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Wasserstein Distributionally Robust Motion Planning and Control with Safety Constraints Using Conditional Value-at-Risk

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Abstract— **In this paper, we propose an optimization-based decision-making tool for safe motion planning and control in an environment with randomly moving obstacles. The unique feature of the proposed method is that it limits the risk of unsafety by a pre-specified threshold even when the true probability distribution of the obstacles' movements deviates, within a *Wasserstein ball*, from an available empirical distribution. Another advantage is that it provides a *probabilistic out-of-sample performance* guarantee of the risk constraint. To develop a computationally tractable method for solving the distributionally robust model predictive control problem, we propose a set of reformulation procedures using (i) the Kantorovich duality principle, (ii) the extremal representation of conditional value-at-risk, and (iii) a geometric expression of the distance to the union of halfspaces. The performance and utility of this distributionally robust method are demonstrated through simulations using a 12D quadrotor model in a 3D environment.**

I. Introduction

Safety is one of the most fundamental challenges in the operation of robots and autonomous systems in practical environments, which are uncertain and dynamic. In particular, the unpredicted motion of objects and agents often risks the collision-free navigation of mobile robots. To gather information about an obstacle's uncertain movement, it is typical to use (historical) sample data of its motion. The main goal of this work is to develop an optimization-based method for risk-aware motion planning and control by incorporating data about moving obstacles into the robot's decision-making in a *distributionally robust* manner.

Several risk-averse decision-making methods have been proposed for collision avoidance in uncertain environments. Chance-constrained methods are one of the most popular approaches as they can directly limit the probability of collision. Because of their intuitive and practical role, chance constraints are extensively used in sampling-based planning [1]–[3] and model predictive control (MPC) [4], [5]. However, it is computationally nontrivial to handle a chance constraint because of its nonconvexity. This often limits the admissible class of probability distributions and system dynamics and/or requires an approximation. To resolve the issue of nonconvexity, a few theoretical and algorithmic tools have been developed using a particle-based approximation [6], and semidefinite programming formulation [7], among others. Another approach is to use a risk measure, which is computationally tractable to handle. In particular, *conditional value-at-risk* (CVaR) has recently drawn a great deal of interest in motion planning and control [8]–[11]. The CVaR of a random loss represents the conditional expectation of the loss within the $(1-\alpha)$ worst-case quantile of the loss distribution, where $\alpha \in (0,1)$ [12]. As claimed in [9], CVaR is suitable for rational risk assessments in robotic applications because of its *coherence* in the sense of Artzner *et al.* [13]. In addition to its computational tractability, CVaR is capable of distinguishing the worst-case tail events, and thus it is effective to take into account rare but unsafe events. We adopt CVaR to measure the risk of unsafety to enjoy these advantages.

The success of such risk-aware tools critically depends on the quality of information about the probability distribution of underlying uncertainties, such as an obstacle's random motion. If a poorly estimated distribution is used, it may cause catastrophic behaviors of the robot, leading to collision. One of the simplest and most popular ways to estimate the probability distribution is to collect the sample data of an obstacle's movement and construct an empirical distribution. The use of an empirical distribution is equivalent to a sample average approximation (SAA) of the stochastic programs [14]. Even though SAA is quite effective with asymptotic optimality, it does not ensure the satisfaction of risk constraints. The violation of risk constraints is likely to occur particularly with a small sample size.

In this paper, we propose a *distributionally robust* motion planning and control method to resolve the issue inherent in such empirical distributions. In the first stage, a reference trajectory is generated by using a rapidly exploring random tree* (RRT*), given the initial position of obstacles. However, this trajectory may not be safe to track anymore, as obstacles randomly move. To limit the risk of unsafety, we use a novel MPC method in the second stage. The proposed MPC method is carefully designed with CVaR constraints that must hold for any perturbation of the empirical distribution within a tunable *Wasserstein* ball. Thus, the resulting control decision guarantees to satisfy the risk constraints for safety in the presence of distribution errors characterized by the Wasserstein ball. Another advantage of the proposed method is that it provides a *probabilistic out-of-sample performance* guarantee, meaning that the risk constraints are satisfied with probability no less than a certain threshold even when evaluated with new sample data

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chosen independently of the training data. However, the distributionally robust MPC (DR-MPC) problem is nontrivial to solve because each worst-case CVaR constraint involves an infinite-dimensional optimization problem over the space of probability distributions. To remove the issue of infinite dimensionality, we propose a set of reformulations by using distributionally robust optimization techniques based on the Kantorovich duality principle [15].

The rest of the paper is organized as follows: In Section II, we introduce the problem setup and the Wasserstein DR-MPC tool with CVaR constraints for safety. In Section III, we propose a method to reformulate the DR-MPC problem into a finite-dimensional problem. In Section IV, the performance of the distributionally robust method is demonstrated through simulations, using a quadrotor model in a 3D environment.

II. Problem Formulation

A. System and Obstacle Models

In this paper, we consider a mobile robot with states $x(t) \in \mathbb{R}^{n_x}$ and control inputs $u(t) \in \mathbb{R}^{n_u}$ evolving with the following discrete-time motion model:[1]

$$x(t+1) = Ax(t) + Bu(t).$$

The robotic system is subject to the following state and control constraints:

$$x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U} \quad \forall t \geq 0,$$

where $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ and $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ are assumed to be convex sets. Since our goal is the motion planning and control of the robotic vehicle, we regard its current position in the n_y-dimensional configuration space as the system output $y(t) \in \mathbb{R}^{n_y}$, which can be modeled by

$$y(t) = Cx(t).$$

For its safety, the robot must avoid L *randomly moving* rigid body obstacles. Let $\mathcal{O}_\ell(t) \subset \mathbb{R}^{n_y}$ denote the region occupied by obstacle ℓ at stage t. For any obstacle ℓ that is not a convex polytope, we over-approximate it as a polytope and choose its convex hull as $\mathcal{O}_\ell(0)$. We assume that the obstacle's movement between two stages is modeled by translations:[2]

$$\mathcal{O}_\ell(t+k) = \mathcal{O}_\ell(t) + w_{\ell,t,k},$$

where $w_{\ell,t,k}$ is a random translation vector in $\mathbb{R}^{n_y}$, as illustrated in Fig. 1. Here, the sum of a set $\mathcal{A}$ and a vector w is defined by $\mathcal{A} + w := \{a + w \mid a \in \mathcal{A}\}$.

It is desirable for the robot to navigate in the *safe region*, which is defined by $\bigcap_{\ell=1}^{L} \mathcal{Y}_\ell(t)$, where

$$\mathcal{Y}_\ell(t) := \mathbb{R}^{n_y} \setminus \mathcal{O}_\ell^o(t) \quad \forall t \geq 0,$$

[1]The reformulation results in Section III remain valid for nonlinear systems. However, we focus on linear systems because the computational costs of nonlinear MPC are often prohibitive.

[2]Our method can also handle the rotation of obstacles by using the model proposed in our previous work [11]. However, for ease of exposition, we only consider translations.

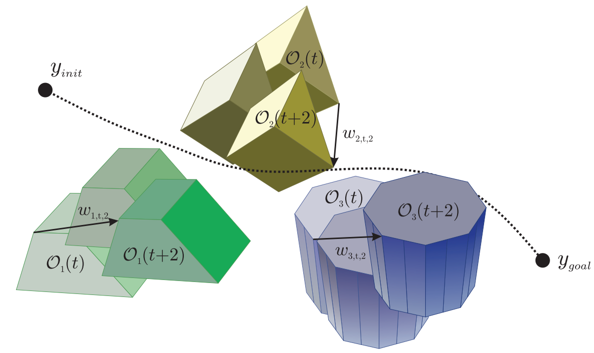


Fig. 1: Robot configuration space with moving obstacles.

where $\mathcal{O}_\ell^o(t)$ denotes the interior of $\mathcal{O}_\ell(t)$. Due to the random motion of the obstacles, $\mathcal{Y}_\ell$ changes over time according to

$$\mathcal{Y}_\ell(t+k) = \mathcal{Y}_\ell(t) + w_{\ell,t,k}.$$

B. Reference Trajectory Planning

Having information about the initial obstacle configuration, we first perform a fast collision-free trajectory planning. The resulting trajectory can then be traced by a motion controller to be introduced later. Among several trajectory generation techniques, we use a simple, yet quite efficient algorithm, called the rapidly exploring random tree* (RRT*) [16]. It is an extension of RRT, which is based on a rapidly growing tree graph, where the nodes are randomly sampled from the configuration space. However, as opposed to RRT, RRT* provides an asymptotically optimal planning solution by using neighbor search and tree rewriring.

C. Measuring Safety Risk Using CVaR

As the obstacles start to randomly move, the reference trajectory generated in the planning stage may no longer be safe to follow. To systematically limit the risk of collision, we use the notion of *safety risk* introduced in our previous work [17].

To begin with, we define the *loss of safety* regarding obstacle ℓ at stage t by the deviation of the robot's position from the safe region $\mathcal{Y}_\ell(t)$:

$$\mathrm{dist}(y(t), \mathcal{Y}_\ell(t)) := \min_{a \in \mathcal{Y}_\ell(t)} \|y(t) - a\|_2.$$

Ideally, it is desirable to ensure $\mathrm{dist}(y(t), \mathcal{Y}_\ell(t)) = 0$ for all t and ℓ. However, such a deterministic guarantee would lead to a very conservative decision for motion control. To systematically resolve this issue, we take a *risk-aware* approach by using CVaR. Specifically, we limit the *safety risk* regarding obstacle ℓ as follows:

$$\mathrm{CVaR}_\alpha[\mathrm{dist}(y(t), \mathcal{Y}_\ell(t))] \leq \delta_\ell \quad \forall t, \ \forall \ell, \tag{1}$$

where δ_ℓ is a user-specified risk-tolerance parameter. Here, $\mathrm{CVaR}_\alpha[X] := \min_{z \in \mathbb{R}} \mathbb{E}[z + (X - z)^+/(1-\alpha)]$,[3] and thus the safety risk measures the conditional expectation of the safety loss within the $(1-\alpha)$ worst-case quantile of the safety loss distribution.

[3]Throughout this paper, let $(\boldsymbol{x})^+ := \max\{\boldsymbol{x}, 0\}$.

D. Wasserstein Distributionally Robust MPC

To compute CVaR in the risk constraint (1), we need the probability distribution of $w_{\ell,t,k}$'s. Unfortunately, it is difficult to obtain a reliable distribution in practice. In many cases, we only have a few sample data $\{\hat{w}_{\ell,t,k}^{(1)}, \ldots, \hat{w}_{\ell,t,k}^{(N_k)}\}$ of $w_{\ell,t,k}$. One of the simplest ways to incorporate these data is to use the following empirical distribution:

$$\nu_{\ell,t,k} := \frac{1}{N_k} \sum_{i=1}^{N_k} \boldsymbol{\delta}_{\hat{w}_{\ell,t,k}^{(i)}}, \tag{2}$$

where $\boldsymbol{\delta}_w$ is the Dirac delta measure concentrated at w. However, when the number of sample data, N_k, is small, the control decision made using the empirical distribution may not satisfy the original risk constraint (1).

To overcome this issue of limited distribution information, we formulate the following *distributionally robust* MPC (DR-MPC) problem:[4]

$$\inf_{\mathbf{u},\mathbf{x},\mathbf{y}} J(x(t), \mathbf{u}) := \sum_{k=0}^{K-1} r(x_k, u_k) + q(x_K) \tag{3a}$$
$$\text{s.t. } x_{k+1} = Ax_k + Bu_k \tag{3b}$$
$$y_k = Cx_k \tag{3c}$$
$$x_0 = x(t) \tag{3d}$$
$$x_k \in \mathcal{X},\ u_k \in \mathcal{U} \tag{3e}$$
$$\sup_{\mu_{\ell,t,k} \in \mathbb{D}_{\ell,t,k}} \mathrm{CVaR}_\alpha^{\mu_{\ell,t,k}}[\mathrm{dist}(y_k, \mathcal{Y}_\ell(t+k))] \le \delta_\ell,\ \forall \ell \tag{3f}$$

where $\mathbf{u} := (u_0, \ldots, u_{K-1})$, $\mathbf{x} := (x_0, \ldots, x_K)$, $\mathbf{y} := (y_0, \ldots, y_K)$, where all the constraints must hold for $k = 0, \ldots, K$, except for (3b) and $u_k \in \mathcal{U}$ in (3e), which should hold for $k = 0, \ldots, K-1$. Once an optimal solution $\mathbf{u}^\star$ is computed given $x(t)$, the first component $u_0^\star$ is selected as the control input at stage t, i.e., $u(t) := u_0^\star$. The MPC problem is defined in a receding horizon manner for each stage. The cost functions are selected so as to penalize the deviation of the states from the reference trajectory and large control inputs. In particular, we let $J(x(t), \mathbf{u}) := \sum_{k=0}^{K-1} \|y_k - y_k^{\mathrm{ref}}\|_Q^2 + \|u_k\|_R^2 + \|y_K - y_K^{\mathrm{ref}}\|_P^2$, where $y_k^{\mathrm{ref}} \in \mathbb{R}^{n_x}$ is the reference trajectory generated in Section II-B; $Q \succeq 0$, $R \succ 0$ are the state and control weighting matrices, respectively; and $P \succeq 0$ is chosen in a way to ensure stability. The constraints (3b) and (3c) are for evolving the system states and outputs over the MPC horizon starting from x_0, which is set to be the current state $x(t)$. The state and input constraints are specified in (3e).

The most important part in this problem is the constraint (3f). The left-hand side of this inequality represents the worst-case value of CVaR when the probability distribution $\mu_{\ell,t,k}$ lies in a given set $\mathbb{D}_{\ell,t,k}$, called an *ambiguity set*. Therefore, by limiting the worst-case risk value, the resulting control action is robust against distribution errors characterized by the ambiguity set.

In this work, we choose the ambiguity set as a Wasserstein ball centered at the empirical distribution (2). More precisely,

$$\mathbb{D}_{\ell,t,k} := \{\mu \in \mathcal{P}(\mathbb{W}) \mid W(\mu, \nu_{\ell,t,k}) \le \theta\}, \tag{4}$$

where $\mathcal{P}(\mathbb{W})$ denotes the set of Borel probability measures on the support $\mathbb{W}$. Here, the Wasserstein metric of order 1 is defined by

$$W(\mu, \nu) := \min_{\kappa \in \mathcal{P}(\mathbb{W}^2)} \Big\{ \int_{\mathbb{W}^2} \|w - w'\| \, \mathrm{d}\kappa(w, w') \mid \Pi^1 \kappa = \mu, \Pi^2 \kappa = \nu \Big\},$$

where $\Pi^i \kappa$ denotes the ith marginal of κ for $i = 1, 2$, and $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^{n_y}$. The Wasserstein distance between two probability distributions represents the minimum cost of redistributing mass from one to another by using a nonuniform perturbation. It is worth mentioning that other types of ambiguity sets can be chosen in the proposed DR-MPC formulation. A popular choice in distributionally robust optimization are moment-based ambiguity sets [25]–[27]. However, such ambiguity sets are often overly conservative and require a large sample size to reliably estimate moment information. Statistical distance-based ambiguity sets have also drawn a great interest, by using phi-divergence [28] and Wasserstein distance [15], [29]–[31], among others. However, compared to other statistical distance-based ones, Wasserstein ambiguity sets contain a richer set of relevant distributions and provide a better finite-sample performance guarantee [15].

III. Finite-Dimensional Reformulation via Kantorovich Duality

The Wasserstein DR-MPC problem (3) is nontrivial to solve because of the worst-case CVaR constraint (3f). The maximization problem in the constraint is over the space of probability distributions, and thus it is infinite dimensional. In this section, we propose a set of procedures to reformulate (3) as a finite-dimensional problem. For ease of exposition, we consider

$$\sup_{\mu \in \mathbb{D}} \mathrm{CVaR}_\alpha^\mu[\mathrm{dist}(y, \mathcal{Y} + w)] \le \delta \tag{5}$$

instead of (3f), by suppressing the subscripts in (3f). Here,

$$\mathcal{Y} := \bigcup_{j=1}^m \{y \mid c_j^\top y \ge d_j\} \tag{6}$$

represents the safe region in the presence of an obstacle, which is the convex polytope $\{y \mid c_j^\top y \le d_j, j = 1, \ldots, m\}$.

A. Distance to the Union of Halfspaces

The first step is to simplify the loss of safety, $\mathrm{dist}(y, \mathcal{Y} + w)$, by noticing that the safe region is a union of halfspaces.

[4]Distributionally robust optimization has been used for sequential decision making in conjunction with Markov decision processes/optimal control [18]–[23], safety/reachability specification [24], and RRT [3], among others. However, we use MPC to revise the control decision online in response to the obstacles' random movements.

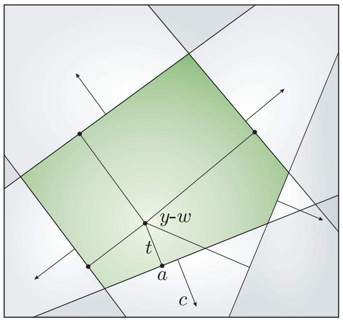


Fig. 2: Illustration of the distance to the union of halfspaces.

Lemma 1. *Suppose that the safe region is given by* (6)*. Then, the following equality holds:*

$$\operatorname{dist}(y, \mathcal{Y} + w) = \min_{j=1,\ldots,m} \left\{ \frac{(d_j - c_j^\top (y-w))^+}{\|c_j\|_2} \right\}.$$

Proof. By (6), we first observe that

$$\operatorname{dist}(y, \mathcal{Y} + w) = \min_{j=1,\ldots,m} \{\operatorname{dist}(y, \mathcal{Y}_j + w)\}. \tag{7}$$

As illustrated in Fig. 2, the distance between y and each halfspace can be represented as

$$\operatorname{dist}(y, \mathcal{Y}_j + w) = \inf_{a,t} \{\|t\|_2 \mid y - w - a = t, c_j^\top a \geq d_j\}. \tag{8}$$

The Lagrangian of this problem is given by

$$L(a, t, \lambda, \gamma) = \|t\|_2 + \lambda(d_j - c_j^\top a) + \gamma^\top (y - w - a - t).$$

The corresponding Lagrange dual function is obtained as

$$\begin{aligned} g(\lambda, \gamma) = &\inf_t (\|t\|_2 - \gamma^\top t) + \inf_a (-\gamma - \lambda c_j)^\top a \\ &+ \lambda d_j + \gamma^\top (y - w). \end{aligned}$$

Note that

$$\inf_t \{\|t\|_2 - \gamma^\top t\} = \begin{cases} 0 & \text{if } \|\gamma\|_2 \leq 1 \\ -\infty & \text{otherwise.} \end{cases}$$

Therefore, the dual problem of (8) can be written as

$$\begin{cases} \max_{\gamma,\lambda} & \lambda d_j + \gamma^\top (y - w) \\ \text{s.t.} & \|\gamma\|_2 \leq 1 \\ & \gamma = -\lambda c_j \\ & \lambda \geq 0 \end{cases} = \begin{cases} \max_\lambda & \lambda d_j - \lambda c_j^\top (y - w) \\ \text{s.t.} & \lambda \|c_j\|_2 \leq 1 \\ & \lambda \geq 0 \end{cases}$$

$$= \begin{cases} \max_\lambda & \lambda (d_j - c_j^\top (y - w)) \\ \text{s.t.} & \lambda \leq \frac{1}{\|c_j\|_2} \\ & \lambda \geq 0 \end{cases} = \left(\frac{d_j - c_j^\top (y - w)}{\|c_j\|_2} \right)^+.$$

Combining this with (7), the result follows directly from strong duality, which holds because the primal problem satisfies the weak Slater's conditions, as the inequality constraints are linear and the primal problem is feasible [32, Section 5.2.2]. $\square$

B. Reformulation of DR-CVaR Constraints

We now use the simplified representation of $\operatorname{dist}(y, \mathcal{Y}+w)$ to reformulate the distributionally robust CVaR (DR-CVaR) constraint (5) in a conservative manner, which is suitable for our purpose of limiting safety risk.

Lemma 2. *Suppose that the safe region is given by* (6)*. The following inequality holds:*

$$\begin{aligned} &\sup_{\mu \in \mathbb{D}} \mathrm{CVaR}_\alpha^\mu [\operatorname{dist}(y, \mathcal{Y} + w)] \leq \\ &\inf_{z \in \mathbb{R}} z + \frac{\sup_{\mu \in \mathbb{D}} \mathbb{E}^\mu \big[\max\{\min_j f_j(y, w), z, 0\} - z \big]}{1 - \alpha}, \end{aligned} \tag{9}$$

where $f_j(y, w) = \frac{d_j - c_j^\top (y-w)}{\|c_j\|_2}$, $j = 1, \ldots, m$.

Its proof is contained in Appendix I, which uses the extremal representation of CVaR and the minimax inequality. Note that the inner maximization problem in (9) still involves optimization over a set of distributions. We now use Kantorovich duality to further reformulate it as a finite-dimensional optimization problem. More specifically, we have the following dual representation of the inner maximization problem in (9):

$$\begin{aligned} &\sup_{\mu \in \mathbb{D}} \mathbb{E}^\mu \Big[\max\Big\{ \min_j f_j(y, w), z, 0 \Big\} - z \Big] = \\ &\inf_{\lambda \geq 0} \Big[\lambda\theta + \frac{1}{N} \sum_{i=1}^N \sup_{w \in \mathbb{W}} \Big[\max\Big\{ \min_j f_j(y, w), z, 0 \Big\} \\ &\qquad\qquad - z - \lambda \|w - \hat{w}^{(i)}\| \Big] \Big]. \end{aligned} \tag{10}$$

It is shown that there is no duality gap [30, Theorem 1].

Proposition 1. *Suppose that* $\mathbb{W} := \{w \in \mathbb{R}^{n_y} \mid Hw \leq h\}$, *where* $H \in \mathbb{R}^{q \times n_y}$ *and* $h \in \mathbb{R}^q$*. Then, the following holds:*

$$\begin{aligned} &\sup_{\mu \in \mathbb{D}} \; \mathbb{E}^\mu \Big[\max\Big\{ \min_{j=1,\ldots,m} f_j(y, w), z, 0 \Big\} - z \Big] = \\ &\inf_{\lambda, \mathbf{s}, \rho, \gamma, \eta, \zeta} \; \lambda\theta + \sum_{i=1}^N s_i \\ &\qquad \textit{s.t.} \; \langle \rho_i, G(y - \hat{w}^{(i)}) + g \rangle + \langle \gamma_i, h - H\hat{w}^{(i)} \rangle \leq s_i + z \\ &\qquad\quad \langle \eta_i, h - H\hat{w}^{(i)} \rangle \leq s_i + z \\ &\qquad\quad \langle \zeta_i, h - H\hat{w}^{(i)} \rangle \leq s_i \\ &\qquad\quad \|H^\top \gamma_i - G^\top \rho_i\|_* \leq \lambda \\ &\qquad\quad \|H^\top \eta_i\|_* \leq \lambda \\ &\qquad\quad \|H^\top \zeta_i\|_* \leq \lambda \\ &\qquad\quad \langle \rho_i, e \rangle = 1 \\ &\qquad\quad \gamma_i \geq 0, \; \rho_i \geq 0, \; \eta_i \geq 0, \; \zeta_i \geq 0, \end{aligned}$$

where all the constraints hold for $i = 1, \ldots, N$*, and the dual norm* $\|\cdot\|_*$ *is defined by* $\|z\|_* := \sup_{\|\xi\| \leq 1} \langle z, \xi \rangle$*, where* $\langle \cdot, \cdot \rangle$ *denotes the inner product. Here,* $G \in \mathbb{R}^{m \times n_y}$ *is a matrix with rows* $\frac{-c_j^\top}{\|c_j\|_2}$, $j = 1, \ldots, m$, $g \in \mathbb{R}^m$ *is a column vector with entries* $\frac{d_j}{\|c_j\|_2}$, $j = 1, \ldots, m$, *and* $e \in \mathbb{R}^m$ *is a vector of all ones.*

A proof of this proposition can be found in Appendix II.

C. Reformulation of DR-MPC Problem and Out-of-Sample Performance Guarantee

By using Lemma 2 and Proposition 1, we can reformulate the Wasserstein DR-MPC problem (3) as follows:

$$\inf_{\substack{\mathbf{u},\mathbf{x},\mathbf{y},\mathbf{z},\lambda,\\ \mathbf{s},\rho,\gamma,\eta,\zeta}} J(x(t),\mathbf{u}) := \sum_{k=0}^{K-1} r(x_k,u_k) + q(x_K) \tag{11a}$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k \tag{11b}$$

$$x_0 = x(t) \tag{11c}$$

$$z_{\ell,k} + \frac{1}{1-\alpha}\Big[\lambda_{\ell,k}\theta + \frac{1}{N_k}\sum_{i=1}^{N_k} s_{\ell,k,i}\Big] \le \delta_\ell \tag{11d}$$

$$\langle \rho_{\ell,k,i}, G_t(Cx_k - \hat{w}_{\ell,t,k}^{(i)}) + g_t\rangle \tag{11e}$$

$$+ \langle \gamma_{\ell,k,i}, h - H\hat{w}_{\ell,t,k}^{(i)}\rangle \le s_{\ell,k,i} + z_{\ell,k} \tag{11f}$$

$$\langle \eta_{\ell,k,i}, h - H\hat{w}_{\ell,t,k}^{(i)}\rangle \le s_{\ell,k,i} + z_{\ell,k} \tag{11g}$$

$$\langle \zeta_{\ell,k,i}, h - H\hat{w}_{\ell,t,k}^{(i)}\rangle \le s_{\ell,k,i} \tag{11h}$$

$$\|H^\top \gamma_{\ell,k,i} - G_t^\top \rho_{\ell,k,i}\|_* \le \lambda_{\ell,k} \tag{11i}$$

$$\|H^\top \eta_{\ell,k,i}\|_* \le \lambda_{\ell,k} \tag{11j}$$

$$\|H^\top \zeta_{\ell,k,i}\|_* \le \lambda_{\ell,k} \tag{11k}$$

$$\langle \rho_{\ell,k,i}, e\rangle = 1 \tag{11l}$$

$$\gamma_{\ell,k,i}, \rho_{\ell,k,i}, \eta_{\ell,k,i}, \zeta_{\ell,k,i} \ge 0 \tag{11m}$$

$$x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ z_{\ell,k} \in \mathbb{R}, \tag{11n}$$

where all the constraints hold for $k = 0,\dots,K$, $\ell = 1,\dots,L$ and $i = 1,\dots,N_k$, except for the constraint (11b) and $u_k \in \mathcal{U}$ in (11n), which should hold for $k = 0,\dots,K-1$. The control decision obtained by solving the reformulated problem satisfies the original DR-CVaR constraint (3f) because of Lemma 2 and Proposition 1. Thus, the safety risk is limited by the pre-specified threshold even when the actual distribution deviates from the empirical distribution (2) within the Wasserstein ball (4). Another advantage is to assure a *probabilistic out-of-sample performance guarantee*, i.e., the safety risk constraint is satisfied with probability no less than a certain threshold, $1-\beta$, even when evaluated under a set of new samples chosen independently of the training data. More precisely, for a carefully chosen Wasserstein ball radius θ an optimal $(\mathbf{u}^\star, \mathbf{x}^\star)$ satisfies the following inequality:[5]

$$\mu^N\big\{\mathrm{CVaR}_\alpha^\mu[\mathrm{dist}(Cx_k^\star, \mathcal{Y}(t+k))] \le \delta\big\} \ge 1-\beta \quad \forall k,$$

This finite-sample guarantee is the result of a measure concentration inequality for Wasserstein ambiguity sets [33, Theorem 2]. More details can be found in [34].

The reformulated Wasserstein DR-MPC problem is nonconvex because the fourth constraint is bilinear; all the other constraints and the objective function are convex. This problem can be solved by efficient nonlinear programming algorithms such as interior-point methods.

[5]It is assumed that the true distribution has a light tail in the sense that $\mathbb{E}^\mu[\exp(\|w\|^c)] \le B$ for $c > 1$ and $B > 0$. The radius has the form of $\theta := [\log(c_1/\beta)/(c_2N)]^{1/2}$ if $N \ge \log(c_1/\beta)/c_2$ and $\theta := [\log(c_1/\beta)/(c_2N)]^{1/c}$ if $N < \log(c_1/\beta)/c_2$ for some constants $c_1, c_2 > 0$.

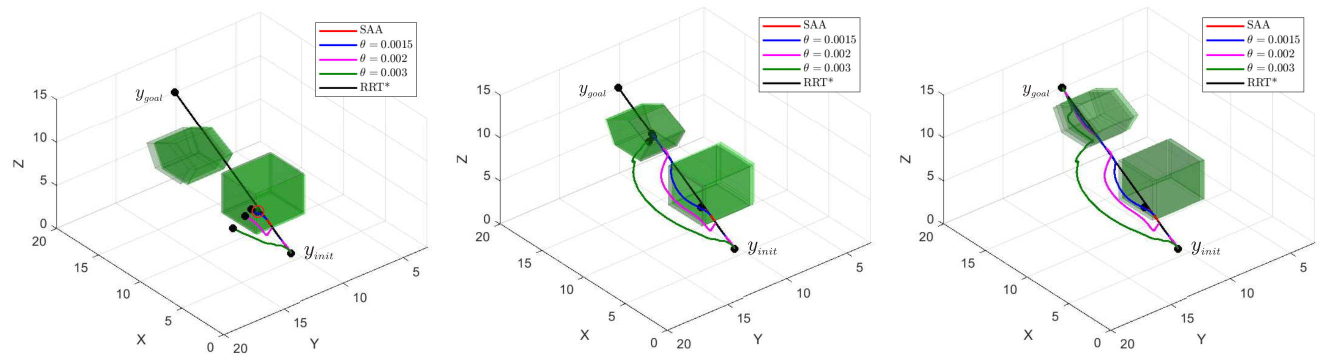
IV. Simulation Results

In this section, we present simulation results, demonstrating the performance and utility of the distributionally robust method. Consider a quadcopter navigating in a 3-D environment with the following linear dynamics: $\ddot{\mathrm{x}} = -g\theta, \ddot{\mathrm{y}} = g\phi, \ddot{\mathrm{z}} = -\frac{1}{m}u_1, \ddot{\phi} = \frac{l}{I_{xx}}u_2, \ddot{\theta} = \frac{l}{I_{yy}}u_3, \ddot{\psi} = \frac{l}{I_{zz}}u_4$, where m is the quadrotor's mass, g is the gravitational acceleration, and I_{xx}, I_{yy} and I_{zz} are the area moments of inertia about the principle axes in the body frame. The quadrotor in 3D space is expressed by it position and orientation—$(x, y, z, \phi, \theta, \psi)$. There are two polytopic obstacles randomly moving in the space as shown in Fig. 3. The following parameters are used in simulations: $m = 0.65\ kg, l = 0.23\ m, I_{xx} = 0.0075\ kg \cdot m^2, I_{yy} = 0.0075\ kg \cdot m^2, I_{zz} = 0.013\ kg \cdot m^2, g = 9.81\ ms^2$, $\delta = 0.02$, and $\alpha = 0.95$.

Using the proposed method, a reference trajectory is first generated by using RRT*. Then, the quadrotor starts following the trajectory while avoiding obstacles by solving the DR-MPC problem for $T = 50$. The MPC horizon and the weights are set to $K = 10$, $Q = P = I$, $R = 0.001I$. In the following experiments, the two obstacles' random movements in each direction for stage $t+k$ are sampled from a normal distribution with mean $\mu_1 = 0.02, \mu_2 = 0$ and variance $\sigma_1^2 = 0.2, \sigma_2^2 = 0.2$. The DR-MPC problem (11) was modeled in AMPL [35] and solved by using the interior-point method-based solver in Artelys Knitro 12.0 [36].

In practice, only a small number of sample data may be available. Thus, only $N = 5$ data are used in all simulations. We compare the standard SAA approach (e.g., [11]) and the proposed distributionally robust method. Fig. 3 shows the resulting trajectories at different time stages. At each stage, the quadrotor observes the current obstacles position and makes a control decision accordingly. As shown in Fig. 3 (a), in early stages, the quadrotor trajectories controlled by the SAA and distributionally robust methods are similar to each other for $\theta = 1.5 \times 10^{-3}, 2 \times 10^{-3}$. However, for $\theta = 3 \times 10^{-3}$, the quadrotor immediately starts avoiding the first obstacle, even when it is not close. At $t = 16$, the quadrotor controlled by the SAA method collides with the first obstacle (marked with a red circle). This indicates that the SAA method with a very small sample size is not able to satisfy the safety risk constraints and leads to collision. Meanwhile, the trajectories generated by DR-MPC remain safe and avoid the first obstacle. This result shows the DR-MPC's practical advantage that it properly limits safety risk even with a very small sample size.

When $t = 36$, the quadrotor has already passed the first obstacle as shown in Fig. 3 (b). It can be noticed that the trajectories for smaller Wasserstein ball radii start following the reference trajectory, while the trajectory generated with $\theta = 3 \times 10^{-3}$ still keeps a safe distance from the first obstacle. The trajectories at $t = 50$ are shown in Fig. 3 (c). Note that, as the Wasserstein ball radius increases, the quadrotor further deviates from the reference trajectory to ensure a bigger margin for safety. This result is consistent with the fact that the DR-MPC makes a more conservative



(a) $t = 16$ (b) $t = 36$ (c) $t = 50$

Fig. 3: Quadrotor trajectories at difference stages controlled by the standard SAA method and the proposed distributionally robust method with Wasserstein ball radii $\theta = 1.5 \times 10^{-3}, 2 \times 10^{-3}, 3 \times 10^{-3}$. The quadrotor controlled by the SAA method collides with the first obstacle at $t = 16$, while the proposed method achieves the collision-free operation of the quadrotor.

control decision by considering larger distribution errors when a bigger Wasserstein ball is used.

V. Conclusions and Future Work

We have proposed a distributionally robust decision-making tool for collision avoidance in an environment with randomly moving obstacles. This method alleviates the issue of using empirical distributions in safety-critical systems or tasks by explicitly allowing distribution errors and making decisions robust against such errors. It also enjoys the proposed finite-dimensional reformulation of associated MPC problems for computational tractability. As a future research, the proposed method can be extended to achieve the distributional robustness of robot's decision-making when a Gaussian process model is used to predict the motion of obstacles.

Appendix I
Proof of Lemma 2

Proof. By the definition of CVaR and Lemma 1, we have $\mathrm{CVaR}_\alpha^\mu[\mathrm{dist}(y, \mathcal{Y}+w)] = \inf_{z\in\mathbb{R}} \mathbb{E}^\mu[z+(\min_j[f_j(y,w)]^+ - z)^+/(1-\alpha)] = \inf_{z\in\mathbb{R}} \mathbb{E}^\mu[z+([\min_j f_j(y,w)]^+ - z)^+/(1-\alpha)]$. Next, we use the minimax inequality to obtain

$$\sup_{\mu\in\mathbb{D}} \mathrm{CVaR}_\alpha^\mu[\mathrm{dist}(y, \mathcal{Y}+w)] \leq \inf_{z\in\mathbb{R}} \sup_{\mu\in\mathbb{D}} \mathbb{E}^\mu\left[z + \frac{1}{1-\alpha}\Big(\big[\min_j f_j(y,w)\big]^+ - z\Big)^+\right],$$

and therefore the result follows. □

Appendix II
Proof of Proposition 1

Proof. By [15, Theorem 4.2], the dual problem in (10) can be rewritten as

$$\begin{cases} \inf_{\lambda,s} \; \lambda\theta + \frac{1}{N}\sum_{i=1}^N s_i \\ \text{s.t.} \quad \sup_{w\in\mathbb{W}} \Big[\max\big\{\min_{j=1,\dots,m} f_j(y,w) - z, -z, 0\big\} \\ \qquad -\lambda\|w - \hat{w}^{(i)}\|\Big] \leq s_i \\ \qquad \lambda \geq 0 \end{cases}$$

$$= \begin{cases} \inf_{\lambda,s,\xi,\nu} \; \lambda\theta + \sum_{i=1}^N s_i \\ \text{s.t.} \quad \sup_w[\langle \xi_{i,1} - \nu_{i,1}, w\rangle + \min_j f_j(y,w) - z] \\ \qquad + \sigma_{\mathbb{W}}(\nu_{i,1}) - \langle \xi_{i,1}, \hat{w}^{(i)}\rangle \leq s_i \\ \qquad \sup_w[\langle \xi_{i,2} - \nu_{i,2}, w\rangle + z] \\ \qquad + \sigma_{\mathbb{W}}(\nu_{i,2}) - \langle \xi_{i,2}, \hat{w}^{(i)}\rangle \leq s_i \\ \qquad \sup_w\langle \xi_{i,3} - \nu_{i,3}, w\rangle \\ \qquad + \sigma_{\mathbb{W}}(\nu_{i,3}) - \langle \xi_{i,3}, \hat{w}^{(i)}\rangle \leq s_i \\ \qquad \|\xi_{i,k}\|_* \leq \lambda, \; k = 1,2,3, \end{cases} \tag{12}$$

where all the constraints hold for $i = 1, \dots, N$, and the support function $\sigma_{\mathbb{W}}$ is defined by $\sigma_{\mathbb{W}}(z) := \sup_{w\in\mathbb{W}}\langle z, w\rangle$. We now note that

$$\begin{aligned} &\sup_{w\in\mathbb{R}^{n_y}} \Big[\langle \xi_{i,1} - \nu_{i,1}, w\rangle + \min_{j=1,\dots,m} f_j(y,w) - z\Big] \\ &= \begin{cases} \sup_{w\in\mathbb{R}^{n_y}, \tau\in\mathbb{R}} & \langle \xi_{i,1} - \nu_{i,1}, w\rangle - z + \tau \\ \text{s.t.} & G(y-w) + g \geq \tau e \end{cases} \\ &= \begin{cases} \inf_{\rho\geq 0} & \langle \rho, g + Gy\rangle - z \\ \text{s.t.} & G^\top \rho = \xi_{i,1} - \nu_{i,1} \\ & \langle \rho, e\rangle = 1, \end{cases} \end{aligned}$$

where the last equality follows from strong duality of linear programming, which holds since the primal maximization problem is feasible. Similarly, we obtain that

$$\sup_{w\in\mathbb{R}^{n_y}} [\langle \xi_{i,2} - \nu_{i,2}, w\rangle - z] = \begin{cases} -z, & \text{if } \xi_{i,2} = \nu_{i,2} \\ \infty, & \text{otherwise,} \end{cases}$$

$$\sup_{w\in\mathbb{R}^{n_y}} \langle \xi_{i,3} - \nu_{i,3}, w\rangle = \begin{cases} 0, & \text{if } \xi_{i,3} = \nu_{i,3} \\ \infty, & \text{otherwise.} \end{cases}$$

By the definition of support function, we also have

$$\sigma_{\mathbb{W}}(\nu) = \begin{cases} \sup_{w\in\mathbb{R}^{n_y}} & \langle \nu, w\rangle \\ \text{s.t.} & Hw \leq h \end{cases} = \begin{cases} \inf_{\gamma\geq 0} & \langle \gamma, h\rangle \\ \text{s.t.} & H^\top\gamma = \nu, \end{cases}$$

where the last equality follows from strong duality of linear programming, which holds because the uncertainty set is nonempty. By substituting the results above into (12), we conclude that the proposed reformulation is exact. □

References

[1] B. Luders, M. Kothari, and J. How, “Chance constrained RRT for probabilistic robustness to environmental uncertainty,” in *AIAA Guidance, Navigation, and Control Conference*, 2010.

[2] A. Bry and N. Roy, “Rapidly-exploring random belief trees for motion planning under uncertainty,” in *IEEE International Conference on Robotics and Automation*, 2011.

[3] T. Summers, “Distributionally robust sampling-based motion planning under uncertainty,” in *IEEE/RSJ International Conference on Intelligent Robots and Systems*, 2018.

[4] L. Blackmore, M. Ono, and B. C. Williams, “Chance-constrained optimal path planning with obstacles,” *IEEE Transactions on Robotics*, vol. 27, no. 6, pp. 1080–1094, 2011.

[5] N. E. Du Toit and J. W. Burdick, “Robot motion planning in dynamic, uncertain environments,” *IEEE Transactions on Robotics*, vol. 28, no. 1, pp. 101–115, 2012.

[6] L. Blackmore, M. Ono, A. Bektassov, and B. C. Williams, “A probabilistic particle-control approximation of chance-constrained stochastic predictive control,” *IEEE Transactions on Robotics*, vol. 26, no. 3, pp. 502–517, 2010.

[7] A. Jasour, N. S. Aybat, and C. M. Lagoa, “Semidefinite programming for chance constrained optimization over semialgebraic sets,” *SIAM Journal on Optimization*, vol. 25, no. 3, pp. 1411–1440, 2015.

[8] Y. Chow, A. Tamar, S. Mannor, and M. Pavone, “Risk-sensitive and robust decision-making: a CVaR optimization approach,” in *Advances in Neural Information Processing Systems*, 2015.

[9] A. Majumdar and M. Pavone, “How should a robot assess risk? towards an axiomatic theory of risk in robotics,” in *International Symposium on Robotics Research*, 2017.

[10] S. Singh, Y.-L. Chow, A. Majumdar, and M. Pavone, “A framework for time-consistent, risk-sensitive model predictive control: Theory and algorithms,” *IEEE Transactions on Automatic Control*, vol. 64, no. 7, pp. 2905–2912, 2019.

[11] A. Hakobyan, G. C. Kim, and I. Yang, “Risk-aware motion planning and control using CVaR-constrained optimization,” *IEEE Robotics and Automation Letters*, vol. 4, no. 4, pp. 3924–3931, 2019.

[12] R. T. Rockafellar and S. Uryasev, “Conditional value-at-risk for general loss distribution,” *Journal of Banking & Finance*, vol. 26, pp. 1443–1471, 2002.

[13] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, “Coherent measures of risk,” *Mathematical Finance*, vol. 9, no. 3, pp. 203–228, 1999.

[14] A. Shapiro, D. Dentcheva, and A. Ruszczyński, *Lectures on Stochastic Programming: Modeling and Theory*, 2nd ed. SIAM, 2014.

[15] P. Mohajerin Esfahani and D. Kuhn, “Data-driven distributionally robust optimization using the Wasserstein metric: performance guarantees and tractable reformulations,” *Mathematical Programming*, vol. 171, pp. 115–166, 2018.

[16] S. Karaman and E. Frazzoli, “Sampling-based algorithms for optimal motion planning,” *The International Journal of Robotics Research*, vol. 30, no. 7, pp. 846–894, 2011.

[17] S. Samuelson and I. Yang, “Safety-aware optimal control of stochastic systems using conditional value-at-risk,” in *American Control Conference*, 2018.

[18] H. Xu and S. Mannor, “Distributionally robust Markov decision processes,” *Mathematics of Operations Research*, vol. 37, no. 2, pp. 288–300, 2012.

[19] P. Yu and H. Xu, “Distributionally robust counterpart in Markov decision processes,” *IEEE Transactions on Automatic Control*, vol. 61, no. 9, pp. 2538–2543, 2016.

[20] I. Yang, “A convex optimization approach to distributionally robust Markov decision processes with Wasserstein distance,” *IEEE Control Systems Letters*, vol. 1, no. 1, pp. 164–169, 2017.

[21] B. P. G. Van Parys, D. Kuhn, P. J. Goulart, and M. Morari, “Distributionally robust control of constrained stochastic systems,” *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 430–442, 2016.

[22] I. Yang, “Wasserstein distributionally robust stochastic control: A data-driven approach,” *arXiv preprint arXiv:1812.09808.*, 2018.

[23] I. Tzortzis, C. D. Charalambous, and T. Charalambous, “Infinite horizon average cost dynamic programming subject to total variation distance ambiguity,” *SIAM Journal on Control and Optimization*, vol. 57, no. 4, pp. 2843–2872, 2019.

[24] I. Yang, “A dynamic game approach to distributionally robust safety specifications for stochastic systems,” *Automatica*, vol. 94, pp. 94–101, 2018.

[25] G. C. Calafiore and L. El Ghaoui, “On distributionally robust chance-constrained linear programs,” *Journal of Optimization Theory and Applications*, vol. 130, no. 1, pp. 1–22, 2006.

[26] E. Delage and Y. Ye, “Distributionally robust optimization under moment uncertainty with application to data-driven problems,” *Operations Research*, vol. 58, no. 3, pp. 595–612, 2010.

[27] W. Wiesemann, D. Kuhn, and M. Sim, “Distributionally robust convex optimization,” *Operations Research*, vol. 62, no. 6, pp. 1358–1376, 2014.

[28] A. Ben-Tal, D. Den Hertog, A. De Waegenaere, B. Melenberg, and G. Rennen, “Robust solutions of optimization problems affected by uncertain probabilities,” *Management Science*, vol. 59, no. 2, pp. 341–357, 2013.

[29] C. Zhao and Y. Guan, “Data-driven risk-averse stochastic optimization with Wasserstein metric,” *Operations Research Letters*, vol. 46, no. 2, 2018.

[30] R. Gao and A. J. Kleywegt, “Distributionally robust stochastic optimization with Wasserstein distance,” *arXiv:1604.02199*, 2016.

[31] A. R. Hota, A. Cherukuri, and J. Lygeros, “Data-driven chance constrained optimization under Wasserstein ambiguity sets,” in *American Control Conference*, 2019.

[32] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[33] N. Fournier and A. Guillin, “On the rate of convergence in Wasserstein distance of the empirical measure,” *Probability Theory and Related Fields*, vol. 162, no. 3–4, pp. 707–738, 2015.

[34] A. Hakobyan and I. Yang, “Wasserstein distributionally robust motion control for collision avoidance using conditional value-at-risk,” *arXiv preprint arXiv:2001.04727*, 2020.

[35] R. Fourer, D. M. Gay, and B. W. Kernighan, “A modeling language for mathematical programming,” *Management Science*, vol. 36, no. 5, pp. 519–554, 1990.

[36] R. H. Byrd, J. Nocedal, and R. A. Waltz, “K nitro: An integrated package for nonlinear optimization,” in *Large-scale nonlinear optimization*. Springer, 2006, pp. 35–59.