COMS10014 Worksheet 8: Sets

1. Introduction to Sets (★)

In the universe $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7,8,9,10\}$, consider the sets $A = \{1, 4,7,10\}$, $B = \{1,2,3,4,5\}$ and $C = \{2,4,6,8\}$. What are the elements of the following sets?

a. $A \cup B$
b. $B \cap C$
c. $A \setminus B$
d. $B \setminus A$
e. $\bar{A}$
f. $\mathcal{U} \setminus C$
g. $\bar{\mathcal{U}}$
h. $A \cup \emptyset$
i. $B \cap \emptyset$
j. $A \cup \mathcal{U}$
k. $B \cap \mathcal{U}$
l. $A \cap (B \cup C)$
m. $\bar{B} \cap (C \setminus A)$
n. $(A \cap B) \setminus C$
o. $\overline{A \cap B} \cup C$
p. $(A \cup B) \setminus (C \setminus B)$

2. Cartesian Product (★)

For the sets $D = \{0,1\}, L = \{a, b, c\}, S = \{+,*\}, M = \{a, b\}$ write down the following sets.

1. $\mathrm{D} \times \mathrm{L}$
2. $\mathrm{L} \times \mathrm{D}$
3. $\mathrm{D} \times \mathrm{D} \times \mathrm{D}$
4. $(\mathrm{D} \times \mathrm{D}) \times \mathrm{D}$
5. $\mathrm{D} \times \mathrm{S} \times \mathrm{D}$
6. $\mathrm{D} \times \emptyset$
7. $(\mathrm{D} \times \mathrm{M}) \cup (\mathrm{D} \times \mathrm{S})$
8. $(\mathrm{D} \cup \mathrm{M}) \times (\mathrm{D} \cup \mathrm{S})$

3. Power Sets (★)

a. For the sets $A = \{a\}$ and $B = \{a, b\}$ write down the following sets.
   1. $\mathcal{P}(A)$
   2. $\mathcal{P}(B)$
   3. $\mathcal{P}(B \setminus A)$
   4. $\mathcal{P}(B) \setminus \mathcal{P}(A)$
   5. $A \times \mathcal{P}(B)$
   6. $\mathcal{P}(A \times B)$

b. For the same sets as above, which of the following are true?
   1. $A \in \mathcal{P}(B)$
   2. $A \subseteq \mathcal{P}(B)$
   3. $\mathcal{P}(A) \in \mathcal{P}(B)$
   4. $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

c. (★★) Let A, B now be arbitrary sets (which could be empty). If $A \subseteq B$, which of the four statements in (b.) hold in general? Prove the ones that do, and find counter-examples for the others.

4. Special Properties (★★)

Here are some properties of sets that do not hold in general. What can we deduce about sets A, B from these properties, if they do hold?

a. $A \cap B = A$
b. $A \cup B = A$
c. $\bar{A} \cap B = \emptyset$
d. $\overline{A \cap B} = \bar{B}$

5. General Properties (★★)

Which of the following laws are true for all sets A, B, C? Prove or disprove each one. Determining whether a law is true or not is easy; the proof in each true case is straightforward resp. "mechanical" too by turning a statement about sets into one about logic, using laws of logic to transform it, and turning it back into one about sets. The difficulty here is the level of abstraction you need to work at.

a. $A \cap B = B \cap A$
b. $A \setminus B = B \setminus A$
c. $(A \setminus B) \cap C = (A \cap C) \setminus B$
d. $A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$
e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
f. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
g. $(A \setminus B) \setminus C = A \setminus (B \setminus C)$
h. $(A \setminus B) \setminus C = (A \setminus C) \setminus B$

6. More on Cartesian Products and Power Sets (★★)

a. Is the Cartesian product commutative? What about associative?
b. Show that for sets A, B, we have $(A \times B = \emptyset) \mathrel{\widehat{=}} (A = \emptyset \lor B = \emptyset)$.
c. What can be said, for any sets A, B, about $(A \times B) \cap (B \times A)$?
d. Can you find example sets A, B so that both $A \subseteq B$ and $A \subseteq \mathcal{P}(B)$, and neither set is empty? Or can you prove that no such sets exist?