HW 6 - SMAI ROLL_NUMBER - 20171213

Q2.

20171213

Ans-2) HW6

(a) $P(\theta|BG)P(BG) = P(\theta|FG)P(FG)$

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} \times \exp\left(-\frac{(\theta-\mu_1)^2}{2\sigma_1^2}\right) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(\frac{-(\theta-\mu_2)^2}{2\sigma_2^2}\right)$$

given $\sigma_1 = \sigma_2$ $\therefore$ $-(\theta-\mu_1)^2 = -(\theta-\mu_2)^2$

$$\therefore \quad (\theta-\mu_1)^2 - (\theta-\mu_2)^2 = 0$$

$$(\cancel{\theta} - \mu_1 - \cancel{\theta} + \mu_2)(\theta - \mu_1 + \theta - \mu_2) = 0$$

$$(\mu_2 - \mu_1)(2\theta - (\mu_1 + \mu_2)) = 0$$

$\because$ $\mu_1 \neq \mu_2$ $\therefore$ $\boxed{\theta = \frac{\mu_1 + \mu_2}{2}}$

(b) $P(\theta|BG) * P(BG) = P(\theta|FG) * P(FG)$

$$P(BG) \times \frac{1}{\sqrt{2\pi\sigma_1^2}} \times \exp\left(-\frac{\left(\frac{\mu_1+\mu_2}{2}-\mu_1\right)^2}{2\sigma_1^2}\right) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \times \exp\left(-\frac{\left(\frac{\mu_1+\mu_2}{2}-\mu_2\right)^2}{2\sigma_2^2}\right) \times P(FG)$$

$$\therefore \quad \frac{P(BG)}{\sigma_1} e^{-\frac{(\mu_1-\mu_2)^2}{8\sigma_1^2}} = \frac{1}{\sigma_2} e^{-\frac{(\mu_1-\mu_2)^2}{8\sigma_2^2}} P(FG)$$

$$\therefore \quad e^{-\left[\frac{(\mu_1-\mu_2)^2}{8}\left[\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right]\right]} = \frac{\sigma_1}{\sigma_2} \cdot \frac{P(FG)}{P(BG)}$$

let $\boxed{\sigma_1 = k\sigma_2}$

$$e^{-\left[\frac{(\mu_1-\mu_2)^2}{8}\left(\frac{1}{\sigma_1^2} - \frac{k^2}{\sigma_1^2}\right)\right]} = k\,\frac{P(FG)}{P(BG)}$$

then $$\boxed{\frac{P(FG)}{P(BG)} = \frac{1}{k} e^{-\left(\frac{(\mu_1-\mu_2)^2}{8\sigma_1^2}\left[1-k^2\right]\right)}}$$ $\boxed{k>0}$

(c) $P(BG) * \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(\theta^* - \mu_1)^2}{2\sigma_1^2}} = P(FG) * \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{(\theta^* - \mu_2)^2}{2\sigma_2^2}}$

given: $P(BG) = 4 * P(FG)$, $\mu_1 = 100$, $\mu_2 = 200$ & $\sigma_1 = \sigma_2 = \sigma$

$\therefore \quad 4 * \cancel{P(FG)} \frac{1}{\cancel{\sqrt{2\pi}\,\sigma}} e^{-\frac{(\theta^* - 100)^2}{2\sigma^2}} = \cancel{P(FG)} \frac{1}{\cancel{\sqrt{2\pi}\,\sigma}} e^{-\frac{(\theta^* - 200)^2}{2\sigma^2}}$

$\ln 4 - \frac{(\theta^* - 100)^2}{2\sigma^2} = -\frac{(\theta^* - 200)^2}{2\sigma^2}$

$\therefore \quad (\theta^* - 200)^2 - (\theta^* - 100)^2 = 2\sigma^2 \ln 4$

$\cancel{2\theta^{*2}} \quad (\theta^* - 200 - \theta^* + 100)(\theta^* - 200 + \theta^* - 100) = 2\sigma^2 \ln 4$

$100\,(300 - 2\theta^*) = 2\sigma^2 \ln 4$

$300 - \frac{2\sigma^2}{100} \ln 4 = 2\theta^*$

$\theta^* = 150 - \frac{1\,\sigma^2 \ln 4}{100}$

$\boxed{\theta^* = 150 - 0.01388\,\sigma^2}$