UNIVERSITY OF BIRMINGHAM

Visualisation

Week 3

Intro to Fourier Analysis

Signals

- A <u>signal</u> is a measure of some physical phenomenon, e.g., an electrical current, the temperature at an airport, etc.
- It is a function of one or more variables.

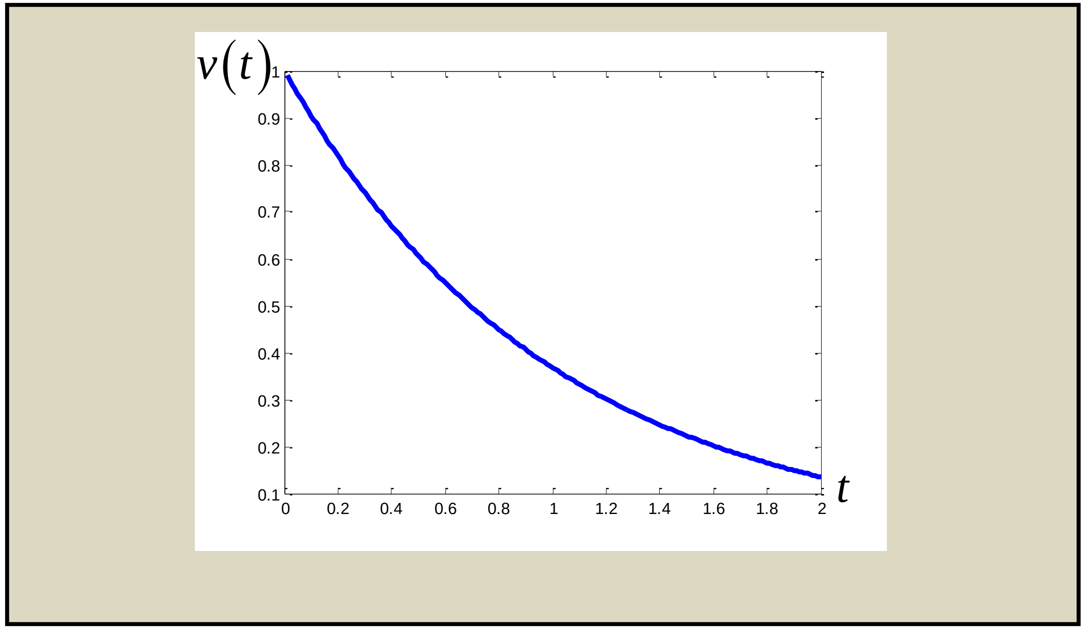
- Many popular signals in electrical and computer engineering are functions of a single variable, usually time.

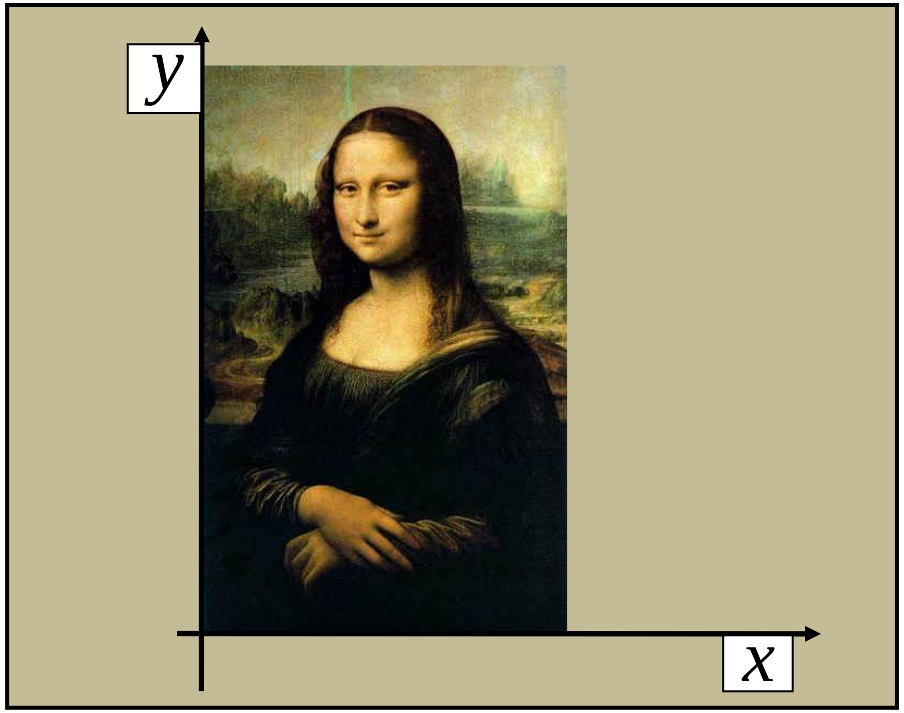
- A signal is a mathematical representation of a physical phenomenon under consideration.
- This physical phenomenon normally changes over time.

Signals

- Many 1-dimensional signals are expressed as functions of time:
$$f(t), g(t), i(t), v(t)$$

- Many 2-dimensional signals are functions of space:
$$r(x,y), g(x,y), b(x,y)$$





Discrete-time Signals

- If a signal is defined (only) over discrete values of time, then we have a discrete-time signal.
- In this case, the signal is not defined over (continuum) segments of time. An example of a discrete-time signal is:

$$x(t) = \exp(-t) \text{ where } t = nT = 0,0.1,0.2,0.3, \ldots$$

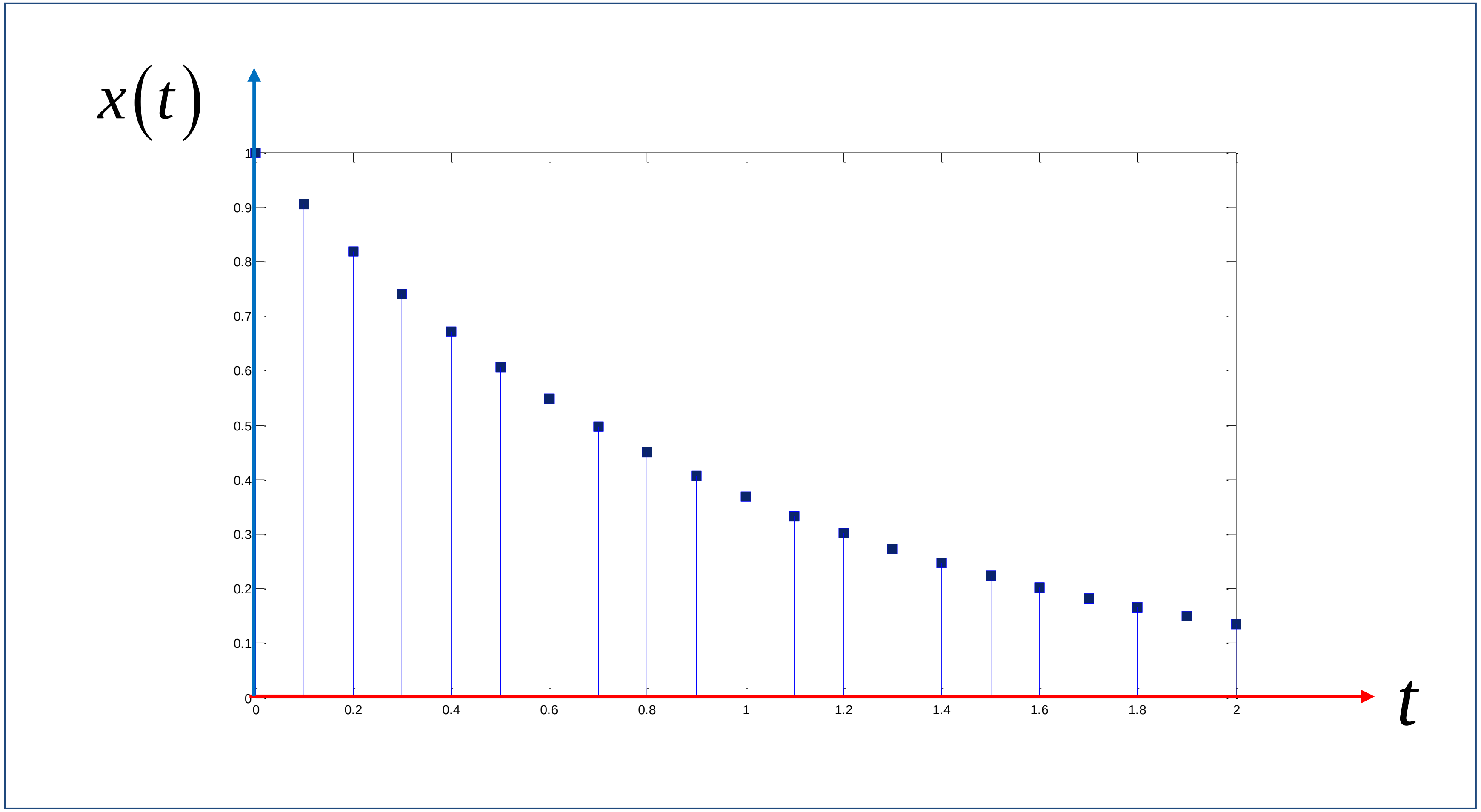
Note:

Discrete-time signals are not defined at other times.

Discrete-time Signals

This is called a stem plot and makes the discrete-time nature of the signal explicit.

<u>Time-series are discrete-time signals.</u>



Periodic and Aperiodic Signals

- A signal is periodic if it meets the following property for all time t:

$$x(t) = x(t + T_0) \quad \forall t, T_0 > 0$$

- The smallest value of T_0 that satisfies this condition is called the period or cycle of the periodic signal $x(t)$.

- If a signal is not periodic then it is aperiodic.

Frequency

- Periodicity of a signal is also captured by another quantity: frequency.
- Frequency of a signal is the multiplicative inverse of its period.

$$f_0 = \frac{1}{T_0}, \qquad \text{units: Cycles per second (cps), Hertz (Hz)}$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}, \qquad \text{units: Radians per second}$$

Signal Energy

- Signal energy can be measured by integrating the signal square $x^2(t)$ over an interval of time.
- The energy of a continuous time signal $x(t)$ in interval $t \in [t_a, t_b]$ is,

$$E = \int_{t_a}^{t_b} x^2(t)dt$$

- For a discrete-time signal,

$$E = \sum_{n=a}^{b} x^2[n]$$

Total Signal Energy

- To measure the total energy E_x of a signal $x(t)$, the time interval is extended from $-\infty$ to ∞, i.e.:

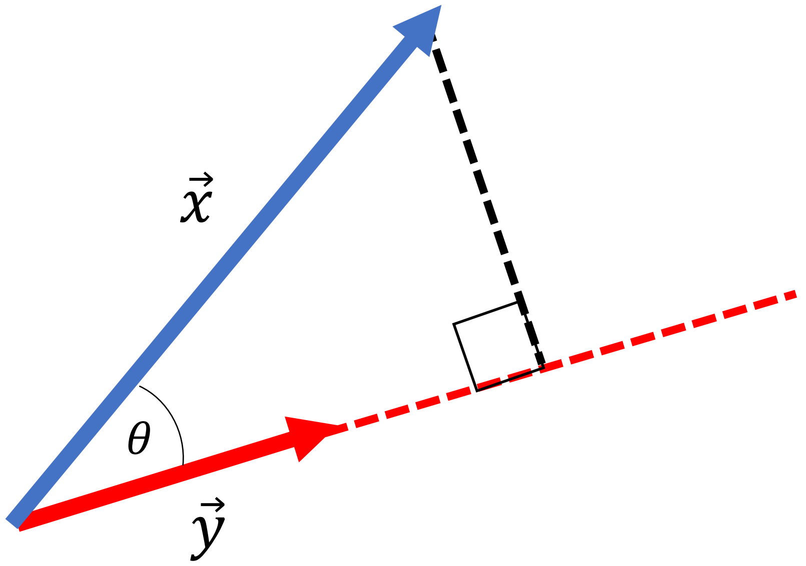
$$E_\infty = \int_{-\infty}^{+\infty} x^2(t)dt\,, \qquad E_\infty = \sum_{n=-\infty}^{+\infty} x^2[n]$$

- A signal $x(t)$ whose total energy is finite, $0 < E_\infty < \infty$, is known as an energy signal.

Quantifying Similarity: Vectors

Similarity of $\vec{x}$ and $\vec{y}$ = length of projection of $\vec{x}$ along $\vec{y}$

$$|\vec{x}|\cos\theta \times |\vec{y}| = \vec{x}\cdot\vec{y}$$



Also known as the Dot product.

Also computable without finding θ first:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y} = x_1y_1 + x_2y_2$$

Quantifying Signal Similarity: Discrete-time

For arbitrary, k-dimensional vectors:

$$\vec{x}^T\vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ky_k = \sum_{i=1}^{k} x_iy_i$$

- Time-series = vector
- Can use dot/scalar product of two vectors to quantify the similarities of two time-series.

Date	Price	Vector $\boldsymbol{x}$	Function $f(t)$
1987-05-20	18.63	x_1	$f(1)$
1987-05-21	18.45	x_2	$f(2)$
1987-05-22	18.55		
1987-05-25	18.60	$\vdots$	$\vdots$
1987-05-26	18.63		
...	...		
2020-08-24	44.43	x_t	$f(t)$
2020-08-25	46.01		
2020-08-26	45.79	$\vdots$	$\vdots$
2020-08-27	44.84		
2020-08-28	45.22	x_n	$f(n)$

Quantifying Signal Similarity: Continuous-time

- What is the equivalent operation for (periodic) continuous-time signals?

Discrete-time:

$$x[n]^T y[n] = \sum_{n=0}^{T_0-1} x[n]y[n]$$

Continuous-time:

$$x(t)y(t) =?$$

Quantifying Signal Similarity: Continuous-time

- What is the equivalent operation for (periodic) continuous-time signals?

Discrete-time:

$$x[n]^T y[n] = \sum_{i=0}^{T_0-1} x[i]y[i]$$

Continuous-time:

$$\text{Similarity}\big(x(t), y(t)\big) = \int_0^{T_0} x(t)y(t)dt$$

Sum of Scaled Sine and Cosine Functions

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

Where,

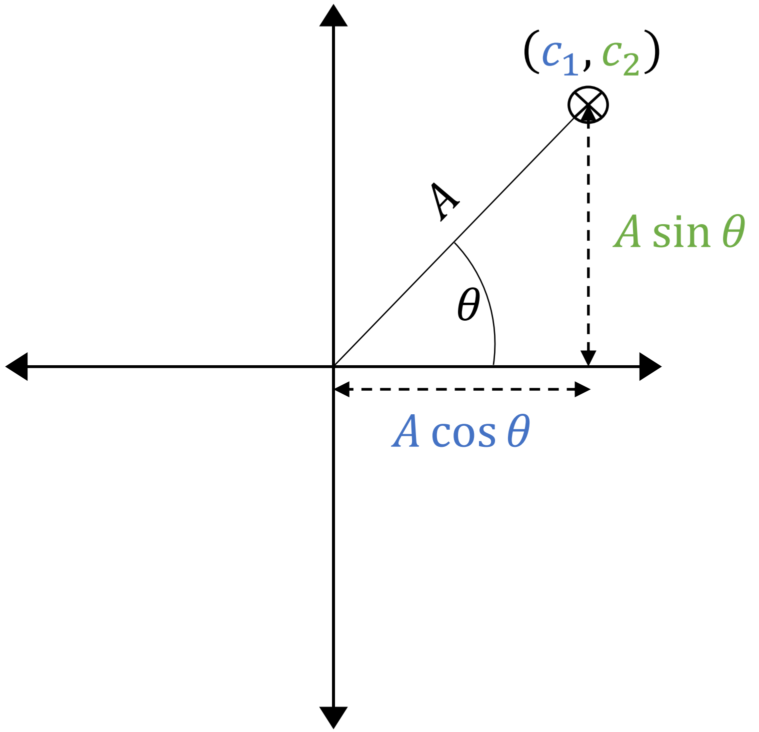
$$A =, \sqrt{c_1^2 + c_2^2}, \qquad \theta = \tan^{-1} \frac{c_2}{c_1}$$

$$= A \cos \theta \cos \omega_0 t + A \sin \theta \sin \omega_0 t$$
$$= A(\cos \theta \cos \omega_0 t + \sin \theta \sin \omega_0 t)$$

Since,

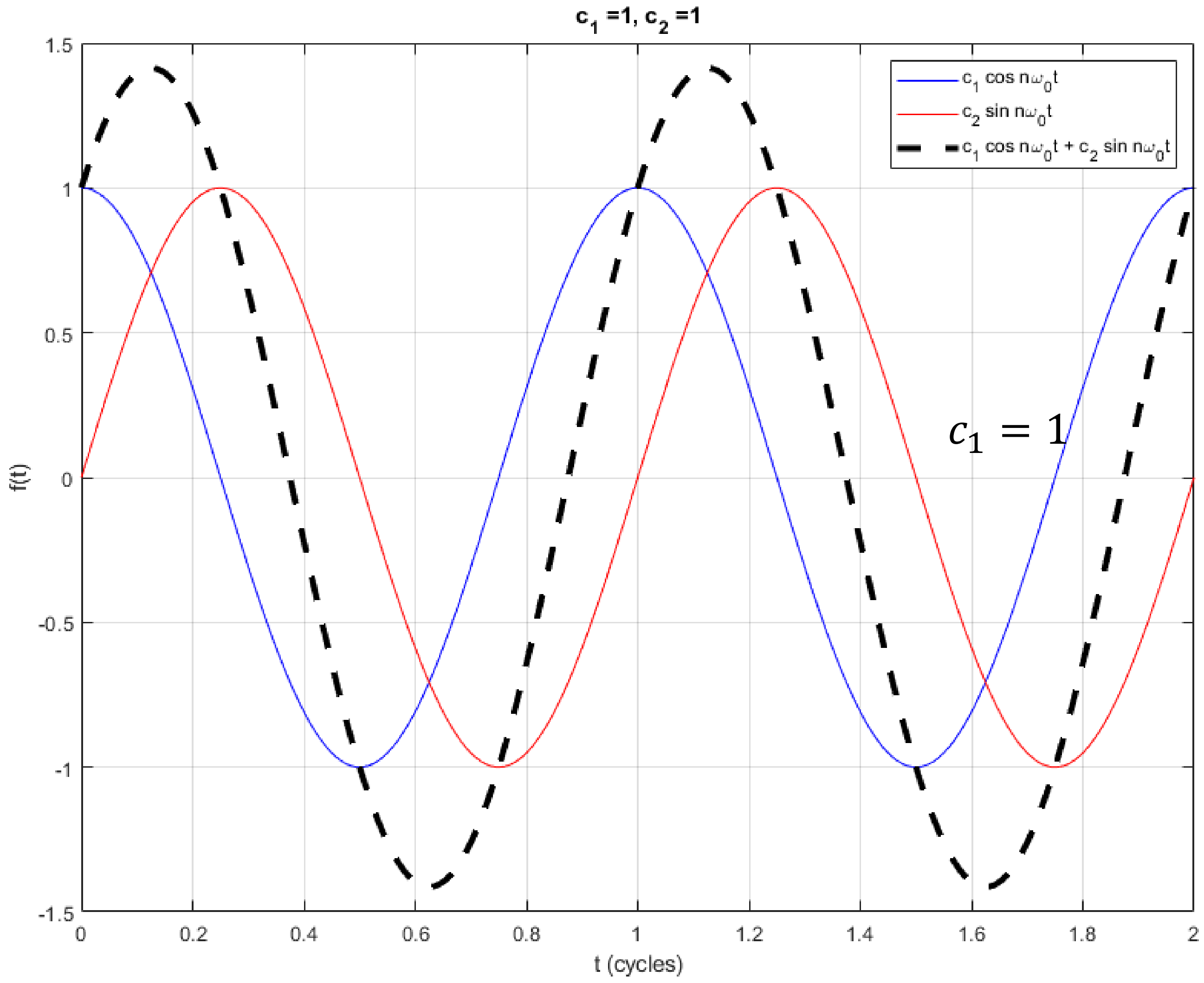
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$x(t) = A \cos(\omega_0 t - \theta)$$



The sum of scaled sine and cosine functions of same frequency is a phase-shifted sinusoid.

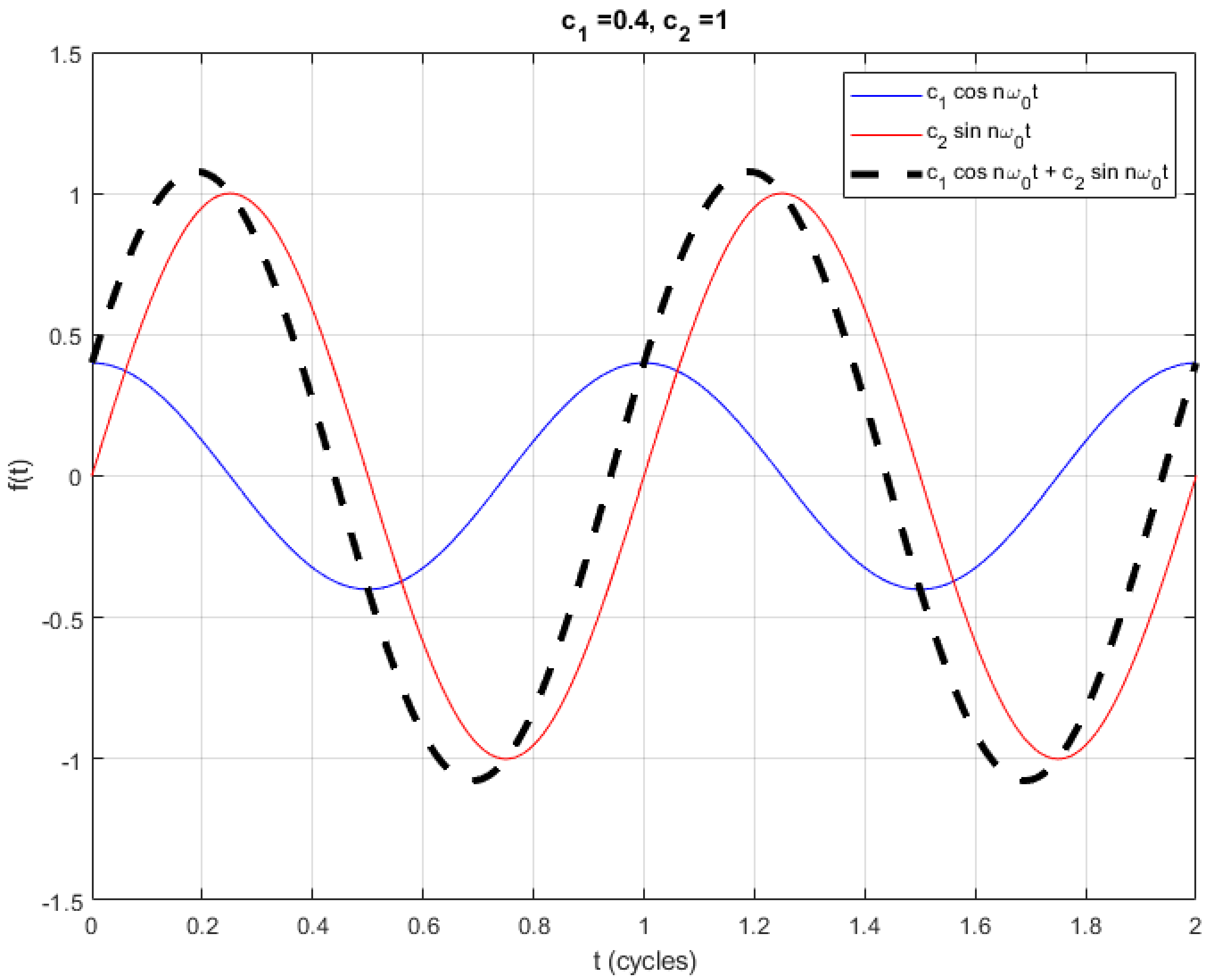
Sum of Scaled Sine and Cosine Functions



$$c_1 \cos n\omega_0 t + c_2 \sin n\omega_0 t$$

- $c_1 = 1$
- $c_2 = 1$

Sum of Scaled Sine and Cosine Functions



$c_1 \cos n\omega_0 t + c_2 \sin n\omega_0 t$

- $c_1 = 0.4$
- $c_2 = 1$

Fourier Theorem

- According to the Fourier theorem, a periodic signal is composed of a series of sinusoidal components whose frequencies are those of the fundamental and its harmonics (multiples), each component having the proper amplitude and phase.
- The sequence of components that form this signal is called its spectrum.

[Britannica]

Big Picture Challenge

- Consider a periodic, continuous-time signal $x(t)$ of frequency f_0 / ω_0 and period T_0.

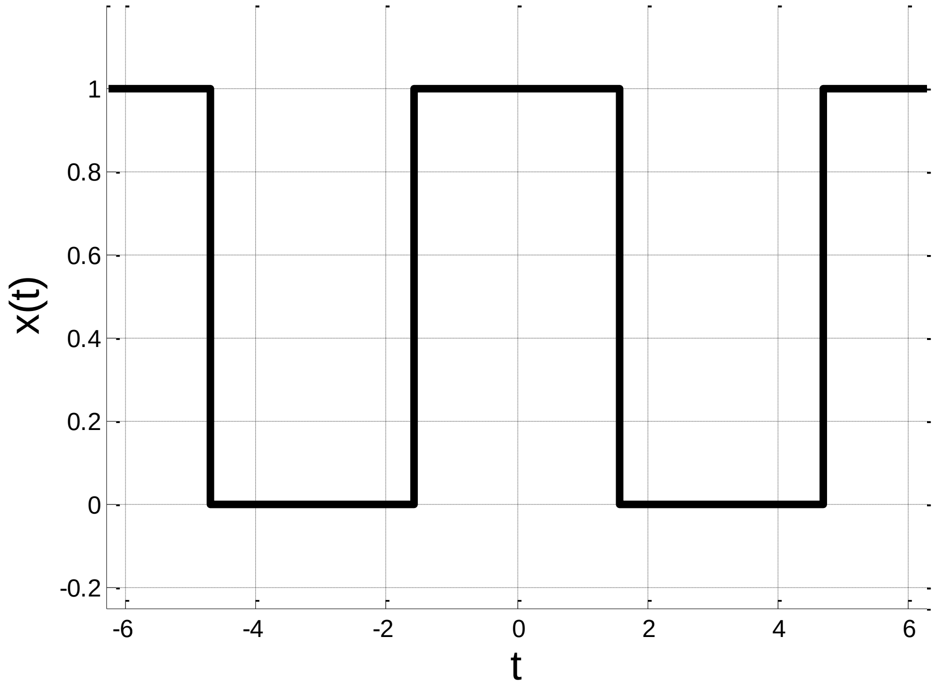
With the tools we have so far, how can we determine the degree to which a sinusoid of a certain frequency is a component of $x(t)$?

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

Example: Fourier Series of Pulse Train Function

- Consider the following periodic, continuous-time function with $T = 2\pi$:

$$x(t) = \begin{cases} 1, \text{for } t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0, \qquad \text{otherwise} \end{cases}$$



- According to the Fourier theorem, this signal can be expressed as the sum of sinusoids:

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

Example: Fourier Series of Pulse Train Function

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

Low-frequency components

High-frequency components

- Let us view partial reconstructions of $x(t)$ by adding successively higher frequency components.

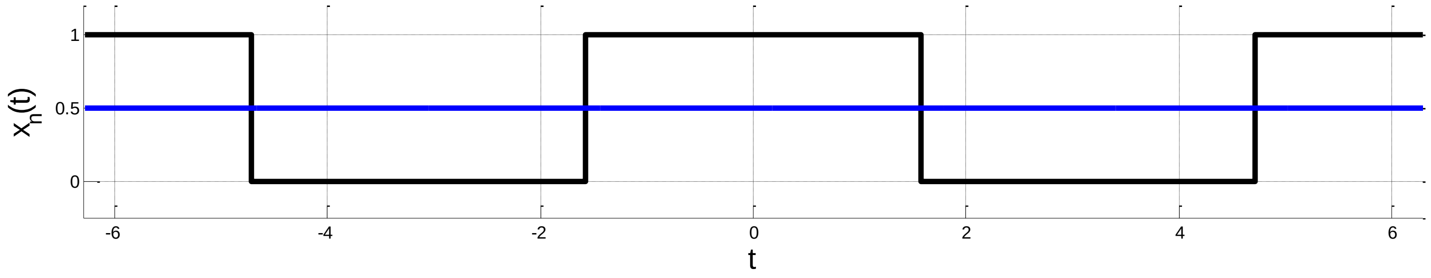
Example: Fourier Series of Pulse Train Function

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

Low-frequency components

High-frequency components

- This is a gross approximation of $x(t)$ using only the lowest frequency component (0 frequency, the DC component).

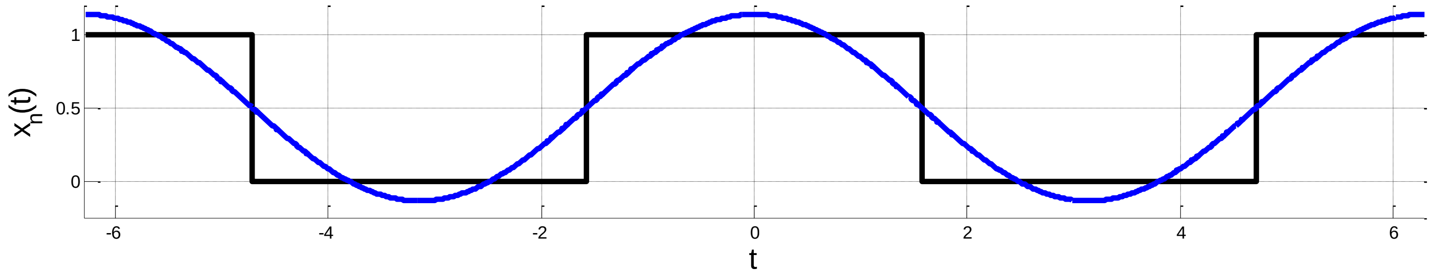


Example: Fourier Series of Pulse Train Function

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

Low-frequency components — High-frequency components

- Now we add the lowest frequency component, the fundamental frequency.



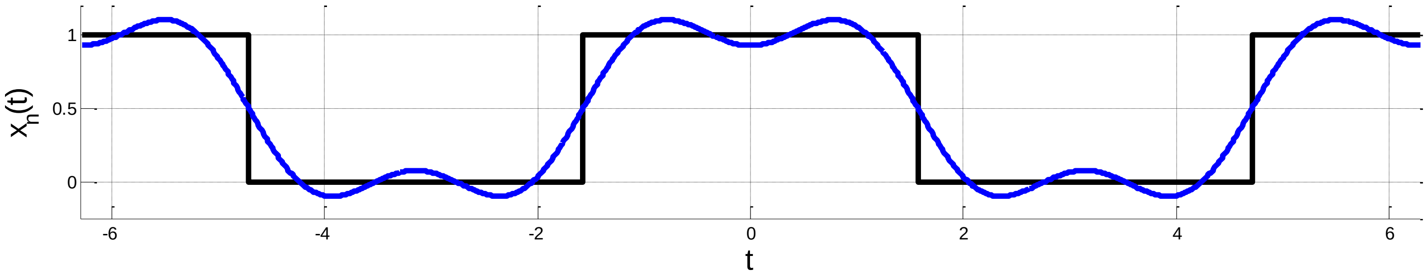
Example: Fourier Series of Pulse Train Function

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

Low-frequency components

High-frequency components

- Now we add the next lowest frequency component, the 3rd harmonic.

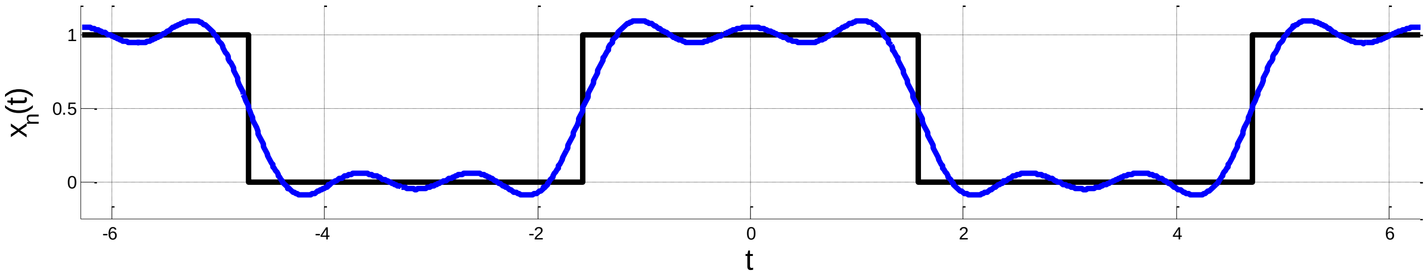


Example: Fourier Series of Pulse Train Function

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

Low-frequency components ⟷ High-frequency components

- Now we add the next lowest frequency component, the 5th harmonic.

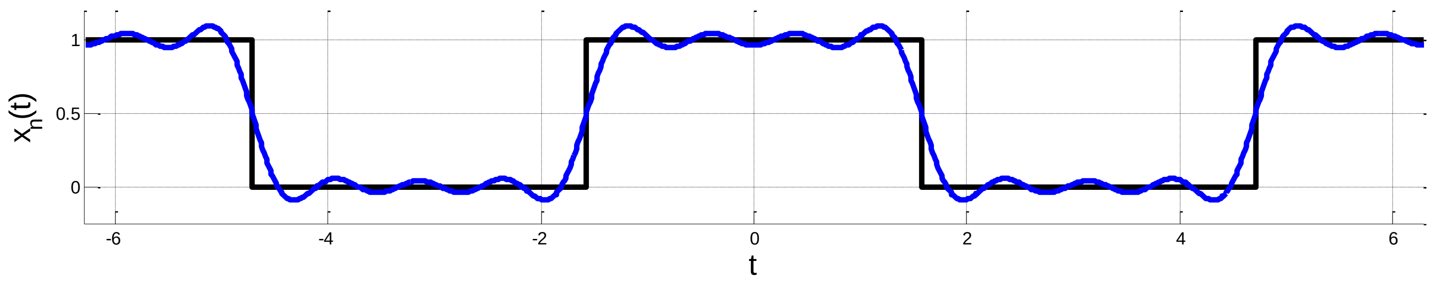


Example: Fourier Series of Pulse Train Function

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

Low-frequency components ⟷ High-frequency components

- Now we add the next lowest frequency component, the 7th harmonic.



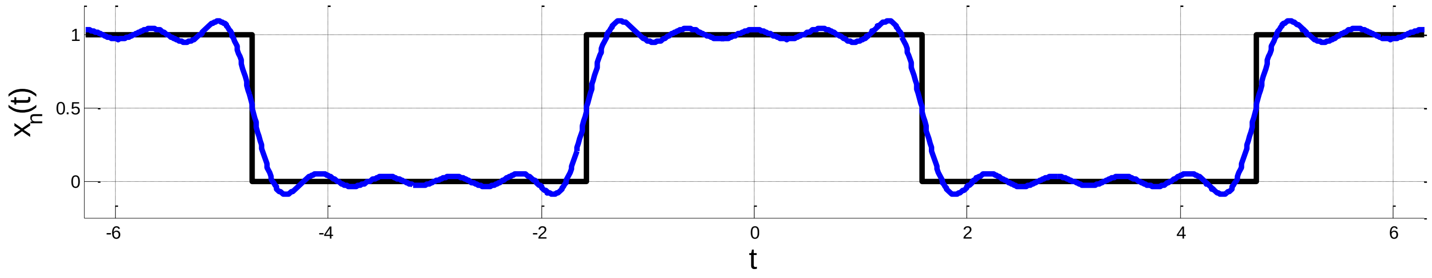
Example: Fourier Series of Pulse Train Function

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

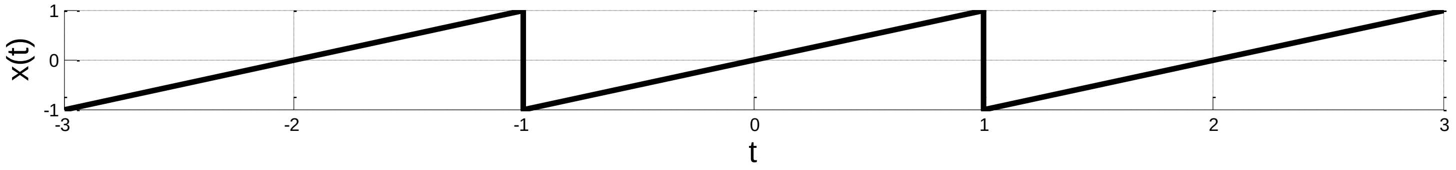
Low-frequency components

High-frequency components

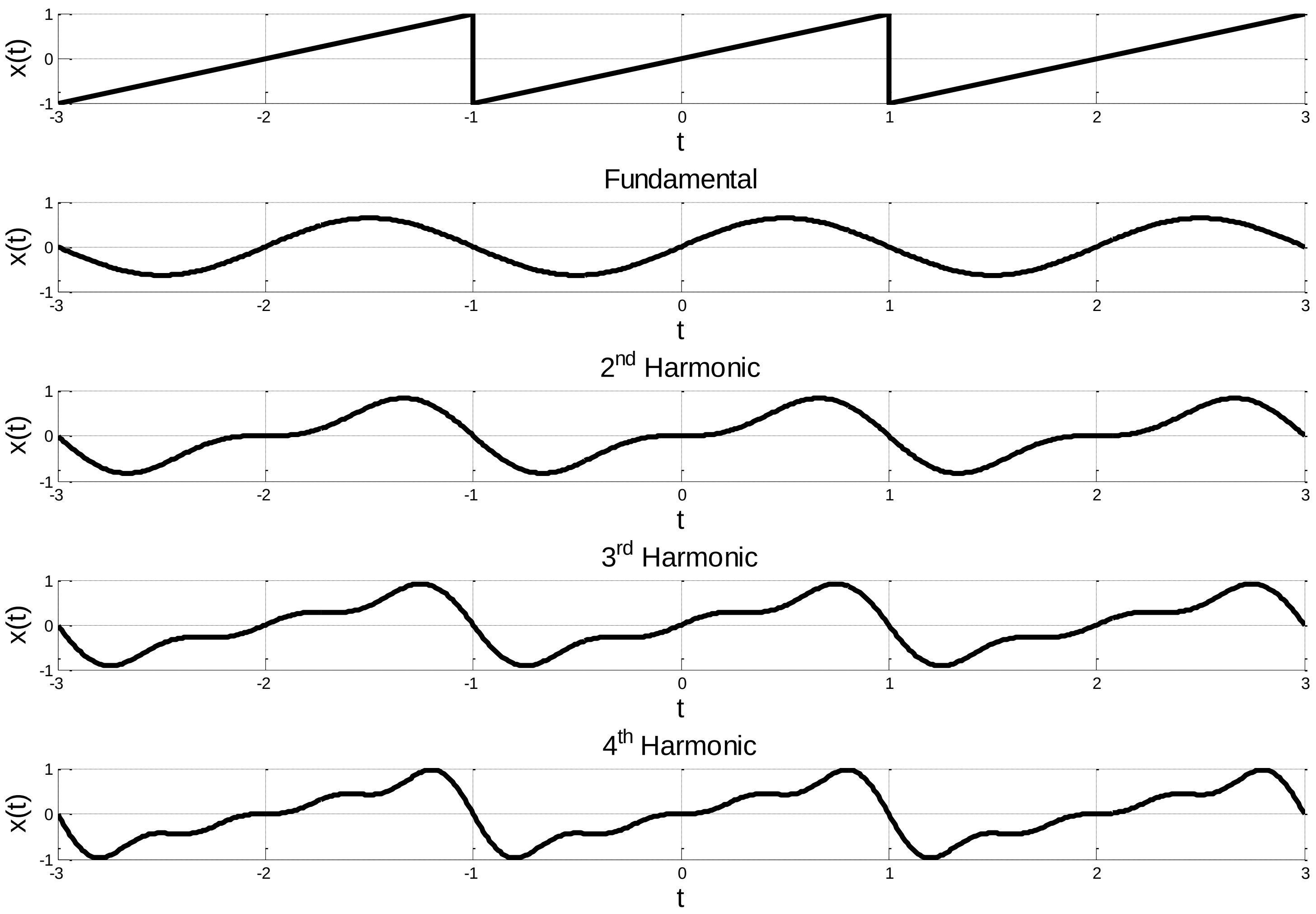
- Now we add the next lowest frequency component, the 9$^{\text{th}}$ harmonic.



Example 2: Reconstructing a Sawtooth Function



Example 2: Reconstructing a Sawtooth Function



The Trigonometric Fourier Series
(for Periodic, Continuous-time Signals)

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

Where,

$$a_0 = \frac{1}{T_0} \int_{T_0} f(t) dt$$

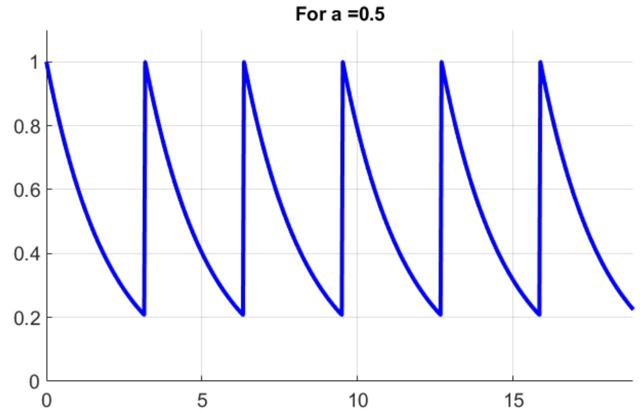
$$a_n = \frac{2}{T_0} \int_{T_0} f(t) \cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T_0} \int_{T_0} f(t) \sin n\omega_0 t \, dt$$

Note:
Frequency coefficients a_n and b_n are similarity measures of signal $f(t)$ with sinusoids of integral multiple n frequencies of fundamental frequency ω_0.

A Simple, From-the-ground-up Example

- Consider the following function $x(t) = e^{-at}$ in the interval $[0, \pi]$ with period $T_0 = \pi$.
- First, let us evaluate ω_0:



$$\omega_0 = \frac{2\pi}{T_0} = 2 \text{ rad/sec}$$

- Therefore,

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

$$= a_0 + \sum_{n=1}^{\infty} [a_n \cos 2nt + b_n \sin 2nt]$$

A Simple, From-the-ground-up Example

- Now, for a_0,

$$a_0 = \frac{1}{\pi} \int_{T_0} x(t)dt = \frac{1}{\pi} \int_0^{\pi} e^{-at} dt$$

$$= \frac{1}{\pi} \left| \frac{-1}{a} e^{-at} \right|_0^{\pi} = \frac{1}{\pi} \left| \frac{-1}{a} e^{-a\pi} - \frac{1}{a} e^{0} \right|_0^{\pi}$$

$$= \frac{1}{a\pi} (1 - e^{-a\pi})$$

A Simple, From-the-ground-up Example

- For a_n,

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-at}\cos 2nt\,dt = \frac{2}{\pi}\left[\left|\frac{e^{-at}}{a^2+4n^2}(-a\cos 2nt + 2n\sin 2nt)\right|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{-a\pi}}{a^2+4n^2}(-a\cos 2n\pi + 2n\sin 2n\pi) - \frac{1}{a^2+4n^2}(-a\cos 0 + 2n\sin 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{-a\pi}}{a^2+4n^2}(-a) - \frac{1}{a^2+4n^2}(-a)\right] = \frac{2}{\pi}\left[\frac{1-e^{-a\pi}}{a^2+4n^2}\right]a$$

A Simple, From-the-ground-up Example

- For b_n,

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-at}\sin 2nt\,dt = \frac{2}{\pi}\left[\left|\frac{e^{-at}}{a^2+4n^2}(-a\sin 2nt + 2n\cos 2nt)\right|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{-a\pi}}{a^2+4n^2}(-a\sin 2n\pi + 2n\cos 2n\pi) - \frac{1}{a^2+4n^2}(-a\sin 0 + 2n\cos 0)\right]$$

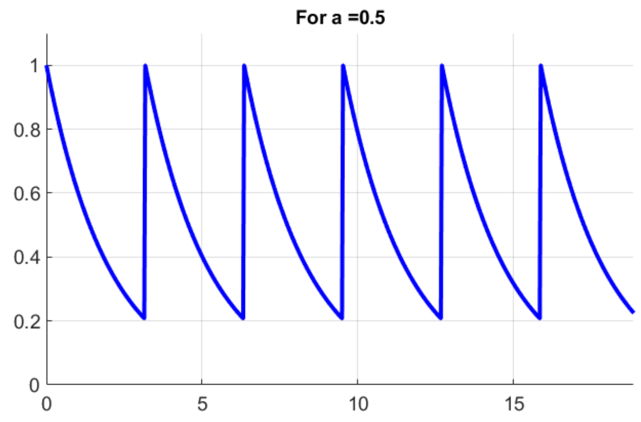
$$= \frac{2}{\pi}\left[\frac{e^{-a\pi}}{a^2+4n^2}(-2n) - \frac{1}{a^2+4n^2}(-2n)\right] = \frac{2}{\pi}\left[\frac{1-e^{-a\pi}}{a^2+4n^2}\right]2n$$

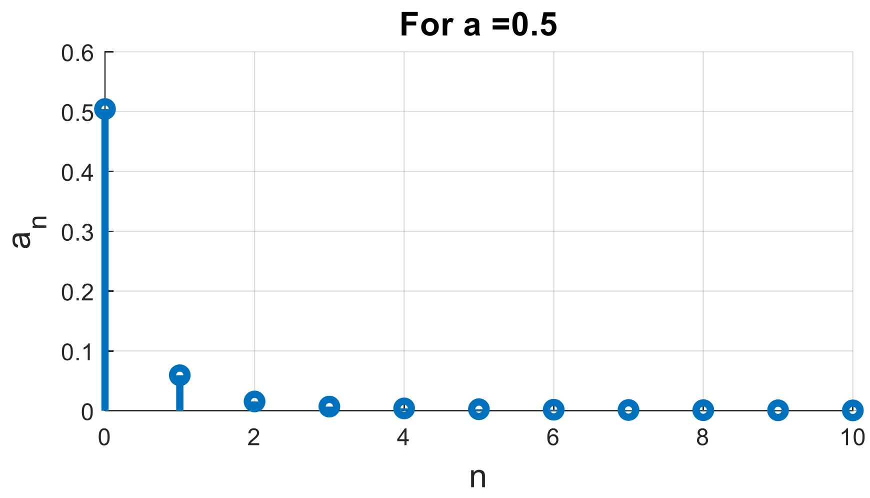
A Simple, From-the-ground-up Example

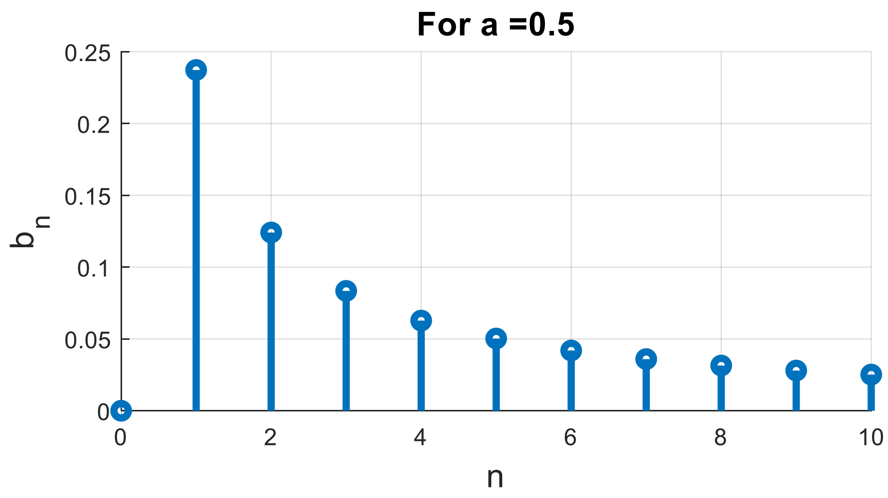
- Then, the Trigonometric Fourier Series form of $x(t)$ is,

$$x(t) = \frac{1}{a\pi}(1 - e^{-a\pi}) + \sum_{n=1}^{\infty}\left[\begin{array}{l} \frac{2}{\pi}\left[\frac{1 - e^{-a\pi}}{a^2 + 4n^2}\right] \mathrm{a}\cos 2nt \\ + \frac{2}{\pi}\left[\frac{1 - e^{-a\pi}}{a^2 + 4n^2}\right] 2n \sin 2nt \end{array}\right]$$

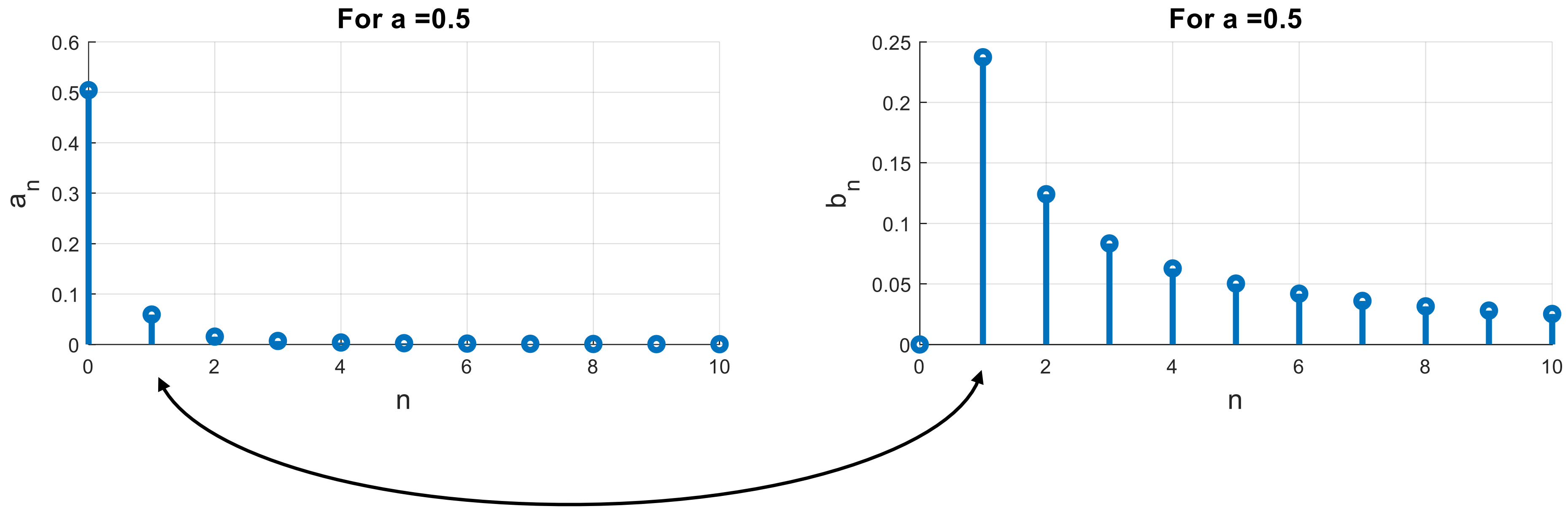
A Simple, From-the-ground-up Example







A Simple, From-the-ground-up Example



- Analyzing frequency components of $n\omega_0$ of a signal requires looking at two numbers, a_0 and b_0 → Complicated.

Is there an alternative form of the Fourier Series that uses a <u>single number</u> to represent the strength of a single frequency component?

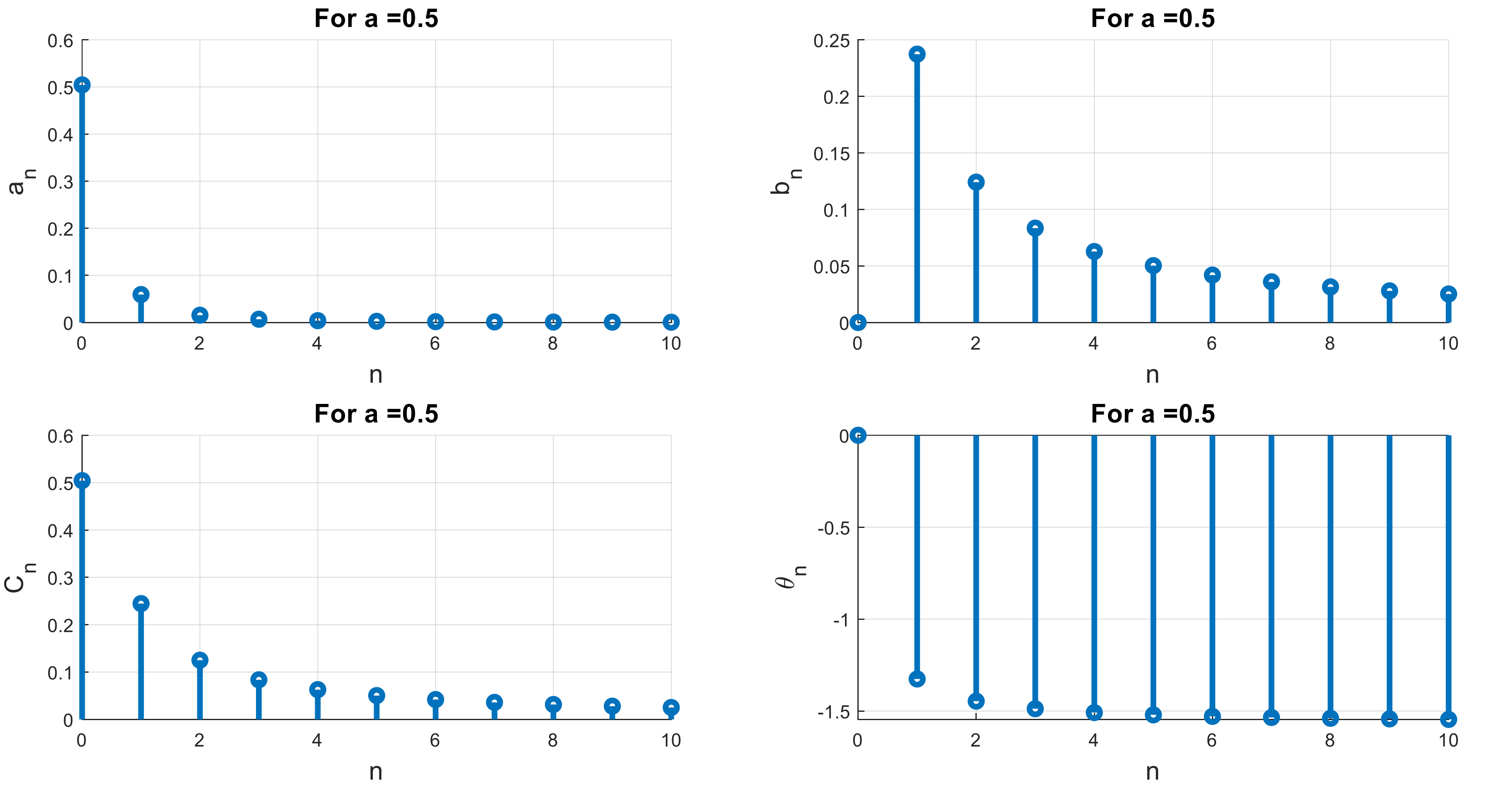
Fourier Series Representations of Periodic Signals

Fourier Series Form	Coefficient Computation
Trigonometric $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$	$a_0 = \frac{1}{T_0}\int_{T_0} f(t)dt$ $a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\, dt$ $b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin n\omega_0 t\, dt$

Fourier Series Representations of Periodic Signals

Fourier Series Form	Coefficient Computation	Conversion Formula
Trigonometric $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$	$a_0 = \frac{1}{T_0}\int_{T_0} f(t)dt$ $a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\, dt$ $b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin n\omega_0 t\, dt$	$a_0 = C_0 = D_0$ $a_n - jb_n = C_n e^{j\theta_n} = 2D_n$ $a_n + jb_n = C_n e^{-j\theta_n} = 2D_{-n}$
Compact Trigonometric $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$	$C_0 = a_0$ $C_n = \sqrt{a_n^2 + b_n^2}$ $\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$	$C_0 = D_0$ $C_n = 2\lvert D_n \rvert \quad n \geq 1$ $\theta_n = \angle D_n$

A Simple, From-the-ground-up Example (Compact Trigonometric Form)

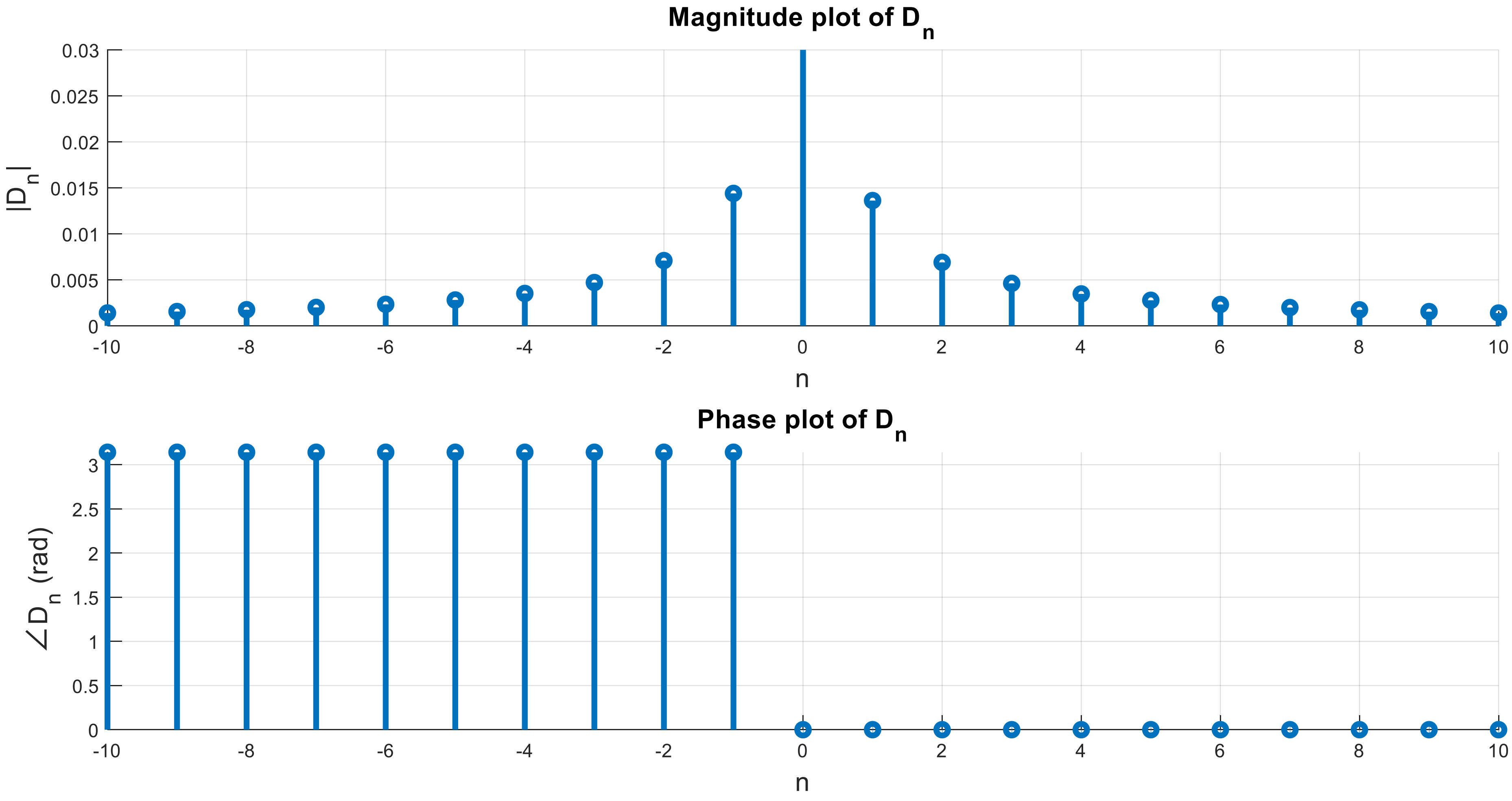


Fourier Series Representations of Periodic Signals

Fourier Series Form	Coefficient Computation	Conversion Formula
Trigonometric $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$	$a_0 = \frac{1}{T_0}\int_{T_0} f(t)dt$ $a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\, dt$ $b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin n\omega_0 t\, dt$	$a_0 = C_0 = D_0$ $a_n - jb_n = C_n e^{j\theta_n} = 2D_n$ $a_n + jb_n = C_n e^{-j\theta_n} = 2D_{-n}$
Compact Trigonometric $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$	$C_0 = a_0$ $C_n = \sqrt{a_n^2 + b_n^2}$ $\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$	$C_0 = D_0$ $C_n = 2\lvert D_n \rvert \quad n \geq 1$ $\theta_n = \angle D_n$
Exponential $f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$	$D_n = \frac{1}{T_0}\int_{T_0} f(t) e^{-jn\omega_0 t} dt$	-

$e^{jn\omega_0 t} = \cos n\omega_0 t + j \sin n\omega_0 t$ $\qquad$ $e^{-jn\omega_0 t} = \cos n\omega_0 t - j \sin n\omega_0 t$

A Simple, From-the-ground-up Example (Complex Exponential Form)



Fourier Analysis of Signals

	Continuous-time signal	Discrete-time signal
Periodic Discrete Spectra	Fourier Series	Discrete Fourier Transform (DFT)
Aperiodic Continuous Spectra	Fourier Transform	Discrete-time Fourier Transform (DTFT)

Discrete Fourier Transform

- The Discrete Fourier Transform (DFT) is the analog of the Fourier Transform for finite, discrete-time signal $x[n]$ of length N.
- The DFT treats the signal *as if it were periodic*.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn}$$ Inverse DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$ DFT

DFT Implementation

- A direct implementation of the DFT relies on matrix-vector multiplication with time-complexity $O(n^2)$.

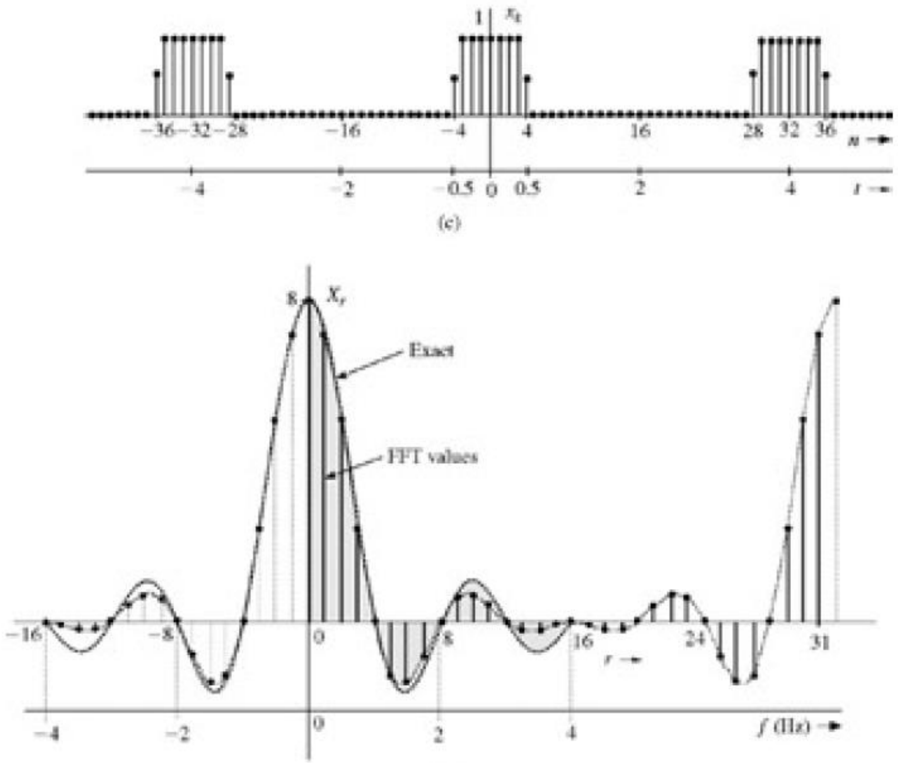
$$
\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{1\cdot 1} & W_N^{1\cdot 2} & W_N^{3} & \cdots & W_N^{1\cdot(N-1)} \\ 1 & W_N^{2\cdot 1} & W_N^{2\cdot 2} & W_N^{2\cdot 3} & \cdots & W_N^{2\cdot(N-1)} \\ 1 & W_N^{3\cdot 1} & W_N^{3\cdot 2} & W_N^{3\cdot 3} & \cdots & W_N^{3\cdot(N-1)} \\ \vdots & & & & & \\ 1 & W_N^{(N-1)\cdot 1} & W_N^{(N-1)\cdot 2} & W_N^{(N-1)\cdot 3} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}^{\text{DFT Matrix}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}
$$

Where,

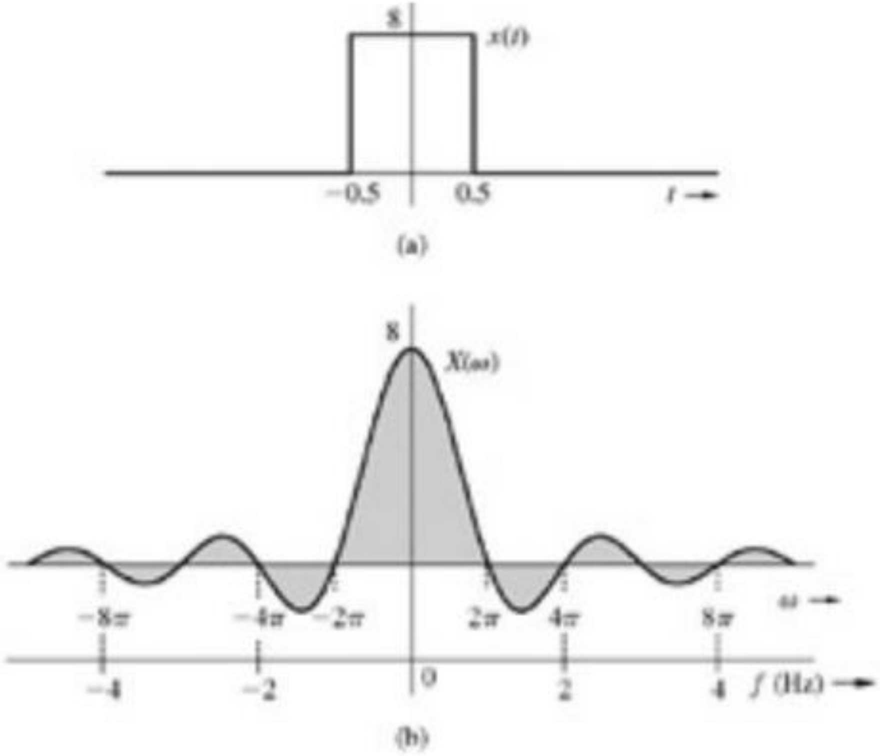
$$W_N = e^{-j\frac{2\pi}{N}}$$

Example: DFT (and the link with CT domain)

DFT



Fourier Transform



Linear Systems and Signals, 2nd ed., B.P. Lathi, 2005, pp. 809, Oxford University Press

DFT Implementation

- There are several algorithms that implement the DFT.
- The most popular by far is called the ***Fast Fourier Transform (FFT)***.
- The FFT's time-complexity is $O(n \log n)$.

Summary

- Signals can be decomposed into sinusoids.
    - True for continuous-time and discrete-time (time series).

- DFT treats a given signal as periodic, resulting a discrete spectral representation.
    - Useful for Fourier analysis of time-series.

- FFT is an efficient $O(n \log n)$ implementation of the otherwise $O(n^2)$ DFT.