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Modeling and optimal control of travel times and traffic emission on freeways

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Abstract: In this paper a modeling method and a control approach is proposed for minimising both travel times and traffic emission on freeways. A simulation-based investigation among emission models is performed to determine the model to be engaged for a model-based control. The chosen model is imposed on a second-order macroscopic traffic model. A constrained LQ control is proposed for optimization. An emission-optimal control and a multicriteria (travel time and emission) optimization are compared to conventional, solely travel-time optimal control. Simulations prove that ramp metering can be used to reduce both emission and travel times on motorways.

Keywords: traffic control, traffic/emission modeling, constrained control

1. INTRODUCTION

The constant increase of global motorization has caused severe environmental impacts. Road traffic of the EU aggregates 25% of CO_2 emission which is considered the gas being principally accountable for the greenhouse effect and thus the global warming (L. Schipper et al.). Further constituents of exhaust gases (NO_X, CO, HC) cause both local and global damages. One way to model the environmental contribution of traffic is to quantify the emitted exhaust gases by emission models with variables of real-time traffic measuerements.

In previous papers, efforts have been made to find control algorithms both minimizing travel times and moderating emission of exhaust gases. Model Predictive Control strategies were presented in (Zegeye et al. a, b) based on two different emission models, and unambigous results proved that emission and travel times can be optimized with the same controller. In (Xia et al.) a Lagrangian approach was proposed for modeling emission in urban areas, based on macroscopic emission model 'Copert' (Ekström et al.) which inspired the mesoscopic groundwork for analysis of models detailed in section 2.2. In (Panis et al.) a microscopic modeling of urban traffic was utilized and effects of dynamic speed limits were evaluated. The contribution of that paper is twofold: simulations showed that emission of urban networks mainly depends on instantaneous accelerations and that speed management can reduce average speed of traffic in network so that overall emissions are reduced.

Edified by the aforementioned studies, this paper concentrates on traffic emissions on freeways and aims at minimising both emissions and travel times by an LTI control approach. In our research, a comparative simulation of potentionally appliable emission models was performed. The comparison groundwork was a microscopic traffic modeling software which had the advantage of providing the reproduction of macroscopic traffic measurements and that it examines each vehicle. A one- and a two-variable mesoscopic emission model was created, and the importance of variables were investigated on freeways by comparative simulations. Based on the chosen emission model and sexond-order macroscopic modeling, an LQ-based ramp metering control was designed. As the system input was strictly positive, a control respecting input constraints was needed. Piecewise LQ control was adopted for this goal and the feedback gain was derived for discrete-time systems.

The outline of the paper is as follows. After a short introductory section, emission and freeway traffic models are reviewed. The following section presents the control design. In the penultimate section simulation results of the controlled system are presented. Finally, in section 5. conclusions are drawn and future work is stated.

2. PRELIMINARIES

This section summarizes the issues of traffic and emission modeling. Several approaches are reviewed and the applied simplifications are detailed.

2.1 Freeway modeling

Freeway traffic is most commonly described by macroscopic traffic models. This approach considers traffic as compressible fluid neglecting individual vehicle dynamics and describes it by aggregated variables such as traffic flow (q), traffic density (ρ) and space mean speed (v) of traffic. The model was originally derived in continuous-time, whereas the aforementioned variables can only be measured in discrete temporal and spatial intervals. Thus, a temporally and spatially discretized description was formed. First order models characterize flow speed as a static function of traffic density, whereas second-order modeling engages a speed momentum equation in addition to the equilibrium relation of flow speed and density. Hence flow speed is a dynamic function of density in second-order modeling.

The nonlinear equations of the second-order model regarding segment i at discrete time step k are the following:

$$\rho_i(k+1)=\rho_i(k)+\frac{T}{L_i\cdot\lambda_i}\left[q_{i-1}(k)-q_i(k)+r_i(k)\right] \quad (1)$$

$$V_i(\rho_i)=v_{free}\cdot\exp\left(-\frac{1}{a}\left(\frac{\rho_i}{\rho_{cr}}\right)^a\right) \quad (2)$$

$$\begin{aligned} v_i(k+1) &= v_i(k)+\frac{T}{\tau}\left[V\left[\rho_i(k)\right]-v_i(k)\right] \\ &+\frac{T}{L}\cdot v_i(k)\cdot\left[v_{i-1}(k)-v_i(k)\right] \\ &-\frac{T\cdot\eta}{\tau\cdot L}\cdot\frac{\rho_{i+1}(k)-\rho_i(k)}{\rho_i(k)+\kappa}-\frac{\delta\cdot T}{L\cdot\lambda_i}\cdot\frac{r_i(k)v_i(k)}{\rho_i(k)+\kappa} \end{aligned} \quad (3)$$

$$q_i(k)=\rho_i(k)\cdot v_i(k)\cdot\lambda_i\,, \quad (4)$$

where λ_i is the number of the lanes, r_i denotes the flow of the on-ramp and a, β, ρ_{cr}, κ, τ, δ, v_{free}, and η are additional constant parameters (Luspay et al. 2009).

Equations (1) - (4) are nonlinear, which raises difficulties when designing a model-based controller. The simplest way to eliminate this trait of the system, is to linearize model equations around setpoints operation points). Thus, system dynamics become linear although biased (described relative to the setpoints), which is called centered dynamics. The state-space form of the centered system describes the dynamics around the setpoints (x^*):

$$\Delta x(k)=x(k)-x^*(k) \quad (5)$$

$$\Delta x(k+1)=A\Delta x(k)+B\Delta u(k)+H\Delta d(k)\,, \quad (6)$$

where $x\in\Re^m$ denotes system states, $u\in\Re^r$ denotes control input and $d\in\Re^d$ denotes disturbances of the system. $A\in\Re^{mxm}, B\in\Re^{mxr}, H\in\Re^{mxd}$ are the first order partial derivatives (Jacobian matrices) of the nonlinear system dynamics. In this particular case, $\Delta x(k)=\left[\Delta\rho_1(k)\quad\Delta v_1(k)\quad\ldots\quad\Delta\rho_n(k)\quad\Delta v_n(k)\right]^T$, $\Delta u(k)=\left[\Delta r_1\ldots\Delta r_r\right]^T$, $\Delta d(k)=\left[\Delta q_0\ \Delta v_0\ \Delta\rho_{n+1}\right]^T$ and matrices A, B and H were determined accordingly. Linearized modeling of traffic process leaves much to be desired in terms of accuracy but has been succesfully used to create model-based control (Papageorgiou et al.).

The disposal of setpoints are detailed in section 3.2.

2.2 Emission modeling

In previous papers (Zegeye et al. 2009. a, b, Xia et al) efforts have been made to create control suites to optimize emission of traffic. Prior to this paper, a number of emission functions have been utilized for the calculation of pollution on freeways, but adaptibility considerations have not been taken. This section aims at filling this gap.

Modeling concepts of road traffic emission can be classified to microscopic and macroscopic approaches. Microscopic modeling is more accurate, characterizing each vehicle's instananeous emission (E_j) as a function of instantaneous speed $v_j(t)$ and acceleration $a_j(t)$ (of vehicle j).

$$E_j(t)=f\left(v_j(t),a_j(t)\right) \quad (7)$$

There are several models handling microscopic emission (Rakha et al., Panis et al.), the drawback of this approach is the lack of data of individual vehicles' dynamic variables. Regarding traffic data, only macroscopic measurements are available for any kind of emission modeling. However, macroscopic emission models also exist (Ekström et al.), in which model variables are obtained from macroscopic traffic variables (8):

$$E_i(k)=f(v_i(k),\rho_i(k)) \quad (8)$$

where v_i denotes space mean speed of traffic at segment i, and ρ_i is the traffic density of segment i.

Mesoscopic traffic modeling approximates individual vehicles' dynamics by using macroscopic variables (9) (mean speeds as individual speeds, reciprocal of density for headways, etc) and coincides with macroscopic modeling as aggregated vehicle emissions equal to the macroscopically modeled emission (10).

$$E_j(k)=f(v_i(k),\rho_i(k)) \quad (9)$$

$$E_i(k)=\sum_j E_j(k) \quad (10)$$

Mesoscopic level of modeling is convenient for a comparison of emission functions. Traffic is modeled on microscopic level, and by reproducing macroscopic measurements, mesoscopic modeling can be obtained – a basis of comparison of different emission functions. Beforehand the model-based control, significance of variables in emission models were analysed by simulations. A two-variable $E(t)=f(v(t),a(t))$ (7) and a single variable $E(t)=f(v(t))$ emission function was compared, both based on the VT-Micro emission and fuel consumption model (Rakha et al.). Acceleration variable of the two-variable model was obtained by using macroscopic acceleration of traffic (Luspay et al. 2010). Behaviour of freeway traffic in complex situations on a detail of single vehicles cannot be modeled analytically (as certain parameters of microscopic models are random varibles, e.g. driver sensitivity, headways, desired speed), so simulations were needed for comparison of models.

The simulations, run in Matlab were based on the nonlinear car-following microscopic model (Gazis et al.) and macroscopic measurements were reproduced. Microscopically calculated emission was considered as reference. Based on the macroscopic measurements, mesoscopic models were built and emissions of three models (microscopic, two variable mesoscopic and a single variable mesoscopic model) were compared.

Simulations were performed in a certain traffic situation: effect of speed limitation was considered an intentional disturbance on traffic resulting in largest changes in vehicles' speed trajectories. During simulations, free-flow traffic of different flow levels reaching and leaving zones of speed limits, thus both positive and negative accelerations were performed. The highest difference between relative errors of single variable and two variable model was 5% which was found diminutive enough to be neglected (Table 1.). Hence, emission on freeways was considered as a function of the only variable of traffic speed. This single variable emission model was used for model-based control alongside with the second-order macroscopic traffic model.

Table 1. Comparison of emission models

Relative errors of mesoscopic models relative to microscopic reference model according to pollutants [%]					
Pollutant		HC	CO	CO_2	NO_X
Deceleration phase	Two-variable model	5,6	8,7	3,6	7,8
	Single-variable model	5,4	7,1	4,7	11,3
Acceleration phase	Two-variable model	-1,4	-3,2	-2,4	-5,9
	Single-variable model	-2,8	-6,1	-3,4	-7,5

3. CONTROL DESIGN

3.1 Control method

Control design is mainly determined by the nature of the linearized system. An LQ based controller is designed that aims at reaching setpoints whilst minimizing both input energy and the energy of the overall system. Exploiting this trait, the linearized system's setpoints are determined as minimums of the broad-sense outputs of the system, which are nonlinear functions of the states, such as 'travel times' and 'distance-specific emission of CO_2'. Control strategies differ regarding state weighting and setpoint choices (detailed in section 3.2 and 3.3). To respect input bounds, a constrained LQ control is applied. (See in section 3.3.b).

Remark: control design is carried out for one pollutant (CO_2) and vehicle class only (Euro3 passanger car, engine displacement under 1.8 l); optimum points of other pollutants and vehicle classes may vary – they may even occur at unstable traffic conditions. This paper only considers the capability of ramp metering of reducing gaseous emission of traffic in case emission optimum and traffic stability is not inconsistent (i.e. traffic density remains below the critical density ρ_{cr}).

Three control strategies were designed and then compared:

- a conventional, travel-time optimizing controller: TT optimal control
- a solely emission minimizing controller: TE (Total Emission) optimal control
- a compromised controller minimizing both travel times and CO_2 emission: TE+TT optimal control.

3.2 Setpoint considerations

During linearization of the second-order model, there are *N* steady-state equations of *N+4* variables to be solved if *N* denotes the number of segments. Thus, only four variables can be set arbitrarily in advance of solving the steady-state equations to get the remaining setpoint values.

a) Ramp inflow

As realized input on the ramp (i.e. ramp inflow) can only be a non-negative number, constrained control design is needed. Hence, setpoint of ramp inflow is positively biased from zero and the designed controller ensures the ramp inflow values to vary in a certain interval (for further details see section 3.3.b and section 4.1). Accordingly, three variables remain to be set arbitrarily during each design strategy.

b) TT optimal control

Minimal total time spent in a network at a discrete time-step (*TTS(k)*) is accomplished by maximizing the outflow of the motorway network – in case of a motorway stretch, the outflow of the last freeway segment (Papageorgiou et al.)

$$TTS(k) = T \cdot \left[N_i(k-1) + T \cdot q_{i-1}(k-1) - T \cdot q_i(k-1) \right] \quad (11)$$

where T denotes the sample time of the discrete system, and $N_i(k-1)$ denotes the number of vehicles in road segment i during discrete time interval $k-1$. Maximum of the $q(\rho)$ function is at ρ_{cr}, the critical density. To achieve minimal travel times, setpoints of density variables are set to ρ_{cr}.

However, during calculation of TTT (Total Travel Time) the waiting time of idling cars at the on-ramp are also considered:

$$TWT(k) = T^2 \cdot \left(d_i(k) - r_i(k) \right) \quad (12)$$

Where d_i denotes the demand emerging at the on-ramp. Thus, total travel time (TTT) arises from the sum of TWT and TTS.

c) TE optimal control

Distant specific emission (13) of traffic at segment i is a function of both v_i and ρ_i. $\alpha_2^{p,c}, \alpha_1^{p,c}, \alpha_0^{p,c}$ are constant model parameters ordered to pollutant p and vehicle class c. Design is carried out for one pollutant (CO_2) vehicle class (Euro3 passanger car, engine displacement under 1.8 l).

$$E_i^{p,c}(k) = \rho_i(k)(\alpha_2^{p,c} \cdot v_i(k)^2 + \alpha_1^{p,c} \cdot v_i(k) + \alpha_0^{p,c}) = [\text{g/km}] \quad (13)$$

In case of second-order macroscopic modeling, v and ρ can be considered independent variables and Fig 1. can be drawn. In this case emission is linear function of density and a convex function of speed. Chosen setpoints of the control strategy equal to the speed value of emission minimum.

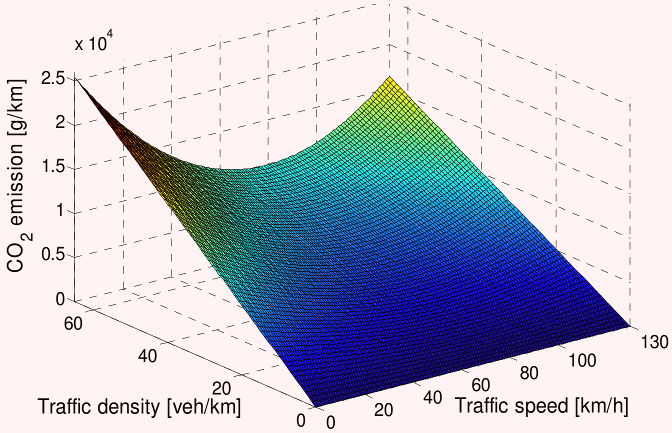

Fig. 1. Distance-specific CO_2 emission of traffic

d) Compromised control

In case of compromised control, setpoints cannot be chosen so that both travel times and emission are minimized so a tradeoff was required. In favour of travel times, last segment's setpoint was chosen the critical density, for optimal pollution level of intermediate segments traffic speed setpoints were emission optimum values.

3.3 LQ design

LQ control defines the functional (14) to minimize and thus optimally control the system.

$$\mathrm{J}\big(y(k),u(k)\big)=\frac{1}{2}\sum_{k=1}^{\infty} y^T(k)Qy(k)+u^T(k)Ru(k) \qquad (14)$$

where $y\in\Re^n$ denotes the output, $u\in\Re^r$ the input of the system, $R\in\Re^{rxr}$ and $Q\in\Re^{nxn}$ are weighting matrices.
In the following, determination of Q and R is presented.

a) State weighting – objective functions

The output of the broad-sense system contains of the nonlinear functions of TT and emission of each segment. After linearization, the following substitution stands:

$$y^T Q y = x^T C^T Q C x \qquad (15)$$

As $Q=I$, $\tilde{Q}$ is introduced:

$$\tilde{Q}=C^T C \qquad (16)$$

Thus, weighting of states instead of outputs can be determined for the system. In case of TT optimal design, $y'=[TT_1\ 0\ TT_2\ 0\ TT_3\ \ 0]^T$ was considered, during emission optimal design $y'=[0\ \ TE_1\ 0\ TE_2\ 0\ TE_3]^T$ and in case of compromised control (TT+TE) design $y'=[TT_1\ TE_1\ TT_2\ TE_2\ TT_3\ TE_3]^T$ was used to determine the value of $\tilde{Q}$.

b) Input weighting

The sole aspect of weighting the input is to design a control that respects input constraints as negative flow on the on-ramp cannot be assigned. In (Wredenhagen et al.) a constrained LQ design method, the Piecewise LQ (PLQ) control was introduced, exploiting the idea that with appropriate input weighting limited input can be obtained on invariant state sets. In previous works, PLQ was only used and derived for continuous-time systems of mechanical engineering, concentrating on the problem of bounded appliable forces as inputs because of the finite tensile strength of mountings. However, traffic systems are described as discrete-time systems, therefore the derivation of weighting in discrete case of PLQ control is required. Derivation of input weighting and optimum feedback gain is detailed in the appendix.

c) Handling of disturbances

The aforementioned disturbances of system (section 2.1) are considered as measured co-states with concomitant random-walk dynamics and are thus attached to states during the design of LQ gain.

4. CLOSED-LOOP SIMULATION

4.1 Simulation setup

To demonstrate the performance of the designed controllers, simulations are run on a 1.5 km freeway stretch. The stretch is divided to three segments of equal length leading to a system of 6 degrees of freedom. Control input affects the system via ramp metering at segment no. 2. Model parameters are the same as those in (Luspay et al. 2009).

Based on the premeditation in section 3.2.a) setpoint of the on-ramp's flow is disposed at 750 veh/h, and an input bound of 450 veh/h is used for further design, thus ramp inflow remains in the interval of [300 veh/h; 1200 veh/h].

4.2 Simulation results

In cases of low traffic, effect of ramp metering is also considerable but most spectacular results are obtained when simulating congested traffic.

The situation of a bottleneck is simulated arising at 300 s and only disappearing at 1000 s. Traffic variables of the situation are summarized in table 2. Congestion is not solved by deploying ramp metering; jam dissolves at 1000 sec by itself but ramp metering appeases its effect.

Table 2. Simulated traffic situation

Interval [sec]	0-300	300-1000	1000-1800
q_{in} [veh/h]	1200	750	1200
v_{in} [km/h]	70	15	70
$\rho_{subsequent}$ [veh/km]	22	50	22
ramp demand	700+ 50·sin(0.02t)		

Analysis of the behaviour of controllers (fig 2-4.) shows that TE optimizing controller reacts even before congestion: the current traffic speed is not optimal optimal in terms of emission (which would be around around 80 km/h – see figure 1.). Lower ramp inflow of the TE control leads to lower traffic densitiy and thus higher mean speed. Controllers 'TT' and' TT+TE' both realize the hold-up of traffic at the same time. The performance of controllers meet the expectations: best travel times are resulted when using 'TT' optimal controller, and best emission performance is attainable with the 'TE' optimal controller. Compromised ('TT+TE') strategy shows very similar values to those of the 'TE' controller, both in reaction and travel times and had a significant advantage in terms of emission. Summarizing simulation results, the following conclusion can be drawn: the utilization of compromised controller design can lead to significant reduction of travel times and emission.

5. CONCLUSION

This paper presented a modeling and control approach of traffic emission on freeways using macroscopic traffic measurements. A simulational analysis in section 2.2 quantified the differences among emission models and justified the utilization of the chosen model – the single-variable emission function proved to estimate the pollution of freeway traffic satisfactorily.

The chosen emission model was imposed on a linearized macroscopic model and an LTI controller was designed, the chosen control input was ramp metering. As control input may only be positive, a constrained control method was needed. The feedback gain and input weight respecting input bounds were derived for discrete-time systems using the framework of Piecewise LQ control. Three design strategies were compared: a conventional travel time optimal control, a solely emission optimizing scenario and a multicriteria, travel times and emission optimizing control design. Simulations on the controlled system resulted a considerable reduction of both travel times and emission in case of compromised control. Nevertheless, control design was presented for one pollutant only; optimum points of other pollutants and vehicle classes vary. In addition, reduction of emission only works in case reaching optimal speed of traffic in terms of emission does not destabilize traffic.

Regarding the future research, several directions are outlined. One direction is to design a control that minimizes emission even if the setpoint that is optimal in terms of emission would destabilize traffic – for this aim different tools of control (e.g. variable speed limit) is needed. Another field is the analysis and research of multicriteria optimization, inspecting the possiblities of optimization of concurrent pollutants. On further investigation, a more sophisticated description of the system (e.g. LPV modeling) is purposed.

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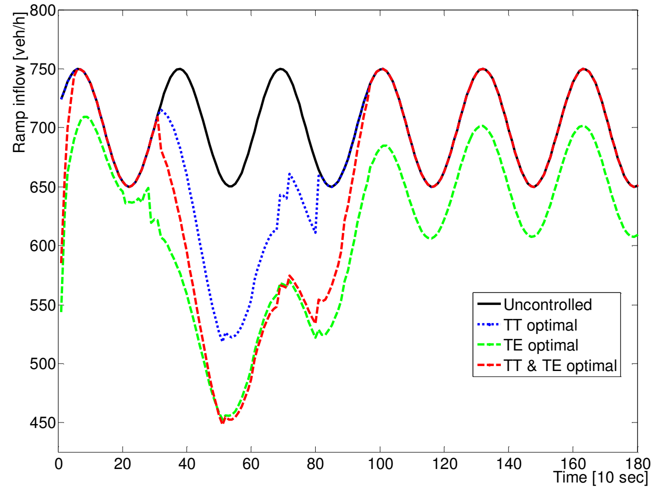

Fig. 2. Commanded inputs on ramp

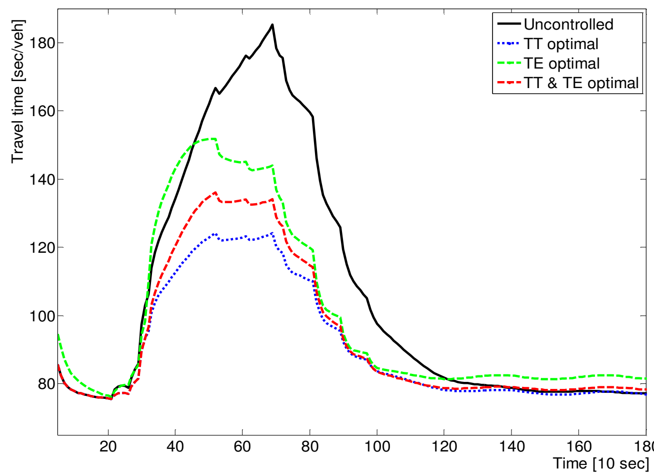

Fig. 3. Performance of controllers: TT values

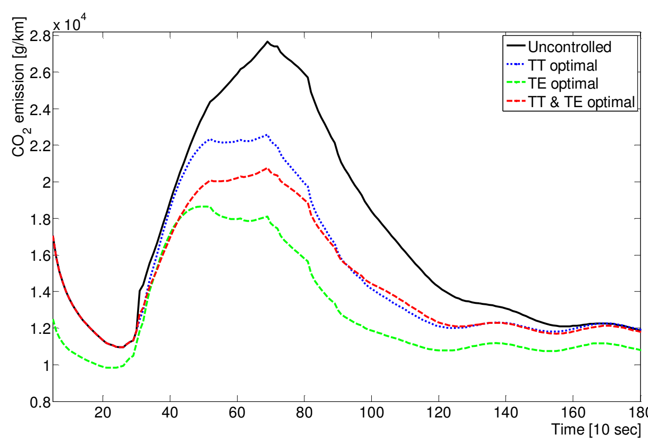

Fig. 4. Performance of controllers: CO_2 emission values

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APPENDIX

Derivation of optimal gain of discrete–time Piecewise LQ

In [Wredenhagen et al.], invariant set of state $x \in \Re^m$ is considered in which bounded feedback gain can be obtained. The invariant set is mapped by an ellipsoid of equation (17). x_0 denotes the boundary of state and $P \in \Re^{mxm}$ denotes the root of the Riccati equation. $\vartheta > 0$ equals to the Lyapunov level of the invariant set.

$$\varepsilon(P,\rho) : x_0^T P x_0 = \vartheta \tag{17}$$

LQ gain K is obtained by solving DARE (Discrete Algebraic Riccati Equation), where A and B are matrices of the system's state-space representation.

$$K = (B^T PB + R)^{-1} \cdot B^T PA \tag{18}$$

Partitioning of B into column vectors and K into row vectors:

$$B = [b_1, ..., b_i, ..., b_n] \tag{19}$$

$$K = \begin{bmatrix} k_1^T \\ ... \\ k_i^T \\ ... \\ k_n^T \end{bmatrix} \tag{20}$$

R input weighting considered as diagonal matrix:

$$R = diag\ (r_i) \tag{21}$$

Partitioned form of feedback gain:

$$k_i^{\ T} = (b_i^{\ T} P b_i + r_i)^{-1} \cdot b_i^{\ T} PA \tag{22}$$

Maximum input value u_{max} is obtained in case of state x_0, the boundary of invariant set (Wredenhagen et al.).

$$u_{\max} = k_i^{\ T} x_0 = \vartheta^{1/2} (k_i P^{-1} k_i^{\ T})^{1/2} \tag{23}$$

Substituting (18) into (23):

$$k_i^{\ T} x_0 = \sqrt{\vartheta}\left[(b_i^{\ T} P b_i + r_i)^{-1} b_i^{\ T} PAP^{-1}((b_i^{\ T} P b_i + r_i)^{-1} b_i^{\ T} PA)^T\right]^{\frac{1}{2}} \tag{24}$$

Inverse of scalar:

$$(b_i^T P b_i + r_i)^{-1} = ((b_i^T P b_i + r_i)^{-1})^T = \frac{1}{b_i^T P b_i + r_i} \tag{25}$$

By further arrangement r_i is equal to

$$r_i = \frac{\sqrt{\vartheta}}{k_i^T x_0} b_i^T PAP^{-1} A^T P^T b_i - b_i^T P b_i \tag{26}$$

Using (23), r_i input weighting can be formulated as:

$$r_i \geq \frac{\sqrt{\vartheta}}{\bar{u}_i} b_i^T PAP^{-1} A^T P^T b_i - b_i^T P b_i \tag{27}$$

The input weighting that provides a feedback gain respecting the input constraints within the current set of states is presented in (26). This equation however supposes the knowledge of ϑ (Lyapunov) level parameter that satisfies (17). Thus, optimal gain can be calculated by an iteration using (26) as outer and (17) as inner iteration condition.