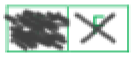
COMS10014 Class Test 2

Instructions

1. Write your 7-digit student number (from the back of your UCard) in the provided box of your answer sheet, one digit per box. Do not write your name or any other identifying information.
2. You must complete your answers on the provided answer sheet.
3. You must use pencil for your answer sheet. If you make a mistake, use your eraser.
4. Only your answers will be marked, not your workings. You may use the blank paper provided for your workings, but this will not be marked.
5. You may use a non-programmable calculator and up to one side A4 of handwritten notes.

Completing the Answer Sheet

	✓ Correct	✗ Incorrect
Where a question asks for a cross, it must fit entirely within a green box.		
Use an eraser if you need to change your answer.		
Make crosses and no other symbols.		

Not following these rules immediately gives zero marks for the question.

Do not turn over until you are told to start the test.

Question 1 **(6 marks)**

Ash, a new Pokémon trainer, gets to pick one of four options for his first Pokémon:
a fire Pokémon, a water Pokémon, a leaf Pokémon, or an electric Pokémon.

Assume that the following statement is logically **false**:

> "If Ash likes one of the four Pokémon, then there is also another one that he likes."

Which of the following statements are satisfiable, assuming the statement above is false?

a. Ash likes the fire Pokémon.
b. Ash likes both the leaf and the water Pokémon.
c. Ash likes none of the four Pokémon.
d. If Ash likes the electric Pokémon, then he does not like the water Pokémon.
e. Ash likes all four Pokémon.
f. Ash does not like the leaf Pokémon.

Make exactly one cross in each of the six pairs of boxes; for each statement cross True (T) if it is satisfiable given that the original statement is false, or False (F) if the statement is not satisfiable given that the original statement is false.

Question 2 **(8 marks)**

The following is a proof in natural deduction, showing that $\neg a \land \neg b \vdash \neg(a \lor b)$, one side of one of DeMorgan's laws. In natural deduction, each line is one of three operations: making an assumption, applying a law of logical reasoning, or discharging the last assumption.

If you assume A, derive B (as the last term before you discharge the assumption) and then discharge, you get the sequent $A \vdash B$ in the current scope.

Line	Rule	Result
1	assume $\neg a \land \neg b$	$\neg a \land \neg b$
2	??? on line 1	$\neg a$
3	??? on line 1	$\neg b$
4	assume a	a
5	??? on lines 4, 2	F
6	discharge	$a \vdash F$
7	assume b	b
8	??? on lines 7, 3	F
9	discharge	$b \vdash F$
10	assume $a \lor b$	$a \lor b$
11	??? on lines 10, 6, 9	F
12	discharge	$a \lor b \vdash F$
13	??? on line 12	$\neg(a \lor b)$
14	discharge	$\neg a \land \neg b \vdash \neg(a \lor b)$

What laws of logical reasoning are used on lines 2, 5, 11 and 13?

Make exactly one cross in each row of the table of your answer sheet, in the column of the law used.

Question 3 **(4 marks)**

Which of the following formulas are tautologies in predicate logic, for all predicates P, Q?

a) $\forall x\, \exists y. \big(P(x,y) \lor Q(x,y)\big) \to \exists y\, \forall x. \big(P(x,y) \lor Q(x,y)\big)$
b) $\forall x\, \exists y. \big(P(x,y) \land Q(x,y)\big) \to \exists y\, \forall x. \big(P(x,y) \land Q(x,y)\big)$
c) $\exists x\, \forall y. \big(P(x,y) \lor Q(x,y)\big) \to \forall y\, \exists x. \big(P(x,y) \lor Q(x,y)\big)$
d) $\exists x\, \forall y. \big(P(x,y) \land Q(x,y)\big) \to \forall y\, \exists x. \big(P(x,y) \land Q(x,y)\big)$

For each formula, mark it as T (true) on your answer sheet if it is a tautology for all P, Q; otherwise as F (false) if it is not a tautology.

Question 4 **(4 marks)**

The following is a proof by induction of (for natural numbers $n \geq 1$)

$$1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$

however, the reasoning for each line has been left out.

Line	**Operation**
1	We prove the equivalent statement $6(1^2 + 2^2 + \cdots + n^2) = n(n+1)(2n+1)$.
2	Let $L(n) = 6(1^2 + 2^2 + \cdots + n^2)$, to be precise, $L(n) = \sum_{i=1}^{n} i^2$.
3	Let $R(n) = n(n+1)(2n+1)$.
4	We have $L(1) = 6$ and $R(1) = 1 \times 2 \times 3 = 6$.
5	Assume that $L(n) = R(n)$ for some natural number $n > 0$.
6	Write out $L(n+1) = 6(1^2 + 2^2 + \cdots + n^2 + (n+1)^2)$.
7	Rewrite this as $L(n+1) = 6(1^2 + 2^2 + \cdots + n^2) + 6(n+1)^2 = L(n) + 6(n+1)^2$.
8	Substitute $L(n) = R(n)$ to get $L(n+1) = R(n) + 6(n+1)^2$.
9	Calculate out $L(n+1) = n(n+1)(2n+1) + 6(n+1)^2$.
10	Calculate further $L(n+1) = (n+1)(2n^2 + 7n + 6) = (n+1)(n+2)(2n+3)$.
11	Write out $R(n+1) = (n+1)(n+2)(2(n+1)+1) = (n+1)(n+2)(2n+3)$.
12	This shows $L(n+1) = R(n+1)$.
13	This proves the statement by induction.

On which line does this proof (a) use the induction hypothesis and (b) show the base case?

Make exactly one cross in each row in the table on your answer sheet.

Question 5 **(8 marks)**

Consider the sets $A = \{1, 3\}$ and $B = \{2, 3\}$ over the universe $\mathcal{U} = \{1,2,3\}$.
What are the elements of the following sets?

a. $\mathcal{P}(B) \cap \mathcal{P}(A)$
b. $\mathcal{P}(\bar{A})$
c. $A \cup \mathcal{P}(B)$
d. $\mathcal{P}(A \setminus B)$

For each set's row in the table, cross all the boxes in columns where the column header is an element of the set. Each row may contain zero, one, or more than one crosses.

Question 6 **(8 marks)**

The following is an algorithm for division with remainder by repeated subtraction for positive integers.

```
1  // input: integers a, b with a, b > 0
2  // output: integers q, r with a = q * b + r and 0 ≤ r < b
3  q = 0
4  r = 0
5  while a ≥ b do
6      q = q + 1
7      a = a - b
8  end while
9  r = ???
```

1. What should r be set to on line 9? *Cross exactly one of the boxes in your answer sheet.*

a.	`r = q`	e.	`r = b - q`
b.	`r = q - a`	f.	`r = q - b`
c.	`r = a - b`	g.	`r = a`
d.	`r = b - a`	h.	`r = b`

2. Find a loop invariant for this algorithm in the form

$$[X] = [Y] \times b_0 \; [\triangle] \; [Z]$$

where b_0 is the initial value of the variable b (which does not change during the algorithm); X, Y, Z are variables from the set $\{q, q_0, a, a_0, b, r\}$ and $\triangle$ is either the $+$ or the $-$ operation.

Enter your result by making exactly one cross in each of the rows of the table for the X, Y, Z variables, in the column of the correct variable. Further, cross exactly one of the $+$ and $-$ boxes for the operation $\triangle$. You must make four crosses in total to answer this question.

This is the end of the class test.