1 General Problems

Configuration and Specification (Sets)

Equivalence Configuration Problem

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$
Query: Whether $\exists K \subseteq C : K \equiv \beta$

Let

$$\mathrm{IMP}(C, \beta) := \{\alpha \mid \alpha \in C \text{ and } \beta \models \alpha\}$$

Then

$$\exists K \subseteq C(K \equiv \beta) \Longleftrightarrow \mathrm{IMP}(C, \beta) \models \beta$$

So, equivalence problem is in $\mathrm{P}^{\mathbf{NP}[\log n]}$ which is the class of problems solvable in polynomial time with $O(\log n)$ queries to an NP oracle.

Next we show hardness. The problem $\mathrm{SAT}^n_{\mathrm{odd}}$ is complete for $\mathrm{P}^{\mathbf{NP}[\log n]}$. Where $\mathrm{SAT}^n_{\mathrm{odd}}$ is the problem of determining whether the number of satisfiable formulas among n given CNF formulas.

We shall construct a reduction from $\mathrm{SAT}^n_{\mathrm{odd}}$ to the equivalence configuration problem.

Suppose $F_1, \cdots, F_n$ are 3CNF formulas such that they have pairwisely distinct vaiables. We assume F_i is not tautolodical, otherwise we consider $F_i \wedge p_i$.

It is easy to see that φ is satsfiable iff the number of satisfiable formulas among $F_1, \cdots, F_n$ is odd.

Please note that there is transformation T such that a 3CNF formula F is satisfiable iff $T(F)$ is minimal unsatisfiable.

Now we have $T(F_1), \cdots, T(F_n)$, and suppose they have $m_1, \cdots, m_n$ clauses, repectively.

By the definition of transformation T, in each $T(F_i)$ we have a clause $c_i = (y_{i,1} \vee \cdots \vee y_{i,m_i})$, where $y_{i,j}$ are introduced new variables by T.

Now we can see F_i is satisfiable iff $T(F_i) - \{c_i\}$ is satisfiable and equivalent to $\neg c_i \wedge F_i$ are equivalent.

Let $T'(F_i) := (T(F_i) - \{c_i\}) \vee_{cl} c_i$

Let $C := \{T'(F_1), \cdots, T'(F_n), c_1, \cdots, c_n\}$.

Let $\beta := (c_1 \vee F_1) \wedge \cdots \wedge (c_n \vee F_n) \wedge (\neg c_1 \otimes \cdots \otimes \neg c_n)$

Suppose there are odd number of sat frmulas. For simplicility, we assume $F_1, \cdots, F_{2k+1}$ are sat, while others are unsat.

Let $K = \{T'(F_i), \cdots, T'(F_n)\}$,

Retricted Equivalence Configuration(R-equivalence)

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$, β, and a set V of variables
Query: Whether $\exists K \subseteq C : K \equiv^V \beta$

the problem of determining whether $F \equiv_V G$ is in Π_2^P (for the complementary problem: we guess a clause γ over V, check that $G \models \gamma$ but $F \not\models \gamma$.

Thus, R-equivalence configuraton is Σ_3^P (guess a $K \subseteq C$, theck that it is satisfiable and $K \equiv_V \beta$.)

Next we show the hardness.

Consider $\Phi := \exists \vec{x} \forall \vec{y} \exists \vec{z} \varphi$ where φ is a CNF formula. We assume w.l.o.g. that φ contains clauses non-tautological calss over $\vec{z}$ (otherwise, we pick new varaiable z' and consider $\exists \vec{x} \forall \vec{y} \exists \vec{z} \exists z' (\varphi \wedge z')$

φ be written as $(c_1' \vee c_1'') \wedge (c_n' \vee c_n'')$ in which c_i' is clause over $\vec{x}$.

Pick a new variable $w_1, \cdots, w_n, w$. Let

Let

$$\psi := \left(\bigwedge_{i=1}^{n} (c_i' \to w_i) \right) \wedge \left(\bigwedge_{i=1}^{n} (\neg c_i' \wedge \neg c_i'' \to \neg w_i) \right)$$

Let

$$C_0 := \{\psi \wedge (c_1'' \to w_1), \cdots, \psi \wedge (c_n'' \to w_n)\}$$

$$C_1 = \{c_1', \neg c_1', \cdots, c_n', \neg c_n'\}$$

$$C_2 = \{x_1, \neg x_1 \cdots, x_n, \neg x_n\}$$

$$C_3 = \{(w_1 \wedge \cdots \wedge w_n \to w), (w \to (c_1' \vee c_1'') \wedge \cdots \wedge (c_n' \vee c_n''))\}$$

Now let

$$C := C_0 \cup C_1 \cup C_2 \cup C_3 \cup \{w\}$$

Let β be

$$w_1 \wedge \cdots \wedge w_n \wedge w$$

Let $V := \vec{y} \cup \{w_1, \cdots, w_n,\} \cup \{w\}$

Claim: Suppose $K \subseteq C$ satisfiable, $c_i \notin K$ and $K \not\models c_i$, then $K \cup \{\neg c_i, \psi \wedge (c_i'' \to w_i)\}$ is still satisfiable.

(Lemma: $F \models_V G$ iff for for any truth assignment t on V, if it can be extended to a satisfying truth assignment for F, it can be extended to an satisfying truth assignment for G.)

We shall show Φ is true if and only if there is satisfiable $K \subseteq C$ s.t. $K \equiv_V \beta$

Implication Problem

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$, β

Query: Whether $\exists K \subseteq C : K \models \beta$

Clearly in Σ_2^P. We shall the hardness

Consider $\Phi = \exists x_1, \cdots, x_n \forall y_1, \cdots, y_m \varphi$ where φ is a DNF formula. Pick a new varibale z.

Let $C := \{\varphi \to z, x_1, \neg x_1, \cdots, x_n, \neg x_n\}$, $\beta := z$

Clearly, Φ is true if and only if $\exists K \subseteq C$ such that $K \models \beta$.

Note: In all problems K is demanded satisfiable.

$$\alpha \equiv_V \beta \Longleftrightarrow \forall \gamma \text{ over } V, (\alpha \models \gamma \Leftrightarrow \beta \models \gamma)$$

$$\alpha \models_V \beta \Longleftrightarrow \forall \gamma \text{ over } V, (\beta \models \gamma \Longrightarrow \alpha \models \gamma)$$

Euivalence Problem is in $\mathrm{P}^{\mathrm{NP}[\log n]}$

R-Equivalence Problem same as above

Implication Σ_2^P?

2 Resticted to DHORN

C: set of DHORN, β: DHORN

Equivalence problem is in PTIME

Idea: Let

Implication problem is trivial

The existence of K is equivalent to $C \models \beta$

The V-equivalence problem

we guess it seems Σ_2^P-complete.
See Hans's book page 251

3 Restricted to HORN

Equivalence Problem: same as the DHORN case (PTIME).

iff $\mathrm{IMP}(C, \beta)$ is satisfiable and $\mathrm{IMP}(C, \beta) \models \beta$

V-Equivalence Problem: Σ_2^P-complete

$$\Phi := \exists y_1, \cdots, y_k \forall x_1, \cdots, x_m \, (c_1 \vee \cdots \vee c_n)$$

Where c_i is a conjunction of literals.

for each variable z, introduce a new variable $\pi(\neg z)$. Let $\pi(z) = z$.

introduce U.

for a clause $c = L_1 \wedge \cdots \wedge L_s$, we write $\pi(c) := \pi(L_1) \wedge \cdots \wedge \pi(L_s)$

$$
\begin{aligned}
C := \;& \{\pi(c_i) \to U \mid i = 1, \cdots, n\} \cup \\
& \{\neg\pi(\neg x_j) \vee \neg x_j \mid j = 1, \cdots, m\} \cup \\
& \{\neg\pi(\neg y_i) \vee \neg y_i \mid i = 1, \cdots, k\} \cup \\
& \{y_i, \pi(\neg y_i) \mid i = 1, \cdots, k\}
\end{aligned}
$$

$$\beta := U \vee \left(\bigvee_{j=1}^{k} (\neg\pi(\neg x_j) \wedge \neg x_j) \right)$$

(Remark: using Tsting algorithm)

Let

$$V := \{U\} \cup \{x_1, \cdots, x_m\}$$

($\exists K \subseteq C$ such that $K \equiv_V \beta$) if and only if Φ is true.

Implication Problem
NP-complete
Given a 3CNF F

$$\bigwedge_{i=1}^{m} (L_{i,1} \vee L_{i,2} \vee L_{i,3}) \;\text{ over } x_1, \cdots, x_n$$

For each $i = 1, \cdots, m$, pick a new variable z_i. For each $j = 1, \cdots, n$ we pick a new variable $\pi(\neg x_j)$. For convenience, we also write x_j as $\pi(x_j)$.

Define C

$$
\begin{aligned}
C := \;& \textstyle\bigcup_{i=1}^{m} \{\pi(L_{i,1}) \to z_i, \pi(L_{i,2}) \to z_i, \pi(L_{i,3}) \to z_i\} \cup \\
& \textstyle\bigcup_{j=1}^{n} \{\to x_j, \to \pi(\neg x_j)\} \cup \\
& \{z_1 \wedge \cdots \wedge z_m \to z\}
\end{aligned}
$$

Define

$$\beta := z \wedge \bigwedge (\neg\pi(\neg x_j) \vee \neg x_j)$$

Impliation problem iff F is satisfiable

Specification Problem

Given a partial configuration K, demand β, a set of variables V, W
Looking for σ over W such that

1. $K \wedge \sigma(W) \equiv \beta$
2. $K \wedge \sigma(W) \equiv_V \beta$
3. $K \wedge \sigma(W) \models \beta$

Query Learning
black box α
equivalence query. Guess a β ask whether $\alpha \equiv \beta$. If the answer is no, output a truth assignment satisfying α and $\neg\beta$ or satisfying $\neg\alpha$ and β.

membership query
guess a truth assignment v answer $v(\alpha) = 0$ or 1.

4 Configuration with constraints

$C = \{\alpha_1, \cdots, \alpha_n\}, \beta$, D is set of formulas over $A_1, \cdots, A_n$ which are propositional atoms. Whether there is $K \subseteq C$ such that

- K is satisfiable,
- $K \equiv \beta$, and
- the truth assignment v_K satisfies D, where $v_K(A_i) = \begin{cases} 1 & \text{if } \alpha_i \in K \\ 0 & \text{if } \alpha_i \notin K \end{cases}$