COMS10014 Solutions 7: Predicates

1. Introduction

a. P(1) and P(2) are false, as $1 > 42$ and $2 > 42$ are false. P(128) is $128 > 42$ which is true.
b. The first one is false, and the other two are true.
c. The first one is true. The second one is false (the 4th planet is Mars). The final one is false, as Pluto is no longer counted among the planets.

2. Quantifiers

a. $\neg\exists x. P(x)$ or equivalently, $\forall x \neg P(x)$.
b. $\exists x. \neg P(x)$.
c. $\forall x. (F(x) \rightarrow P(x))$. Note that we need the outer brackets due to precedence of $\forall$.
d. $\exists x. \big(F(x) \land P(x)\big)$. Again, the outer brackets are needed.
e. $\forall x. (F(x) \land P(x))$. The brackets are still needed.
f. $\neg\forall x. F(x) \lor \exists x. \neg P(x)$.

3. Negation of Quantifiers

a. Assuming the domain of all animals, and $D(x)$: x is a dog; $O(x)$: x is old; $T(x)$: x can learn new tricks, $\exists x. (O(x) \land D(x) \land T(x))$. If you assume the domain of all dogs, you can leave off the D part, or even for the domain of all animals you could write the abbreviated form $\exists x: D. (O(x) \land T(x))$.
The negation is, using DeMorgan, $\neg\exists x. \big(O(x) \land D(x) \land T(x)\big) \equiv \forall x. \big(\neg O(x) \lor \neg D(x) \lor \neg T(x)\big)$ which reads: every animal is either not old, not a dog, or cannot learn new tricks. Of course, using $\neg x \lor y \equiv (x \rightarrow y)$ and another use of first-order DeMorgan, you could rewrite this as $\forall x. \big((O(x) \land D(x)) \rightarrow \neg T(x)\big)$, which one could reasonably translate as "you can't teach old dogs new tricks."

b. Assuming $R(x)$: x is a rabbit and $C(x)$: x knows calculus, we can write $\forall x. (R(x) \rightarrow \neg C(x))$ or the equivalent $\forall x. (\neg R(x) \lor \neg C(x))$. In both cases, the outermost brackets are required.
Negating the second form gives $\neg\forall x. \big(\neg R(x) \lor \neg C(x)\big) \equiv \exists x. (R(x) \land C(x))$, which translates as "There is a rabbit that knows calculus."

c. Assuming $F(x)$: x can fly, this is simply $\forall x. F(x)$. Negating gives $\neg\forall x. F(x) \equiv \exists x. \neg F(x)$, "Some bird cannot fly".

d. Here we need a second predicate $B(x)$: x is a bird. Then we have $\forall x. (B(x) \rightarrow F(x))$. Negating gives $\neg\forall x. \big(B(x) \rightarrow F(x)\big) \equiv \exists x. \Big(\neg\big(B(x) \rightarrow F(x)\big)\Big) \equiv \exists x. (B(x) \land \neg F(x))$, "There is some bird that cannot fly.", or simply "Some bird cannot fly."

e. This is exactly the same schema as above, if $D(x)$: x is a dog and $T(x)$: x can talk then $\forall x. \big(D(x) \rightarrow \neg T(x)\big)$ or $\neg\exists x. (D(x) \land T(x))$ both work. The negation in either case is $\exists x. (D(x) \land T(x))$, "Some dog can talk."

f. Assuming the domain is the class and $F(x)$: x knows French and $S(x)$: x knows Spanish, $\neg\exists x.(F(x) \land S(x))$. The negation is simply $\exists x.(F(x) \land S(x))$: “Someone in the class knows both French and Spanish.”

4. Predicates in English

This exercise was written back in the days when we handed out printed paper copies of all worksheets to all students in the class, and on one particular day, printing the worksheets proved to be a problem.

a. If some printer is both out of service and busy, then some job is lost.
b. If all printers are busy, then some print job is queued.
c. If some job is both queued and lost, then some printer is out of service.
d. If all printers are busy, and some job is queued, then some job is lost.

5. Quantifiers and Integers

1. The statements are:

a. True, as $1 > 0$.
b. True, as $0 > (-2)$.
c. False, as $2 > 2$ is false.
d. True, for example we have just seen that $Q(0)$ is true.
e. False, for example we have just seen that $Q(1)$ is not true.
f. True, for example $x = 1$.
g. False, since for $x = 0$ the statement $\neg Q(x)$ is false, because $Q(0)$ is true.

2. The statements:

a. For any number x, there is a number y that is smaller than x. This is true over the integers (one could take $y = x - 1$). Note, it would be false over the natural numbers.
b. For any two numbers, if they are both zero then their product is also zero. This is true over any set of numbers. It would even still be true if we replaced the ∧ with a ∨.
c. There is a single number z so that, for all other numbers x, y, their product is z. This is false.
d. For all numbers x, y, there is a number z that is the product of x and y. With the quantifiers this way round, this is true.

6. Oliver!

1. With the constant O for Oliver:

a. $\forall p. L(p, O)$
b. $\forall p \exists q. L(p, q)$
c. $\exists p \forall q. L(q, p)$

d. $\neg\exists p\forall q.\, L(p,q)$.
You could DeMorgan this to get $\forall p\exists q.\, \neg L(p,q)$ "Each person has somebody that they do not love."
e. $\exists p.\, \neg L(O,p)$
f. $\exists p\forall q.\, \neg L(q,p)$

2.

An English language note for non-native speakers: many European languages make strict distinctions between different grammatical cases and indicate this with different word endings.

English has largely abandoned different word endings for cases, however there are two places where English still makes the distinction. The first is pronouns, for example he/him (I see him, he sees me – "I see he" is an error). You use the first version (he, she, they) for the subject and the second (him, her, them) for the object of the sentence. If you see someone with a "my pronouns are" sticker or email signature, they will generally mention both the subject and object versions, such as they/them.

The point of this digression, apart from encouraging respect for non-binary people, is that the second place where English can make a distinction is the relative pronoun who/whom, with the first version for the subject and the second for the object. In everyday speech, it is perfectly understandable to use "who" in both cases, but we are being precise here, so we write "There is somebody WHOM everybody loves" for $\exists p\forall q.\, L(q,p)$ as opposed to $\exists p\forall q.\, L(p,q)$ "There is somebody WHO loves everybody." It is not a mathematical mistake if you got this grammatical point wrong, and the who/whom difference will not be tested in the exam.

a. $\neg\forall p.\, L(p,O) \equiv \exists p.\, \neg L(p,O)$ There is a person who does not love Oliver.
b. $\neg\forall p\exists q.\, L(p,q) \equiv \exists p\forall q.\, \neg L(p,q)$ There is a person who does not love anybody.
c. $\neg\exists p\forall q.\, L(q,p) \equiv \forall p\exists q.\, \neg L(q,p)$ Every person has someone who does not love them.
d. $\neg\neg\exists p\forall q.\, L(p,q) \equiv \exists p\forall q.\, L(p,q)$ There is someone who loves everybody.
e. $\neg\exists p.\, \neg L(O,p) \equiv \forall p\; L(O,p)$ Oliver loves everybody.
f. $\neg\exists p\forall q\; .\, \neg L(q,p) \equiv \forall p\exists q.\, L(q,p)$ Everyone has someone who loves them.

3. The harder ones:

a. $\exists p.\, (\forall q.\, L(q,p) \wedge \forall r.\, (\forall q.\, L(q,r) \rightarrow p = r))$
"There is a person p as follows: every person q loves p [this includes p themselves] and, for every person r who also has the property that everyone loves r, that person is the same one as p."
Generally, to show in mathematics that there is exactly one element with a particular property, we show two things: (a) there is an element with this property, and (b1) any two elements with this property are identical. Sometimes it is easier to do the second part by contradiction: (b2) if there are two distinct elements with the property, then we get a contradiction.
b. $\exists p\exists q.\Big(L(O,p) \wedge L(O,\; q) \wedge p \neq q \wedge \forall r.\, \big(L(O,r) \rightarrow (r = p \vee r = q)\big)\Big)$.
"There are people p, q as such: Oliver loves both of them, they are not the same person, and any person whom Oliver loves is one of the two."
c. $\exists p.\, (L(p,p) \wedge \forall q\; L(p,q) \rightarrow p = q)$.
"There is a person p who loves themselves and, every person whom they love is p again."