CHAPTER 7 □ TRANSPORT THEORY OF WAVE PROPAGATION IN RANDOM PARTICLES

In Chapters 4–6, we discussed scattering and propagation characteristics of continuous and pulse waves in a tenuous distribution of particles. In this case, it was possible to employ the single scattering or the first order multiple scattering or the Rytov approximations. These approximations become inadequate as the particle density is increased and the coherent intensity becomes comparable to or less than the incoherent intensity. The multiple scattering effects then become dominant in determining the fluctuation characteristics of a wave. Historically, two distinct theories have been developed in dealing with multiple scattering problems. One may be called analytical theory and the other transport theory.

In analytical theory (see Chapters 14 and 15 for reference citations), we start with basic differential equations such as the Maxwell equations or the wave equation, introduce the scattering and absorption characteristics of particles, and obtain appropriate differential or integral equations for the statistical quantities such as variances and correlation functions. This is mathematically rigorous in the sense that in principle all the multiple scattering, diffraction, and interference effects can be included. However, in practice, it is impossible to obtain a formulation which completely includes all these effects, and various theories which yield useful solutions are all approximate, each being useful in a specific range of parameters. Twersky's theory, the diagram method, and the Dyson and Bethe–Salpeter equations belong to this analytical theory. In Chapters 14 and 15, we outline these theories.

Transport theory (Chandrasekhar, 1950; Sobolev, 1963; Preisendorfer, 1965; see Menzel, 1966, for selected papers), on the other hand, does not start with the wave equation. It deals directly with the transport of energy through a medium containing particles. The development of the theory is heuristic and it lacks the mathematical rigor of the analytical theory. Even though diffraction and interference effects are included in the description of the scattering and absorption characteristics of a single particle, transport theory itself does not include diffraction effects. It is assumed in transport theory that there is no correlation between fields, and therefore, the addition of powers rather than the addition of fields holds.

Transport theory, also called radiative transfer theory, was initiated by Schuster in 1903. The basic differential equation is called the equation of transfer and is equivalent to Boltzmann's equation (also known as the Maxwell–Boltzmann collision equation) used in the kinetic theory of gases (Sommerfeld, 1956, Chapter V) and in neutron transport theory.† The formulation is flexible and capable of treating many physical phenomena. It has been successfully employed for the problems of atmospheric and underwater visibility, marine biology, optics of papers and photographic emulsions, and the propagation of radiant energy in the atmospheres of planets, stars, and galaxies.

Since these two theories deal with the same phenomena, even though their starting points are different, we may expect that there may be some fundamental relationships between them. This is indeed the case, and we will show later that the specific intensity used in transport theory and the mutual coherence function used in analytical theory are related through a Fourier transform. This also means that even though transport theory was developed on the basis of the addition of powers, it contains information about the correlation of the fields (See Ishimaru, 1975, 1977.)

In transport theory, the polarization effect can be included through the Stokes matrix. However, in most cases, polarization is neglected mainly for mathematical convenience. In this chapter, all sections except Section 7-7 do not include the polarization effect. This must be kept in mind, particularly when interpreting experimental data.

In this chapter, we present definitions of important quantities and clarify their characteristics in free space and homogeneous and inhomogeneous random media containing particles. We then derive the basic differential equation called the equation of transfer for the intensity in random particles, discuss power conservation in such a medium, and clarify the boundary conditions. We also derive general integral formulations and give their physical interpretations. Partially polarized waves are incorporated into the equation of transfer by means of the Stokes parameters. In the last section, we present a point of view relating the specific intensity in transport theory and the Poynting vector in Maxwell's theory.

7-1 SPECIFIC INTENSITY, FLUX, AND ENERGY DENSITY

We start with the basic definitions of important quantities in transport theory. These quantities are specific intensity, flux, energy density, and average intensity. Among these, the most important and fundamental is specific intensity.

† Because of this equivalence, neutron transport theory is directly applicable to radiative transfer theory. For neutron transport theory, see Davison (1958), Case and Zweifel (1969), Bell and Glasstone (1970), and Williams (1971).

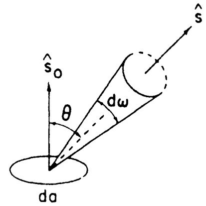


FIG. 7-1 Specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ and the power dP given in (7-1).

Let us consider a flow of wave energy at a point $\mathbf{r}$ in a random medium. The frequency, phase, and amplitude of the wave undergo some random variations in time, and therefore the magnitude and direction of its power flux density vector vary continuously in time. For a given direction defined by a unit vector $\hat{\mathbf{s}}$, we can find the average power flux density within a unit frequency band centered at frequency ν within a unit solid angle. This quantity $I(\mathbf{r}, \hat{\mathbf{s}})$ is called the specific intensity (also radiance or brightness) and is measured in W m^{-2} sr^{-1} Hz^{-1} (sr = steradian = unit solid angle). It is one of the fundamental quantities in the transport theory of wave propagation (for relationship with Poynting vector and mutual coherence function, see Sections 7-8 and 14-7).

The amount of power dP flowing within a solid angle $d\omega$ through an elementary area da oriented in a direction of unit vector $\hat{\mathbf{s}}_0$ in a frequency interval $(\nu, \nu + d\nu)$ is, therefore, given by (Fig. 7-1)

$$dP = I(\mathbf{r}, \hat{\mathbf{s}}) \cos\theta \, da \, d\omega \, d\nu \quad \text{(watts)}. \tag{7-1}$$

The specific intensity describes the radiation characteristic of the flux emitted from a surface (Fig. 7-2a). However, we can also take a point $\mathbf{r}$ on a hypothetical surface A in space and consider the power flux $I_- \, da \times d\omega \, d\nu$ flowing into da. This flux should be identical to the flux $I_+ \, da \, d\omega \, d\nu$

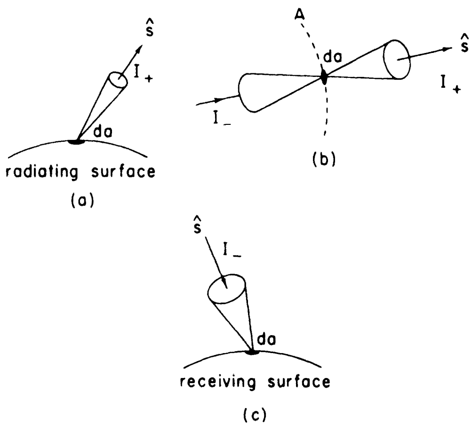


FIG. 7-2 Surface intensity I_+ and field intensity I_-.

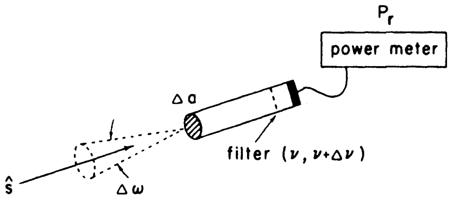


FIG. 7-3 Measurement of specific intensity.

emitted by da in the opposite direction $\hat{\mathbf{s}}$ (Fig. 7-2b). Similarly, we can consider the flux flowing into a surface from outside (Fig. 7-2c).

The specific intensity $I_+(\mathbf{r}, \hat{\mathbf{s}})$ describes the radiation emanating from a surface whether it is an actual radiating surface or a hypothetical surface, and is called the surface intensity. The radiation $I_-(\mathbf{r}, \hat{\mathbf{s}})$ received by a surface, whether it is a hypothetical surface in space or an actual surface, is called the field intensity. These two represent somewhat different concepts, but numerically they are identical and, therefore, no mathematical distinction needs to be made between them.

The measurement of a specific intensity at $\mathbf{r}$ in the direction $\hat{\mathbf{s}}$ can be made in the following manner: We take a receiver with a small receiving area Δa and a receiving solid angle $\Delta\omega$, and point it in the direction $-\hat{\mathbf{s}}$ (Fig. 7-3). The filter passes that portion of the wave in the frequency range $(\nu, \nu + \Delta\nu)$. The specific intensity is then given by the total received power P_r divided by $\Delta a\ \Delta\omega\ \Delta\nu$ when Δa, $\Delta\omega$, and $\Delta\nu$ are taken sufficiently small.

Let us consider the total flux passing through a small area da on a surface A. Let $\hat{\mathbf{s}}_0$ be a unit vector normal to the surface da (Fig. 7-4). This flux is given by integrating (7-1) over a solid angle 2π in the forward range $(0 \le \theta \le \pi/2)$ and can be written as $F_+\ da$ where F_+, the forward flux density, is defined by

$$F_+(\mathbf{r}, \hat{\mathbf{s}}_0) = \int_{(2\pi)+} I(\mathbf{r}, \hat{\mathbf{s}})\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_0\, d\omega, \qquad \hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_0 = \cos\theta. \tag{7-2}$$

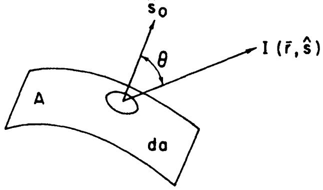


FIG. 7-4 Flux through da on a surface A.

Similarly, we can define the backward flux density F_- for the flux flowing through da in the backward $(-\hat{\mathbf{s}}_0)$ direction. This is given by

$$F_-(\mathbf{r}, \hat{\mathbf{s}}_0) = \int_{(2\pi)-} I(\mathbf{r}, \hat{\mathbf{s}})\hat{\mathbf{s}} \cdot (-\hat{\mathbf{s}}_0)\, d\omega \tag{7-3}$$

where the integration is over the solid angle 2π in the range $\pi/2 \le \theta \le \pi$. Both F_+ and F_- are measured in W m^{-2} Hz^{-1}. For a radiating surface, the forward flux density F_+ is often called the radiant emittance or the radiant excitance (Fig. 7-5a). When the flux is incident on the surface, F_- is called the irradiance. At a point $\mathbf{r}$ in space, we can consider the forward flux density F_+ in the direction $\hat{\mathbf{s}}_0$ perpendicular to a hypothetical surface A and the backward flux density F in the direction $-\hat{\mathbf{s}}_0$ (Fig. 7-5c). The vector sum of F_+ and F_- is the total flux density in the direction $\hat{\mathbf{s}}_0$.

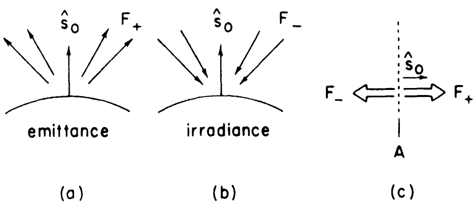


FIG. 7-5 Forward flux density F_+ and backward flux density F_-.

The total flux density can be expressed as that component of the flux density vector $\mathbf{F}(\mathbf{r})$ along $\hat{\mathbf{s}}_0$:

$$F_+(\mathbf{r}, \hat{\mathbf{s}}_0) - F_-(\mathbf{r}, \hat{\mathbf{s}}_0) = \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{s}}_0, \qquad \mathbf{F}(\mathbf{r}) = \int_{4\pi} I(\mathbf{r}, \hat{\mathbf{s}})\hat{\mathbf{s}}\, d\omega \tag{7-4}$$

where the integration is taken over a complete solid angle 4π. The vector $\mathbf{F}(\mathbf{r})$ represents the amount and the direction of the net flow of power and is used in Section 7-3 in connection with the conservation of power.

Let us consider the energy density $u(\mathbf{r})$ at $\mathbf{r}$. The amount of energy in time dt leaving a small area da in a direction normal to it within a solid angle $d\omega$ and a frequency interval $(\nu, \nu + d\nu)$ is $I\, da\, d\omega\, d\nu\, dt$. This energy should occupy a volume $da\, c\, dt$ where c is the velocity of wave propagation. Therefore, the energy density $du(\mathbf{r})$ in a unit frequency interval is given by

$$du(\mathbf{r}) = \frac{I\, da\, d\omega\, d\nu\, dt}{da\, c\, dt\, d\nu} = \frac{I(\mathbf{r}, \hat{\mathbf{s}})\, d\omega}{c}. \tag{7-5}$$

Adding the energy due to the radiation in all directions, we get the energy density:

$$u(\mathbf{r}) = \frac{1}{c}\int_{4\pi} I(\mathbf{r}, \hat{\mathbf{s}})\, d\omega. \tag{7-6}$$

Sometimes, we find it convenient to define the average intensity $U(\mathbf{r})$ by

$$U(\mathbf{r}) = \frac{1}{4\pi}\int_{4\pi} I(\mathbf{r}, \hat{\mathbf{s}})\, d\omega. \tag{7-7}$$

The average intensity does not in general represent the power flow, but is proportional to the energy density.

If a specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ is independent of the direction $\hat{\mathbf{s}}$, then the radiation is said to be "isotropic." If the specific intensity radiated from a surface da is isotropic, then the power P radiated from this surface da in a direction $\hat{\mathbf{s}}$ is given by

$$P\ (\mathrm{W\ sr^{-1}\ Hz^{-1}}) = (I\ da)\cos\theta = P_0 \cos\theta \tag{7-8}$$

where θ is the angle between the direction $\hat{\mathbf{s}}$ and the normal to the surface da. This relationship (7-8) is called Lambert's cosine law (Born and Wolf, 1964, p. 182).

7-2 SPECIFIC INTENSITY IN FREE SPACE AND AT BOUNDARIES BETWEEN HOMOGENEOUS MEDIA

Even though we are concerned with the problem of wave propagation in a random medium, it is instructive to examine how the quantities defined in the preceding section behave in free space and in a homogeneous medium. First, we will show that the specific intensity is invariant along the ray path in free space. At first sight, this appears to be contrary to our usual concept of power flux spreading in free space although it is not. Let us prove this invariance.

Consider the specific intensities I_1 and I_2 at two points $\mathbf{r}_1$ and $\mathbf{r}_2$ separated by a distance r along the direction $\hat{\mathbf{s}}$ and two small areas da_1 and da_2 perpendicular to $\hat{\mathbf{s}}$ (Fig. 7-6). We can express the power received by da_2 in two different ways. In terms of I_1, it is $I_1\ da_1\ d\omega_1$ according to (7-1). But in terms of I_2, it should be $I_2\ da_2\ d\omega_2$. These two should be equal. But $da_1\ d\omega_1 = da_2\ d\omega_2$ because $da_1 = r^2\ d\omega_2$ and $da_2 = r^2\ d\omega_1$, and, therefore, $I_1 = I_2$, proving the invariance of the specific intensity along the ray in free space. (For a more satisfactory proof, see Section 14-7.)

As an example, let us consider the radiation from a sphere of radius a into surrounding free space (Fig. 7-7). We assume that the radiation emitted

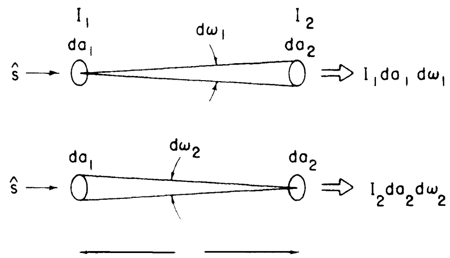


FIG. 7-6 Proof of invariance of specific intensity in free space.

from the surface of the sphere is independent of direction outward and thus at $r = a$, $I(\mathbf{r}, \hat{\mathbf{s}}) = I_0 = \text{const}$ when $\hat{\mathbf{s}}$ is directed in any direction outward from the surface, and $I(\mathbf{r}, \hat{\mathbf{s}}) = 0$ when $\hat{\mathbf{s}}$ is directed in any direction inward. Let us calculate the specific intensity at a distance r from the center of the sphere. Because of the invariance of I, we have $I(\mathbf{r}, \hat{\mathbf{s}}) = I_0$ within the angle $0 \le \theta \le \theta_0$ where $\theta_0 = \sin^{-1}(a/r)$. The flux density in the radial direction is given by

$$F_r(\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{r}} = \int I(\mathbf{r}, \hat{\mathbf{s}})\hat{\mathbf{s}} \cdot \hat{\mathbf{r}}\, d\omega = \int_0^{2\pi} d\phi \int_0^{\theta_0} \sin\theta\, d\theta\, I_0 \cos\theta = \pi I_0 \left(\frac{a}{r}\right)^2. \tag{7-9}$$

Note that even though I is constant, $F_r(\mathbf{r})$ decreases as r^{-2}, as would be expected from consideration of power conservation. The total power radiated is given by

$$P_t = F_r 4\pi r^2 = 4\pi^2 a^2 I_0 \tag{7-10}$$

which is of course independent of the distance r. The energy density $u(\mathbf{r})$ is given by

$$u(\mathbf{r}) = \frac{1}{c} \int_0^{2\pi} d\phi \int_0^{\theta_0} \sin\theta\, d\theta\, I_0 = \frac{2\pi I_0}{c} \left[1 - \sqrt{1 - \left(\frac{a}{r}\right)^2}\,\right] \tag{7-11}$$

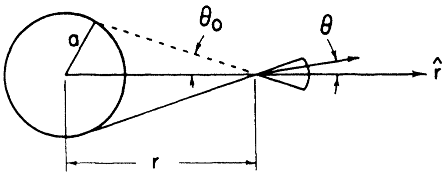


FIG. 7-7 Radiation from a sphere.

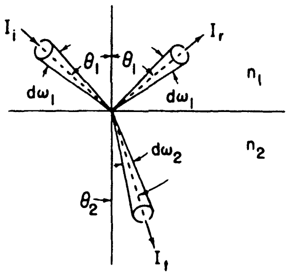


FIG. 7-8 Specific intensities at a plane boundary between two homogeneous media.

and the average intensity $U(\mathbf{r})$ is

$$U(\mathbf{r}) = \frac{I_0}{2}\left[1 - \sqrt{1 - \left(\frac{a}{r}\right)^2}\,\right]. \tag{7-12}$$

Next, we consider the condition to be satisfied by the specific intensity at a plane boundary between two media with indices of refraction n_1 and n_2 (Fig. 7-8).

For a plane wave incident on a plane boundary, the reflection coefficient $R = E_r/E_i$ and the transmission coefficient $T = E_t/E_r$ are given by

$$\begin{aligned} R_\parallel &= \frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}, & T_\parallel &= \frac{2n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \\ R_\perp &= \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}, & T_\perp &= \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \end{aligned} \tag{7-13}$$

where $R_\parallel$ and $T_\parallel$ are for electric fields polarized in the plane of incidence and $R_\perp$ and $T_\perp$ are for electric fields polarized perpendicular to the plane of incidence.

It should be obvious that the reflected specific intensity I_r is related to I_i through

$$I_r = |R|^2 I_i \tag{7-14}$$

where R is $R_\parallel$ or $R_\perp$, depending on the polarization. If the wave is completely unpolarized, $|R|^2$ should be equal to $\frac{1}{2}(|R_\parallel|^2 + |R_\perp|^2)$.

Now the question is what I_t/I_i should be. From Fig. 7-8, it is seen that in order to relate I_t to I_i, we should take into account the relationship between $d\omega_1$ and $d\omega_2$. In fact, the incident power flux into a small area da on the boundary must be equal to the sum of the reflected and transmitted fluxes:

$$I_i\, da \cos\theta_1\, d\omega_1 = I_r\, da \cos\theta_1\, d\omega_1 + I_t\, da \cos\theta_2\, d\omega_2. \tag{7-15}$$

We note that $d\omega_1 = \sin\theta_1\, d\theta_1\, d\phi_1$ and $d\omega_2 = \sin\theta_2\, d\theta_2\, d\phi_2$. From Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$, we get $n_1 \cos\theta_1\, d\theta_1 = n_2 \cos\theta_2\, d\theta_2$ and noting $d\phi_1 = d\phi_2$, we get

$$I_i/n_1{}^2 = I_r/n_1{}^2 + I_t/n_2{}^2. \tag{7-16}$$

We can express the relationship between I_t and I_i by using the power transmissivity T_p and the power reflectivity R_p defined by the ratio of the transmitted and the reflected power to the incident power normal to the surface:

$$T_p = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} |T|^2, \qquad R_p = |R|^2, \qquad R_p + T_p = 1. \tag{7-17}$$

From (7-14), (7-16), and (7-17), we have

$$I_t = \frac{n_2{}^2}{n_1{}^2}(1 - R_p)I_i = \frac{n_2{}^2}{n_1{}^2} T_p I_i = \frac{n_2{}^3 \cos\theta_2}{n_1{}^3 \cos\theta_1} |T|^2 I_i. \tag{7-18}$$

$|T|^2$ in (7-18) is equal to $|T_\parallel|^2$ or $|T_\perp|^2$, depending on the polarization, or to $\frac{1}{2}(|T_\parallel|^2 + |T_\perp|^2)$ for a completely unpolarized wave.

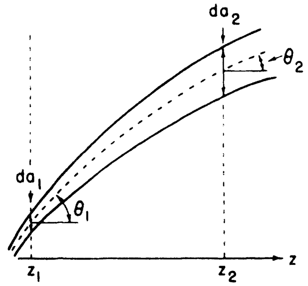


FIG. 7-9 Specific intensities I_1 and I_2 at two points along the ray.

If the index of refraction is a slowly varying function in one coordinate: $n(\mathbf{r}) = n(z)$, then the ratio of the specific intensities at two points $\mathbf{r}_1$ and $\mathbf{r}_2$ along the ray may be obtained by generalizing (7-15) and (7-16):

$$(I_1\, da_1)/n_1{}^2 = (I_2\, da_2)/n_2{}^2 \tag{7-19}$$

where da_1 and da_2 are the cross sections perpendicular to the z axis of the tube of rays at $\mathbf{r}_1$ and $\mathbf{r}_2$ (see Fig. 7-9).

7-3 DIFFERENTIAL EQUATION FOR SPECIFIC INTENSITY

In the preceding section we defined the specific intensity and other fundamental quantities. We also discussed the characteristics of the specific intensity in free space and at boundaries between two homogeneous media. In this section, we examine the fundamental characteristics of the specific intensity in a medium containing random particles. The particles scatter and absorb the wave energy, and these characteristics should be included in a differential equation to be satisfied by the specific intensity. This equation is called the equation of transfer in radiative transfer theory. It is identical to the Boltzmann equation used in neutron transport theory.

Let us consider a specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ incident upon a cylindrical elementary volume with unit cross section and length ds. The volume ds contains $\rho\, ds$ particles where ρ is the number of particles in a unit volume and is called the number density. Each particle absorbs the power $\sigma_a I$ and scatters the power $\sigma_s I$, and therefore, the decrease of the specific intensity $dI(\mathbf{r}, \hat{\mathbf{s}})$ for the volume ds is expressed as

$$dI(\mathbf{r}, \hat{\mathbf{s}}) = -ds(\sigma_a + \sigma_s)I = -\rho\, ds\, \sigma_t I. \tag{7-20}$$

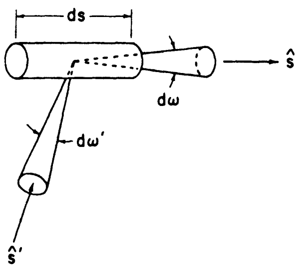


FIG. 7-10 Scattering of specific intensity incident upon the volume ds from the direction $\hat{\mathbf{s}}'$ into the direction $\hat{\mathbf{s}}$.

At the same time, the specific intensity increases because a portion of the specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ incident on this volume from other directions $\hat{\mathbf{s}}'$ is scattered into the direction $\hat{\mathbf{s}}$ and is added to the intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ (Fig. 7-10).

In order to determine this contribution, let us consider a wave incident in the direction $\hat{\mathbf{s}}'$ on a particle. The incident flux density through a small solid angle $d\omega'$ is given by $S_i = I(\mathbf{r}, \hat{\mathbf{s}}')\, d\omega'$. This flux is incident on particles in the volume ds. The power flux density S_r of the wave scattered by a single particle in the direction $\hat{\mathbf{s}}$ at a distance R from the particle is then given by $S_r = [|f(\hat{\mathbf{s}}, \hat{\mathbf{s}}')|^2/R^2]S_i$, where $f(\hat{\mathbf{s}}, \hat{\mathbf{s}}')$ is the scattering amplitude defined in Section 2-1. The scattered specific intensity in the direction $\hat{\mathbf{s}}$ due to S_i is therefore

$$S_r R^2 = |f(\hat{\mathbf{s}}, \hat{\mathbf{s}}')|^2 S_i = |f(\hat{\mathbf{s}}, \hat{\mathbf{s}}')|^2 I(\mathbf{r}, \hat{\mathbf{s}}')\, d\omega'.$$

Adding the incident flux from all directions $\hat{\mathbf{s}}'$, the specific intensity scattered into the direction $\hat{\mathbf{s}}$ by $\rho\, ds$ particles in the volume ds is given by

$$\int_{4\pi} \rho\, ds |f(\hat{\mathbf{s}}, \hat{\mathbf{s}}')|^2 I(\mathbf{r}, \hat{\mathbf{s}}')\, d\omega' \tag{7-21}$$

where the integration over all ω' is taken to include the contributions from all directions $\hat{s}'$. We can express (7-21) using the phase function† $p(\hat{s}, \hat{s}')$:

$$p(\hat{s}, \hat{s}') = \frac{4\pi}{\sigma_t} |f(\hat{s}, \hat{s}')|^2, \qquad \frac{1}{4\pi} \int_{4\pi} p(\hat{s}, \hat{s})\, d\omega = W_0 = \frac{\sigma_s}{\sigma_t} \tag{7-22}$$

where W_0 is the albedo of a single particle.

The specific intensity also may increase due to the emission from within the volume ds. Denoting by $\varepsilon(\mathbf{r}, \hat{s})$ the power radiation per unit volume per unit solid angle in the direction $\hat{s}$, the increase of the specific intensity is given by

$$ds\, \varepsilon(\mathbf{r}, \hat{s}). \tag{7-23}$$

Adding the contributions (7-20), (7-21), and (7-23), we get the "equation of transfer":

$$\frac{dI(\mathbf{r}, \hat{s})}{ds} = -\rho\sigma_t I(\mathbf{r}, \hat{s}) + \frac{\rho\sigma_t}{4\pi} \int_{4\pi} p(\hat{s}, \hat{s}') I(\mathbf{r}, \hat{s}')\, d\omega' + \varepsilon(\mathbf{r}, \hat{s}). \tag{7-24}$$

The left-hand side of this equation can also be written using a gradient or divergence operator as follows:

$$\frac{dI(\mathbf{r}, \hat{s})}{ds} = \hat{s} \cdot \operatorname{grad} I(\mathbf{r}, \hat{s}) = \operatorname{div}[I(\mathbf{r}, \hat{s})\hat{s}] \tag{7-25}$$

where we made use of the fact that $\hat{s}$ is a constant vector and thus $\operatorname{div} \hat{s} = 0$.

In this equation the particle density and size can be different at different locations, and therefore $\rho\sigma_t$ and p can be functions of $\mathbf{r}$. It is sometimes convenient to measure the distance in terms of a nondimensional "optical distance τ" defined by‡

$$\tau = \int \rho\sigma_t\, ds. \tag{7-26}$$

The optical distance $\tau = 1$ means that over this distance the power flux diminishes by scattering and absorption according to (7-20) to the value equal to $\exp(-1)$ of the incident flux. Using (7-26), (7-24) becomes

$$\frac{dI(\tau, s)}{d\tau} = -I(\tau, \hat{s}) + \frac{1}{4\pi} \int_{4\pi} p(\hat{s}, \hat{s}') I(\tau, \hat{s}')\, d\omega' + J(\tau, \hat{s}) \tag{7-27}$$

where $J(\mathbf{r}, \hat{s}) = \varepsilon(\mathbf{r}, \hat{s})/\rho\sigma_t$ is called the source function.

† The name "phase function" has its origin in astronomy where it refers to lunar phases. It has no relation to the phase of a wave. See Section 2-1 and Van de Hulst (1957, p. 12).

‡ In Chapters 4–6, we used γ to denote optical distance and τ for time difference. In this chapter, we use τ for optical distance following Chandrasekhar (1950).

Let us next examine the conservation of power. We write the left-hand side of (7-24) using the divergence form in (7-25) and integrate over all 4π of solid angle. Noting the definitions of the flux vector $\mathbf{F}$ in (7-4) and of the average intensity $U(\mathbf{r})$ in (7-6), and using (7-22), we get

$$\operatorname{div} \mathbf{F}(\mathbf{r}) = -\rho\sigma_a \int_{4\pi} I(\mathbf{r}, \hat{\mathbf{s}})\, d\omega + \int_{4\pi} \varepsilon(\mathbf{r}, \hat{\mathbf{s}})\, d\omega \tag{7-28a}$$

where

$$E(\mathbf{r}) = \int_{4\pi} \varepsilon(\mathbf{r}, \hat{\mathbf{s}})\, d\omega \tag{7-28b}$$

is the power generated per unit volume per unit frequency interval ($\mathrm{W\ m^{-3}\ Hz^{-1}}$), and where

$$E_a(\mathbf{r}) = \rho\sigma_a \int_{4\pi} I(\mathbf{r}, \hat{\mathbf{s}})\, d\omega = 4\pi\rho\sigma_a U(\mathbf{r}) \tag{7-28c}$$

is the total power absorbed per unit volume per unit frequency interval. We note that in (7-28c), $\sigma_a I$ is the power absorbed by a single particle when illuminated by I, and the total absorbed power is the summation of $\sigma_a I$ over the total solid angle, and for all particles, irrespective of the direction of I.

Physically, (7-28a) means that the outflow of flux $\mathbf{F}$ per unit volume is equal to the power generated per unit volume minus the power absorbed per unit volume. If the medium is lossless, $\sigma_a = 0$, and if there is no source $[\varepsilon(\mathbf{r}, \hat{\mathbf{s}}) = 0]$, then the conservation of the power flux holds:

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = 0. \tag{7-28d}$$

7-4 REDUCED INCIDENT INTENSITY, DIFFUSE INTENSITY, BOUNDARY CONDITION, AND SOURCE FUNCTION

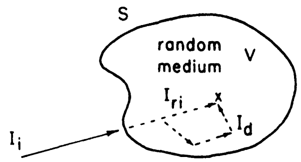
It is often convenient to divide the total intensity† into two parts, the reduced incident intensity I_{ri} and the diffuse intensity I_d.

As a wave enters a volume V containing many particles, the incident flux diminishes and increases according to (7-20) and (7-21). That part of the flux which decreases due to scattering and absorption according to (7-20) is called the reduced incident intensity and satisfies the equation

$$\frac{dI_{ri}(\mathbf{r}, \hat{\mathbf{s}})}{ds} = -\rho\sigma_t I_{ri}(\mathbf{r}, \hat{\mathbf{s}}). \tag{7-29}$$

† In this book, $I(\mathbf{r}, \hat{\mathbf{s}})$ always represents specific intensity. Sometimes it is simply called intensity.

FIG. 7-11 Incident intensity I_i, reduced incident intensity I_{ri}, and diffuse intensity I_d.



The other part, which is created within the medium due to scattering, is called the diffuse intensity (Fig. 7-11).

Since the total intensity $I(\mathbf{r}, \hat{\mathbf{s}}) = I_{ri}(\mathbf{r}, \hat{\mathbf{s}}) + I_d(\mathbf{r}, \hat{\mathbf{s}})$ satisfies (7-24), the diffuse intensity I_d satisfies the equation

$$\frac{dI_d(\mathbf{r}, \hat{\mathbf{s}})}{ds} = -\rho\sigma_t I_d(\mathbf{r}, \hat{\mathbf{s}}) + \frac{\rho\sigma_t}{4\pi}\int_{4\pi} p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')I_d(\mathbf{r}, \hat{\mathbf{s}}')\, d\omega' + \varepsilon_{ri}(\mathbf{r}, \hat{\mathbf{s}}) + \varepsilon(\mathbf{r}, \hat{\mathbf{s}}) \tag{7-30}$$

where ε_{ri} is the equivalent source function

$$\varepsilon_{ri}(\mathbf{r}, \hat{\mathbf{s}}) = \frac{\rho\sigma_t}{4\pi}\int_{4\pi} p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')I_{ri}(\mathbf{r}, \hat{\mathbf{s}}')\, d\omega' \tag{7-31}$$

generated by the reduced incident intensity.

The boundary condition at the surface S of a medium containing many particles (Fig. 7-11) is obtained by noting that the diffuse intensity $I_d(\mathbf{r}, \hat{\mathbf{s}})$ is generated only within the medium. We therefore require that at the surface S, there should be no diffuse radiation entering the medium, and the diffuse intensity should always be pointed outward.† Mathematically, the boundary condition is stated as

$$I_d(\mathbf{r}, \hat{\mathbf{s}}) = 0 \quad \text{on } S \text{ when } \hat{\mathbf{s}} \text{ is pointed inward.} \tag{7-32}$$

If the medium extends to infinity, we require that the diffuse intensity I_d must diminish at infinity. This condition at infinity is discussed in Chapter 11 with examples. Equation (7-30) together with the boundary condition (7-32) constitutes the complete mathematical statement of the problem.

The reduced incident intensity I_{ri} may be "collimated" or "diffuse." For example, the incident wave may be well collimated in a particular direction $\hat{\mathbf{s}}_0$ as in the case of a laser beam or in the case of a plane wave. We call this the collimated incident intensity I_{ci}. We can express $I_{ci}(\mathbf{r}, \hat{\mathbf{s}})$ pointed in the direction $\hat{\mathbf{s}}_0$ using a delta function

$$I_{ci}(\mathbf{r}, \hat{\mathbf{s}}) = F_0\,\delta(\hat{\omega} - \hat{\omega}_0) \tag{7-33}$$

† Strictly speaking, this holds only for a convex surface where the intensity does not reenter the surface.

where F_0 is the flux density (W m^{-2} Hz^{-1}), $\delta(\hat{\omega} - \hat{\omega}_0)$ a solid angle delta function (sr^{-1}), and $\hat{\omega}$ and $\hat{\omega}_0$ the unit vectors representing solid angles in the directions $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}_0$. For example, in the spherical coordinate system,

$$\delta(\hat{\omega} - \hat{\omega}_0) = \frac{\delta(\theta - \theta_0)\,\delta(\phi - \phi_0)}{\sin\theta}, \qquad \int_{4\pi} \delta(\hat{\omega} - \hat{\omega}_0)\,d\omega = 1 \tag{7-34}$$
$$d\omega = \sin\theta\,d\theta\,d\phi.$$

In contrast with the collimated incident intensity, the diffuse incident intensity comes from various directions with different magnitudes. For example, consider the radiation scattered through a cloud which is incident on an ocean surface. In this case, the wave incident on the water is not collimated, but is already diffused. We call this the diffuse incident intensity.

Let us next examine the source function $J(\mathbf{r}, \hat{\mathbf{s}}) = \varepsilon(\mathbf{r}, \hat{\mathbf{s}})/\rho\sigma_t$. If a point source is located at $\mathbf{r}_0$ and radiates the total power P_0 (W/Hz) uniformly in all directions, we write

$$\varepsilon(\mathbf{r}) = (P_0/4\pi)\,\delta(\mathbf{r} - \mathbf{r}_0)\ \text{W m}^{-3}\,\text{sr}^{-1}\,\text{Hz}^{-1}. \tag{7-35}$$

If the medium is in local thermodynamic equilibrium at a temperature T (degrees Kelvin), then Kirchhoff's law for a blackbody gives a reasonable approximation to the radiating energy (Kondratyev, 1969, Chapter 1). We then have approximately

$$\varepsilon(\mathbf{r}) = \rho\sigma_a B(T), \qquad \text{where} \quad B(T) = (2h\nu^3/c^2)[\exp(h\nu/KT) - 1]^{-1} \tag{7-36}$$

and K and h are the Boltzmann and Planck constants, respectively.

7-5 INTEGRAL EQUATION FORMULATION

In general, there are two ways to formulate a given problem. One is to start with a differential equation, obtain a general solution with unknown coefficients, and then apply appropriate boundary conditions to determine these coefficients. The other approach is to combine differential equations and boundary conditions into integral equations for appropriate unknown functions. In simple geometries such as a plane-parallel atmosphere, the differential equation approach has been used extensively. For more complex geometries, it is often convenient to start with integral equations and obtain approximate solutions. In this section, we discuss the general integral equation formulation.

Let us consider a volume V containing random particles (see Fig. 7-12).

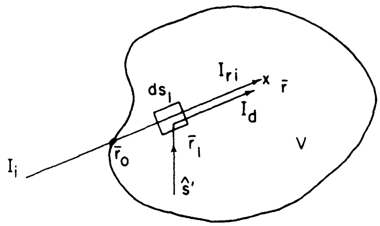


FIG. 7-12 Reduced incident intensity I_{ri} and diffuse intensity I_d.

We start with the differential equation (7-24):

$$\frac{dI(\mathbf{r}, \hat{\mathbf{s}})}{ds} = -\rho\sigma_t I(\mathbf{r}, \hat{\mathbf{s}}) + \frac{\rho\sigma_t}{4\pi}\int_{4\pi} p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')I(\mathbf{r}, \hat{\mathbf{s}}')\, d\omega' + \varepsilon(\mathbf{r}, \hat{\mathbf{s}}). \quad (7\text{-}37)$$

This is a first order differential equation with respect to s. Noting that (7-37) has the form

$$dy/dx + Py = Q \quad (7\text{-}38a)$$

we get a general solution

$$I(\mathbf{r}, \hat{\mathbf{s}}) = ce^{-\tau} + e^{-\tau}\int Q(\hat{\mathbf{s}}_1)e^{\tau_1}\, ds_1 \quad (7\text{-}38b)$$

where $\tau = \int \rho\sigma_t\, ds$ is the optical distance, c is a constant, and

$$Q(s) = \frac{\rho\sigma_t}{4\pi}\int_{4\pi} p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')I(\mathbf{r}, \hat{\mathbf{s}}')\, d\omega' + \varepsilon(\mathbf{r}, \hat{\mathbf{s}}). \quad (7\text{-}38c)$$

Now we apply the boundary condition that at a point of incidence $\mathbf{r} = \mathbf{r}_0$, the diffuse intensity I_d is zero, and thus the total intensity is equal to the incident intensity $I_i(\mathbf{r}_0, \hat{\mathbf{s}})$. Measuring the distance $\hat{\mathbf{s}}$ from $\mathbf{r} = \mathbf{r}_0$, we get

$$I(\mathbf{r}, \hat{\mathbf{s}}) = I_{ri}(\mathbf{r}, \hat{\mathbf{s}}) + I_d(\mathbf{r}, \hat{\mathbf{s}}), \qquad I_{ri}(\mathbf{r}, \hat{\mathbf{s}}) = I_i(\mathbf{r}_0, \hat{\mathbf{s}}) \exp(-\tau)$$

$$I_d(\mathbf{r}, \hat{\mathbf{s}}) = \int_0^s \exp[-(\tau - \tau_1)]\left[\left(\frac{\rho\sigma_t}{4\pi}\right)\int_{4\pi} p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')I(\mathbf{r}_1, \hat{\mathbf{s}}')\, d\omega' + \varepsilon(\mathbf{r}_1, \hat{\mathbf{s}})\right] ds_1 \quad (7\text{-}39)$$

where I_{ri} and I_d are the reduced incident intensity and the diffuse intensity, $\tau = \int_0^s \rho\sigma_t ds$, and $\tau_1 = \int_0^{s_1} \rho\sigma_t ds$. If the medium V containing the scatterers has an index of refraction n which is different from that outside, then the incident intensity $I_i(\mathbf{r}_0, \hat{\mathbf{s}})$ must be replaced by the transmitted intensity $I_t(\mathbf{r}_0, \hat{\mathbf{s}})$ and $\hat{\mathbf{s}}$ is the unit vector in the direction of propagation for the transmitted field as discussed in Section 7-2.

Equation (7-39) gives an integral equation for the specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ along the ray $\hat{\mathbf{s}}$, but since the integrand contains contributions from the intensity coming from a different direction $\hat{\mathbf{s}}'$, it is obviously not complete. A more complete description should include the contributions from all points in V.

To obtain such an integral equation, we integrate (7-39) over the total solid angle and divide it by 4π. The left-hand side becomes the average intensity $U(\mathbf{r})$:

$$\frac{1}{4\pi}\int_{4\pi} I(\mathbf{r}, \hat{\mathbf{s}})\, d\omega = U(\mathbf{r}). \tag{7-40a}$$

The reduced intensity becomes

$$\frac{1}{4\pi}\int_{4\pi} I_{ri}(\mathbf{r}, \hat{\mathbf{s}})\, d\omega = \frac{1}{4\pi}\int_{4\pi} I_i(\mathbf{r}_0, \hat{\mathbf{s}})e^{-\tau}\, d\omega = U_{ri}(\mathbf{r}). \tag{7-40b}$$

To calculate the integral of the diffuse intensity, we note that $d\omega = da/|\mathbf{r} - \mathbf{r}_1|^2$ and $da\, ds_1 = dV_1$. Therefore, we obtain

$$U(\mathbf{r}) = U_{ri}(\mathbf{r}) + \int_V \left[\frac{\rho\sigma_t}{4\pi}\int_{4\pi} p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')I(\mathbf{r}_1, \hat{\mathbf{s}}')\, d\omega' + \varepsilon(\mathbf{r}_i, \hat{\mathbf{s}})\right] \frac{\exp[-(\tau - \tau_1)]}{4\pi|\mathbf{r} - \mathbf{r}_1|^2}\, dV_1. \tag{7-41}$$

The physical meaning of (7-41) can be appreciated by examining Fig. 7-13. The first term $U_{ri}(\mathbf{r})$ is the average intensity due to the reduced incident specific intensity coming from all directions. To interpret the second term, we note that $I(\mathbf{r}_1, \hat{\mathbf{s}}')$ is incident on the volume dV_1 from the direction $\hat{\mathbf{s}}'$ and, therefore, the first term of the integrand is the contribution from $I(\mathbf{r}_1, \hat{\mathbf{s}}')$ in the direction of $\hat{\mathbf{s}}$, diminished by the spherical expansion $|\mathbf{r} - \mathbf{r}_1|^{-2}$ and the attenuation $\exp[-(\tau - \tau_1)]$, and this contribution is integrated over all the solid angle ω' and over all V. The second term of the integrand is the contribution to the field at $\mathbf{r}$ from the source function at $\mathbf{r}_1$.

Equation (7-41) gives the average intensity $U(\mathbf{r})$ which is proportional to the energy density. We can also follow a similar procedure to obtain an equation for the flux vector $\mathbf{F}(\mathbf{r})$:

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}_{ri}(\mathbf{r}) + \int_V \left[\frac{\rho\sigma_t}{4\pi}\int_{4\pi} p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')I(\mathbf{r}_1, \hat{\mathbf{s}}')\, d\omega' + \varepsilon(\mathbf{r}_1, \hat{\mathbf{s}})\right] \times \frac{\exp[-(\tau - \tau_1)]}{|\mathbf{r} - \mathbf{r}_1|^2}\, \hat{\mathbf{s}}\, dV_1 \tag{7-42}$$

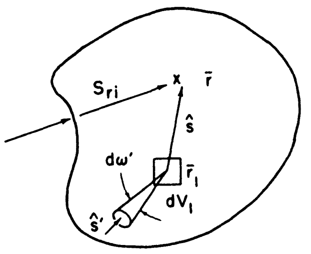


FIG. 7-13 Physical meaning of integral equation (7-41).

where $\mathbf{F}(\mathbf{r}) = \int_{4\pi} I(\mathbf{r}, \hat{\mathbf{s}})\hat{\mathbf{s}}\, d\omega$. It will be seen in Chapter 14 that these equations are in fact compatible with Twersky's multiple scattering theory and the Dyson and Bethe–Salpeter equations.

7-6 RECEIVING CROSS SECTION AND RECEIVED POWER

In the preceding sections we have discussed general formulations for specific intensity in terms of differential and integral equations. In actual measurement of these quantities, however, we need to take into account the characteristics of the receiver. The receiving characteristics can be most conveniently described in terms of the "receiving cross section" $A_r(\hat{\mathbf{s}}_r, \hat{\mathbf{s}})$. When a wave with the power density $S_i(\hat{\mathbf{s}})$ (W/m^2) is incident on the receiver in the direction $\hat{\mathbf{s}}$ and the receiver is pointed in the $-\hat{\mathbf{s}}_r$ direction (see Fig. 7-14), the receiving cross section is defined by the ratio of the received power P_r (watts) to the incident power flux $S_i(\hat{\mathbf{s}})$:

$$P_r = A_r(\hat{\mathbf{s}}_r, \hat{\mathbf{s}})S_i(\hat{\mathbf{s}}). \tag{7-43}$$

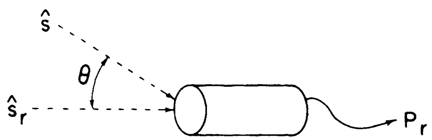


FIG. 7-14 Receiving cross section $A_r(\hat{\mathbf{s}}_r, \hat{\mathbf{s}})$ and received power P_r.

A_r has the dimensions of area and it depends on the direction of the incident wave $\hat{\mathbf{s}}$ and the orientation of the receiver $\hat{\mathbf{s}}_r$. In terms of the specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ incident on the receiver, we have

$$P_r = \int_{\Omega} A_r(\hat{\mathbf{s}}_r, \hat{\mathbf{s}})I(\mathbf{r}, \hat{\mathbf{s}})\, d\omega \quad \text{(W/Hz)} \tag{7-44}$$

where the integration is taken over all the solid angle to include all waves incident on the receiver. For example, in Fig. 7-14, the integration should include the range $0 \le \theta \le \pi/2$. In general, the receiving cross section can have a narrow ($\theta \simeq$ small) or a broad receiving angle. Only in the special case in which the receiver accepts the incident power equally from all directions do we have $A_r(\hat{\mathbf{s}}_r, \hat{\mathbf{s}}) = A_0 \cos\theta$. The receiving characteristics can be incorporated into the general integral formulation of the preceding section. The power P_R received by the receiver at $\mathbf{r}$ pointed toward $-\hat{\mathbf{s}}_r$ is then given in analogy to (7-41):

$$P_R(\mathbf{r}, \hat{\mathbf{s}}_r) = P_{Rri}(\mathbf{r}, \hat{\mathbf{s}}_r) + P_{Rd}(\mathbf{r}, \hat{\mathbf{s}}_r). \tag{7-45}$$

$P_{Rri}(\mathbf{r}, \hat{\mathbf{s}}_r)$ is the received power due to the reduced incident intensity and is given by

$$P_{Rri}(\mathbf{r}, \hat{\mathbf{s}}_r) = \int_{\Omega} A_r(\hat{\mathbf{s}}_r, \hat{\mathbf{s}}) I_{ri}(\mathbf{r}, \hat{\mathbf{s}})\, d\omega. \tag{7-46}$$

$P_{Rd}(\mathbf{r}, \hat{\mathbf{s}}_r)$ is the component due to the diffuse intensity and is given by

$$P_{Rd}(\mathbf{r}, \hat{\mathbf{s}}_r) = \int_V A_r(\hat{\mathbf{s}}_r, \hat{\mathbf{s}}) \left[\frac{\rho\sigma_t}{4\pi}\int_{4\pi} p(\hat{\mathbf{s}}, \hat{\mathbf{s}}') I(\mathbf{r}_1, \hat{\mathbf{s}}')\, d\omega' + \varepsilon(\mathbf{r}_1, \hat{\mathbf{s}})\right] \times \frac{\exp[-(\tau - \tau_1)]}{|\mathbf{r} - \mathbf{r}_1|^2}\, dV_1. \tag{7-47}$$

In most practical applications the receiving cross section A_r can be assumed to be equal to that in free space. Strictly speaking, however, the receiving characteristics can be affected by the presence of random particles. In general, the fluctuation of a wave incident upon a receiver causes the increase of beamwidth of the average receiving pattern and the decrease of gain. This effect is sometimes called the medium-to-antenna coupling loss (Shifrin, 1971, p. 259).

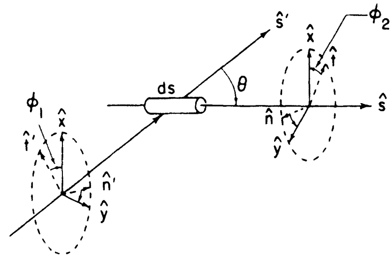
7-7 TRANSPORT EQUATION FOR A PARTIALLY POLARIZED ELECTROMAGNETIC WAVE

Up to this point, we have not included any polarization effects. This is justified only for acoustic waves. For electromagnetic waves, it represents only an approximation. In this section, we present a general formulation of transport theory of electromagnetic waves including polarization. All electromagnetic waves in random media are, in general, partially polarized because even if the incident wave is linearly polarized, the scattered wave is generally elliptically polarized, and its polarization should vary randomly due to the randomness of the medium.

In dealing with partially polarized waves, the specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ must be replaced by a vector $\mathbf{I}(\mathbf{r}, \hat{\mathbf{s}}, \hat{\mathbf{t}})$ whose components are the Stokes parameters (I_1, I_2, U, V) for specific intensity (W m^{-2} sr^{-1} Hz^{-1}). The Stokes parameters are defined in the rectangular coordinate system where the z axis is the direction of wave propagation and I_1 and I_2 are the average intensities of the x and y components of the electric fields (see Sections 2-9 and 2-10). To conform to the notations in this chapter, we denote unit vectors in the x and z directions by $\hat{\mathbf{t}}$ and $\hat{\mathbf{s}}$ (see Fig. 7-15). In analogy to (7-24), we write

$$\frac{d\mathbf{I}(\mathbf{r}, \hat{\mathbf{s}}, \hat{\mathbf{t}})}{ds} = -\rho\sigma_t \mathbf{I}(\mathbf{r}, \hat{\mathbf{s}}, \hat{\mathbf{t}}) + \int_{4\pi} \bar{\mathbf{P}}(\hat{\mathbf{s}}, \hat{\mathbf{t}}; \hat{\mathbf{s}}, \hat{\mathbf{t}}') \mathbf{I}(\mathbf{r}, \hat{\mathbf{s}}', \hat{\mathbf{t}}') d\omega' + \boldsymbol{\varepsilon}(\mathbf{r}, \hat{\mathbf{s}}, \hat{\mathbf{t}}). \tag{7-48}$$

FIG. 7-15 Rotation of the Stokes parameters through ϕ_1, scattering from the direction $\hat{s}'$ to the direction $\hat{s}$, and rotation through ϕ_2.



Here, we have made use of the principle of the addition of Stokes parameters for independent waves. $\mathbf{I}$ is expressed by a column matrix of four elements (I_1, I_2, U, V) and $\bar{\mathbf{P}}$ is a 4×4 matrix called the phase matrix, which relates the incident Stokes parameter $\mathbf{I}(\mathbf{r}, \hat{s}', \hat{t}')$ to the scattered Stokes parameter in the direction defined by $\hat{s}$ and $\hat{t}$. $\varepsilon(\mathbf{r}, \hat{s}, \hat{t})$ is a source Stokes parameter representing the radiation per unit solid angle in the direction $\hat{s}$.

The phase matrix $\bar{\mathbf{P}}(\hat{s}, \hat{t}, \hat{s}, \hat{t}')$ can be derived in the following manner. In Section 2-12, we derived the Stokes matrix $\bar{\boldsymbol{\sigma}}(\hat{s}, \hat{x}; \hat{s}', \hat{x})$ relating the scattered Stokes parameter $\mathbf{I}_s(\hat{s}, \hat{x})$ to the incident Stokes parameter $\mathbf{I}_i(\hat{s}', \hat{x})$, where the plane defined by the vectors $\hat{s}$ and $\hat{s}'$ is called the plane of scattering and the x axis is perpendicular to this plane. They are the scattered and incident power density (W/m^2) for a monochromatic wave. We can define the scattered and incident Stokes parameters for specific intensity (W m^{-2} sr^{-1} Hz^{-1}) for a volume of a cylinder with unit cross section and length ds containing $\rho\, ds$ particles. We then have

$$\mathbf{I}_s(\hat{s}, \hat{x}) = \rho\, ds\, \bar{\boldsymbol{\sigma}}(\hat{s}, \hat{x}; \hat{s}', \hat{x})\mathbf{I}_i(\hat{s}', \hat{x}) \tag{7-49}$$

where $\mathbf{I}_s$ and $\mathbf{I}_i$ are the Stokes parameters for specific intensity. Now, we must relate $\mathbf{I}_i(\hat{s}', \hat{x})$ to $\mathbf{I}(\hat{s}', \hat{t}')$. This is simply a transformation of the Stokes parameter due to rotation of the axis. This has already been discussed in Section 2-13:

$$\mathbf{I}_i(\hat{s}', \hat{x}) = \bar{\mathbf{L}}(\phi_1)\mathbf{I}(\hat{s}', \hat{t}'). \tag{7-50}$$

Similarly, we have

$$\mathbf{I}(\hat{s}, \hat{t}) = \bar{\mathbf{L}}(-\phi_2)\mathbf{I}_s(\hat{s}, \hat{x}). \tag{7-51}$$

Therefore, the phase matrix $\bar{\mathbf{P}}$ is given by

$$\bar{\mathbf{P}}(\hat{s}, \hat{t}; \hat{s}', \hat{t}') = \bar{\mathbf{L}}(-\phi_2)\bar{\boldsymbol{\sigma}}(\hat{s}, \hat{x}; \hat{s}', \hat{x})\bar{\mathbf{L}}(\phi_1) \tag{7-52}$$

which combines the rotation from $\hat{t}'$ to $\hat{x}$, the scattering from $\hat{s}'$ to $\hat{s}$, and the rotation from $\hat{x}$ to $\hat{t}$.

Further details are not included in this section as they are adequately covered in other textbooks. See, for example, Chandrasekhar (1950, Chapter 1) and Sekera (1966) for scattering matrix representations for a plane-parallel medium with vertical and horizontal reference axes.

7-8 RELATIONSHIP BETWEEN SPECIFIC INTENSITY AND POYNTING VECTOR

In Section 7-1, we defined the specific intensity in terms of the amount of power flux density within a unit solid angle in a frequency range $(\nu, \nu + d\nu)$. In this section, we present another definition of specific intensity which will relate it to the fields of Maxwell's equations.

Consider a Poynting vector $\mathbf{S}$ for a wave in a frequency range $(\nu, \nu + d\nu)$:

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t). \tag{7-53}$$

The Poynting vector $\mathbf{S}$ at a given point $\mathbf{r}$ is a random function of time, and as each Cartesian component of $\mathbf{S}$ varies in time, the tip of the vector $\mathbf{S}$ moves randomly. Let us consider a three-dimensional space whose Cartesian coordinates are the Cartesian components of $\mathbf{S}$, namely S_x, S_y, and S_z. We let $W(S_x, S_y, S_z)\, dS_x\, dS_y\, dS_z$ be the probability that the tip of $\mathbf{S}$ will be found in a volume element $dS_x\, dS_y\, dS_z$. Using spherical coordinates (S, θ, ϕ), where $S_x = S \sin\theta \cos\phi$, $S_y = S \sin\theta \sin\phi$, and $S_z = S \cos\theta$, we can write this probability as

$$W(S_x, S_y, S_z)\, dS_x\, dS_y\, dS_z = W(S, \theta, \phi) S^2\, dS\, d\omega \tag{7-54}$$

where $d\omega = \sin\theta\, d\theta\, d\phi$. The probability density function W satisfies the normalization condition

$$\int_{4\pi} \int_0^\infty W(S, \theta, \phi) S^2\, dS\, d\omega = 1. \tag{7-55}$$

We can now define the specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ by

$$I(\mathbf{r}, \hat{\mathbf{s}}) = \int_0^\infty S W(S, \theta, \phi) S^2\, dS. \tag{7-56}$$

This means that I is the sum of all Poynting vectors whose tips are located within a unit solid angle in the direction $\hat{\mathbf{s}}$ defined by (θ, ϕ).

The average intensity U is then given by

$$U = \frac{1}{4\pi} \int_{4\pi} d\omega \int_0^\infty S W(S, \theta, \phi) S^2\, dS \tag{7-57}$$

and the flux vector $\mathbf{F}$ is given by

$$\mathbf{F} = \int_{4\pi} d\omega \int_0^\infty S \hat{\mathbf{s}} W(S, \theta, \phi) S^2\, dS. \tag{7-58}$$

For acoustic waves, the Poynting vector must be replaced by $\mathbf{S} = p\mathbf{V}$, and the same argument shown above can be made.

The definition given in (7-56) shows the specific intensity as a statistical average of the randomly varying Poynting vector, and gives some physical insight into the nature of the specific intensity.

The specific intensity as defined in this chapter represents the averaged power flow and no consideration is given to the fluctuations of a wave associated with this power flow. One may ask whether the specific intensity is related in any way to the wave characteristics of the field.

Since the equation of transfer is heuristically derived from the power consideration, it appears that the wave characteristics are not included in the formulation except in the scattering and absorption characteristics of particles. However, since the wave equation has been used to calculate the cross sections and the scattering amplitude, the specific intensity cannot be determined without the knowledge of the interactions of the fields with the medium. There have been some recent investigations relating the radiative transfer theory with the analytical theory, and some aspects of this interesting development are discussed in Chapter 14. It will be shown that (7-57) can be generalized to show that the mutual coherence function is a Fourier transform of the specific intensity and that (7-57) is a special case of this general relationship (see Section 14-7).