Histogram Flattening

- Because the values $\mathbf{p}(k)$ represent the normalized frequencies of occurrence of each gray-level k, they can be thought of as the **probabilities** of the gray level k occurring at a given pixel.

$$\mathbf{p}(k) \approx \text{discrete probability density of gray-level k}$$

- We now define an intermediate image $\mathbb{J}_1$:

$$J_1(i, j) = \sum_{k=0}^{I(i, j)} \mathbf{p}(k)$$

the sum of the probabilities of all gray levels less than I(i,j)'s. Note that

$$0 \le J_1(i, j) \le 1$$

and

$$J_1(i, j) \le J_1(m, n) \quad \text{if} \quad I(i, j) \le I(m, n)$$

- The elements of the **cumulative probability** image $\mathbb{J}_1$ will be approximately linearly distributed between 0 and 1.

- We then linearly scale the intermediate image to cover the range 0, ..., K-1, produce the histogram-flattened image $\mathbb{J}$:

$$J(i, j) = \text{INT}\left[\ (K-1)\cdot J_1(i, j) + 0.5\ \right]$$

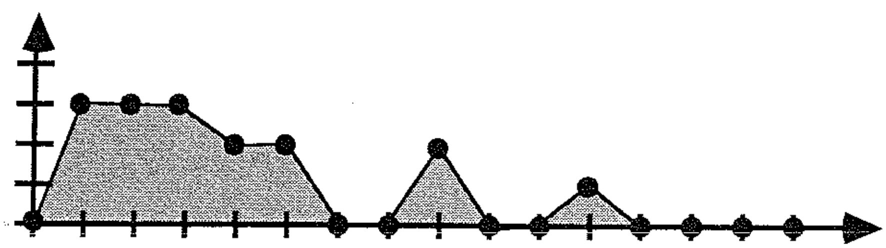
- This is best understood by an example:

Example - Let the 4 x 4 image $\mathbb{I}$ with allowable gray-level range $\{0, \ldots, 15\}$(i.e., K-1 = 15) be given by

$$\mathbb{I} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 4 \\ \hline 2 & 5 & 3 & 2 \\ \hline 8 & 1 & 8 & 2 \\ \hline 4 & 5 & 3 & 11 \\ \hline \end{array}$$

- It's histogram is

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
H(k)	0	3	3	3	2	2	0	0	2	0	0	1	0	0	0	0



- The normalized histogram is

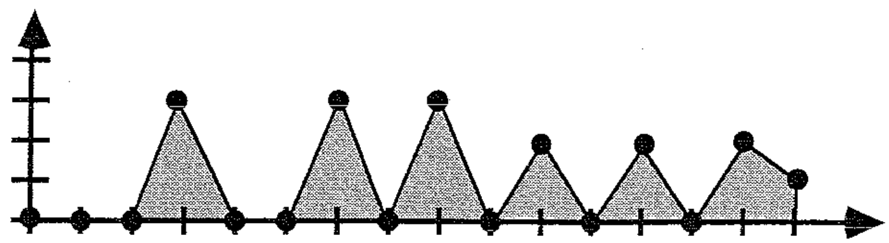
k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p(k)	0	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	0	0	$\frac{2}{16}$	0	0	$\frac{1}{16}$	0	0	0	0

- From which we can compute the intermediate image $\mathbb{J}_1$ and finally the "flattened" image $\mathbb{J}$:

$$\mathbb{J}_1 = \begin{array}{|c|c|c|c|} \hline 3/16 & 3/16 & 9/16 & 11/16 \\ \hline 6/16 & 13/16 & 9/16 & 6/16 \\ \hline 15/16 & 3/16 & 15/16 & 6/16 \\ \hline 11/16 & 13/16 & 9/16 & 16/16 \\ \hline \end{array} \qquad \mathbb{J} = \begin{array}{|c|c|c|c|} \hline 3 & 3 & 8 & 10 \\ \hline 6 & 12 & 8 & 6 \\ \hline 14 & 3 & 14 & 6 \\ \hline 10 & 12 & 8 & 15 \\ \hline \end{array}$$

- The new histogram looks like this:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
H(k)	0	0	0	3	0	0	3	0	3	0	2	0	2	0	2	1



- Histogram flattening doesn't really "flatten" the image - it just makes it "flatter" by **spreading out** the compressed histogram values **without changing them**.

- The spaces between values is highly characteristic of a "flattened" histogram (compare to the original).

Histogram Shaping

- We can generate an image $\mathbb{J}$ having an approximate **specified** histogram shape, such as a triangle or bell-shaped curve.
- Let **H**$'(k)$ be the desired histogram shape, with corresponding normalized values (probabilities) **p**$'(k)$.
- We define the cumulative probability image as before

$$J_1(i, j) = \sum_{k=0}^{I(i, j)} \mathbf{p}(k)$$

- We also define the cumulative probabilities:

$$\mathbf{P}'_n = \sum_{k=0}^{n} \mathbf{p}'(k)$$

- Algorithm:

  - Let $n(i, j)$ denote the **minimum value** of n such that

$$\mathbf{P}'_n \geq J_1(i, j)$$

  - Then $J(i, j) = n(i, j)$!

- This works since $n(i, j)$ is essentially the **inverse** of $\mathbf{P}'_n$:

$$n(i, j) = \mathbf{P}'^{-1}_n[J_1(i, j)]$$

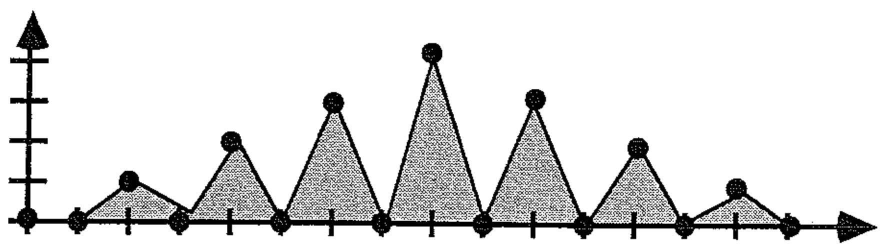
- By example:

Example - Consider the same image as in the last example. We had

$$\mathbb{I} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 5 & 3 & 2 \\ \hline 8 & 1 & 8 & 2 \\ \hline 4 & 5 & 3 & 11 \\ \hline \end{array} \qquad \mathbb{J}_1 = \begin{array}{|c|c|c|c|}\hline 3/16 & 3/16 & 9/16 & 11/16 \\ \hline 6/16 & 13/16 & 9/16 & 6/16 \\ \hline 15/16 & 3/16 & 15/16 & 6/16 \\ \hline 11/16 & 13/16 & 9/16 & 16/16 \\ \hline \end{array}$$

- Let's fit this to the following (triangular) histogram:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$H'(k)$	0	0	1	0	2	0	3	0	4	0	3	0	2	0	1	0
$p'(k)$	0	0	$\frac{1}{16}$	0	$\frac{2}{16}$	0	$\frac{3}{16}$	0	$\frac{4}{16}$	0	$\frac{3}{16}$	0	$\frac{2}{16}$	0	$\frac{1}{16}$	0



- Here's the cumulative (summed) probabilities associated with it:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P'_n	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{6}{16}$	$\frac{6}{16}$	$\frac{10}{16}$	$\frac{10}{16}$	$\frac{13}{16}$	$\frac{13}{16}$	$\frac{15}{16}$	$\frac{15}{16}$	$\frac{16}{16}$	$\frac{16}{16}$

- **Careful** visual inspection of $\mathbb{J}_1$ let's us form the new image:

$$\mathbb{J} = \begin{array}{|c|c|c|c|} \hline 4 & 4 & 8 & 10 \\ \hline 6 & 10 & 8 & 6 \\ \hline 12 & 4 & 12 & 6 \\ \hline 10 & 10 & 8 & 14 \\ \hline \end{array}$$

- Here's the new histogram:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
H(k)	0	0	0	0	3	0	3	0	3	0	4	0	2	0	1	0