Statistical Signal Processing

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1 Basic

1.1 Mean Square Error

$$
\begin{aligned}
MSE(\hat{\theta}) &= E[(\hat{\theta}-\theta)^2] \\
&= E[(\hat{\theta}-E[\hat{\theta}]+E[\hat{\theta}]-\theta)^2] \\
&= E[((\hat{\theta}-E[\hat{\theta}])^2-2(\hat{\theta}-E[\hat{\theta}])(E[\hat{\theta}]-\theta)+(E[\hat{\theta}]-\theta))^2]
\end{aligned}
$$

As you may notice, since the Expectation of sum is same as the sum of expectation, so the second term becomes $E[2(\hat{\theta}-E[\hat{\theta}])(E[\hat{\theta}]-\theta)]$, and since $(E[\hat{\theta}]-\theta)$ is constant, so we can move it out which becomes$(2E[\hat{\theta}]-\theta)E[(\hat{\theta}-E[\hat{\theta}])]$. And $E[(\hat{\theta}-E[\hat{\theta}])]=0$

$$= var(\hat{\theta})+(bias(\theta))^2$$

Intuitively, the Mean Square Error of an estimator is the square of the bias ,which make sense that if a estimator is biased itself then the error will not be small. And since the estimator is not a constant, its variance will also influence the MSE, which contributes to make more error.

1.2 Minimum Variance Unbiased

1.2.1 Unbiased

Definition:

$$E[\hat{\theta}=\theta], \theta \in (a,b)$$

This has to be true for *all* values
This estimation *does not always* exists

1.2.2 Minimum Variance

For whatever θ, the variance has to be minimum, which also makes this not always satisfiable.

2 Cramer Rao Lower Bound

Prerequisite **for calculating the lower bound**:

$$E[\hat{\theta}] = \theta$$

Prerequisite **for find the MVU or when the variance achieve the lower bound**:

$$\exists, I(\theta), s.t. \frac{\partial \ln p((x);\theta)}{\partial \theta} = I(\theta)(\hat{\theta} - \theta)$$

Conclusion:
For a scalar estimator, the minimum variance it could achieve is :

$$Var(\hat{\theta}) \geq \frac{1}{E[(\frac{\partial \ln p(\boldsymbol{x};\theta)}{\partial \theta})^2]} = \frac{1}{-E[\frac{\partial^2 \ln p(\boldsymbol{x};\theta)}{\partial^2 \theta}]}$$

If the lowerbound exists,

$$I(\theta) = -E[\frac{\partial^2 \ln p(\boldsymbol{x};\theta)}{\partial \theta^2}]$$

2.1 Tips

- if an estimator has same variance as CRLB, then it is called **Fully Efficient**
- if an estimator $\hat{A}$ is fully efficient for a parameter A, and another parameter B is linear to it(i.e.$B = 3A + 1$). The linear transformed estimator (i.e.$3\hat{A} + 1$)will be fully efficient for B too

- However, if $A = B^2$ and $\hat{A}$ is fully efficient for A, $\hat{B}^2$ is NOT fully efficient for B. It is actually not even unbiased. It is called asymptotic unbiasedness.This is the case for both mean and variance