

# **Dynamic Programming and Applications**

Stochastic Dynamic Programming in Continuous Time

Lectures 5 – 6

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# Outline

Part 1: Stochastic processes, Brownian motion, and stochastic differential equations

1. Stochastic processes in continuous time
2. Continuous time Markov chains
3. Brownian motion
4. Diffusion processes
5. Ito's Lemma
6. Poisson processes
7. The generator of a stochastic process

# Outline

## Part 2: Optimization with stochastic dynamics

1. Stochastic neoclassical growth model
2. Stochastic neoclassical growth with diffusion process
3. Stochastic neoclassical growth with Poisson process

# Outline

## Part 3: Applications

1. Consumption-savings with stochastic income fluctuations
2. Firm profit maximization
3. Capital investment with adjustment cost
4. Investing in stocks
5. Real business cycles
6. AK technology

# Part 1: Stochastic Processes

# 1. Stochastic processes in continuous time

**Definition.** A **stochastic process** is a time-indexed sequence of random variables.

- A random variable maps an “event” into a scalar, a stochastic process maps an event into a path
- Formally, an event is  $\omega \in \Omega$  and a random variable  $X : \Omega \rightarrow \mathbb{R}$  (learn some basic measure theory!)
- A stochastic process is a sequence  $\{X_t\}_{t \geq 0}$  such that each  $X_t$  is a random variable. In discrete time  $t \in \{0, 1, \dots\}$  and in continuous time  $t \in [0, \infty)$
- We can also think of a stochastic process as a random variable, but one that maps into a different space—a function space!

**Definition.** Let  $X = \{X_t\}_{t \geq 0}$  be a sequence of random variables taking values in a finite or countable state space  $\mathcal{X}$ . Then  $X$  is a *continuous-time Markov chain* if it satisfies the *Markov property*: For any sequence  $0 \leq t_1 < t_2 < \dots < t_n$  of times

$$\mathbb{P}(X_{t_n} = x \mid X_{t_1}, \dots, X_{t_{n-1}}) = \mathbb{P}(X_{t_n} = x \mid X_{t_{n-1}})$$

**Definition.** Process  $X$  is **time-homogeneous** or **stationary** if the conditional probability does not depend on the current time, i.e., for  $x, y \in \mathcal{X}$ :

$$\mathbb{P}(X_{t+s} = x \mid X_s = y) = \mathbb{P}(X_t = x \mid X_0 = y)$$

- Alternatively: If all possible joint distributions are independent of time  $t$ .

- The *transition density* of process  $X$  is denoted  $p(t, x | s, y)$  and is defined as

$$\mathbb{P}(X_t \in A | Y_s = y) = \int_A p(t, x | s, y) dx$$

for any (Borel) set  $A \subset \mathcal{X}$ . In words:  $p(t, x | s, y)$  is the probability (density) that process  $X_t$  ends up at  $X_t = x$  at time  $t$  if it started at  $X_s = y$  at time  $s$

- *Conditional expectation* can be written as:  $\mathbb{E}[f(X_t) | X_0 = y] = \int p(t, x | 0, y) f(x) dx$

**Definition.** A process  $X$  is a **martingale** if it satisfies

$$\mathbb{E}\left[X_{t+s} \mid \text{information available at } t\right] = X_t$$

- The most important martingale in macroeconomics: Suppose  $\beta R_t = 1$  then

$$\mathbb{E}_t\left[u'(C_{t+s})\right] = u'(C_t).$$

- Efficient markets hypothesis in finance  $\approx \mathbb{E}_t(P_{t+s}) = P_t$
- A random walk is a martingale (but not every martingale is a random walk)

## Example:

- Consider the two-state employment process  $z_t \in \{z^L, z^H\}$  with transition rates  $\lambda^{LH}$  (from L to H) and  $\lambda^{HL}$  (from H to L)
- The associated transition matrix (*generator*) is

$$\mathcal{A}^z = \begin{pmatrix} -\lambda^{LH} & \lambda^{LH} \\ \lambda^{HL} & -\lambda^{HL} \end{pmatrix}$$

- Interpretation: households transition *out of* state  $i$  at rate  $\lambda^{ij}$
- Notice: In discrete time, Markov transition matrix rows sum to 1. Here, rows sum to 0 (*mass preservation*)

## 2. Brownian motion

- Brownian motion is the most
- Einstein (1905) uses Brownian motion to model motion of particles
- We will introduce / discuss Brownian motion from 3 perspectives

## Brownian motion: perspective #1

- Consider a process  $X(t)$ . Every  $\Delta$  time step, the process either goes up or down by  $h$

$$\Delta X \equiv X(t + \Delta) - X(t) = \begin{cases} +h & \text{with probability } p \\ -h & \text{with probability } q = 1 - p \end{cases}$$

- Then we have:

$$\mathbb{E}(\Delta X) = ph - qh = (p - q)h$$

$$\mathbb{E}((\Delta X)^2) = ph^2 + qh^2 = h^2$$

$$Var(\Delta X) = \mathbb{E}[\Delta X - \mathbb{E}(\Delta X)]^2 = 4pqh^2$$

- Notice that  $X(t) - X(0)$  is a binomial random variable with

$$\mathbb{E}[X(t) - X(0)] = n(p - q)h = t(p - q)\frac{h}{\Delta}$$

$$Var[X(t) - X(0)] = n4pqh^2 = t4pq\frac{h^2}{\Delta}$$

where  $n = t/\Delta$  is the number of jumps in interval  $[0, t]$

- Next, let

$$h = \sigma\sqrt{\Delta} \quad \text{and} \quad p = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right]$$

- This implies

$$(p - q) = \frac{\mu}{\sigma}\sqrt{\Delta}$$

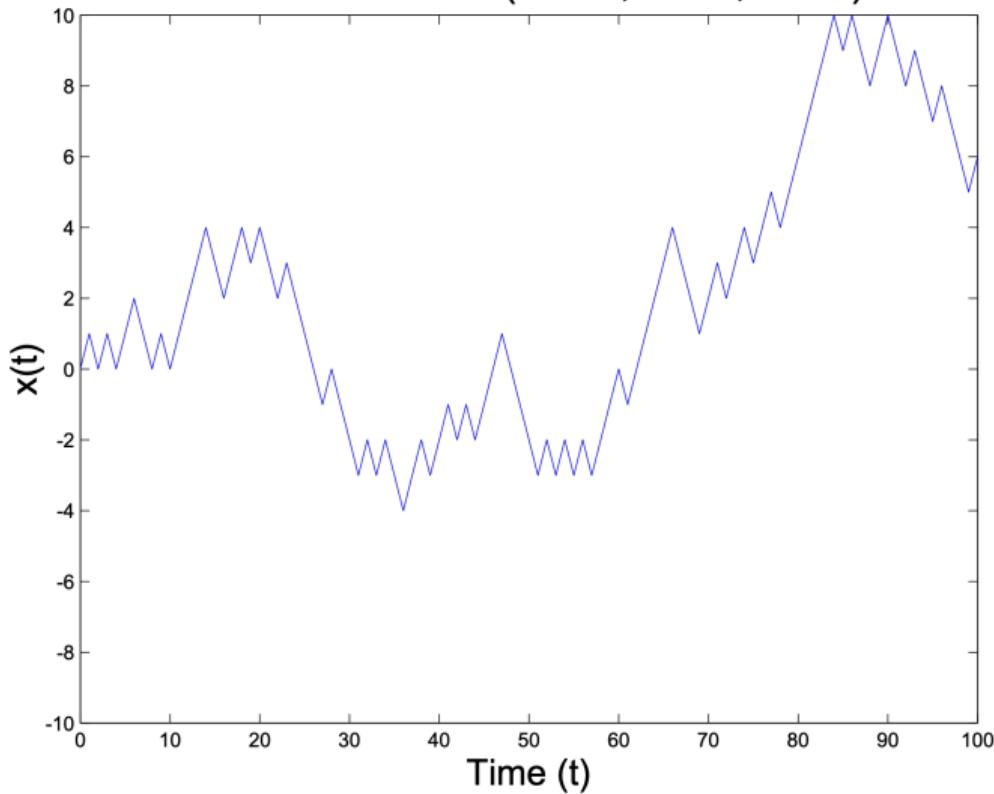
$$\mathbb{E}[X(t) - X(0)] = t\frac{\mu}{\sigma}\sqrt{\Delta}\sigma\frac{\sqrt{\Delta}}{\Delta} = \mu t$$

$$Var[X(t) - X(0)] \xrightarrow{\text{as } \Delta \rightarrow 0} \sigma^2 t$$

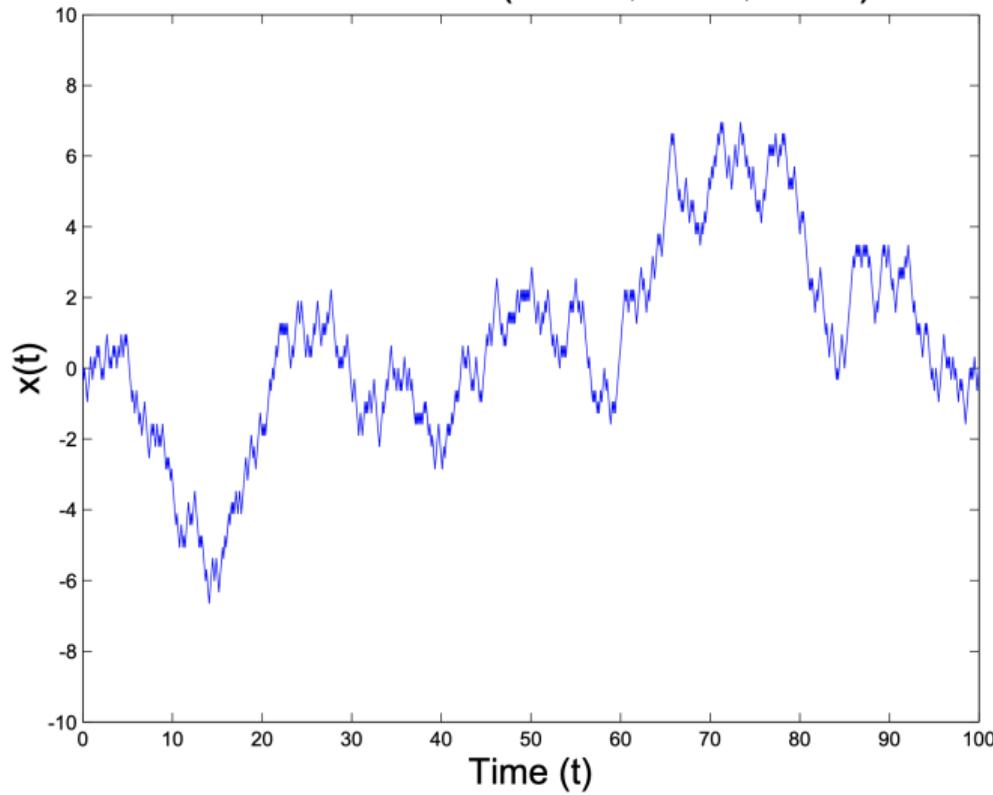
## Implications:

- Vertical movements proportional to  $\sqrt{\Delta}$  not  $\Delta$
- Convergence in distribution:  $X(t) - X(0) \rightarrow^D \mathcal{N}(\mu t, \sigma t)$  since Binomial  $\rightarrow^D$  Normal
- Distance traveled (length of curve) during  $t \in [0, 1]$  is  $= nh = \frac{1}{\Delta} \sigma \sqrt{\Delta} = \sigma \frac{1}{\sqrt{\Delta}} \rightarrow \infty$
- Time derivative  $\mathbb{E} \frac{dX}{dt}$  doesn't exist:  $\frac{\Delta X}{\Delta} = \frac{\pm \sigma \sqrt{\Delta}}{\Delta} = \frac{\pm \sigma}{\sqrt{\Delta}} \rightarrow \infty$
- $\frac{\mathbb{E} \Delta X}{\Delta} = \frac{\mu h^2 / \sigma^2}{\Delta} = \mu$  so we can write  $\mathbb{E}(dX) = \mu dt$
- $\frac{Var(\Delta X)}{\Delta} \rightarrow \sigma^2$  so we can write  $Var(dX) = \sigma^2 dt$

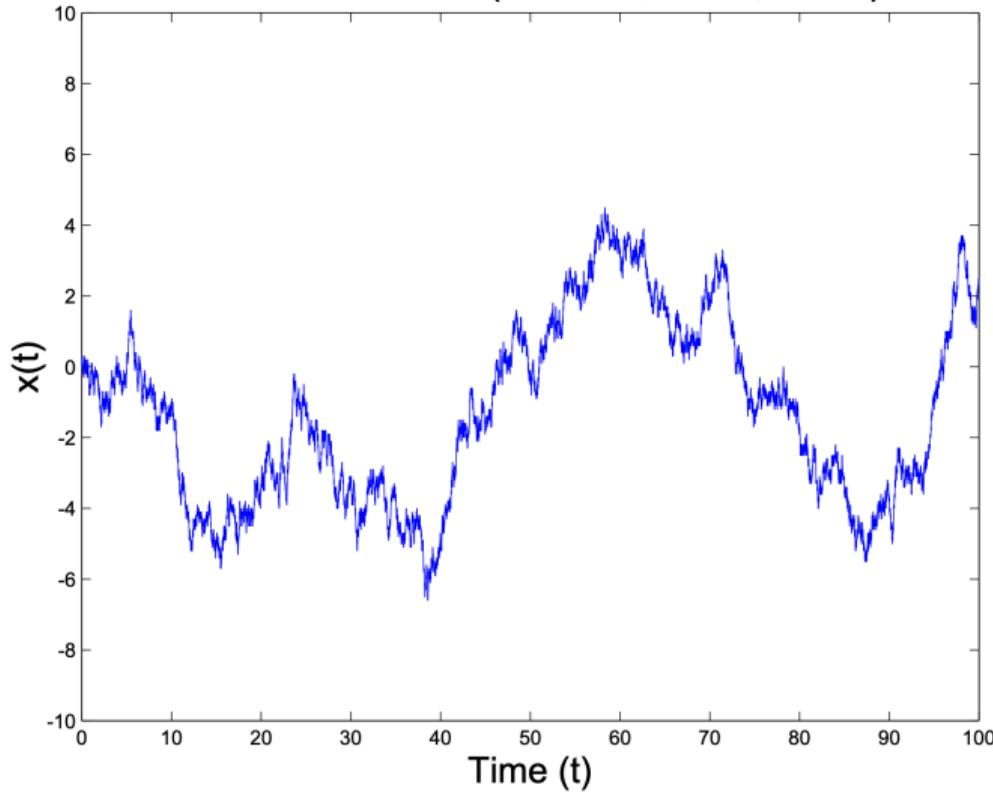
### Brownian Motion ( $\Delta t = 1$ , $\sigma = 1$ , $\alpha = 0$ )



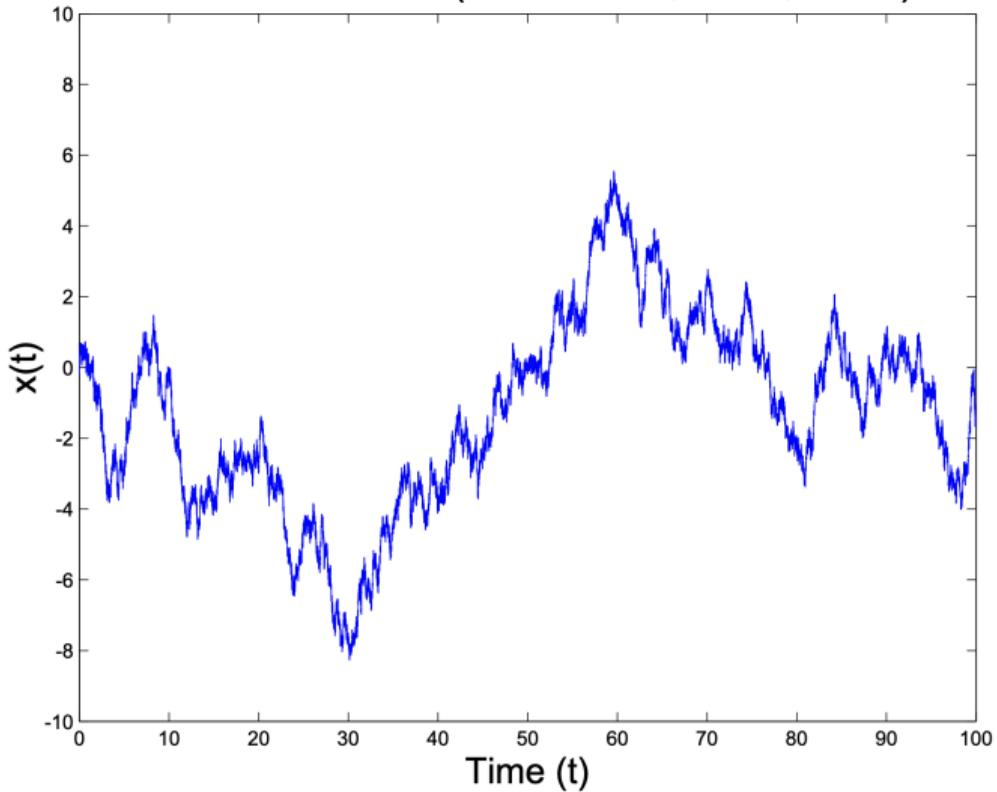
### Brownian Motion ( $\Delta t = .1$ , $\sigma = 1$ , $\alpha = 0$ )



Brownian Motion ( $\Delta t = .01$ ,  $\sigma = 1$ ,  $\alpha = 0$ )



Brownian Motion ( $\Delta t = .00001$ ,  $\sigma = 1$ ,  $\alpha = 0$ )



## Brownian motion: perspective #2

- Consider a random walk process in discrete time:

$$W_{t+1} = W_t + \epsilon_t, \quad \text{where } W_0 = 0 \text{ and } \epsilon_t \sim \mathcal{N}(0, 1)$$

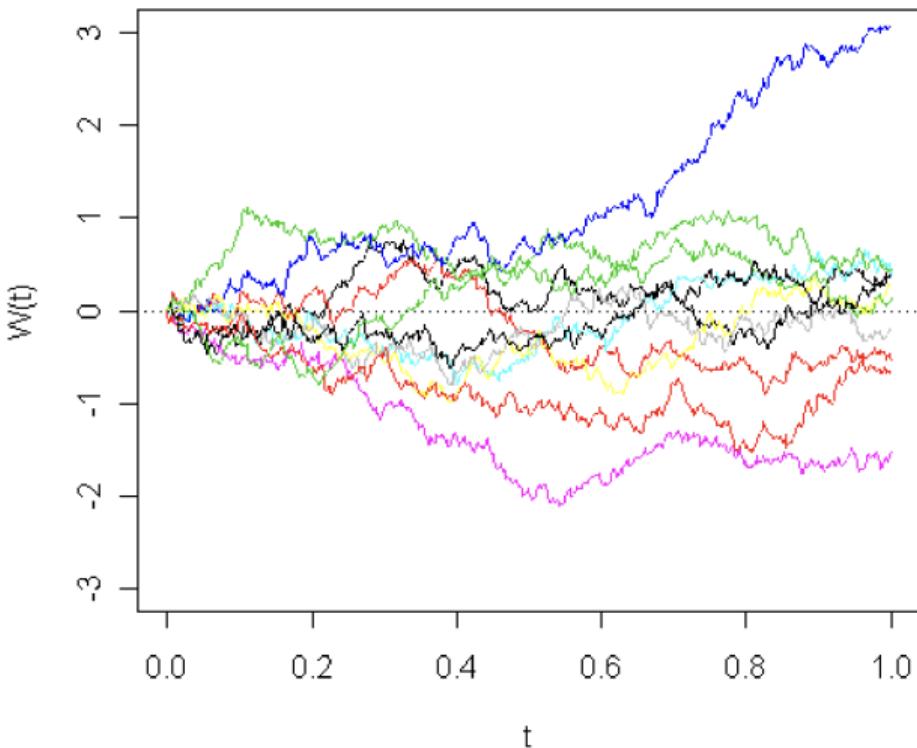
- We can extend this to a generalized time step:

$$W_{t+\Delta} = W_t + \sqrt{\Delta} \epsilon_t, \quad \text{where } W_0 = 0 \text{ and } \epsilon_t \sim \mathcal{N}(0, 1)$$

- Why the square root?

$$\text{Var}(W_{t+\Delta} - W_t) = \text{Var}(\sqrt{\Delta} \epsilon_t) = \Delta \text{Var}(\epsilon_t) = \Delta \implies \text{Var}(W_t) = t$$

- You get Brownian motion by taking limit  $\Delta \rightarrow 0$ : Brownian motion = continuous time random walk



## Brownian motion: perspective #3

**Definition.** Brownian motion  $\{B_t\}_{t \geq 0}$  is a stochastic process with properties:

- (i)  $B_0 = 0$
- (ii) (*Independent increments*) For non-overlapping  $0 \leq t_1 < t_2 < t_3 < t_4$ , we have  $B_{t_2} - B_{t_1}$  independent from  $B_{t_4} - B_{t_3}$
- (iii) (*Normal, stationary increments*)  $B_t - B_s \sim \mathcal{N}(0, t - s)$  for any  $0 \leq s < t$
- (iv) (*Continuity of paths*) The sample paths of  $B_t$  are continuous

- Brownian motion is the only stochastic process with stationary and independent increments that's also continuous
- Brownian motion is a Markov process
- Brownian motion is nowhere differentiable
- Brownian motion is a martingale
- Summary: Brownian motion is a continuous time random walk with zero drift and unit variance

## Other properties of Brownian motion

- $B_t \sim B_t - B_0 \sim \mathcal{N}(0, t)$
- Important:  $dB_t \sim \mathcal{N}(0, dt)$  because

$$dB_t \approx B_{t+\Delta} - B_t \sim \mathcal{N}(0, t + \Delta - t) = \mathcal{N}(0, \Delta)$$

and now take  $\Delta \rightarrow dt$  (continuous-time limit)

- Alternatively:  $B_{t+\Delta} - B_t \sim \mathcal{N}(0, \Delta) \sim \epsilon_t \sqrt{\Delta}$  where  $\epsilon_t \sim \mathcal{N}(0, 1)$ . So as  $\Delta \rightarrow dt$ ,

$$\mathbb{E}(dB_t) = \mathbb{E}(\epsilon_t \sqrt{dt}) = 0$$

$$\mathbb{E}[(dB_t)^2] = \mathbb{E}[(\epsilon_t \sqrt{dt})^2] = dt$$

## 4. Ito's Lemma

- Q: How do functions / transformations of stochastic processes evolve?
- Warm-up: Consider the process  $x_t$  characterized by the ODE

$$\dot{x}_t = \frac{dx_t}{dt} = \mu_t \quad \Rightarrow \quad dx_t = \mu_t dt$$

- Let  $y_t = f(x_t)$ . What do we know about  $\dot{y}_t$  or  $dy_t$ ? Using calculus, we simply have:

$$\frac{dy_t}{dt} = f'(x_t) \frac{dx_t}{dt} \quad \Rightarrow \quad dy_t = f'(x_t) dx_t = f'(x_t) \mu_t dt$$

- Next, suppose  $y_t = f(t, x_t)$ . Then using  $f_x = \frac{\partial f}{\partial x}$  we have:

$$\frac{dy_t}{dt} = f_x(t, x_t) \frac{dx_t}{dt} + f_t(t, x_t) \quad \Rightarrow \quad dy_t = f_x(t, x_t) \mu_t dt + f_t(t, x_t) dt$$

- Now what about functions of Brownian motion or other diffusion processes?
- What to remember for Ito's lemma: (i) take a  $2^{nd}$ -order Taylor expansion and (ii) remember that

$$(dt)^2 = 0 \quad \text{and} \quad (dt)(dB) = 0 \quad \text{and} \quad (dB)^2 = 0$$

- Suppose  $Y_t = f(B_t)$  where  $B_t$  is Brownian motion. Show that:

$$dY_t = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

- Suppose  $Y_t = f(t, B_t)$ . Show that:

$$dY_t = f_t(t, B_t)dt + f_b(t, B_t)dB_t + \frac{1}{2}f_{bb}(t, B_t)dt$$

- This is **Ito's lemma**, a core building block of **stochastic calculus**

## 5. Diffusion processes

- Consider the class of stochastic processes  $\{X_t\}$  characterized by

$$dX_t = \mu_t(X_t)dt + \sigma_t(X_t)dB_t$$

- These are called **diffusion processes** (continuous sample paths)
- For any twice differentiable function  $Y_t = f(t, X_t)$ , we have

$$\begin{aligned} dY &= f_t dt + f_x dX + \frac{1}{2} f_{xx} (dX)^2 \\ &= f_t dt + f_x (\mu(t, X) dt + \sigma(t, X) dB) + \frac{1}{2} f_{xx} \sigma(t, X)^2 dt \\ &= \left( f_t + f_x \mu(t, X) + \frac{1}{2} f_{xx} \sigma(t, X)^2 \right) dt + f_x \sigma(t, X) dB \end{aligned}$$

- Powerful result: Any (twice differentiable) function of a diffusion is also a diffusion

## 6. Stochastic differential equations

### Geometric Brownian motion:

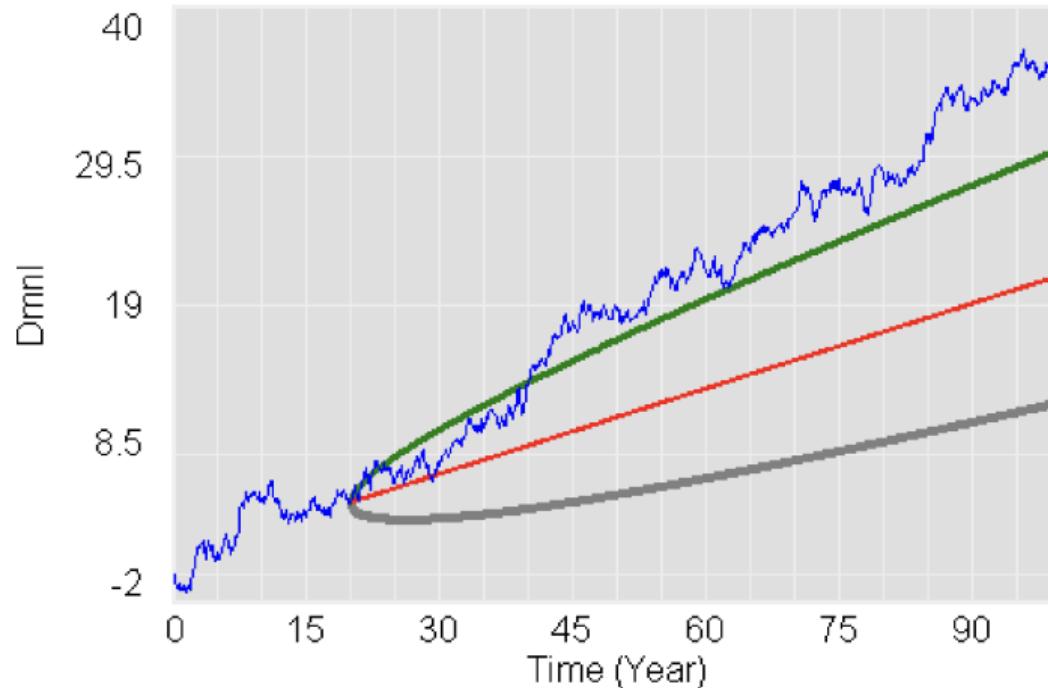
- Stochastic differential equations (SDEs) add noise / uncertainty to ordinary differential equations (ODEs)
- Start with  $\dot{X}_t = \mu X_t$  with solution  $X_t = X_0 e^{\mu t}$
- Rewrite as  $dX_t = \mu X_t dt$  and “add noise” (using Brownian motion):

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

- Solution to geometric Brownian motion (work this out on homework):

$$X_t = X_0 e^{\mu t - \frac{\sigma^2}{2} t + \sigma B_t}$$

## Brownian Motion with Drift



"x(t)" : Current3

Forecast x : Current3

"Forecast x + 1 SD" : Current3

"Forecast x - 1 SD" : Current3

## Ornstein-Uhlenbeck (OU) process:

- Continuous time analog of the AR(1) process

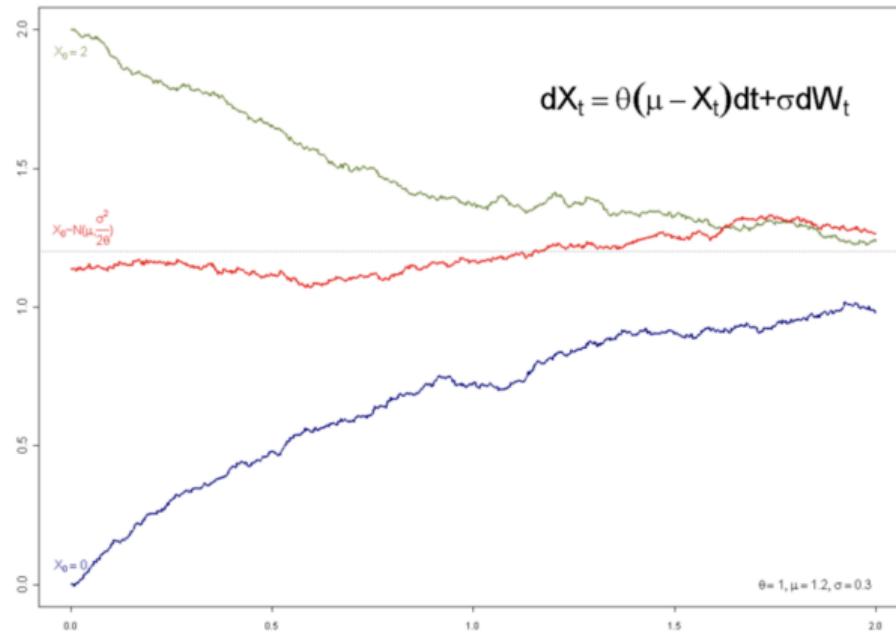
$$dz_t = \theta(\bar{z} - z_t)dt + \sigma dB_t$$

- Popular model for earnings risk and income fluctuations
- Auto-correlation of  $e^{-\theta t}$   $\approx 1 - \theta t$ . Compare to:

$$x_{t+1} = \theta \bar{x} + (1 - \theta)x_t + \sigma \epsilon_t$$

- Stationary distribution is  $\mathcal{N}(\bar{z}, \frac{\sigma^2}{2\theta})$

## Ornstein - Uhlenbeck



- Diffusion processes allow us to construct many other processes with desired properties by simply choosing  $\mu(t, X)$  and  $\sigma(t, X)$
- Process that stays in interval  $[0, 1]$  and mean-reverts around 1/2:

$$dX = \theta\left(\frac{1}{2} - X\right)dt + \sigma X(1 - X)dB$$

- Feller square root process (finance: “Cox-Ingersoll-Ross”):

$$dX = \theta(\bar{X} - X)dt + \sigma\sqrt{X}dB$$

with stationary distribution  $\sim \Gamma\left(\frac{2\theta\bar{X}}{\sigma^2}, \frac{\sigma^2}{2\theta\bar{X}}\right)$

## **Part 2: Optimization with stochastic dynamics**

# 1. The generator of a stochastic process

**Definition.** The **generator** of a stochastic process  $X$  is defined (for any test function  $f$ ) as

$$\mathcal{A}f = \lim_{\Delta t \rightarrow 0} \mathbb{E}_t \frac{f(t + \Delta t, X(t + \Delta t)) - f(t, X(t))}{\Delta t}$$

- The generator  $\mathcal{A}$  tells us how the stochastic process is *expected* to evolve
- The generator  $\mathcal{A}$  is a functional operator

# Generator for diffusion processes

- We start with the diffusion process

$$dX = \mu(t, X)dt + \sigma(t, X)dB$$

- On homework, you will show that:

$$\mathcal{A}f = \partial_t f(t, X) + \mu(t, X)\partial_X f(t, X) + \frac{1}{2}\sigma(t, X)^2\partial_{XX}f(t, X)$$

- For the general / multi-dimensional version see Oksendal

## Generator for Poisson processes

- Next, we consider the poisson process  $\{Y_t\}$  where  $Y_t \in \{Y^1, Y^2\}$ . This is a two-state Markov chain in continuous time.
- We assume that the Poisson intensity / arrival rate / hazard rate is  $\lambda$
- The generator is now given by

$$\mathcal{A}f(Y^j) = \lambda \left[ f(Y^{-j}) - f(Y^j) \right]$$

- Intuition: at rate  $\lambda$  you transition, so you lose the value of your current state,  $f(Y^j)$ , and obtain the value of the new state,  $f(Y^{-j})$
- Again see Oksendal for general version of this and more details

## 2. Stochastic Neoclassical Growth Model

**Preferences.** Lifetime utility of representative household

$$\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

**Technology.** Final consumption good produced using technology

$$y_t = F(k_t, z_t) \quad \text{where } F(0, z) = 0, \text{ and } F_k, F_z > 0, \text{ and } F_{kk} < 0$$

where  $z_t$  is exogenous productivity, and capital accumulation technology is

$$dk_t = (i_t - \delta k_t) dt.$$

**Resource constraint** for the final good is:  $y_t = c_t + i_t$

**Endowment.** At time  $t = 0$ , economy's initial state is  $(k_0, z_0)$

**Planning problem.** Taking as given  $(k_0, z_0)$ , choose an allocation to maximize lifetime utility of representative household subject to technologies and resource constraints:

$$V(k_0, z_0) = \max \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

subject to

$$y_t = F(k_t, z_t)$$

$$y_t = c_t + i_t$$

$$dk_t = (i_t - \delta k_t)dt$$

$dz_t$  exogenous

and taking as given  $(k_0, z_0)$

Combining:

$$dk_t = [F(k_t, z_t) - c_t - \delta k_t]dt$$

### 3. Productivity as a diffusion process

- We start with diffusion process:

$$dz_t = -\theta z_t dt + \sigma dB_t,$$

where  $\theta$  and  $\sigma$  are constants

- This is a continuous-time, mean-reverting AR(1) process called the Ornstein-Uhlenbeck process
- State space is now given by

$$\left\{ (k, z) \mid k \in [0, \bar{k}] \text{ and } z \in [\underline{z}, \bar{z}] \right\}$$

- In discrete time, we would have

$$V(k_t, z_t) = \max_c \left\{ u(c)\Delta t + \frac{1}{1+\rho\Delta t} \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) \right\}$$

- Difference from previous lecture:  $\mathbb{E}$  because there is uncertainty

$$(1 + \rho\Delta t)V(k_t, z_t) = \max_c \left\{ (1 + \rho\Delta t)u(c)\Delta t + \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) \right\}$$

$$\rho\Delta t V(k_t, z_t) = \max_c \left\{ u(c)\Delta t + \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) - V(k_t, z_t) \right\}$$

$$\rho V(k_t, z_t) = \max_c \left\{ u(c) + \mathbb{E}_t \frac{V(k_{t+\Delta t}, z_{t+\Delta t}) - V(k_t, z_t)}{\Delta t} \right\}$$

- Take limit  $\Delta t \rightarrow 0$  and drop time subscripts:

$$\rho V(k, z) = \max_c \left\{ u(c) + \mathbb{E} \frac{dV(k, z)}{dt} \right\}$$

- What remains? Characterizing continuation value  $\frac{d}{dt} V(k, z)$  (i.e., characterizing how process  $dV$  evolves)

- The generator  $\mathcal{A}$  is exactly the answer to this question! I.e.,

$$\begin{aligned}\mathbb{E} \frac{dV(k, z)}{dt} &= \mathcal{A}V(k, z) \\ &= (F(k, z) - \delta z - c)\partial_k V(k, z) - \theta z \partial_z V(k, z) + \frac{\sigma^2}{2} \partial_{zz} V(k, z)\end{aligned}$$

- Therefore, we arrive at the Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\rho V(k, z) &= \max_c \left\{ u(c) + (F(k, z) - \delta z - c)\partial_k V(k, z) \right. \\ &\quad \left. - \theta z \partial_z V(k, z) + \frac{\sigma^2}{2} \partial_{zz} V(k, z) \right\}\end{aligned}$$

with first-order condition

$$u'(c(k, z)) = \partial_k V(k, z)$$

## 4. Productivity as a Poisson process

- Next, consider Poisson process for  $\{z_t\}$  with  $z_t \in \{z^L, z^H\}$
- Generator now given by

$$\mathcal{A}V(k, z^j) = \left( F(k, z) - \delta z - c \right) \partial_k V(k, z) + \lambda \left[ V(k, z^{-j}) - V(k, z^j) \right]$$

- Note: derivation of HJB exactly as before *up to* characterizing  $\mathbb{E}[dV]$
- With Poisson process, HJB becomes

$$\begin{aligned} \rho V(k, z^j) &= \max_c \left\{ u(c) + \left( F(k, z) - \delta z - c \right) \partial_k V(k, z) \right. \\ &\quad \left. + \lambda \left[ V(k, z^{-j}) - V(k, z^j) \right] \right\} \end{aligned}$$

with first-order condition

$$u'(c(k, z^j)) = \partial_k V(k, z^j)$$

# Part 3: Applications

# 1. Consumption-savings with income fluctuations

- Economy is populated by representative household that faces income risk
- Household accumulates wealth according to

$$\dot{a}_t = r a_t + e^{z_t} - c_t$$

subject to borrowing constraint  $a_t \geq 0$

- Preferences again:  $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$
- Income follows diffusion process:  $dy_t = -\theta y_t dt + \sigma dB_t$
- Away from borrowing constraint, HJB given by

$$\rho V = \max_c \left\{ u(c) + (ra + e^z - c)V_a - \theta z V_z + \frac{\sigma^2}{2} V_{zz} \right\}$$

with  $V_a = \partial_a V(a, z)$  (you'll see this often)

## 2. Firm profit maximization

- Firm maximizes NPV of profit:  $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-rt} \pi_t dt$
- For now, profit given by:  $\pi_t = A_t n_t^\alpha - w_t n_t$  where firm chooses labor  $n_t$   
Assume  $\alpha < 1$ , so this is a decreasing-returns production function
- Firm is small and takes wage  $\{w_t\}$  as given (wages determined in general equilibrium)
- Productivity follows two-state high-low process, with  $A_t \in \{A^{\text{rec}}, A^{\text{boom}}\}$
- Recursive representation:  $A$  is only state variable,  $w_t = w(A_t)$

$$rV(A^{\text{boom}}) = \max_n \left\{ A^{\text{boom}} n^\alpha - w(A^{\text{boom}})n + \lambda [V(A^{\text{rec}}) - V(A^{\text{boom}})] \right\}$$

with first-order condition

$$n = \left( \frac{\alpha A^j}{w(A^j)} \right)^{\frac{1}{1-\alpha}}$$

### 3. Capital investment with adjustment cost

- Firm again maximizes NPV of profit:  $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-rt} \pi_t dt$
- Now: let  $\psi(\cdot)$  denote an adjustment cost

$$\begin{aligned}\pi_t &= e^{A_t} k_t^\alpha - Q_t \iota_t - \psi(\iota_t, k_t) \\ dk_t &= (\iota_t - \delta k_t) dt \\ dA_t &= -\theta A_t dt + \sigma dB_t\end{aligned}$$

- Firm is small and takes capital price as given
- Recursive representation in terms of  $(k, A)$ , i.e.,  $Q_t = Q(k_t, A_t)$

$$\begin{aligned}rV(k, A) &= \max_{\iota} \left\{ e^{A_t} k_t^\alpha - Q(A) \iota_t - \psi(\iota_t, k_t) + (\iota - \delta k) \partial_k V(k, A) \right. \\ &\quad \left. - \theta A \partial_A V(k, A) + \frac{\sigma^2}{2} \partial_{AA} V(k, A) \right\}\end{aligned}$$

with first-order condition:  $Q(k, A) + \partial_\iota \psi(\iota(k, A), k) = \partial_k V(k, A)$

## 4. Investing in stocks

- Suppose you optimize lifetime utility  $V_0 = \mathbb{E}_0 \int_0^\infty u(c_t)dt$
- You can trade two assets: riskfree bond (return  $r dt$ ), and risky stock

$$dR = (r + \pi)dt + \sigma dB, \text{ where } \pi \text{ is the equity premium}$$

- You have wealth  $a_t$  and invest a share  $\theta_t$  in stocks, thus,

$$da_t = \theta_t a_t dR_t + (1 - \theta_t) a_t r_t dt + y - c_t$$

or, rearranging, and dropping  $t$  subscripts

$$da = ra + \theta a \pi dt + y - c + \theta a \sigma dB$$

- HJB becomes:

$$\rho V(a) = \max_{c,\theta} \left\{ u(c) + (ra + \theta a \pi dt + y - c)V'(a) + \frac{1}{2}(\sigma \theta a)^2 V''(a) \right\}$$

with FOCs: (i)  $u'(c) = V'(a)$  and (ii)  $\theta = -\frac{\pi}{\sigma^2} \frac{V'(a)}{a V''(a)}$

## 5. Real business cycles

- Next semester, Yuriy will teach the Real Business Cycle model. This is basically the stochastic neoclassical growth model, estimated to match business cycle moments
- Recall that this model is efficient so we can look at the planning problem
- Preferences are:

$$\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

- And combining technologies and resource constraints yields:

$$dk_t = [f(k_t, z_t) - \delta k_t - c_t] dt$$

- Now suppose TFP follows a Feller process:  $dz_t = \theta(\bar{z} - z_t)dt + \sigma \sqrt{z_t} dB_t$

- With  $dz_t = \theta(\bar{z} - z_t)dt + \sigma\sqrt{z_t}dB_t$ , the HJB is then given by:

$$\rho V(k, z) = \max_c \left\{ u(c) + [f(k, z) - \delta k - c]V_k(k, z) + \theta(\bar{z} - z)V_z(k, z) + \frac{1}{2}\sigma^2 z V_{zz}(k, z) \right\}$$

- We have now seen 3 variants of this model with 3 different assumptions for the process  $dz_t$
- This model is the foundation for business cycle macro

## 6. AK technology and log utility

- Assume that  $u(c_t) = \log c_t$  and  $f(k_t, z_t) = z_t k_t$
- Assuming that  $dz_t$  follows a stationary diffusion process:

$$\rho V(k, z) = \max_c \left\{ \log c + [f(k, z) - \delta k - c] V_k(k, z) + \mu(z) V_z(k, z) + \frac{1}{2} \sigma(z)^2 V_{zz}(k, z) \right\}$$

- Show that the consumption policy function is:

$$c(k, z) = \rho k$$

- As a result, model solution characterized by the two *forward* equations:

$$dk_t = (z_t - \rho - \delta) k_t dt$$

$$dz_t = \mu(z_t) dt + \sigma(z_t) dB_t$$

## Proof:

- Guess and verify:

$$V(k, z) = v(z) + A \log k$$

- FOC:

$$u'(c(k, z)) = V_k(k, z) \implies \frac{1}{c(k, z)} = A \frac{1}{k} \implies c(k, z) = \frac{1}{A} k$$

- Plug back into HJB:

$$\rho v(z) + \rho A \log k = \log \frac{1}{A} + \log k + (z - \frac{1}{A} - \delta) k A \frac{1}{k} + \mu(z) v'(z) + \frac{1}{2} \sigma(z)^2 v''(z)$$

- Collect terms in  $\log k$  and confirm  $\rho A = 1$ . Solve ODE for  $v(z)$ !
- Economics: log preferences  $\implies$  income and substitution effects of future  $z_t$  changes cancel  $\implies$  constant savings rate  $\rho$