ACSE-2

Lecture 8

Conservation Equations & Rheology

Outline

- Conservation equations
- Energy equation
- Rheology
- Elasticity and Wave Equation
- Newtonian Viscosity and Navier Stokes

Learning Objectives

- Learn main conservation equations used in continuum mechanics modelling and understand what different terms in these equations represent
- Be able to solve conservation equations for basic analytical solutions given boundary/initial conditions.
- Understand basic properties of elastic and viscous rheology and understand how the choice of rheology leads to different forms of the momentum conservation equation
- Using tensor analysis to obtain relations between the main isotropic elastic parameters

Continuum Mechanics Equations

General:

1. Kinematics – describing deformation and velocity without considering forces
2. Dynamics – equations that describe force balance, conservation of linear and angular momentum
3. Thermodynamics – relations temperature, heatflux, stress, entropy

Material-specific

4. Constitutive equations – relations describing how material properties vary as a function of T,P, stress,…. Such material properties govern dynamics (e.g., ***density***), response to stress (***viscosity***, ***elastic parameters***), heat transport (***thermal conductivity***, ***heat capacity***)

Conservation equations

- Conservation of mass
  - Kinematics

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

- Conservation of linear momentum
  - Dynamics
  - Newton's second law

$$\rho \mathbf{a} = \nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}$$

- Conservation of energy
  - First law of thermodynamics

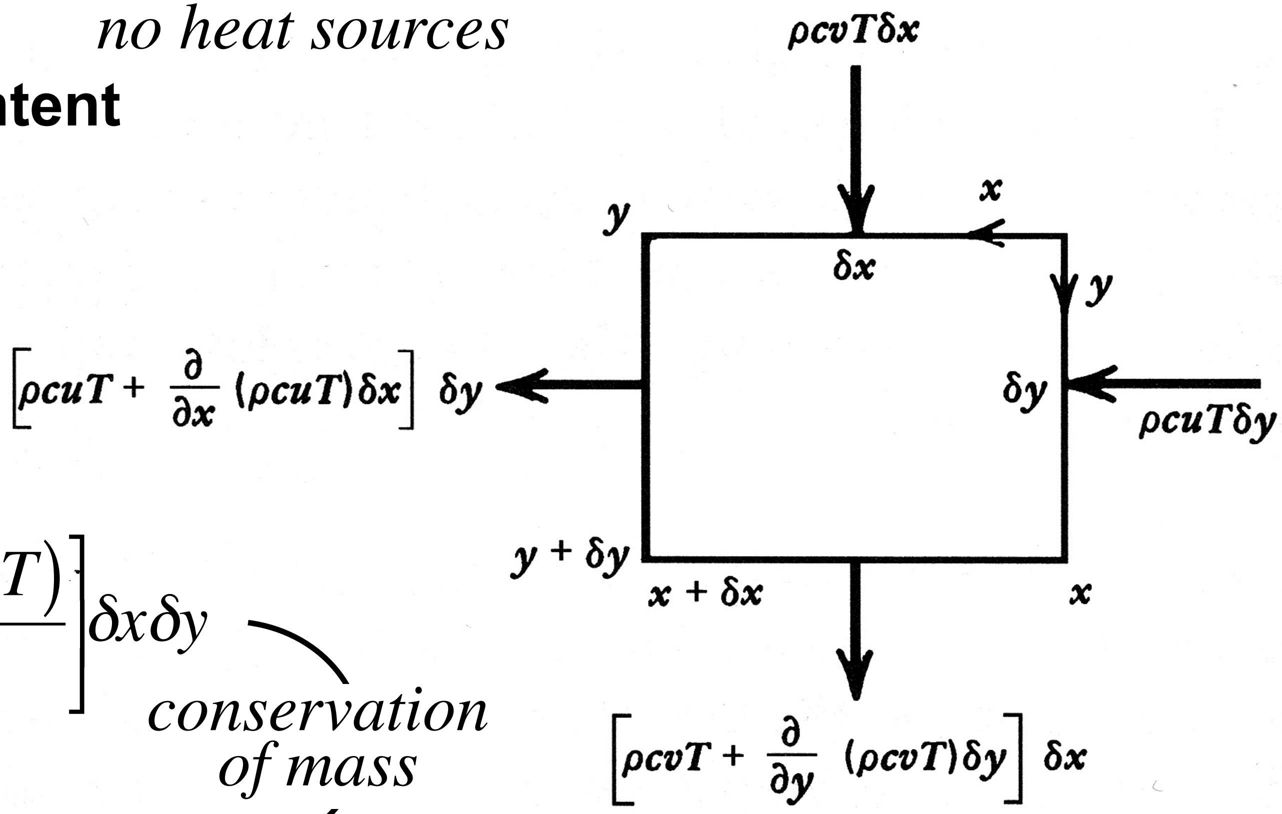
$$\frac{D(K+U)}{Dt} = W + Q$$

K- kinetic energy, U- internal energy, W – power input, Q – heat input

2-D energy equation

Spatial, constant ρ, C_P, k, incompressible

no heat sources



- **Change in heat content**

$$\frac{\partial(\rho C_P T)}{\partial t}\delta x \delta y$$

- **Advection**

$$\left[\frac{\partial(\rho C_P u T)}{\partial x} + \frac{\partial(\rho C_P v T)}{\partial y}\right]\delta x \delta y$$

conservation of mass

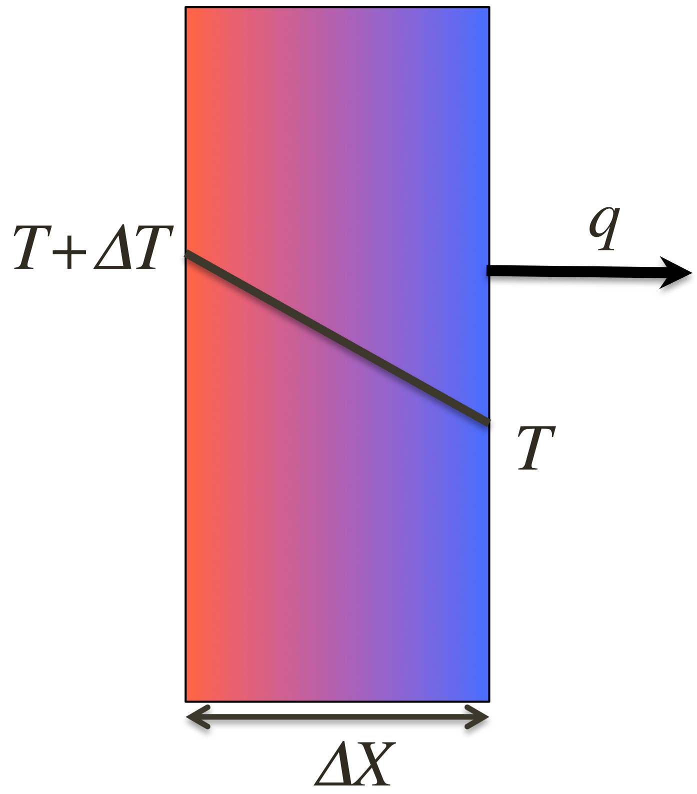
How does this simplify?

- **Conduction**

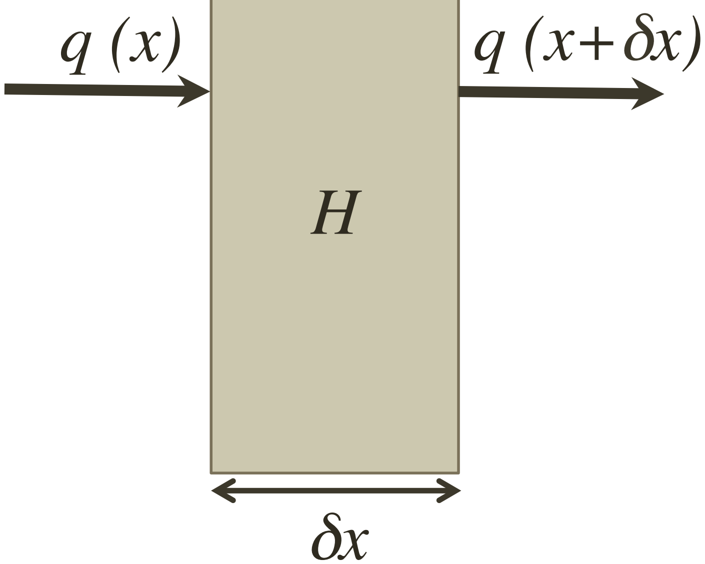
Fourier's Law for conduction

$$q = -k \frac{dT}{dx}$$

- ***Heat flux***, q, = heat/area = energy/time/area, unit: J/s/m^2 = W/m^2
- Heat flux proportional to ***temperature gradient***
- Minus sign because heat flows from hot to cold
- Constant of proportionality: ***thermal conductivity***, k, unit: W/m/K



1-D Steady State Conduction



- **net heat flow/unit area/unit time =**

  $q(x+\delta x)\text{-}q(x)$

$$q(x+\delta x) = q(x) + \delta x\frac{dq}{dx} + \ldots.$$

$$q(x+\delta x) - q(x) \approx \delta x\frac{dq}{dx}$$

$$\delta x\frac{dq}{dx} = \delta x\left[\frac{d}{dx}\left(-k\frac{dT}{dx}\right)\right]$$

$$\delta x\frac{dq}{dx} = \delta x\left[-k\frac{d^2T}{dx^2}\right] \qquad \textit{for constant } k$$

- **heat produced** = $\rho H\delta x = A\delta x$

  H - heat production rate/unit mass (W/kg)

  A – heat production/unit volume (W/m^3)

$$\boxed{-k\frac{d^2T}{dx^2} = \rho H = A}$$

2-D energy equation

Spatial, constant ρ, C_P, k, , incompressible

- **Change in heat content**

$$\frac{\partial(\rho C_P T)}{\partial t}\delta x \delta y$$

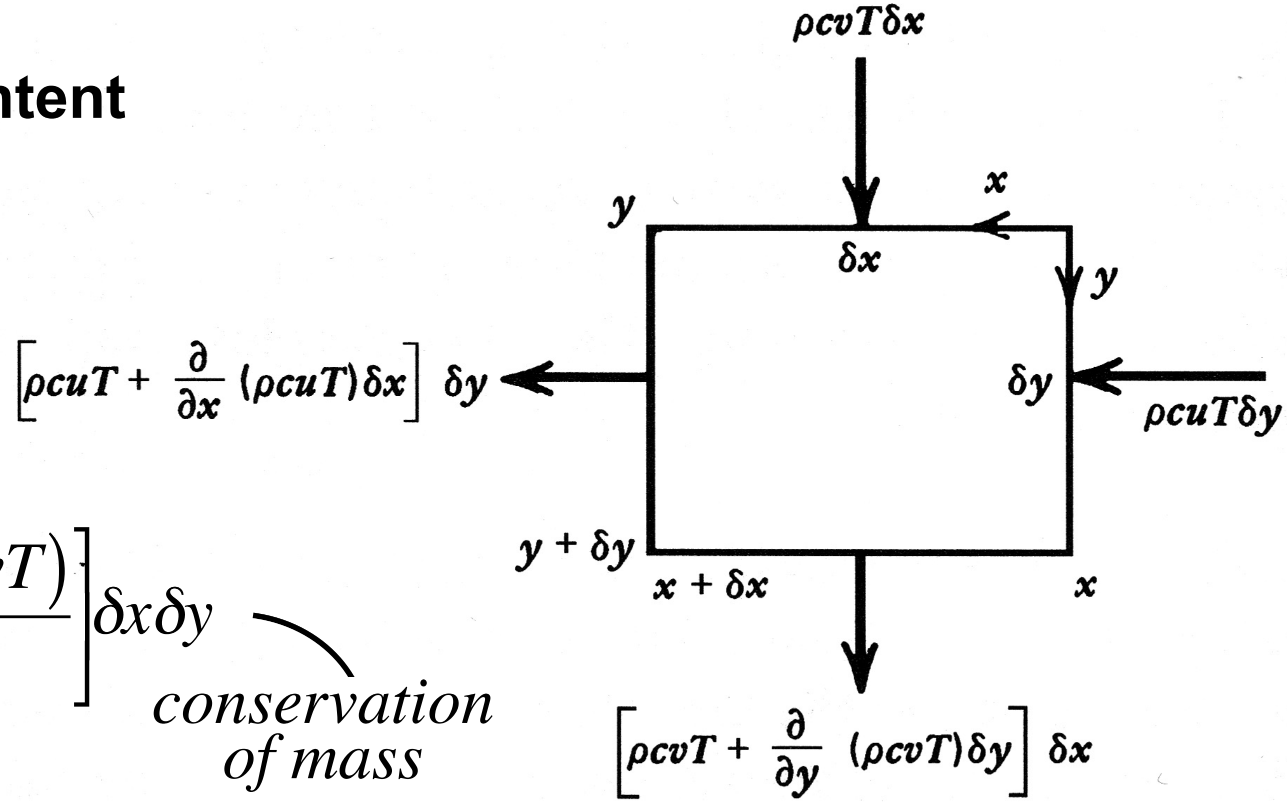
- **Advection**

$$\left[\frac{\partial(\rho C_P uT)}{\partial x}+\frac{\partial(\rho C_P vT)}{\partial y}\right]\delta x \delta y$$

conservation of mass

- **Conduction**

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)\delta x \delta y$$



$$\rho C_P\left[\frac{\partial T}{\partial t}+u\cdot\nabla T\right]=k\nabla^2 T$$

Energy equation

$$\frac{D(K+U)}{Dt} = W + Q$$

- **Material derivative internal heat**

$$\rho C_P\left[\frac{\partial T}{\partial t} + u \cdot \nabla T\right] = \rho C_P \frac{DT}{Dt} \Rightarrow \frac{D\left(\rho C_P T\right)}{Dt}$$

Allowing for spatial variations of material parameters

- **Heat input**

$$k\nabla^2 T \Rightarrow \nabla \cdot k\nabla T \qquad +A$$

Conduction *Internal heat production*

- **Work done**

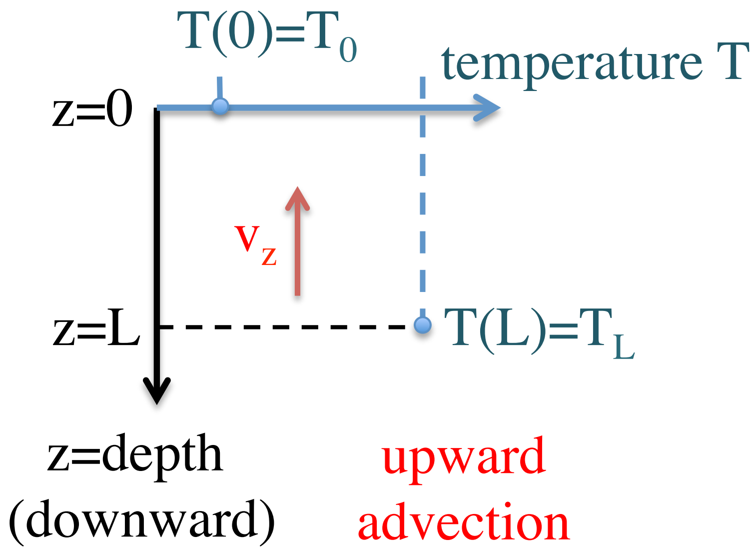
$\Rightarrow$ Changes in *motion* (kinetic energy) and *internal deformation*

Net effect of $W - \dfrac{DK}{Dt}$ becomes $\boldsymbol{\sigma} : \mathbf{D}$

D – strain rate

1-D advection-diffusion solution

$$-v_z \frac{\partial T}{\partial z} = \kappa \frac{\partial^2 T}{\partial z^2}$$



Take $f(z) = \dfrac{\partial T}{\partial z}$ and $c = \dfrac{-v_z}{\kappa}$

Then $\dfrac{\partial f}{\partial z} = -cf(z)$

$\Rightarrow$ This yields $f(z) = f(0)\mathrm{e}^{-cz}$, i.e. $\dfrac{\partial T}{\partial z}(z) = A\mathrm{e}^{-v_z z/\kappa}$

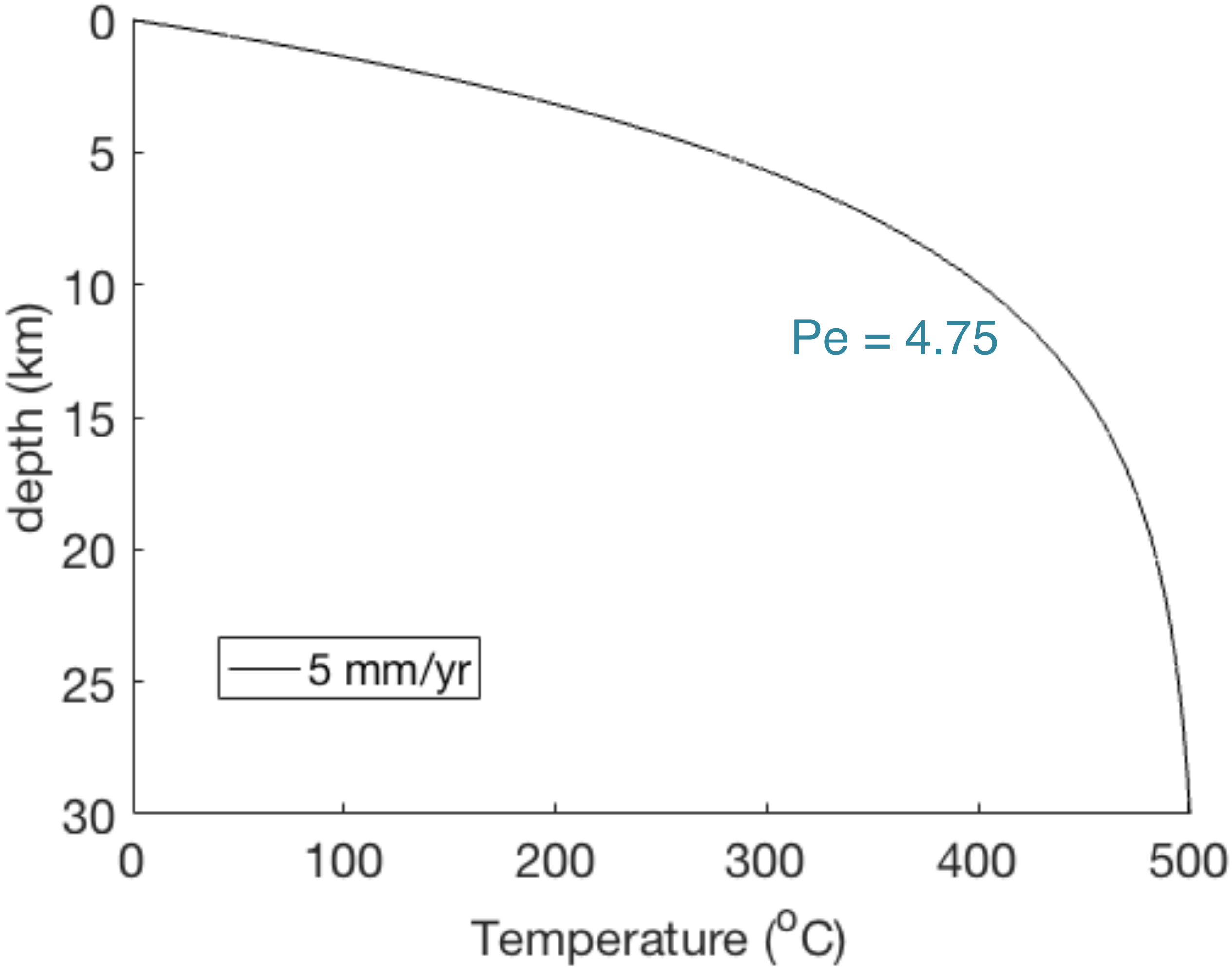
$$T(z) = B - \frac{A}{v_z/\kappa}\mathrm{e}^{-v_z z/\kappa}$$

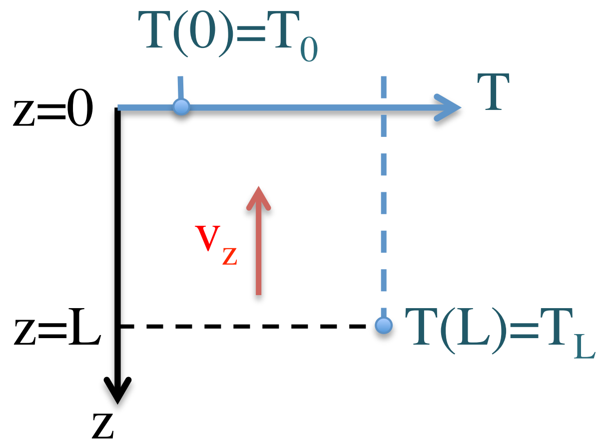
where A, B are integration constants

For constant temperature boundary conditions at z=0 and z=L

$\Rightarrow$ Integration gives: $T(z) = T_L \left[\dfrac{1-\mathrm{e}^{-v_z z/\kappa}}{1-\mathrm{e}^{-v_z L/\kappa}}\right]$

1-D advection-diffusion solution





$$T(z) = T_L\left[\frac{1-e^{-v_z z/\kappa}}{1-e^{-v_z L/\kappa}}\right]$$

advdiff.ipynb

Peclet number, measure of relative importance advection/diffusion

$$Pe = \frac{v_z L}{\kappa} = \frac{[(m/s)m]}{[m^2/s]}$$

Energy equation

conservation of heat

I	II	III	IV	V	VI
$D(\rho C_p T)/Dt =$	$\nabla \cdot k \nabla T$	$+ A$	$+ \boldsymbol{\sigma} \mathbf{:D}$	$(+ \alpha T \mathbf{v} \cdot \nabla P$	$\ldots)$

I - change in temperature with time
II - heat transfer by conduction (and radiation)
III - heat production (including latent heat)
IV - heat generated by internal deformation

V - heat generated by adiabatic compression
VI - other heat sources, e.g. latent heat

Conservation equations

- Conservation of mass $$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0$$
- Conservation of linear momentum $$\rho\frac{D\mathbf{v}}{Dt} = \nabla \cdot \underline{\underline{\sigma}} + \mathbf{f}$$
- Conservation of angular momentum: $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{\mathrm{T}}$
- Conservation of energy $$\frac{D\left(\rho C_P T\right)}{Dt} = \nabla \cdot k\nabla T + A + \boldsymbol{\sigma} : \mathbf{D}$$
- Entropy inequality *Rate of entropy increase of a particle always ≥ entropy supply*

  Which law is this?

Continuum Mechanics Equations

General:

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Material-specific

4. Constitutive equations – relations describing how material properties vary as a function of T,P, stress,…. Such material properties govern dynamics (e.g., ***density***), response to stress (***viscosity***, ***elastic parameters***), heat transport (***thermal conductivity***, ***heat capacity***)

Thermal parameters

Can you name 4 material parameters that affect temperatures or how material responds to changes in temperature

Each of these may depend on T, P, phase, composition,…

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Rheology

$$\boxed{\text{deformation } (\mathbf{E}) = \textit{rheology} \cdot \text{stress } (\boldsymbol{\sigma})}$$

material response to stress, depends on material, P,T, time, deformation history, environment (volatiles, water)

- *elastic*
- *viscous*
- *brittle*
- *plastic*

• experiments under simple stress conditions
⇒ strain evolution under constant stress, stress-strain rate diagrams

• thermodynamics + experimental parameters

• ab-initio calculations

Recap Fluid - Solid

- What is a solid?

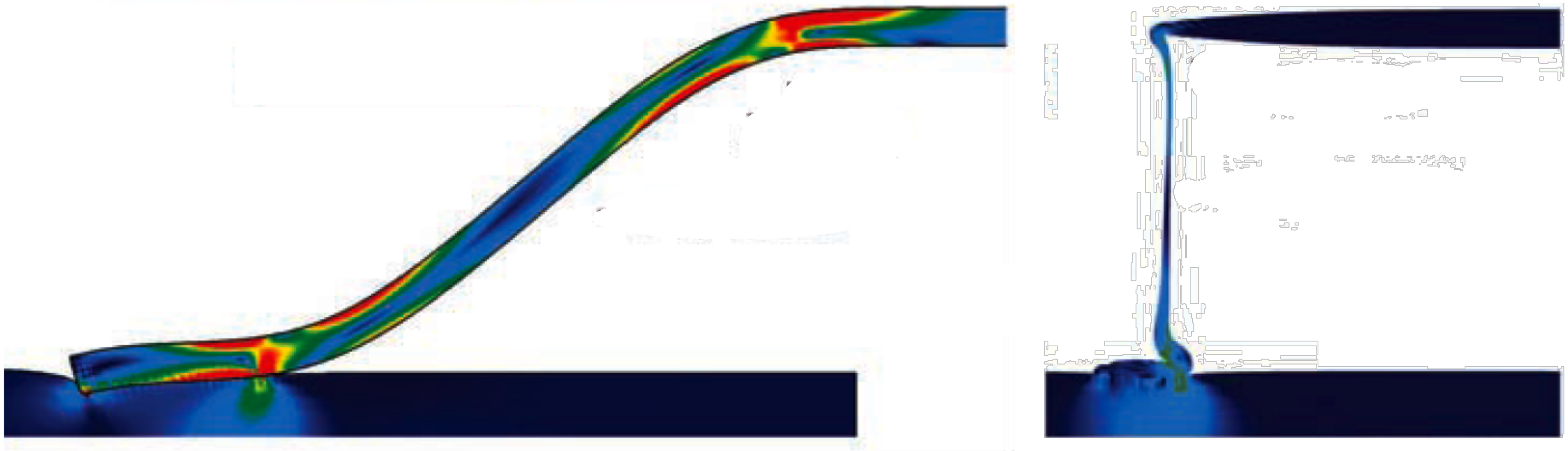
  A solid acquires finite deformation under stress

  stress σ ~ *strain* ε

- What is a fluid?

  A material that flows in response to applied stress

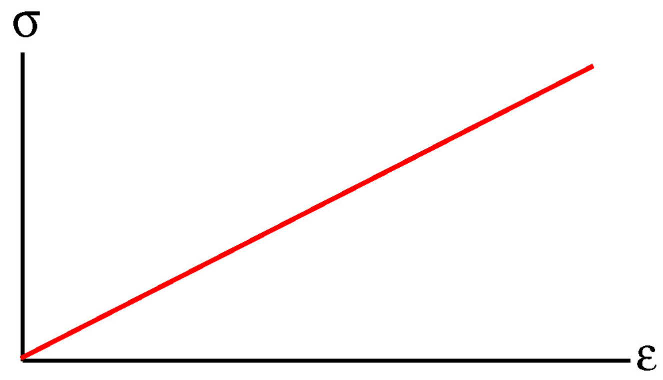
  stress σ ~ *strain rate* $D\varepsilon/Dt$

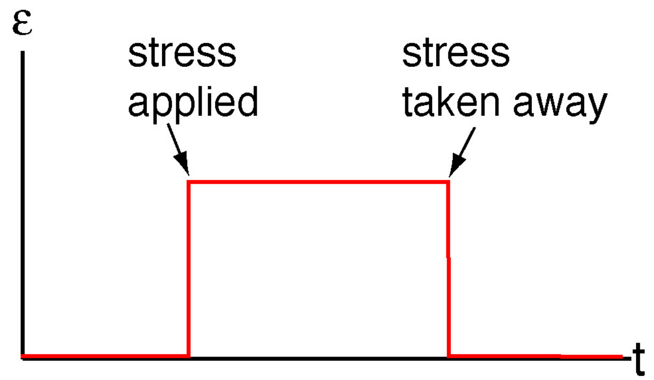


Figures from Funiciello et al. (2003a)

Elasticity

- linear response to load applied
- instantaneous
- completely recoverable
- below threshold (yielding) stress
- *dominates behaviour of coldest part of the lithosphere on time scales of up to 100 m.y. -> fault loading*
- *on time scale of seismic waves the whole Earth is elastic*





- $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ - Hooke's law
  C_{ijkl} - rank 4 elasticity tensor
  3^4 elements, up to 21 independent

ε=E, to avoid confusion with Young's modulus E

Elasticity tensor

C_{ijkl} $3^4=81$ elements (for n=3)

- symmetry of σ_{ij} and ε_{kl}

  $\Rightarrow$ only 36 independent elements

Why 36?

- conservation of elastic energy $U=\boldsymbol{\sigma}:\varepsilon=\mathbf{C}:\varepsilon:\varepsilon \geq 0$

  $\Rightarrow C_{ijkl}=C_{klij}$

  $\Rightarrow$ only 21 independent elements - most general form of **C**

- other symmetries further reduce the number of independent elements

Elasticity tensor

- for example for *isotropic* media

  Only 2 independent elements (λ, μ):

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl}+\alpha\delta_{ik}\delta_{jl}\varepsilon_{kl}+\beta\delta_{il}\delta_{jk}\varepsilon_{kl}$$
$$= \lambda\delta_{ij}\varepsilon_{kk}+\alpha\varepsilon_{ij}+\beta\varepsilon_{ji}$$
$$= \lambda\delta_{ij}\theta+(\alpha+\beta)\varepsilon_{ij}$$

$$\Rightarrow \sigma_{ij}= \lambda\theta\delta_{ij}+2\mu\varepsilon_{ij}$$

What is isotropic?

3 isotropic rank

4 tensors:

$\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$

Hooke's law for isotropic material: 2 independent coefficients

Lamé constants

λ and **μ** : $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

Bulk and shear modulus

K and **μ=G** : $\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$

where: $p = \dfrac{\sigma_{kk}}{3} = \left(\lambda + \dfrac{2}{3}\mu\right)\theta = K\theta$ *hydrostatic*

$\sigma'_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - p\delta_{ij} = 2\mu\varepsilon'_{ij}$ *deviatoric*

Young's modulus and Poisson's ratio

E and **ν** : $E = \sigma_{11}/\varepsilon_{11}$, $\nu = -\varepsilon_{33}/\varepsilon_{11}$ (uniaxial stress)

Determine in problem set

Wave equation

For infinitesimal deformation:
spatial coordinates ≈ material coordinates

v_i (spatial) $\approx \partial u_i/\partial t$
a_i (spatial) $\approx \partial v_i/\partial t = \partial^2 u_i/\partial t^2$

Equation of motion: $f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$ (1)

Elastic rheology: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$ (2)

Substitute (2) in (1) if (infinitesimal) deformation is consequence of force balance

Wave equation

Equation of motion: $f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

Elastic rheology: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

$$\partial\sigma_{ji}/\partial x_j = \lambda\partial\varepsilon_{kk}/\partial x_i + \mu\partial(\partial u_i/\partial x_j + \partial u_j/\partial x_i)/\partial x_j$$

$$= \lambda\partial(\partial u_k/\partial x_k)/\partial x_i + \mu\partial^2 u_i/\partial^2 x_j + \mu\partial(\partial u_j/\partial x_j)/\partial x_i$$

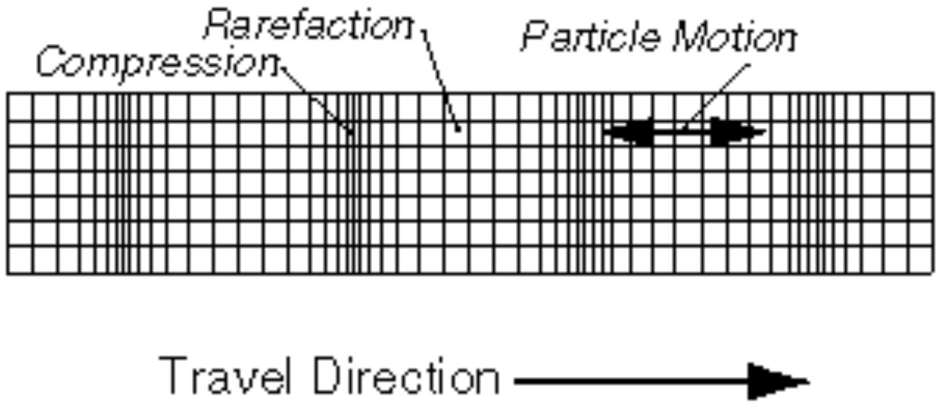
$\nabla \cdot \sigma \;=$ Write vector equation

Using: $\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times\nabla\times\mathbf{u}$

=> $\boxed{\rho\partial^2\mathbf{u}/\partial t^2 = \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times\nabla\times\mathbf{u}}$

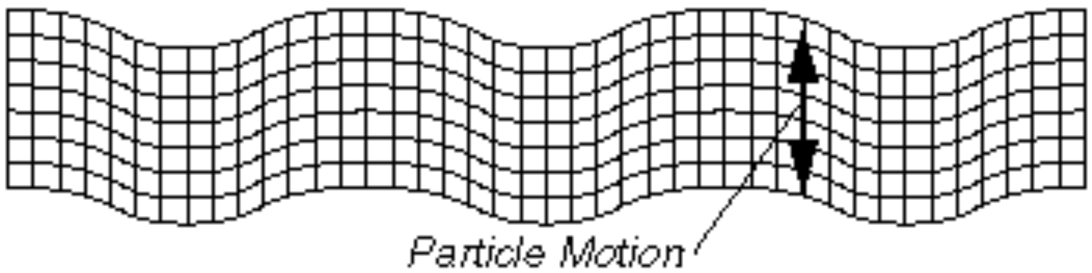
what type of deformation do the two terms represent?

P wave



Rarefaction
Compression
Particle Motion
Travel Direction

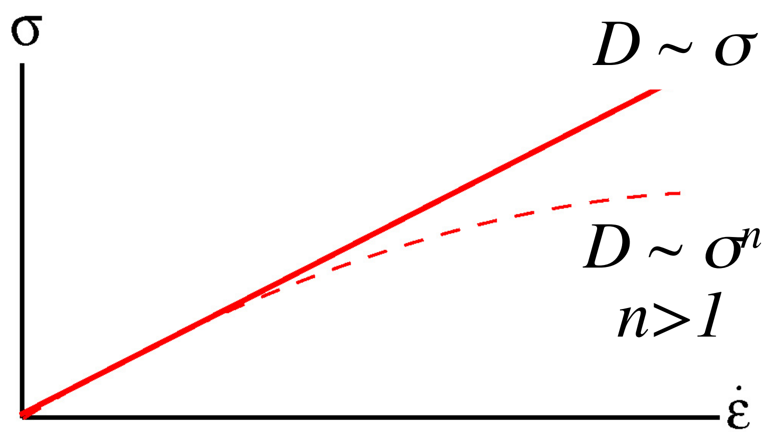
S wave

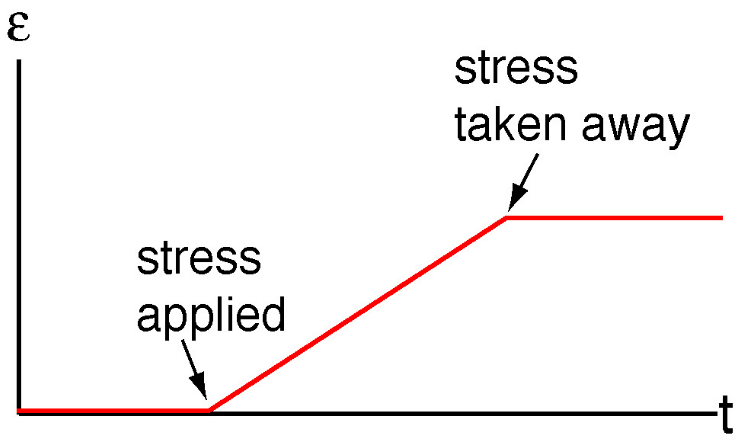


Particle Motion

Viscous Flow

- steady state flow at constant stress
- permanent deformation
- linear (Newtonian) or non-linear (e.g., Powerlaw) relation between strain rate and stress
- isotropic stress does not cause flow
- *on timescales > years lower lithosphere and mantle deform predominantly viscously -> plate motions, postseismic deformation, but also glaciers, magmas*





Hydrostatics

Fluids can not support shear stresses

i.e. if in rest/rigid body motion: $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \lambda \hat{\mathbf{n}}$

and this normal stress is the same on any plane: $\boldsymbol{\sigma} = -p\mathbf{I}$

p is *hydrostatic pressure*

In force balance: $\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0$

$$-\nabla p = -\mathbf{f}$$

In gravity field $\dfrac{\partial p}{\partial z} = \rho g \quad \Rightarrow \quad p_2 - p_1 = \rho g h$

Newtonian Fluids

In general motion: $\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}'$

In Newtonian fluids,
deviatoric stress varies ***linearly*** with ***strain rate***, $\mathbf{D}$
$D_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$

For *isotropic*, Newtonian fluids, 2 *material parameters*:

Viscous stress tensor $\sigma'_{ij} = -\zeta D_{kk}\delta_{ij} + 2\eta D_{ij}$

where ζ is ***bulk viscosity*** and η (***shear***) ***viscosity***, $\Delta = D_{kk}$

$$\boldsymbol{\sigma} = \left(-p + \varsigma\Delta\right)\mathbf{I} + 2\eta\mathbf{D}$$

p not always mean normal stress: $\sigma_{kk} = -3p + (3\zeta + 2\mu)D_{kk}$

Consider a Newtonian shear flow with velocity field $v_1(x_2)$, v_2=v_3=0

What is $\mathbf{D}$? What is $\boldsymbol{\sigma}$?

Navier-Stokes for incompressible Newtonian Flow

For incompressible fluids Δ=0, so that: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $$\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}$$

Show that: $$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

Assuming constant η

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \eta \nabla^2 \mathbf{v}$$

Navier-Stokes for incompressible Newtonian Flow

For incompressible fluids Δ=0, so that: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \dfrac{D\mathbf{v}}{Dt}$

Navier Stokes equation of motion:

$$-\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{f} = \rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right]$$

Assuming constant η

Together with continuity, 4 equations, 4 unknowns (p, v_x, v_y, v_z)

$$\nabla \cdot \mathbf{v} = 0$$

Navier-Stokes for compressible Newtonian Flow

$$\boldsymbol{\sigma} = \left(-p + \varsigma\Delta\right)\mathbf{I} + 2\eta\mathbf{D}$$

$$\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho\frac{D\mathbf{v}}{Dt}$$

Navier Stokes equation of motion:

$$-\nabla p + \left(\zeta + \eta\right)\nabla\mathit{\Delta} + \eta\nabla^2\mathbf{v} + \mathbf{f} = \rho\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right]$$

Assuming constant ζ, η

+ Conservation of mass:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$$

6 equations
6 unknowns
$(p, v_x, v_y, v_z, \rho, T)$

+ Energy equation

+ Equation of state for ρ(T,p)

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Outline

- Conservation equations
- Energy equation
- Rheology
- Elasticity and Wave Equation
- Newtonian Viscosity and Navier Stokes

More reading on the topics covered in this lecture can be found in, for example: Lai et al. Ch 4.14-4-16, 6.18, Ch 5.1-5.6, Ch 6.1-6.7; Reddy parts of Ch 5 & Ch 6

Outline of course

- **1.** Mathematical essentials – *Matthew Piggott*
- **2.** Linear Algebra I – *Matthew Piggott*
- **3.** Linear Algebra II, ODEs– *Matthew Piggott*
- **4.** Verifying models– *Matthew Piggott*
- **5.** Vector and Tensor Calculus - *Saskia Goes*
- **6.** Stress principles - *Saskia Goes*
- **7.** Kinematics and strain - *Saskia Goes*
- **8.** Rheology and conservation equations - *Saskia Goes*
- **9.** Potential flow - *Stephen Neethling*
- **10.** Fluid flow I - *Stephen Neethling*
- **11.** Fluid flow II - *Stephen Neethling*
- **12.** Wave propagation - *Adrian Umpleby*

Exam

- 1.5 hours
- Analytical/essay-style questions
- From first 8 lectures
- Questions as shorter non-numerical ones from the problem sets/lecture examples
- An example exam will be released before Christmas break
- Only most basic equations expected to remember. Others will be given.