Index of Lectures

01: Terms

- Number sets ($\mathbb{N}, \mathbb{Z}$ etc.)
- Terms (in arithmetic) as mathematical objects; identity $=$
- Syntax rules, $\vdash$ and $\widehat{=}$
- Term strings, parsing
- Printing - infix, fully bracketed infix, RPN
- Parsing RPN
- Semantics / evaluation, environments, $\perp$, equivalence $\equiv$
- Completeness and Soundness

02: Logic

- Propositional logic, propositions as sentences
- Implication $\rightarrow$, bi-implication $\leftrightarrow$, operations $\wedge, \vee, \neg, \oplus$
- Truth Tables
- Terms / Formulas vs Statements (inner and outer logic)
- Laws of logic

03: More Logic

- Tautology, contingency, contradiction; (un)satisfiable
- Entailment $\models$
- Converse, Contrapositive, Inverse
- Functional Completeness
- CNF and DNF
- NAND $\uparrow$ and NOR $\downarrow$

04: Proofs

- Elements of proof style
- Rules of arithmetic
- Unpack / calculate / pack principle
- Direct, indirect and contradiction proofs
- Case distinction
- Euclid's theorem (statement), odd and even numbers
- Proofs of uniqueness

05: More on proofs

- Laws of reasoning: modus ponens, modus tollens, disjunctive and hypothetical syllogisms
- Scope in proofs
- Natural deduction, laws of introduction and elimination

06: Induction

- Induction over $\mathbb{N}$
- Well-ordering
- Structural induction
- Loop variant and invariant proofs
- Existence proof for Euclid's theorem with a loop invariant

07: Predicate Logic

- Free and bound variables
- Quantifiers
- Proof techniques
- Predicate DeMorgan
- α-renaming
- Nesting quantifiers
- Expressing 'at most one' and 'exactly one'

08: Sets

- Element-of $\in$ (and $\notin$), empty set $\varnothing$
- Set notation $\{\ldots\}$ and equality
- Set restriction and projection
- Subsets $\subseteq$
- Set operations $\cup, \cap, \setminus$ and complement in a universe
- Set theory laws
- Partitions
- Sets of sets
- Powerset $\mathcal{P}$, cardinality $|\cdot|$, Cartesian product $\times$

09: Functions

- Definition in set theory
- Injective, surjective, bijective
- Inverse
- Preimage, (direct) image
- Composition $\circ$
- Cardinality of function sets

10: Relations

- Definition in set theory
- Inverse, composition
- Partial and total orders, lexicographic order
- Equivalence relations
- Number theory applications

Procedures for Logic & Proof

1. Identify and distinguish strings representing terms/formulas, statements, equations, and other types of mathematical objects; this includes identifying invalid strings.
2. Convert arithmetic and logic terms between infix, tree, and RPN forms.
3. Identify and eliminate unnecessary brackets in terms.
4. Evaluate simple and compound logic terms in all environments over the variables involved, producing a truth table. Logic terms can include the operators $\wedge, \vee, \neg, \oplus, \rightarrow, \leftrightarrow, \uparrow, \downarrow$ and can be written as trees, infix (precedence rules apply), or RPN.
5. Apply laws of logic to transform logic terms, and identify their use in a provided derivation.
6. Determine if logic terms are semantic implications ($\models$) or equivalences ($\equiv$).
7. Determine if logic terms are any of: tautology, contradiction, contingency, (un)satisfiable.
8. Convert propositions between English and symbolic form.
9. Determine the converse, inverse and contrapositive of logical implications, in both English and symbolic forms.
10. Put logic terms into CNF and DNF forms, and identify if terms are already in these forms (including degenerate cases).
11. Write small-ish logic terms using only NAND, or only NOR.
12. Evaluate and transform formluas in predicate logic with the $\forall$ and $\exists$ quantifiers, including working with nested and negated quantifiers.
13. Identify free and bound variables in a term, and α-rename bound variables resp. determine when and why a renaming would be invalid.
14. Prove simple statements in elementary number theory (related to Euclid's theorem) and set theory using direct, indirect and by-contradiction proof techniques.
15. Use syntax and/or semantics to do simple proofs in logic, arithmetic, set and number theory.
16. Identify and use case distinction in proofs.
17. Perform simple proofs by natural, structural and invariant induction.
18. Identify uses of the various types of syllogism in a proof.
19. Identify and use introduction and elimination laws for logical operators in simple natural deduction proofs.
20. Evaluate and transform terms in set theory, including membership $\in$, set operations $\cup, \cap, \setminus$, powersets $\mathcal{P}$, cardinalities, Cartesian products $\times$ and all of this including on sets of sets.
21. Use, and prove using logic, laws of basic set theory, e.g. 'prove that the subset relation is antisymmetric'.
22. Find the domain and image of a function represented as a set or an arrow diagram, transform functions between these representations, and compose functions.
23. Identify if a function is injective, surjective or bijective, and compute the inverse where it exists.
24. Compute the preimage and (direct) image of a function on a value or a set of values.
25. Compute with relations, including inverses and composing, and representing relations as sets.
26. Identify and prove whether a relation is a partial/total order, or an equivalence, showing working knowledge of all the individual concepts involved (reflexive, (anti)symmetric, transitive).

11: Combinatorics

- Factorial $n!$
- Binomial $\binom{n}{k}$ and associated formulas
- Multinomial $\binom{n}{k_1,\ldots,k_m}$
- Double-counting
- Inclusion-exclusion

12: Probability

- Definition of a discrete probability space and an event
- small-p and capital-P and their properties
- Uniform distribution (introduction to)
- Drawing with and without replacement
- Independence of events
- Binomial distribution (introduction to)

13: Random Variables

- Definition, distribution p_F
- Indicator variables
- Joint probabilities / distributions
- Independence of RV
- Event (alternative) notation $P(X = x)$

14: Conditional Probability

- Definition
- Conditional distribution
- Conditional sum rule
- Total probability rule
- Joint distribution tables
- Tree diagrams

15: Bayes' Rule

- Definition
- Total probability version
- Base-rate trade-off against test accuracy
- True/False Positives/Negatives
- Simpson's Paradox

16: Expected Values

- Two (equivalent) definitions of E.V.
- E.V. of functions
- Variance and S. D.
- Linearity of Expectation
- Application to Coupon Collector problem
- E.V. of products of independent RVs

17: Discrete Distributions

- Bernoulli distribution
- Binomial distribution
- Geometric distribution (mean time to failure)
- 3-way geometric distribution: win, lose, re-roll
- Poisson distribution
- Cumulative distributions

18: Continuous Distributions

Lecture 18 is split over two lectures in 2023–24.

- Two definitions: subsets (on sigma algebra) and cumulative
- Continuous uniform distributions
- PDFs and relation to calculus (integrals everywhere)
- Expected Values
- Piece-wise PDFs
- Piece-wise cumulative (no PDF exists)

19: The Normal Distribution

- Formula and graph
- Sources of Normal Distributions
- Properties and transformations
- z-scores and calculations

Procedures for Combinatorics and Probability

1. Calculate factorials $n!$, binomial coefficients $(n$ choose $k)$, multinomials and exponentials n^k.
2. Model combinatorial questions as drawing with or without replacement, and use combinatorial formulas (as in the point above) to solve counting problems.
3. Count using conditinality and independence of steps to multiply, and case distinction to add (e.g. the cups and saucers example).
4. Use double-counting and the inclusion/exclusion principle to show simple equivalences and solve counting problems.
5. Define and model discrete probability spaces, events and random variables and solve related questions by reducing to combinatorics, e.g. 'good cases over total cases'.
6. Work with random variables, know the different standard notations $P(X = x)$ vs $p_X(x)$, and compute their distributions.
7. Compute joint distribution table for two random variables and use them to solve problems relating to joint, conditional or marginal probabilities.
8. Define and solve problems using conditional probability, using as appropriate any of: joint tables, definition of conditionals, Bayes' law, total probability theorem, sums that must equal 1, inclusion/exclusion.
9. Define and check for independence of events and random variables, and use this correctly in calculations involving conditionals.
10. Calculate expected values of distributions, random variables, and functions of random variables; this includes the special case of calculating variances.
11. Use linearity of expectation to model and solve problems, including the special case of Coupon Collector problems.
12. Use recurrence relations to solve problems as an alternative to infinite series.
13. Work with iterated/repeated Bernoulli experiments, with the Binomial and Geometric distributions as particularly important cases.
14. Calculate continuous probabilities from cumulative (CDF) and density (PDF) functions and transform between these, usually involving taking integrals.
15. Compute with piecewise continuous distributions by splitting the calculation (usually an integral) for each piece.
16. Draw and interpret graphs of PDFs and CDFs.
17. Solve joint and conditional probability questions on simple continuous distributions.
18. Create a probability distribution from a discrete or continuous function by normalising.