve these two ODEs independently with the proviso that the two solutions are zero everywhere.
- Since they are independent and operate in different directions, this also implies that each of their solutions must be constant or else the sum would vary in different directions:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \lambda \qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\lambda$$

Analytical Solutions to Laplace's Equation

Continued…

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \lambda \qquad\qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\lambda$$

These have general solutions of the form:

$$X = \gamma\, cosh(\lambda x) + \delta\, sinh(\lambda x)$$

$$Y = \alpha\, cos(\lambda y) + \beta\, sin(\lambda y)$$

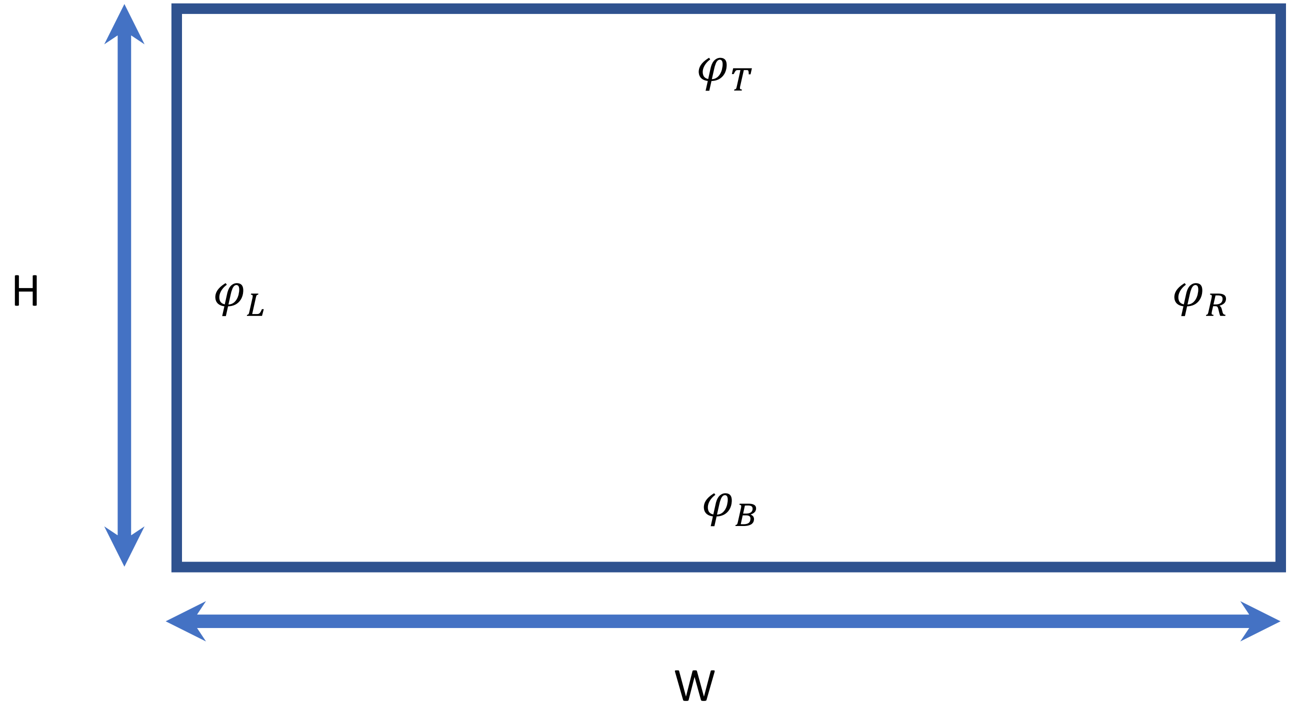
Analytical Solutions to Laplace’s Equation

- Note that because we have assumed the form for the potential, this is A solution, but not necessarily THE solution

- For a specific case, though, we can use the fact that Laplace’s equation is linear to arrive at the THE solution by <u>summing solutions of the previous form that match the various boundaries</u>

Analytical Solutions to Laplace's Equation

- Example: Rectangle in which the potential at each side is specified. Note that this method works for potentials that are arbitrary functions of position along the boundaries



Analytical Solutions to Laplace's Equation

$$\varphi = \left(\gamma\ cosh(\lambda x) + \delta\ sinh(\lambda x)\right)\left(\alpha\ cos(\lambda y) + \beta\ sin(\lambda y)\right)$$

- What we need to do is find values for the constants that satisfy the boundaries:

- Because we can sum solutions, we can find a solution that satisfies one of the boundaries, while having all the other boundaries zero:

- φ_1 will be the solution where φ_L is a function of position along the left hand boundary, while $\varphi_R = \varphi_T = \varphi_B = 0$

Analytical Solutions to Laplace's Equation

Since φ_L has a value:

$$\varphi_1(0, y) = \varphi_L = \gamma\big(\alpha\, cos(\lambda y) + \beta\, sin(\lambda y)\big)$$

… this implies that γ is non-zero for arbitrary values of φ_L

The bottom boundary is assumed zero:

$$\varphi_1(x, 0) = 0 = \alpha\,\big(\gamma\, cosh(\lambda x) + \delta\, sinh(\lambda x)\big)$$

Since we know that γ is non-zero, this implies that $\alpha = 0$

Analytical Solutions to Laplace’s Equation

- The other two boundaries are a bit more complex:

Top boundary:

$$\varphi_1(x, H) = 0 = \beta\, sin(\lambda H)\big(\gamma\, cosh(\lambda x) + \delta\, sinh(\lambda x)\big)$$

- This means that λ must be chosen such that λH is a half period of the *sin* wave:

$$\lambda = \frac{n\pi}{H}$$ where n must be an integer

Analytical Solutions to Laplace's Equation

RHS boundary:

$$\varphi_1(W, x) = 0 = \beta\ sin\left(\frac{n\pi y}{H}\right)\left(\gamma\ cosh\left(\frac{n\pi W}{H}\right) + \delta\ sinh\left(\frac{n\pi W}{H}\right)\right)$$

Since $\beta\ sin\left(\frac{n\pi y}{H}\right)$ is non-zero for an arbitrary y, this implies that the second term must be zero and therefore:

$$\delta = -\gamma\ coth\left(\frac{n\pi W}{H}\right)$$

Analytical Solutions to Laplace's Equation

- Going back to the first boundary and substituting for what we know:

$$\varphi_1(0,y) = \varphi_L = \gamma\, sin\left(\frac{n\pi y}{H}\right)$$

- This is not able to be satisfied for an arbitrary value of y using a single value of n, but we can luckily sum solutions as it is linear and therefore use a Fourier series, which is trivial for a constant φ_L and readily solvable when φ_L is a known function of position:

We need for γ for each value of n:

$$\varphi_1(0,y) = \varphi_L = \sum_{n=1}^{\infty} \gamma_n\, sin\left(\frac{n\pi y}{H}\right) \qquad \gamma_n = \frac{2}{H}\int_0^H \varphi_L sin\left(\frac{n\pi y}{H}\right) dy$$

Analytical Solutions to Laplace’s Equation

- Substituting everything together:

$$\varphi_1 = \sum_{n=1}^{\infty} \gamma_n sin\left(\frac{n\pi y}{H}\right)\left(cosh\left(\frac{n\pi x}{H}\right) - coth\left(\frac{n\pi W}{H}\right) sinh\left(\frac{n\pi x}{H}\right)\right)$$

$$\gamma_n = \frac{2}{H}\int_0^H \varphi_L sin\left(\frac{n\pi y}{H}\right) dy$$

- This is for one boundary only. You need to do a similar thing for each of the other boundaries and sum the solutions
  - Either follow this method for each boundary or, more easily, transform the coordinates to get the equations for the other boundaries

Deviations from Potential Flow

- Thus far we have assumed that the proportionality between the flux and the potential is constant
  - Constant conductivity in heat and electrical flow systems
  - Constant permeability in flow in porous media
  - Constant diffusivity in diffusive systems

- The proportionality can be a function of position
  - The material through which the flux is occurring varies spatially
  - Still a linear problem and readily solved

- The proportionality can be anisotropic
  - “Conductivity” is different in different directions
  - Arises when the material has a complex and anisotropic microstructure
  - Still a linear problem and thus also readily solvable
    - Depending on the form of the anisotropy can sometimes back to Laplace’s equation by suitable scaling of the position

- The proportionality is a function of the potential or other coupled dependent variable in the system
  - An example would be multi-phase flow in porous media where the permeability is a function of the saturation
  - Non-linear and therefore more complex to solve

Prefactor that is a function of position

$$\mathbf{v} = -k\nabla\varphi \qquad \nabla \cdot \mathbf{v} = 0$$

- Looks like an advection-diffusion equation:

$$\nabla^2\varphi + \frac{1}{k}\nabla k \cdot \nabla\varphi = 0$$

Prefactor that is Anisotropic

- Prefactor can be represented by a rank 2 tensor
  - Tensor will be symmetric
  - Diagonal terms are equivalent to conventional permeability/conductivity, but with different values in different directions
  - Off diagonal terms are for fluxes induced by potential gradients in a different direction (e.g. flux in y direction induced by a potential gradient in the x direction)

The Workshop

- This afternoon we will be solving a problem defined in terms of stream functions
- Will need to implement an SOR based solution of the problem
- We will also calculate and implement the infinite series analytical solution