	$\equiv$	$\equiv_V$	$\models$
DHORN	PTIME	Σ_2^P, hardness is unknown	PTIME
HORN	PTIME	Σ_2^P-complete	NP-complete
CNF	$P^{NP[\log n]}$, hardness unknown	Σ_3^P-complete	Σ_2^P-complete

1 General Problems

Configuration and Specification (Sets)

Equivalence Configuration Problem

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$ and β
Query: Whether $\exists K \subseteq C : K \equiv \beta$

Let

$$\mathrm{IMP}(C, \beta) := \{\alpha \mid \alpha \in C \text{ and } \beta \models \alpha\}$$

Then

$$\exists K \subseteq C(K \text{ is satisfiable and } K \equiv \beta) \Longleftrightarrow \mathrm{IMP}(C, \beta) \text{ is satisfiable and } \mathrm{IMP}(C, \beta) \models \beta$$

So, equivalence problem is in $\mathrm{P}^{\mathrm{NP}[\log n]}$ which is the class of problems solvable in polynomial time with $O(\log n)$ queries to an NP oracle.

Next we show hardness. The problem $\mathrm{SAT}^n_{\mathrm{odd}}$ is complete for $\mathrm{P}^{\mathrm{NP}[\log n]}$. Where $\mathrm{SAT}^n_{\mathrm{odd}}$ is the problem of determining whether the number of satisfiable formulas among n given CNF formulas.

We shall construct a reduction from $\mathrm{SAT}^n_{\mathrm{odd}}$ to the equivalence configuration problem.

Suppose $F_1, \cdots, F_n$ are 3CNF formulas such that they have pairwisely distinct vaiables. We assume n is even. Otherwise we add formula $p \wedge \neg p$.

For F_i introduce a new variable w_i. We claim that F_i is unsatisfiable iff $T_i := (\neg F_i \to w_i) \leftrightarrow w_i$ is tautological. In other word, F is satisfiable iff T_i is not a tautology (i.e., $\neg T_i$ is satisfiable.

Let $C := \{\neg w_1 \wedge F_1 \wedge \neg T_1, \cdots, F_n \wedge \neg T_i)\}$.

Let $\beta := (F_1 \wedge \neg T_1) \otimes \cdots \otimes (F_n \wedge \neg T_n)$.

Suppose there are odd number of satisfiable formulas, say $F_1, \cdots, F_{2k+1}$

Let $K := \{\neg T_1, \cdots, \neg T_{2k+1}\}$.
Clearly, $K \models \beta$.

Retricted Equivalence Configuration(R-equivalence)

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$, β, and a set V of variables
Query: Whether $\exists K \subseteq C : K \equiv^V \beta$

the problem of determining whether $F \equiv_V G$ is in Π_2^P (for the complementary problem: we guess a clause γ over V, check that $G \models \gamma$ but $F \not\models \gamma$, or $G \not\models \gamma$ but $F \models \gamma$).

Thus, R-equivalence configuraton is Σ_3^P (guess a $K \subseteq C$, check that it is satisfiable and $K \equiv_V \beta$.)

Next we show the hardness.
Consider $\Phi := \exists\vec{x}\forall\vec{y}\exists\vec{z}\varphi$ where φ is a CNF formula.

We assume w.l.o.g. that φ contains a non-tautological clause over $\vec{z}$ (otherwise, we pick new varaiable z' and consider $\exists\vec{x}\forall\vec{y}\exists\vec{z}\exists z'(\varphi \wedge z')$).
By this assumption, $\forall\vec{x}\exists\vec{y}\exists\vec{z}\neg\varphi$ is true

We assume that each clause contains a positive occurrence of a variable from $\vec{y} \cup \vec{z}$. If otherwise pick a new variable u, and consider $\exists\vec{x}\forall\vec{y}\forall u\exists\vec{z}\varphi^u$, where φ^u is obtained from φ by adding u to clauses containing no positive literal from $\vec{y} \cup \vec{z}$. Clearly, Φ and the resulting formula has the same truth.
By this assumption, $\forall\vec{x}\exists\vec{y}\exists\vec{z}\phi$ is true. (we will use this later).

φ can be written as $(c'_1 \vee c''_1) \wedge \cdots \wedge (c'_n \vee c''_n)$ in which c'_i is over $\vec{x}$, while c''_i is over $\vec{y} \cup \vec{z}$.
Pick new variables $w_1, \cdots, w_n, w$. Let
Let

$$\psi := \left(\bigwedge_{i=1}^{n}(c'_i \rightarrow w_i)\right) \wedge \left(\bigwedge_{i=1}^{n}(\neg c'_i \wedge \neg c''_i \rightarrow \neg w_i)\right)$$

Let

$$C_0 := \{\psi \wedge \neg c'_1 \wedge (c''_1 \rightarrow w_1), \cdots, \psi \wedge \neg c'_n \wedge (c''_n \rightarrow w_n)\}$$

$$C_1 = \{c'_1, \cdots, c'_n, \}$$

$$C_2 = \{x_1, \neg x_1 \cdots, x_n, \neg x_n\}$$

$$C_3 = \{(w \to (c'_1 \vee c''_1) \wedge \cdots \wedge (c'_n \vee c''_n))\}$$

Now let

$$C := C_0 \cup C_1 \cup C_2 \cup C_3 \cup \{w\}$$

Let β be

$$w_1 \wedge \cdots \wedge w_n \wedge w$$

Let $V := \vec{y} \cup \{w_1, \cdots, w_n, \} \cup \{w\}$

(Lemma: $F \models_V G$ iff for for any truth assignment t on V, if it can be extended to a satisfying truth assignment for F, it can be extended to an satisfying truth assignment for G.)

We shall show Φ is true if and only if there is satisfiable $K \subseteq C$ s.t. $K \equiv_V \beta$

From right to left:

Given a such $K \equiv_V \beta$. Clearly, $K \cap C_0$ is non-empty.

It must be that $w \in K$. Let t satisfy K. By our assumption we know $t \upharpoonright \vec{x}$ can be extended to s which falsifies φ. Now we assign truth values to w_i according to the tuth values $s(c'_i)$ and $s(c''_i)$ to make formulas ψ and $(c''_i \to w_i)$ to be true. There must be some w_i which is false. Then if we set w to be false, K is still be satisfied by s (because clauses in C_3 are satisfied). This contradict the V-euivalence.

It must be that $C_3 \subseteq K$, i.e., $w \to (c'_1 \vee c''_1) \wedge \cdots \wedge (c'_n \vee c''_n)$ is in K. Suppose it is not the case. Similar as above, there would be a satisfying truth assigment of K which makes some w_i false, contradicts the V-equivalence.

Suppose $K \not\models c'_i$. Then $\psi \wedge \neg c'_i \wedge (c''_i \to w_i)$ must be in K. Otherwise, $K \cup \{\neg c'_i\}$ is consitent, then there is a satisfying truth assignment t for K such that $t(c'_i) = 0, t(w) = 1, t(w_i) = 0$. (In fact by our assumption, t can be assumed to satisfiy $(c_1' \vee c_1'') \wedge \cdots \wedge (c_n' \vee c_n'')$ because every clause contains positive literal from $\vec{y} \cup \vec{z}$). This contradict the fact that $K \equiv_V \beta$.

Now we can see for each i, either $K \models \neg c_i'$ or $K \models c_i'$. That is, for any two satisfying truth assignments t_1, t_2 of K, each c_i' has the same truth under $t_1 \uparrow \vec{x}$ and $t_2 \uparrow \vec{x}$.

Now fix a trut assigment e on $\vec{x}$ which can be extended to a satisfying truth assigment of K.

Consider any truth assigment s on $\vec{y}$. Since s can be extended to satisfy β, it can be extended to a truth assigment t which satisfies K. Please note we can assmue that $t \uparrow \vec{x}$ is e. Since w and $w \to (c_1' \vee c_1'') \wedge \cdots \wedge (c_n' \vee c_n'')$ are in K. We can see t satisfy φ.

Consequently, for any truth assigment s on $\vec{y}$, $e * s$ can be extended to a satisfying truth assigment of φ. Thus, Φ is true.

From left to right.

Suppose $\Phi = \exists \vec{x} \forall \vec{y} \exists \vec{z} \varphi$ is true.

Let e be a truth assignment on $\vec{x}$ such that $\forall \vec{y} \exists \vec{x} \varphi[x/e]$ is true

Let

$$K_0 := \{\psi \wedge \neg c_i' \wedge (c_i'' \to w_i) \mid e(c_i') = 0, i = 1, \cdots, n\}$$

$$K_1 := \{c_i' \mid e(c_i') = 1, i = 1, \cdots, n\}$$

$$K_2 := \{x_i \mid e(x_i) = 1, i = 1, \cdots, n\} \cup \{\neg x_i \mid e(x_i) = 0, i = 1, \cdots, n\}$$

$$K := K_0 \cup K_1 \cup K_2 \cup C_3 \cup \{w\}$$

Consider any truth assignment s on V.

Suppose s can be extended to satisfy K. Say the extenson is t. Since $w \in K$, $(c_1' \vee c_1'') \wedge \cdots \wedge (c_n' \vee c_n')$ be be true under t. By formulas in K_0, we can see each w_i is true under t. That means s make β true. Thus, $K \models_V \beta$.

Now suppose s satisfies β. Since Φ is true, $e * (s \uparrow \vec{y})$ can be extended to satisfy Φ. Let t be such an extension. Next we show t can be extended to satisies K. Since t makes φ true, either c_i' is true or c_i'' is true. We set each w_i to be true. Then all clauses in K_0 are true. Set w to be true. Then clauses in C_3 are true. Please note formulas in $K_1 \cup K_2$ are already atisfied by e. Consequently, s can be extended to satisfy K. Hence, $\beta \models_V K$.

Altogether we obtain $K \equiv_V \beta$.

Implication Problem

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$, β
Query: Whether $\exists K \subseteq C : K \models \beta$

Clearly in Σ_2^P. We shall the hardness
Consider $\Phi = \exists x_1, \cdots, x_n \forall y_1, \cdots, y_m \varphi$ where φ is a DNF formula. Pick a new varibale z.
Let $C := \{\varphi \to z, x_1, \neg x_1, \cdots, x_n, \neg x_n\}$, $\beta := z$
Clearly, Φ is true if and only if $\exists K \subseteq C$ such that K is satisfiable and $K \models \beta$.

Note: In all problems K is demanded satisfiable.

$$\alpha \equiv_V \beta \Longleftrightarrow \forall \gamma \text{ over } V, (\alpha \models \gamma \Leftrightarrow \beta \models \gamma)$$

$$\alpha \models_V \beta \Longleftrightarrow \forall \gamma \text{ over } V, (\beta \models \gamma \Longrightarrow \alpha \models \gamma)$$

2 Resticted to DHORN

C: set of DHORN, β: DHORN

Equivalence problem is in PTIME

Idea: just checked $\text{IMP}(C, \beta) \equiv \beta$.

Implication problem is trivial

The existence of K is equivalent to $C \models \beta$

The V-equivalence problem

we guess it seems Σ_2^P-complete.
See Hans's book page 251

3 Restricted to HORN

Equivalence Problem: same as the DHORN case (PTIME).

Desired K exists iff $\mathrm{IMP}(C, \beta)$ is satisfiable and $\mathrm{IMP}(C, \beta) \models \beta$

V-Equivalence Configuration Problem : Σ_2^P-complete

$$\Phi := \exists y_1, \cdots, y_k \forall x_1, \cdots, x_m \, (c_1 \vee \cdots \vee c_n)$$

Where c_i is a conjunction of literals.

for each variable z, introduce a new variable $\pi(\neg z)$. Let $\pi(z) = z$.

introduce U.

Let

$$\psi_0 := \left(\bigwedge_{j=1}^{m} (\neg \pi(\neg x_j) \vee \neg x_j) \right), \ \psi_1 := \left(\bigwedge_{i=1}^{k} (\neg \pi(\neg y_i) \vee \neg y_i) \right)$$

$$\theta := \left(\bigvee_{i=1}^{m} (\neg x_i \wedge \neg \pi(\neg x_i)) \right)$$

Please note that θ is not HORN when $n > 1$. Forturnately, by using Tseiting algorithm, one can transform θ to an equivalent HORN formula when restricted to old variables.

For a conjunction $c = L_1 \wedge \cdots \wedge L_s$, we write $\pi(c) := \pi(L_1) \wedge \cdots \wedge \pi(L_s)$

$$\begin{array}{lcl} C_0 & := & \{\psi_0 \wedge \psi_1 \wedge (\bigwedge_{i=1}^{n}(\pi(c_i) \to U \vee \theta))\} \\ C & := & C_0 \cup \{y_i, \pi(\neg y_i) \mid i = 1, \cdots, k\} \\ \beta & := & (U \vee \theta) \wedge \psi_0 \\ V & := & \{U\} \cup \{x_1, \cdots, x_m, \pi(\neg x_1), \cdots, \pi(\neg x_m)\} \end{array}$$

We shall show

$$(\exists K \subseteq C \text{ such that } K \equiv_V \beta) \text{ if and only if } \Phi \text{ is true.}$$

!!!! I suddenly find that $U \vee \theta$ may not be HORN. Forturnately we can change U to $\neg U$ in C and β. Then the following proof should be changed accordingly

Suppose Φ is true. There is truth assignment e on $\{y_1, \cdots, y_m\}$ such that $\forall x_1, \cdots, x_m \varphi[\vec{y}/e]$ is true, where $\vec{y} = y_1 \cdots, y_k$. Define

$$K = C_0 \cup \{y_i \mid t(y_i) = 1, 1 \leq i \leq k\} \cup \{\pi(\neg y_i) \mid t(y_i) = 0, 1 \leq i \leq k\}$$

For any satisfying truth assignment t for K, if it makes θ true then β is true under t. Suppose t makes θ false, then t corresponds a truth assignment on $x_1, \cdots, x_n$ accoding the truth of x_i and $\pi(\neg x_i)$. Since Φ is true, some $\pi(c_i)$ must be true under t, then $T(U) = 1$. Therefore, $t(\beta) = 1$.

Suppose s is a satisfying truth assignment for β. Then $s * e$ satisfies $\psi_0 \wedge \psi_1$. If s makes θ true, then $s * e$ satisfies K. Suppose $s(\theta) = 0$. Then $s(U) = 1$, hence K is still satisfied by $s * e$.

Altogether, we have $K \equiv_V \beta$.

For the inverse direction, suppose there is satisfiable $K \subseteq C$ such that $K \equiv_V \beta$.

Clearly, the formula in C_0 must be in K. Then by formula ψ_1, either y_i or $\pi(\neg y_i)$ is not in K.

Pick a truth assignment e on $\{y_1, \cdots, y_k\}$ such that if $y_i \in K$ then $e(y_i) = 1$, and $e(y_i) = 0$ if else.

We need to show $\forall x_1, \cdots, x_m(c_1 \vee \cdots \vee c_n)$ is true.

Consider any truth assigment s on $\{x_1, \cdots, x_m\}$. Let s' be the assignment defined by $s'(\pi(\neg x_i) = 1$ iff $s(x_i) = 0$. Then $e' * s'$ satisfies $\psi_0 \wedge \psi_1$, where e' is obtained from e is the same way as s'. We claim that, $(e' * s')$ makes $\pi(c_i)$ true for some i. Otherwise, we could set U to be false, and get a truth assignment satisfying K but bot satisfying β (please note that $s'(\theta) = 0$), contradict the V-quivalence.

Consequently, Φ is true.

Implication Problem

NP-complete

Given a 3CNF F

$$\bigwedge_{i=1}^{m}(L_{i,1} \vee L_{i,2} \vee L_{i,3}) \text{ over } x_1, \cdots, x_n$$

For each $i = 1, \cdots, m$, pick a new variable z_i. For each $j = 1, \cdots, n$ we pick a new variable $\pi(\neg x_j)$. For convenience, we also write x_j as $\pi(x_j)$.

Define C

$$\begin{array}{rl} C := & \bigcup_{i=1}^{m}\{\pi(L_{i,1}) \to z_i, \pi(L_{i,2}) \to z_i, \pi(L_{i,3}) \to z_i\} \cup \\ & \bigcup_{j=1}^{n}\{\to x_j, \to \pi(\neg x_j\} \cup \\ & \{z_1 \wedge \cdots \wedge z_m \to z\} \end{array}$$

Define

$$\beta := z \wedge \bigwedge(\neg\pi(\neg x_j) \vee \neg x_j)$$

Impliation problem iff F is satisfiable

Specification Problem

Given a partial configuration K, demand β, a set of variables V, W

Looking for σ over W such that

1. $K \wedge \sigma(W) \equiv \beta$
2. $K \wedge \sigma(W) \equiv_V \beta$

3. $K \wedge \sigma(W) \models \beta$

Query Learning

black box α

equivalence query. Guess a β ask whether $\alpha \equiv \beta$. If the answer is no, output a truth assignment satisfying α and $\neg\beta$ or satisfying $\neg\alpha$ and β.

membership query

guess a truth assignment v answer $v(\alpha) = 0$ or 1.

4 Configuration with constraints

$C = \{\alpha_1, \cdots, \alpha_n\}, \beta$, D is set of formulas over $A_1, \cdots, A_n$ which are propositional atoms. Whether there is $K \subseteq C$ such that

- K is satisfiable,
- $K \equiv \beta$, and
- the truth assignment v_K satisfies D, where $v_K(A_i) = \begin{cases} 1 & \text{if } \alpha_i \in K \\ 0 & \text{if } \alpha_i \notin K \end{cases}$