COMS10014 Worksheet 10: Functions and Relations

1. Image and Preimage (★)

Consider the function $f: \{0, 1, \ldots, 9\} \to \{a, b, \ldots, z\}$ that maps digits to the first letter of the digit, when spelled out as an English word. For example, $f(0) = z$ because 0 is spelled 'zero'.

1. What is the image of f?
2. What is the preimage of the letter 's' under f?
3. What is the (direct) image of {1,2,3} under f?
4. Is f injective, surjective, both (e.g. bijective) or neither?

2. Injective and Surjective (★)

For each of the following functions, determine whether they are injective, surjective, or both.

1. $f(x) = x + 1$ on the natural numbers.
2. $f(x) = x^2$ on the natural numbers.
3. $f(x) = x^2$ on the real numbers, e.g. $f: \mathbb{R} \to \mathbb{R}$.
4. $f(x) = e^x$ as a function $\mathbb{R} \to \mathbb{R}$.
5. $f(x) = e^x$ as a function $\mathbb{R} \to \mathbb{R}^{>0}$, that is the range is all positive real numbers.
6. Logical negation ($\neg$) as a function $V \to V$ where $V = \{T, F\}$.

3. Function Composition (★)

1. On $\mathbb{R} \to \mathbb{R}$, consider the functions $f(x) = 2x$, $g(x) = x + 1$ and $h(x) = x^2$. What is the function $k = g \circ h \circ f$, both as a function, and evaluated at $x = 4$?
2. The same as above, but for the function $j = f \circ h \circ g$?
3. On the set $S = \{1, 2, 3, 4\}$, consider the functions f, g with signature $S \to S$ defined as $f = \{(1,2), (2,4), (4,1), (3,3)\}$ and $g = \{(1,3), (3,1), (2,2), (4,2)\}$. What is $g \circ g \circ f$, both as a function, and evaluated at 2?

4. Trigrams (☯☯)

The ancient Chinese book I Ching[1] (易經) contains eight trigrams[2]:

☰	☱	☲	☳	☴	☵	☶	☷
heaven	lake	fire	thunder	wind	water	mountain	earth

A broken line (⚋) is called a 'yin' (陰) and a solid line (⚊) a 'yang' (陽) line.

It is said in the West that binary numbers are partly the invention of the German Philosopher and Mathematician (among many other things) Gottfried Wilhelm Leibniz. However, Leibniz was definitely aware of the I Ching and even came up with the analogy yin=0, yang=1, so the invention of binary might be better credited to China some time around 1000 B.C.[3]

1. The relation R on the set of eight trigrams is defined as follows: $R(A, B)$ if all the places where you put ink to draw A, you also put ink to draw B; equivalently, you could transform A into B by possibly adding ink, but not removing any. For example, $R($☵, ☱$)$ holds becuase you can turn 'water' into 'lake' by adding ink to complete the lowest line, but $\neg R($☱, ☵$)$ because once you have drawn 'lake', the bottom line is solid so you cannot make 'water' without erasing.

What kind of relation is R? Prove your answer, and draw the relation as a diagram in a way that makes sense to you.

2. The relation S on the set of eight trigrams is defined as follows: $S(A, B)$ if you can change A into B by changing at most one line (that is, changing a yin into a yang line, or the other way round).

What kind of relation is S, which of the properties of relations does it have?

3. The relation T on the set of eight trigrams is defined as follows: $T(A, B)$ if A and B have the same number of yin lines.

What kind of relation is T, which of the properties of relations does it have?

[1] Sometimes also spelled Yi Jing, or many other ways.
[2] Often used in pairs in the book, to form 64 Hexagrams.
[3] Or, according to legend, to the creator-god Fu Xi (伏羲).

5. Cardinality on Infinite Sets

For finite sets S, we define $|S|$ to be the number of elements in S, and call this the cardinality of S. We could then prove that a bijective function between two sets S, T exists if and only if $|S| = |T|$. In fact, an injective function $f: S \to T$ exists if and only if $|S| \leq |T|$, which is part of the proof.

If there are two sets S, T with injective functions $f: S \to T$ and $g: T \to S$, even if these functions are not surjective (and so not inverses of each other), then there is a general theorem saying that there is also some bijective function $h: S \to T$, even if the sets involved are infinite (this is called the Cantor-Schröder-Bernstein theorem[4]).

If we say that two infinite sets have the same cardinality if there is a bijective function between them, then we can define cardinalities for infinite sets. For example, the cardinality of the natural numbers $\mathbb{N}$ is usually called '*countably infinite*' or $\aleph_0$ (aleph-zero). Of course, this only makes sense if 'same cardinality' is something like an equivalence:

1. (★★) Show that the property[5] "there is a bijective function from S to T" is reflexive, symmetric, and transitive. You can use the fact that composing bijective functions gives another bijective function.

Any set that has a bijection to $\mathbb{N}$ can now safely also be called *countably infinite.* The rest of this exercise is about exploring the implications of this, by looking at other countably infinite sets.

2. Find injective or bijective functions as follows:
   a. (★) A bijective function from $\mathbb{N}$ to the set of even natural numbers.
   This means that there are 'as many' even numbers as natural numbers, not 'half as many'!

   b. (★★) An injective function from $\mathbb{Z}$ to $\mathbb{N}$ (*you will most likely find a bijective one*).

   c. (★★★) A sketch of how one could build an injective function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, or a surjective function from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ (in both cases it will probably turn out to be bijective, too). If you want a ★★★★ exercise, find a formula!
   Since there is an obvious injective function $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$, resp. a surjective function in the other direction, this shows that there are 'as many' natural numbers as pairs of natural numbers. Infinite cardinalities are not always intuitive.

   d. (★★) Argue from the above why there must be a bijection $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ (again, you are not asked for a formula). Note, composing injective functions remains injective.

   e. (★) Argue from the above that there is an injective function $\mathbb{Q} \to \mathbb{N}$, and why this implies a bijection $\mathbb{Q} \to \mathbb{N}$ too. *This shows that there are 'as many' natural numbers as fractions.*

In summary, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and any finite products you can build out of these sets (proof: by induction) all have the same cardinality of 'countably infinite'. None of these is 'bigger' or 'smaller', or 'more infinite' or 'less infinite' than any other.

Note however that $\mathbb{R}$, or even the real interval $[0,1]$, are strictly larger than countably infinite (they are called *uncountably infinite*). But that is for another time in your degree.

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[4] There are actually different 'models' of set theory in pure mathematics, and the theorem is not true in all of them. But this goes well beyond the content of an undergraduate degree and we ignore it here.
[5] To be really pedantic, this property is not a relation, as 'all sets' is not a set. But for practical purposes we can treat the property as if it were an equivalence relation on all sets.