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LINEAR-QUADRATIC OPTIMAL CONTROL WITH INTEGRAL QUADRATIC CONSTRAINTS†

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SUMMARY

We derive closed-form solutions for the linear-quadratic (LQ) optimal control problem subject to integral quadratic constraints. The optimal control is a non-linear function of the current state and the initial state. Furthermore, the optimal control is easily calculated by solving an unconstrained LQ control problem together with an optimal parameter selection problem. Gradient formulae for the cost functional of the optimal parameter selection problem is derived. Application to minimax problems is given. The method is illustrated in a numerical example. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: LQ optimal control; integral quadratic constraints; feedback

1. INTRODUCTION

Many properties of a system can be expressed as quadratic performance costs. This motivates the study of LQ optimal control problems with several quadratic costs.[1–6] The infinite-horizon, time-invariant quadratically constrained LQ optimal control problem is equivalent to an eigenvalue problem (EVP);[1] that is, an optimization problem with linear cost subject to a linear matrix inequality (LMI) which can be solved using interior point methods. In this paper, we consider a class of constrained LQ optimal control problems which cannot be solved in this manner, namely quadratically constrained finite-horizon time-varying problems. Although we shall focus on the deterministic problem, our techniques can be used to solve (time varying) LQG problems. In the unconstrained deterministic case,[7] the optimal control is a linear state feedback. In the constrained deterministic case, the software package MISER 3.1[8] can be used to obtain an optimal control numerically. However, the obtained solution will be open loop. Thus, the challenge remains to find the optimal control as a feedback control law. In this paper, we approach the quadratically constrained LQ problem as a convex optimization problem. We show that the optimal control law is a non-linear function of the current state and the initial state. Although this is not a true feedback control (it is impossible to remove the dependence on the initial state for this is a characteristic of integral constraints), there is a feedback structure and for this reason, we shall refer to it as a hybrid open-loop/closed-loop control. This feedback structure gives the optimal control greater robustness to unmodelled perturbations of the state trajectory,

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a property that is clearly illustrated in our examples which show that the hybrid open-loop/closed-loop optimal control is more robust to disturbances than the optimal control expressed in open-loop form. Furthermore, the optimal control can be calculated by solving the dual problem. This approach is similar to the one developed in Reference 6 in which the dual problem associated with a time-invariant infinite-horizon quadratically constrained LQG problem is solved using Newton's method. However, since we allow for time-varying differential equations and constraints over a finite horizon, the dual problem we obtain is a finite-dimensional optimization problem subject to a differential equation constraint, that is, an optimal parameter selection problem (see Reference 9). Furthermore, the numerical solution of this optimal parameter selection problem is easily obtained using MISER 3.1.[8] We end with a brief discussion of the minmax LQ problem considered in Reference 3. It is very similar to the one encountered in the quadratically constrained LQ problem.

2. PROBLEM STATEMENT

Consider the linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}^0 \tag{1}$$

where $\mathbf{x}^0$ is a given vector in $\mathbf{R}^n$. In the finite-horizon case ($T < \infty$) we shall consider $\mathbf{u} \in L_2^m[0, T]$ and $\mathbf{x}$: $[0, T] \to \mathbf{R}^n$ an absolutely continuous function on $[0, T]$ such that $\dot{\mathbf{x}} \in L_2^n[0, t]$ where $L_2^n[0, T]$ is the Hilbert space of measurable square integrable functions on $[0, T]$ with values in $\mathbf{R}^n$. The matrices $\mathbf{A}(t) \in \mathbf{R}^{n\times n}$ and $\mathbf{B}(t) \in \mathbf{R}^{n\times m}$ are continuous matrices on $[0, T]$. The problem may now be stated as follows:

Given the dynamical system (1), find a control $\mathbf{u} \in L_2^m[0, T]$ such that the cost functional $g_0(\mathbf{u})$ is minimized subject to the functional constraints

$$g_i(\mathbf{u}) \leqslant c_i, \quad i = 1, 2, \ldots, N \tag{2}$$

where g_i, $i = 0, 1, \ldots, N$ are defined by

$$\begin{aligned} g_i(\mathbf{u}) = {} & \frac{1}{2}\int_0^T (\mathbf{x}'(t)\mathbf{Q}_i(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}_i(t)\mathbf{u}(t))\,\mathrm{d}t + \frac{1}{2}\mathbf{x}'(T)\mathbf{S}_i(T)\mathbf{x}(T) \\ & + \int_0^T (\mathbf{a}_i'(t)\mathbf{x}(t) + \mathbf{b}_i'(t)\mathbf{u}(t))\,\mathrm{d}t + \mathbf{h}_i'\mathbf{x}(T) \end{aligned} \tag{3}$$

and c_i, $i = 1, 2, \ldots, N$ are given constants. In (3), $\mathbf{R}_0(t)$ is symmetric positive definite, and $\mathbf{R}_i(t)$ ($i = 1, \ldots, N$), $\mathbf{Q}_i(t)$ and $\mathbf{S}_i$ ($i = 0, \ldots, N$) are symmetric positive semi-definite for each $t \in [0, T]$. Note that this allows for the case of linear integral constraints.

3. MAIN RESULTS

3.1. Optimal control

Let $L_2^m[0, T]$ denote the Hilbert space of all square integrable functions defined on $[0, T]$ with values in $\mathbf{R}^m$. We assume that the following condition holds.

Assumption

For every $\lambda_i \geqslant 0$, $i = 1, \ldots, N$ (not all equal to 0) there exists $\mathbf{u} \in L_2^m[0, T]$ such that

$$\sum_{i=1}^{N} \lambda_i \left(g_i(\mathbf{u}) - c_i\right) < 0$$

(Note that a sufficient condition for the validity of this condition is the existence of a control $\mathbf{u} \in L_2^m[0, T]$ such that $g_i(\mathbf{u}) < c_i$, $i = 1, \ldots, N$.)

Let $\mathbf{u}^*$ be the optimal control and define

$$g(\lambda, \mathbf{u}) = g_0(\mathbf{u}) + \sum_{i=1}^{N} \lambda_i g_i(\mathbf{u}) \tag{4}$$

It follows from the Kuhn–Tucker conditions that there exists a $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_N^*)$ such that $(\boldsymbol{\lambda}^*, \mathbf{u}^*)$ is the optimal solution of the problem

$$\max_{\lambda \geqslant 0} [\min_{\mathbf{u}} \{g(\boldsymbol{\lambda}, \mathbf{u})\} - \boldsymbol{\lambda}' c] \tag{5}$$

Let

$$\mathbf{Q}(\boldsymbol{\lambda}, t) = \mathbf{Q}_0(t) + \sum_{i=1}^{N} \lambda_i \mathbf{Q}_i(t)$$

while $\mathbf{R}(\boldsymbol{\lambda}, t)$, $\mathbf{S}(\boldsymbol{\lambda})$, $\mathbf{a}(\boldsymbol{\lambda}, t)$, $\mathbf{b}(\boldsymbol{\lambda}, t)$ and $\mathbf{h}(\lambda)$ are defined similarly. It follows from (3) that $g(\boldsymbol{\lambda}, \mathbf{u})$ is given by

$$\begin{aligned} g(\boldsymbol{\lambda}, \mathbf{u}) = {} & \frac{1}{2} \int_0^T (\mathbf{x}'(t)\mathbf{Q}(\boldsymbol{\lambda}, t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(\boldsymbol{\lambda}, t)\mathbf{u}(t))\, \mathrm{d}t + \frac{1}{2}\mathbf{x}'(T)\mathbf{S}(\boldsymbol{\lambda}, T)\mathbf{x}(T) \\ & + \int_0^T (\mathbf{a}'(\boldsymbol{\lambda}, t)\mathbf{x}(t) + \mathbf{b}'(\boldsymbol{\lambda}, t)\mathbf{u}(t))\, \mathrm{d}t + \mathbf{h}'(\boldsymbol{\lambda})\mathbf{x}(T) \end{aligned}$$

Theorem 3.1

For every $\boldsymbol{\lambda} \geqslant 0$, the optimal control law for the problem

$$\min_{\mathbf{u}} g(\boldsymbol{\lambda}, \mathbf{u}) \tag{6}$$

subject to the linear system (1) is

$$\mathbf{u}(\lambda, t) = -\mathbf{R}^{-1}(\lambda, t)\, [\mathbf{B}'(t)\mathbf{P}(\lambda, t)\mathbf{x}(t) + \mathbf{B}'(t)\,\mathbf{d}(\lambda, t) + \mathbf{b}(\lambda, t)] \tag{7}$$

and the resulting cost is

$$g(\boldsymbol{\lambda}, \mathbf{u}(\boldsymbol{\lambda}, t)) = \tfrac{1}{2}\mathbf{x}'(0)\mathbf{P}(0)\mathbf{x}(0) + \mathbf{d}'(0)\mathbf{x}(0) + \tfrac{1}{2}p(0) \tag{8}$$

where

$$\dot{\mathbf{P}} = -\mathbf{PA} - \mathbf{A}'\mathbf{P} + \mathbf{PBR}^{-1}(\lambda)\mathbf{B}'\mathbf{P} - \mathbf{Q}(\lambda), \qquad \mathbf{P}(T) = \mathbf{S}(\lambda, T) \tag{9}$$

$$\dot{\mathbf{d}} = -[\mathbf{A} - \mathbf{BR}^{-1}(\lambda)\mathbf{B}'\mathbf{P}]'\mathbf{d} - \mathbf{a}(\lambda) + \mathbf{PBR}^{-1}(\lambda)\mathbf{b}(\lambda), \qquad \mathbf{d}(T) = \mathbf{h}(\lambda) \tag{10}$$

$$\dot{p} = [\mathbf{B}'\mathbf{d} + \mathbf{b}(\lambda)]^{-1}\mathbf{R}^{-1}(\lambda)[\mathbf{B}'\mathbf{d} + \mathbf{b}(\lambda)], \qquad p(T) = 0 \tag{11}$$

Proof. The optimal control law (7) and the resulting optimal cost (8) are derived by solving the Hamilton–Jacobi equation associated with (6). □

The following theorem is an immediate consequence of the Lagrangian duality (see Reference 10) and Theorem (3.1).

Theorem 3.2

Let $\boldsymbol{\lambda}(\mathbf{x}(0))$ be the optimal solution of

$$\max_{\boldsymbol{\lambda} \geqslant 0} \{\tfrac{1}{2}\,\mathbf{x}'(0)\,\mathbf{P}(0)\,\mathbf{x}(0) + \mathbf{d}'(0)\,\mathbf{x}(0) + \tfrac{1}{2}\,p(0) - \boldsymbol{\lambda}'\mathbf{c}\} \tag{12}$$

subject to (9)–(11). Then the optimal control law for the problem (1)–(3) is

$$\mathbf{u}(\boldsymbol{\lambda}(\mathbf{x}(0)), t) = -\mathbf{R}^{-1}(\boldsymbol{\lambda}(\mathbf{x}(0)), t)[\mathbf{B}'(t)\,\mathbf{P}(t)\,\mathbf{x}(t) + \mathbf{B}'(t)\,\mathbf{d}(t) + \mathbf{b}(\lambda(\mathbf{x}(0)), t)] \tag{13}$$

Once again, it is easily observed that the optimal control law (13) is calculated by solving the optimal parameter selection problem (9)–(12) and an unconstrained LQ optimal control problem. The optimal control law has a feedback structure. However, it is not a true feedback control law because the optimal solution $\boldsymbol{\lambda}(\mathbf{x}(0))$ of the optimal parameter selection problem (9)–(12) is dependent on the value of the initial state $\mathbf{x}(0)$. Nevertheless, it still possesses certain robustness properties due to its feedback structure. We shall demonstrate this point by means of a numerical example in Section 4.

3.2. Optimal parameter selection problem

In view of Theorem 3.2, it is clear that we need to solve the optimal parameter selection problem (9)–(12). In fact, optimal parameter selection problems can be solved as mathematical programming problems. In other words, so long as the value of the cost functional and the gradient of the cost functional can be calculated for any given $\boldsymbol{\lambda}$, algorithms for solving mathematical programming problems can be used to solve optimal parameter selection problems. In this section, we consider a slightly more general optimal parameter selection problem. We then derive the gradient formulas for its cost functional. On this basis, the numerical optimal control software package MISER 3.1[8] can be used to solve this problem.

The optimal parameter selection problem considered in this section is

$$\max_{\boldsymbol{\lambda} \geqslant 0} \{J(\boldsymbol{\lambda}) \equiv \Phi(\mathbf{P}(0), \mathbf{d}(0), p(0), \boldsymbol{\lambda})\} \tag{14}$$

subject to

$$\dot{\mathbf{P}}(t) = \boldsymbol{\Gamma}(t, \mathbf{P}(t), \boldsymbol{\lambda}), \qquad \mathbf{P}(T) = \mathbf{P}_T(\boldsymbol{\lambda}) \tag{15}$$

$$\dot{\mathbf{d}}(t) = \mathbf{f}(t, \mathbf{P}(t), \mathbf{d}(t), \boldsymbol{\lambda}), \quad \mathbf{d}(T) = \mathbf{d}_T(\boldsymbol{\lambda}) \tag{16}$$

$$\dot{p}(t) = q(t, \mathbf{d}(t), \boldsymbol{\lambda}), \qquad p(T) = p_T(\boldsymbol{\lambda}) \tag{17}$$

where Φ, $\boldsymbol{\Gamma}$, $\mathbf{f}$ and q as well as $\mathbf{P}_T$, $\mathbf{d}_T$ and p_T are given functions. We assume that $\boldsymbol{\Gamma}$ and $\mathbf{P}_T$ are symmetric, so that the solution $\mathbf{P}(t)$ of (15) is symmetric.

Define the Hamiltonian

$$\begin{aligned} &H(t, \mathbf{P}(t), \mathbf{d}(t), p(t), \boldsymbol{\lambda}, \boldsymbol{\Lambda}(t), \beta(t), \gamma(t)) \\ &\quad = \operatorname{tr}\{\boldsymbol{\Lambda}(t)\boldsymbol{\Gamma}(t, \mathbf{P}(t), \boldsymbol{\lambda})\} + [\boldsymbol{\beta}(t)]' \, \mathbf{f}(t, \mathbf{P}(t), \mathbf{d}(t), \boldsymbol{\lambda}) + \gamma(t)\, q(t, \mathbf{d}(t), \boldsymbol{\lambda}) \end{aligned} \tag{18}$$

where $\boldsymbol{\Lambda}(t) \in \mathbf{R}^{n\times n}$, $\boldsymbol{\beta}(t) \in R^n$ and $\gamma(t) \in \mathbf{R}$ are defined, for each given $\lambda \in \mathbf{R}^N$, by the costate equations

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\Lambda}(t)]' = -\frac{\partial H(t, \mathbf{P}(t), \dots)}{\partial \mathbf{P}(t)}, \qquad [\boldsymbol{\Lambda}(0)]' = -\frac{\partial \Phi(\mathbf{P}(0), \dots)}{\partial \mathbf{P}(0)} \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\beta}(t)]' = -\frac{\partial H(t, \dots, \mathbf{d}(t), \dots)}{\partial \mathbf{d}(t)}, \qquad [\boldsymbol{\beta}(0)]' = -\frac{\partial \Phi(\dots, \mathbf{d}(0), \dots)}{\partial \mathbf{d}(0)} \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = -\frac{\partial H(t, \dots, p(t), \dots)}{\partial p(t)}, \qquad \gamma(0) = -\frac{\partial \Phi(\dots, p(0), \dots)}{\partial p(0)} \tag{21}$$

The gradient formula for the cost functional (14) subject to the system of equations (15)–(17) is given as follows. The derivation is in the appendix.

Theorem 3.3

Given the optimal parameter selection problem (14)–(17), the gradient formula of the cost functional $J(\boldsymbol{\lambda})$ with respect to $\boldsymbol{\lambda}$ is

$$\begin{aligned} \frac{\partial J(\boldsymbol{\lambda})}{\partial \lambda} = {} & \frac{\partial \Phi(\dots, \boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} - [\operatorname{vec} \boldsymbol{\Lambda}(T)]' \frac{\partial(\operatorname{vec} \mathbf{P}_T(\boldsymbol{\lambda}))}{\partial \boldsymbol{\lambda}} - [\boldsymbol{\beta}(T)]' \frac{\partial \mathbf{d}_T(\boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} \\ & - \gamma(T) \frac{\partial p_T(\boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} + \int_0^T \frac{\partial H(t, \dots, \lambda, \dots)}{\partial \boldsymbol{\lambda}} \, \mathrm{d}t \end{aligned} \tag{22}$$

where, for a given matrix $\mathbf{M}$ with columns $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k$, $\operatorname{vec} \mathbf{M}$ denotes the column vector $(\mathbf{m}_1', \mathbf{m}_2', \dots, \mathbf{m}_k')'$.

When $\mathbf{P}(t)$, $\mathbf{d}(t)$ and $p(t)$ are given by (9)–(11) and $J(\boldsymbol{\lambda})$ is the cost functional in (12), it follows from Theorem 3.3 that the gradient of the cost functional (12) can be expressed in the following way. For a derivation of this result, see the appendix. Another way of calculating the gradients can be found in Reference 11.

Theorem 3.4

Let $\lambda \geqslant 0$ be given. The gradient of the cost functional $J(\boldsymbol{\lambda})$ of the optimal parameter selection problem (9)–(12) is

$$\frac{\partial J(\boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} = \left[\frac{\partial J(\boldsymbol{\lambda})}{\partial \lambda_1}, \dots, \frac{\partial J(\boldsymbol{\lambda})}{\partial \lambda_N}\right]$$

$$\begin{aligned} \frac{\partial J(\boldsymbol{\lambda})}{\partial \lambda_p} = {} & \frac{1}{2}\int_0^T [\boldsymbol{\beta}'(t)\mathbf{Q}_p\boldsymbol{\beta}(t) + \eta'(t)\,\mathbf{R}_p\boldsymbol{\eta}(t)]\,\mathrm{d}t + \frac{1}{2}\boldsymbol{\beta}'(T)\mathbf{S}_p\boldsymbol{\beta}(T) \\ & + \int_0^T [\mathbf{a}_p'(t)\boldsymbol{\beta}(t) + \mathbf{b}_p'(t)\boldsymbol{\eta}(t)]\,\mathrm{d}t + \mathbf{h}_p'(T)\,\boldsymbol{\beta}(T) - c_p \end{aligned} \tag{23}$$

where $\boldsymbol{\beta}(t)$ is the solution of the costate equation

$$\dot{\boldsymbol{\beta}}(t) = \mathbf{A}(t)\boldsymbol{\beta}(t) + \mathbf{B}(t)\boldsymbol{\eta}(t), \quad \boldsymbol{\beta}(0) = \mathbf{x}_0 \tag{24}$$

while $\boldsymbol{\eta}(t)$ is given by

$$\boldsymbol{\eta}(t) = -\mathbf{R}^{-1}(\boldsymbol{\lambda}, t)[\mathbf{B}'(t)\mathbf{P}(t)\boldsymbol{\beta}(t) + \mathbf{B}'(t)\mathbf{d}(t) + \mathbf{b}(\boldsymbol{\lambda}, t)] \tag{25}$$

and $\mathbf{P}(t)$, $\mathbf{d}(t)$, $p(t)$ are the solutions of the differential equations (9)–(11).

3.3. Application to minmax LQ problems

The difficulty in dealing with the minimax LQ problems is solving the corresponding dual problems or the problems associated with the Riccati equations. We have seen that the quadratically constrained LQ problem (1)–(3) can be solved by solving the optimal parameter selection problem (9)–(12). In fact, the quadratically constrained LQ problem can be reduced to an optimal parameter selection problem very similar to the one considered in previous sections. Consider the following min–max problem

$$\max_{\mathbf{u}\in L_2^m[0,T]} \; \max_{1\leqslant i\leqslant N} \; g_i(\mathbf{u}) \tag{26}$$

where $g_i(\mathbf{u})$ is given by (3) and the underlying systems equations are given by (1). The following theorem is proven in Reference 3.

Theorem 3.5

If $\boldsymbol{\lambda}^*$ is the optimal solution of the optimal parameter selection problem

$$\max_{\lambda} J(\boldsymbol{\lambda}) = \tfrac{1}{2}\,\mathbf{x}_0'\mathbf{P}(0)\mathbf{x}_0 + \mathbf{d}'(0)\mathbf{x}_0 + \tfrac{1}{2}\,p(0) \tag{27}$$

subject to

$$\sum_{i=1}^{N} \lambda_i = 1, \quad \lambda_i \geqslant 0 \tag{28}$$

and (9)–(11), then

$$\mathbf{u}(\boldsymbol{\lambda}^*, t) = -\mathbf{R}^{-1}(\boldsymbol{\lambda}^*, t)[\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}(t) + \mathbf{B}'(t)\mathbf{d}(t) + \mathbf{b}(t)] \tag{29}$$

is the minimax solution.

Problem (9)–(11), (27) and (28) is an optimal parameter selection problem and can be solved using the control parameterization technique[9] once the gradient of (27) is given. With the techniques used to derive the gradient formula given in Theorem 3.3 we can show that the gradient of the cost functional (27) is as given in the following theorem.

Theorem 3.6

Let $\lambda > 0$ be given. The gradient of the cost functional $J(\boldsymbol{\lambda})$ of the optimal parameter selection problem (9)–(11) and (27) and (28) is

$$\frac{\partial J(\boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} = \left[\frac{\partial J(\boldsymbol{\lambda})}{\partial \lambda_1}, \ldots, \frac{\partial J(\boldsymbol{\lambda})}{\partial \lambda_N}\right]$$

where

$$\frac{\partial J(\boldsymbol{\lambda})}{\partial \lambda_p} = \frac{1}{2}\int_0^T [\boldsymbol{\beta}'(t)\mathbf{Q}_p\boldsymbol{\beta}(t) + \boldsymbol{\eta}'(t)\mathbf{R}_p\boldsymbol{\eta}(t)]\,\mathrm{d}t + \frac{1}{2}\boldsymbol{\beta}'(T)\mathbf{S}_p\boldsymbol{\beta}(T)$$
$$+ \int_0^T [\mathbf{a}_p'(t)\boldsymbol{\beta}(t) + \mathbf{b}_p'(t)\boldsymbol{\eta}(t)]\,\mathrm{d}t + \mathbf{h}_p'(T)\boldsymbol{\beta}(T)$$

where $\boldsymbol{\beta}(t)$ is the solution of the costate equation

$$\dot{\boldsymbol{\beta}}(t) = \mathbf{A}(t)\boldsymbol{\beta}(t) + \mathbf{B}(t)\boldsymbol{\eta}(t), \quad \beta(0) = \mathbf{x}_0$$

$\boldsymbol{\eta}(t)$ is given by

$$\boldsymbol{\eta}(t) = -\mathbf{R}^{-1}(\boldsymbol{\lambda}, t)\,[\mathbf{B}'(t)\mathbf{P}(t)\boldsymbol{\beta}(t) + \mathbf{B}'(t)\mathbf{d}(t) + \mathbf{b}(\boldsymbol{\lambda}, t)]$$

and $\mathbf{P}(t)$, $\mathbf{d}(t)$, $p(t)$ are the solutions of the differential equations (9)–(11).

In Reference 2, an algorithm based on convex optimization is proposed to solve (26). They prove that the vector $\mathbf{y}^* \in \mathbf{R}^N$ such that $y_i^* = g_i(\mathbf{u}^*)$ satisfies a certain infinite-dimensional inequality constraint. The algorithm they propose attempts to satisfy this constraint by updating current estimates $\mathbf{y}^n \in \mathbf{R}^N$ of $\mathbf{y}^*$. A bi-product of each iteration is a Lagrange multiplier $\boldsymbol{\lambda}^n$ and sub-optimal control $\mathbf{u}(\boldsymbol{\lambda}^n, t)$.

4. EXAMPLE

For illustration, we consider the following example. The system equations are given by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$$
$$\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}4\\ -4\end{bmatrix} \tag{30}$$

while cost and the constraint functionals are given by

$$g_0(\mathbf{u}) = \frac{1}{2}x_1(1)^{\;2} + \frac{1}{2}\int_0^1 (x_1(t)^2 + u_1(t)^2 + u_2(t)^2)\,\mathrm{d}t \tag{31}$$

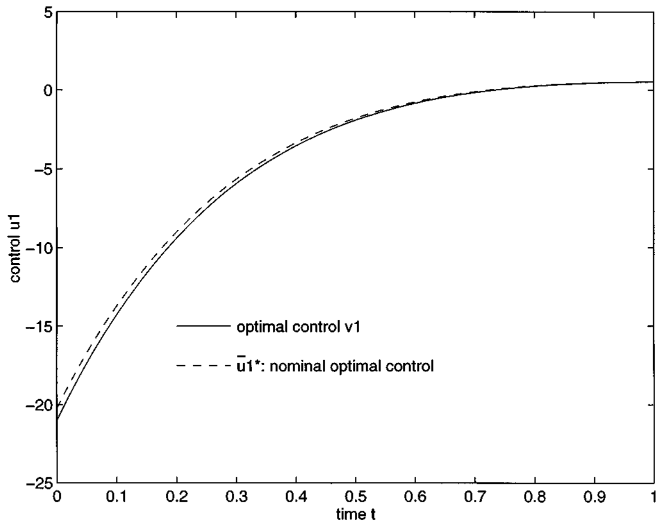

Figure 1. Optimal hybrid feedback control v_1 and optimal open-loop control $\bar{u}_1^*$

and

$$g_1(\mathbf{u}) = \frac{1}{2} x_2(1)^2 + \frac{1}{2} \int_0^1 (x_2(t)^2 + u_1(t)^2 + u_2(t)^2)\, \mathrm{d}t \leqslant 8 \tag{32}$$

respectively. For comparison, we shall solve (30)–(32) using two methods: by the control parameterization method[7] 1 and the method developed in this paper. To use the control parameterization technique, we approximate each of the two controls by a piecewise constant functions, taking a constant value in each of the 50 equally divided subintervals. The optimal control software corresponding to the control parameterization technique, MISER 3.1 [8], is then used to calculate the suboptimal control $\bar{\mathbf{u}}^*(t) = (\bar{\mathbf{u}}_1^*(t), \bar{\mathbf{u}}_2^*(t))'$ (the corresponding state is denoted by $\bar{\mathbf{x}}^*(t) = (\bar{\mathbf{x}}_1^*(t), \bar{\mathbf{x}}_2^*(t))'$. An approximate optimal cost value of $g_0(\bar{\mathbf{u}}^*) = 62{\cdot}69162$ is obtained. As for the method developed in this paper, we can readily write down the corresponding version of the optimal parameter selection problem (9)–(12). This maximization problem with respect to the parameter λ can also be solved by using MISER 3.1. The optimal parameter $\lambda^* = 0{\cdot}709887$ is obtained together with the corresponding behaviours of $\mathbf{P}(t, \lambda^*)$, $d(t, \lambda^*)$, and $p(t, \lambda^*)$. We can then calculate the optimal control law according to (13). For convenience , we denote the hybrid feedback control law obtained at this stage by $\mathbf{u}^*(\mathbf{x}(t))$. Using the NAG routine D02BAF[12], the optimal control $\mathbf{v} = (v_1, v_2)'$, the optimal states $\mathbf{y} = (y_1, y_2)'$, and the optimal cost value $g(\mathbf{v})$ are obtained by solving the differential equations corresponding to (30)–(32) with $\mathbf{u}(t)$ replaced by $\mathbf{u}^*(\mathbf{x}(t))$ which is now a function of the unknown state $\mathbf{x}(t)$. The cost value obtained here is $g(\mathbf{v}) = 62{\cdot}66103$. This result is slightly better than the one obtained by the control parameterization technique. The control policies and the state trajectories corresponding to $\bar{\mathbf{u}}^*$, $\mathbf{v}$, $\bar{\mathbf{x}}^*$ and y are plotted in Figures 1–4. The states $\bar{\mathbf{x}}^*$ and $\mathbf{y}$ are so close to each other that they are hardly distinguishable from their plots (Figures 3 and 4).

In general, the control law $\mathbf{u}^*(\mathbf{x}(t))$ is preferable to an open-loop control law $\bar{\mathbf{u}}^*$ due to its closed form which may make it possess some of the advantages of a closed-loop control law. For example, numerical experiment support the claim that the control law $\mathbf{u}^*(\mathbf{x}(t))$ is more robust to unmodelled state perturbations than the pure open-loop control law $\bar{\mathbf{u}}^*$.

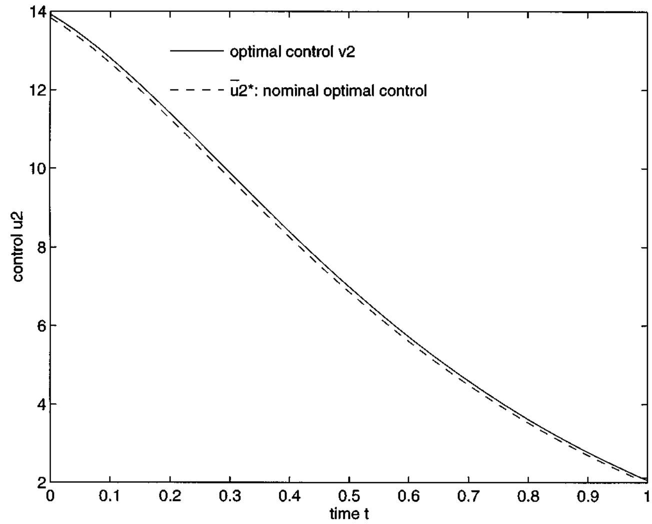

Figure 2. Optimal hybrid feedback control v_2 and optimal open-loop control $\bar{u}_2^*$

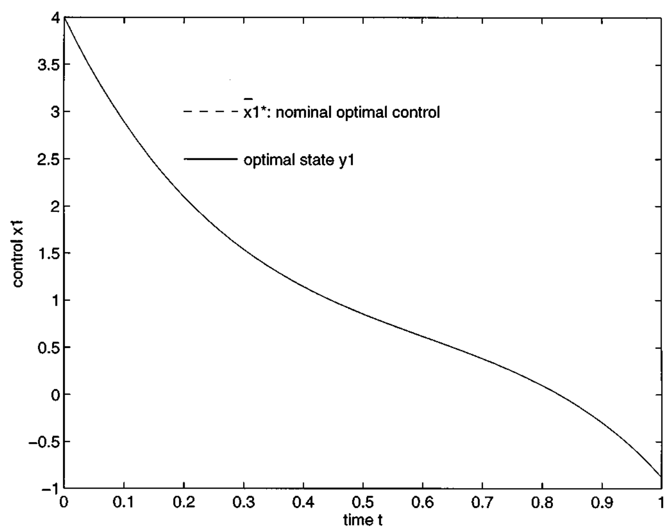

Figure 3. State trajectory x_1 corresponding to v and $\bar{u}^*$

5. CONCLUSION

We have derived the optimal control for the LQ optimal control problem subject to integral quadratic constraints. We have shown that the optimal control is a non-linear function of the current state and initial state, and is calculated by solving an optimal parameter selection problem together with an unconstrained LQ optimal control problem. Furthermore, the optimal parameter selection problem is easily solved using the numerical optimal control package MISER 3·1. We have also shown that the solution of the minmax LQ control problem can be obtained in a similar manner. In both the quadratically constrained LQ problem and the minmax LQ problem, solving an optimal parameter selection problem is crucial.

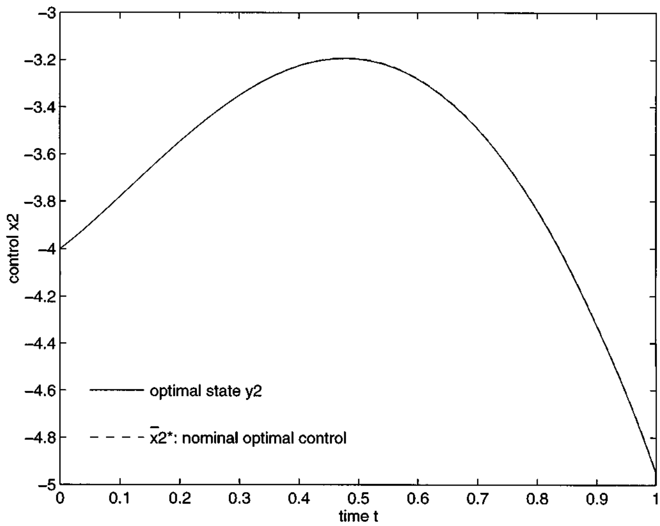

Figure 4. State trajectory x_2 corresponding to v and $\bar{u}^*$

APPENDIX I: PROOFS

Proof of Theorem (3.3). For any given fixed $\boldsymbol{\rho} \in \mathbf{R}^N$, let $\boldsymbol{\lambda}(\varepsilon) = \boldsymbol{\lambda} + \varepsilon\boldsymbol{\rho}$ where $0 < \varepsilon \ll 1$. For any function $g(\boldsymbol{\lambda})$, define $\Delta g(\boldsymbol{\lambda})$ by

$$\Delta g(\boldsymbol{\lambda}) = \left.\frac{\mathrm{d}g(\boldsymbol{\lambda}(\varepsilon))}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}$$

For the moment, let $\boldsymbol{\Lambda}(t)$, $\boldsymbol{\beta}(t)$ and $\gamma(t)$ be arbitrary functions. By the definitions of $J(\boldsymbol{\lambda})$ and $H(t)$ in (14) and (18), respectively, we have

$$J(\boldsymbol{\lambda}) = \Phi(\mathbf{P}(0), \mathbf{d}(0), p(0), \lambda) + \int_0^T [H(t) - [\operatorname{vec}\boldsymbol{\Lambda}(t)]'[\operatorname{vec}\boldsymbol{\Gamma}(t)] - [\boldsymbol{\beta}(t)]'\mathbf{f}(t) - \gamma(t)\,q(t)]\,\mathrm{d}t$$

it follows that

$$\Delta J(\boldsymbol{\lambda}) = \Delta\Phi + \int_0^T [\Delta H(t) - [\operatorname{vec}\boldsymbol{\Lambda}(t)]'\Delta[\operatorname{vec}\boldsymbol{\Gamma}(t)] - [\boldsymbol{\beta}(t)]'\Delta\mathbf{f}(t) - \gamma(t)\Delta q(t)]\,\mathrm{d}t \qquad (33)$$

We note two things. First,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\operatorname{vec}\mathbf{P}(t)] = \operatorname{vec}\dot{\mathbf{P}}(t) = \operatorname{vec}\boldsymbol{\Gamma}(t, \mathbf{P}(t), \boldsymbol{\lambda})$$

Therefore,

$$\operatorname{vec}\mathbf{P}(t) = \operatorname{vec}\mathbf{P}(T) + \int_T^t [\operatorname{vec}\boldsymbol{\Gamma}(s, \mathbf{P}(s), \boldsymbol{\lambda})]\,\mathrm{d}s$$

and

$$\Delta(\operatorname{vec}\mathbf{P}(t)) = \Delta(\operatorname{vec}\mathbf{P}(T)) + \int_T^t [\Delta(\operatorname{vec}\boldsymbol{\Gamma}(s, \mathbf{P}(s), \boldsymbol{\lambda}))]\,\mathrm{d}s$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\Delta(\operatorname{vec}\mathbf{P}(t))] = \Delta(\operatorname{vec}\boldsymbol{\Gamma}(t, \mathbf{P}(t), \boldsymbol{\lambda})) \tag{34}$$

Similarly, it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\Delta\mathbf{d}(t)] = \Delta\mathbf{f}(t, \mathbf{P}(t), \mathbf{d}(t), \boldsymbol{\lambda}) \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\Delta p(T)] = \Delta\mathbf{h}(t, \mathbf{d}(t), \boldsymbol{\lambda}) \tag{36}$$

Second, from the chain rule, we obtain

$$\Delta H(t) = \frac{\partial H(t)}{\partial(\operatorname{vec}\mathbf{P}(t))}\Delta(\operatorname{vec}\mathbf{P}(t)) + \frac{\partial H(t)}{\partial\mathbf{d}(t)}\Delta\mathbf{d}(t) + \frac{\partial H(t)}{\partial p(t)}\Delta p(t) + \frac{\partial H(t)}{\partial\boldsymbol{\lambda}}\boldsymbol{\rho} \tag{37}$$

Substituting (34)–(37) into (33), we obtain

$$\Delta J(\boldsymbol{\lambda}) = \Delta\Phi + \int_0^T \left[\frac{\partial H(t)}{\partial(\operatorname{vec}\mathbf{P}(t))}\Delta(\operatorname{vec}\mathbf{P}(t)) - (\operatorname{vec}\boldsymbol{\Lambda}(t))'\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\operatorname{vec}\mathbf{P}(t)) \right.$$
$$\left. + \frac{\partial H(t)}{\partial\mathbf{d}(t)}\Delta\mathbf{d}(t) - [\boldsymbol{\beta}(t)]'\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\mathbf{d}(t)) + \frac{\partial H(t)}{\partial p(t)}\Delta p(t) - \gamma(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta p(t)) + \frac{\partial H(t)}{\partial\boldsymbol{\lambda}}\boldsymbol{\rho}\,\mathrm{d}t\right.$$

Up until now, we have assumed that $\boldsymbol{\Lambda}(t)$, $\boldsymbol{\beta}(t)$ and $\gamma(t)$ are arbitrary functions. Choosing $\boldsymbol{\Lambda}(t)$, $\boldsymbol{\beta}(t)$ and $\gamma(t)$ to satisfy (15)–(17) and noting that the solution $\boldsymbol{\Lambda}(t)$ of (15) is symmetric, we obtain

$$\begin{aligned}
\Delta J(\boldsymbol{\lambda}) &= \Delta\Phi + \int_0^T \left[-\frac{\mathrm{d}}{\mathrm{d}t}([\operatorname{vec}\boldsymbol{\Lambda}(t)]'[\Delta(\operatorname{vec}\mathbf{P}(t))]) - \frac{\mathrm{d}}{\mathrm{d}t}([\boldsymbol{\beta}(t)]'[\Delta\mathbf{d}(t)]) \right. \\
&\qquad \left. - \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t)\Delta p(t)) + \frac{\partial H(t)}{\partial\boldsymbol{\lambda}}\boldsymbol{\rho}\right]\mathrm{d}t \\
&= \frac{\partial\Phi}{\partial(\operatorname{vec}\mathbf{P}(0))}\Delta(\operatorname{vec}\mathbf{P}(0)) + \frac{\partial\Phi}{\partial\mathbf{d}(0)}\Delta\mathbf{d}(0) + \frac{\partial\Phi}{\partial p(0)}\Delta p(0) + \frac{\partial\Phi}{\partial\boldsymbol{\lambda}}\boldsymbol{\rho} \\
&\qquad - [\operatorname{vec}\boldsymbol{\Lambda}(T)]'[\Delta(\operatorname{vec}\mathbf{P}(T))] + [\operatorname{vec}\boldsymbol{\Lambda}(0)]'[\Delta(\operatorname{vec}\mathbf{P}(0))] \\
&\qquad - [\boldsymbol{\beta}(T)]'[\Delta\mathbf{d}(T)] + [\boldsymbol{\beta}(0)]'[\Delta\mathbf{d}(0)] - [\gamma(T)]'[\Delta p(T)] \\
&\qquad + [\gamma(0)][\Delta p(0)] + \int_0^T \frac{\partial H(t, \dots, \boldsymbol{\lambda}, \dots)}{\partial\boldsymbol{\lambda}}\mathrm{d}t\,\boldsymbol{\rho} \\
&= \frac{\partial\Phi(\dots, \boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}\boldsymbol{\rho} - [\operatorname{vec}\boldsymbol{\Lambda}(T)]'[\Delta(\operatorname{vec}\mathbf{P}_T(\lambda))] - [\beta(T)]'[\Delta\mathbf{d}_T(\boldsymbol{\lambda})] \\
&\qquad - [\gamma(T)][\Delta p_T(\boldsymbol{\lambda})] + \int_0^T \frac{\partial H(t, \dots, \boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}\mathrm{d}t\,\boldsymbol{\rho}
\end{aligned}$$

Since

$$\Delta(\operatorname{vec}\mathbf{P}_T(\boldsymbol{\lambda})) = \frac{\partial(\operatorname{vec}\mathbf{P}_T(\boldsymbol{\lambda}))}{\partial\boldsymbol{\lambda}}\,\boldsymbol{\rho},\quad \Delta\mathbf{d}_T(\boldsymbol{\lambda}) = \frac{\partial\mathbf{d}_T(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}\,\boldsymbol{\rho},\quad \Delta p_T(\boldsymbol{\lambda}) = \frac{\partial p_T(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}\,\boldsymbol{\rho}$$

we obtain

$$\begin{aligned}\Delta J(\boldsymbol{\lambda}) &= \frac{\partial J(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}\,\boldsymbol{\rho}\\ &= \left[\frac{\partial\Phi(\ldots,\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}} - [\operatorname{vec}\boldsymbol{\Lambda}(T)]'\,\frac{\partial(\operatorname{vec}\mathbf{P}_T(\boldsymbol{\lambda}))}{\partial\boldsymbol{\lambda}} - [\boldsymbol{\beta}(T)]'\,\frac{\partial\mathbf{d}_T(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}\right.\\ &\qquad\left. - [\gamma(T)]'\,\frac{\partial p_T(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}} + \int_0^T \frac{\partial H(t,\ldots,\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}\,\mathrm{d}t\right]\boldsymbol{\rho}\end{aligned}$$

The result follows from the fact that $\boldsymbol{\rho}$ is arbitrary. □

Proof of Theorem 3.4. We consider now the cost functionals for the optimal parameter selection problem (9)–(12). This is of the form

$$J(\boldsymbol{\lambda}) = \tfrac{1}{2}\,\boldsymbol{\xi}'\mathbf{P}(0,\boldsymbol{\lambda})\,\boldsymbol{\xi} + \mathbf{d}'(0,\boldsymbol{\lambda})\,\boldsymbol{\xi} + \tfrac{1}{2}\,p(0,\boldsymbol{\lambda}) - \boldsymbol{\lambda}'\mathbf{c} \tag{38}$$

The Hamiltonian (18) becomes

$$\begin{aligned}&H(t,\mathbf{P}(t),\mathbf{d}(t),p(t),\boldsymbol{\lambda},\boldsymbol{\Lambda}(t),\boldsymbol{\beta}(t),\gamma(t))\\ &\quad= \operatorname{tr}\{(-\tfrac{1}{2}\,\boldsymbol{\Lambda}(t))(-\mathbf{P}\mathbf{A} - \mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\,\mathbf{B}'\mathbf{P} - \mathbf{Q}(\boldsymbol{\lambda}))\}\\ &\qquad + [-\boldsymbol{\beta}(t)]'\,[-(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\,\mathbf{B}'\mathbf{P})'\,\mathbf{d} - \mathbf{a}(\boldsymbol{\lambda}) + \mathbf{P}\mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\,\mathbf{b}(\boldsymbol{\lambda})]\\ &\qquad + [-\gamma(t)]\,[(\mathbf{B}'\mathbf{d} + \mathbf{b}(\boldsymbol{\lambda}))'\mathbf{R}^{-1}(\boldsymbol{\lambda})\,(\mathbf{B}'\mathbf{d} + \mathbf{b}(\boldsymbol{\lambda}))]\end{aligned} \tag{39}$$

It is clear from (21), (38) and (39) that

$$\dot{\gamma}(t) = 0,\quad \gamma(0) = \tfrac{1}{2}$$

Hence,

$$\gamma(t) = \tfrac{1}{2}$$

From (20), we obtain

$$\dot{\boldsymbol{\beta}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\,\mathbf{B}'\mathbf{P})\,\boldsymbol{\beta}(t) - \mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\,(\mathbf{B}'\mathbf{d} + \mathbf{b}(\boldsymbol{\lambda})),\quad \boldsymbol{\beta}(0) = \boldsymbol{\xi} \tag{40}$$

Using the identities

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{P}}\operatorname{tr}\{\mathbf{F}\mathbf{P}\} = \mathbf{F}'\quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}\mathbf{P}}\operatorname{tr}\{\mathbf{F}\mathbf{P}\mathbf{G}\mathbf{P}\} = \mathbf{F}\mathbf{P}\mathbf{G} + \mathbf{G}\mathbf{P}\mathbf{F}$$

when $\mathbf{F}$ and $\mathbf{G}$ are symmetric matrices of appropriate dimension, we obtain from (19) that

$$\begin{aligned}\dot{\mathbf{\Lambda}}(t) &= (\mathbf{A}-\mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{B}'\mathbf{P})\,\mathbf{\Lambda}(t)+\mathbf{\Lambda}(t)(\mathbf{A}-\mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{B}'\mathbf{P})' \\ &\quad -\boldsymbol{\beta}(t)(\mathbf{B}'\mathbf{d}+\mathbf{b}(\boldsymbol{\lambda}))'\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{B}'-\mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})(\mathbf{B}'\mathbf{d}+\mathbf{b}(\boldsymbol{\lambda}))\boldsymbol{\beta}'(t) \\ \mathbf{\Lambda}(0) &= \xi\xi'\end{aligned} \tag{41}$$

It is easily shown that the solution $\mathbf{\Lambda}(t)$ satisfies

$$\mathbf{\Lambda}(t)=\boldsymbol{\beta}(t)\,\boldsymbol{\beta}'(t) \tag{42}$$

We shall now evaluate the gradient of the cost functional (38) using the result stated in Theorem 3.3. For this we need the following identities. First, if $\mathbf{F}$ and $\mathbf{G}$ are matrices independent of $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_N)$ and $\mathbf{R}^{-1}(\boldsymbol{\lambda})$ is defined as before, then

$$\frac{\partial}{\partial\lambda_j}\operatorname{tr}\{\mathbf{F}\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{G}\} = -\operatorname{tr}\{\mathbf{F}\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{R}_j\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{G}\}$$

Second, if $\mathbf{x}$ is a column vector of appropriate dimension which is independent of $\boldsymbol{\lambda}$, and $\mathbf{R}(\boldsymbol{\lambda})$, $\mathbf{b}(\boldsymbol{\lambda})$ are defined as before, then

$$\frac{\partial}{\partial\lambda_j}[\mathbf{x}'\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{b}(\boldsymbol{\lambda})] = -\mathbf{x}'\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{R}_j\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{b}(\boldsymbol{\lambda})+\mathbf{x}'\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{b}_j$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_j}[\mathbf{b}'(\boldsymbol{\lambda})\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{b}(\boldsymbol{\lambda})] = -\mathbf{b}'(\boldsymbol{\lambda})\mathbf{R}^{-1}\mathbf{R}_j\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{b}(\boldsymbol{\lambda})+2\mathbf{b}'(\boldsymbol{\lambda})\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{b}_j$$

Using $\mathbf{\Lambda}(t)=\boldsymbol{\beta}(t)\boldsymbol{\beta}'(t)$ and the identities above, it follows that

$$\begin{aligned}\frac{\partial H}{\partial\lambda_j} &= \tfrac{1}{2}\boldsymbol{\beta}'(t)(\mathbf{P}\mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{R}_j\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{B}'\mathbf{P}+\mathbf{Q}_j)\,\boldsymbol{\beta}(t)+\boldsymbol{\beta}'(t)\mathbf{P}\mathbf{B}\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{R}_j\mathbf{R}^{-1}(\boldsymbol{\lambda})\,(\mathbf{B}'\mathbf{d}+\mathbf{b}(\boldsymbol{\lambda})) \\ &\quad +\tfrac{1}{2}(\mathbf{B}'\mathbf{d}+\mathbf{b}(\boldsymbol{\lambda}))'\mathbf{R}^{-1}(\boldsymbol{\lambda})\mathbf{R}_j\mathbf{R}^{-1}(\boldsymbol{\lambda})(\mathbf{B}'\mathbf{d}+\mathbf{b}(\boldsymbol{\lambda})) \\ &\quad +\mathbf{a}_j'\boldsymbol{\beta}(t)-\mathbf{b}_j'(\mathbf{R}^{-1}(\boldsymbol{\lambda})\,[\mathbf{B}'\mathbf{P}\boldsymbol{\beta}(t)+\mathbf{B}'\mathbf{d}+\mathbf{b}(\boldsymbol{\lambda})])\end{aligned}$$

Putting $\boldsymbol{\eta}(t)=-\mathbf{R}^{-1}(\boldsymbol{\lambda})\,[\mathbf{B}'\mathbf{P}\boldsymbol{\beta}(t)+\mathbf{B}'\mathbf{d}+\mathbf{b}(\boldsymbol{\lambda})]$ gives

$$\frac{\partial H}{\partial\lambda_j}=\tfrac{1}{2}\,[\boldsymbol{\beta}'(t)\,\mathbf{Q}_j\boldsymbol{\beta}(t)+\boldsymbol{\eta}'(t)\,\mathbf{R}_j\boldsymbol{\eta}(t)]+[\mathbf{a}_j'(t)\,\boldsymbol{\beta}(t)+\mathbf{b}_j'(t)\,\boldsymbol{\eta}(t)] \tag{43}$$

It is straightforward to show that

$$\left[-\frac{\operatorname{vec}\mathbf{\Lambda}(T)}{2}\right]'\frac{\partial(\operatorname{vec}\mathbf{P}_T(\boldsymbol{\lambda}))}{\partial\lambda_j} = -\frac{1}{2}\operatorname{tr}\{\mathbf{\Lambda}(T)\mathbf{S}_j\} = -\frac{1}{2}\boldsymbol{\beta}'(T)\mathbf{S}_j\boldsymbol{\beta}(T) \tag{44}$$

$$[-\boldsymbol{\beta}(T)]'\,\frac{\partial\mathbf{d}_T(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}_j} = -\boldsymbol{\beta}'(T)\,\mathbf{h}_j(T) \tag{45}$$

$$[-\gamma(T)]\,\frac{\partial p_T(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}=0 \tag{46}$$

$$\frac{\partial\Phi}{\partial\lambda_j}=-c_j \tag{47}$$

By substituting (43)–(45) into (22), we obtain the desired result.

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