

# **Dynamic Programming and Applications**

## Deterministic Dynamic Programming in Continuous Time

Lectures 3 – 4

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# Outline

## Part 1: Differential equations

1. The continuous time limit
2. Ordinary differential equations (ODEs)
3. Boundary conditions
4. Linear first-order ODEs
5. Examples of ODEs in macro
6. Application: solving the Solow growth model
7. Partial differential equations (PDEs)

# Outline

## Part 2: Optimization with deterministic dynamics

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4. Simple example
5. Hamilton-Jacobi-Bellman (HJB) equation
6. First-order condition for consumption
7. Envelope condition and Euler equation
8. Connection between calculus of variations / optimal control and HJBs
9. Boundary conditions: no-borrowing in the wealth / capital dimension

# Outline

## Part 3: Applications

1. Labor: search and matching
2. Urban / trade / dynamic spatial: migration
3. Macro: sticky prices
4. IO: duopoly
5. Public finance: tax competition

# Part 1: Differential Equations

# 1. Continuous time limit

- Consider the two key difference equations:

$$K_{t+1} = I_t + (1 - \delta)K_t$$

and

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- On the board: (i) generalized discrete time step  $\Delta$  and (ii) continuous time limit

## 2. Ordinary differential equations

- Consider the “discrete-time” equation

$$X_{t+\Delta t} - X_t = G(X_t, t, \Delta t)$$

- Continuous-time limit*: consider the limit as  $\Delta t \rightarrow 0$

$$\dot{X}_t \equiv \frac{dX}{dt} \equiv \lim_{\Delta t \rightarrow 0} \frac{X_{t+\Delta t} - X_t}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} G(X_t, t, \Delta t) \equiv g(X_t, t)$$

- $\dot{X}_t = g(X_t)$  is *autonomous* and dropping subscripts:  $\dot{X} = g(X)$
- This is a *first-order (ordinary) differential equation*, second-order equations are:

$$\frac{d^2 X_t}{dt^2} = g\left(\frac{dX_t}{dt}, X_t, t\right)$$

- We often consider ODEs in the *time dimension* but ODEs can be defined on any state space (e.g., space dimensions)

### 3. Boundary conditions

- Boundary conditions are critical for characterizing differential equations
- Consider an ODE on the time interval  $t \in [0, 1]$ . We call  $[0, 1]$  the *state space*.  $(0, 1)$  is the *interior of the state space* and  $\{0, 1\}$  is the *boundary*
- The way to think about it: differential equations are defined on the interior of the state space but not on the boundary
- To characterize the function that satisfies the ODE on the interior on the *full* state space, we need a set of boundary conditions to also characterize the behavior on the boundary
- Heuristically: we need as many boundary conditions as the order of the differential equation

- Similar to discrete-time difference equations: forward equations have initial conditions, backward equations have terminal conditions
- For ODEs, you will often see the terminology:
  - *Initial value problems* specify a differential equation for  $X_t$  with some *initial condition*  $X_0$
  - *Terminal value problems* instead specify  $X_T$
- More broadly: We need sufficient information to characterize the function of interest along the boundary
- Types of boundary conditions: Dirichlet ( $X_0 = c$ ), von-Neumann ( $\frac{dX_0}{dt} = c$ ), reflecting boundaries, ...
- Boundary conditions are very important and can be very subtle (especially for PDEs)

## 4. Linear first-order ODEs

- Consider the equation:

$$\dot{X}(t) = a(t)X(t) + b(t) \quad (1)$$

- If  $b(t) = 0$ , (1) is a *homogeneous* equation, if  $a(t) = a$  and  $b(t) = b$  we say (1) has *constant coefficients*
- Start with  $\dot{X}(t) = aX(t)$ , divide by  $X(t)$  and integrate with respect to  $t$

$$\int \frac{\dot{X}(t)}{X(t)} dt = \int a dt$$

$$\log X(t) + c_0 = at + c_1$$

$$X(t) = Ce^{at}$$

where  $C = e^{c_1 - c_0}$

- Pin down constant  $C$  by using the boundary condition (we need 1)

- Consider time-varying coefficient with  $\dot{X}(t) = a(t)X(t)$  with initial condition  $X(0) = \bar{x}$
- Dividing by  $X(t)$ , integrating, and exponentiating yields

$$X(t) = Ce^{\int_0^t a(s)ds}$$

- Constant of integration again pinned down by boundary condition:  $C = \bar{x}$
- Finally, for  $\dot{X}(t) = aX(t) + b$ , we find

$$X(t) = -\frac{b}{a} + Ce^{at}$$

after using change of variables  $Y(t) = X(t) + \frac{b}{a}$

- Many results for systems of linear differential equations:  $\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t)$

## 5. Examples of differential equations in macro

Capital accumulation:

$$\dot{K}_t = I_t - \delta K_t$$

- We can always map back and forth between DT and CT
- In discrete time with *unit* time steps,  $K_{t+1} = I_t + (1 - \delta)K_t$
- With arbitrary  $\Delta$  time step,  $K_{t+\Delta} = K_t + \Delta(I_t - \delta K_t)$
- Continuous-time limit:

$$\begin{aligned} K_{t+\Delta} &= K_t + \Delta(I_t - \delta K_t) \\ \frac{K_{t+\Delta} - K_t}{\Delta} &= I_t - \delta K_t \\ \dot{K}_t &= I_t - \delta K_t \end{aligned}$$

- Suppose  $\{I_t\}_{t \geq 0}$  exogenously given
- Solving this *inhomogeneous equation*, we use *integrating factor*:

$$\begin{aligned}\dot{K}_t + \delta K_t &= I_t \\ e^{\int_0^t \delta ds} \dot{K}_t + e^{\int_0^t \delta ds} \delta K_t &= e^{\int_0^t \delta ds} I_t\end{aligned}$$

- Notice that  $\int_0^t \delta ds = \delta \int_0^t ds = \delta[s]_0^t = \delta(t - 0) = \delta t$ , so

$$e^{\delta t} \dot{K}_t + e^{\delta t} \delta K_t = e^{\delta t} I_t$$

- We have  $e^{\delta t} \dot{K}_t + e^{\delta t} \delta K_t = \frac{d}{dt}(K_t e^{\delta t})$ , integrating:

$$\begin{aligned}K_t e^{\delta t} &= \tilde{C} + \int_0^t e^{\delta s} I_s ds \\ K_t &= C + \int_0^t e^{-\delta(t-s)} I_s ds\end{aligned}$$

- Integrating constant solves initial condition:  $C = K_0$

**Wealth dynamics** (*very important equation in this course*):

$$\dot{a}_t = r_t a_t + y_t - c_t$$

- $r_t$  is the real rate of return on wealth,  $y_t$  is income, and  $c_t$  is consumption
- Structure of the equation similar to capital accumulation equation

## Consumption Euler equation:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho$$

- The Euler equation typically takes the form of a *backward equation* and comes with a terminal condition ( $C_T$ ) or transversality condition ( $\lim_{T \rightarrow \infty} C_T$ )
- Stationary point only if  $r_t = \rho$
- Suppose we are at  $r_t = r = \rho$  and a shock is realized.  $r_0 > r$  what happens?  $r_0 < r$  what happens?

## 6. Example: Solow growth model

- As before,  $Y_t = C_t + I_t$  and

$$\dot{K}_t = Y_t - C_t - \delta K_t$$

- Representative firms operates neoclassical production function

$$Y_t = F(K_t, L_t, A_t)$$

- Normalize labor to  $L_t = 1$  and hold TFP constant  $A_t = A$
- We again assume constant savings rate:  $Y_t - C_t = I_t = sY_t$
- Assume Cobb-Douglas  $Y_t = AK_t^\alpha$  so equilibrium allocation

$$\dot{K}_t = sAK_t^\alpha - \delta K_t$$

- Steady state is given by

$$K_{ss} = \left( \frac{sA}{\delta} \right)^{\frac{1}{1-\alpha}}$$

- Key equilibrium condition in  $\dot{K}_t$  is *non-linear* — how to proceed?
- Let  $X_t = K_t^{1-\alpha}$ , then

$$\begin{aligned}\dot{X}_t &= (1-\alpha)K_t^{-\alpha}\dot{K}_t \\ &= (1-\alpha)K_t^{-\alpha}(sAK_t^\alpha - \delta K_t) \\ &= (1-\alpha)sA - (1-\alpha)K_t^{1-\alpha}\delta \\ &= (1-\alpha)sA - (1-\alpha)\delta X_t\end{aligned}$$

- Solution with initial condition  $X_0$  (work this out):

$$X_t = X_{ss} + e^{-(1-\alpha)\delta t} \left[ X_0 - X_{ss} \right], \quad \text{where } X_{ss} = \frac{sA}{\delta}$$

- Transition dynamics (rate of convergence) governed by  $-(1-\alpha)\delta$

## 7. What are partial differential equations?

- Partial differential equations (PDEs) generalize ODEs to higher-dimensional state spaces
- PDEs are at the heart of (i) continuous-time **dynamic programming** and (ii) heterogeneous-agent models in macro
- PDEs have long been a core tool in physics, applied math, ...  
     $\implies$  increasingly used in economics

- Consider a function  $u(x_1, x_2, \dots, x_n)$  where  $x_1, \dots, x_n$  are coordinates in  $\mathbb{R}^n$
- Partial derivatives of  $u(\cdot)$

$$\frac{\partial u}{\partial x_i} \equiv \partial_{x_i} u \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j} u$$

- A PDE is an equation in  $u$  and its partial derivatives — fully generally:

$$0 = G(u, \partial_{x_1} u, \dots, \partial_{x_n} u, \partial_{x_1 x_1} u, \dots)$$

- The *order* of the PDE, is the order of the highest partial derivative
- Examples from physics
  - Heat equation:  $\partial_t u = \partial_{xx} u$  (second-order, linear, homogeneous)
  - Wave equation:  $\partial_{tt} u = \partial_{xx} u$  (second-order, linear, homogeneous)
  - Transport equation:  $\partial_t u = \partial_x u$  (first-order, linear, homogeneous)
- Income distribution “solves heat equation”, wealth dynamics “solve transport equations”, dynamic programming often transport + heat

## **Part 2: Optimization with Deterministic Dynamics**

# 1. Neoclassical growth model in continuous time

- The lifetime value of the representative household is

$$v(k_0) = \max_{\{c_t\}_{t \geq 0}} \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

subject to

$$\begin{aligned} \dot{k}_t &= F(k_t) - \delta k_t - c_t \\ k_0 &\text{ given ,} \end{aligned}$$

where  $\dot{x}_t = \frac{d}{dt}x_t$ ,  $\rho$  is the discount rate,  $c_t$  is the rate of consumption,  $u(\cdot)$  is instantaneous utility flow, and  $\dot{k}_t$  is the rate of (net) capital accumulation

- No uncertainty for now
- This is the **sequence problem** in continuous time

## 2. Calculus of variations

- Resources:
  - LeVeque: Finite Difference Methods for Ordinary and Partial Differential Equations
  - Kamien and Schwartz: Dynamic Optimization
  - Gelfand and Fomin: Calculus of Variations
- This dynamic optimization problem is associated with the Lagrangian

$$L = \int_0^{\infty} e^{-\rho t} \left[ u(c_t) + \mu_t \left( F(k_t) - \delta k_t - c_t - \dot{k}_t \right) \right] dt$$

- $\mu_t$  is the Lagrange multiplier on the capital accumulation ODE
- What do we do with  $\dot{k}_t$ ??

- Integrate by parts:

$$\begin{aligned}\int_0^\infty e^{-\rho t} \mu_t \dot{k}_t dt &= e^{-\rho t} \mu_t k_t \Big|_0^\infty - \int_0^\infty \frac{d}{dt} \left( e^{-\rho t} \mu_t \right) k_t dt \\ &= -\mu_0 k_0 + \int_0^\infty e^{-\rho t} \rho \mu_t k_t dt - \int_0^\infty e^{-\rho t} \dot{\mu}_t k_t dt\end{aligned}$$

- Plugging into Lagrangian:

$$L = \int_0^\infty e^{-\rho t} \left[ u(c_t) + \mu_t \left( F(k_t) - \delta k_t - c_t \right) - \rho \mu_t k_t + \dot{\mu}_t k_t \right] dt + \mu_0 k_0$$

- What have we accomplished?
- Notice  $\mu_0 k_0$ , this is crucial. What's intuition?

$$L = \int_0^{\infty} e^{-\rho t} \left[ u(c_t) + \mu_t \left( F(k_t) - \delta k_t - c_t \right) - \rho \mu_t k_t + \dot{\mu}_t k_t \right] dt + \mu_0 k_0$$

- The planner optimizes over paths  $\{c_t\}$  and  $\{k_t\}$
- At an optimum, there cannot be *any* small perturbation in these paths that the planner finds preferable
- Let  $\{c_t\}$  and  $\{k_t\}$  be *candidate* optimal paths. Consider  $\hat{c}_t = c_t + \alpha h_t^c$  and  $\hat{k}_t = k_t + \alpha h_t^k$  for arbitrary functions  $h_t^c$  and  $h_t^k$

$$L(\alpha) = \int_0^{\infty} e^{-\rho t} \left[ u(c_t + \alpha h_t^c) + \mu_t \left( F(k_t + \alpha h_t^k) - \delta k_t - \delta \alpha h_t^k - c_t - \alpha h_t^c \right) - \rho \mu_t (k_t + \alpha h_t^k) + \dot{\mu}_t (k_t + \alpha h_t^k) \right] dt + \mu_0 (k_0 + \alpha h_0^k)$$

- What about *boundary conditions*? At  $t = 0$ , capital stock is fixed ( $k_0$  given) while consumption is free. So must have:  $h_0^k = 0$  while  $h_0^c$  is free

Necessary condition for optimality:  $\frac{d}{d\alpha}L(0) = 0$

$$L(\alpha) = \int_0^\infty e^{-\rho t} \left[ u(c_t + \alpha h_t^c) + \mu_t \left( F(k_t + \alpha h_t^k) - \delta k_t - \delta \alpha h_t^k - c_t - \alpha h_t^c \right) \right. \\ \left. - \rho \mu_t (k_t + \alpha h_t^k) + \dot{\mu}_t (k_t + \alpha h_t^k) \right] dt + \mu_0 (k_0 + \alpha h_0^k)$$

Work this out yourselves (many times, in many applications!)

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t} \left[ u'(c_t) h_t^c + \mu_t \left( F'(k_t) h_t^k - \delta h_t^k - h_t^c \right) \right. \\ \left. - \rho \mu_t h_t^k + \dot{\mu}_t h_t^k \right] dt + \mu_0 h_0^k$$

where  $h_0^k = 0$  because  $k_0$  is fixed

Group terms:

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t} \left[ \left( u'(c_t) - \mu_t \right) h_t^c + \left( \mu_t \left( F'(k_t) - \delta \right) - \rho \mu_t + \dot{\mu}_t \right) h_t^k \right] dt$$

**Fundamental Theorem of the Calculus of Variations:** Since  $h_t^c$  and  $h_t^k$  were arbitrary, we must have *pointwise*

$$0 = u'(c_t) - \mu_t$$

$$0 = \mu_t \left( F'(k_t) - \delta \right) - \rho \mu_t + \dot{\mu}_t$$

**Proposition.** (Euler equation for marginal utility)

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{\dot{u}_{c,t}}{u_{c,t}} = \rho - F'(k_t) + \delta = \rho - r_t$$

- We have now solved the neoclassical growth model in continuous time. Its solution is given by a system of two ODEs.
- Suppose  $u(c) = \log(c)$  and  $F(k) = k^\alpha$ , then:

$$\begin{aligned}\frac{\dot{c}_t}{c_t} &= \alpha k_t^{\alpha-1} - \delta - \rho \\ \dot{k}_t &= k_t^\alpha - \delta k_t - c_t\end{aligned}$$

with  $k_0$  given

- Derive the consumption Euler equation yourselves!
- What are the boundary conditions? (Always ask about BCs!)
  - Initial condition on capital:  $k_0$  given
  - Terminal condition on consumption :  $\lim_{T \rightarrow \infty} c_T = c_{ss}$

### 3. Optimal control theory

- Optimal control theory emerged from the calculus of variations
- Applies to dynamic optimization problems in continuous time that feature (ordinary) differential equations as constraints
- Again the neoclassical growth model:

$$v(k_0) = \max_{\{c_t\}_{t \geq 0}} \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t, \quad k_0 \text{ given}$$

- Three new terms:
  - **State variable:**  $k_t$
  - **Control variable:**  $c_t$
  - **Hamiltonian:**  $H(c_t, k_t, \mu_t) = u(c_t) + \mu_t [F(k_t) - \delta k_t - c_t]$

- With Hamiltonian in hand, *copy-paste* formula that we can always use:
  - **Optimality condition:**  $\frac{\partial}{\partial c} H = 0$
  - **Multiplier condition:**  $\rho\mu_t - \dot{\mu}_t = \frac{\partial}{\partial k} H$
  - **State condition:**  $\dot{k}_t = \frac{\partial}{\partial \mu} H$
- This gives us the same equations that we derived using calc of variations:

$$\begin{aligned} u'(c_t) &= \mu_t \\ \rho\mu_t - \dot{\mu}_t &= \mu_t(F'(k_t) - \delta) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

- We again get system of Euler equation and capital accumulation:

$$\begin{aligned} \dot{c}_t &= \frac{u'(c_t)}{u''(c_t)} (\rho - F'(k_t) + \delta) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

## 4. Simple example [*skip*]

- Credit: Kamien-Schwartz p. 129
- Simple problem: not much intuition, but illustrates mechanics

$$\max \int_0^1 (x + u) dt$$

subject to  $\dot{x} = 1 - u^2$  and initial condition  $x_0 = 1$

- Step 1: form Hamiltonian  $H(t, x, u, \lambda) = x + u + \lambda(1 - u^2)$
- Step 2: necessary conditions (note: no discounting here)

$$\begin{aligned} 0 &= H_u = 1 - 2\lambda u \\ -\dot{\lambda} &= H_x = 1 \end{aligned}$$

and terminal condition  $\lambda_1 = 0$  (because  $u_1$  is *free*)

- Step 3: manipulate necessary conditions:

$$\lambda = 1 - t$$

$$u = \frac{1}{2\lambda}$$

and therefore:  $u = \frac{1}{2}(1 - t)$

- Finally: solve for all paths (control, state, multiplier)

$$x_t = t - \frac{1}{4}(1 - t) + \frac{5}{4}$$

$$\lambda_t = 1 - t$$

$$u_t = \frac{1}{2}(1 - t)$$

## 5. Hamilton-Jacobi-Bellman equation

- Recall the neoclassical growth model in continuous time

$$v(k_0) = \max_{\{c_t\}_{t \geq 0}} \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t$$

$k_0$  given ,

where  $\dot{x}_t = \frac{d}{dt}x_t$ ,  $\rho$  is the discount rate,  $c_t$  is the rate of consumption,  $u(\cdot)$  is instantaneous utility flow, and  $\dot{k}_t$  is the rate of (net) capital accumulation

- No uncertainty for now
- This is the infinite-horizon sequence problem,  $t \in [0, \infty)$
- A function  $v(\cdot)$  that solves this problem is a solution to the neoclassical growth model

- We will now work towards a recursive representation (good reference: Stokey textbook)
- The discrete-time Bellman equation would be

$$v(k_t) = \max_c \left\{ u(c)\Delta + \frac{1}{1 + \rho\Delta} v(k_{t+\Delta}) \right\}$$

where  $\beta = \frac{1}{1 + \rho\Delta}$

- Next: multiply by  $1 + \rho\Delta$

$$(1 + \rho\Delta)v(k_t) = \max_c \left\{ (1 + \rho\Delta)u(c)\Delta + v(k_{t+\Delta}) \right\}$$

$$\rho\Delta v(k_t) = \max_c \left\{ u(c)\Delta + \rho u(c)\Delta^2 + v(k_{t+\Delta}) - v(k_t) \right\}$$

$$\rho v(k_t) = \max_c \left\{ u(c) + \rho u(c)\Delta + \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta} \right\}$$

$$\rho v(k_t) = \max_c \left\{ u(c) + \rho u(c)\Delta + \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta} \right\}$$

- Now we take limit  $\Delta \rightarrow 0$
- Notice  $\rho u(c)\Delta \rightarrow 0$  and also

$$\lim_{\Delta \rightarrow 0} \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta} = \lim_{\Delta \rightarrow 0} \frac{v(k(t+\Delta)) - v(k(t))}{\Delta} = \frac{dv(k(t))}{dt}$$

- Therefore we arrive at

$$\rho v(k(t)) = \max_c \left\{ u(c) + \frac{dv(k(t))}{dt} \right\}$$

- Only step left for us to do: What is  $\frac{dv(k(t))}{dt}$ ?
- Simply use Chain rule!  $\frac{dv(k(t))}{dt} = \frac{dv}{dk} \frac{dk}{dt}$  and recall  $\frac{dk}{dt} = F(k) - \delta k - c$
- Therefore, we arrive at the **Hamilton-Jacobi-Bellman equation**:

$$\rho v(k) = \max_c \left\{ u(c) + \left( F(k) - \delta k - c \right) v'(k) \right\}$$

- We drop  $t$  subscripts for clarity: this equation holds for all possible  $k$
- Notice: We conjectured a stationary value function (what does this mean?)

## 6. First-order condition for consumption

- HJB still has “max” operator:

$$\rho v(k) = \max_c \left\{ u(c) + \left( F(k) - \delta k - c \right) v'(k) \right\}$$

- To get rid of this, we have to resolve optimal consumption choice
- First-order condition:

$$u'(c(k)) = v'(k)$$

- This defines the **consumption policy function**
- We can now plug back in, obtaining an ODE in  $v'(k)$

$$\rho v(k) = u(c(k)) + \left( F(k) - \delta k - c(k) \right) v'(k)$$

- Why is this a “stationary” value function and ODE? What would a time-dependent ODE look like? When would we get one?

## 7. Envelope condition and Euler equation

- We now derive the Euler equation in continuous time
- We start with the **HJB envelope condition**. Differentiating in  $k$ :

$$\rho v'(k) = u'(c(k))c'(k) + \left(F'(k) - \delta - c'(k)\right)v'(k) + \left(F(k) - \delta k - c(k)\right)v''(k)$$

$$\rho v'(k) = \left( \underbrace{F'(k) - \delta}_{\text{interest rate } r} \right)v'(k) + \left(F(k) - \delta k - c(k)\right)v''(k)$$

$$(\rho - r)v'(k) = \left(F(k) - \delta k - c(k)\right)v''(k)$$

- Next, we characterize *process*  $dv'(k)$ . Using Ito's lemma (even though no uncertainty):

$$\begin{aligned} dv'(k) &= v''(k)dk \\ &= v''(k)(F(k) - \delta k - c(k))dt \\ &= (\rho - r)v'(k)dt. \end{aligned}$$

- Recall first-order condition  $u'(c(k)) = v'(k)$ .
- The **Euler equation for marginal utility** is given by

$$\frac{du'(c)}{u'(c)} = (\rho - r)dt.$$

- To go from marginal utility to consumption, we use CRRA utility:  $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$ .  $u'(c) = c^{-\gamma}$  is a function of *process*  $c$ , so by Ito's lemma:

$$\begin{aligned} du'(c) &= -\gamma c^{-\gamma-1}dc \\ &= -\gamma u'(c) \frac{dc}{c} \end{aligned}$$

- Plugging in yields **Euler equation for consumption** in continuous time:

$$\frac{dc}{c} = \frac{r - \rho}{\gamma} dt$$

or (you'll often see this notation when no uncertainty):  $\frac{\dot{c}}{c} = \frac{r - \rho}{\gamma}$

Connection between calculus of variations and HJB:

- What is the connection between costate / multiplier  $\mu_t$  and marginal value of wealth  $V'(k)$ ?
- What is the connection between multiplier equation and envelope condition?

## 9. Boundary conditions

- Important: everything we have done so far is only valid in the **interior of the state space**
- What's the state space of a model?
- For the neoclassical growth model without uncertainty, state space is  $k \in [0, \infty)$ , or

$$\left\{ k \mid k \in [0, \bar{k}] \right\}$$

where we impose an upper boundary  $\bar{k}$  (think about putting this on the computer)

- This is like the domain of the function  $v(k)$  that will be valid
- We say  $\{0, \bar{k}\}$  is the **boundary** of the state space and  $(0, \bar{k})$  is the **interior**
- As is the case **for all differential equations**, the HJB holds on the interior and we need **boundary conditions** to characterize  $v(k)$  along the boundary

- What differential equation is HJB in this model? How many boundary conditions do we need?
- In terms of the economics, what is correct economic behavior at the boundary  $k \in \{0, \bar{k}\}$ ?

- What differential equation is HJB in this model? How many boundary conditions do we need?
- In terms of the economics, what is correct economic behavior at the boundary  $k \in \{0, \bar{k}\}$ ?
- We want households to not leave the state space, so we impose that they do not dissave / borrow as  $k \rightarrow 0$  and save as  $k \rightarrow \bar{k}$
- Nice intuition: 2 boundary inequalities do same job as 1 boundary equality

- This implies:

$$\begin{aligned} v'(0) &\geq u'(F(0)) \\ v'(\bar{k}) &\leq u'(F(\bar{k}) - \delta\bar{k}) \end{aligned}$$

- Why? Implies  $v'(0) = u'(c(0)) \geq u'(F(0))$  which implies  $c(0) \leq F(0)$ . Also implies  $v'(\bar{k}) = u'(c(\bar{k})) \leq u'(F(\bar{k}) - \delta\bar{k})$  which implies  $c(\bar{k}) \geq F(\bar{k}) - \delta\bar{k}$
- In neoclassical growth model, Inada conditions take care of this (you never reach boundary)

# Part 3: Applications

# 1. Labor: search and matching

- One of most important ideas in labor: frictional search and matching  
Diamond-Mortensen-Pissarides (DMP) model
- This is just a simple application of dynamic programming

**Firms:** Can post vacancies at cost (rate)  $c$ , vacancy filled at rate  $q$ . Value of a vacancy is given by HJB

$$rV = -c + q(J - V).$$

Assume firms post until  $V = 0$ . Once matched, workers produce revenue at rate  $p$  and cost wage  $w$ . Match separates at rate  $s$ . Value of job is given by HJB

$$rJ = p - w - sJ$$

**Labor demand:** In equilibrium, labor demand schedule given by

$$p - w = (r + s) \frac{c}{q}$$

Profit  $p - w$  equalized with amortized cost of search / posting vacancies

**Workers:** When worker is unemployed, gets benefit  $b$  and can search at intensity  $\lambda$ , which costs  $\psi(\lambda)$ . When employed, gets wage  $w$  but separates at rate  $s$ . Let  $U$  be value of unemployment and  $E$  value of employment:

$$\begin{aligned} rU &= \max_{\lambda} \left\{ b - \psi(\lambda) + \lambda(E - U) \right\} \\ rE &= w + s(U - E) \end{aligned}$$

**Labor supply schedule** characterized by FOC for search intensity

$$\psi'(\lambda) = E - U$$

where  $E$  and  $U$  solve the coupled system of HJBs

## 2. Urban / trade / dynamic spatial: migration

- One of most important themes in urban, trade, international and dynamic spatial literatures: people move (migrate) in response to shocks  
For example: To what extent do households migrate in response to China trade shock or climate change?
- Turns out: state-of-the-art dynamic migration model (Caliendo-Dvorkin-Parro) is a simple application of our tools

**Households:** There are  $N$  regions indexed by  $j$ . Consider a household  $i$  and denote her region  $j_{i,t}$ . Lifetime utility is

$$V_{i,0} = \max \mathbb{E} \int_0^{\infty} e^{-\rho t} u(c_{i,t}) dt$$

- Household inelastically supplies 1 unit of labor, earns wage  $w_{j_{i,t}}$
- For simplicity: They cannot save or borrow, so  $c_{i,t} = w_{j_{i,t}}$  (“hand-to-mouth”)
- Problem will be stationary because  $w_j$  are time-invariant

## Migration: discrete-choice optimal stopping problem

- Households face fixed cost  $\kappa_{jk}$  to move from  $j$  to  $k$
- Key trick:** At rate  $\mu$ , household draws opportunity and **extreme-value taste shock**  $\epsilon_k$  with shape parameter  $\theta$  for possible destinations  $k$

## Recursive representation:

$$\rho V_j = u(w_j) + \mu \left( \mathbb{E} \left[ \max_k V_k - \kappa_{jk} + \epsilon_k \right] - V_j \right)$$

- Using extreme-value taste shocks  $\rightarrow$  nice expression for migration flows
- Incredibly easy to solve on computer using tools you're learning in Section

### 3. Macro: sticky prices

- In the data, firms do not adjust prices instantly
- Price stickiness (nominal rigidities) is at the heart of the New Keynesian model
- We will derive the New Keynesian Phillips Curve as simple application of our tools  
⇒ much easier to derive in continuous time!!
- Consider a continuum of firms indexed by  $j$  that compete monopolistically
- Firm  $j$  faces demand function

$$Y_{j,t} = \left( \frac{P_{j,t}}{P_t} \right)^{-\epsilon} Y_t$$

where  $Y_t$  and  $P_t$  are aggregate (industry) demand and price index

- Firms produce intermediate varieties with the linear production function

$$Y_{j,t} = A_t N_{j,t}$$

- $A_t$  is aggregate productivity and  $N_{j,t}$  firm  $j$ 's labor demand
- Firm  $j$  sells at price  $P_{j,t}$ , profit = revenue net of operating expenses

$$\Pi_{j,t} = P_{j,t} Y_{j,t} - W_t N_{j,t}$$

- Firms maximize NPV of future profit streams, discounted at interest rate  $r$

- Firms set prices optimally over time by choosing inflation  $\dot{P}_{j,t} = P_{j,t}\pi_{j,t}$
- Firms pay quadratic adjustment cost  $\frac{\delta}{2}\pi_{j,t}^2 P_t Y_t$  to adjust nominal price
- Firm problem:

$$\max_{\{\pi_{j,t}, N_{j,t}\}_{t \geq 0}} \int_0^{\infty} e^{-rds} \left( P_{j,t} Y_{j,t} - W_t N_{j,t} - \frac{\delta}{2} \pi_{j,t}^2 P_t \right) dt,$$

- Firms are small and take as given  $\{W_t, Y_t, P_t\}_{t \geq 0}$  and initial condition  $P_{j,0}$
- Any two firms  $j$  and  $j'$  with same initial price  $P_{j,0} = P_{j',0}$  adopt identical inflation and production policies  $\implies$  we get back to representative firm

- Hamiltonian (state:  $P_{j,t}$ , control:  $\pi_{j,t}$ , multiplier:  $\eta_{j,t}$ ):

$$\mathcal{H}_t(P_{j,t}, \pi_{j,t}, \eta_{j,t}) = P_{j,t}^{1-\epsilon} P_t^\epsilon Y_t - \frac{W_t}{A_t} P_{j,t}^{-\epsilon} P_t^\epsilon Y_t - \frac{\delta}{2} \pi_{j,t}^2 P_t Y_t + \eta_{j,t} P_{j,t} \pi_{j,t}$$

- Conditions for optimum:

$$\dot{\eta}_{j,t} - \dot{\eta}_{j,t} = (1 - \epsilon) P_{j,t}^{-\epsilon} P_t^\epsilon Y_t + \epsilon \frac{W_t}{A_t} P_{j,t}^{-\epsilon-1} P_t^\epsilon Y_t + \eta_{j,t} \pi_{j,t}$$

$$0 = -\delta \pi_{j,t} P_t Y_t + \eta_{j,t} P_{j,t},$$

as well as the initial condition for the multiplier  $\eta_{j,0} = 0$

- Now we can impose symmetric equilibrium:  $P_{j,t} = P_t$  for all  $j$

$$i_t \eta_t - \dot{\eta}_t = (1 - \epsilon) P_t^{-\epsilon} P_t^{\epsilon} Y_t + \epsilon \frac{W_t}{A_t} P_t^{-\epsilon-1} P_t^{\epsilon} Y_t + \eta_t \pi_t$$

$$0 = -\delta \pi_t P_t Y_t + \eta_t P_t$$

- Or simply:

$$i_t \eta_t - \dot{\eta}_t = (1 - \epsilon) Y_t + \epsilon \frac{w_t}{A_t} Y_t + \eta_t \pi_t$$

$$\eta_t = \delta \pi_t Y_t$$

- Differentiating eq. 2 ( $\dot{\eta}_t = \delta \dot{\pi}_t Y_t + \delta \pi_t \dot{Y}_t$ ) yields:

$$\dot{\pi}_t = \pi_t \left( i_t - \pi_t - \frac{\dot{Y}_t}{Y_t} \right) - \frac{\epsilon}{\delta} \left( \frac{w_t}{A_t} - \frac{\epsilon - 1}{\epsilon} \right)$$

- From previous slide:

$$\dot{\pi}_t = \pi_t \left( i_t - \pi_t - \frac{\dot{Y}_t}{Y_t} \right) - \frac{\epsilon}{\delta} \left( \frac{w_t}{A_t} - \frac{\epsilon - 1}{\epsilon} \right)$$

- Last step: recall Euler equation of the representative household

$$\frac{\dot{C}_t}{C_t} = r_t - \rho$$

and use goods market clearing

$$Y_t = C_t$$

- **NKPC:**

$$\dot{\pi}_t = \rho \pi_t - \frac{\epsilon}{\delta} \left( \frac{w_t}{A_t} - \frac{\epsilon - 1}{\epsilon} \right)$$

## 4. IO: duopoly

- Consider continuous-time variant of Ericson-Pakes (1995) quality-ladder model
- Duopolistic competition: 2 firms  $i \in \{A, B\}$  produce good with quality  $q_t^i$  and maximize NPV of profits:  $\max \int_0^\infty e^{-rt} \pi_t^i dt$ . They compete over investments  $\iota_t^i$ :

$$\dot{q}^i = \iota_t^i - \delta q_t^i$$

- Profits  $\pi_t^i$  depend on both firms' product qualities  $\rightarrow$  state variables for recursive representation are  $\omega \equiv (\omega^A, \omega^B)$
- Best-response of firm  $A$  to firm  $B$  characterized by HJB

$$rV^A(\omega) = \pi^A(\omega) + \max_{\iota} \left\{ (\iota - \delta\omega^A)V_{\omega^A}^A(\omega) - \Phi(\iota) \right\} + (\iota^B - \delta\omega^B)V_{\omega^B}^A(\omega)$$

where  $\Phi(\cdot)$  is cost of investment, and best-response takes  $\iota^B$  as given

## 5. Public finance: tax competition

- Two countries,  $i \in \{A, B\}$ , setting corporate tax rates  $\tau_t^i$  on firms operating / headquartered in country  $i$
- Mass of multinational firms  $j$ , with  $\mu_t$  denoting % in country  $A$  at time  $t$
- Firms relocate activity / headquarters at rate  $\theta$  towards low-tax country:

$$d\mu_t = \theta\mu_t(\tau_t^B - \tau_t^A)^\gamma dt$$

- Country  $A$  maximizes tax revenue:  $\max \int_0^\infty e^{-\rho t} \tau_t^A \mu_t dt$ . Countries compete over taxes  $\{\tau_{it}\}$
- Dynamic Nash: country  $A$  sets  $\tau_t^A$  as best response taking  $\tau_t^B$  as given
- Recursive representation: the only state variable is  $\mu_t$

$$\rho V^A(\mu) = \max_{\tau^A} \left\{ \tau^A \mu + \theta \mu \left( \tau^B(\mu) - \tau^A \right)^\gamma \partial_\mu V^A(\mu) \right\}$$

Best response strategies:  $0 = \mu + \gamma \theta \mu (\tau^B(\mu) - \tau^A)^{\gamma-1} V_\mu^A(\mu)$