

PTC (Fall 2018) – Assignment 3

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Problem 1

Consider the (deterministic) Turing machine M given by

$$M = (\{q_0, q_1, q_2\}, \{a, b\}, \{a, b, B\}, \delta, q_0, B, \{q_2\})$$

which has exactly four transitions defined in it, as described below.

$$1. \delta(q_0, a) = (q_0, B, R)$$

$$2. \delta(q_0, b) = (q_1, B, R)$$

$$3. \delta(q_1, b) = (q_1, B, R)$$

$$4. \delta(q_1, B) = (q_2, B, R)$$

Please answer the following questions:

- Specify the execution trace of M on the input string abb .
- Provide a regular expression for the language of the Turing machine.
- Suppose we added the transition $\delta(q_1, a) = (q_0, B, R)$ to the above machine, provide a regular expression for the language of the resulting Turing machine.

a

$$q_0aab \vdash q_0ab \vdash q_0b \vdash q_1B \vdash q_2B$$

b

$$a^*b^+$$

c

$$a^*b^+(a^+b^+)^*$$

Problem 2

Please design TM's to decide following languages:

- $L_1 = \{1^m \times 1^n = 1^{mn} \mid m, n \in \mathbb{N}^+\}$ (e.g. $11 \times 111 = 111111 \in L_1$, but $1 \times 1 = 11 \notin L_1$)
- $L_2 = \{ww \mid w \in \{a, b\}^*\}$

a

First, let's define a TM M_1 which will convert $\times 1^n = 1^{n'}$ to $\times 1^n = 1^{n'-n}$ where $n' \geq n > 0$

$$M_1 = (\{p_0, p_R, p_a, p_{a'}, p_d, p_L, p_f, p_e\}, \{1, 0, \times, =\}, \{1, 0, \times, =, B, a\}, \delta, p_0, B, \{p_e\})$$

At the beginning, the head should at \times . Then we will enter p_a , which will change the first 1 to a when the head goes right.

$$\delta(p_0, \times) = (p_a, \times, R)$$

$$\delta(p_a, a) = (p_a, a, R)$$

$$\delta(p_a, 1) = (p_R, a, R)$$

The state p_R moves the head to the first blank at the right of the input, and then switches to state p_d , which will delete a 1.

$$\delta(p_R, X) = (p_R, X, R), \text{ where } X = \{=, 1\}$$

$$\delta(p_R, B) = (p_d, B, L)$$

$$\delta(p_d, 1) = (p_L, B, L)$$

The state p_L will move to \times and then enters $p_{a'}$

$$\delta(p_L, X) = (p_L, X, L), \text{ where } X = \{1, =, a\}$$

$$\delta(p_L, \times) = (p_a, \times, R)$$

Then the state $p_{a'}$ will act as p_a , change a 1 to a, then move to the right and delete a 1 except that $p_{a'}$ will go to the state p_f if it sees no 1 before the $=$, under which condition p_a would halt without accepting.

$$\delta(p_{a'}, a) = (p_{a'}, a, R)$$

$$\delta(p_{a'}, 1) = (p_R, a, R)$$

$$\delta(p_{a'}, =) = (p_f, =, L)$$

The state p_f will change all the a back to 1.

$$\delta(p_f, a) = (p_f, 1, L)$$

$$\delta(p_f, \times) = (p_e, \times, R)$$

Then we construct the TM M to describe the language L_1

$$M = (M_1.states \cup \{q_0, q_R, q_{call}, q_L, q_1, q_e, q_{check}, q_f\}, \{1, 0, \times, =\}, \{1, 0, \times, =, B, a\}, M_1.\delta \cup \delta, q_0, B, \{q_f\})$$

The state q_0 will remove the first 1, and goes to state q_R

$$\delta(q_0, 1) = (q_R, B, R)$$

The state q_R will move to the \times and then enters q_{call}

$$\delta(q_R, 1) = (q_R, 1, R)$$

$$\delta(q_R, \times) = (q_{call}, \times, L)$$

The state q_{call} will call the M_1

$$\delta(q_{call}, 1) = (p_0, 1, R)$$

When the M returns from M_1 , state q_L will move the head to the blank at the left end of the input and then enters q_1

$$\delta(q_L, 1) = (q_L, 1, L)$$

$$\delta(q_L, \times) = (q_L, \times, L)$$

$$\delta(q_L, B) = (q_1, B, R)$$

q_1 acts just like q_0 except that when it finds no 1 before \times , it will switch to q_e

$$\delta(q_1, 1) = (q_R, B, R)$$

$$\delta(q_1, \times) = (q_e, \times, R)$$

q_e and q_{check} will check whether the RHS of the = is blank.

$$\delta(q_e, 1) = (q_e, 1, R)$$

$$\delta(q_e, =) = (q_{check}, =, R)$$

$$\delta(q_{check}, B) = (q_f, B, L)$$

b

We will use multi-track TMs in this problem.

First, define a TM M_1 which will compare whether two strings are equal to each other. The beginning of the second string will be marked as $\langle x, c \rangle$, the beginning before the first string is marked as $\langle z, B \rangle$, where $c \in \{a, b\}$. Other related cells on the second track is initialized to $*$.

$$M_1(\{q_0, q_c, q_a, q_b, q_{da}, q_{db}, q_L, q_f, q_e\}, \{a, b\}, \{a, b, x, *, z, B\}, \delta, q_0, B, \{q_e\})$$

q_0 will move the head to the end of the first string. $\delta(q_0, \langle *, X \rangle) = (q_0, \langle *, X \rangle, R)$, where $X \in \{a, b\}$

$$\delta(q_0, \langle x, X \rangle) = (q_c, \langle x, X \rangle, L), \text{ where } X \in \{a, b\}$$

q_c will remove a character from the tail of the first string and remember this string in its state.

$$\delta(q_c, \langle *, a \rangle) = (q_a, \langle *, B \rangle, R)$$

$$\delta(q_c, \langle *, b \rangle) = (q_b, \langle *, B \rangle, R)$$

$$\delta(q_a, \langle *, X \rangle) = (q_a, \langle *, X \rangle, R)$$

$$\delta(q_b, \langle *, X \rangle) = (q_b, \langle *, X \rangle, R)$$

q_a and q_b will move the head to the end of the second string and remember to delete a or b respectively.

$$\delta(q_a, \langle *, B \rangle) = (q_{da}, \langle *, B \rangle, L)$$

$$\delta(q_b, \langle *, B \rangle) = (q_{db}, \langle *, B \rangle, L)$$

q_{da} and q_{db} will delete one a or one b respectively.

$$\delta(q_{da}, \langle *, a \rangle) = (q_L, \langle *, B \rangle, L)$$

$$\delta(q_{da}, \langle x, a \rangle) = (q_L, \langle x, B \rangle, L)$$

$$\delta(q_{db}, \langle *, b \rangle) = (q_L, \langle *, B \rangle, L)$$

$$\delta(q_{db}, \langle x, b \rangle) = (q_L, \langle x, B \rangle, L)$$

q_L will move the head to the ending blanks of the first string.

$$\delta(q_L, \langle *, X \rangle) = (q_L, \langle *, X \rangle, L), \text{ where } X \in \{a, b\}$$

$\delta(q_L, \langle x, X \rangle) = (q_c, \langle x, X \rangle, L)$, where $X \in \{a, b\}$

q_c will find the next char to remove, when it meets z on the second track, the TM goes to a new state

q_f

$\delta(q_c, \langle *, B \rangle) = (q_c, \langle *, B \rangle, L)$

$\delta(q_c, \langle z, B \rangle) = (q_f, \langle z, B \rangle, R)$

q_f will check whether the second string has been reduced to blanks, if so, it will change to the accepting state q_e .

$\delta(q_f, \langle *, B \rangle) = (q_f, \langle *, B \rangle, R)$

$\delta(q_f, \langle x, B \rangle) = (q_e, \langle x, B \rangle, R)$

Then we define the TM M for language L_2 .

$M(M_1.states \cup \{p_0, p_1, p_{odd}, p_{even}, p_{back}, p_{fwd}, p_R, p_{last}, p_{next}, p_{markZ}\}, \{a, b\}, \{a, b, x, *, t, z, B\}, M_1.\delta \cup \delta, p_0, B, \{q_e\})$

M will accept the empty strings.

$\delta(p_0, \langle B, B \rangle) = (q_e, \langle B, B \rangle, R)$

Firstly, marked the first blank at the left of the input as x .

$\delta(p_0, \langle B, X \rangle) = (p_1, \langle B, X \rangle, L)$

$\delta(p_1, \langle B, B \rangle) = (p_{odd}, \langle x, B \rangle, R)$

If this is the odd indexed (start from 1) char from the input, then the state will change to even and vice versa.

$\delta(p_{odd}, \langle B, X \rangle) = (p_{even}, \langle *, X \rangle, R)$, where $X \in \{a, b\}$

$\delta(p_{even}, \langle B, X \rangle) = (p_{back}, \langle t, X \rangle, L)$, where $X \in \{a, b\}$

When M makes every two steps right, it will move x forward for one step. Then when the first blank at right is met, the x will be moved to the end of the first string.

$\delta(p_{back}, \langle *, X \rangle) = (p_{back}, \langle *, X \rangle, L)$, where $X \in \{a, b\}$

$\delta(p_{back}, \langle x, X \rangle) = (p_{fwd}, \langle *, X \rangle, R)$, where $X \in \{a, b, B\}$

$\delta(p_{fwd}, \langle *, X \rangle) = (p_R, \langle x, X \rangle, R)$, where $X \in \{a, b\}$

$\delta(p_R, \langle *, X \rangle) = (p_R, \langle *, X \rangle, R)$, where $X \in \{a, b\}$

$\delta(p_R, \langle t, X \rangle) = (p_{odd}, \langle *, X \rangle, R)$, where $X \in \{a, b\}$

Then we move x forward for one more step.

$\delta(p_{odd}, \langle B, B \rangle) = (p_{last}, \langle *, B \rangle, R)$

$\delta(p_{last}, \langle *, X \rangle) = (p_{last}, \langle *, X \rangle, L)$, where $X \in \{a, b\}$

$\delta(p_{last}, \langle x, X \rangle) = (p_{next}, \langle *, X \rangle, R)$, where $X \in \{a, b\}$

$\delta(p_{next}, \langle *, X \rangle) = (p_{markZ}, \langle x, X \rangle, L)$, where $X \in \{a, b\}$

$\delta(p_{markZ}, \langle *, X \rangle) = (p_{markZ}, \langle *, X \rangle, L)$, where $X \in \{a, b\}$

Finally, we marked the blank at the left of the input as z , and call M_1 .

$\delta(p_{markZ}, \langle *, B \rangle) = (q_0, \langle z, B \rangle, R)$

Problem 3

A *useless state* in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a state in a Turing machine is useless. Formulate this problem as a language and show it is decidable or undecidable. (**Hint:** consider the language E_{TM})

Construct a TM M' as follows. Given any input s , if s can be interpreted as $\langle M, w \rangle$, where M is a TM, w is a string, then we input it to the algorithm for ATM. If ATM accepts $\langle M, w \rangle$, M' will enter a state QF , where $QF \notin$ the state set of ATM.

It's obvious that the TM M' will enter the state QF if and only if the ATM accepts $\langle M, w \rangle$. So if we can decide whether or not QF is a useless state, then we can decide ATM. Since ATM is undecidable, so useless state is undecidable.

Problem 4

Show that the following questions are decidable:

- The set L of codes for TM's M such that, when started with the blank tape will eventually write some nonblank symbol on its tape. (**Hint:** If M has m states, consider the first m transitions that it makes)
- The set L of codes for TM's that never make a move left on any input.
- The set L of pairs (M, w) such that TM M , started with input w , never scans any tape cell more than once.

a

Let's construct a directed graph (N, E, P) , where N is the set of all the nodes, E is the set of edges and P is a set for some special nodes. For each transition $\delta(p, X) = (q, Y, D)$, if X is B, we add a node pXq . Also, if Y is not B, then we also add this node to the set P . Specially, we add a node p_0Bp_0 , where p_0 is the start state of M . For each pair of nodes, say, p_1Bq_1, p_2Bq_2 , if $q_1 = p_2$ and p_2 is not a final state, add an edge (q_1, p_2) .

Then we traverse the sub-graph which can be access from p_0Bp_0 . When we meet any node in P , then we accept this M . Otherwise, if we never meet any node in P after visiting all the reachable node, we reject this M .

b

Let's construct a directed graph (N, E, P) , where N is the set of all the nodes, E is the set of edges and P is a set for some special nodes. Specially, we add a node p_0Bp_0 , where p_0 is the start state of M . For each transition $\delta(p, X) = (q, Y, D)$, we add a node pXq . Also, if D is L, then we also add this

node to the set P. For each pair of nodes, say, $p_1 X q_1, p_2 Y q_2$, if $q_1 = p_2$ and p_2 is not a final state, add an edge (q_1, p_2) .

Then we traverse the sub-graph which can be access from $p_0 B p_0$. When we meet any node in P, then we reject this M. Otherwise, if we never meet any node in P after visiting all the reachable node, we accept this M.

c

Suppose that the ID of M is $q_0 w$ at the beginning.

case 1

If there exists $\delta(q_0, c) = (q', Y, L)$, where c is the first character of w, q' is an arbitrary state and Y is an arbitrary tape character, then we construct a graph(N,E,P) as follows. For all $\delta(p, B) = (q, Y, D)$, add a node pBq in N. If D is R, also add this node in P. Specially, we add a node $q_0 X q'$. For each pair of nodes, say, $p_1 X q_1, p_2 Y q_2$, if $q_1 = p_2$ and p_2 is not a final state, add an edge (q_1, p_2) .

Then we traverse the sub-graph which can be access from $q_0 B q'$. If we meet any node in P, we reject this (M,w).

case2

If there exists $\delta(q_0, c) = (q', Y, R)$, where c is the first character of w, q' is an arbitrary state and Y is an arbitrary tape character, then M cannot move any step to left. Suppose the length of w is n. Then we simulate M for n steps. If M moves to left in any step, we reject this (M,w). If M halts within n steps not moving to left, then we accept.

case3

If M doesn't halt within n steps, the head must point to the B at the right of w, suppose the state now is q_x .

We construct a graph(N,E,P) as follows. For all $\delta(p, B) = (q, Y, D)$, add a node pBq in N. If D is L, also add this node in P. Specially, we add a node $q_x B q_x$. For each pair of nodes, say, $p_1 X q_1, p_2 Y q_2$, if $q_1 = p_2$ and p_2 is not a final state, add an edge (q_1, p_2) .

Then we traverse the sub-graph which can be access from $q_x B q_x$. If we meet any node in P, we reject this (M,w). Otherwise, we accept this (M,w).

Problem 5

If a pushdown automaton has k stacks, we call it k -PDA. Clearly, 0-PDA is NFA, 1-PDA is PDA, and 1-PDA is more powerful than 0-PDA.

1. What is the difference between the express ability of 2-PDA and 1-PDA. Please clarify your argument. Prove the (un)equivalence.
2. How about 3-PDA and 2-PDA.

1

It's obvious that 2-PDA and 1-PDA are not equivalence. Since 2-PDA can accept languages like $a^n b^n c^n$ while 1-PDA cannot. Also, it's trivial to show that 2-PDA can express any language 1-PDA can accept. So 2-PDA is more powerful than 1-PDA.

2

3-PDA and 2-PDA are equivalence. I'll show that both 3-PDA and 2-PDA are equivalence to TM. Firstly, I'll prove that every language accepted by 2-PDA can be accepted by TM.

We choose a 2-tape TM. The first tape contains the input, and the second tape is a simulation for the stack. For any transition $\delta'(q, a, X) = (p, Y)$, supposed that the length of Y is k , we will get these transitions for the TM:

If $Y = \epsilon$, for the first tape: $\delta(q, a) = (p, a, R)$; for the second tape: $\delta(q, X) = (p, B, L)$

If $Y \neq \epsilon$, for the first tape:

$\delta(q, a) = (p, a, R)$ for the second tape: $\delta(q, X) = (q_{k-1}, Y[k-1], R)$

Also, for each integer i from 0 to $k-2$, exists $\delta(q_{i+1}, B) = (q_i, Y[i], R)$

finally, add $\delta(q_0, B) = (p, B, L)$

Similarly, we can simulate 3 stack using a 3-tape TM.

Now we have shown that any language accepted by 2-PDA and 3-PDA can be accepted by a TM.

Then we'll prove that 2-PDA can simulate a TM.

At the beginning, we just push all the input into stack A, and pop the character from stack A and push this character into stack B one by one, until we meet the stack-bottom marker of A. Then the stack B holds characters at the right of the TM head, the stack A holds characters at the left of the TM head.

For each transition in TM noted as $\delta'(p, X) = (q, Y, D)$:

If D is L, add $\delta(p, \epsilon, A, X) = (q, \epsilon, AY)$, where A is any symbol at the top of A except for the stack-bottom marker of A, noted as \perp_A .

When the top of A is \perp_A , add $\delta(p, \epsilon, \perp_A, X) = (q, \perp_A, BY)$

If D is R, add $\delta(p, \epsilon, A, X) = (q, YA, \epsilon)$.

In addition, if X is B(the blank symbol in M), also add $\delta(p, \epsilon, A, \perp_B) = (q, YA, \perp_B)$ ($D=R$) or $\delta(p, \epsilon, A, \perp_B) = (q, \epsilon, AY \perp_B)$ ($D=L$ & $A \neq \perp_A$) and $\delta(p, \epsilon, \perp_A, \perp_B) = (q, \perp_A, BY \perp_B)$ ($D=L$ & $A = \perp_A$);

Finally, change all the occurrences like $B \perp_B$, $B \perp_A$ to \perp_B , \perp_A respectively.

Now we have shown that 2-PDA can simulate a TM. Since 3-PDA can simulate 2-PDA by using only 2 stack, 3-PDA can simulate a TM. Then both 2-PDA and 3-PDA are equivalent to TM, which means they are equivalent.

Problem 6

Suppose we have an encoding of context-free grammars using some finite alphabet. Consider the following two languages:

1. $L_1 = \{(G, A, B) \mid G \text{ is a (coded) CFG, } A \text{ and } B \text{ are (coded) variables of } G, \text{ and the sets of terminal strings derived from } A \text{ and } B \text{ are the same}\}.$
2. $L_2 = \{(G_1, G_2) \mid G_1 \text{ and } G_2 \text{ are (coded) CFG's, and } L(G_1) = L(G_2)\}.$

Answer the following questions:

- a. Show that L_1 is polynomial-time reducible to L_2 .
- b. Show that L_2 is polynomial-time reducible to L_1 .

a

Just copy the language G twice, but mark A as the start symbol of G_1 , mark B as the start symbol of G_2 . The copy action will consume just $O(n)$ time, and the modification will consume constant time. So the reduction is polynomial-time. Also, we should prove that when L_2 accepts, L_1 accepts and when L_2 rejects, L_1 rejects.

If L_2 accepts, that means the terminal strings derived from the start symbol of G_1 and G_2 are equivalent. Since G_1 and G_2 share the same productions in G , then they will surely derivate the same terminal strings when they're two variables in G .

If L_2 rejects, we can assume that there is a string w , which is in $L(G_1)$ but is not in $L(G_2)$, then variable A in G can derivate the terminal string w but variable B cannot. Then (G, A, B) should be rejected.

b

Suppose that the start symbol of G_1 and G_2 are S_1, S_2 respectively. Then construct G as $S \rightarrow S_1 | S_2$. Before we copy productions from G_1 and G_2 , we just rename variables in G_1 such that variables in G_1 are disjoint from variables in G_2 . Then we get an instance of L_1 which will be (G, S_1, S_2) . It's obvious that the reduction takes $O(n)$ time. Now we're going to prove that when L_1 accepts, so does L_2 , and when L_1 rejects, so does L_2 .

If L_1 accepts, that means S_1 and S_2 can derivate the exact same terminal string set. Also, notice that all the production they can use are belongs to G_1 and G_2 respectively since we have guarantee that the variables in G_1 are disjoint with variables in G_2 . So for each terminal string they can derivate in G , they can also derivate it in G_1 or G_2 . So we can conclude that $L(G_1) = L(G_2)$.

If L_1 rejects, we can assume that there is a terminal string w that belongs to the terminal strings of S_1 but not belongs to the terminal strings of S_2 . It's obvious that S_1 can derivate w in G_1 . So $L(G_1) \neq L(G_2)$.

Problem 7

As classes of languages, \mathcal{P} and \mathcal{NP} each have certain closure properties. Prove or disprove that \mathcal{P} and \mathcal{NP} are closed under each of the following operations:

- Union.
- Concatenation.
- Complementation.

a

Both \mathcal{P} and \mathcal{NP} are closed under union.

\mathcal{P} : For any \mathcal{P} languages P_1 and P_2 , noted the related TM as M_1, M_2 . Construct a two-tape TM M' as follows:

Suppose that we have a subroutine S that will copy the input from tape 1 to tape 2 in $O(n)$ time. The accepting states, alphabet, tape symbols of M' are the union of their counterparts in M_1, M_2 and S . The start state is the start state of S .

1. For each transition in M_1 , noted as $\delta(p, X) = (q, Y, D)$, add $\delta(p, X, *) = (q, Y, *, D, S)$, where $*$ is any tape symbol on tape 2. Similarly, for each transition in M_2 , noted as $\delta(p, X) = (q, Y, D)$, add $\delta(p, *, X) = (q, *, Y, S, D)$, where $*$ is any tape symbol on tape 1.

2. The transitions will also include transitions that will copy the input to tape 2.

3. Also, for every non-accepting state p and every tape symbols c , if (p, c) is not defined in the transitions of M_1 , add $\delta(p, c, *) = (q_2, c, *, R, S)$, where q_2 is the start state of M_2 .

Since M_1 and M_2 will halt within polynomial time, the maximum time M' consumes will in polynomial times.

\mathcal{NP} : Similarly, if M_1 and M_2 are NTMs halt in polynomial time, then M' will be a NTM. The maximum time M' consumes will still in polynomial times.

b

\mathcal{P} and \mathcal{NP} are closed under concatenation.

We'll reuse the notations of problem a. \mathcal{P} :

The input can be divided as two substrings with length $[0, n], [2, n-2], \dots, [n, 0]$

Construct a TM with $2n$ tapes. Group the tapes into n group. For each group, there will be a division of the input string with the first part on one tape and the second one for another.

The division will takes $O(n^2)$ time since we will copy the string of length n into n groups.

Then for each group, we use M_1 and M_2 to decide them respectively. If both M_1 and M_2 are accepted, then we can say M_1 and M_2 accepts this group. If any group is accepted, then M' accepts the input, otherwise rejects.

The sum of the decide time for one group should be polynomial. For n groups, it will still be polynomial.

So M' can decide the input in polynomial time.

NP: The NTM M' will choose one division, then we use M_1 and M_2 to decide these two parts respectively. If both M_1 and M_2 are accepted, then NTM accepts this input.

If under no circumstances M' can accept this input, then M' will reject it. The decision will be made in polynomial time.

c

P and NP are closed under complementation.

P: First, copy the input to another tape. ($O(n)$)

Second, see whether M_1 accepts this input. If not, M' rejects this input. (polynomial)

(If M_1 accepts), third, see whether M_2 accepts this input, if so, the M' rejects this input, otherwise accepts. (polynomial)

So the total time would be polynomial.

NP: The time analysis for NTM is similar to that for TM.