ASSIGNMENT - 2

Analysis of FitzHugh-Nagumo Neuron Model

BT6270 Computational Neuroscience
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Introduction

The FitzHugh–Nagumo model is a simplified, abstraction of the Hodgkin–Huxley neuronal model that captures the essential features of neuronal behavior while reducing mathematical complexity.

The resulting system has only two variables (v, w). After transformation to dimensionless variables, and some approximations, the resulting FN model may be defined as:

$$\frac{dv}{dt} = f(v) - w + I_{app},$$

$$\frac{dw}{dt} = bv - rw,$$

where $\quad f(v) = v(a - v)(v - 1)$

and the values chosen for the parameters are, $\quad a = 0.5, \quad b = 0.1, \quad r = 0.1.$

For this analysis, we use the τ vs Δ map derived in class to classify the fixed points obtained in different cases.

where

$$\frac{dv}{dt} = F(v, w), \qquad \frac{dw}{dt} = G(v, w),$$

and

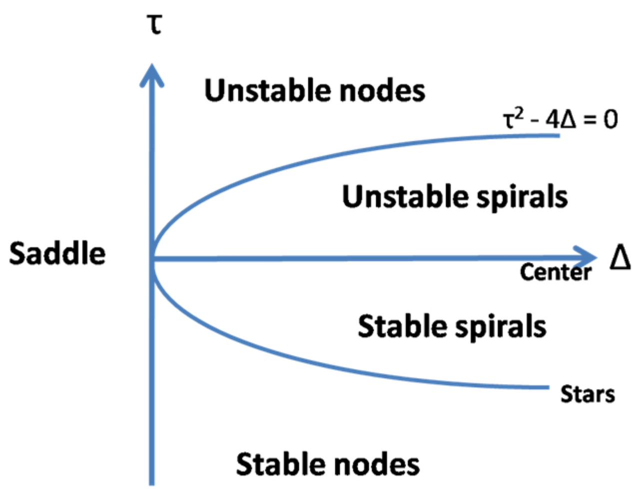
$$F(v, w) = f(v) - w + I_{app}, \quad G(v, w) = bv - rw.$$

Thus, the Jacobian matrix is given by:

$$A = \begin{bmatrix} \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \end{bmatrix} = \begin{bmatrix} f'(v) & -1 \\ b & -r \end{bmatrix}.$$

and

$$\tau = f'(v) - r, \quad \Delta = -rf'(v) + b.$$



Map of τ vs Δ

Stable point identification

The type of the stationary point can be expressed in terms of the determinant (Δ) and trace (τ) of the Jacobian, using the following rules:

- If $\Delta < 0$, the stationary point is a **saddle** irrespective of the value of τ.
- If $\Delta > 0$, $\tau < 0$, the point is **stable**.
- If $\Delta > 0$, $\tau > 0$, the point is **unstable**.

For $\Delta > 0$,

$$f'(v)(-r) > -b$$
$$f'(v)r < b$$
$$f'(v) < \frac{b}{r}$$

That is, $\Delta > 0$ when the slope of the f-nullcline is lesser than the slope of the w-nullcline.

$$\tau > 0 \Rightarrow f'(v) - r > 0$$

Characterizing System Response to Input Currents

Question I, $I_{app} = 0$ Excitatory Behavior of the Neuron

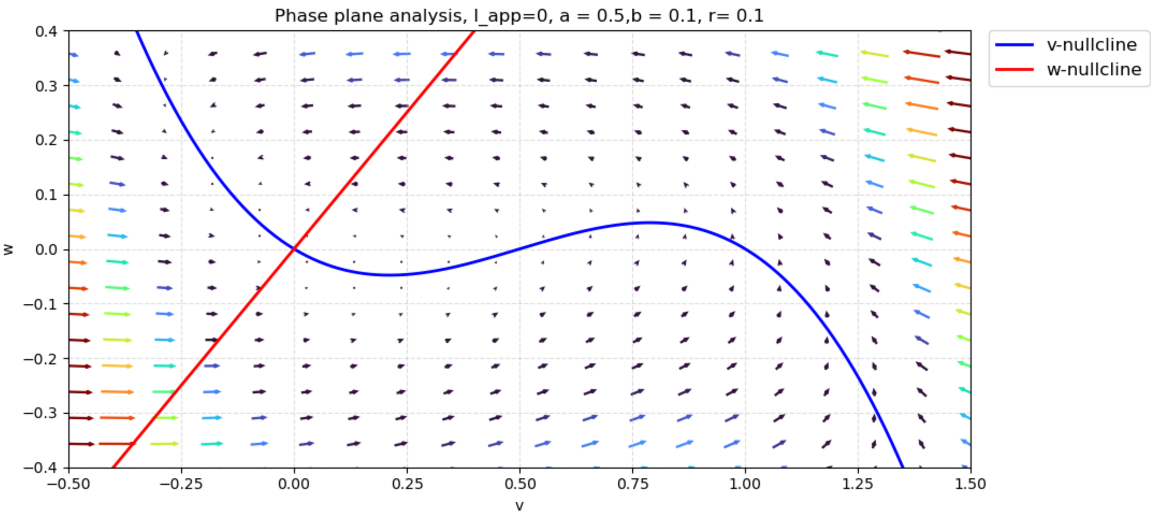


Figure 1

The **slope of the f-nullcline** is greater than that of the w**-nullcline** This condition implies that $\Delta > 0$ and $\tau < 0$, corresponding to a **stable fixed point**.Consequently, the **origin is stable**, and serves as the resting state of the neuron.

Excitability refers to the system's ability to generate a large transient response (an action potential) in response to a sufficiently strong perturbation, followed by a return to the resting state.

As shown in Figures 1.1.A and 1.1.B, small perturbations cause the neuron to return immediately to its stable resting state. In this case no action potential is generated

However, for slightly larger perturbations, the system exhibits **excitability**, as illustrated in Figures 1.2.A and 1.2.B. In this case, the trajectory rises toward the v-nullcline, moves leftward, reaches the leftmost branch of the v-nullcline, and finally returns to the origin, representing the firing of an action potential before relaxing back to rest.

Case A, V(0) < a and W (0)= 0

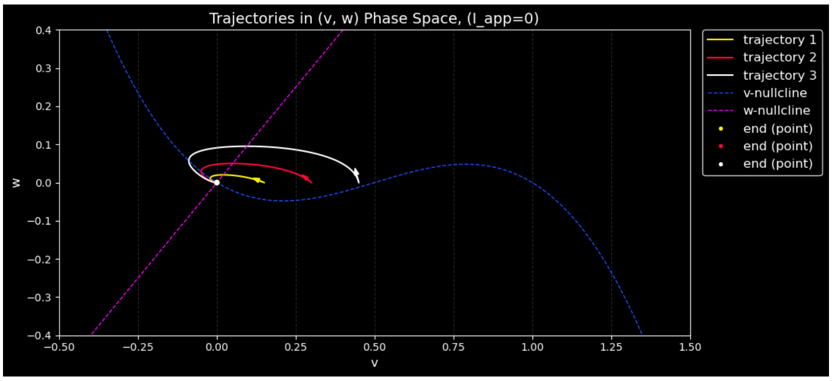


Figure 1.1.A

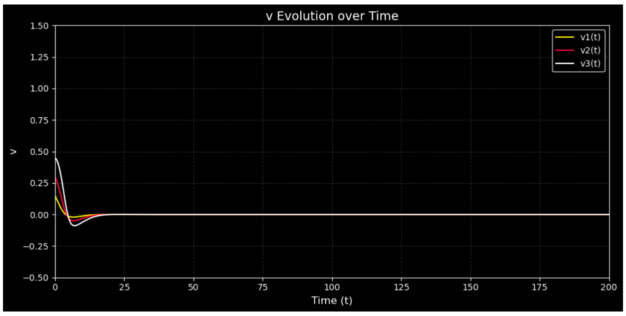


Figure 1.1.B

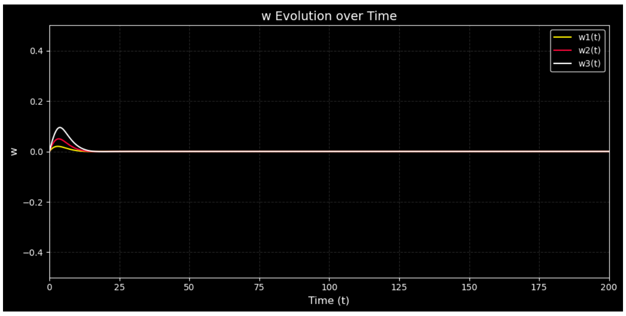


Figure 1.1.C

Case B, V(0) > a and W (0)= 0

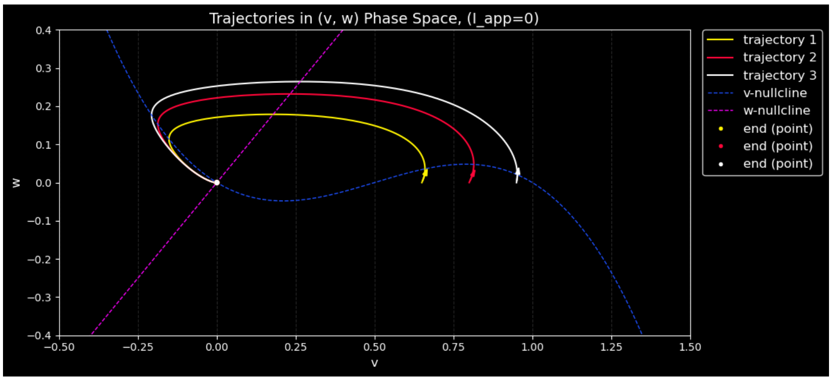


Figure 1.2.A

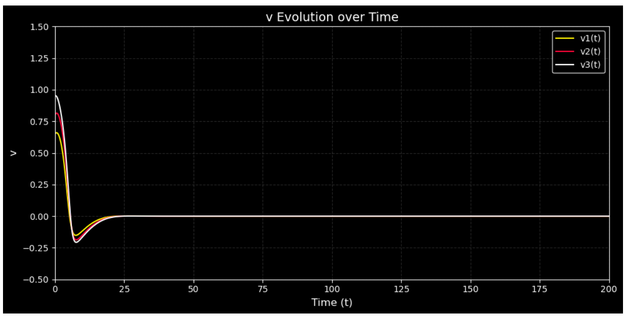


Figure 1.2.B

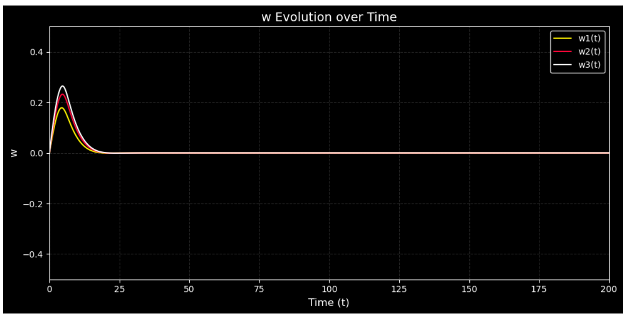


Figure 1.2.C

Question II, $I_{app} \neq 0$; Oscillatory Behavior of the Neuron

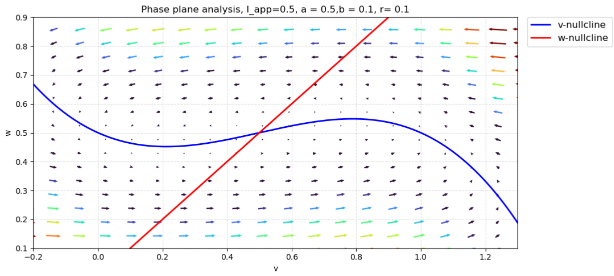


Figure 2.1.A

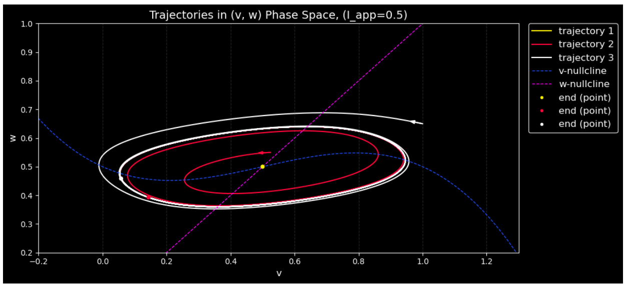


Figure 2.1.B

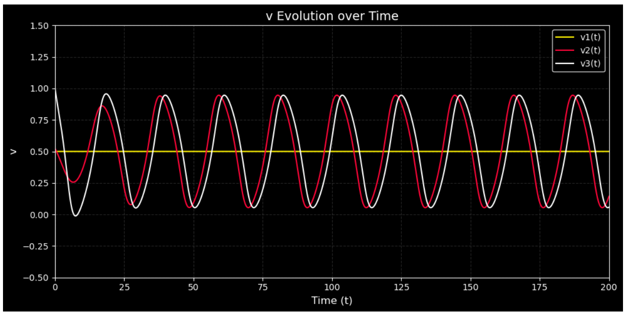


Figure 2.1.C

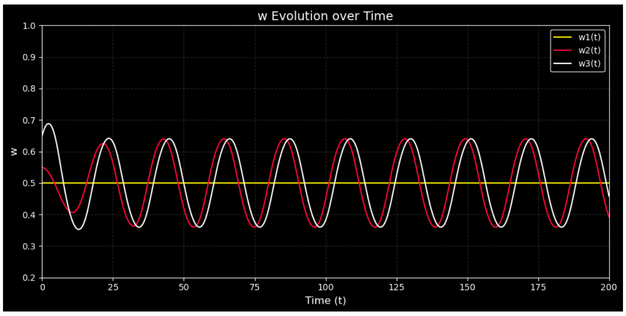


Figure 2.1.D

As shown in the above Figures 2.1.A, 2.1.B, 2.1.C, 2.1.D, if the initial (v,w) is the fixed point, the Neuron shows no dynamics (yellow trajectory in the above figures).

However, for small (red trajectory) and large (white trajectory) perturbations from this fixed point, it spirals into the stable limit cycle and stays within the closed loop once it reaches it.

Computing I_1 and I_2

Computing τ and Δ

$$\begin{aligned} f(v) &= v(a-v)(v-1) \\ &= v\Big(av - a - v^2 + v\Big) \\ &= -v^3 + v^2(a+1) - av. \end{aligned}$$

For $a = 0.5$:

$$f(v) = -v^3 + 1.5v^2 - 0.5v.$$

Thus,

$$f'(v) = -3v^2 + 3v - 0.5.$$

Now

$$\tau = f'(v) - r = -3v^2 + 3v - 0.5 - 0.1 = -3v^2 + 3v - 0.6,$$
$$\Delta = -rf'(v) + b = -0.1(-3v^2 + 3v - 0.5) + 0.1 = 0.1(3v^2 - 3v + 1.5).$$

Limit Cycle Conditions

The system exhibits a **limit cycle behavior** when:

$$\tau > 0 \quad \text{and} \quad \tau^2 - 4\Delta < 0.$$

Plotting the equations for:

$$\tau \quad and \quad \tau^2 - 4\Delta$$

From Figure 2.2 the trace τ is positive in the range:

$$v \in [0.28,\ 0.72]$$

and $\tau^2 - 4\Delta < 0$ for all v in this range

.

Therefore, the system exhibits a **limit cycle behavior** if the intersection of nullclines lies within:

$$v \in [0.28,\ 0.72].$$

Nullcline Intersection Points

At the intersection of the nullclines, $\frac{dw}{dt} = 0 \Rightarrow w = \frac{b}{r}v = v$.

At $v = 0.28$:

$$\begin{gathered}(0.28)(0.5 - 0.28)(0.28 - 1) - 0.28 + I_1 = 0,\\ (0.28)(-0.22)(-0.72) - 0.28 + I_1 = 0,\\ I_1 \approx 0.3244.\end{gathered}$$

At $v = 0.72$:

$$\begin{gathered}(0.72)(0.5 - 0.72)(0.72 - 1) - 0.72 + I_2 = 0,\\ (0.72)(-0.22)(-0.28) - 0.72 + I_2 = 0,\\ I_2 \approx 0.6756.\end{gathered}$$

Thus for

$$I_1(\approx 0.33) < I_{app} < I_2(\approx 0.67)$$

The neuron shows oscillatory behavior.

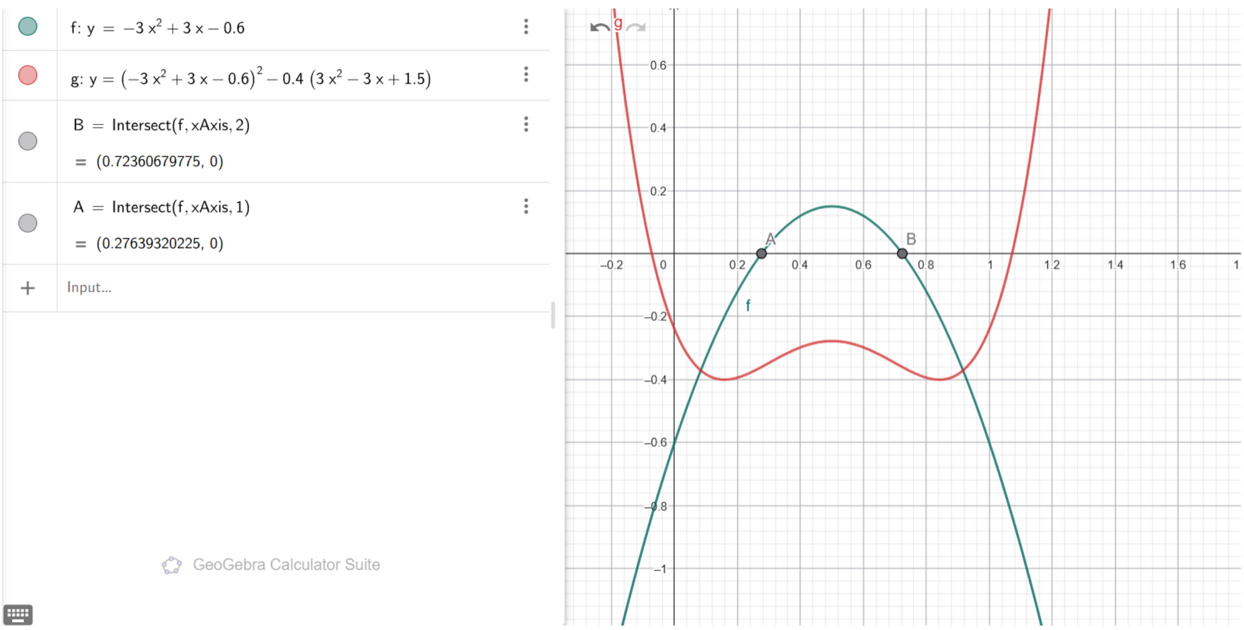


Figure 2.2

Limit cycles at I_1

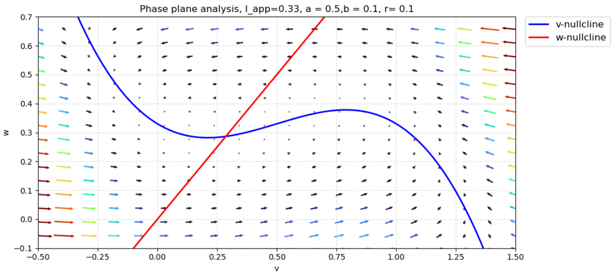


Figure 2.3.A

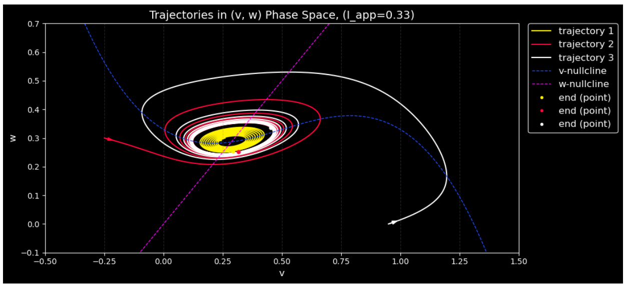


Figure 2.3.B

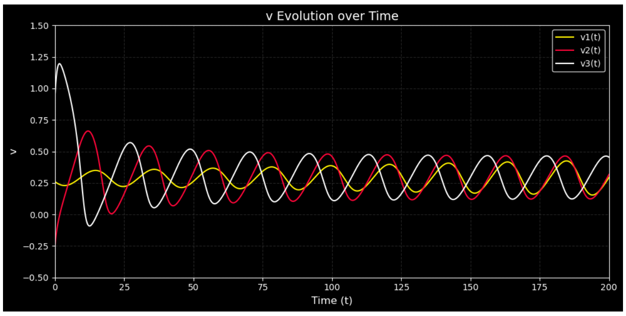


Figure 2.3.C

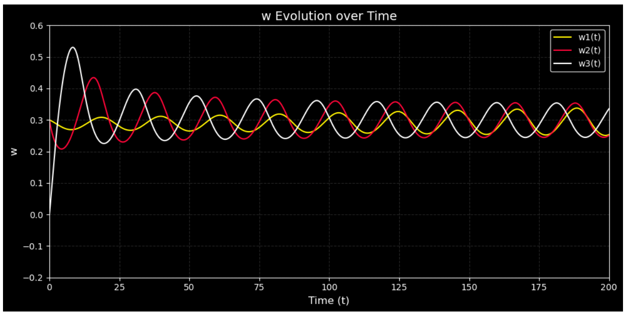


Figure 2.3.D

Limit cycles at I_2

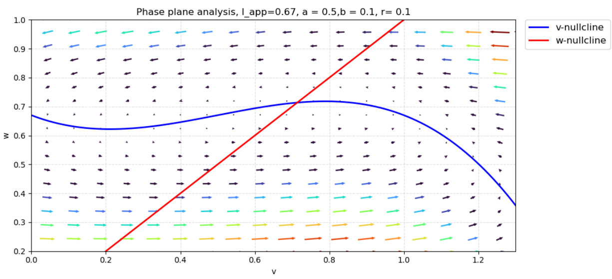


Figure 2.4.A

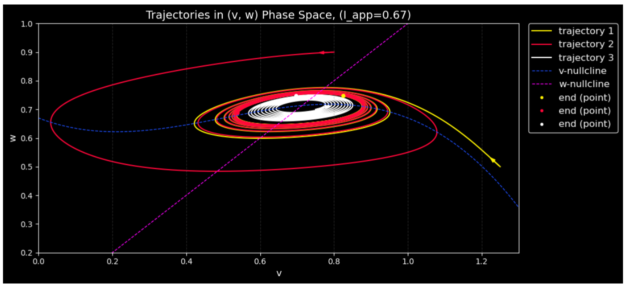


Figure 2.4.B

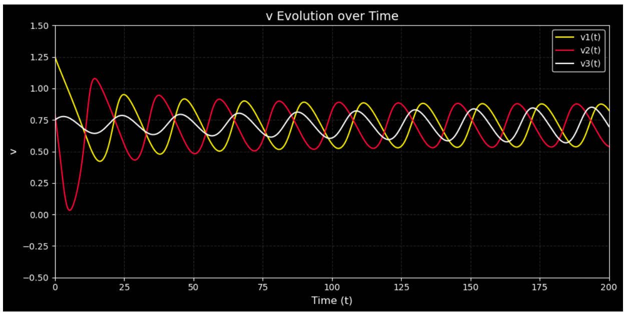


Figure 2.4.C

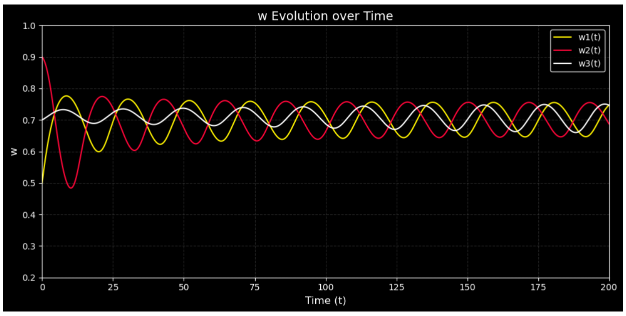


Figure 2.4.D

Across three distinct initial conditions chosen (starting from both inside and outside the closed orbit in the phase plane), the trajectories converge onto the **same periodic orbit** in both the edge cases.

The trajectory starting from inside the closed limit cycle region, converges onto the path of the trajectories starting from outside the closed limit cycle, which is made apparent when looking at the v(t) and w(t) plot in both the cases (Figures 2.3.C, 2.3.D and Figures 2.4.C, 2.4.D).

Question III, $I_{app} > I_2$; Depolarization of the Neuron

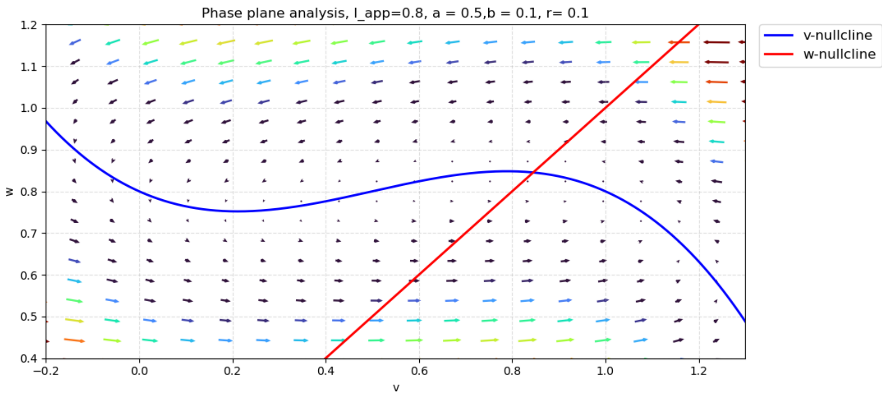


Figure 3.1

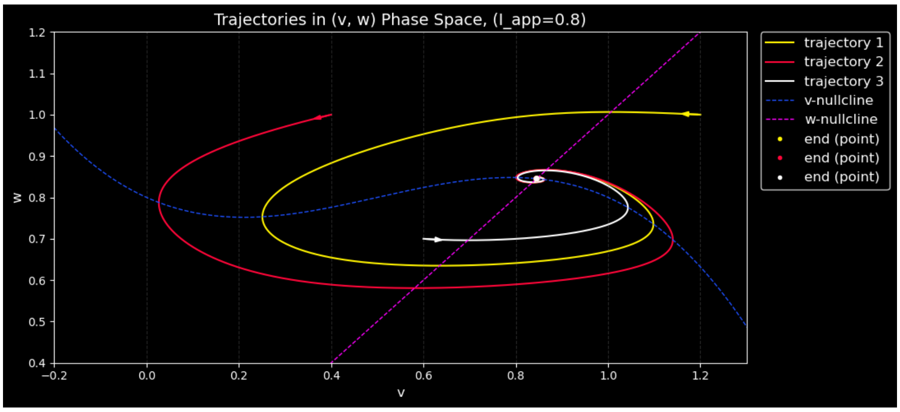


Figure 3.2

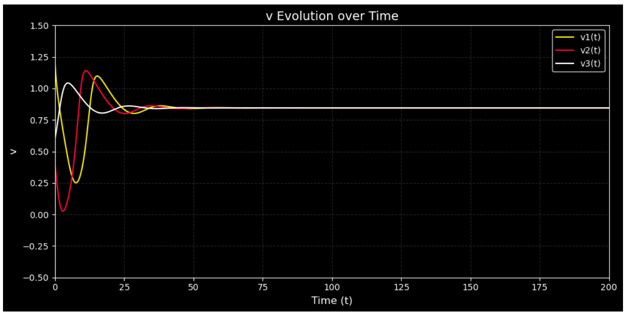


Figure 3.3

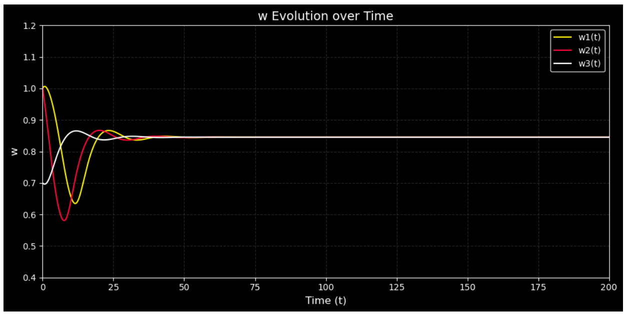


Figure 3.4

In this regime, the slope of the f-nullcline is negative, whilst the w-nullcline has a positive slope, the condition

$$f'(v) < \frac{b}{r}$$

is satisfied, implying that the determinant $\Delta > 0$ and the trace $\tau < 0$. Consequently, the fixed point is **stable**, and no oscillatory behavior is observed.

The phase-plane trajectory confirms this stability: regardless of the initial condition, the system relaxes monotonically toward the fixed point without exhibiting any closed orbits or limit-cycle tendencies. The membrane potential $v(t)$ stabilizes at a higher steady value, representing a **tonically depolarized state**.

Question IV; Bistable Behavior of the Neuron

At the intersection point of the nullclines for a particular set of values of (v, w), both time derivatives become nil:

$$\frac{dw}{dt} = 0, \qquad \frac{dv}{dt} = 0.$$

From the second equation:

$$\frac{dw}{dt} = bv - rw = 0 \quad \Rightarrow \quad w = \left(\frac{b}{r}\right) v.$$

Substitute into

$$\frac{dv}{dt} = f(v) - w + I_{\text{app}},$$

to get

$$\frac{dv}{dt} = v(a - v)(v - 1) - \left(\frac{b}{r}\right) v + I_{\text{app}}.$$

Simplifying,

$$\frac{dv}{dt} = v\Big[(a + 1)v - v^2 - a\Big] - \left(\frac{b}{r}\right) v + I_{\text{app}}.$$

$$\Rightarrow \frac{dv}{dt} = -v^3 + (a + 1)v^2 - \left(a + \frac{b}{r}\right) v + I_{\text{app}}.$$

This is a **cubic equation** in v. For the system to exhibit bistability, this cubic must have three real roots. For a cubic to have three real roots, its discriminant (D) must satisfy:

$$D > 0.$$

Assume $\frac{b}{r} = \alpha$ and $a = 0.5$. The cubic can be written as

$$pv^3 + qv^2 + rv + s = 0,$$

with

$$p = -1, \qquad q = 1.5, \qquad r = -(\alpha + 0.5), \qquad s = I_{\text{app}}.$$

The discriminant of a cubic is given by:

$$D = 18pqrs - 4q^3 s + q^2 r^2 - 4pr^3 - 27p^2 s^2.$$

Substituting the coefficients and simplifying:

$$D = 27(\alpha + 0.5)I_{\text{app}} - 13.5 I_{\text{app}} + 2.25(\alpha + 0.5)^2 - 4(\alpha + 0.5)^3 - 27(I_{\text{app}})^2 > 0.$$

Region of bistability

To determine the region of bistability, we plot the curve of α versus I_{app}. We assume:

$$I_{\text{app}} > 0, \qquad b > 0, \qquad r > 0.$$

The discriminant was plotted along the z-axis, with the x-axis denoting depicting I_{app} and the y axis depicting α, as shown in Figure 4.1, and it's projection an the xy plane was taken to obtain the region of interest as shown in Figure 4.2.

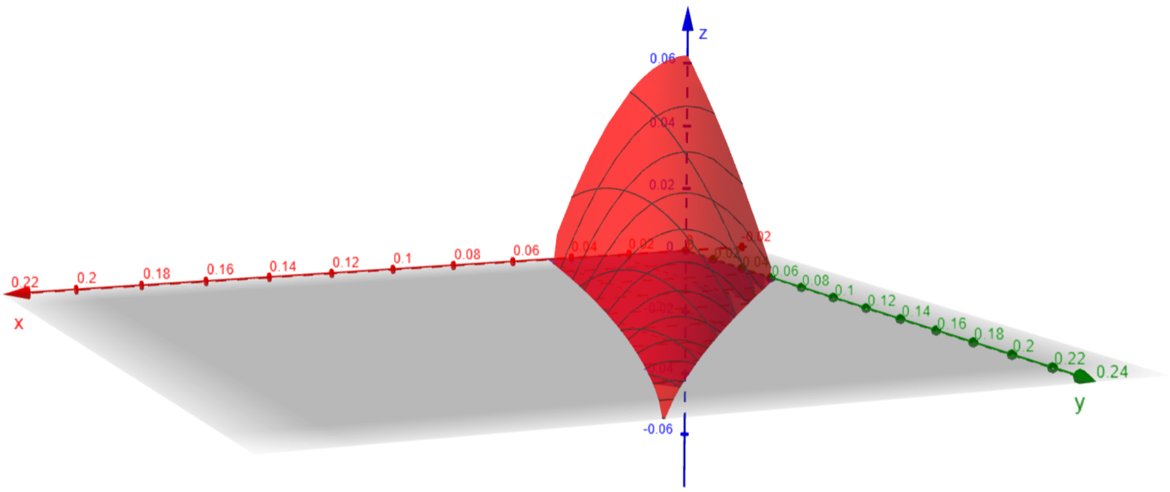


Figure 4.1

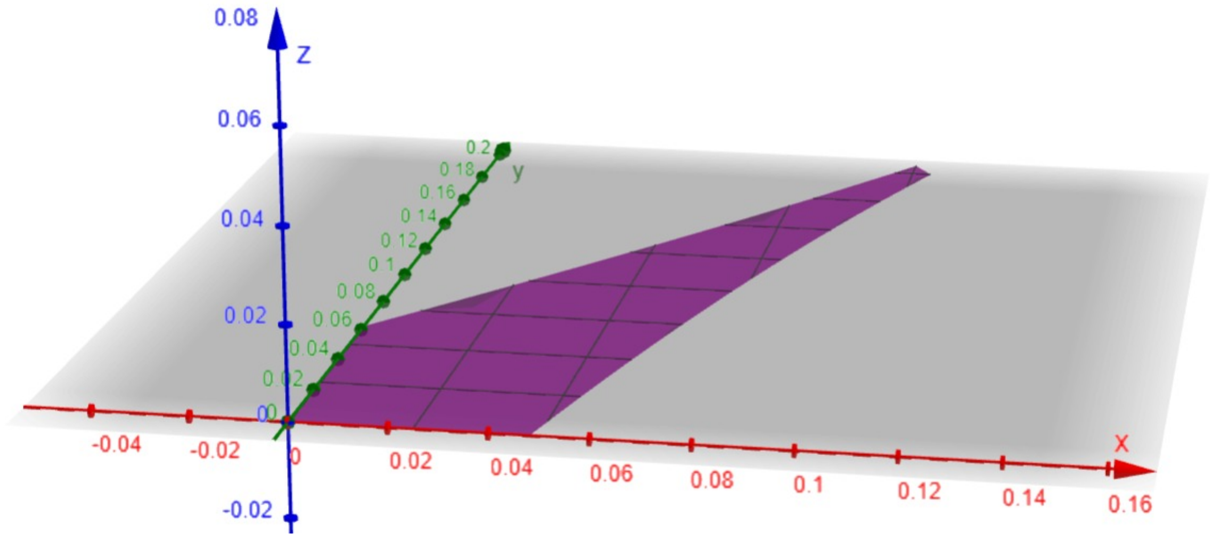


Figure 4.2

From the region $I_{app} = 0.05$ and $\alpha = \frac{b}{r} = 0.1$ was chosen and analyzed as shown below.

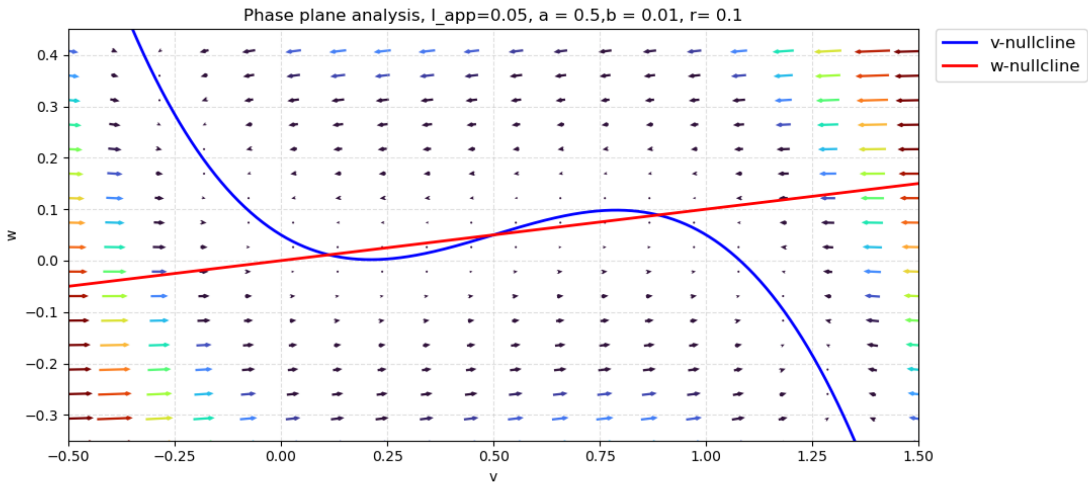


Figure 4.3

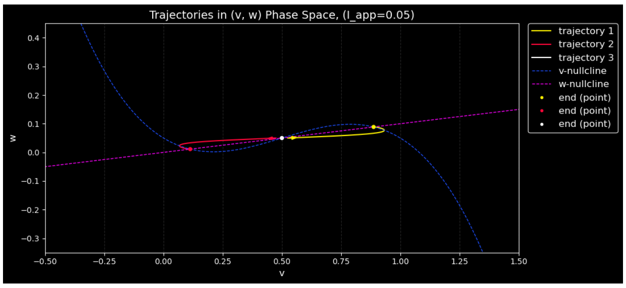


Figure 4.4.A

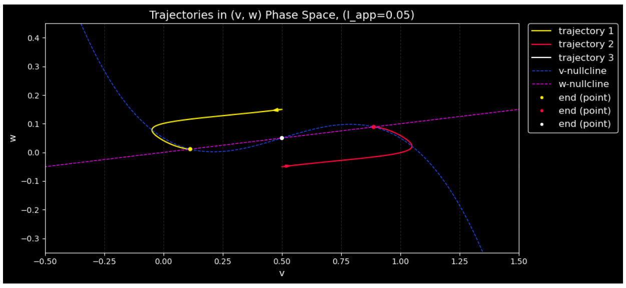


Figure 4.5.A

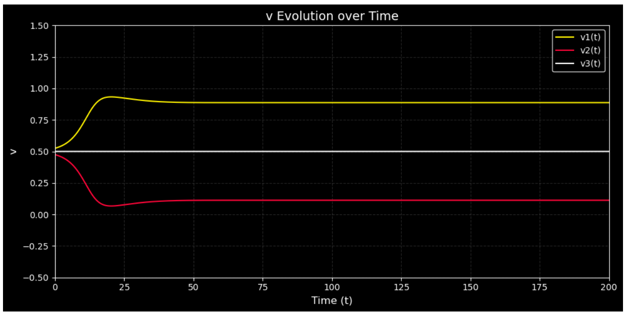


Figure 4.4.B

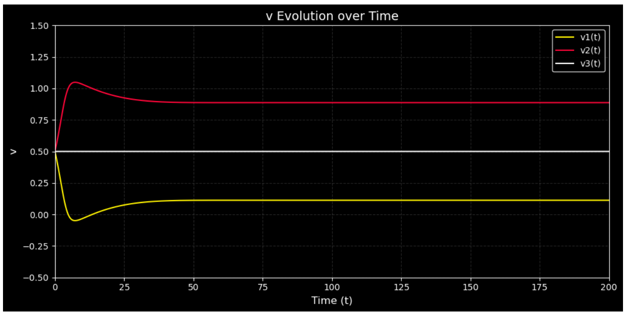


Figure 4.5.B

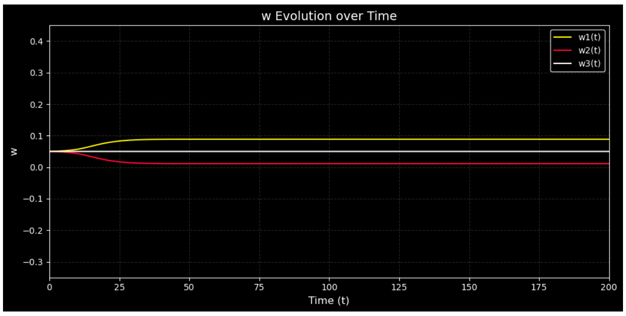


Figure 4.4.C

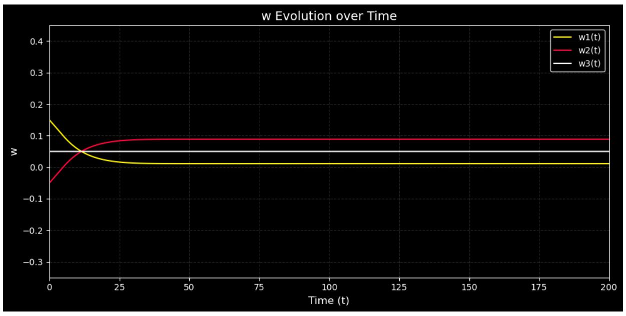


Figure 4.5.C

Fixed point Analysis

Using Figure 4.3 as a reference for the intersection points.

p1: $(v, w) = (0.113, 0.0113)$

$$f'(v) < 0 < \frac{b}{r}, \quad \Delta > 0$$
$$\tau = f'(v) < 0, \quad \text{stable}$$

p2: $(v, w) = (0.5, 0.05)$

$$f'(v) > \frac{b}{r}, \quad \Delta < 0$$
$$\tau = f'(v) > 0, \quad \text{saddle node}$$

p3: $(v, w) = (0.887, 0.0887)$

$$f'(v) < 0 < \frac{b}{r}, \quad \Delta > 0$$
$$\tau = f'(v) < 0, \quad \text{stable}$$

Note: Figures 4.4 and 4.5 illustrate perturbations from the saddle node (P2) along the v-axis and w-axis, respectively. In both cases, the trajectories evolve toward the stable nodes at P1 and P3.