1 General Problems

Configuration and Specification (Sets)

Equivalence Problem

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$
Query: Whether $\exists K \subseteq C : K \equiv \beta$

Let

$$\mathrm{IMP}(C, \beta) := \{\alpha \mid \alpha \in C \text{ and } \beta \models \alpha\}$$

Then

$$\exists K \subseteq C(K \equiv \beta) \Longleftrightarrow \mathrm{IMP}(C, \beta) \models \beta$$

So, equivalence problem is in $\mathrm{P}^{\mathbf{NP}[\log n]}$ which is the class of problems solvable in polynomial time with $O(\log n)$ queries to an NP oracle.

Next we show hardness. The problem $\mathrm{SAT}^n_{\mathrm{odd}}$ is complete for $\mathrm{P}^{\mathbf{NP}[\log n]}$. Where $\mathrm{SAT}^n_{\mathrm{odd}}$ is the problem of determining whether the number of satisfiable formulas among n given CNF formulas.

We shall construct a reduction from $\mathrm{SAT}^n_{\mathrm{odd}}$ to the equivalence configuration problem.

Suppose $F_1, \cdots, F_n$ are 3CNF formulas such that they have pairwisely distinct vaiables. We assume F_i is not tautolodical, otherwise we consider $F_i \wedge p_i$.

It is easy to see that φ is satsfiable iff the number of satisfiable formulas among $F_1, \cdots, F_n$ is odd.

Please note that there is transformation T such that a 3CNF formula F is satisfiable iff $T(F)$ is minimal unsatisfiable.

Now we have $T(F_1), \cdots, T(F_n)$, and suppose they have $m_1, \cdots, m_n$ clauses, repectively.

By the definition of transformation T, in each $T(F_i)$ we have a clause $c_i = (y_{i,1} \vee \cdots \vee y_{i,m_i})$, where $y_{i,j}$ are introduced new variables by T.

Now we can see F_i is satisfiable iff $T(F_i) - \{c_i\}$ is satisfiable and equivalent to $\neg c_i \wedge F_i$ are equivalent.

Let $T'(F_i) := (T(F_i) - \{c_i\}) \vee_{cl} c_i$

Let $C := \{T'(F_1), \cdots, T'(F_n), c_1, \cdots, c_n\}$.

Let $\beta := (c_1 \vee F_1) \wedge \cdots \wedge (c_n \vee F_n) \wedge (\neg c_1 \otimes \cdots \otimes \neg c_n)$

Suppose there are odd number of sat frmulas. For simplicility, we assume $F_1, \cdots, F_{2k+1}$ are sat, while others are unsat.

Let $K = \{T'(F_i), \cdots, T'(F_n)\}$,

Retricted Equivalence(R-equivalence)

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$, β, and a set V of variables
Query: Whether $\exists K \subseteq C : K \equiv^V \beta$

Σ_3^P

Implication Problem

Input: $C = \{\alpha_1, \cdots, \alpha_n\}$, β
Query: Whether $\exists K \subseteq C : K \models \beta$

Note: In all problems K is demanded satisfiable.

$$\alpha \equiv_V \beta \Longleftrightarrow \forall \gamma \text{ over } V, (\alpha \models \gamma \Leftrightarrow \beta \models \gamma)$$

$$\alpha \models_V \beta \Longleftrightarrow \forall \gamma \text{ over } V, (\beta \models \gamma \Longrightarrow \alpha \models \gamma)$$

Euivalence Problem is in $\mathrm{P}^{\mathrm{NP}[\log]n}$
R-Equivalence Problem same as above
Implication Σ_2^P?

2 Resticted to DHORN

C: set of DHORN, β: DHORN

Equivalence problem is in PTIME

Idea: Let

Implication problem is trivial

The existence of K is equivalent to $C \models \beta$

The V-equivalence problem

we guess it seems Σ_2^P-complete.
See Hans's book page 251

3 Restricted to HORN

Equivalence Problem: same as the DHORN case (PTIME).

iff $\mathrm{IMP}(C, \beta)$ is satisfiable and $\mathrm{IMP}(C, \beta) \models \beta$

V-Equivalence Problem: Σ_2^P-complete

$$\Phi := \exists y_1, \cdots, y_k \forall x_1, \cdots, x_m \, (c_1 \vee \cdots \vee c_n)$$

Where c_i is a conjunction of literals.

for each variable z, introduce a new variable $\pi(\neg z)$. Let $\pi(z) = z$.

introduce U.

for a clause $c = L_1 \wedge \cdots \wedge L_s$, we write $\pi(c) := \pi(L_1) \wedge \cdots \wedge \pi(L_s)$

$$
\begin{aligned}
C := \quad & \{\pi(c_i) \to U \mid i = 1, \cdots, n\} \cup \\
& \{\neg\pi(\neg x_j) \vee \neg x_j \mid j = 1, \cdots, m\} \cup \\
& \{\neg\pi(\neg y_i) \vee \neg y_i \mid i = 1, \cdots, k\} \cup \\
& \{y_i, \pi(\neg y_i) \mid i = 1, \cdots, k\}
\end{aligned}
$$

$$\beta := U \vee \left(\bigvee_{j=1}^{k} (\neg\pi(\neg x_j) \wedge \neg x_j) \right)$$

(Remark: using Tsting algorithm)

Let

$$V := \{U\} \cup \{x_1, \cdots, x_m\}$$

($\exists K \subseteq C$ such that $K \equiv_V \beta$) if and only if Φ is true.

Implication Problem
NP-complete
Given a 3CNF F

$$\bigwedge_{i=1}^{m} (L_{i,1} \vee L_{i,2} \vee L_{i,3}) \ \text{ over } x_1, \cdots, x_n$$

For each $i = 1, \cdots, m$, pick a new variable z_i. For each $j = 1, \cdots, n$ we pick a new variable $\pi(\neg x_j)$. For convenience, we also write x_j as $\pi(x_j)$.

Define C

$$
\begin{aligned}
C := \quad & \textstyle\bigcup_{i=1}^{m} \{\pi(L_{i,1}) \to z_i, \pi(L_{i,2}) \to z_i, \pi(L_{i,3}) \to z_i\} \cup \\
& \textstyle\bigcup_{j=1}^{n} \{\to x_j, \to \pi(\neg x_j)\} \cup \\
& \{z_1 \wedge \cdots \wedge z_m \to z\}
\end{aligned}
$$

Define

$$\beta := z \wedge \bigwedge (\neg\pi(\neg x_j) \vee \neg x_j)$$

Impliation problem iff F is satisfiable

Specification Problem

Given a partial configuration K, demand β, a set of variables V, W
Looking for σ over W such that

1. $K \wedge \sigma(W) \equiv \beta$
2. $K \wedge \sigma(W) \equiv_V \beta$
3. $K \wedge \sigma(W) \models \beta$

Query Learning
black box α
equivalence query. Guess a β ask whether $\alpha \equiv \beta$. If the answer is no, output a truth assignment satisfying α and $\neg\beta$ or satisfying $\neg\alpha$ and β.

membership query
guess a truth assignment v answer $v(\alpha) = 0$ or 1.