Fluid Flow

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Outline of Lecture

- Fluid flow in simple geometries
- Reynolds number and the onset of turbulence
- Modelling turbulence
- Strain-rate dependent rheologies

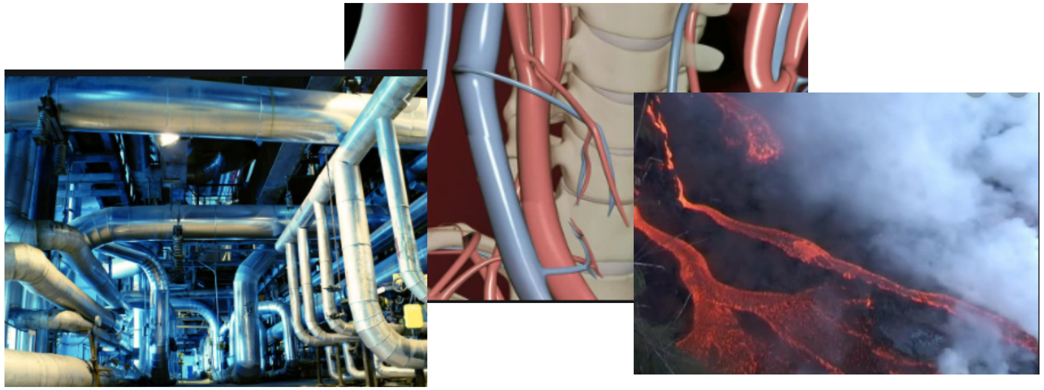
Learning Objectives

- Be able to set up Navier-Stokes equations and find analytical solutions for simple steady flow problems:
  - channel flow, pipe flow and flow down a slope,
  - with Newtonian or non-Newtonian viscosities,
  - gain understanding in physical behaviour of such flows.

- Understanding different common methods to model turbulence

Why study flow in simple geometries?

- Channel flow
- Pipe flow
- Flow down a slope



Analytical solutions - first estimates of flow characteristics, benchmarks

Numerical solutions => complex geometries, complex rheology, non-steady problems

The Navier-Stokes Equation

- Momentum balance:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \nabla \cdot \boldsymbol{\tau} + \rho\, \mathbf{g}$$

- Mass balance (continuity equation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Where:
u – velocity
P – pressure
τ – deviatoric stress tensor
ρ – density
g – gravitational acceleration

Material derivative:

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

or equivalently

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u}\mathbf{u}$$

Use index notation to check that these are equivalent

Where:
S – strain rate tensor
μ – (shear) viscosity

Assumptions in this lecture

- We will only be considering incompressible flow – Constant density:

$$\nabla \cdot \mathbf{u} = 0$$

- Start by considering Newtonian Flow:
  - Shear stress is proportional to the strain rate

$$\boldsymbol{\tau} = 2\mu\,\mathbf{S} \qquad\qquad \mathbf{S} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

- This assumes viscosity is isotropic and also ignores the bulk viscosity, which is associated with changes in the volume/density of the fluid (we are making an incompressible assumption here). Bulk viscosity can be important in, for instance shocks

Incompressible Newtonian Fluid

- In an incompressible Newtonian fluid the gradient of viscous stress can be further simplified:

$$\nabla \cdot \boldsymbol{\tau} = \mu \, \nabla \cdot (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) = \mu \nabla^2 \boldsymbol{u}$$

- This requires the implication of the incompressible assumption:

$$\nabla \cdot \mathbf{u} = 0$$

Check, e.g. using index notation

Side note: Dynamic and Kinematic Viscosity

Be careful when talking about viscosity as to what is meant

Dynamic viscosity μ (sometimes called absolute viscosity)

- (units: Pa·s)
- Proportionality between shear stress and strain rate

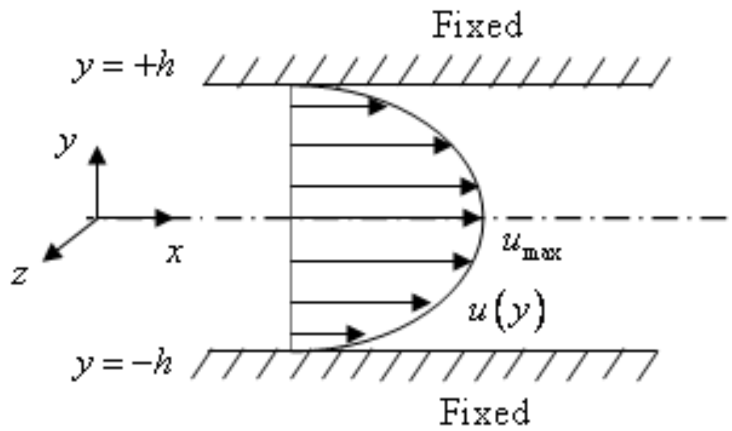
Kinematic viscosity $\nu = \mu/\rho$

- (units: m^2/s)
- Equivalent to a diffusivity for momentum

Pressure Driven Steady Flow between Parallel Plates

- Can be derived easily either from the Navier-Stokes equation or by considering a force balance on a unit of fluid:
  - No time dependency
  - Only non-zero velocity components are in the direction
  - Ignore body forces
  - Flow in direction of pressure drop

$$\frac{dP}{dx} = -\frac{\Delta P}{L} = \mu \frac{d^2 u_x}{dy^2}$$



- Integrate and note that u_x at $y=h$ and $y=-h$:

$$u_x = \frac{\Delta P}{2\mu L}(h^2 - y^2)$$

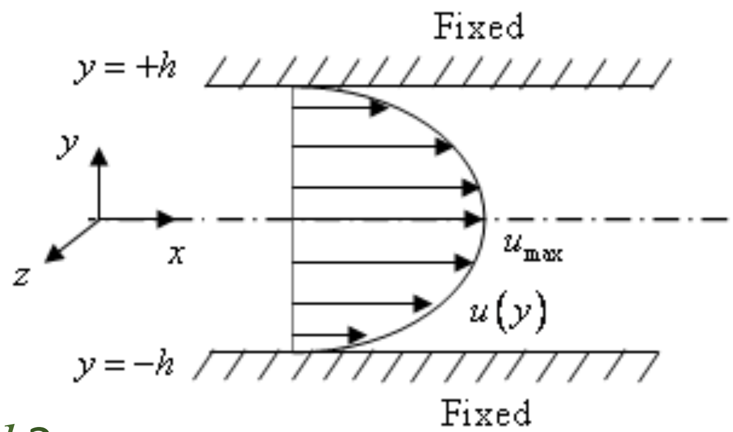
Check this

Pressure Driven Steady Flow between Parallel Plates

- For boundary condition u_x at $y=h$ and $y=-h$:

$$u_x = \frac{\Delta P}{2\mu L}(h^2 - y^2)$$

What is the volumetric flow rate (per meter of width) ?



- What is the solution for the boundary condition $u_x=U$ at $y=h$ and $u_x=0$ at $y=-h$?

This type of flow is called Couette flow.

Pressure Driven Steady Flow in a pipe

- Similar derivation as before, but done in cylindrical coordinates:
  - Navier Stokes momentum equation in cylindrical coordinates:
- r direction:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}\right)=$$
$$-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_r$$

- θ direction:

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\theta u_r}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=$$
$$-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]+\rho g_\theta$$

- z direction:

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=$$
$$-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_z$$

Pressure Driven Steady Flow in a pipe

- Need only consider momentum balance in z direction
- z direction only non-zero velocity and no time dependency:

$$\frac{dP}{dz} = -\frac{\Delta P}{L} = \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)$$

- Integrate once:

$$\frac{du_z}{dr} = -\frac{\Delta P}{2\,\mu\,L}r$$

Constant of integration is zero as symmetry means that there is a zero velocity gradient at the centre of the pipe

- Integrate again and note that velocity is zero at the wall ():

$$u_z = \frac{\Delta P}{4\,\mu\,L}(R^2 - r^2)$$

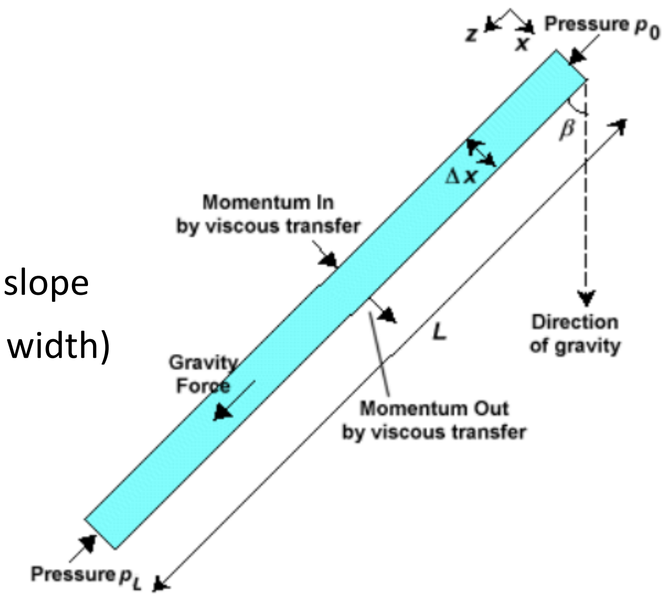
What is the volumetric flow rate in this case?

Calculating flow and pressure

- For simple geometries either simplify the Navier-Stokes equation or do your own momentum balance

- Free surface flow on an inclined plane using a momentum balance
    - For a given slope angle, volumetric fluid flow-rate, fluid density and fluid viscosity, how thick will the fluid layer be (assuming a large enough plane that the thickness is constant)?

Calculations:



- Assume x is distance from free surface and z is distance down slope
- Force balance on control in direction parallel to slope (for unit width)

$$p_0 \Delta x - p_L \Delta x + \rho g \cos(\beta) \Delta x L + \tau_{xz}(x + \Delta x) L - \tau_{xz}(x) L = 0$$

Pressure force on ends of control volume

Gravity force down the slope

Shear force on sides of control volume (note that τ_{xz} refers to the stress on the x direction face acting in the z direction)

- As the system is at steady state, there are no viscous stresses exerting a force in the direction normal to the slope
  - Pressure with depth below free surface purely hydrostatic – no pressure gradient down the slope $(p_0 = p_L)$
- Rearranging the equation:

$$\frac{\tau_{xz}(x + \Delta x) - \tau_{xz}(x)}{\Delta x} = -\rho g \cos(\beta)$$ Taking the limit as Δx goes to zero: $$\frac{d\tau_{xz}}{dx} = -\rho g \cos(\beta)$$

Calculations continued:

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos(\beta)$$

- Integrate to get shear stress as a function of the depth below the surface:

$$\tau_{xz} = -\rho g \cos(\beta) x + A$$

- As the air above the free surface is much lower viscosity than the liquid we can assume that it exerts negligible viscous stress:

$$\tau_{xz} = 0 \text{ at } x = 0 \text{ therefore } A = 0$$

- If we assume a Newtonian fluid then the shear stress is proportional to the strain rate, with viscosity being the proportionality:
  - Note that the sign depends on whether you consider the stress to be the force exerted by the fluid or on the fluid – Either can be used, but it must be used consistently

$$\tau_{xz} = \mu \frac{dv_z}{dx}$$

- Substituting and integrating again yields:

$$v_z = -\frac{\rho g \cos(\beta)}{2\mu} x^2 + B$$

Calculations continued:

$$v_z = -\frac{\rho g \cos(\beta)}{2\mu} x^2 + B$$

- Constant of integration, obtain from bottom boundary condition:
- If h is the thickness of the slide, then $v_z = 0$ at $x = h$ (noting that we have defined x as being the distance below the free surface)

$$B = \frac{\rho g \cos(\beta)}{2\mu} h^2$$

- The velocity profile is therefore parabolic with the following shape

$$v_z = \frac{\rho g \cos(\beta)}{2\mu} (h^2 - x^2)$$

- Note that this is the same velocity profile as would be obtained for similar flow between 2 parallel plates $2h$ apart
  - Planes of symmetry and free surfaces are physically very similar as they involve no shear stress

Question 1 - worksheet

- A fluid is flowing between two large parallel plates (density ρ and viscosity μ). The plates are a distance h apart and the pressure drop is ΔPL. The plates are inclined at an angle ϑ. Assuming that the flow is laminar, calculate the volumetric flowrate through the system per meter of width as a function of the inclination angle. Simplify the Navier-Stokes equation in order to carry out this calculation.
- At what angle does the flow stop in terms of the other variables (you can also check this based on a force balance over the entire system)?

Not the end of the story – Why?

- An explicit assumption in these derivations is that the flow is steady (no time dependency)

- What happens if there is a perturbation in the flow?
  - If this perturbation grows, it means that the flow can not be steady and that this assumption is invalid

- Perturbation will be damped if viscous forces are stronger than the inertial force associated with the perturbation

Reynolds Number

- Reynolds number represents the balance between inertial and viscous forces

$$Re \propto \frac{Inertial\ Force}{Viscous\ Force} \propto \frac{\rho v \frac{dv}{dx}}{\mu \frac{d^2 v}{dx^2}} \propto \frac{\rho v \frac{v}{d}}{\mu \frac{v}{d^2}} \qquad Re = \frac{\rho d v}{\mu}$$

- If Reynolds number is small perturbations are suppressed – Laminar Flow
- If Reynolds number is large perturbations grow – Turbulent Flow

Reynolds Number – Flow in Pipe

- Pipe diameter as characteristic length scale
- Average velocity as characteristic velocity

Laminar	Re < 2100
Transition	2100 < Re < 4000
Turbulent	Re > 4000

$$Re = \frac{\rho d v}{\mu}$$

Flow in Pipe

- As Reynolds number increases beyond transition average flow profile goes from parabolic to closer to plug flow
  - Laminar boundary layer near the wall – Gets thinner as the Reynolds number increases
  - Thickness of boundary layer also influenced by surface roughness

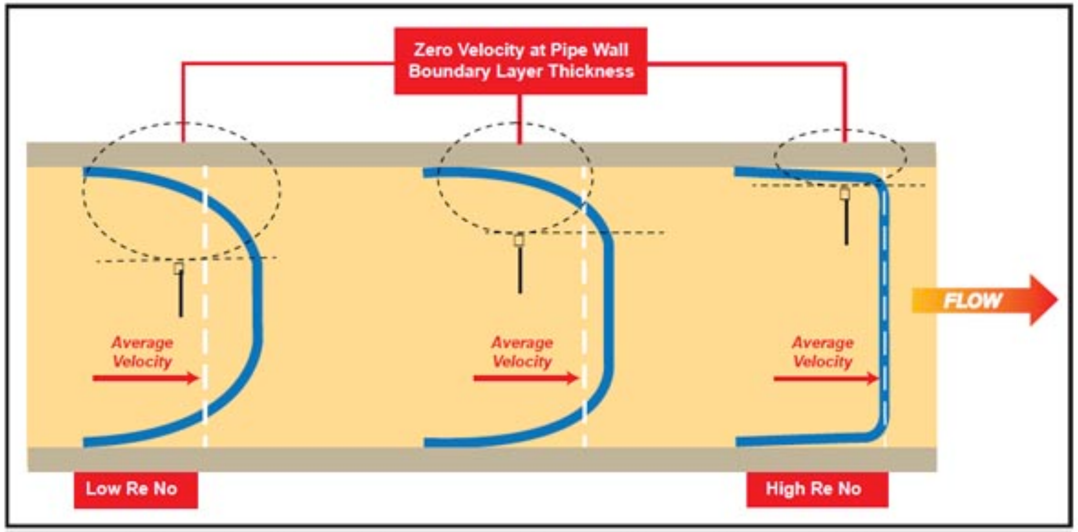


Figure 1. *Boundary layer influence on flow profile and Reynolds Number*

Flow in Pipes – The Engineering Approach

- Fanning Friction factor:
  - Ratio of shear stress exerted on the fluid to its kinetic energy:

$$f = \frac{\tau}{\rho \frac{v^2}{2}}$$

  - For fully developed flow the shear stress on the wall must balance the pressure drop:

$$\Delta P \, \pi \, R^2 = \tau \, 2 \, \pi \, R \, L$$

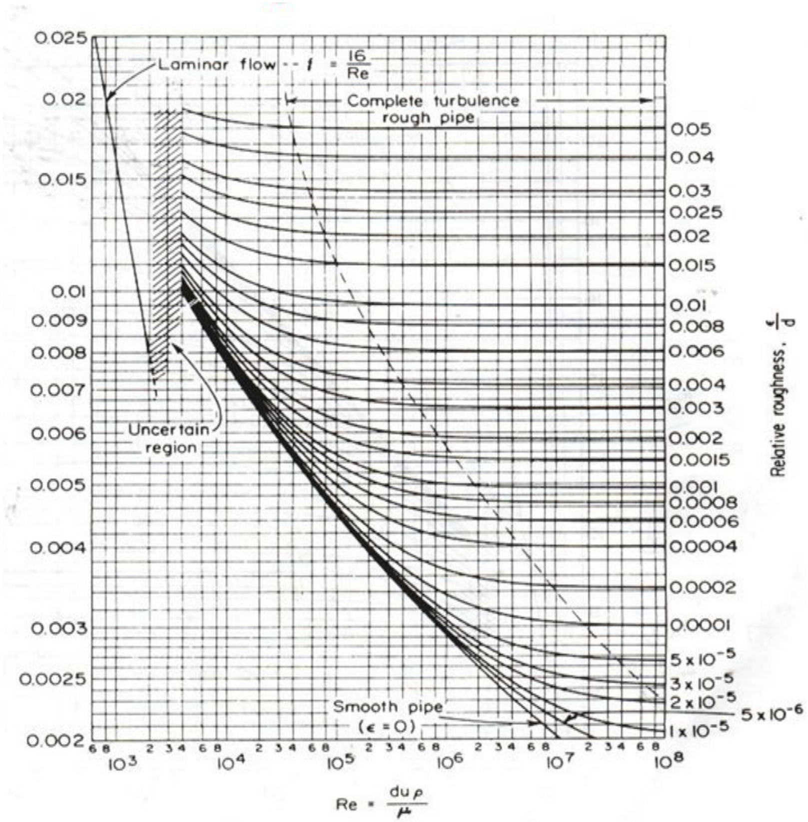
  - Combine to give Fanning friction factor: $f = \dfrac{\Delta P \, R}{L \, \rho \, v^2}$

Flow in Pipes – The Engineering Approach

- Friction factor is a function of Reynolds Number and pipe roughness:
  - For laminar flow $f = 16/Re$
  - For turbulent flow empirical relationships or charts are use

$$f = \frac{\Delta P\ R}{L\ \rho\ v^2}$$

Note that that Moody friction factor is often used instead of the Fanning friction factor, but it is simply 4 times larger



Non-dimensionalisation

Use characteristic length and velocity: $r' = \dfrac{r}{R} \qquad u' = \dfrac{u}{\bar{u}}$

Equation of Motion

$$\frac{du}{dr} = \frac{r}{2\mu}\frac{dp}{dx} \quad \Rightarrow \quad \frac{du'}{dr'} = \frac{R^2}{\bar{u}}\frac{r'}{2\mu}\frac{dp}{dx} = -\frac{f\,\mathrm{Re}}{4}r'$$

dimensionless pressure gradient

Friction factor	Reynolds number
$f = \dfrac{-R}{\rho\bar{u}^2}\dfrac{dp}{dx}$	$\mathrm{Re} = \dfrac{\rho\bar{u}D}{\mu}$

$$\frac{inertial\ \ forces}{viscous\ \ forces}$$

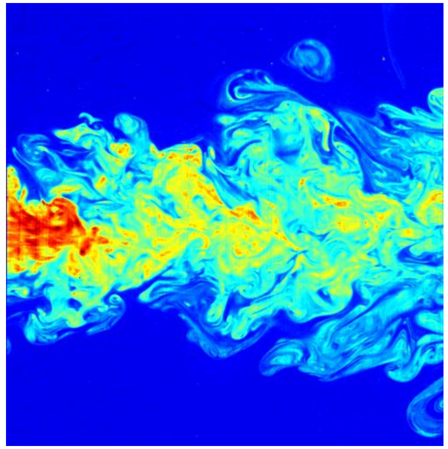
Solution laminar flow

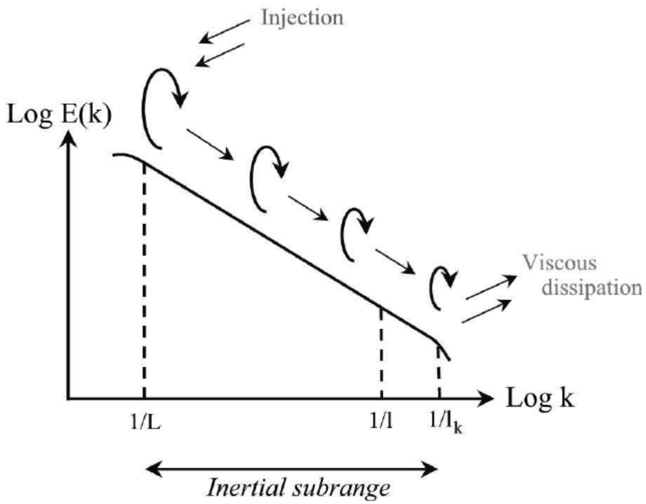
$$u'(r') = \frac{f\,\mathrm{Re}}{8}\left(1 - r'^2\right) \qquad u'_{\max} = 2\bar{u}' = \frac{f\,\mathrm{Re}}{8}$$

$$\bar{u}' = 1 = \frac{f\,\mathrm{Re}}{16} \quad \Rightarrow \quad f = 16/\mathrm{Re}$$

What happens in Turbulent Flow?

- At high Reynolds numbers, large scale flow structures and instabilities will develop smaller scale perturbations
- This cascade of larger perturbations generating smaller flow structures will continue down until they are small enough that viscosity can "win"
  - The Kolmogorov length scale
- At this finest level the energy in the flow is dissipated as heat via the viscosity





Direct Numerical Simulation (DNS)

- Solve Navier-Stokes equation with enough resolution to resolve all the eddies down to their laminar cores
  - Very computationally expensive – prohibitively so for all but the lowest Reynolds numbers and smallest systems

- Can get a rough estimate of the resolution required based on Kolmogorov length scales and dimensional arguments

$$\Delta x \sim \left(L \left(\frac{\mu}{\rho |\boldsymbol{v}|} \right)^3 \right)^{\frac{1}{4}}$$

Water (μ=1·10^{-3} Pa s, ρ = 1·10^{3} kg/m^{3} at 1 m/s in a 1 m channel would require a resolution of about 10 µm (about 10^{15} elements in cubic box)!

Check dimensions

Turbulence Modelling

- As DNS is only useable on a small range of problems, we need to model the effect of turbulence
- Turbulence dissipates more energy than the macroscopic strain rate would suggest
  - Need to calculate this extra dissipation

3 main approaches

- Simply increase the viscosity!
  - Trivial to do – often used in large scale simulations such as ocean modelling
- Assume that you can separate the turbulent and mean components of the flow
  - Reynold Average Navier Stokes (RANS) type models
  - We will look at the k-ε model as it is the most commonly used of this type
- Resolve as much as you can and then use a model for the bits you can't resolve
  - Large Eddy Simulations (LES)

Both RANS and LES approaches involve the calculation of an additional turbulent viscosity that needs to be added to the underlying fluid viscosity

Mean and Fluctuating Flow Components

- Turbulent structures are typically embedded within macroscopic flow structures
- In analysing and modelling turbulence it is thus useful to distinguish between these flow components
- Assume that the flow at any point can be decomposed into a mean component and a fluctuating component:

where $\mathbf{u}$ is the instantaneous velocity, $\mathbf{U}$ is the average velocity and $\mathbf{u}'$ is the fluctuating component of the velocity

RANS version of the Navier Stokes Equation

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}$$

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u}\mathbf{u}$$

- Assume that there is a mean and fluctuating component of the velocity and pressure
  - Fluctuating component of pressure has no impact on final equation form

$$\mathbf{u} = \mathbf{U} + \mathbf{u}' \qquad p = P + p'$$

- Substituting and applying the RANS averaging rules:

$$\rho \frac{D\mathbf{U}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{U} + \rho \mathbf{g} - \rho \nabla \cdot \overline{\mathbf{u}'\mathbf{u}'}$$

Why only a single term with ***u****’?*

Reynolds Stress Tensor

- The averaged equation is similar to the original except for the inclusion of the divergence of a rank 2 tensor:

$$\rho \nabla \cdot \overline{\mathbf{u}'\mathbf{u}'}$$

- Known as the Reynolds Stress Tensor

$$\rho \overline{\mathbf{u}'\mathbf{u}'} = \rho \begin{bmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'v'} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'w'} \end{bmatrix} \qquad \mathbf{u}' = \{u', v', w'\}$$

Reynolds Stress Tensor

- Need to relate the Reynolds Stress Tensor to the mean flow
  - Know as the turbulence closure problem
  - No right way to do this, but some approximations are better than others

- The Reynolds Stress Tensor represents the transfer of momentum due to the fluctuations
- In the standard Navier-Stokes equation the viscous stress term represents the "diffusion" of momentum
  - Can think of Reynolds Stress Tensor in terms of a diffusion of momentum due to velocity fluctuations – a turbulent eddy viscosity

Turbulent Eddy Viscosity

- Write the Reynolds Stress Tensor in terms of the macroscopic strain rate:

???

$$\overline{\mathbf{u}'\mathbf{u}'} = -\upsilon_T \left(\nabla\mathbf{U} + \nabla\mathbf{U}^T\right)$$

  - Where ν_T is a kinematic viscosity
- This means that you can combine the liquid viscosity and the turbulent losses into a single effective viscosity:
  - In turbulence modelling it is often more convenient to use the kinematic viscosity

$$\upsilon_{effective} = \upsilon + \upsilon_T$$

- We now need to calculate this turbulent eddy viscosity

Zero Equation Models

- The simplest type of turbulence model
  - Assume no explicit transport of turbulence
- Mainly based on dimensional arguments
  - l_0 is a turbulent length scale, t_0 is a turbulent time scale

$$\upsilon_T = \frac{{l_0}^2}{t_0}$$

- Zero equation models are not very good in steady state RANS type models where the turbulent length scale is not well defined
  - Often adequate in LES models where the turbulent structures are resolved down to the grid resolution
    - Unresolved turbulent length scale similar to mesh resolution
    - Less transport of small scale turbulence before it is dissipated

Zero Equation Models

- If we assume that we know the turbulent length scale, we still need the turbulent time scale from some macroscopic flow property
  - Two obvious candidates
    - Magnitude of the strain rate or magnitude of the vorticity
    - Both have units of inverse time
  - Based on strain rate – Smagorinsky model

    $$\upsilon_T = (C_S l_0)^2 |\mathbf{S}|$$

    $$\mathbf{S} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

    - C_S is a tuneable constant. Typical values are 0.1-0.2
  - Based on vorticity – Baldvin-Lomaz model

    $$\upsilon_T = (k l_0)^2 |\boldsymbol{\omega}|$$

    $$\omega = \nabla \times \mathbf{U}$$

    - Less widely used than Smagorinsky – Can be used for boundary layers
    - In boundary layers is a function of distance from the wall

Two Equation Model

- We will revisit zero equation models in the context of LES
- For RANS modelling zero equation models are inadequate

- Need to consider the generation, transport and dissipation of turbulence
  - Two quantities can be used to model this
    - Turbulent kinetic energy - k
    - Turbulent dissipation rate - ε
  - So called turbulence model
    - Other models exist, but this is one of the most widely used and illustrates the idea

$$\upsilon_T = C_\mu \frac{k^2}{\varepsilon}$$

Turbulent Kinetic Energy - k

- The fluctuating components of the turbulence stores energy
  - This is known as the turbulent kinetic energy, which, per mass of fluid, can be written as follows:

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \qquad \mathbf{u}' = \{u', v', w'\}$$

  - Note that the turbulent kinetic energy is therefore closely related to the Root Mean Square (RMS) of the velocity distribution

$$u_{rms} = \sqrt{\overline{|\mathbf{u}'|^2}} = \sqrt{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}$$

- This turbulent kinetic energy is generated at the large scale and dissipated at the small scale
  - It can also be transported around the system by the mean flow

Turbulent Dissipation Rate - ε

- This is equal to the viscous losses associated with the fluctuating component of the velocity
  - I.e. the rate at which the turbulent kinetic energy is converted into thermal energy

$$\varepsilon = \upsilon\, \overline{\mathbf{S}' \cdot \mathbf{S}'} \qquad\qquad \mathbf{S}' = \frac{1}{2}(\nabla \boldsymbol{u}' + \nabla \boldsymbol{u}'^{T})$$

ν – *kinematic viscosity*

  - While this is how the energy is dissipated, this equation is not directly useful as we do not know the fluctuation components (or their gradients)

Equations for k and ε

- Both the turbulent kinetic energy (k) and the dissipation rate (ε) can be transported, generated and destroyed and are represented by a pair of coupled ODEs
  - Don't worry too much about how these are derived – note, though, the additional complexity associated with modelling turbulence

$$\mathbf{S}' = \frac{1}{2}(\nabla \boldsymbol{u}' + \nabla \boldsymbol{u}'^{T})$$

$$\frac{\partial k}{\partial t} + \mathbf{U} \cdot \nabla k = \nabla \cdot \left(\frac{\upsilon_t}{\sigma_k} \nabla k \right) + 2\,\upsilon_t\,\mathbf{S} \cdot \mathbf{S} - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \mathbf{U} \cdot \nabla \varepsilon = \nabla \cdot \left(\frac{\upsilon_t}{\sigma_\varepsilon} \nabla \varepsilon \right) + C_{1\varepsilon} \frac{\varepsilon}{k}\,\mathbf{S} \cdot \mathbf{S} - C_{2\varepsilon} \frac{\varepsilon^2}{k}$$

$$\upsilon_T = C_\mu \frac{k^2}{\varepsilon}$$

The semi-empirical constants are typically assigned the following values:

$$C_\mu = 0.09,\ \sigma_k = 1.00,\ \sigma_\varepsilon = 1.30,\ C_{1\varepsilon} = 1.44,\ C_{2\varepsilon} = 1.92$$

Boundaries in Turbulence modelling

- At inlets both k and ε need to be assigned values
  - If appropriate values are not assigned, long inlet regions will be required

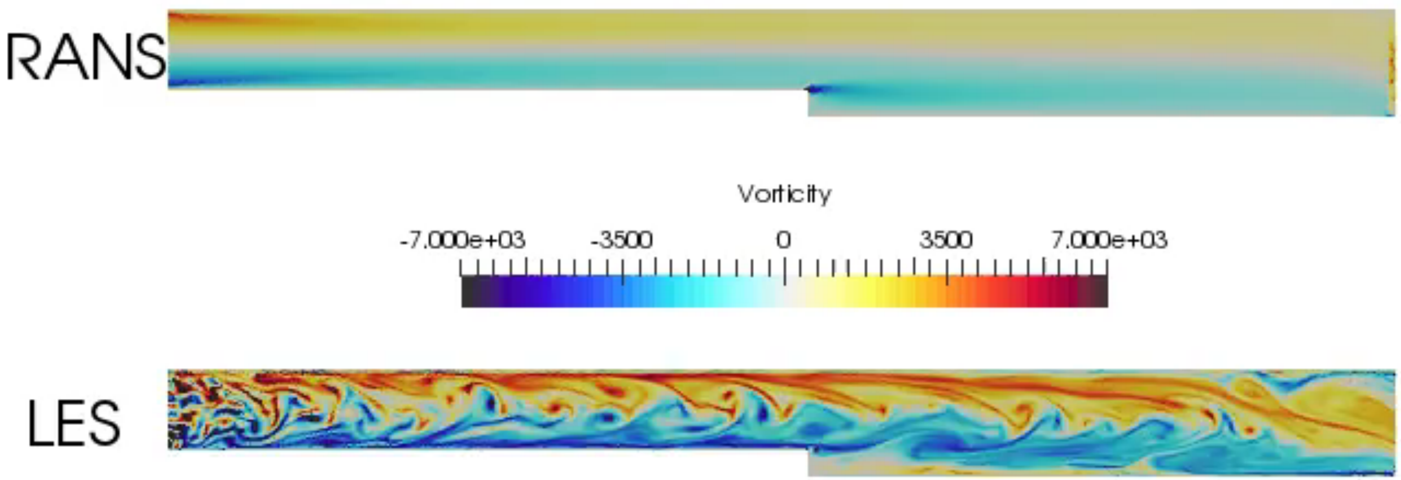
- At walls k is zero (no slip), but the transition typically occurs over a very narrow boundary layer
  - Typically need to model this boundary layer as it is usually computationally very expensive to explicitly resolve

Large Eddy Simulations - LES

- Transient simulations that resolve as much of the dynamics as possible
- Zero equation models are usually appropriate
  - No time averaging or smoothing of equations above the grid resolution
  - If unresolved resolution is small then the assumption of no turbulent transport between generation and dissipation is appropriate
- Can use Smagorinsky model with length scale as the grid resolution

$$\upsilon_T = (C_S \Delta x)^2 |\mathbf{S}|$$

RANS vs LES



Question 4 - worksheet

- Using an approximation for the relation between friction factor and Reynolds number for laminar and turbulent flow, write a program that returns the volumetric flow rate for given pressure drop, pipe length, pipe radius, pipe roughness, fluid density and fluid viscosity.
- See detailed instructions on the worksheet

Other Rheologies

- Thus far we have only considered Newtonian rheology
  - Shear stress proportional to the strain rate
  - A good approximation for many fluids, importantly including water and virtually all gases

- Other Rheologies are possible
  - Shear stress a more complex function of strain rate
  - Can also be a function of strain history
    - Thixotropic fluids

Shear thinning

- Apparent viscosity decreases with strain rate
  - Quite common
    - Dense suspensions, emulsions, foams etc…
  - Sometimes called pseudo-plastic fluids

- Some physical origins:
  - Structures in the fluid can break down at high shears
    - E.g. some suspensions where particles can form flocs
  - Molecules align with or are extended in the shear direction
    - E.g. molten plastics

Shear-thinning viscosity

"Frog saliva switches between being thin and watery as the whip-like tongue hits its target, to thick and sticky as the insect is reeled in"

Yield Stress

- An extreme example of shear thinning are materials with a yield stress

- Below a yield stress a material does not flow and acts like a solid

- Above the yield stress the material flows and acts like a liquid

- Quite common
  - It is really a matter of the size of the yield stress
  - Logical extension of plastic deformation in a solid

Shear thickening

- Apparent viscosity increases with strain rate
  - Much less common
    - Classic example is corn starch in water

- Some physical origins:
  - Structures in the fluid jam and can't pass one another as easily at high strain rates

- Lots of current interest in shear-thickening fluids
  - Flexible protection and armouring

Modelling non-Newtonian Rheologies

- Need to incorporate these rheologies into the Navier-Stokes equation
  - Require a relationship between shear stress and strain rate tensors
- Often easier to incorporate rheology as an apparent viscosity that depends on strain rate
  - Note that in most solvers it is the velocity (and thus strain rate) that is assumed to be known

Power law fluid

- The simplest model for both shear thinning and shear thickening
  - The separation of the strain rate into two terms is so that stress tensor points in the correct direction

$$\boldsymbol{\tau} = 2k\left|2\mathbf{S}\right|^{n-1}\mathbf{S}$$

$$\mathbf{S} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)$$

  - If n=1 this is Newtonian
  - If n>1 this is shear thickening
  - If n<1 this is shear thinning

$$\mu_{eff} = k\left|2\mathbf{S}\right|^{n-1}$$

- If the velocity only changes in one direction

$$\tau_{xy} = k\frac{du_y}{dx}\left|\frac{du_y}{dx}\right|^{n-1}$$

Question 2 - worksheet

- A power law fluid is flowing in a cylindrical pipe driven by a pressure gradient of $\Delta P/L$. The flow is occurring in the z direction, with the shear stress thus being the result of a velocity gradient in the r direction:

$$\tau = k\frac{\partial u_y}{\partial x}\left|\frac{\partial u_y}{\partial x}\right|^{n-1}$$

- Use a force/momentum balance to calculate the velocity profile as a function of the power law exponent n.
- Plot these results in order to see how the velocity profile is influenced by changes in the rheology.

Bingham Plastic

$$\mathbf{S} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

- The simplest type of yield behaviour is to assume that there is a yield stress followed by a linearly increasing stress with increasing strain rate

$$if\ |\tau| > \tau_0: \quad \tau = (2\mu_\infty|\mathbf{S}| + \tau_0)\frac{\mathbf{S}}{|\mathbf{S}|}$$

$$if\ |\tau| < \tau_0: \quad \mathbf{S} = 0$$

$$if\ |\tau| > \tau_0: \quad \mu_{eff} = 2\mu_\infty + \frac{\tau_0}{|\mathbf{S}|}$$

$$if\ |\tau| < \tau_0: \quad \mu_{eff} = \infty$$

- If the velocity only changes in one direction

$$if\ |\tau| > \tau_0: \quad \tau = sign\left(\frac{\partial u_y}{\partial x}\right)\left(\mu_\infty\left|\frac{\partial u_y}{\partial x}\right| + \tau_0\right)$$

$$if\ |\tau| < \tau_0: \quad \frac{\partial u_y}{\partial x} = 0$$

Herschel-Bulkley Fluid

- Combines yield stress with power law post-yield behaviour:

$$if\ |\tau| > \tau_0: \qquad \tau = (k(2|\mathbf{S}|)^n + \tau_0)\frac{\mathbf{S}}{|\mathbf{S}|}$$

$$if\ |\tau| < \tau_0: \qquad \mathbf{S} = 0$$

$$if\ |\tau| > \tau_0: \qquad \mu_{eff} = k(2|\mathbf{S}|)^{n-1} + \frac{\tau_0}{|\mathbf{S}|}$$

$$if\ |\tau| < \tau_0: \qquad \mu_{eff} = \infty$$

- If the velocity only changes in one direction

$$if\ |\tau| > \tau_0: \quad \tau = sign\left(\frac{\partial u_y}{\partial x}\right)\left(k\left(\left|\frac{\partial u_y}{\partial x}\right|\right)^n + \tau_0\right)$$

$$if\ |\tau| < \tau_0: \qquad \frac{\partial u_y}{\partial x} = 0$$

Free surface flow on an inclined plane

- This is the problem that we solved earlier for a Newtonian fluid
- Let us consider a Bingham plastic with a yield stress τ_0

Calculation

- The shear stress distribution in this problem is the same as that for the Newtonian fluid and can be obtained in the same manner:

$$\tau_{xz} = -\rho g \cos(\beta) x$$

Note that again is defined as the distance below the free surface

$$\tau_{xz} = -\rho g \cos(\beta) x$$

Calculation continued:

- As the component is the only non-zero shear stress:

$$if\ |\tau| > \tau_0: \qquad \tau = \ sign\left(\frac{\partial u_z}{\partial x}\right)\left(\mu_\infty \left|\frac{\partial u_z}{\partial x}\right| + \tau_0\right)$$

$$if\ |\tau| < \tau_0: \qquad \frac{\partial u_z}{\partial x} = 0$$

- If we assume that the geometry is defined such that is a positive number, then τ_{xz} is negative, so $|\tau_{xz}| = -\tau_{xz}$ and $|\mathrm{d}u_z/\mathrm{d}x| = -\mathrm{d}u_z/\mathrm{d}x$ (note that if τ_{xz} was positive, which would have been the case if we had defined as being the distance above the bottom of the flow, $|\tau_{xz}| = \tau_{xz}$ and $|\mathrm{d}u_z/\mathrm{d}x| = \mathrm{d}u_z/\mathrm{d}x$)

$$if\ |\tau| > \tau_0: \qquad \tau_{xz} = -\left(\mu_\infty \frac{-du_z}{dx} + \tau_0\right) = \mu_\infty \frac{du_z}{dx} - \tau_0$$

$$if\ |\tau| < \tau_0: \qquad \frac{du_z}{dx} = 0$$

Calculation continued:

$$if\ |\tau| > \tau_0:\ \tau_{xz} = \mu_\infty \frac{du_z}{dx} - \tau_0$$
$$if\ |\tau| < \tau_0:\ \frac{du_z}{dx} = 0$$

$$\tau_{xz} = -\rho g \cos(\beta) x$$

- We need to do this in two regions
  - A deep region when $\tau_{xz} > \tau_0$ and a shallow region where $\tau_{xz} < \tau_0$
  - Let x_0 be the depth below the liquid surface of this transition:

$$x_0 = \frac{\tau_0}{\rho g \cos(\beta)}$$

- In the deep region ($x > x_0$):

$$\mu_\infty \frac{du_z}{dx} - \tau_0 = -\rho g \cos(\beta) x$$

- Which can be integrated to give:

$$u_z = -\frac{\rho g \cos(\beta)}{2\mu_\infty} x^2 + \frac{\tau_0}{\mu_\infty} x + B$$

- B can be obtained from no slip at the bottom ($u_z = 0$ at $x = h$)

$$u_z = \frac{\rho g \cos(\beta)}{2\mu_\infty}\left(h^2 - x^2\right) - \frac{\tau_0}{\mu_\infty}(h - x)$$

Calculation continued:

- From previous slide:

$$\text{if } x > x_0: \quad u_z = \frac{\rho g\cos(\beta)}{2\mu_\infty}(h^2 - x^2) - \frac{\tau_0}{\mu_\infty}(h - x)$$

- The velocity in the top portion is constant and can be obtained by setting

$$\text{if } x > x_0: \quad u_z = \frac{\rho g\cos(\beta)}{2\mu_\infty}(h^2 - x_0^2) - \frac{\tau_0}{\mu_\infty}(h - x_0)$$

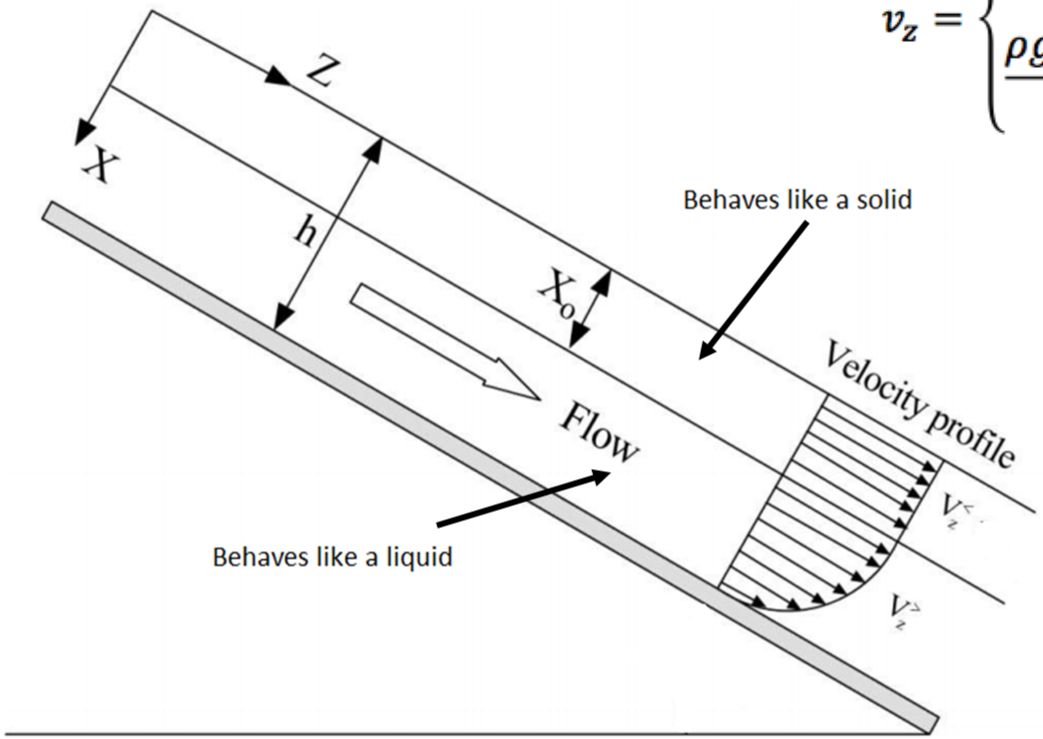
- Yielding following overall velocity profile:

$$u_z = \begin{cases} \dfrac{\rho g\cos(\beta)}{2\mu_\infty}(h^2 - x_0^2) - \dfrac{\tau_0}{\mu_\infty}(h - x_0) & \text{if } x < x_0 \\ \dfrac{\rho g\cos(\beta)}{2\mu_\infty}(h^2 - x^2) - \dfrac{\tau_0}{\mu_\infty}(h - x) & \text{if } x > x_0 \end{cases}$$

$$x_0 = \frac{\tau_0}{\rho g\cos(\beta)}$$

Bingham plastic velocity profile

$$v_z = \begin{cases} \dfrac{\rho g \cos(\beta)}{2\mu_\infty}(h^2 - x_0^2) - \dfrac{\tau_0}{\mu_\infty}(h - x_0) & \text{if } x < x_0 \\ \dfrac{\rho g \cos(\beta)}{2\mu_\infty}(h^2 - x^2) - \dfrac{\tau_0}{\mu_\infty}(h - x) & \text{if } x > x_0 \end{cases}$$



Question 3 - worksheet

- Assume that a soil flow can be approximated as a Bingham plastic. Consider a layer of soil with density 2500 kg/m^3 on a slope that can be assumed to be infinitely wide and long. The layer is 1 m thick and is at an angle of 30° from the horizontal.
- Assume that it is initially at its angle of repose, which implies that the yield stress is just able to balance the gravitational down the slope. force Gravitational acceleration =9.8 m/s^2. After heavy rain, the yield stress of the material decreases by 20%.
- Assuming that the μ_{∞}*=100 Pa.s*, what is the maximum velocity that the slide achieves down the slope?

Outline of Lecture

- Fluid flow in simple geometries
- Reynolds number and the onset of turbulence
- Modelling turbulence
- Strain-rate dependent rheologies