**ACSE-2 Exam Study Guide Nov. 2019**

**Notation:**

Scalars – a or $a$
Vectors – $\mathbf{v}$ or $\vec{\mathbf{v}}$ or $\bar{\mathbf{v}}$, vector length $|\mathbf{v}|$
Tensors – $\mathbf{T}$ or (if rank 2) $\underline{\mathbf{T}}$
Unit vector along direction of $\mathbf{v}$: $\hat{\mathbf{e}}_v = \frac{\mathbf{v}}{|\mathbf{v}|}$
Unit outward normal for a plane: $\hat{\mathbf{n}}$

**Equations/concepts you are expected to know:**
*Examples given here all for 3-D, orthonormal Cartesian reference frame*

- Taylor series expansion
- Vector norms
- Linear systems, solvability of systems, etc
- Eigenvalues, eigenvectors, matrix diagonalization, use in matrix exponential etc
- Index notation: vector or tensor components written as $v_i$ or $T_{ij}$ with i,j=1,2,3 or i,j =x,y,z
- Einstein convention – implied summation of the same index repeated twice within a single term, e.g. $v_i w_i = \sum_{i=1}^{3} v_i w_i$
- Vector and tensor products:
  - dot product: $\mathbf{v}\cdot\mathbf{w}=v_i w_i$ or $\mathbf{T}\cdot\mathbf{v}=T_{ij}v_j$
  - multiple contraction, e.g. $\boldsymbol{\sigma} = \mathbf{C}:\boldsymbol{\varepsilon} = C_{ijkl}\,\varepsilon_{kl}$
  - cross product: $\mathbf{v}\times\mathbf{w} = \varepsilon_{ijk} v_i w_j \hat{\mathbf{e}}_k$
  - tensor product: $\mathbf{vw}=v_i w_j$
- Transpose: $T_{ji}=T^{T}_{ij}$
- Tensor symmetry:
  - Symmetric in i,j: $T_{ji}=T_{ij}$,
  - Antisymmetric in i,j: $T_{ji}=-T_{ij}$
- Tensor trace: for rank 2 tensor $\mathrm{tr}(\mathbf{T})=T_{11}+T_{22}+T_{33}=T_{ii}$.
- Kronecker delta $\delta_{ij}=1$ if i=j, =0 if i≠j
- Levi-Civita tensor $\varepsilon_{ijk}=1$ for even permutations of 1,2,3, $\varepsilon_{ijk}=-1$ for odd permutations of 1,2,3, $\varepsilon_{ijk}=0$ if any i,j,k are equal
- Determinant rank 2 tensor:
  - $\det(\mathbf{T})=\varepsilon_{ijk}T_{1i}T_{2j}T_{3k}$, where $\varepsilon_{ijk}$ is the Levi-Civita tensor.

- det(**T**)≠0 means that the columns of **T** are linearly independent and the inverse operation $\mathbf{T}^{-1}$ exists
- To find eigenvalues λ of a symmetric tensor **T**, solve det(**T**- λ**I**)=0, where **I** is the unit tensor. Eigenvectors **x** satisfy **T·x**=λ**x.**

- Lagrangian or material – description of motion following a 'particle', all fields described as a function of position **ξ** at a reference time $t_0$ and time *t*.

- Eulerian or spatial – description of motion from a fixed observation point. All fields described as a function of position **x** and time *t*.

- Material Derivative
  - in spatial description, the full time derivative of a field P(**x**,*t*) becomes: $\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P$, i.e., contains a time and advective term
  - in material description, the time derivative of the field P(**ξ**,*t*) is $\frac{DP}{Dt} = \frac{\partial P}{\partial t}$

- Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$
  - represents source/sink of a **v** field
  - can also be applied to tensors, e.g. $(\nabla \cdot \mathbf{T})_i = \frac{\partial T_{1i}}{\partial x_1} + \frac{\partial T_{2i}}{\partial x_2} + \frac{\partial T_{3i}}{\partial x_3}$

- Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)$
  - represents vorticity of a **v** field

- Gradient:
  - of a scalar $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$
  - or of a vector $(\nabla \mathbf{v})_{ij} = \frac{\partial v_j}{\partial x_i}$

- Laplacian: $\nabla \cdot \nabla f = \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x_i \partial x_i}$

- Cauchy Stress tensor:
  - stress tensor component $\sigma_{ij}$ represents a force in $\hat{\mathbf{e}}_j$ direction on a plane with normal in $\hat{\mathbf{e}}_i$ direction. Positive normal stress corresponds to extension.
  - stress tensor is symmetric: $\sigma_{ij} = \sigma_{ji}$ (conservation of angular momentum)
  - traction **t** on a plane with normal $\hat{\mathbf{n}}$ is $\mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$
  - the stress tensor can be diagonalised, with principal components $\sigma_1$, $\sigma_2$, $\sigma_3$ which include maximum and minimum normal stress
  - Can be decomposed into isotropic stress (pressure p = $-\sigma_{kk}/3$) and deviatoric stress **σ**' such that $\sigma_{ij}=-p\delta_{ij}+\sigma'_{ij}$

- Conservation of linear momentum (per unit volume): $\rho \frac{D^2\mathbf{u}}{Dt^2} = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$, where ρ is density, **u** is displacement and **f** is body force.

- Infinitesimal strain tensor:

- Infinitesimal strain tensor component $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$, where **u** is the displacement field. Applicable if $\nabla\mathbf{u} \ll 1$.
- An original line segment described by vector **x** deforms to a new line segment $\mathbf{x}'$ as: $\mathbf{x}' = \boldsymbol{\varepsilon} \cdot \mathbf{x}$
- Diagonal components of $\varepsilon_{ij}$ represent fractional length changes, i.e., if **x** is a vector in $\hat{\mathbf{e}}_1$ direction than $\boldsymbol{\varepsilon}_{11} = \frac{|\mathbf{x}'| - |\mathbf{x}|}{|\mathbf{x}|}$. Similarly, for a given vector **s**, the product $\mathbf{s} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{s}$ corresponds to the fractional change in |**s**| by the strain $\boldsymbol{\varepsilon}$.
- Off-diagonal components represent changes in angle (i.e., shape), such that $2\boldsymbol{\varepsilon}_{12}$ equals the change in angle between a line segment originally in $\hat{\mathbf{e}}_1$ direction and one originally in $\hat{\mathbf{e}}_2$ direction. Given two originally perpendicular vectors **s** and **p**, 2 x the product $\mathbf{p} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{s}$ corresponds to the change in the angle between **s** and **p** by the strain $\boldsymbol{\varepsilon}$.
- $tr(\boldsymbol{\varepsilon}) = \nabla \cdot \mathbf{u}$ and represents the fractional change in volume.
- $\varepsilon_{ij}$ is symmetric and can be diagonalised, such that principal strain components $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ include the maximum and minimum fractional length changes in the strain field described by $\boldsymbol{\varepsilon}$.
- Can be decomposed into isotropic and deviatoric strain, like the stress tensor

- Strain rate tensor
  - Strain rate tensor $\mathbf{D}=D\boldsymbol{\varepsilon}/Dt$ has same kind of properties as the infinitesimal strain tensor, but depends on the velocity field **v**: $D_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)$.
  - $tr(\mathbf{D}) = \nabla \cdot \mathbf{v} = 0$ means no change in volume, and is the conservation of mass equation for an incompressible material.

- Energy equation – if given the equation, understand the different terms (conduction, advection, heat production, power dissipated by deformation) and be able to use (e.g., to solve for temperature for simple case)

- Rheology – know difference between elastic and viscous rheology. Be able to use if equations are given.

  - Elasticity – $\boldsymbol{\sigma}=\mathbf{C}:\boldsymbol{\varepsilon}$, linear relationship between stress and infinitesimal strain. For an isotropic medium, only two independent parameters, e.g. Lamé parameters, bulk and shear moduli, or Young's modulus and Poisson's ratio.

  - Newtonian Viscosity – linear relationship between deviatoric stress $\boldsymbol{\sigma}'$ and strain rate **D**. If isotropic => bulk and shear viscosity as the two material parameters.

- Equations of motion – wave equation for elastic media and Navier Stokes for fluids, understand terms and derive simple solutions if equations given.