Lecture 9: Functions

A function is something where you can put an element in and get an element out, and if you put the same element in several times then you will always get the same result. We define functions more formally in this lecture.

You are probably used to functions defined mathematically such as $f(x) = x^2 + 1$ or $g(x, y) = x^2 - y^2$ or (different notation) $f : x \mapsto x^2 + 1$. Set theory makes very precise what a function is.

Definition of Functions

Definition 1 (function)**.** For non-empty sets A, B, a function from A to B is a subset of $A \times B$ where every element $a \in A$ appears exactly once on the left-hand side of a pair (a, b).

We call A the domain of the function, and B the range (or codomain) of the function.

For example, consider the function with domain $A = \{0, 1, 2, 3\}$, range $B = \mathbb{Z}$, given by the formula $f(x) = 2x - 1$. We could write this function as a table:

x	0	1	2	3
$f(x)$	-1	1	3	5

Note that each element of A appears exactly once in its row. The definition of the function as a set is simply an encoding of this table:

$$f = \{(0, -1), (1, 1), (2, 3), (3, 5)\}$$

You can do the same with infinite sets, for example if you extend the domain of this function to $\mathbb{Z}$, you get $\{(z, 2z - 1) \mid z \in \mathbb{Z}\}$. Indeed, the set projection $\{(x, f(x)) \mid x \in D\}$ where D is the domain of a function, is exactly the definition of a function as a set. The notation $x \mapsto y$ can be read as an alternative way of writing the pair (x, y) or the set of all such pairs, projected over the domain.

For another example, suppose you have a function $g : \mathbb{N} \to \{a, b, \ldots, z\}$ that maps a natural number to the first letter of its English spelling, for example $g(0) = z, g(1) = o, g(2) = t, \ldots$ for the words z(ero), o(ne), t(wo), $\ldots$. As a set, this would be $g = \{(0, z), (1, o), (2, t), (3, t), (4, f), \ldots\}$. It does not matter that the same letter appears more than once on the right of pairs (both two and three start with t) as long as each natural number appears exactly once on the left.

Arrow Notation

You might already be used to the notation $f : A \to B$ to denote that f is a function with domain A and range B. We can see that $\to$, like $\cap$ or $\times$, is in fact an operation on sets that constructs new sets from existing ones. Thus we can write $f \in A \to B$ to mean the same thing as $f : A \to B$. (This means we are using the same $\to$ notation for two different things, logical implication and functions. It should be clear from the context which one is meant.)

Definition 2 (function arrow)**.** For sets A, B, the notation $A \to B$ refers to the set of functions from A to B. If either of A, B is empty, then so is the function set.

Injective, Surjective, and Bijective

A function $A \to B$ is a set of pairs where each element a of A appears exactly once on the left of a pair. What about elements of B?

Definition 3. A function $f \in A \to B$ is called

- **injective**, if no element $b \in B$ appears more than once on the right-hand side of a pair,
- **surjective**, if every element $b \in B$ appears at least once on the right-hand side of a pair,
- **bijective**, if it is both injective and surjective. In this case, every $b \in B$ appears exactly once on the right-hand side of a pair.

As formulas, a function $f \in A \to B$ is injective if

$$\forall x, y : A.\ (f(x) = f(y) \to x = y)$$

or equivalently, the contrapositive

$$\forall x, y : A.\ (x \neq y \to f(x) \neq f(y))$$

Note that the two $\to$ are actually $\leftrightarrow$, as the other direction follows from the definition of a function.

A function $f \in A \to B$ is surjective if

$$\forall b : B \exists a : A.\ f(a) = b$$

It is important to write out 'on the right-hand side' each time in the English definitions because we could have $A = B$. For example, in a bijective function $\mathbb{Z} \to \mathbb{Z}$, each integer will appear exactly twice in the defintion, once on the left and once on the right of a pair. Or at least, that would be the case if we could write out the whole infinite set.

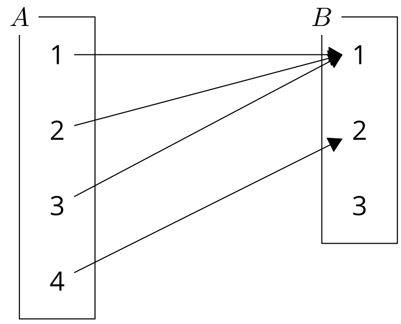
Whether a function is injective depends not only on its 'formula' but also on its domain. For example, $f : \mathbb{N} \to \mathbb{Z}, f(x) = x^2$ is injective but the same formula for a function $\mathbb{Z} \to \mathbb{Z}$ would not be injective. Similarly, whether a function is surjective depends on its range: the function $g(x) = x + 1$ as a function $\mathbb{N} \to \mathbb{N}$ is not surjective, but as a function $\mathbb{N} \to \mathbb{N}^+$ it would be surjective (remember, $\mathbb{N}^+$ excludes zero).

Arrow Diagrams

Just like we can use Venn diagrams to visualise sets, we can use arrow diagrams to visualise functions. An arrow diagram for a function $f \in A \to B$ contains two areas that do not intersect, one for A and one for B with each element written out (so elements in $A \cap B$ are written twice) and for each element $(a, b) \in f$, an arrow from a in the A-area to b in the B-area. Consider for example $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$ and the function given by this table:

x	1	2	3	4
$f(x)$	1	1	1	2

As a set, this is $f = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$. Its arrow diagram is



We can see that there is exactly one arrow coming out of every $a \in A$, which is the definition of a function. However, this function is not injective, as $1 \in B$ has more than one arrow coming in, nor is it surjective as $3 \in B$ has no arrow coming in.

Indeed, in an arrow diagram, a function is injective if no element in the B-area (that is, the range) has more than one incoming arrow, and surjective if every element in the range has at least one incoming arrow.

Inverses

The inverse of a function, if it exists, is 'the same thing with the arrows backwards', so the inverse of a function $f \in A \to B$ would be a function in $B \to A$. The problem is that turning the arrows around in an arrow diagram only gives a function again if the original function was bijective, since the inverse function otherwise does not have the property anymore that every element of its domain B has exactly one arrow coming out.

> **Definition 4** (inverse). If $f \in A \to B$ is a bijective function, then its inverse $f^{(-1)} \in B \to A$ is the function $\{(b, a) \mid (a, b) \in f\}$ or equivalently, $\forall a : A . \forall b : B . \ \big(f(a) = b \leftrightarrow f^{(-1)}(b) = a\big)$.

For example, if $f \in \mathbb{Z} \to \mathbb{Z}$ with $f(z) = z + 1$ then written out,

$$f = \{\ldots, (-2, -1), (-1, 0), (0, 1), (1, 2), (2, 3), \ldots\}$$

This function is injective, as $a + 1 = b + 1 \vDash a = b$ and it is surjective as for any $w \in \mathbb{Z}$, there is a z with $w = z + 1$ namely $z = w - 1$. So the function has an inverse, and its inverse has the pairs reversed: $f^{(-1)} = \{\ldots, (-1, -2), (0, -1), (1, 0), (2, 1), (3, 2), \ldots\}$. This is of course $f^{(-1)}(z) = z - 1$.

For another example, if $f \in \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$ then this function is injective as $2x + 1 = 2y + 1 \vDash x = y$ and it is surjective, as for any $y \in \mathbb{R}$ we can find an x such that $y = 2x + 1$ namely $x = (y - 1)/2$. Therefore, $f^{(-1)}(y) = (y - 1)/2$ is the inverse function, as $y = 2x + 1 \equiv x = (y - 1)/2$. In this case, f written out as a set would contain the elements $(1, 3)$ and $(2, 5)$ among others, and $f^{(-1)}$ would contain the elements $(3, 1)$ and $(5, 2)$ among others.

Preimage

If a function $f \in A \to B$ is not bijective, we cannot form an inverse function that, on input a single element $b \in B$, gives us a single element $a \in A$ as an output. After all, if there are no arrows coming in to b then we do not have any candidates for such an a, and if there is more than one incoming arrow we have no way of deciding which one to pick. Mathematically, if we want to build something like a function, except that it sometimes returns more than one element, and sometimes none, we can model this with sets.

Definition 5 (preimage)**.** If $f \in A \to B$ is a function, then the function $f^* \in B \to \mathcal{P}(A)$ with $f^*(b) = \{a \in A \mid f(a) = b\}$ is called the preimage function of f.

Equivalently, $\forall a : A. \forall b : B.\ (f(a) = b \leftrightarrow a \in f^*(b))$.

The preimage function exists for every function, whether bijective or not, and its range is the powerset of the original function's domain. For example, for $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3\}$ and the function given by the table

x	1	2	3	4
$f(x)$	1	1	1	2

we have $f^*(1) = \{1, 2, 3\}$ since these three elements map to 1; $f^*(2) = \{4\}$ since 4 is the only element with $f(4) = 2$ and $f^*(3) = \varnothing$ since there are no $a \in A$ with $f(a) = 3$. So,

$$f^* = \{(1, \{1, 2, 3\}), (2, \{4\}), (3, \varnothing)\}$$

If the original function f happens to be bijective and so has an inverse $f^{(-1)}$, then the preimage function f^* just returns sets containing one element, which is the inverse. For example, if $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x + 1$ then $f^{(-1)}(y) = (y - 1)/2$ and $f^*(y) = \{(y - 1)/2\}$. As a general formula, $f^*(y) = \{f^{(-1)}(y)\}$ for bijective functions.

In an arrow diagram of a function $f \in A \to B$, the preimage $f^*(b)$ of a $b \in B$ is the set of all $a \in A$ whose arrow points at b. This set might be empty if there are no such arrows.

Image

The image of a function is the set of all elements in the range that the function can actually reach. For example, if $f \in \mathbb{Z} \to \mathbb{Z}$ with $f(z) = 2z$ then the image of f is the set of even numbers.

Definition 6 (image)**.** The image of a function $f \in A \to B$ is the following set (described two different ways):

- $\{f(a) \mid a \in A\}$
- $\{b \in B \mid \exists a : A.\ f(a) = b\}$

If the function is surjective, then the image is simply the range B.

The above definition is also a good example of the difference between set restriction and set projection.

The term image is also used in a second way when discussing functions: the image of a point under a function is the function evaluated at that point, for example the image of 2 under $f(x) = 2x + 1$ is 5 because $f(2) = 5$. Other books call this 'the value of f at 2' instead.

Some books take this idea further and define, for $f \in A \to B$, the image of a subset $S \subseteq A$ to be $\{f(s) \mid s \in S\}$, that is the function evaluated at every point in the subset (equivalently, the image of the function when you replace the domain by the subset). For example, still with $f \in \mathbb{Z} \to \mathbb{Z}$ with formula $f(x) = 2x + 1$, the image of $\{1, 2\}$ would be $\{3, 5\}$.

If you want a notation for this, you can use a star as a subscript, that is $f_*(\{1, 2\}) = \{3, 5\}$. That means that for $f \in A \to B$ we are defining a new function $f_* \in \mathcal{P}(A) \to \mathcal{P}(B)$ called the image function (often also the *direct image* function). Note that it would be an error to simply write $f(\{1,2\})$ as f takes inputs in $\mathbb{Z}$, that is integers, not sets of integers.

The definition of functions has a small technical problem with 'types'. The terms *domain* and *range* are related to each other, which is why the range is also called the codomain; in a similar way the terms *image* and *preimage* are related, and so are *injective* and *surjective*. But, if you have the definition of a function as a set, although you can find the domain by taking the set of all elements on the left of a pair, if you do the same thing on the right it gives you the image, not the range. This is annoying. It is even more annoying that you can tell if a function is injective by looking at its set definition, but not if it is surjective, since you do not know if any range elements are 'missing': the function $\{(x, 2x) \mid x \in \mathbb{N}\}$ is not surjective as a function into the naturals, but it is as a function into the even naturals.

One way to fix this would be to give each function a 'type' defined by its domain and range (not image), similar to how programming languages might define a `Function<A, B>` type. Another fix, used in some parts of mathematics, is to view a function as a triple (D, f, R) where D, R are the domain and range, and f is the set as we defined earlier. This correctly gets you that (D, f, R) and (D, f, R') for two different ranges $R \neq R'$ are different functions, even if they use the same mapping set f. This way you can also tell from the definition whether a function is surjective or not. But for most of mathematics, we do not have to be this pedantic.

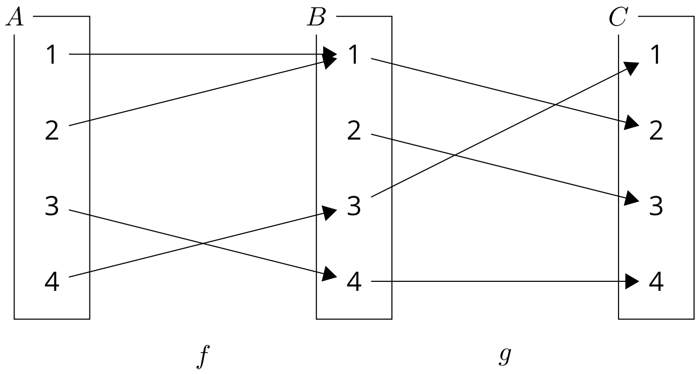
Composing Functions

If we have a function $f \in A \to B$ and another function $g \in B \to C$, then for any $a \in A$ we can compute $b = f(a)$ in B and from this, $c = g(b)$ in C. Written out in one go, $c = g(f(a))$. This procedure defines a new function $h \in A \to C$ which is called the composition of f and g.

Definition 7 (composition)**.** If A, B, C are non-empty sets and $f \in A \to B$ and $g \in B \to C$ are functions, then the composition $h = f \circ g$ of these functions is a function $h \in A \to C$ given by $h(a) = g(f(a))$.

Let us look at the situation with arrow diagrams. Suppose that $A = B = C = \{1, 2, 3, 4\}$ and we have these functions:

x	1	2	3	4
$f(x)$	1	1	4	3
$g(x)$	2	3	1	4



The composed function $h = f \circ g$ is the one that 'follows the arrows', which will always work because both f and g, being functions, are guaranteed to have an arrow coming out of every element of their domain. For example, for $h(4)$ you follow the f-arrow from 4 to 3, then the g-arrow from 3 to 1 so $h(4) = 1$. Indeed, as a set,

$$h = f \circ g = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$$

Although we define $h = f \circ g$, we compute it 'backwards', that is $h(x) = g(f(x))$ so that we apply f first. Indeed, to compute $h(4) = 1$, we first compute $f(4) = 3$ and then $g(3) = 1$. The backwards bit comes from how functions are nested in the notation: $g(f(x))$ means compute f first, then g, so the order of computing the functions in the usual notation is right-to-left.

Some properties of functions are preserved when we compose them. Specifically,

- Composing two (or more) injective functions produces another injective function.
- Composing two (or more) surjective functions produces another surjective function.
- Therefore, composing bijective functions must produce another bijective function too.

Cardinality

If A, B are finite sets, how many functions are there from A to B, that is, what is $|A \to B|$?

If we write out a function table with two columns x and $f(x)$, then the number of rows in the table (minus the header) is the number of elements in the domain A. For each of these elements a, every possible choice of the value b for $f(a)$ gives a different function. Therefore, using some combinatorics, we have

$$|A \to B| = |B|^{|A|}$$

Looking at functions on $\mathbb{B} = \{0, 1\}$, the formula tells us that $|\mathbb{B} \to \mathbb{B}| = 2^2$, that is there are four such functions. They are the identity function, the negation function, and the constant-0 and constant-1 functions.

Looking at truth-functions with two inputs, that is functions in $(\mathbb{B} \times \mathbb{B}) \to \mathbb{B}$, we have

$$|(\mathbb{B} \times \mathbb{B}) \to \mathbb{B}| = |\mathbb{B}|^{|\mathbb{B} \times \mathbb{B}|} = 2^4 = 16$$

as any textbook on logic will tell you. For example, logical and, or, not, xor, nand, nor etc. are some of these functions.

How many possible truth-functions are there with n inputs, or equivalently, how many different truth tables can you make with n variables? The answer here is

$$|(\mathbb{B})^n \to B| = 2^{(2^n)}$$

where $(\mathbb{B})^n$ stands for $\mathbb{B} \times \ldots \times \mathbb{B}$ with n operands.

On finite sets, cardinality interacts with functions in an interesting way. The *pigeonhole principle* in mathematics says that if you have more pigeons than holes, and you put the pigeons in to the holes, then at least one hole has at least two pigeons in it. For example, you cannot make an injective function from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ because you need three arrows coming out of A, but there are only two elements in B for the arrows to end up, so one element must have at least two arrows coming in. In general,

- If A and B are finite sets with $|A| > |B|$ then there cannot be an injective function from A to B; conversely, if we have two finite sets C, D with an injective function $f \in C \to D$ then we must have $|C| \leq |D|$.
- If A and B are finite sets with $|A| < |B|$ then there cannot be a surjective function from A to B; conversely, if we have two finite sets C, D with a surjective function $f \in C \to D$ then we must have $|C| \geq |D|$.
- Therefore, bijective functions between finite sets can only exist between sets of exactly the same cardinality, but there is a bijective function between any two sets of the same finite cardinality (in fact, for any two sets of cardinality n, there are exactly $n!$ bijective functions between them).

The word *finite* is important in all the above properties. For infinite sets, you define cardinality by looking at the presence of injective or surjective functions between the sets in the first place. For example $\mathbb{N}, \mathbb{Z}$ and $\mathbb{N} \times \mathbb{N}$ all have bijective functions between them, so they are all the same cardinality which is called *countably infinite* or $\aleph_0$ (that symbol is the Hebrew letter aleph, so the cardinality is pronounced 'aleph-zero').

But, there cannot be an injective function from $\mathbb{R}$ to $\mathbb{N}$, a fact which is important for Computer Science because it implies that there cannot be an algorithm that, on input a computer program in a 'powerful enough' language, checks whether the program halts or runs for ever — this is Alan Turing's *Halting Problem*.