

$$(1) (a) \quad \text{tr}(\underline{S}) = 12$$

$$(b) \quad \underline{S} : \underline{S} = 281$$

$$(c) \quad \underline{S} : \underline{S}^T = 240$$

$$(d) \quad \underline{\hat{a}} \cdot \underline{S} = \begin{pmatrix} 31 \\ 25 \\ 30 \end{pmatrix}$$

$$(e) \quad \underline{S} \cdot \underline{\hat{a}} = \begin{pmatrix} 18 \\ 15 \\ 34 \end{pmatrix}$$

$$(f) \quad \underline{S}^T \cdot \underline{\hat{a}} = \begin{pmatrix} 31 \\ 25 \\ 30 \end{pmatrix} = \underline{\hat{a}} \cdot \underline{S}$$

Problem Set 5

ACSE-2

Solutions

$$(g) \quad \underline{\hat{a}} \underline{\hat{b}} = \begin{bmatrix} 16 & 10 & -14 \\ -8 & -5 & 7 \\ 32 & 20 & -28 \end{bmatrix}$$

$$(h) \quad \underline{\hat{b}} \underline{\hat{a}} = \begin{bmatrix} 16 & -8 & 32 \\ 10 & -5 & 20 \\ -14 & 7 & -28 \end{bmatrix} = (\underline{\hat{a}} \underline{\hat{b}})^T$$

(2) orthonormal base transformation

$$\text{check } \underline{\hat{e}}'_i \cdot \underline{\hat{e}}'_j = \delta_{ij}$$

$$\text{transformation matrix } L = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 3 & -3 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

(3) (a) simplify $\delta_{ij} \delta_{jk} \delta_{kp} \delta_{pi}$

$$\delta_{kp} \delta_{pi} \xrightarrow{\text{sum}} \delta_{ki}$$

similarly for $\delta_{jk} \delta_{ki}$

$$= \delta_{ij} \delta_{ji} = 3$$

↳ for dimension=3

$$(b) \quad \varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{lm} \delta_{ji}$$

For any given i, j, l, m

$$\varepsilon_{ijk} \varepsilon_{lmk} = \sum_{k=1}^3 \varepsilon_{ijk} \varepsilon_{lmk}$$

↳ only one non-zero term

if $i \neq j$ and either

$i=l$ and $j=m \rightarrow +1$

or $i=m$ and $j=l \rightarrow -1$

then for one term

$i \neq j \neq k$

other two will be = 0

The case $i=l, j=m \Rightarrow \delta_{il} \delta_{jm}$, case $i=m, j=l \Rightarrow -\delta_{im} \delta_{jl}$

$$(c) \quad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

$$\text{call } \vec{v} = (\vec{b} \times \vec{c})$$

$$\vec{a} \times \vec{v} = \epsilon_{ijk} a_i v_j \hat{e}_k$$

$$v_j = \epsilon_{lmj} b_l c_m$$

$$[\vec{a} \times (\vec{b} \times \vec{c})]_k = \epsilon_{ijk} \epsilon_{lmj} a_i b_l c_m$$

$$\epsilon_{ijk} = \epsilon_{kij} \quad = \epsilon_{kij} \epsilon_{lmj} a_i b_l c_m$$

use (b)

$$\begin{aligned} \rightarrow [\vec{a} \times (\vec{b} \times \vec{c})]_k &= (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) a_i b_l c_m \\ &= (a_i c_i b_k) - (a_l b_l c_k) \\ &= (\vec{a} \cdot \vec{c}) b_k - (\vec{a} \cdot \vec{b}) c_k \end{aligned}$$

(d) in similar way can use index notation to prove other triple product identities

$$(4) (a) \quad \nabla \times (\nabla F) = 0$$

• either write out all components

$$+ \text{realise } \frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i} (*) \Rightarrow \text{all components} = 0$$

• or use index notation

$$= \epsilon_{ijk} \hat{e}_k \frac{\partial^2 F}{\partial x_i \partial x_j} \quad (i)$$

$$= -\epsilon_{jik} \hat{e}_k \frac{\partial^2 F}{\partial x_j \partial x_i} \quad (ii)$$

however from symmetry (*) + renaming dummy indices

$$\rightarrow = \epsilon_{ijk} \hat{e}_k \frac{\partial^2 F}{\partial x_i \partial x_j} \quad (iii) \quad \begin{matrix} i \rightarrow j \\ j \rightarrow i \end{matrix}$$

$\Rightarrow (i) = (iii)$ expression equals own negative $\Rightarrow = 0$