Dimensional Analysis Worksheet Solutions

Question 1

a) There are 5 variables (d, γ, g, ρ_h and ρ_l) and 3 dimensions (M, L, T) and so **TWO** dimensionless groups are required.
b) Doing it the proper Buckingham Pi way:

$$\begin{array}{cc} & \begin{array}{ccccc} \gamma & g & \rho_h & \rho_l & d \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 1 \\ -2 & -2 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Setting $x_\gamma = 1$ and $x_{\rho_l} = 0$ results in the following dimensionless group.

The resulting dimensionless group is:

$$\boldsymbol{N = \frac{\gamma}{\rho_h g d^2}}$$

(This is actually the inverse of the Bond number).

c) This dimensionless group represents the ratio of the capillary to gravity force.

Question 2

a)

$$\frac{\partial \phi}{\partial t} = \frac{2k_1}{\lambda}\phi\frac{\partial \phi}{\partial z} + \frac{k_2}{2\sqrt{\lambda}\sqrt{\phi}}\left(\frac{\partial \phi}{\partial z}\right)^2 + \frac{k_2\sqrt{\phi}}{\sqrt{\lambda}}\frac{\partial^2 \phi}{\partial z^2}$$

where $k_1 = \frac{\rho g}{3\mu C_{PB}}$ and $k_2 = \frac{\gamma\sqrt{3-\frac{\pi}{2}}}{6\mu C_{PB}}$

b) $z^* = z\sqrt{\frac{\rho g}{\gamma}}$ and

$$t^* = t\frac{\sqrt{\gamma\rho g}}{\mu}$$

(other ways of non-dimensionalising the problem are possible).

c)

$$\frac{\partial \phi}{\partial t^*} = Bo\frac{2}{3k_\lambda C_{PB}}\phi\frac{\partial \phi}{\partial z^*} + \frac{\sqrt{3-\frac{\pi}{2}}}{6C_{PB}\sqrt{k_\lambda}}\sqrt{Bo}\left(\frac{1}{2\sqrt{\phi}}\left(\frac{\partial \phi}{\partial z^*}\right)^2 + \sqrt{\phi}\frac{\partial^2 \phi}{\partial z^{*2}}\right)$$

Where $Bo = \frac{\rho g {d_b}^2}{\gamma}$

d) Simply replace the derivatives with their approximations and also note that ϕ should be replaced with $\phi_{i,j}$:

$$\phi_{i,j+1} = \phi_{i,j} + \Delta t^*\left(Bo\frac{2}{3k_\lambda C_{PB}}\phi_{i,j}\frac{\phi_{i+1,j}-\phi_{i-1,j}}{2\Delta z^*} + \frac{\sqrt{3-\frac{\pi}{2}}}{6C_{PB}\sqrt{k_\lambda}}\sqrt{Bo}\left(\frac{1}{2\sqrt{\phi_{i,j}}}\left(\frac{\phi_{i+1,j}-\phi_{i-1,j}}{2\Delta z^*}\right)^2 + \sqrt{\phi_{i,j}}\frac{\phi_{i+1,j}+\phi_{i-1,j}-2\phi_{i,j}}{\Delta z^{*2}}\right)\right)$$

For finding the time step:

$$D_{max} = \frac{\sqrt{3-\frac{\pi}{2}}}{6C_{PB}\sqrt{k_\lambda}}\sqrt{Bo}\sqrt{\phi_{max}}$$

$$v_{max} = \max\left(Bo\frac{2}{3k_\lambda C_{PB}}\phi_{max}, \frac{\sqrt{3-\frac{\pi}{2}}}{6C_{PB}\sqrt{k_\lambda}}\sqrt{Bo}\frac{1}{2\sqrt{\phi_{min}}}\frac{\partial \phi}{\partial z^*}_{max}\right)$$

The maximum gradient occurs on the first timestep, which given the resolution, is $\frac{\partial \phi}{\partial z^*}_{max} = \frac{\phi_{max}-\phi_{min}}{\Delta z^*}$. Ignoring this gradient condition won't actually cause proper divergence, but can add wiggles in the formation of the capillary boundary layer.

$\phi_{max} = 0.3$ and $\phi_{min} = 0.01$ as the liquid contents cannot go outside these limits if the code is working correctly.