LSAA#1774390, VOL 0, ISS 0

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STOCHASTIC ANALYSIS AND APPLICATIONS
https://doi.org/10.1080/07362994.2020.1774390

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Optimal dynamic reinsurance strategies in multidimensional portfolio

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ABSTRACT
The present paper addresses the issue of choosing an optimal dynamic reinsurance policy, which is state-dependent, for an insurance company that operates under multiple insurance business lines. For each line, the Cramer–Landberg model is adopted for the risk process and one of the contracts such as Proportional reinsurance, excess-of-loss reinsurance (XL) and limited XL reinsurance (LXL) is intended for transferring a portion of the risk to the reinsurance. In the optimization method used in this paper, the survival function is maximized with respect to to the dynamic reinsurance strategies. The optimal survival function is characterized as the unique nondecreasing viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation with limit one at infinity. The finite difference method (FDM) has been utilized for the numerical solution of the optimal survival function and optimal dynamic reinsurance strategies and the proof for the convergence of the numerical solution to the survival probability function is provided. The findings of this paper provide insights for the insurance companies as such that based upon the lines in which they are operating, they can choose a vector of the optimal dynamic reinsurance strategies and consequently transfer some part of their risks to several reinsurers. With the numerical examples, the significance of the elicited results in reducing the probability of ruin is demonstrated in comparison with the previous findings.

ARTICLE HISTORY
Received 18 March 2019
Accepted 21 May 2020

KEYWORDS
Cramer–lundberg process; optimal reinsurance; Hamilton–Jacobi–Bellman equation; viscosity solution; dynamic programing principle

1. Introduction

An effective way for an insurance firm to manage its risk is to buy reinsurance. According to the reinsurance contract, some parts of the claim are shared with the reinsurer, in return, the insurance firm pays a part of its income premium to the reinsurance. Determination of the optimal reinsurance contracts has been discussed extensively in the literature. Dynamic proportional reinsurance in the classical risk model for the minimization of the ruin probability was first studied by Schmidli [1]. Hipp and Vogt [2] utilized the concept of dynamic excess of loss reinsurance to extend the Schimidli approach. Schmidli [3], have suggested the optimal investment and reinsurance strategies to minimize the ruin probability and have concluded that the investment and reinsurance decrease the ruin probability for larger initial surplus under

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the Pareto claim sizes. In this direction, Taksar and Markussen [4] and Schmidli [5] developed the above approach to the diffusion model. Subsequently, Irgens and Paulsen [6] discussed the maximization of the expected utility of the asset of an insurance company under the reinsurance and investment constraints in a diffusion classical risk model. A general presentation on ruin probability minimization by means of reinsurance in a classical and diffusion risk models can be found in Schmidli [7]. Some additional results with a focus on non-proportional reinsurance contracts are given in Hipp and Taksar [8]. Recently, Cani and Thonhauser [9] studied a dynamic reinsurance problem obtained from an economical valuation criterion in risk theory introduced by Højgaard and Taksar [10].

In this paper, we assume that the insurance company produces multiple types of coverage, where customers may purchase different types of insurance policies (such as fire, health, vehicle, etc.). Due to the different risk processes in different lines, it is reasonable that the insurance companies use several reinsurances to share their risk. For instance, it is possible for an insurance company to purchase a proportional reinsurance in one line and an excess-of-loss reinsurance in another line or buy one type of excess-of-loss in one line and a different type of excess-of-loss insurance in another line. The survival function, in this paper, is considered as the objective function, and the vector of reinsurance strategies is obtained in such a way that the objective function is maximized; therefore, the results presented in Azcue and Muler [11], which use a dynamic reinsurance strategy for transferring risk to reinsurers, have been generalized in such a way that the insurer uses the vector of dynamic reinsurance strategies to transfer risk to several reinsurers.

In the second section of the paper, a brief introduction of the model with the presence of the reinsurer and the statement of the problem are provided. In the third section, the main results and in the fourth section, the numerical examples are presented.

2. Model formulation

In the classical Cramer–Lundberg process, the reserve X_t of an insurance company can be described by

$$X_t = x + pt - \sum_{i=1}^{N_t} U_i, \tag{1}$$

where N_t is a Poisson process with claim arrival intensity $\beta > 0$ and the claims size U_i are i.i.d random variables with distribution F. The premium rate p is calculated using the expected value principle with relative safety loading $\eta > 0$; that is, $p = (1+\eta)\beta E(U_i)$. The limitation of this model is the assumption that insurers produce only one type of insurance, but in practice, most insurers produce different types of coverage. (e.g. automobile insurance, fire insurance, workers' compensation insurance, etc.). The idea for modeling the surplus process for a company that produces multiple types of coverage is as follows: consider the process $X_t = (X_t^1, \ldots, X_t^n)$ defined as;

$$X_t^{(k)} = p_k t - \sum_{i=1}^{N_t^{(k)}} U_i^{(k)}, \qquad k = 1, \ldots, n \tag{2}$$

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where $N_t^{(k)}$ is a Poisson process with claim arrival intensity $\beta_k > 0$ and the claims size $U_i^{(k)}$'s are i.i.d random variables with distribution F_k. Let the risk process of the kth line of insurance company be modeled by $X_t^{(k)}$. The premium rate for line k, p_k, is calculated using the expected value principle with relative safety loading $\eta_k > 0$; that is, $p_k = (1+\eta_k)\beta_k E(U_i^{(k)})$. Given an initial surplus x, the surplus Y_t of the insurance company at time t can be written as $Y_t = x + \sum_{k=1}^{n} X_t^{(k)}$ and if $X_t^{(1)}, \ldots, X_t^{(n)}$ are independent random variables, then Y_t has a compound Poisson distribution, that is,

$$Y_t = x + \left(\sum_{k=1}^{n} p_k\right)t - \sum_{i=1}^{N_t} U_i, \qquad k = 1, \ldots, n \tag{3}$$

where N_t is a Poisson process with claim arrival intensity $\beta = \sum_{i=1}^{n} \beta_i$ and the claims size U_i's are i.i.d random variables with distribution $\sum_{i=1}^{n} \frac{\beta_i}{\sum_{i=1}^{n} \beta_i} F_i$. Let $(\Omega_k, \Sigma_k, (\mathcal{F}_{kt})_{t\geq 0}, P_k)$ be the filtered probability space corresponds to line k, then, we can describe filtered probability space model by

$$(\Omega, \Sigma, (\boldsymbol{\mathcal{F}}_t)_{t\geq 0}, \boldsymbol{P}) = (\Omega_1, \Sigma_1, (\mathcal{F}_{1t})_{t\geq 0}, P_1) \times \cdots \times (\Omega_n, \Sigma_n, (\mathcal{F}_{nt})_{t\geq 0}, P_n). \tag{4}$$

Reinsurance can be an effective way to manage risk by transferring risk from an insurer to a second insurer (referred to as the reinsurer). A reinsurance contract is an agreement between an insurer and a reinsurer under which, claims that arise are shared between the insurer and reinsurer.

Let a Borel measurable function $R : [0, \infty) \to [0, \infty)$, called retained loss function, describing the part of the claim that the company pays and satisfies $0 \leq R(\alpha) \leq \alpha$. The reinsurance company covers $\alpha - R(\alpha)$, where the size of the claim is α. Now assume that in order to reduce the risk exposure of the portfolio, the insurer has the possibility to take reinsurances in a dynamic way for some insurance lines, each of these reinsurances is indexed by $\{1, \ldots, n\}$. We denote by $\boldsymbol{\mathcal{R}}$ the vector $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$, in which $\mathcal{R}_i$ is the family of retained loss functions associated to the reinsurance policy in i'th line, and denote by $\Pi_x^{\boldsymbol{R}}$ the set of all control strategies with initial surplus $x \geq 0$. So, the reinsurances control strategy is a collection $\boldsymbol{R} = (\boldsymbol{R}_t)_{t\geq 0} = (R_{1t}, \ldots, R_{nt})_{t\geq 0}$ of the vector functions $\boldsymbol{R}_t : \Omega \to \boldsymbol{\mathcal{R}}$ for any $t \geq 0$.

Well-known reinsurance types are:

(1) Proportional reinsurance with $R_P(\alpha) = b\alpha$,
(2) Excess-of- loss reinsurance (XL) with $R_{XL}(\alpha) = \min(\alpha, M), 0 \leq M \leq \infty$,
(3) Limited XL reinsurance (LXL) with $R_{LXL}(\alpha) = \min\{(\alpha, M)\} + (\alpha - M - L)^+$, $0 \leq M,\ L \leq \infty$.

In this paper, we assume that in line k, $k = 1, \ldots, n$, the reinsurance calculates its premium using the expected value principle with reinsurance safety loading factor $\eta_{1k} \geq \eta_k > 0$,

$$q_{R_k} = (1+\eta_{1k})\beta_k E(U_i^{(k)} - R(U_i^{(k)}))$$

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and so $p_{R_k} = p_k - q_{R_k}$. Now, the surplus process in the presence of reinsurance strategies can be written as

$$X_t^{\boldsymbol{R}} = x + \sum_{k=1}^{n} \left(\int_0^t p_{R_{ks}} ds - \sum_{i=1}^{N_t^{(k)}} R_{k\tau_i}(U_i^{(k)}) \right) \tag{4}$$

$$= x + \int_0^t \sum_{k=1}^{n} p_{R_{ks}} ds - \sum_{k=1}^{n} \sum_{i=1}^{N_t^{(k)}} R_{k\tau_i}(U_i^{(k)}). \tag{5}$$

Without losing the generality, we can write

$$\begin{aligned} X_t^{\boldsymbol{R}} &= x + \int_0^t \sum_{k=1}^{n} p_{R_{ks}} ds - \sum_{i=1}^{N_t} Z_i \\ &= x + \int_0^t p_{\boldsymbol{R}_s} ds - \sum_{i=1}^{N_t} Z_i, \end{aligned} \tag{6}$$

where Z_i's are i.i.d random variables with distribution $F_{\boldsymbol{R}} = \sum_{i=1}^{n} \frac{\beta_i}{\sum_{i=1}^{n} \beta_i} F_{R_i}$. The time of ruin for this process is defined by

$$\tau^{\boldsymbol{R}} = \inf\{t \geq 0 : X_t^{\boldsymbol{R}} < 0\}. \tag{7}$$

In this paper, we aim to identify the reinsurance strategies $\boldsymbol{R} = (\boldsymbol{R}_t)_{t\geq 0} = (R_{1t}, \ldots, R_{nt})_{t\geq 0}$ that maximize the survival probability $\delta^{\boldsymbol{R}}(x) = P(\tau^{\boldsymbol{R}} = \infty | X_0^{\boldsymbol{R}} = x)$; in other words, we are looking for the optimal survival function

$$\delta(x) = \sup_{\boldsymbol{R} \in \Pi_x^{\boldsymbol{R}}} \delta^{\boldsymbol{R}}(x). \tag{8}$$

From now forward, where we use $\delta(.)$, we take it to mean the optimal survival function. It is easy to show, similar to section 2.1.1 of Azcue and Muler [11], that the HJB equation of this problem can be written as

$$\sup_{\mathcal{R}} \quad \mathcal{L}_{\boldsymbol{R}}(\delta)(x) = 0, \tag{9}$$

where

$$\mathcal{L}_{\boldsymbol{R}}(\delta)(x) = p_{\boldsymbol{R}} \delta'(x) - \left(\sum_{i=1}^{n} \beta_i \right) \delta(x) + \left(\sum_{i=1}^{n} \beta_i \right) \int_0^x \delta(x - \alpha) dF_{\boldsymbol{R}}(\alpha). \tag{10}$$

3. Main results

The dynamic programing method is a cogent means to scrutinize the stochastic control problems through the HJB equation. In (9), we have obtained the associated equation to the value function (8). Nonetheless, in the classical approach, this method is adopted only when it is assumed *a priori* that the optimal value functions are smooth enough. In general, the optimal value function is not expected to be smooth enough to satisfy these equations in the classical sense. These highlight the need for a week notation of

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solution of the HJB equation: the theory of viscosity solutions. Let us define this notion (see [11]).

Definition 3.1. We say that a function $\underline{u} : [0, \infty) \to \mathbb{R}$ is a viscosity subsolution of (9) at $x \in (0, \infty)$ if it is locally Lipschitz and any continuously differentiable function $\varphi : (0, \infty) \to \mathbb{R}$ (called test function) with $\varphi(x) = \underline{u}(x)$ such that $\underline{u} - \varphi$ reaches the maximum at x and satisfies $sup_{\mathcal{R}} \ \ \mathcal{L}_{\boldsymbol{R}}(\varphi)(x) \geq 0$. We say that a continuous function $\bar{u} : [0, \infty) \to \mathbb{R}$ is a viscosity supersolution of (9) at $x \in (0, \infty)$ if it is locally Lipschitz any continuously differentiable function $\phi : (0, \infty) \to \mathbb{R}$ (called test function) with $\phi(x) = \bar{u}(x)$ such that $\bar{u} - \phi$ reaches the maximum at x and satisfies $sup_{\mathcal{R}} \ \ \mathcal{L}_{\boldsymbol{R}}(\phi)(x) \leq 0$. Finally, we say that a continuous function $u : [0, \infty) \to \mathbb{R}$ is a viscosity solution of (9) if it is both a viscosity subsolution and a viscosity supersolution at any $x \in (0, \infty)$.

The methods which will be used later are gleaned from Azcue and Muler [11] and Nozadi [12].

3.1. Viscosity solution

In chapter 3 of Azcue and Muler [11], there is an example that the survival probability function in problem with reinsurance can be non-differentiable. Hence, in general, we cannot expect for δ to have the smoothness properties needed to regard it as a solution (in the classical sense) for the corresponding HJB Equation (9). We prove instead that δ is a viscosity solution of the corresponding HJB equation. Before stating the main results, the following lemmas are needed.

Lemma 3.1. *Consider an arbitrary admissible strategy* $\boldsymbol{R} = (\boldsymbol{R}_t)_{t \geq 0} \in \Pi_x^{\boldsymbol{R}}$ *and set* $x \geq 0$. *Then* $\delta^{\boldsymbol{R}}(X^{\boldsymbol{R}}_{t \wedge \tau^{\boldsymbol{R}}})$ *is a martingale.*

Lemma 3.2. $0 \leq \delta(x) \leq 1$ *for all* $x \geq 0$, $\lim_{x \to \infty} \delta(x) = 1$, δ *is increasing, and it is Lipschitz with Lipschitz constant* $K = \beta / (\sup_{\boldsymbol{\mathcal{R}} \in \mathcal{R}} \sum_{i=1}^{n} p_{R_i})$.

Lemma 3.3. *Let a nonnegative continuously differentiable function* $u : \mathbb{R}_+ \to \mathbb{R}$. *Then for any finite stopping time* $\tau \leq \tau^{\boldsymbol{R}}$, *the following equality is true,*

$$u(X^{\boldsymbol{R}}(t)) - u(x) = \int_0^{\tau} \mathcal{L}_{\mathcal{R}}(u)(X^{\boldsymbol{R}}_{s^-}) ds + M_{\tau},$$

where $X^{\boldsymbol{R}}_{\tau}, \tau^{\boldsymbol{R}}$ *and* $\mathcal{L}_{\boldsymbol{\mathcal{R}}}$ *are defined in (4), (7) and (10) and* M_{τ} *is a martingal with zero expectation.*

Lemma 3.4. *Let the vector* $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, ..., \mathcal{R}_n)$, *where* $\mathcal{R}_i$ *is one of the reinsurance families* $\mathcal{R}_P, \mathcal{R}_{XL}$ *and* $\mathcal{R}_{LXL}$. *If u is a nonnegative and a twice continuously differentiable function defined in* $\boldsymbol{R}_+$ *(extended as* $u(x) = 0$ *for* $x < 0$*), then* $H = sup_{\boldsymbol{R} \in \boldsymbol{\mathcal{R}}} \mathcal{L}_{\boldsymbol{R}}(u)(.)$ *is upper semicontinuous. Moreover, for any* $A > 0$ *and* $h \in (0, 1)$, *there exists* $K_A > 0$ *such that*

$$|H(y) - H(x_0)| \leq K_A(y - x_0 + F(y) - F(x_0)), \qquad 0 \leq x_0, \quad y \leq A.$$

Now the following theorem is obtained.

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Theorem 3.1. *Let the vector $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$, where $\mathcal{R}_i$ is one of the reinsurance families $\mathcal{R}_P, \mathcal{R}_{XL}$ and $\mathcal{R}_{LXL}$. Then, the function δ is a viscosity solution of (9).*

Proof. The proof of this theorem is quite similar to the Proposition 3.2 of Azcue and Muler [11]. Indeed, applying the Lemmas 3.1, 3.2, 3.3 and 3.4 and repeating the proof of the Proposition 3.2 of Azcue and Muler [11] provide the proof. □

3.2. Characterization

In this section, it will be proved that the probability function defined in (8) is the unique viscosity solution of the HJB (9) with limit one at infinity. In order to prove the uniqueness result, we state the three following lemmas.

Lemma 3.5. *Suppose that $\bar{u}$ is a nondecreasing Lipschitz viscosity supersolution of (9) such that*

$$\lim_{x\to\infty} \bar{u}(x) = 1.$$

A sequence of positive functions $\bar{u}_n : \mathbb{R}_+ \to \mathbb{R}$ are found such that

(1) *$\bar{u}_n$ is infinitely continuously differentiable and $\bar{u}_n' \leq K$, where K is the Lipschitz constant of $\bar{u}$,*
(2) $\lim_{x\to\infty} \bar{u}_n(x) = 1$,
(3) *$\bar{u}_n \searrow \bar{u}$ uniformly in $\mathbb{R}_+$ and $\bar{u}_n'(x)$ converges to $\bar{u}'(x)$ a.e,*
(4) *There is a sequence $c_n > 0$ with $\lim_{n\to\infty} c_n = 0$ such that*

$$\sup_{\boldsymbol{R}\in\boldsymbol{\mathcal{R}},\, x\geq 0} \mathcal{L}_{\boldsymbol{R}}(\bar{u}_n)(x) \leq c_n.$$

Lemma 3.6. *Suppose that $\underline{u}$ is a nondecreasing Lipschitz viscosity supsolution for (9), such that*

$$\lim_{x\to\infty} \underline{u}(x) = 1.$$

A sequence of positive functions $\underline{u}_n : \mathbb{R}_+ \to \mathbb{R}$ are found such that

(1) $\underline{u}_n$ is infinitely continuously differentiable and $\underline{u}_n' \leq K$, where K is the Lipschitz constant of $\underline{u}$,
(2) $\lim_{x\to\infty} \underline{u}_n(x) = 1$,
(3) $\underline{u}_n \nearrow \underline{u}$ uniformly in $\mathbb{R}_+$ and $\underline{u}_n'(x)$ converges to $\underline{u}'(x)$ a.e,
(4) There is a sequence $c_n > 0$ with $\lim_{n\to\infty} c_n = 0$ such that

$$\sup_{\boldsymbol{R}\in\boldsymbol{\mathcal{R}},\, x\geq 0} \mathcal{L}_{\boldsymbol{R}}(\underline{u}_n)(x) \geq -c_n.$$

Lemma 3.7. *Consider twice continuously differentiable function $\bar{u}$. Then, for $\epsilon > 0$, there exists a stationary reinsurance strategy $\boldsymbol{r} = (r_1, \ldots, r_n)$ satisfying the following,*

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$$\sup_{\boldsymbol{R}\in\mathcal{R}} \mathcal{L}_{\boldsymbol{R}}(x) - \mathcal{L}_{\boldsymbol{r}}(x) < \epsilon, \qquad x \geq 0.$$

Proposition 3.1. *$\delta(.)$ is both the smallest viscosity supersolution and the largest viscosity sub-solution of HJB (9), with limit one at infinity.*

Proof. To prove this theorem, it suffices to repeat the proof of Theorem 4.3 of Azcue and Muler [11] by replacing $\boldsymbol{R} = (\boldsymbol{R}_t)_{t\geq 0} = (R_{1t}, \ldots, R_{nt})_{t\geq 0}$ with $\bar{R}$. At first, it is shown that $\delta(.)$ is smaller or equal to that any supersolution. Assume that $\bar{u}$ is a nondecreasing viscosity supersolution satisfying in (9), with $\lim_{x\to\infty} \bar{u}(x) = 1$. Consider an arbitrary admissible strategy $\boldsymbol{R} = (\boldsymbol{R}_t)_{t\geq 0} \in \Pi_x^{\boldsymbol{R}}$ and $x \geq 0$. Denote by X_t, the controlled risk process with initial surplus x corresponding to $\boldsymbol{R}$ and let τ be its ruin time. For any $M > x$, define the following stopping time

$$\tau_M = \inf\{t \geq 0 : X_t^{\boldsymbol{R}} \geq M\}.$$

Consider the function $\bar{u}_n$ defined in Lemma 3.5, and set $\bar{u}_n = 0$ in $(-\infty, 0)$. From Lemma 3.3 and part (4) of Lemma 3.5, it follows that:

$$\begin{aligned}\bar{u}_n(X_{\tau_M\wedge\tau\wedge t}) - \bar{u}_n(x) &= \int_0^{\tau_M\wedge\tau\wedge t} \mathcal{L}_{\boldsymbol{R}_s}(\bar{u}_n)(X_{s^-})ds + M_{\tau_M\wedge\tau\wedge t} \\ &\leq c_n(\tau_M\wedge\tau\wedge t) + M_{\tau_M\wedge\tau\wedge t},\end{aligned}$$

wherein M_t is martingale with zero mean. Thus, the following inequality is obtained

$$E_x(\bar{u}_n(X_{\tau_M\wedge\tau\wedge t})) - \bar{u}_n(x) \leq c_n t.$$

Now, taking limit from both sides of the above relation when $n \to \infty$, for fix t, the next inequality is established:

$$\limsup_{n\to\infty} E_x(\bar{u}_n(X_{\tau_M\wedge\tau\wedge t})) - \bar{u}_n(x) \leq 0,$$

and so, since $\lim_{n\to\infty} \bar{u}_n(x) = \bar{u}(x)$ and $\bar{u}(x) \leq \bar{u}_n(x)$, $E_x(\bar{u}(X_{\tau_M\wedge\tau\wedge t})) - \bar{u}(x) \leq 0$. Therefore, when $t \to \infty$, the following inequality is obtained

$$E_x(\bar{u}(X_{\tau_M\wedge\tau\wedge t})) - \bar{u}(x) = \bar{u}(M)P(\tau_M < \tau) - \bar{u}(x) \leq 0.$$

Now, taking $M \to \infty$, and using the fact that, $\lim_{M\to\infty} \bar{u}(M) = 1$ and $\lim_{M\to\infty} P(\tau_M < \tau) = \delta^{\boldsymbol{R}}(x)$ it follows that

$$\delta^{\boldsymbol{R}}(x) \leq \bar{u}(x) \qquad \textit{for} \quad \textit{all} \quad \boldsymbol{R} \in \Pi_x^{\boldsymbol{R}},$$

which in turn results in $\delta(x) \leq \bar{u}(x)$.

We shall now prove that $\delta(.)$ is greater or equal than any subsolution. Assume that $\underline{u}$ is a nondecreasing subsolution in (9) with $\lim_{x\to\infty} \underline{u}(x) = 1$. Consider the functions $\underline{u}_n$ defined in Lemma 3.6, and set $\underline{u}_n = 0$ in $(-\infty, 0)$. Based on Lemma 3.7, for any $y \geq 0$ and $n \geq 1$, there exists a stationary reinsurance control $\boldsymbol{r}_n$ such that,

$$\sup_{\boldsymbol{R}\in\mathbb{R}} \mathcal{L}_{\boldsymbol{R}}(\underline{u})(y) - \mathcal{L}_{\boldsymbol{r}_n}(\underline{u}_n)(y) \leq \frac{1}{n}.$$

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Consider the controlled process $(X_t^n)_{t\geq 0}$ with initial surplus x and admissible reinsurance strategy $\boldsymbol{R}^n = (\boldsymbol{R}_t^n)_{t\geq 0}$ and denote by τ^n, the corresponding ruin time. For each $M > 0$, define the following stopping time:

$$\tau_M^n = \inf\{t \geq 0 : X_t^n \geq M\}.$$

From Lemma 3.3 and Lemma 3.6, it follows that for each n,

$$\begin{aligned}\underline{u}_n(X^n_{\tau_M^n\wedge\tau^n\wedge t}) - \underline{u}_n(x) &= \int_0^{\tau_M^n\wedge\tau^n\wedge t} \mathcal{L}_{\boldsymbol{R}_s^n}(\underline{u}_n)(X_{s^-}^n)ds + M_{\tau_M^n\wedge\tau^n\wedge t}\\ &\geq \int_0^{\tau_M^n\wedge\tau^n\wedge t} \sup_{\boldsymbol{R}\in\mathbb{R}}\left(\mathcal{L}_{\boldsymbol{R}}(\underline{u}_n)(X_{s^-}^n) - \frac{1}{n}\right)ds + M_{\tau_M^n\wedge\tau^n\wedge t}\\ &\geq \left(-c_n - \frac{1}{n}\right)(\tau_M^n\wedge\tau^n\wedge t) + M_{\tau_M^n\wedge\tau^n\wedge t},\end{aligned}$$

wherein M_t is a martingale with mean zero. So, the following inequality is obtain by

$$E_x\left(\underline{u}_n(X^n_{\tau_M^n\wedge\tau^n\wedge t}) - \underline{u}_n(x)\right) \geq \left(-c_n t - \frac{1}{n}\right)E_x\left(\tau_M^n\wedge\tau^n\wedge t\right).$$

Now, taking limit from both sides of the above relation when $n\to\infty$, for a fix t, the next inequality is established as,

$$\limsup_{n\to\infty} E_x\left(\underline{u}_n(X^n_{\tau_M^n\wedge\tau^n\wedge t})\right) \geq \underline{u}(x),$$

and so, since $\underline{u}(x) \geq \underline{u}_n(x)$,

$$\limsup_{n\to\infty} E_x\left(\underline{u}(X^n_{\tau_M^n\wedge\tau^n\wedge t})\right) \geq \underline{u}(x).$$

For $\epsilon > 0$, take n_0 large enough so that

$$E_x\left(\underline{u}(X^{n_0}_{\tau_M^{n_0}\wedge\tau^{n_0}\wedge t})\right) \geq \underline{u}(x) - \epsilon.$$

Thus,

$$\begin{aligned}\underline{u}(x) - \epsilon &\leq E_x\left(\underline{u}(X^{n_0}_{\tau_M^{n_0}\wedge\tau^{n_0}\wedge t})\right)\\ &= \underline{u}(M)P(\tau_M^{n_0}\wedge\tau^{n_0}\wedge t) + E_x\left(\underline{u}(X^{n_0}_{\tau_M^{n_0}\wedge\tau^{n_0}\wedge t})I_{t\leq\tau_M^{n_0}\wedge\tau^{n_0}}\right).\end{aligned}$$

Note that $P(\tau_M^{n_0}\wedge\tau^{n_0}\wedge t)$ is nondecreasing in t, $0 \leq \underline{u} \leq 1$, and $\lim_{t\to\infty} P(\tau_M^{n_0}\wedge\tau^{n_0} > t) = 0$, which result in

$$\underline{u}(x) - \epsilon \leq \liminf_{t\to\infty} E_x\left(\underline{u}(X^{n_0}_{\tau_M^{n_0}\wedge\tau^{n_0}\wedge t})\right) = \underline{u}(M)P(\tau_M^{n_0} < \tau^{n_0}).$$

Now, taking $M\to\infty$, the following result is established:

$$\underline{u}(x) - \epsilon \leq \delta^{\boldsymbol{R}^{n_0}} \leq \delta(x),$$

and hence $\underline{u}(x) \leq \delta(x)$. □

In the next theorem, the optimal survival function is characterized. The theorem provided below is the cospicuously obvious result of Theorem 3.1 and Proposition 3.1.

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Theorem 3.2. *$\delta(.)$ is the unique nondecreasing viscosity solution of (9) with limit one at infinity.*

We summarize these results in the following corollary.

Corollary 3.1. *(Verification) If the survival probability function of a vector of reinsurance admissible strategies is a viscosity solution of the HJB equation (9) with limit one at infinity, then the vector of reinsurance admissible strategies and its survival probability function are optimal.*

3.3. Numerical solution

Let us define the function

$$\hat{\delta}(x) = \inf_{\boldsymbol{R}\in\mathcal{R}} \frac{(\sum_{i=1}^{n}\beta_i)\delta(x) - (\sum_{i=1}^{n}\beta_i)\int_0^x \delta(x-\alpha)dF_{\boldsymbol{R}}(\alpha)}{p_{\boldsymbol{R}}}.$$

Similarly to the Lemma 5.6 of Azcue and Muler [11], we can show that $\hat{\delta}(x)$ is well defined, nonnegative, Borel measurable, $\hat{\delta} = \delta'$ a.e., and

$$\sup_{\boldsymbol{R}\in\mathcal{R}}\{p_{\boldsymbol{R}}\hat{\delta}(x) - \left(\sum_{i=1}^{n}\beta_i\right)\delta(x) + \left(\sum_{i=1}^{n}\beta_i\right)\int_0^x \delta(x-\alpha)dF_{\boldsymbol{R}}(\alpha)\} = 0.$$

In this section, using FDM, we try to solve numerically the problem of optimal reinsurance and optimal survival function. The utilization of the FDM is prevalent for solving the HJB equations in stochastic control problem and these numerical solutions are usually converges to the viscosity solution (see [13–1512]). A numerical solution for $sup_{\boldsymbol{R}\in\mathcal{R}}\ \ \mathcal{L}_{\mathcal{R}}(f)(x) = 0$, can be obtained by the use of FDM and adaptation of the boundary condition $\lim_{x\to\infty} f(x) = 1$. Similar to the procedure described in Fleming and Soner [14], section IX.3 or Nozadi [12], section 3.3, we discretize the state space with sufficiently small step size h and define a family of functions f_h in the following procedure: starting with

$$f_h(0) = 1, \text{ and } f'_h(0) = \inf_{\boldsymbol{R}\in\mathcal{R}} \beta\frac{1 - p(\boldsymbol{R}(Y) = 0)}{p_{\boldsymbol{R}}},$$

and for $s = ih,\ i = 1, 2, \ldots,$ we approximate $\int_0^x f(x-\alpha)dF_R(\alpha)$ by

$$G_{\boldsymbol{R}}(s) = \sum_{\{j\le i\}} f_h((i-j)h)P\{(j-1)h < \boldsymbol{R}(Y) \le jh\}.$$

It is easy to show that $G_{\boldsymbol{R}}(s)$ converges to $\int_0^s f(x-\alpha)dG_R(\alpha)$ as h tends zero. Then we define $f'_h(s)$ by

$$f'_h(s) = \inf_{\boldsymbol{R}} \frac{\beta h(f_h(s-h) - G_{\boldsymbol{R}}(s))}{p_{\boldsymbol{R}}}, \tag{11}$$

and set $f_h(s) = f_h(s-h) + hf'_h(s)$.

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Lemma 3.8. *Let some small step size h such that $p_R \geq 0$, $i = 1, 2, \ldots$, and let $D_h = \{ih, i = 0, 1, \ldots\}$. Then*

(i) $f\prime_h(s) \geq 0$ for all $s \in D_h$,
(ii) for all $k \geq 0$, the following inequalities are true,

$$f_h(kh) \leq (1 - \frac{\beta}{p_R} h)^{-k} \leq e^{\frac{\beta}{p_R} kh},$$

and

$$f'_h(kh) \leq \frac{\beta}{p_R} f_h(kh).$$

In the next Proposition, we use the same argument as in Nozadi [12], section 3.3, to demonstrate that the function $f_h/f_h(\infty)$ converges to the unique viscosity δ.

Proposition 3.2. *In the setting of the above lemma, and define*

$$f^*(s) = \lim_{h \to 0} \sup_{ih \to s} f_h(ih), \tag{12}$$

and

$$f_*(s) = \lim_{h \to 0} \inf_{ih \to s} f_h(ih). \tag{13}$$

Then the functions $f^*(s)$ and $f_*(s)$ are respectively, subviscosity and superviscosity solution of (9).

Proof. First, we show that the function f^* is locally Lipschitz and a viscosity subsolution of (9). Fix $M > 0$ and let $0 \leq x < y \leq M$ be arbitrary and take a sequence $h_n \to 0$ as $n \to \infty$ such that: for any number $\epsilon > 0$ there exists some number n_0 and two sequences $k_n^{(1)} \in \mathbb{N}$ and $k_n^{(1)} \in \mathbb{N}$ such that for all $n \geq n_0$ we have $|f_{h_n}(k_n^{(2)} h_n) - f^*(y)| < \epsilon$, $|k_n^{(1)} h_n - x| < \epsilon$ and $|k_n^{(2)} h_n - y| < \epsilon$. It is easy to see that

$$\begin{aligned} f^*(y) - f^*(x) &= \lim_{n \to \infty} f_{h_n}(k_n^{(2)} h_n) - f^*(x) \\ &\leq \lim_{n \to \infty} f_{h_n}(k_n^{(2)} h_n) - \lim_{n \to \infty} f_{h_n}(k_n^{(1)} h_n) \\ &\leq \lim_{n \to \infty} (f_{h_n}(k_n^{(2)} h_n) - f_{h_n}(k_n^{(1)} h_n)). \end{aligned}$$

Now, according to the above inequality and Lemma 3.8, the following relation is obtained:

$$f^*(y) - f^*(x) \leq K(y - x),$$

where K is a common Lipschitz constant of $f_{h_n}(t), 0 \leq t \leq M$.

To show that f^* is a viscosity subsolution, suppose that w is a test function such that $f^*(x) - w(x)$ has a maximum at $s > 0$. Therefore, for h_h sufficiently small, we have

$$\begin{aligned} f_{h_n}(k_n h_n + h_n) - w(k_n h_n + h_n) &\leq f_{h_n}(k_n h_n) - w(k_n h_n), \\ f_{h_n}(k_n h_n + h_n) - f_{h_n}(k_n h_n) &\leq w(k_n h_n + h_n) - w(k_n h_n). \end{aligned}$$

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By definition,

$$\begin{aligned}
0 &= \sup_{\mathcal{R}} \quad \{p_{\boldsymbol{R}} f'_{h_n}(k_n h_n + h_n) - \beta f_{h_n}(k_n h_n + h_n) + \beta G_{\boldsymbol{R}}(k_n h_n + h_n)\} \\
&= \sup_{\mathcal{R}} \quad \{p_{\boldsymbol{R}} \frac{f_{h_n}(k_n h_n + h_n) - f_{h_n}(k_n h_n)}{h_n} - \beta f_{h_n}(k_n h_n + h_n) + \beta G_{\boldsymbol{R}}(k_n h_n + h_n)\} \\
&\leq \sup_{\mathcal{R}} \quad \{p_{\boldsymbol{R}} \frac{w(k_n h_n + h_n) - w(k_n h_n)}{h_n} - \beta f_{h_n}(k_n h_n + h_n) + \beta G_{\boldsymbol{R}}(k_n h_n + h_n)\} \\
&= \sup_{\mathcal{R}} \quad \{p_{\boldsymbol{R}} w'(k_n h_n) - \beta f_h(k_n h_n + h_n) + \beta G_{\boldsymbol{R}}(k_n h_n + h_n)\}.
\end{aligned}$$

Then by Fatou's lemma,

$$\limsup_{n \Rightarrow \infty} G_{\boldsymbol{R}}(k_n h_n + h_n) \leq E_{\boldsymbol{R}}(f^*(s - Y)), \qquad f_{h_n}(k_n h_n + h_n) \to f^*(k_n h_n).$$

So,

$$\begin{aligned}
0 &\leq \sup_{\mathcal{R}} \quad \{p_{\boldsymbol{R}} w'(k_n h_n) - \beta f_{h_n}(k_n h_n + h_n)(k_n h_n) + \beta G_{\boldsymbol{R}}(k_n h_n + h_n)\} \\
&\leq \sup_{\mathcal{R}} \quad \{p_{\boldsymbol{R}} w'(k_n h_n) - \beta f^*(k_n h_n) + \beta E_{\boldsymbol{R}}(f^*(x - Y))\} \\
&\leq \sup_{\mathcal{R}} \quad \{p_{\boldsymbol{R}} w'(s) - \beta w(s) + \beta E_{\boldsymbol{R}}(w(s - Y))\}.
\end{aligned}$$

Thus, f^* is a viscosity subsolution. Similarly, f_* is locally Lipschitz and a viscosity supersolution of (9). □

Theorem 3.3. *The sequence $f_h/f^*(\infty)$ converges to the unique viscosity δ.*

Proof. Define $g^*(s) = f^*(s)/f^*(\infty)$ and $g^*(s) = f_*(s)/f_*(\infty)$. It is obvious that $\lim_{s\to\infty} g^*(s) = 1$ and by Lemma 3.8 $g^*(.)$ and $g_*(.)$ are nondecreasing functions. Now by the Proposition 3.1, $\delta \geq g^*$ and $\delta \leq g_*$ and so $g^* \leq g_*$. Since $g^* \geq g_*$ by definition, we have convergence. Define $g(s) = g^*(s)$ $(= g_*(s))$. Now, by Proposition 3.2, the function $g(s)$ nondecreasing viscosity solution of (9) with limit one at infinity. Therefore, by Theorem 3.2, $\delta(s) = g(s)$. □

4. Examples

Suppose an insurance company is operating on three lines. In the line i, $i = 1, 2, 3$, the reinsurance is displayed as R_i and the distribution function is demonstrated as F_i. Furthermore, for the claim numbers, Poisson distribution with the parameter β_i is used. Suppose further that R_i is one of the proportional reinsurance strategies (R_p) or excess-of-loss (R_{XL}). In this condition, there are different states imaginable for choosing the type of reinsurance contract; some of the states will be dealt with later on.

Example 4.1. Let $F_i(x) = 1 - e^{-\lambda_i x}$, $\mathrm{i} = 1, 2, 3$, and $\lambda_1 = 0.15$, $\lambda_2 = 0.8$, $\lambda_3 = 2$, $\beta_1 = 1$, $\beta_2 = 13$, $\beta_3 = 30$, $\eta_1 = \eta_2 = \eta_3 = 3$ and $\eta_{11} = \eta_{12} = \eta_{13} = 3.5$. The reinsurance strategy in ith line is depicted by $\mathrm{R_i}$. As was mentioned before, if $R_i \in \boldsymbol{\mathcal{R}}_p$ then $R_i(y) = b_i(.)y$ and if $R_i \in \boldsymbol{\mathcal{R}}_{XL}$ then $R_i(y) = \min(y, M(.))$, where $b_i(.)$ and $M_i(.)$

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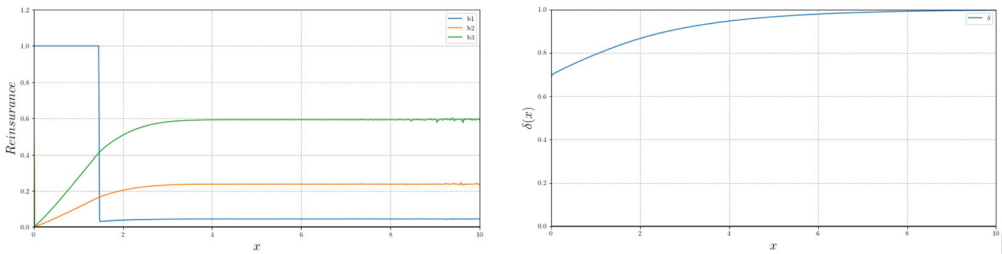

(a) Optimal retained Proportion. (b) A slightly shorter sub-caption.

Figure 1. Three lines of insurance with three proportional reinsurance.

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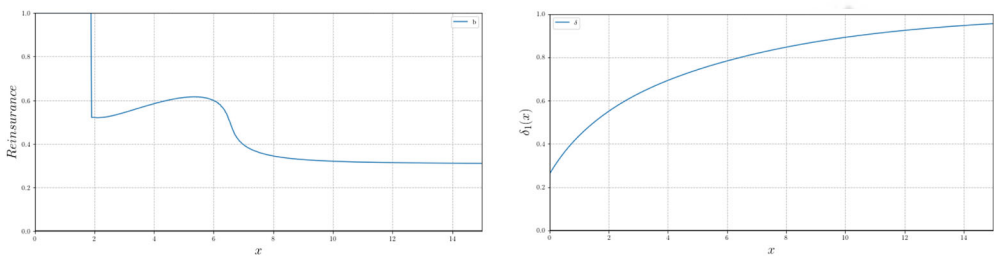

(a) Optimal retained Proportion. (b) Survival probability function with reinsurance.

Figure 2. Three lines of insurance with one proportional reinsurance.

are functions of the company's capital. If the insurance company considers a reinsurance contract for three lines, the optimization issue will be equal with the uni-dimensional model scrutinized by Azcue and Muler(11]. Figure 1, illustrates rules for the state in which three proportional reinsurance strategies for three lines have been used (this contract could vary from one line to another), followed by Figure 2, displaying the state in which one proportional reinsurance is used for all three lines. Furthermore, if we supplant the proportional reinsurance contract with an excess-of-loss reinsurance, the results would be as Figure 3, where 3(a), illustrates three reinsurance strategies in three insurance lines, and 3(b) and 3(c) indicate the charts for reinsurancee strategies of line 2 and 3 (which are the same). In both states, in the sense of increasing the survival function, the use of three appropriate reinsurance contracts would culminate in better results than the use of one contract for all three lines.

In Figure 5, the diagrams for the optimal survival function in the states of three excess-of-loss reinsurance strategies for three lines, one excess-of-loss reinsurance strategy for three lines, three proportional reinsurance strategies for three lines, one proportional reinsurance strategy for three lines and without reinsurances have been juxtaposed.

As evident from the Figures 1–5, one could note that these results could be useful for the insurer such that he/she can draw different contracts. By doing this, the insurer can decrease the probability of bankruptcy *vis-à-vis* the state that he/she uses only one contract. In the previous example, the exponential distribution is considered for the claims size, and only one type of reinsurance strategy is used to transfer the risk to the secondary insurer on all lines. To further explicate the point, suppose the insurance company aims to incorporate the best reinsurance strategy which was given in the

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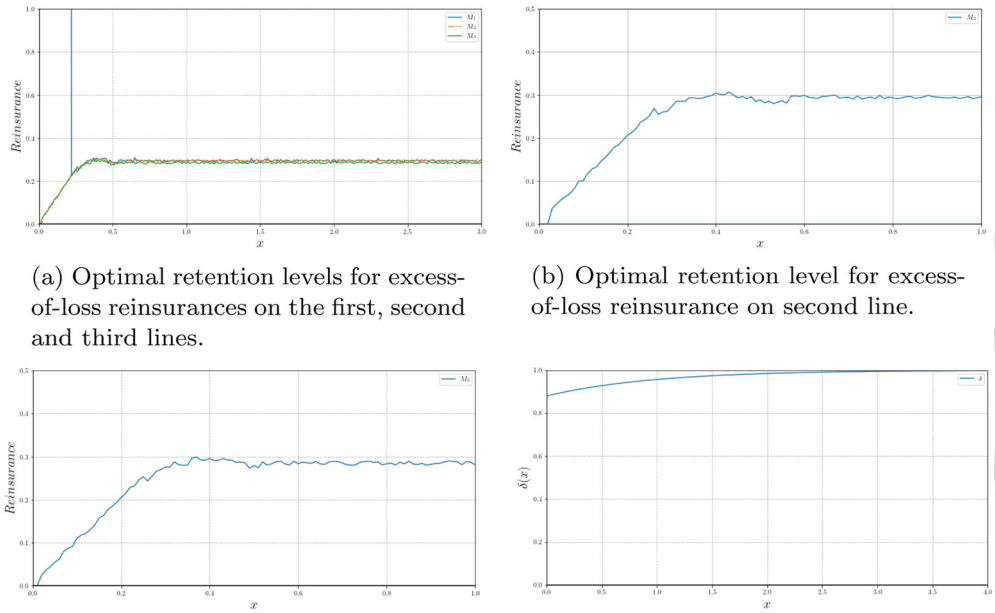

(a) Optimal retention levels for excess-of-loss reinsurances on the first, second and third lines.

(b) Optimal retention level for excess-of-loss reinsurance on second line.

(c) Optimal retention level for excess-of-loss reinsurance on third line.

(d) Survival probability function with excess-of-loss reinsurance.

Figure 3. Three lines of insurance with three excess-of-loss reinsurance.

aforementioned case viz., in a state at which in the three lines, three XL reinsurances are considered. Here, after any alteration in the capital, the insurance company shall upgrade the reinsurances contract. For instance, if a damage is claimed in the second line, the insurance manager and reinsurance managers must adjust the reinsurance contracts based in the three lines considering the current state of the capital of the insurance company. In practice, however, the distribution of claims in each line may be completely different from the other one and the insurance company may use different types of reinsurance contracts.

Example 4.2. In this example, we consider the light-tailed distribution $F_1(x) = 1 - e^{-0.5x}$ for the claims size in the first line, the heavy-tailed distribution $F_2(x) = 1 - (\frac{3}{3+x})^3$ for claims size in the second line and the mixture of heavy-tailed distribution and light-tailed distribution $F_3(x) = 0.7F_1(x) + 0.3F_2(x)$ for claims size in the last line. Also, let $\beta = 8$, $\beta_2 = 0.4$, $\beta_3 = 5, \eta_1 = 0.3$, $\eta_{11} = 0.35$, $\eta_2 = 0.2$, $\eta_{12} = 0.25$, $\eta_3 = 0.25$, and $\eta_{13} = 0.3$. If one of the excess-of-loss or proportional reinsurance strategies is used to control the risk of each line, then all possible settings are as follows:

i. $R_i \in \mathcal{R}_p$, $i = 1, 2, 3$,
ii. $R_i \in \mathcal{R}_p$, $i = 1, 2$ and $R_3 \in \mathcal{R}_{XL}$,
iii. $R_i \in \mathcal{R}_p$, $i = 1, 3$ and $R_2 \in \mathcal{R}_{XL}$,
iv. $R_i \in \mathcal{R}_p$, $i = 2, 3$ and $R_1 \in \mathcal{R}_{XL}$,
v. $R_i \in \mathcal{R}_{XL}$, $i = 1, 2$ and $R_3 \in \mathcal{R}_p$,

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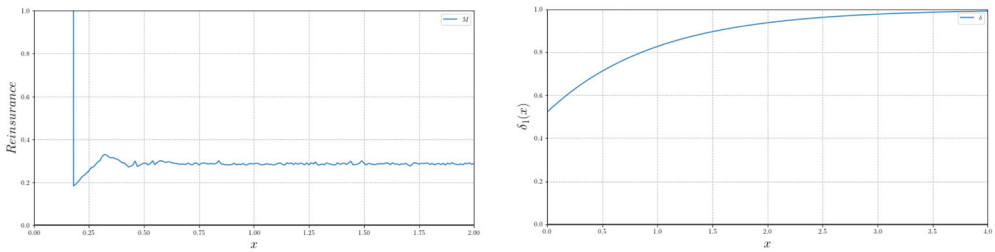

(a) Optimal retention levels for excess-of-loss reinsurance.

(b) Survival probability function with excess-of-loss reinsurance.

Q8 **Figure 4.** Three lines of insurance with one excess of loss reinsurance.

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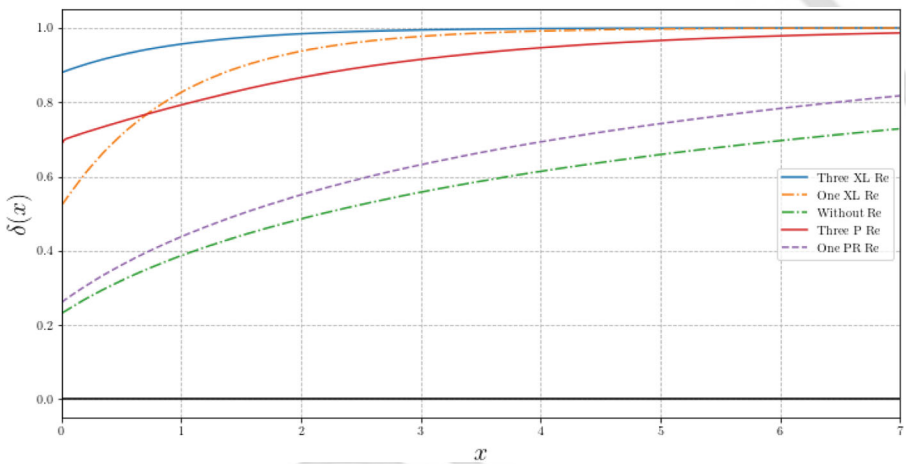

Figure 5. Survival functions.

vi. $R_i \in \mathcal{R}_{XL}$, $i = 1, 3$ and $R_2 \in \mathcal{R}_p$,
vii. $R_i \in \mathcal{R}_{XL}$, $i = 2, 3$ and $R_1 \in \mathcal{R}_p$,
viii. $R_i \in \mathcal{R}_{XL}$, $i = 1, 2, 3$.

The optimization results for all of the above states are presented in Figure 6. In Figure 7, the graph pertained to survival probability is reported in the all 8 above states. As it is displayed in the Figure, the best state is the one in which all lines resort to the XL reinsurance contract. As seen in Figure 7(b), if we decide to resort to the XL strategy only in one line, the best line would be line 2, in which the probability of bigger claim sizes occurring would be higher than the other two lines; furthermore, as it is observed in the Figure 7(c), if we opt for using the XL strategy in two lines, lines 2 and 3 would be our choices.

In Table 1, the survival function for the most critical state of the insurance company, namely, the state in which the capital of the company is zero, is represented for various reinsurance strategies. The state in which the insurance company does not purchase any reinsurance, the survival function in the zero-capital situation is 0.22. Through the selection of appropriate reinsurance, it is possible to immune the insurance company in

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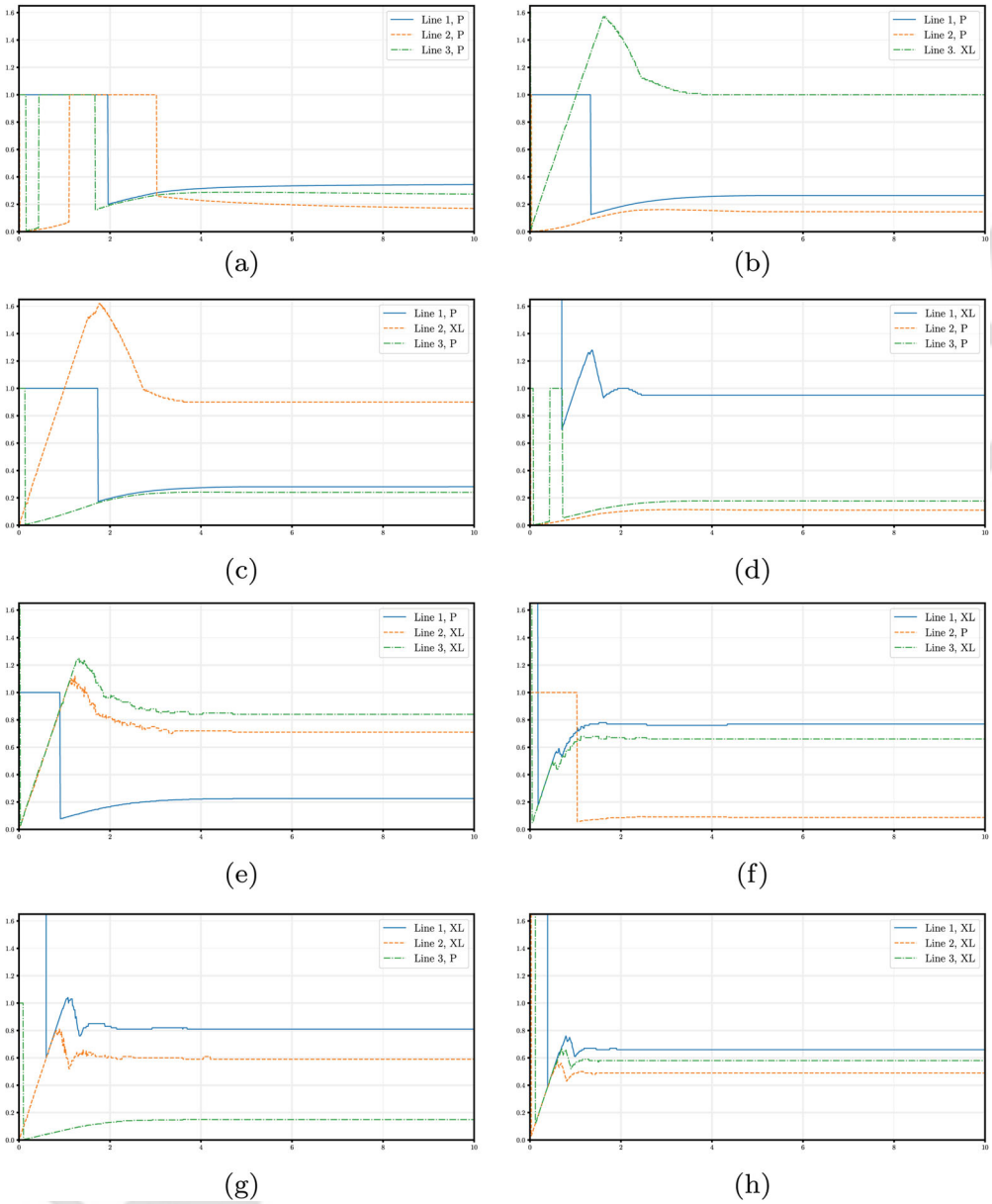

Figure 6. Optimal reinsurance strategies for states $(i) - (viii)$.

the critical conditions and thereby facilitate the access to the ideal conditions by less amount of capital. For instance, in the state that there are three XL contracts considered for the three lines, the survival function in the situation that the capital of the company is zero, increases to 0.66.

The results and findings of this paper reveal that the use of optimal dynamic reinsurance strategies increases the survival function in contrast to the use of one optimal reinsurance strategy in all lines, where a plausible question was raised on the practicality

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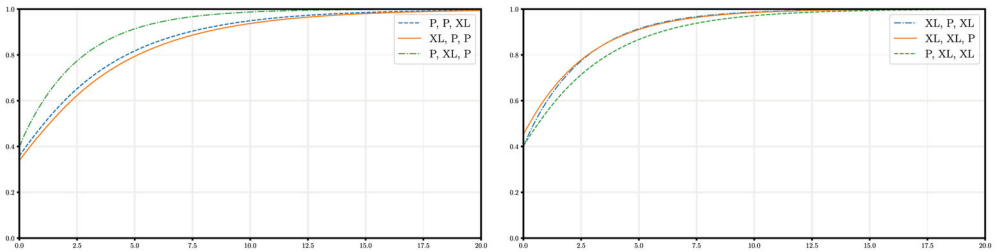
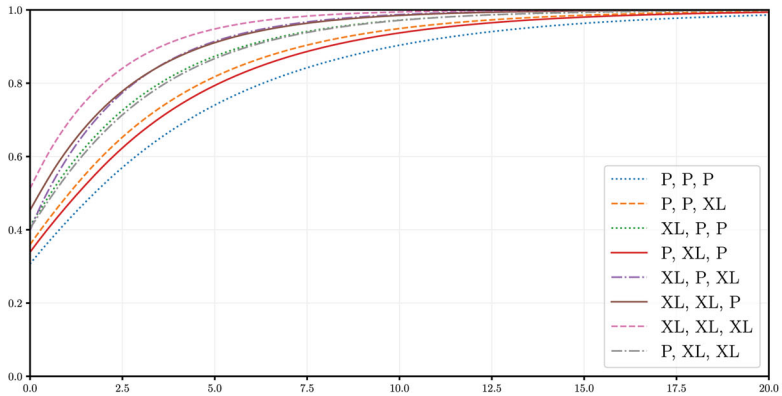


(a) Optimal survival functions for cases where only one XL contract is used.

(b) Optimal survival functions for cases that use two XL contracts.



(c) Survival functions for all possible cases.

Q9 **Figure 7.**

Table 1. Ideal and critical situation.

Reinsurance contracts	$\delta(0)$
Without reinsurance	0.222
P, P, P	0.299
XL, P, P	0.3
XL, P, XL	0.392
P, XL, P	0.393
XL, XL, P	0.447
P, P, XL	0.464
P, XL, XL	0.547
XL, XL, XL	0.641

at resorting to the optimal reinsurance. Perhaps at first glance the answer could be a simple no. Indeed, for an insurance company and the reinsurer, implementing a contract which is dependent on the state of the surplus process and at any moment it may change could be extremely difficult, if not impossible. Nonetheless, it can be concluded that there is an interesting point discernible in Figures, indicating the fact that the XL strategy is a more viable option than the proportional strategy in terms of increasing the survival function. Moreover, ostensibly the XL reinsurance contracts oscillate under the circumstances that the capital of the company dwindle and the company is in the critical condition; they seem constant for increased capital. For instance, consider

the previous example; if three XL reinsurances for three lines have been used, as long as the company's assets belong to the interval $(1, \infty)$, the reinsurance contracts are constant and they only change when the assets of the company belong to the interval (0, 1).

5. Discussion

The issue of dynamic reinsurer strategy is an interesting and efficacious approach for augmenting the survival of an insurance company and one can find a great number of studies in literature on the topic of maximizing the survival function with respect to the reinsurance strategy. In order to solve this optimization issue, an HJB equation associated with the survival function is adapted. Often, the survival function does not have the smoothness properties needed to interpret it as a solution for the corresponding HJB equation in the classical sense, but it is satisfying in this equation in a weaker concept. The present paper deals with the issue of maximization of the survival function with respect to the dynamic reinurance strategy with this difference that the insurance company shares the potential risk with several reinsurers each pertained to a specified line. The optimal survival function is characterized as the unique nondecreasing viscosity solution of (2.9) with limit one at infinity. Unfortunately, obtaining a closed form for the survival function or the reinsurance strategy in the issue discussed at this paper is fiendishly complicated or well-nigh impossible. Therefore, it was more feasible to adopt the numerical solution. For constructing a numerical solution, the FDM has been employed due to the fact that the convergence of the numerical solution to the survival function can be proved through the techniques prevalent in the literature. The convergent findings are displayed in section 4.

The results of the present paper give the insurances companies this stupendous opportunity to share their risk with the reinsurers. In section 5, there are some examples that indicate the fact that using this approach, the survival function will be increased. In a nutshell, with the implementation of this dynamic method for drawing the vector of the reinsurance contracts, the probability of bankruptcy might diminish significantly.

References

[1] Schmidli, H. (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scand. Act. J.* 2001(1):55–68. DOI: 10.1080/034612301750077338.

Q2

[2] Hipp, C., Vogt, M. (2003). Optimal dynamic xl reinsurance. *ASTIN Bull.* 33(2):193–207. DOI: 10.1017/S051503610001343X.

[3] Schmidli, H. (2002). On minimizing the ruin probability by investment and reinsurance. *Ann. Appl. Probab.*. 12(3):890–907. DOI: 10.1214/aoap/1031863173.

[4] Taksar, M. I., Markussen, C. (2003). Optimal dynamic reinsurance policies for large insurance portfolios. *Fin Stoch.* 7(1):97–121. DOI: 10.1007/s007800200073.

[5] Schmidli, H. (2004). *On Cramér-Lundberg Approximations for Ruin Probabilities under Optimal Excess of Loss Reinsurance*. Laboratory of Actuarial Mathematics, University of Copenhagen.

Q4

[6] Irgens, C., Paulsen, J. (2004). Optimal control of risk exposure, reinsurance and investments for insurance portfolios. *Insurance Math Econ.* 35(1):21–51. DOI: 10.1016/j.insmatheco.2004.04.004.

PROOF ONLY

Q5 [7] Schmidli, H. (2007). *Stochastic Control in Insurance*. Springer Science & Business Media.
[8] Hipp, C., Taksar, M. (2010). Optimal non-proportional reinsurance control. *Insurance Math. Econ*. 47(2):246–254. DOI: 10.1016/j.insmatheco.2010.04.001.
[9] Cani, A., Thonhauser, S. (2017). An optimal reinsurance problem in the Cramér–Lundberg model. *Math. Meth. Oper. Res.* . 85(2):179–205. DOI: 10.1007/s00186-016-0559-8.
[10] Højgaard, B., Taksar, M. (1998a). Optimal proportional reinsurance policies for diffusion models. *Scand Actl J*. 1998(2):166–180. DOI: 10.1080/03461238.1998.10414000.
[11] Azcue, P., Muler, N. (2014). *Stochastic Optimization in Insurance: A Dynamic Programming Approach*, Springer.
Q3 [12] Nozadi, A. E. (2014). Optimal constrained investment and reinsurance in Lundberg insurance models.
[13] Crandall, M. G., Ishii, H., Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc*. 27(1):1–67. DOI: 10.1090/S0273-0979-1992-00266-5.
[14] Fleming, W. H., Soner, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, Vol. 25. Springer Science & Business Media.
Q6 [15] Pham, H. (2009). *Continuous-Time Stochastic Control and Optimization with Financial Applications*, Vol. 61, Springer Science & Business Media.
Q7 [16] Højgaard, B., Taksar, M. (1998b). Optimal proportional reinsurance policies for diffusion models with transaction costs. *Insurance Math Econ*. 22(1):41–51. DOI: 10.1016/S0167-6687(98)00007-9.

Appendix

Proofs of Lemma 3.1, 3.2 and 3.3 are very similar to Lemma 1.1, Proposition 2.4 and Proposition 2.12 in Azcue and Muler [11], respectively, we therefore omit them.

Proof of Lemma 3.4. Let us prove first that H is left upper semicontinuous. For assumed x_0, $x_k \nearrow x_0$, consider reinsurance strategies $\boldsymbol{R}^{(k)} \in \mathcal{R}$ such that

$$\sup_{\boldsymbol{R}\in\mathcal{R}} \tilde{\mathcal{L}}_{\boldsymbol{R}}(u)(x_k) \leq \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_k) + \frac{1}{k}. \tag{14}$$

Then, the following is straightforward,

$$\begin{aligned}\mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_0) &= \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_k) + p_{\boldsymbol{R}^{(k)}}(u'(x_0) - u'(x_k)) - \beta(u(x_0) - u(x_k)) \\ &\quad + \int_0^{x_k} (u(x_0-\alpha) - u(x_k-\alpha))dF_{\boldsymbol{R}}^{(k)}(\alpha) \\ &\quad + \int_{x_k}^{x_0} u(x_0-\alpha)dF_{\boldsymbol{R}}^{(k)}(\alpha) \\ &\geq \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_k) + p_{\boldsymbol{R}^{(k)}}(u'(x_0) - u'(x_k)) - \beta(u(x_0) - u(x_k)) \\ &\quad + \int_0^{x_k} (u(x_0-\alpha) - u(x_k-\alpha))dF_{\boldsymbol{R}^{(k)}}(\alpha).\end{aligned}$$

So, we have,

$$\limsup_{k\to\infty} \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_0) \geq \limsup_{k\to\infty} \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_k). \tag{15}$$

Then, from (14) and (15), the following result can be derived;

$$\sup_{\boldsymbol{R}\in\mathcal{R}} \mathcal{L}_{\boldsymbol{R}}(u)(x_0) \geq \limsup_{k\to\infty} \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_0) \geq \limsup_{k\to\infty} \left(\sup_{\boldsymbol{R}\in\mathcal{R}} \mathcal{L}_{\boldsymbol{R}}(u)(x_k)\right).$$

Consequently, the following relation is dominant $\limsup_{x\to x_0^-}(\sup_{\boldsymbol{R}\in\mathcal{R}} \mathcal{L}_{\boldsymbol{R}}(u)(x)) \leq \sup_{\boldsymbol{R}\in\mathcal{R}} \mathcal{L}_{\boldsymbol{R}}(u)(x_0)$. Now, we must prove the following $\limsup_{x\to x_0^+}(\sup_{\boldsymbol{R}\in\mathcal{R}} \mathcal{L}_{\boldsymbol{R}}(u)(x)) \leq \sup_{\boldsymbol{R}\in\mathcal{R}} \mathcal{L}_{\boldsymbol{R}}(u)(x_0)$. Given any sequence $x_k \searrow x_0$, take reinsurance strategies $\boldsymbol{R}^{(k)} \in \mathcal{R}$ such that

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$$\sup_{\boldsymbol{R}\in\mathcal{R}} \mathcal{L}_{\boldsymbol{R}}(u)(x_k) \le \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_k) + \frac{1}{k}.$$

If one of the reinsurance contracts is LXL reinsurance, for example $\mathcal{R}_1 = \mathcal{R}_{LXL}$, take

$$\bar{\boldsymbol{R}}^{(k)} = (\bar{R}_1^{(k)}, \bar{R}_2^{(k)}, \ldots, \bar{R}_n^{(k)}) \in \mathcal{R},$$

such that

$$\bar{R}_1^{(k)}(\alpha) = \begin{cases} R_1^{(k)}(\alpha) & if \quad R_1^{(k)}(\alpha) = \alpha \quad for \quad all \quad \alpha \\ R_1^{(k)}(\alpha) & if \quad R_1^{(k)}(\alpha) = a_k \wedge \alpha + (\alpha - L - a_k)^+ \quad with \quad a_k \notin (x_0, x_k) \\ \alpha \wedge x_0 + (\alpha - L - x_0)^+ & if \quad R_1^{(k)}(\alpha) = a_k \wedge \alpha + (\alpha - L - a_k)^+ \quad with \quad a_k \in (x_0, x_k) \end{cases}$$

and $\bar{R}_2^{(k)} = R_2^{(k)}, \ldots, \bar{R}_n^{(k)} = R_n^{(k)}$. If $a_k \le x_0$ then

$$\begin{aligned} &\int_0^\infty (u(x_k - \alpha) - u(x_0 - \alpha)) dF_{\bar{\boldsymbol{R}}^{(k)}}(\alpha) \\ &= \int_0^\infty (u(x_k - \alpha) - u(x_0 - \alpha)) dF_{\boldsymbol{R}^{(k)}}(\alpha) \\ &\le \sup_{x\in[0,A]} |u'(x)|(y - x_0). \end{aligned}$$

Let us define $\boldsymbol{R}_{n-1}^{(k)} = (R_2^{(k)}, \ldots, R_n^{(k)})$. If $a_x \in (x_0, x_k)$, then

$$\begin{aligned}
&\int_0^\infty u(x_k - \alpha) dF_{\boldsymbol{R}^{(k)}}(\alpha) - \int_0^\infty u(x_0 - \alpha) dF_{\bar{\boldsymbol{R}}^{(k)}}(\alpha) \\
&= \int_0^\infty \int_0^\infty u(x_k - R_1^{(k)}(\alpha_1) - \alpha_2) dF_{\boldsymbol{R}_{n-1}^{(k)}}(\alpha_2) dF_1(\alpha_1) - \int_0^\infty \int_0^\infty u(x_0 - \bar{R}_1^{(k)}(\alpha_1) - \alpha_2) dF_{\boldsymbol{R}_{n-1}^{(k)}}(\alpha_2) dF_1(\alpha_1) \\
&= \int_0^\infty \int_0^\infty (u(x_k - R_1^{(k)}(\alpha_1) - \alpha_2) - u(x_0 - \bar{R}_1^{(k)}(\alpha_1) - \alpha_2) dF_{\boldsymbol{R}_{n-1}^{(k)}}(\alpha_2)) dF_1(\alpha_1) \\
&= \int_0^{x_0} \int_0^\infty (u(x_k - R_1^{(k)}(\alpha_1) - \alpha_2) - u(x_0 - R_1^{(k)}(\alpha_1) - \alpha_2) dF_{\boldsymbol{R}_{n-1}^{(k)}}(\alpha_2)) dF_1(\alpha_1) \\
&+ \int_{x_0}^{a_k} \int_0^\infty (u(x_k - \alpha_1 - (\alpha_1 - L - a_k)^+ - \alpha_2) - u(-(\alpha_1 - L - x_0)^+ - \alpha_2) dF_{\boldsymbol{R}_{n-1}^{(k)}}(\alpha_2)) dF_1(\alpha_1) \\
&+ \int_{a_k}^{x_k} \int_0^\infty (u(x_k - \alpha_k - (\alpha_1 - L - a_k)^+ - \alpha_2) - u(-(\alpha_1 - L - x_0)^+ - \alpha_2) dF_{\boldsymbol{R}_{n-1}^{(k)}}(\alpha_2)) dF_1(\alpha_1) \\
&+ \int_{x_k}^\infty \int_0^\infty (u(x_k - \alpha_k - (\alpha_1 - L - a_k)^+ - \alpha_2) - u(-(\alpha_1 - L - x_0)^+ - \alpha_2) dF_{\boldsymbol{R}_{n-1}^{(k)}}(\alpha_2)) dF_1(\alpha_1) \\
&\le \sup_{x\in[0,A]} |u'(x)|(x_k - x_0) \\
&+ \sup_{\alpha\in[x_0,a_k]} (u(x_k - \alpha - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+)) p(x_0 \le \alpha \le a_k) \\
&+ \sup_{\alpha\in[a_k,x_k]} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+)) p(a_k \le \alpha \le x_k) \\
&+ \sup_{\alpha\in[x_k,\infty)} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+)) p(\alpha \ge x_k).
\end{aligned}$$

According to the property of right-continuously of distribution function; if $x_k \searrow x_0$, then

$$p(x_0 \le \alpha \le a_k) \to 0 \qquad \text{and} \qquad p(a_k \le \alpha \le x_k) \to 0.$$

Now, the term $\sup_{\alpha\in[x_k,\infty)}(u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+))$ should become the focus of attention. In this case, there are two situations as outlined below:

I. If, there is a finite value m satisfying the following,

$$\begin{aligned} &\sup_{\alpha\in[x_k,\infty)} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+)) \\ &= u(x_k - a_k - (m - L - a_k)^+) - u(-(m - L - x_0)^+). \end{aligned}$$

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So,

$$\lim_{x_k \searrow x_0} \sup_{\alpha \in [x_k, \infty)} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+))$$
$$= u(-(m - L - x_0)^+) - u(-(m - L - x_0)^+) = 0.$$

II. If

$$\sup_{\alpha \in [x_k, \infty)} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+))$$
$$= \lim_{\alpha \to \infty} u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+),$$

then

$$\sup_{\alpha \in [x_k, \infty)} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+)) = u(x_k - a_k) - u(0).$$

So,

$$\lim_{x_k \searrow x_0} \sup_{\alpha \in [x_k, \infty)} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+))$$
$$= \lim_{x_k \searrow x_0} (u(x_k - a_k) - u(0)) = 0.$$

It should be noted that $R_1^{(k)}(\alpha) - (x_k - x_0) \leq \bar{R}_1^{(k)}(\alpha) \leq R_1^{(k)}(\alpha)$. So $p_{\bar{R}_1^{(k)}} \nearrow p_{R_1^{(k)}}$. Thus

$$\begin{aligned}
&H(x_k) - H(x_0) \leq \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_k) - \mathcal{L}_{\boldsymbol{R}^{(k)}}(u)(x_0) + \epsilon \\
&\leq p_{\boldsymbol{R}^{(k)}} \sup_{x \in [0, A]} |u''(x)|(x_k - x_0) + 2\beta \sup_{x \in [0, A]} |u'(x)|(x_k - x_0) \\
&+ \sup_{\alpha \in [x_0, a_k]} (u(x_k - \alpha - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+))p(x_0 \leq \alpha \leq a_k) \\
&+ \sup_{\alpha \in [a_k, x_k]} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+))p(a_k \leq \alpha \leq x_k) \\
&+ \sup_{\alpha \in [x_k, \infty)} (u(x_k - a_k - (\alpha - L - a_k)^+) - u(-(\alpha - L - x_0)^+))p(\alpha \geq x_k),
\end{aligned}$$

and so we get that H is right upper semicontinuous. The proof for the case $\boldsymbol{\mathcal{R}}_1 = \boldsymbol{\mathcal{R}}_{XL}$ and $\boldsymbol{\mathcal{R}}_1 = \boldsymbol{\mathcal{R}}_p$ are simpler, we therefore omit them. Now, repeating the arguments presented in the proof of the Lemma 3.2 of Azcue and Muler [11] (replacing $\boldsymbol{R} = (\boldsymbol{R}_t)_{t \geq 0} = (R_{1t}, \ldots, R_{nt})_{t \geq 0}$ with $\bar{R}$), the following relation is obtained,

$$|H(y) - H(x_0)| \leq K_A(y - x_0 + F(y) - F(x_0)).$$

The proof is complete. □

If in Lemma 4.2 and Lemma 4.3 of Azcue and Muler [11], replace

$$\mathcal{L}_R(\delta)(x) = p_R \delta'(x) - \beta\delta(x) + \beta \int_0^x \delta(x - \alpha) dF_R(\alpha),$$

with

$$\mathcal{L}_{\boldsymbol{R}}(\delta)(x) = p_{\boldsymbol{R}} \delta'(x) - \left(\sum_{i=1}^n \beta_i\right)\delta(x) + \left(\sum_{i=1}^n \beta_i\right) \int_0^x \delta(x - \alpha) dF_{\boldsymbol{R}}(\alpha),$$

then two lemmas Lemma 3.5 and Lemma 3.6 are derived and to prove the Lemma 3.7, it suffices to repeat the proof of Lemma 4.4 of Azcue and Muler [11] by replacing $\boldsymbol{R} = (\boldsymbol{R}_t)_{t \geq 0} = (R_{1t}, \ldots, R_{nt})_{t \geq 0}$ with $\bar{R}$. Therefore, proofs of these Lemmas are omitted.

Proof of Lemma 3.8. (i) Assume that i is a positive integer with $f'_h(kh) \geq 0, k = 1, \ldots, i - 1$. Then $f_h(0) \leq f_h(h) \leq \cdots \leq f_h((i - 1)h)$ and thus

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$$G_{\boldsymbol{R}}(ih) = \sum_{\{j \le i\}} f_h((i-j)h) P\{(j-1)h < \boldsymbol{R}(Y) \le jh\} \le f_h((i-1)h).$$

So for $s = ih$

$$\frac{\beta f_h(s-h) - \beta G_{\boldsymbol{R}}(s)}{p_{\boldsymbol{R}}} \ge 0,$$

and thus obviously

$$f'_h(ih) = f'_h(s) = \inf_{\boldsymbol{R} \in \mathcal{R}} \frac{\beta f_h(s-h) - \beta G_{\boldsymbol{R}}(s)}{p_{\boldsymbol{R}}} \ge 0,$$

which completes the induction.

(ii) The proof of this part is similar to the proof of Lemma 9 in Nozadi [12], section 3.3. □

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