A Mathematical Experiments and Formal Validation

This appendix provides a formal mathematical characterization of the experimental procedures used to validate the Adaptive Quantum-Inspired Kolmogorov–Arnold Network (AQ-KAN). In accordance with the Design Science Research (DSR) methodology adopted throughout this work, the experiments are designed as *constructive numerical experiments* rather than empirical benchmarking exercises.

The objective is to validate AQ-KAN as a deterministic, bounded, interpretable, and scalable adaptive computational artifact, consistent with the methodology specified in Section 4 and the operator formalization in Appendix B.

A.1 Experimental Paradigm and Scope

All experiments satisfy the following criteria:

Architectural Verification Each experiment evaluates properties that must hold for any valid instantiation of AQ-KAN, independent of task semantics or dataset choice.

Deterministic Numerical Execution All update rules follow the Adaptive Grid Update Algorithm (AGUA) defined in Section 4.4 and Appendix B, ensuring exact reproducibility.

Minimal-Assumption Environments Synthetic grid- and lattice-based environments are employed to isolate adaptive mechanisms without confounding semantic complexity.

This paradigm aligns with standard numerical analysis practices for validating stability, invariance, and convergence of discrete dynamical systems.

A.2 State Space and Norm Preservation

Each grid cell C_{ij} maintains a normalized state

$$\psi_{ij}(t) = \begin{pmatrix} \alpha_{ij}(t) \\ \beta_{ij}(t) \end{pmatrix} \in \mathbb{R}^2, \qquad \|\psi_{ij}(t)\|_2 = 1. \tag{1}$$

The global system state satisfies

$$\mathcal{S}(t) \in (\mathbb{S}^1)^N, \tag{2}$$

where N is the number of grid cells.

Norm Invariance Experiment Across long-horizon executions, the observed quantity

$$\max_{i,j,t} |\|\psi_{ij}(t)\|_2 - 1|$$

remains bounded by machine precision, confirming Proposition C.1.

A.3 Determinism and Measurement Stability

Identically initialized AQ-KAN instances evolved under identical input sequences exhibit identical trajectories, confirming Lemma C.2.

Small perturbations applied immediately prior to measurement do not alter outputs away from decision boundaries, confirming robustness of the deterministic measurement operator (Lemma C.4).

A.4 Adaptive and Dependency Experiments

Localized input perturbations produce spatially localized adaptive responses that decay smoothly with distance, consistent with Lemma C.3.

Entanglement-inspired dependency weights exhibit monotonic decay with spatial and state distance. Ablation of entanglement mechanisms results in fragmented preference landscapes, establishing necessity of structured coupling.

A.5 Dimensional Generalization

All invariants observed in two-dimensional grids persist in three-dimensional grids, confirming Proposition C.6.

A.6 Summary

Appendix A establishes that AQ-KAN:

- defines a bounded, deterministic dynamical system,
- preserves numerical invariants by construction,
- adapts locally under non-stationarity,
- encodes explicit and interpretable dependencies, and
- scales linearly with grid size.

B Formalization of the Adaptive Grid Update Algorithm (AGUA)

This appendix provides a formal operator-level specification of the *Adaptive Grid Update Algorithm* (AGUA) underlying the Adaptive Quantum-Inspired Kolmogorov–Arnold Network (AQ-KAN). The formulation corresponds directly to the methodological description in Section 4.4 and establishes AGUA as a deterministic, bounded, and well-posed discrete-time update operator.

B.1 State Space Definition

Let the adaptive grid be defined as

$$\mathcal{G} = \{C_{ij} \mid i = 1, \ldots, I,\ j = 1, \ldots, J\}, \tag{3}$$

with total size $N = I \times J$.

Each grid cell maintains a normalized quantum-inspired state

$$\psi_{ij}(t) = \begin{pmatrix} \alpha_{ij}(t) \\ \beta_{ij}(t) \end{pmatrix} \in \mathbb{R}^2, \qquad \|\psi_{ij}(t)\|_2 = 1. \tag{4}$$

The global system state at iteration t satisfies

$$\mathcal{S}(t) \in (\mathbb{S}^1)^N. \tag{5}$$

B.2 Inputs and Neighborhood Structure

Each cell C_{ij} receives a bounded scalar input

$$u_{ij}(t) \in \mathbb{R}. \tag{6}$$

Each cell interacts only with a finite local neighborhood

$$\mathcal{N}(ij) \subset \mathcal{G}, \qquad |\mathcal{N}(ij)| = d, \tag{7}$$

where d is constant for structured grids.

B.3 AGUA Update Operator

AGUA defines a deterministic discrete-time operator

$$\mathcal{S}(t+1) = \mathcal{T}(\mathcal{S}(t), \mathcal{U}(t)), \tag{8}$$

where $\mathcal{U}(t) = \{u_{ij}(t)\}_{i,j}$.

Locality is enforced by construction: each update $\psi_{ij}(t+1)$ depends only on $\psi_{ij}(t)$, $u_{ij}(t)$, and states in $\mathcal{N}(ij)$.

B.4 Local Input Embedding

Each cell computes a bounded input embedding

$$\widetilde{\psi}_{ij}(t+1) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(u_{ij}(t)) \\ \sin(u_{ij}(t)) \end{pmatrix}. \tag{9}$$

This mapping is smooth and Lipschitz-continuous in $u_{ij}(t)$.

B.5 Entanglement Weight Computation

For neighboring cells C_{ij} and $C_{i'j'}$, dependency strength is encoded by

$$\omega_{(ij),(i'j')}(t) = f_\omega\left(|u_{ij}(t) - u_{i'j'}(t)|,\ \psi_{ij}(t),\ \psi_{i'j'}(t)\right), \tag{10}$$

where f_ω is bounded and Lipschitz-continuous.

B.6 Quantum-Inspired Gate Transformations

Each embedded state undergoes a linear transformation

$$\hat{\psi}_{ij}(t+1) = \mathbf{G}_{ij}(t)\, \widetilde{\psi}_{ij}(t+1), \tag{11}$$

where $\mathbf{G}_{ij}(t) \in \mathbb{R}^{2\times 2}$ is orthonormal.

B.7 Normalization and Projection

To ensure numerical robustness, each updated state is normalized:

$$\psi_{ij}(t+1) = \frac{\hat{\psi}_{ij}(t+1)}{\|\hat{\psi}_{ij}(t+1)\|_2}. \tag{12}$$

B.8 Deterministic Measurement

The measurement operator is defined as

$$\hat{s}_{ij}(t+1) = \arg\max\{\alpha_{ij}^2(t+1),\ \beta_{ij}^2(t+1)\}. \tag{13}$$

Measurement is read-only and does not influence subsequent updates.

B.9 Summary

Appendix B defines AGUA as a deterministic, norm-preserving, locality-constrained update operator that directly realizes the adaptive methodology described in Section 4.

C Formal Propositions and Lemmas

This appendix states the formal mathematical properties of AQ-KAN and the Adaptive Grid Update Algorithm (AGUA). Each result corresponds directly to architectural constraints defined in Section 4 and Appendix B.

C.1 State Norm Invariance

Proposition 1 (State Norm Invariance)**.** *If*

$$\|\psi_{ij}(0)\|_2 = 1 \quad \forall\, i, j, \tag{14}$$

then

$$\|\psi_{ij}(t)\|_2 = 1 \quad \forall\, t \geq 0. \tag{15}$$

C.2 Determinism

Lemma 1 (Deterministic Evolution)**.** *For fixed initial state $\mathcal{S}(0)$ and input sequence $\mathcal{U}(t)$, the evolution*

$$\mathcal{S}(t+1) = \mathcal{T}(\mathcal{S}(t), \mathcal{U}(t)) \tag{16}$$

is unique and deterministic.

C.3 Locality

Lemma 2 (Locality of Updates)**.** *Each update $\psi_{ij}(t+1)$ depends only on $\psi_{ij}(t)$, $u_{ij}(t)$, and states in $\mathcal{N}(ij)$.*

C.4 Stability

Lemma 3 (Lipschitz Continuity)**.** *There exists $L > 0$ such that*

$$\|\mathcal{S}(t+1) - \widetilde{\mathcal{S}}(t+1)\|_2 \leq L\, \|\mathcal{S}(t) - \widetilde{\mathcal{S}}(t)\|_2. \tag{17}$$

C.5 Well-Posedness and Scalability

Proposition 2 (Well-Posedness)**.** *AGUA defines a well-posed discrete-time dynamical system on $(\mathbb{S}^1)^N$.*

Proposition 3 (Linear-Time Implementability)**.** *Each AGUA iteration admits*

$$\mathcal{O}(N(d+M)) \tag{18}$$

time complexity and $\mathcal{O}(N)$ space complexity.

D Proof Sketches

This appendix provides proof sketches for the results stated in Appendix C. The arguments rely on standard results from linear algebra, numerical analysis, and discrete dynamical systems.

D.1 Proof Sketch for State Norm Invariance

AGUA updates consist of bounded nonlinear input embeddings, orthonormal linear transformations, and explicit normalization. Orthonormal operators preserve the ℓ_2 norm, and normalization enforces unit norm at each iteration. Induction over t establishes invariance.

D.2 Proof Sketch for Determinism

All update rules are explicit deterministic functions. No stochastic or probabilistic operations are present. Identical inputs therefore yield identical trajectories.

D.3 Proof Sketch for Locality

Each update references only bounded neighborhood states by construction. No global aggregation terms appear in the update equations.

D.4 Proof Sketch for Stability

Each component of the update operator is Lipschitz-continuous on a compact domain. Composition preserves Lipschitz continuity, yielding a global bound.

D.5 Proof Sketch for Scalability

All operations are per-cell or per-neighborhood with constant size. Summation over N cells yields linear time and space complexity.

D.6 Summary

Appendices C and D establish AQ-KAN as a deterministic, bounded, stable, locality-preserving, and linearly scalable adaptive computational framework.