ASSIGNMENT - 3

Coupling of Hopf Oscillators

BT6270 Computational Neuroscience
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Introduction

In this assignment, we couple two Hopf oscillators in order to achieve a given phase difference between the two oscillators. Two different methods are used to achieve this:

1. Complex Coupling
2. Power Coupling

Since oscillatory behaviour is commonly observed in neural spiking activities, it is useful to model these dynamics using oscillators that exhibit limit-cycle behaviour. The Hopf oscillator is the simplest system with such behaviour.

Understanding the coupling between multiple Hopf oscillators enables us use a network of these oscillators to study the properties of more neural oscillations.

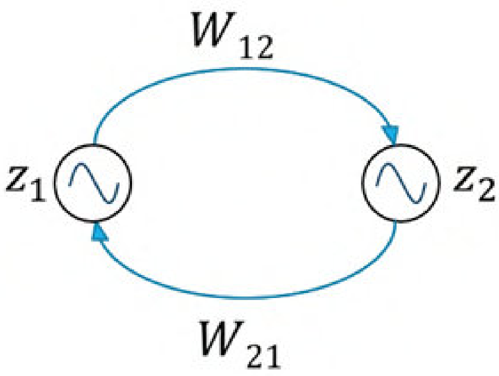

Figure 1 - Coupled Hopf Oscillators

In the first section, we examine the behaviour of oscillators that operate at the same frequency and attempt to achieve a constant phase difference through Complex Coupling.

In the second section, we extend the analysis to oscillators with different operating frequencies that aim to reach a constant "normalized" phase difference, through Power Coupling.

The motivation behind this approach is to explore whether such steady-state relationships, captured in terms of normalized phase differences, can serve as a mechanism for encoding and retaining system memory within larger networks of coupled oscillators.

Complex Coupling

The differential equations for two coupled Hopf oscillators in polar coordinates are:

$$\dot{r}_1 = (\mu - r_1^2)r_1 + Ar_2\cos(\theta_2 - \theta_1 + \phi),$$

$$\dot{r}_2 = (\mu - r_2^2)r_2 + Ar_1\cos(\theta_1 - \theta_2 - \phi),$$

$$\dot{\theta}_1 = \omega_1 + \frac{Ar_2}{r_1}\sin(\theta_2 - \theta_1 + \phi),$$

$$\dot{\theta}_2 = \omega_2 + \frac{Ar_1}{r_2}\sin(\theta_1 - \theta_2 - \phi).$$

where ϕ is the coupling phase offset, and A is the coupling strength (W_{21} and W_{12} in Figure 1).

$$\dot{\psi} = \dot{\theta}_1 - \dot{\theta}_2 = (\omega_1 - \omega_2) - \frac{A\sin(\psi - \phi)}{r_1 r_2}(r_1^2 + r_2^2)$$

At Steady State $\dot{\psi} = 0$;
Since $\omega_1 = \omega_2$, at steady state we obtain:

$$\dot{\psi} = A\sin(\phi - \psi) = 0$$

This ensures that the phase difference $(\theta_1 - \theta_2)$ between the two oscillators equals ϕ, the angle of the complex coupling weight.

The table below shows the values of the variables taken for each case of the simulation.

Parameter	**Steady State:** $\psi = -47°$	**Steady State:** $\psi = 98°$
$A = A_{21} = A_{12}$	0.2	0.2
μ	1	1
r_1 (initial)	0.4	1.4
r_2 (initial)	0.2	0.6
θ_1 (initial)	15°	15°
θ_2 (initial)	−45°	−45°
w1	5 radians/s	5 radians/s
w2	5 radians/s	5 radians/s
ϕ	$-47° \approx -0.8203$ radians	$98° \approx 1.7104$ radians

Table 1: Parameters for Simulating Complex Coupling Cases

Case I : Steady State $\psi = -47°$

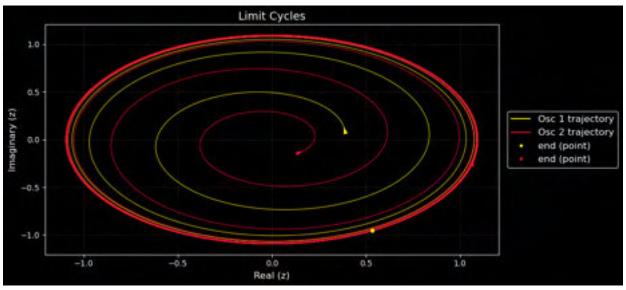

Figure 2.1.1

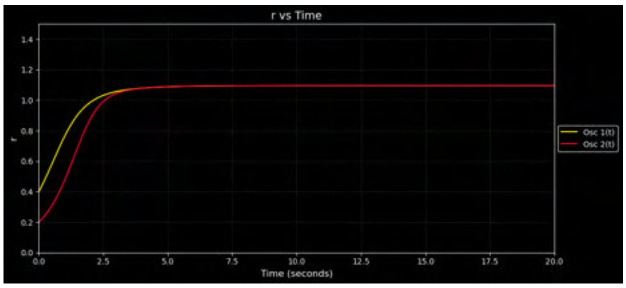

Figure 2.1.2

As expected, since the coupling constant A is relatively small in this case,the amplitude of the limit-cycle oscillations remains approximately equal to $\sqrt{\mu}$, as observed in Figures 2.1.1 and 2.1.2.

Furthermore, from Figure 2.1.3, it can be seen that, regardless of the initial phase difference, the two oscillators synchronize within approximately 12 seconds and subsequently maintain a constant phase difference of $-47°$.

Figure 2.1.4 illustrates the evolution of the phase difference over time. The white line in the figure denotes ϕ, indicating that the steady-state phase difference converges to this value, as further evidenced by Figure 2.1.5, where $\psi - \phi$ tends to 0.

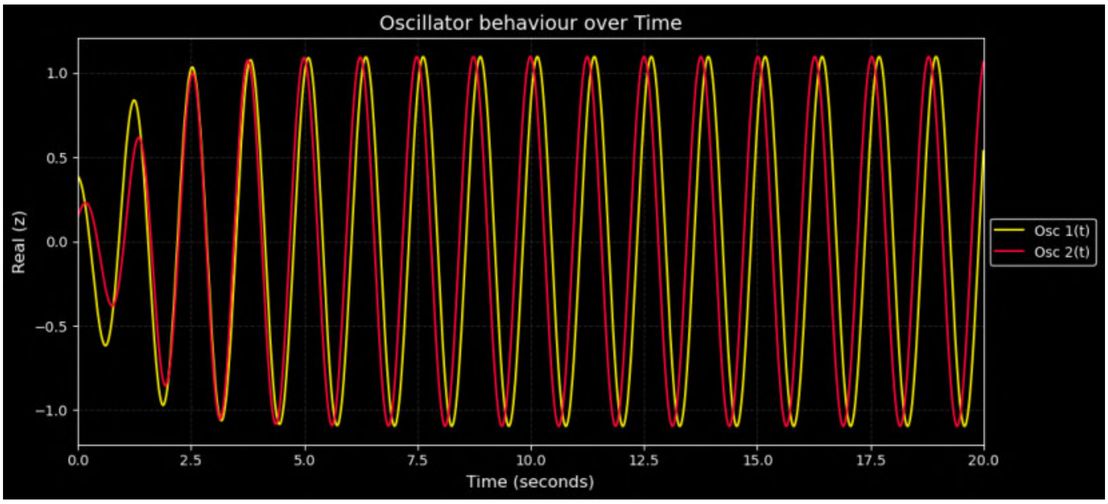

Figure 2.1.3

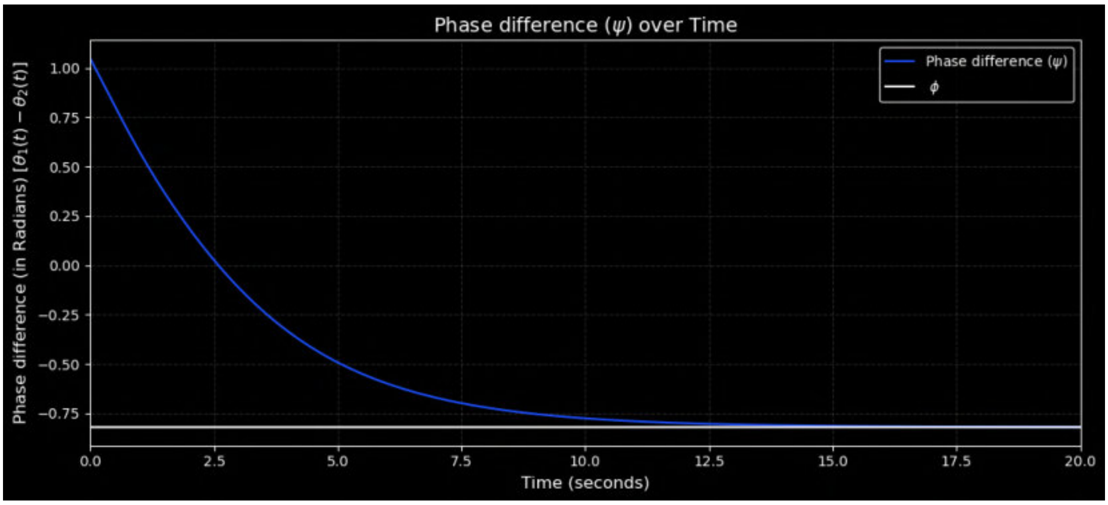

Figure 2.1.4

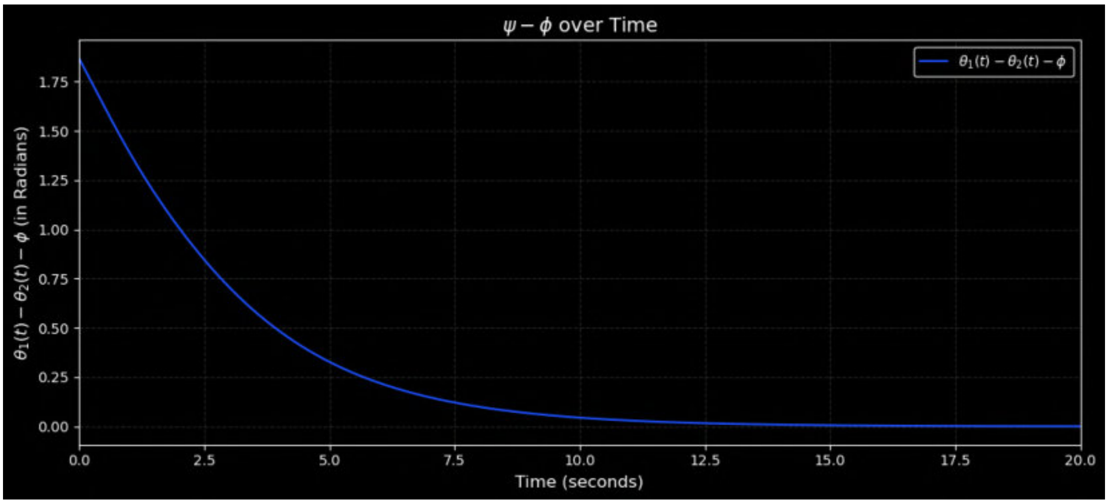

Figure 2.1.5

Case II : Steady State $\psi = 98°$

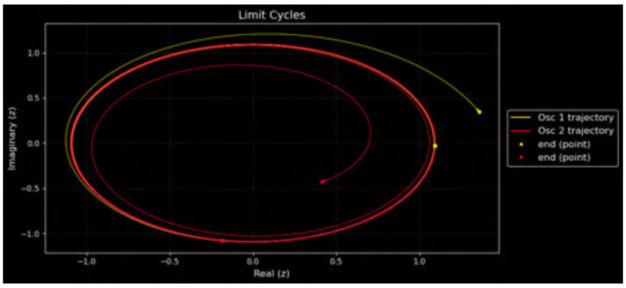

Figure 2.2.1

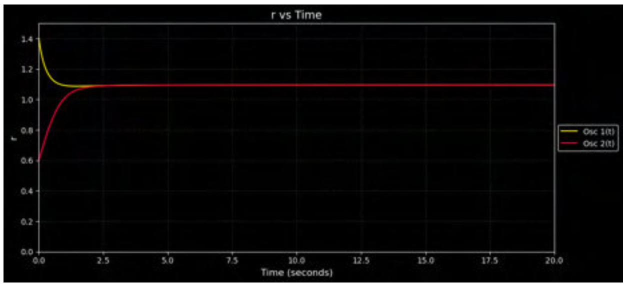

Figure 2.2.2

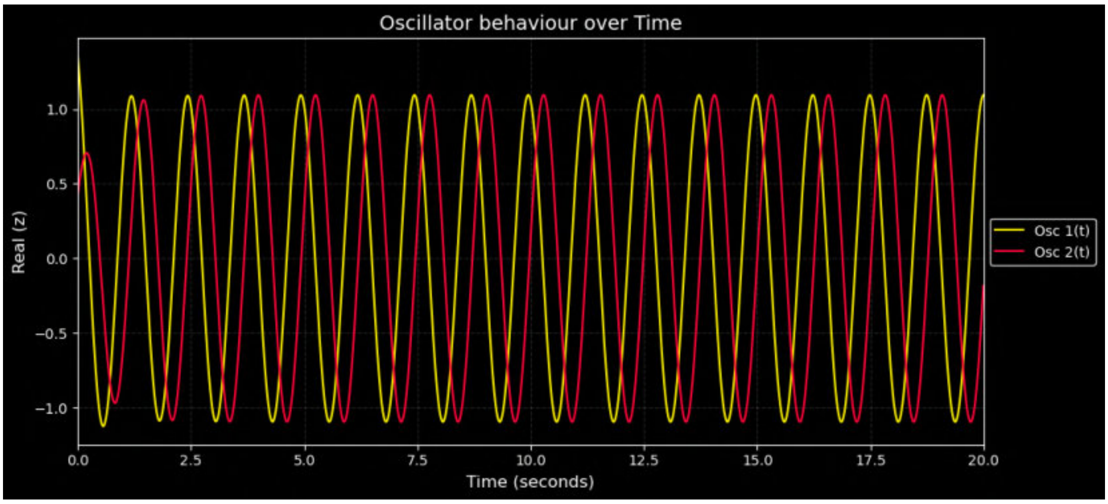

Figure 2.2.3

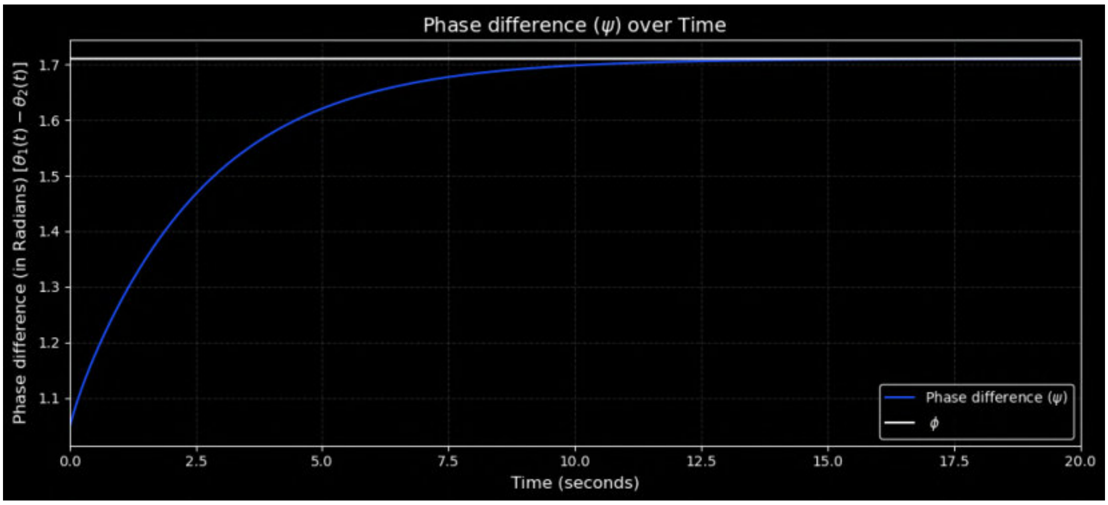

Figure 2.2.4

Similar to the previous case since the coupling constant A is relatively small in this case,the amplitude of the limit-cycle oscillations remains approximately equal to $\sqrt{\mu}$, which is observed in Figures 2.2.1 and 2.2.2.

This time we see that in Figure 2.2.3, the two oscillators once again synchronize within approximately 12 seconds and subsequently maintain a constant phase difference of 98 °.

Figure 2.2.4 illustrates the evolution of the phase difference over time. The white line in the figure denotes ϕ, indicating that the steady-state phase difference converges to this value, as further evidenced by Figure 2.2.5, where $\psi - \phi$ tends to 0.

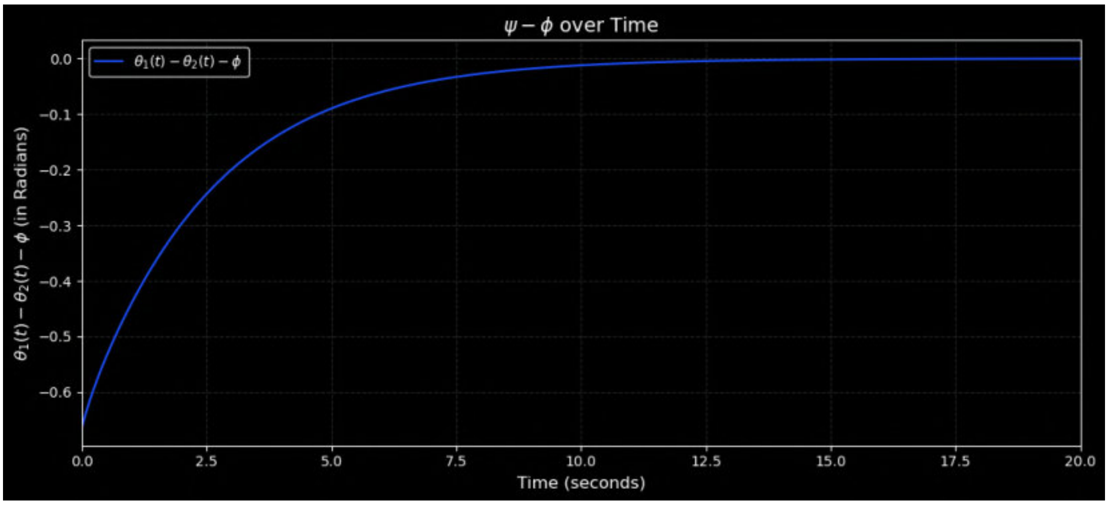

Figure 2.2.5

Conclusion

The complex coupling model successfully demonstrates synchronization between two Hopf oscillators operating at identical natural frequencies, to maintain a constant phase difference (in steady state).

At relatively weak coupling strength at steady state, the oscillators converge an amplitude of oscillation corresponding to $\sqrt{\mu}$, and maintain a constant phase difference that matches the imposed coupling phase offset ϕ confirming that the coupling primarily influences the phase dynamics without significantly perturbing the limit-cycle stability.

This behaviour establishes complex coupling as an effective mechanism for phase tuning within large oscillator networks. Such a tuning principle can serve as a foundation for encoding and retaining memory in these networks.

The primary limitation of using such networks is that they cannot accurately model large-scale biological oscillatory systems, as the operating frequencies in real biological systems are rarely identical. To address this limitation, we next explore the Power Coupling approach, which enables synchronization between oscillators with differing natural frequencies.

Power Coupling

We define a normalized phase difference ψ as shown below

$$\psi = \frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2}$$

The differential equations obtained for the two coupled oscillators in polar coordinates are:

$$\dot{r}_1 = (\mu - r_1^2)r_1 + A r_2^{\frac{\omega_1}{\omega_2}} \cos\left[\omega_1\left(\frac{\theta_2}{\omega_2} - \frac{\theta_1}{\omega_1} + \frac{\varphi}{\omega_1\omega_2}\right)\right],$$

$$\dot{\theta}_1 = \omega_1 + A\frac{r_2^{\frac{\omega_1}{\omega_2}}}{r_1} \sin\left[\omega_1\left(\frac{\theta_2}{\omega_2} - \frac{\theta_1}{\omega_1} + \frac{\varphi}{\omega_1\omega_2}\right)\right],$$

$$\dot{r}_2 = (\mu - r_2^2)r_2 + A r_1^{\frac{\omega_2}{\omega_1}} \cos\left[\omega_2\left(\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} - \frac{\varphi}{\omega_2\omega_1}\right)\right],$$

$$\dot{\theta}_2 = \omega_2 + A\frac{r_1^{\frac{\omega_2}{\omega_1}}}{r_2} \sin\left[\omega_2\left(\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} - \frac{\varphi}{\omega_2\omega_1}\right)\right].$$

where ϕ is the coupling phase offset, and A is the coupling strength (W_{21} and W_{12} in Figure 1).

The derivative of the normalized phase difference with respect to time is given by:

$$\dot{\psi} = \frac{\dot{\theta}_1}{\omega_1} - \frac{\dot{\theta}_2}{\omega_2}$$

Substituting the expressions for $\dot{\theta}_1$ and $\dot{\theta}_2$, we obtain:

$$\dot{\psi} = \frac{Ar_2^{\frac{\omega_1}{\omega_2}}}{\omega_1 r_1} \sin\left[\omega_1\left(\frac{\phi}{\omega_1\omega_2} - \psi\right)\right] + \frac{Ar_1^{\frac{\omega_2}{\omega_1}}}{\omega_2 r_2} \sin\left[\omega_2\left(\frac{\phi}{\omega_1\omega_2} - \psi\right)\right]$$

At steady state, $\psi =$ constant, $\dot{\psi} = 0$, and $r_1 = r_2 \approx \sqrt{\mu}$:

$$\dot{\psi} = \frac{A}{\omega_1} \sin\left[\omega_1\left(\frac{\phi}{\omega_1\omega_2} - \psi\right)\right] + \frac{A}{\omega_2} \sin\left[\omega_2\left(\frac{\phi}{\omega_1\omega_2} - \psi\right)\right] = 0$$

Thus, the steady-state normalized phase difference satisfies:

$$\psi = \frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} = \frac{\phi}{\omega_1\omega_2}$$

We define another quantity σ as shown below to show the convergence $\psi - \phi$ to 0 as steady state is reached:

$$\sigma = \frac{\theta_1(t)}{\omega_1} - \frac{\theta_2(t)}{\omega_2} - \frac{\phi}{\omega_1\omega_2}$$

The variation of $\psi(t)$ and $\dot{\psi}(t)$ with respect to time is examined in the analysis below as well.

The table below shows the values of the variables taken for each case of the simulation.

Parameter	**Steady State:** $\psi = -47°$	**Steady State:** $\psi = 98°$
$A = A_{21} = A_{12}$	0.2	0.2
μ	1	1
r_1 (initial)	0.4	1.3
r_2 (initial)	0.2	0.7
θ_1 (initial)	$-45°$	390°
θ_2 (initial)	45°	90°
w1	5 radians/s	5 radians/s
w2	15 radians/s	15 radians/s
ϕ	-61.523 radians	128.282 radians

Table 2: Parameters for Simulating Power Coupling Cases

Case I : Steady State $\psi = -47°$

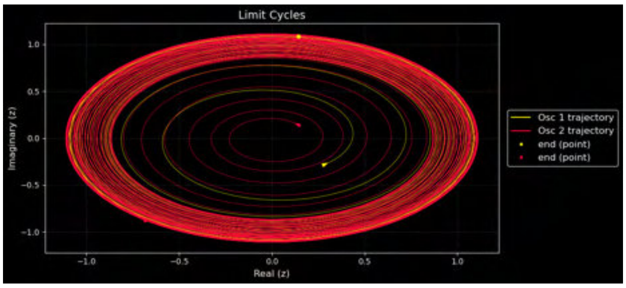

Figure 3.1.1

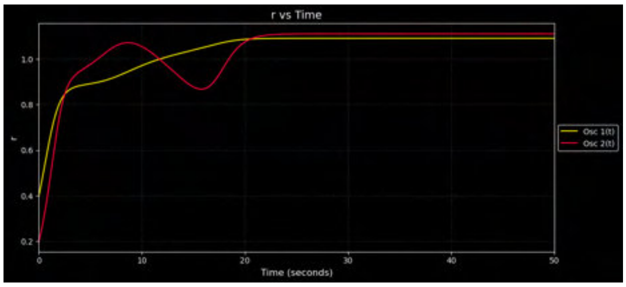

Figure 3.1.2

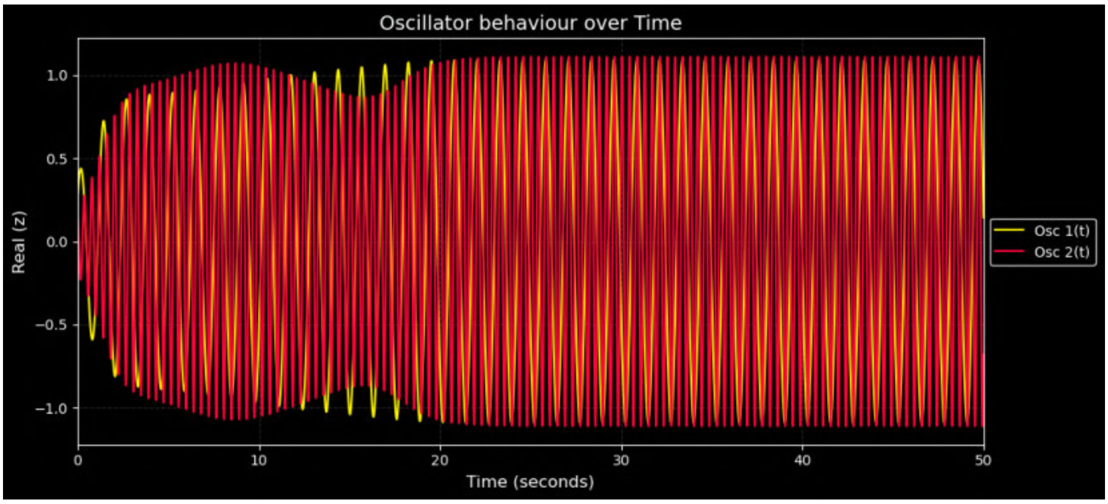

Figure 3.1.3

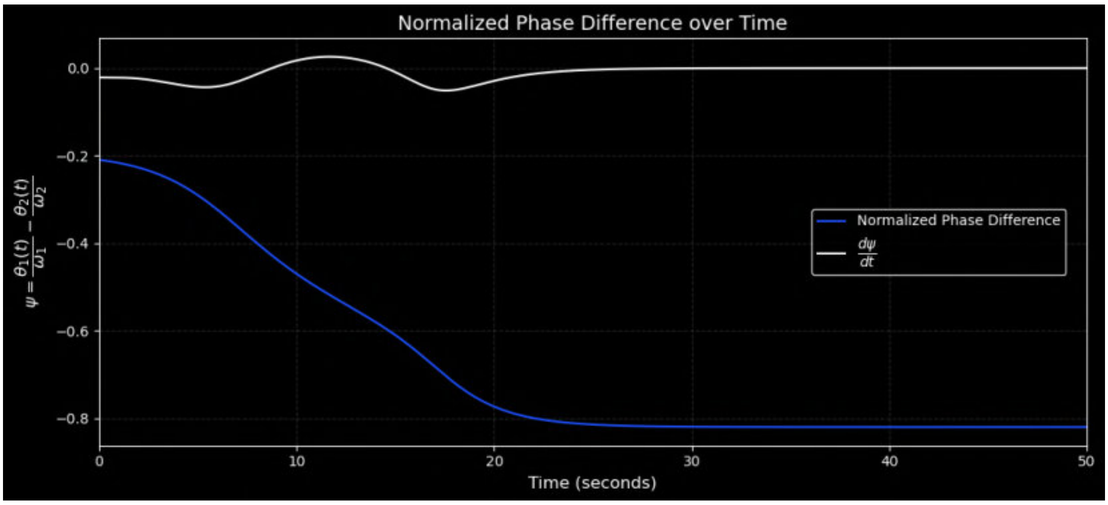

Figure 3.1.4

Right of the bat in power coupling we see that both oscillators converge to an amplitude close to $\sqrt{\mu}$. However, unlike in complex coupling, the oscillators do not converge to the exact same value of r, as illustrated in Figures 3.1.1 and 3.1.2.

It also takes significantly longer for synchronization to occur, i.e., for the system to reach a steady state, compared to complex coupling. In this case, it takes approximately 23 seconds for ψ or σ to converge and remain at their steady state values, as shown in Figures 3.1.4 and 3.1.5 respectively.

Figure 3.1.4 shows that the oscillators reach a constant normalized phase difference of $-47°$ or $\approx$ -0.82 radians.

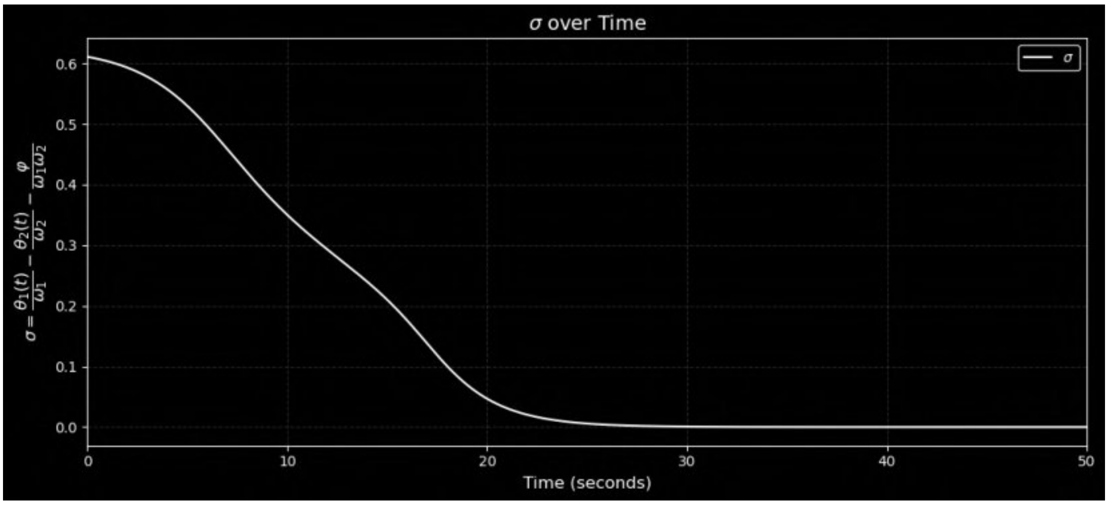

Figure 3.1.5

Case II : Steady State $\psi = 98°$

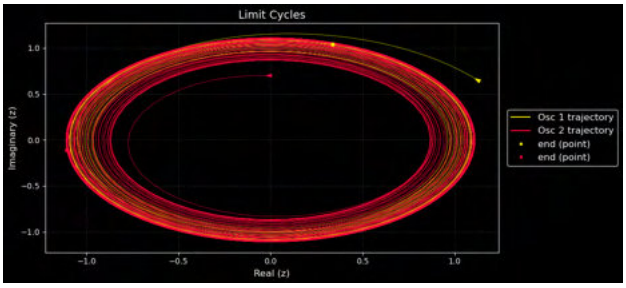

Figure 3.2.1

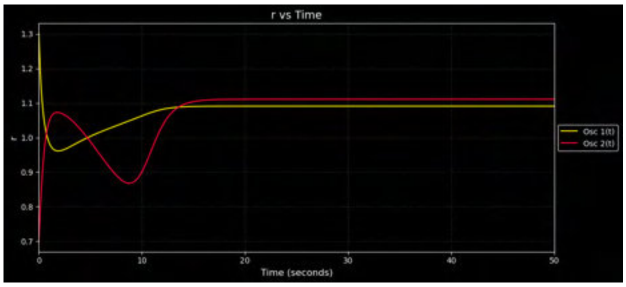

Figure 3.2.2

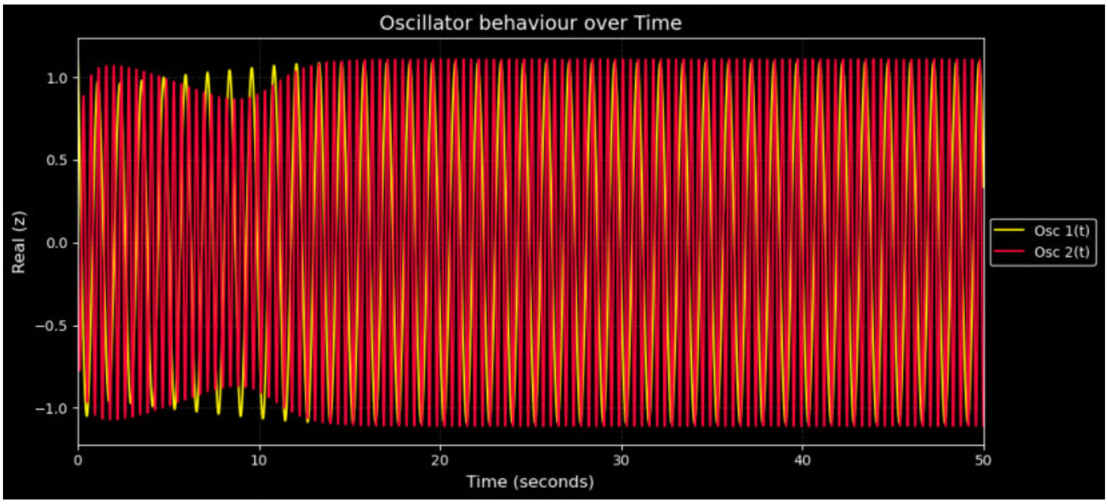

Figure 3.2.3

Both oscillators converge to an amplitude close to $\sqrt{\mu}$ again and similar to the previous case (power coupling) the oscillators do not converge to the exact same value of r, as illustrated in Figures 3.2.1 and 3.2.2.

In this case, it takes approximately 21 seconds for the system to reach steady state where the oscillators reach a constant normalized phase difference of 98° or ≈ 1.71 radians as shown in Figure 3.1.4.

Figure 3.1.5 shows σ takes 21 seconds to converge and remain at zero, after which $\psi = \phi/(\omega_1 \cdot \omega_2)$.

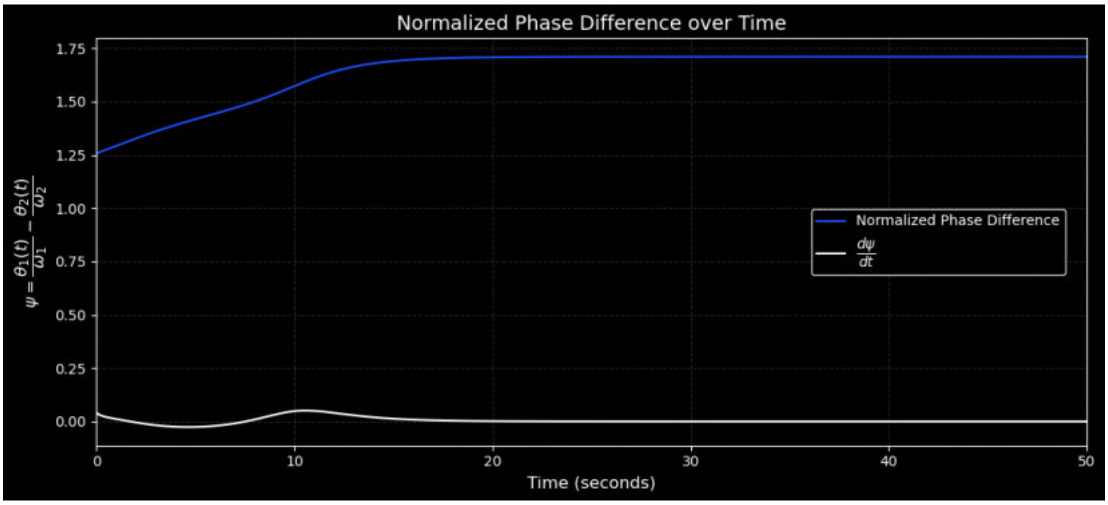

Figure 3.2.4

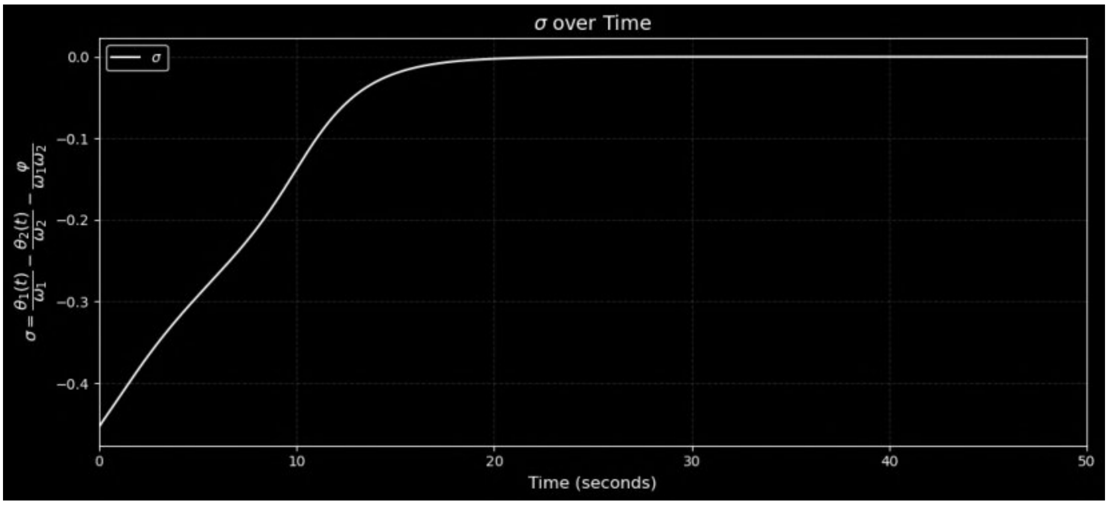

Figure 3.2.5

Conclusion

As demonstrated in the preceding cases, power coupling serves as an effective mechanism for tuning interactions between neurons operating at different frequencies. This form of phase tuning enables the encoding of information between any pair of neurons within a network, independent of their operating frequencies.

From a biological perspective, neuronal populations often form localized clusters that oscillate at common frequencies, while other clusters in the brain operate at distinct frequency bands - a behaviour analogous to magnetic domains in ferromagnets. Within such clusters, complex coupling allows neurons to encode and retain information critical to their local network dynamics.

When information must be transferred across clusters with differing operating frequencies, power coupling provides a means for effective communication and synchronization.

Together, these mechanisms establish a robust framework for modelling, analysing, and understanding oscillatory neuronal networks and their information-processing capabilities.