COMS10014 Solutions 10: Functions and Relations

1. Image and Preimage

1. The image of f is the set of all the letters that appear as a first letter of a digit, so $\{z, o, t, t, f, f, s, s, e, n, t\}$ which normalises to $\{z, o, t, f, s, e, n\}$ (you could sort these letters alphabetically, but there is no concept of an order in a set – either something is an element, or it is not).
2. The preimage of ‘s’ is all digits that map to ‘s’, so {6,7}.
3. The direct image of a set S is the set $\{f(s)|s \in S\}$ so $\{o, t\}$ (after normalising, since both 2 and 3 map to ‘t’).
4. f is not injective, because $f(2) = f(3) =$’t’ and f is not surjective, because nothing maps to q for example.

2. Injective and Surjective

1. This function is injective, as if $x + 1 = y + 1$ then we have $x = y$. However, it is not surjective, as there is no x such that $f(x) = 0$.
2. This function is injective, if $x^2 = y^2$ then $x = y$ or $x = (-y)$ for all real numbers, but no natural numbers are negative so the second case cannot occur. The function is not surjective however, as for example there is no x with $f(x) = 3$, that would be $\pm\sqrt{3}$ over the reals but these are not natural numbers.
3. This function is no longer injective as for example 1 and (-1) both map to 1. It is also not surjective as there is no x over the reals with $f(x) = (-1)$.
4. This function is injective. If $e^x = e^y$ then we can take a logarithm on both sides, noting that $e^x > 0$ for all real x, to get $x = y$. However, it is not surjective as it cannot reach any negative numbers.
5. This one is bijective. It is injective as before, but now for any $y \in \mathbb{R}^{>0}$ we can find $x = \log y$ with $f(x) = y$ so it is surjective too.
6. This function is bijective, as one can see from the truth table. No two inputs (there are only two) map to the same output, making it injective; and every possible output has one input that maps to it, making it surjective too.

3. Function Composition

1. The function is $k(x) = 2(x + 1)^2 = 2x^2 + 4x + 2$ so $k(4) = 50$.
2. $j(x) = (2x)^2 + 1 = 4x^2 + 1$. Therefore, $j(4) = 65$.
3. Taking for example the element 2, we have $(ggf)(2) = (gf)(1) = f(3) = 1$ and doing the same for all elements gives $g \circ g \circ f = \{(1,4), (2,1), (3,2), (4,2)\}$.
   (Draw the arrow diagram, and make sure you get the order right, if this is not clear.)

4, Trigrams

1. R is a partial order (and not a total one). As a diagram:

$$\begin{tikzcd}
\text{☴} \arrow[rrr] & & & \text{☰} \\
& \text{☵} \arrow[ul] \arrow[r] & \text{☱} \arrow[ur] & \\
& \text{☷} \arrow[u] \arrow[r] \arrow[dl] & \text{☳} \arrow[u] \arrow[dr] & \\
\text{☶} \arrow[uuu] \arrow[rrr] & & & \text{☲} \arrow[uuu]
\end{tikzcd}$$

This is a 'reduced' relation diagram: loop arrows on each trigram are not included, and $R(A, B)$ if there is a path from A to B, not just a direct arrow.

- Reflexive: for each character A, if you have a drawing of A you do not need to change any ink to make A. Therefore, $\forall A. R(A, A)$.
- Antisymmetric: If you have to add ink to turn A into B, then you would have to remove the same ink again to turn B into A, so (unless $A = B$) you cannot have both (A, B) and (B, A) in the relation. Therefore, $\forall A, B. (R(A, B) \wedge A \neq B \rightarrow \neg R(B, A))$.
- Transitive: if you can turn A into B by adding ink (this includes 'adding no ink') and B into C by adding ink, then you can turn A into C by composing these two steps. Therefore,$\forall A, B, C. (R(A, B) \wedge R(B, C) \rightarrow R(A, C))$.

This partial order is not total because, for example, neither (☳, ☶) nor (☶, ☳) are in R.

2. This one is

- reflexive, because you can change any A into itself by changing at most one line (namely, by not changing any lines at all).
- symmetric, because if you can change one line to turn A into B then you can change the same one line to turn it back again.
- not antisymmetric (as ☷ and ☵ for example are both related to each other under S)
- not transitive. We have S(☷, ☳) and S(☳, ☱) but $\neg S$(☷, ☱) as you would have to change two lines to get from one to the other.

So, it is neither a partial order nor an equivalence relation since both require transitivity.

3. This is an equivalence relation. You can see this directly from the fact it produces a function $f: X \rightarrow \{0,1,2,3\}$ where X is the set of trigrams, namely $f(x)$ is the number of yin lines in x; the relation is then defined as $T(A, B) \leftrightarrow f(A) = f(B)$ and relations like this are always equivalences (and all equivalences have a hidden function like this). But let's check individually:

- Reflexive: every trigram has the same number of yin lines as itself.
- Symmetric: if A has the same number of yin lines as B, then B has the same number as A.
- Transitive: if A has the same number of yin lines as B, and B has the same number as C, then A also has the same number as C.

5. Cardinality on Infinite Sets

1. Reflexive: on any set S, there is a bijective function $S \to S$ namely the identity function $f(s) = s$.

Symmetric: If $f: S \to T$ is bijective, then so is $f^{(-1)}: T \to S$.

Transitive: if $f: S \to T$ and $g: T \to U$ are bijective, then because the composition of bijective functions is bijective again, so is $(g \circ f): S \to U$.

2. The functions are:

a. $f(n) = 2n$.

b. The idea is to map the positive numbers to the even numbers, and the negative numbers to the odd numbers (this function is in fact bijective):

$$f(z) = \begin{cases} 2z & \text{if } z \geq 0 \\ -2z - 1 & \text{if } z < 0 \end{cases}$$

As a table:

z	…	–4	–3	–2	–1	0	1	2	3	4	…
$f(z)$	…	7	5	3	1	0	2	4	6	8	…

c. One idea is to 'fill the diagonals' going top right to bottom left, which is a bijection:

	0	1	2	3	4	…
0	0	1	3	6	10	
1	2	4	7	11		
2	5	8	12			
3	9	13				
4	14					
…						

Getting a formula is a matter of doing a 'coordinate transformation'. Every point in $\mathbb{N} \times \mathbb{N}$ has coordinates (x, y) where in the table above, x is the row and y the column. For example, the point $(3, 0)$ maps to 9. We can introduce a new coordinate system (u, v) with $u = x + y$ and $v = x$, which is reversible $(x = v, y = u - v)$ so the mapping from one coordinate system to another is bijective (this will become clearer after you have done Maths B Linear Algebra and can write the transformation as a matrix).
The point being, in the new coordinate system the coordinates are:

	0	1	2	3
0	(0,0)	(1,0)	(2,0)	(3,0)
1	(1,1)	(2,1)	(3,1)	(4,1)
2	(2,2)	(3,2)	(4,2)	(5,2)
3	(3,3)	(4,3)	(5,3)	(6,3)

where the u coordinate is the diagonal the element is on (all the points with the same u lie on the same diagonal, going top right to bottom left) and the v coordinate is the position on that diagonal, with the top-most element starting at 0.
To compute the mapping at a value (x, y), we first transform into (u, v) coordinates, then we can easily compute the function value by summing all the points on previous diagonals (with lower u-value) plus the number of previous points on the same diagonal.
For example, take the point $(1,2)$, that is $(x = 1, y = 2)$. Transformed, this point is $(u = 3, v = 1)$ which means we are on the 3rd diagonal (the count starts at 0), so we have to add up the number of points on the diagonals 0–2, and then add 1 for $v = 1$.

How many points are there on the first n diagonals (starting from 0)? There is 1 on the zeroth diagonal, 2 on the first, 3 on the second etc. so the sum is something we have already seen:

$$1 + 2 + \cdots + n = n(n + 1)/2$$

So for $u = 3$ there are $1 + 2 + 3 = 6$ points on lower diagonals, and adding $v = 1$ gets us $f(1,2) = 7$ which checks out against the table above. Therefore, we can write

$$f(x, y) = \frac{u(u + 1)}{2} + v = \frac{(x + y)(x + y + 1)}{2} + x$$

which is our bijective mapping $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. We check that this does all we want:

- Since we divide by 2, we have to make sure the result is actually an integer, but the numerator is a product of two successive numbers so one of them will always be even. And since both inputs are non-negative, the result will be a natural number, not just an integer, too.
- Proving that this function is bijective goes well beyond Maths A. The idea is that we can build an inverse g as follows: for any $n \in \mathbb{N}$, set $\phi(m) = m(m + 1)/2$. This function will definitely have a value greater than n, as $\phi(n) > n$ for $n > 0$ (one could prove this by induction). By the well-ordering principle, we can find a smallest m' such that $\phi(m') > n$; this means that the value $u = m' - 1$ will be the number of the diagonal on which n lies and $v = n - \phi(u)$ will be the position on that diagonal. From this we can recover $x = v, y = u - v$. But there is still more work to do to prove that this function definitely is an inverse, that is $\forall n. f\big(g(n)\big) = n$ and $\forall x, y. g\big(f(x, y)\big) = (x, y)$.
  (We cannot simply 'calculate out', as the function f would not be bijective as a function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, just like the function $t(x) = x^2$ is injective on $\mathbb{N}$ but not $\mathbb{Z}$.)

d. We know there are bijections $\mathbb{Z} \to \mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
The only part that still needs work is showing that if $f: \mathbb{Z} \to \mathbb{N}$ is injective then so is the function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N} \times \mathbb{N}$ that maps $(x, y) \mapsto \big(f(x), f(y)\big)$; some mathematicians would write this function as $f \times f$. So, imagine that $(f(x), f(y)) = (f(x'), f(y'))$. Then, by the definition of equals for pairs, we have $f(x) = f(x')$ and $f(y) = f(y')$. But f is injective, so $x = x'$ and $y = y'$ follows. We now have an injective function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N} \times \mathbb{N}$, which we can compose with the injective function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ from above to get an injective function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$, and then invoke Cantor-Schröder-Bernstein to get our bijection (an injection in the other direction is for example $n \mapsto (n, 0)$).
Alternatively, we could have shown directly that if f is bijective then so is $f \times f$, which is only slightly more work.

e. We define an injective function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ by mapping the fraction a/b to the pair (a, b). (This function is not surjective, because there is no preimage for $(2,4)$ for example, because the 'fraction' 2/4 is in fact 1/2 so it maps to $(1,2)$ – in $\mathbb{Q}$, each fraction appears only once in its normalised form. But no-one said the function had to be surjective.)
We compose with the injective function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ from before to get an injective function $\mathbb{Q} \to \mathbb{N}$, the function $f(n) = n$ is injective in the opposite direction, and by Cantor-Schröder-Bernstein we can summon a bijection.