COMS10014 Solutions 8: Sets

1. Introduction to Sets

a. $\{1,2,3,4,5,7,10\}$
b. $\{2,4\}$
c. $\{7, 10\}$
d. $\{2,3,5\}$
e. $\{2,3,5,6,8,9\}$
f. $\{1,3,5,7,9,10\}$
g. $\emptyset$
h. $\{1,4,7,10\}$
i. $\emptyset$
j. $\{1,2,3,4,5,6,7,8,9,10\}$
k. $\{1,2,3,4,5\}$
l. $\{1,4\}$
m. $\{6,8\}$
n. $\{1\}$
o. $\{2,3,4,5,6,7,8,9,10\}$
p. $\{1,2,3,4,5,7,10\}$

2. Cartesian Product

The sets are

1. $\{(1,a),(0,a),(1,b),(0,b),(1,c),(0,c)\}$
2. $\{(a,1),(a,0),(b,1),(b,0),(c,1),(c,0)\}$
3. $\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$
4. $\{((0,0),0),((0,0),1),((0,1),0),((0,1),1),((1,0),0),((1,0),1),((1,1),0),((1,1),1)\}$
5. $\{(0,+,0),(0,+,1),(0,*,0),(0,*,1),(1,+,0),(1,+,1),(1,*,0),(1,*,1)\}$
6. $\emptyset$. Any product involving the empty set is itself empty.
7. $\{(1,a),(1,+),(0,a),(0,b),(0,+),(1,*),(1,b),(0,*)\}$
8. $\{(1,+),(0,1),(0,+),(b,*),(1,0),(1,1),(b,1),(a,0),(b,+),(0,0),(1,*),(a,*),(a,+),$ $(0,*),(b,0),(a,1)\}$

3. Power Sets

a. The sets are
   1. $\{\{a\},\emptyset\}$
   2. $\{\{a\},\{b\},\{a.b\}.\emptyset\}$
   3. $\{\{b\},\emptyset\}$
   4. $\{\{b\},\{a,b\}\}$
   5. $\{(a,\emptyset),(a,\{a\}),(a,\{b\}),(a,\{a,b\})\}$
   6. $\{\emptyset,\{(a,a)\},\{(a,b)\},\{(a,a),(a,b)\}\}$

b. We have
   1. True. There is an element $\{a\}$ in $\mathcal{P}(B)$.
   2. False. $A \subseteq \mathcal{P}(B)$ would mean that you can remove elements from $\mathcal{P}(B)$ until you are left with $\{a\}$. You can get $\{\{a\}\}$ this way, but not $\{a\}$ because a on its own is not an element of the powerset.
   3. False. There is no element of $\mathcal{P}(B)$ that is equal to $\mathcal{P}(A)$.
   4. True. You can remove elements from $\mathcal{P}(B)$, namely all the ones with a b, to be left over with $\mathcal{P}(A)$.

c. In general,
   1. True. We can write $\mathcal{P}(B) = \{S \mid S \subseteq B\}$. Since $A \subseteq B$, it matches the condition for the powerset of B, so $A \in \mathcal{P}(B)$.
   2. False. We have already seen a counter-example in (b.) above.

3. False. We have already seen a counter-example in (b.) above.
4. True. We have $A \subseteq B \equiv \forall x.(x \in A \to x \in B)$. If A is the empty set, then $\mathcal{P}(A) = \{\emptyset\}$ which is a subset of *every* powerset, just like the empty set is a subset of empty set. If A is not empty, then pick any $S \in \mathcal{P}(A)$, and it will satisfy the condition $S \subseteq A$. This means that $\forall x.(x \in S \to x \in A)$, so from $A \subseteq B$ we get $\forall x.(x \in S \to x \in B)$. Therefore, $S \subseteq B$, so $S \in \mathcal{P}(B)$. We have shown $\forall S.(S \in \mathcal{P}(A) \to S \in \mathcal{P}(B))$, proving $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

4. Special Properties

a. Since $x \in A \vDash x \in A \cap B \vDash x \in B$, we can conlcude $A \subseteq B$. In fact, $A \cap B = A \equiv A \subseteq B$.
b. Here, $x \in B \vDash x \in A \cup B \vDash x \in A$, with the last step allowed because the two sets are the same. Therefore, $B \subseteq A$. Here too, in fact we have $A \cup B = A \equiv B \subseteq A$.
c. $\bar{A} \cap B = \emptyset \equiv B \subseteq A$. For the $\leftarrow$ direction, suppose $x \in \bar{A} \cap B$. Then $x \in B \wedge x \notin A$, which contradicts $x \in B \to x \in A$ as $B \subseteq A$. Therefore, there can be no such x, and $\bar{A} \cap B = \emptyset$. (One way of showing a set is empty is picking an element and deriving a contradiction.) For the $\to$ direction, let $x \in B$. Since $\bar{A} \cap B = \emptyset$, x cannot be in $\bar{A}$ or we would get a contradiction, therefore $x \in A$. This proves $x \in B \vDash x \in A$, hence $B \subseteq A$.
d. Using DeMorgan, this is equivalent to $A^C \cup B^C = B^C$, so by (b.) we have $A^C \subseteq B^C$. Therefore, $x \notin A \vDash x \notin B$ so taking the contrapositive, $x \in B \vDash x \in A$. This is $B \subseteq A$, and since all steps were reversible in this argument, $\overline{A \cap B} = \bar{B} \equiv B \subseteq A$.

5. General Properties

a. This one is true, and is the commutative law for intersection. We can also prove it directly. To show that two sets X, Y are identical, we have to show either both $\forall x.(x \in X \to x \in Y)$ and $\forall y.(y \in Y \to y \in X)$, or if we can manage it in one go, $\forall x\,.(x \in X \leftrightarrow x \in Y)$.
So: let $x \in A \cap B$. Then $x \in A \wedge x \in B$ as that is the definition of the intersection. Therefore, $x \in B \wedge\ x \in A$ as $\wedge$ commutes. But this is the definition of $x \in B \cap A$. All steps in this argument are reversible, so we have shown "$\leftrightarrow$" and are done.

b. This one is false. For example, if $A = \{1\}$ and $B = \{2\}$ then $A \setminus B = \{1\}$ and $B \setminus A = \{2\}$.

c. This one is true. We could either make a truth table for three variables A, B, C and check that the definitions match up in all 8 cases, or we can show as follows.
Suppose $x \in (A \setminus B) \cap C$. This means $x \in A \setminus B \wedge x \in C$, which in turn means $x \in A \wedge x \notin B \wedge x \in C$. Reordering with the commutative and associative laws for $\wedge$ gives $x \in A \cap C \wedge x \notin B$, so $x \in (A \cap C) \setminus B$. Now either make sure all these steps are reversible (they are), or do the proof in the other direction too and you will also get $x \in (A \cap C) \setminus B \equiv x \in A \wedge x \notin B \wedge x \in C$.

d. This one is false. (It would be true that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ with a union symbol on the right-hand side. A counter-example is $A = \{1,2\}, B = \{2\}, C = \emptyset$. Then $A \setminus (B \cap C) = \{1,2\}$ since $B \cap C = \emptyset$, but $A \setminus B = \{1\}$ and intersecting that with something will not get the 2 back.

e. This one is true, and the distributive law for intersection. Writing the proof out, $x \in \big(A \cap (B \cup C)\big) \equiv x \in A \wedge (x \in B \vee \ x \in C) \equiv^{(*)} (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \equiv$ $x \in (A \cap B) \cup (A \cap C)$. The key step marked (*) is the use of the distributive law for $\wedge$.

f. This one is true. $x \in A \setminus (B \cup C)$ means $x \in A \wedge x \notin B \wedge x \notin C$ after one use of DeMorgan, and we can rewrite this as $(x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)$ by doubling one of the clauses, which is fine (and reversible) since $A \equiv A \wedge A$. This gets us $x \in (A \setminus B) \cap (A \setminus C)$.

g. This one is not true. For a counter-example, take $A = B = C = \{1\}$. Then $(A \setminus B)$ is the empty set, and so is $(A \setminus B) \setminus C$; but $B \setminus C$ is also empty, so $A \setminus (B \setminus C) = A \setminus \emptyset = A = \{1\}$.

h. This one is true again. Both sides expand to $x \in A \wedge x \notin B \wedge x \notin C$.

6. More on Cartesian Products and Power Sets

a. No, it is neither. $A \times B$ is a set of pairs (a, b) with the elements from A in the first position, and $B \times A$ is a set of pairs with the elements from B in the first position. The two sets have equal cardinality, and there is an obvious bijective function between them namely $(a, b) \mapsto (b, a)$, but they are not the same set.
For associativity, the answer is "technically no". Formally, $(A \times B) \times C$ is a set of pairs, the first elements of which are pairs again, so they have the form $\big((a, b), c\big)$. $A \times (B \times C)$ is a set of pairs whose second elements are pairs again, of the form $\big(a, (b, c)\big)$. The two sets are not identical, even though there is again an obvious bijection $\big((a, b), c\big) \mapsto \big(a, (b, c)\big)$. However, even mathematicians in "everyday use" will often treat the two as equivalent, or pretend that both of them are actually sets of triples (a, b, c).

b. We have $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. If $A = \emptyset$ then there are no such a, and if $B = \emptyset$ then there are no such b, so if $A = \emptyset \vee B = \emptyset$ then there can be no elements in $A \times B$ and so this is the empty set too. Conversely, if $A \neq \emptyset$ then there is some $a \in A$, and if $B \neq \emptyset$ then there is some $b \in B$, and so the pair (a, b) is in $A \times B$ too, making this set non-empty.

c. Elements in the intersection must be of the form (a, b) where both $a \in A$ and $a \in B$, and also both $b \in A$ and $b \in B$. Therefore, both a, b must be in $A \cap B$, and so we have $(A \times B) \cap (B \times A) \subseteq (A \cap B) \times (A \cap B)$. Conversely, if $(a, b) \in (A \cap B) \times (A \cap B)$, then since $a \in A$ and $b \in B$ we have $(a, b) \in A \times B$, and since $a \in B$ and $b \in A$ we have $(a, b) \in B \times A$. Therefore, $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

d. Yes, for example $A = \big\{\{a\}\big\}$ and $B = \big\{a, \{a\}\big\}$. Then $A \subseteq B$ since $\forall x. (\, x \in A \to x \in B)$ (there is only one such value namely $x = \{a\}$). Further, $\mathcal{P}(B) = \Big\{\emptyset, \{a\}, \big\{\{a\}\big\}, \big\{a, \{a\}\big\}\Big\}$ so $\forall x. (x \in A \to x \in \mathcal{P}(B))$, as the only element in question is again $x = \{a\}$ which is also an element (not just a subset) of $\mathcal{P}(B)$. The trick here is that the plain a in B is giving the $\{a\}$ in the powerset that we want, and the $\{a\}$ in B is making sure that $A \subseteq B$.
Indeed, you will only get both $A \in X$ and $A \subseteq X$ for the same set X if X contains elements of different "nestedness" – which is a concept that needs a lot of mathematical groundwork do define in the first place without running into paradoxes.