ACSE-2

(1) $f(x)=f(a)+f(a)'(x-a)+f(a)''\frac{(x-a)^2}{2!}+f(a)'''\frac{(x-a)^3}{3!}+\cdots$

we choose $a=0$

$f(x)=f(0)+f(0)'x+f(0)''\frac{x^2}{2!}+f(0)'''\frac{x^3}{3!}+\cdots$

$f(x)=e^x \quad f(0)^{(n)}=1$

thus $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

(2)(a) $x=Sz$

$\frac{d(Sz)}{dt}=S\frac{dz}{dt}=ASz \qquad \frac{dz}{dt}=S^{-1}ASz$

we choose $S^{-1}AS=\Lambda \qquad A=S\Lambda S^{-1}$

$A=\begin{bmatrix}-1&3\\3&-1\end{bmatrix} \qquad A-\lambda I=\begin{bmatrix}-1-\lambda&3\\3&-1-\lambda\end{bmatrix} \quad \det(A-\lambda I)=\begin{vmatrix}-1-\lambda&3\\3&-1-\lambda\end{vmatrix}=0$

$(1+\lambda)^2-3^2=0 \qquad \lambda_1=2 \quad \lambda_2=-4$

$\Lambda=\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}=\begin{bmatrix}2&0\\0&-4\end{bmatrix} \qquad \frac{dz}{dt}=\Lambda z \qquad \begin{cases}\frac{dz_1}{dt}=\lambda_1 z_1 \Rightarrow z_1(t)=z_1(0)e^{\lambda_1 t}\\ \frac{dz_2}{dt}=\lambda_2 z_2 \Rightarrow z_2(t)=z_2(0)e^{\lambda_2 t}\end{cases}$

$\begin{cases}z_1(t)=z_1(0)e^{2t}\\ z_2(t)=z_2(0)e^{-4t}\end{cases}$ when $\lambda_1=2$ $\begin{pmatrix}-3&3\\3&-3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ we choose $\begin{pmatrix}1\\1\end{pmatrix}$

when $\lambda_2=-4$ $\begin{pmatrix}3&3\\3&3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ we choose $\begin{pmatrix}-1\\1\end{pmatrix}$

$S=\begin{pmatrix}1&-1\\1&1\end{pmatrix} \qquad Sz=\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}z_1(0)e^{2t}\\z_2(0)e^{-4t}\end{pmatrix}=\begin{pmatrix}z_1(0)e^{2t}-z_2(0)e^{-4t}\\z_1(0)e^{2t}+z_2(0)e^{-4t}\end{pmatrix}$

we know $x(t)=Ce^{(tA)} \qquad C=[C_1\ C_2]$

$\begin{cases}z_1(0)-z_2(0)=C_1\\ z_1(0)+z_2(0)=C_2\end{cases} \qquad \begin{cases}z_1(0)=\frac{1}{2}(C_1+C_2)\\ z_2(0)=\frac{1}{2}(-C_1+C_2)\end{cases}$

Thus $x(t)=Sz=z_1(0)\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+z_2(0)\begin{pmatrix}-1\\1\end{pmatrix}e^{-4t}$

$=\frac{1}{2}(C_1+C_2)\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+\frac{1}{2}(C_2-C_1)\begin{pmatrix}-1\\1\end{pmatrix}e^{-4t}$

(b) when $t \to \infty$, $e^{(2t)} \to \infty$, $e^{(-4t)} \to 0$

So $\frac{1}{2}(C_1+C_2)=0$ thus $x(t) \to 0$ as $t \to \infty$

$C_1+C_2=0$

(3) one norms $\|v\|_1 = |v_1|+|v_2|+\cdots+|v_n| = \sum_{i=1}^{n}|v_i|$

two norms $\|v\|_2 = \sqrt{v_1^2+v_2^2+\cdots+v_n^2} = \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}$

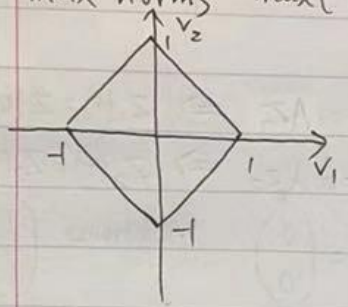
max norms $\|v\|_\infty = \max(|v_1|, |v_2|, \ldots, |v_n|) = \max_{i=1}^{n}|v_i|$

2D

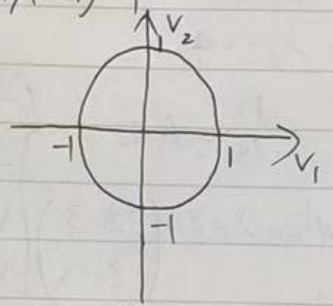
one norms $|v_1|+|v_2|=1$

two norms $v_1^2+v_2^2=1$

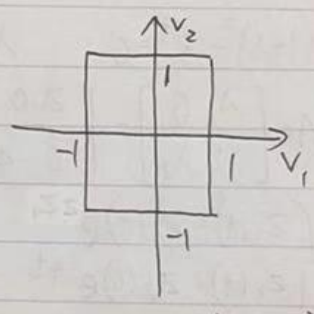
max norms $\max(|v_1|, |v_2|)=1$



$|v_1|+|v_2|=1$

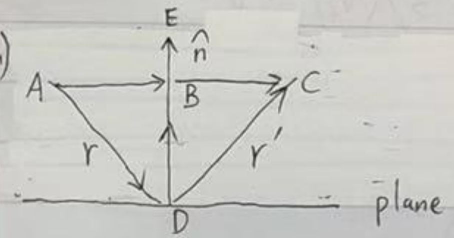


$v_1^2+v_2^2=1$



$\max(|v_1|, |v_2|)=1$

(4)(a)



we choose $\overrightarrow{DE} = \hat{n}$, $\overrightarrow{AD} = r$, $\overrightarrow{DC} = r'$ (reflected vector)

$|\overrightarrow{AB}| = |\overrightarrow{BC}|$

$r + r' = \overrightarrow{AC} = 2\overrightarrow{AB}$ $\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = r + \overrightarrow{DB}$

$r + r' = 2r + 2\overrightarrow{DB}$ $r' = r + 2\overrightarrow{DB}$

$\overrightarrow{DB} = -\frac{r \cdot \hat{n}}{|\hat{n}|^2} \cdot \hat{n}$

since $\hat{n}$ is the unit normal vector, then $|\hat{n}|^2 = 1$

$\overrightarrow{DB} = -(r \cdot \hat{n}) \cdot \hat{n}$

$r' = r - 2(r \cdot \hat{n}) \cdot \hat{n}$ we can choose $r' = T \cdot r$ (T is the transformation matrix)

thus $T \cdot r = r - 2(r \cdot \hat{n}) \cdot \hat{n}$

(b) we choose $r = [x\ y\ z]$

$$\hat{n} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad r\cdot\hat{n} = (x\ y\ z)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}(x+y+z)$$

$$(r\cdot\hat{n})\cdot\hat{n} = \left(\frac{1}{\sqrt{3}}\right)^2(x+y+z)\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}x+y+z\\x+y+z\\x+y+z\end{pmatrix}$$

$$r - 2(r\cdot\hat{n})\cdot\hat{n} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \frac{2}{3}\begin{pmatrix}x+y+z\\x+y+z\\x+y+z\end{pmatrix} = \begin{pmatrix}\frac{1}{3}x-\frac{2}{3}y-\frac{2}{3}z\\-\frac{2}{3}x+\frac{1}{3}y-\frac{2}{3}z\\-\frac{2}{3}x-\frac{2}{3}y+\frac{1}{3}z\end{pmatrix} = \begin{pmatrix}\frac{1}{3}&-\frac{2}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\-\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Thus $T = \begin{pmatrix}\frac{1}{3}&-\frac{2}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\-\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\end{pmatrix}$

(5) $V = x_1^2\hat{e}_1 + x_3^2\hat{e}_2 + x_2^2\hat{e}_3 = (x_1^2, x_3^2, x_2^2) \qquad x_1=1\quad x_2=1\quad x_3=0$

(i)
$$\nabla = \begin{pmatrix}\frac{\partial}{\partial x_1}\\\frac{\partial}{\partial x_2}\\\frac{\partial}{\partial x_3}\end{pmatrix} \qquad \nabla V = \begin{pmatrix}\frac{\partial}{\partial x_1}\\\frac{\partial}{\partial x_2}\\\frac{\partial}{\partial x_3}\end{pmatrix}\begin{pmatrix}x_1^2\\x_3^2\\x_2^2\end{pmatrix} = \begin{bmatrix}2x_1&0&0\\0&0&2x_2\\0&2x_3&0\end{bmatrix} = \begin{bmatrix}2&0&0\\0&0&2\\0&0&0\end{bmatrix}$$

(ii)
$$\nabla\cdot V = \begin{pmatrix}\frac{\partial}{\partial x_1}\\\frac{\partial}{\partial x_2}\\\frac{\partial}{\partial x_3}\end{pmatrix}\cdot\begin{pmatrix}x_1^2\\x_3^2\\x_2^2\end{pmatrix} = 2x_1 = 2$$

(iii)
$$\nabla\times V = \begin{vmatrix}i&j&k\\\frac{\partial}{\partial x_1}&\frac{\partial}{\partial x_2}&\frac{\partial}{\partial x_3}\\x_1^2&x_3^2&x_2^2\end{vmatrix} = (2x_2-2x_3,\ 0,\ 0) = (2, 0, 0)$$

(iv)
$$\frac{dv}{dx} = J = \begin{bmatrix}2x_1&0&0\\0&0&2x_3\\0&2x_2&0\end{bmatrix} = \begin{bmatrix}2&0&0\\0&0&0\\0&2&0\end{bmatrix} \qquad \frac{dx}{ds} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\frac{dv}{ds} = \frac{dv}{dx}\cdot\frac{dx}{ds} = \begin{bmatrix}2&0&0\\0&0&0\\0&2&0\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}2\\0\\2\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{3}}\\0\\\frac{2}{\sqrt{3}}\end{bmatrix}$$

6 (i) $\hat{e_1} = (1,0,0)$ $\hat{e_2} = (0,1,0)$

$$\varepsilon = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times 10^{-5}$$

change of angle $= 2 \times \hat{e_1} \cdot (\varepsilon \cdot \hat{e_2})$

$$\varepsilon \cdot \hat{e_2} = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times 10^{-5} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \times 10^{-5}$$

$$\hat{e_1} \cdot (\varepsilon \cdot \hat{e_2}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \times 10^{-5} = 4 \times 10^{-5}$$

$$2 \cdot \hat{e_1} \cdot (\varepsilon \cdot \hat{e_2}) = 8 \times 10^{-5}$$

(ii)

$$\varepsilon - \lambda I = \begin{bmatrix} 5-\lambda & 4 & 0 \\ 4 & -1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \qquad \det(\varepsilon - \lambda I) = 0$$

$$(3-\lambda)[(5-\lambda)(-1-\lambda) - 16] = 0$$

$$(3-\lambda)(\lambda-7)(\lambda+3) = 0$$

$\lambda_1 = 3$ $\lambda_2 = 7$ $\lambda_3 = -3$

there are 3 principal strains $-3, 3, 7$

(iii) $\lambda_1 = 3$

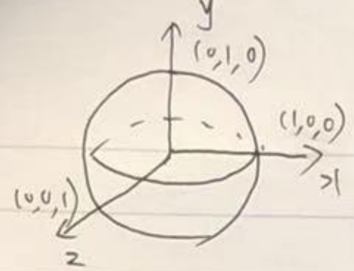
$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ we choose } V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = 7$

$$\begin{bmatrix} -2 & 4 & 0 \\ 4 & -8 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ we choose } V_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$\lambda_3 = -3$

$$\begin{bmatrix} 8 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ we choose } V_3 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$$



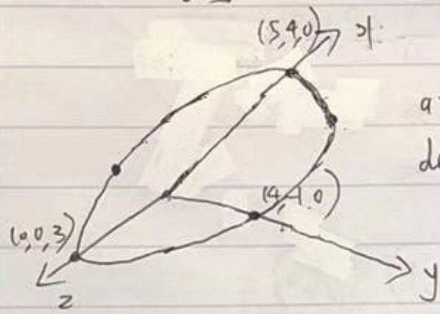
before deformation

$\lambda V = \varepsilon V$

$|V_1| = |V_2| = |V_3| = 1$

$$\lambda_1 \cdot V_1 = 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \lambda_2 \cdot V_2 = 7 \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \quad \lambda_3 V_3 = -3 \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$\varepsilon \cdot \hat{e}_1 = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \quad \varepsilon \cdot \hat{e}_2 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \quad \varepsilon \cdot \hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$



after deformation

(7)(a) for general Navier-stokes Equation

$$\rho \frac{Du}{Dt} = -\nabla P + \nabla \cdot \tau + \rho g$$

In our case $\nabla \cdot \tau = \nabla P \qquad -\nabla P + \nabla \cdot \tau = 0$

this means that $\frac{Du}{Dt} = 0, \ \rho g = 0$

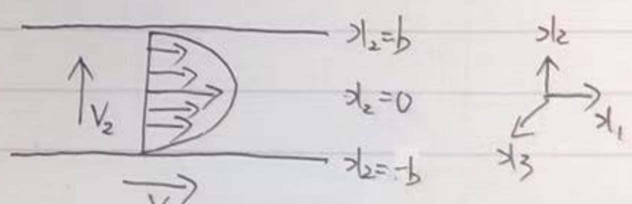
In our case, fluids are incompressible, $\nabla \cdot u = 0$

$$\nabla \cdot \tau = \nabla \cdot (2\eta D) = \eta \nabla \cdot (\nabla u + \nabla u^T)$$
$$= \eta \left[\frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right]$$
$$= \eta \left[\frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial}{\partial x_j} \frac{\partial v_j}{\partial x_i} \right]$$
$$= \eta \left[\frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial}{\partial x_i} \cancel{\frac{\partial v_j}{\partial x_j}} \right] \quad \text{since } \nabla \cdot u = 0, \text{ then } \frac{\partial v_j}{\partial x_j} = 0$$
$$= \eta \frac{\partial^2 v_i}{\partial x_j^2} = \eta \nabla^2 V$$

Thus the force balance $\quad -\nabla P + \eta \nabla^2 V = 0$

(b) $\nabla P = \eta \nabla^2 V = \eta \frac{\partial^2 v_1}{\partial x_2^2}$

$$\frac{dP}{dx} = \eta \frac{\partial^2 v_1}{\partial x_2^2} = -\frac{\Delta P}{L}$$



$$\frac{\partial^2 v_1}{\partial x_2^2} = -\frac{\Delta P}{\eta L} \qquad \frac{dv_1}{\partial x_2} = -\frac{\Delta P}{\eta L} x_2 + C$$

when $x_2 = 0 \quad \frac{dv_1}{dx_2} = 0 \quad C = 0$

$$v_1 = -\frac{\Delta P}{2\eta L} x_2^2 + D$$

when $x_2 = b \quad v_1 = 0 \quad D = -\frac{\Delta P}{2\eta L} b^2$

$$v_1(x_2) = \frac{1}{2\eta} \frac{\Delta P}{L} (b^2 - x_2^2)$$

(c) $v_1 = v(x_2)$ $V_2 = 0$ $V_3 = 0$

we choose $x(t) = (x_1(\xi,t), x_2(\xi,t), x_3(\xi,t))$

$$\frac{\partial x}{\partial t} = (\dot{x}_1(\xi,t), \dot{x}_2(\xi,t), \dot{x}_3(\xi,t))$$

$$= (V_1, V_2, V_3)$$

Thus $X_1(\xi,t) = \xi_1 + V_1 t = \frac{1}{2\eta}\frac{\Delta P}{L}(b^2 - x_2^2)t + \xi_1 = \frac{1}{2\eta}\frac{\Delta P}{L}(b^2 - \xi_2^2)t + \xi_1$

$X_2(\xi,t) = \xi_2$

$x_3(\xi,t) = \xi_3$

$x_1(0) = \xi_1$ $x_1(1) = \frac{1}{2\eta}\frac{\Delta P}{L}(b^2 - \xi_2) + \xi_1$ $x_1(2) = \frac{1}{\eta}\frac{\Delta P}{L}(b^2 - \xi_2) t \xi_1$

$x_2(0) = \xi_2$ $x_2(1) = \xi_2$ $x_2(2) = \xi_2$

$x_3(0) = \xi_3$ $x_3(1) = \xi_3$ $x_3(2) = \xi_3$

choose $(\xi_1, \xi_2, \xi_3) \to (0,0,0)$ $(0,1,0)$ $(0,0,1)$ $(1,0,0)$

$(0,1,1)$ $(1,1,0)$ $(1,0,1)$ $(1,1,1)$