Aprendizagem 2025/26

Homework II

Group 21 Report
Pen&Paper

See notebook for programming and critical analysis

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1.

a)

D	Y1	Y2	Y3	Y4	Y5	Y6
x_1	0,52	0,80	0	1	1	N
x_2	0,53	0,92	0	0	0	N
x_3	0,42	0,48	0	1	1	N
x_4	0,49	0,58	1	0	1	P
x_5	0,62	0,31	0	0	1	P
x_6	0,44	0,38	0	0	1	P
x_7	0,45	0,80	0	0	1	P
x_8	0,50	0,70	0	1	1	N

$P(P) = \frac{3}{6} = 0,5$ $\qquad$ $P(N) = \frac{3}{6} = 0,5$

Y_1, Y_2 set:

$$\mu_P = \left[\frac{0,49+0,62+0,44}{3}; \frac{0,58+0,31+0,38}{3}\right] = [0,5167; 0,4233]$$

x_4 $(0,49-0,5167; 0,58-0,4233) = (-0,0267; 0,1567)$

x_5 $(0,62-0,5167; 0,31-0,4233) = (0,1033; -0,1133)$

x_6 $(0,44-0,5167; 0,38-0,4233) = (-0,0767; -0,0433)$

$$x_4 \begin{bmatrix} (-0,0267)^2 & -0,0267 \times 0,1567 \\ -0,0267 \times 0,1567 & 0,1567^2 \end{bmatrix} = \begin{bmatrix} 0,000713 & -0,004184 \\ -0,004184 & 0,024555 \end{bmatrix}$$

$$x_5 \begin{bmatrix} 0,1033^2 & 0,1033 \times (-0,1133) \\ 0,1033 \times (-0,1133) & 0,1133^2 \end{bmatrix} = \begin{bmatrix} 0,010671 & -0,011704 \\ -0,011704 & 0,012837 \end{bmatrix}$$

$$x_6 \begin{bmatrix} (-0,0767)^2 & -0,0767 \times (-0,0433) \\ -0,0767 \times (-0,0433) & (-0,0433)^2 \end{bmatrix} = \begin{bmatrix} 0,005883 & 0,003327 \\ 0,003327 & 0,001875 \end{bmatrix}$$

Sum of all: $\begin{bmatrix} 0,017267 & -0,012567 \\ -0,012567 & 0,039267 \end{bmatrix}$

$$\Sigma_P = \frac{1}{2}\begin{bmatrix} 0,017267 & -0,012567 \\ -0,012567 & 0,039267 \end{bmatrix} = \begin{bmatrix} 0,008634 & -0,006283 \\ -0,006283 & -0,019633 \end{bmatrix}$$

$$\mu_N \cdot \left[\frac{0,52+0,53+0,42}{3}, \frac{0,80+0,92+0,48}{3}\right] = [0,49; 0,7333]$$

$x_1 \ (0,52-0,49, 0,80-0,7333) = (0,03; 0,0667)$
$x_2 \ (0,53-0,49, 0,92-0,7333) = (0,04; 0,1867)$
$x_3 \ (0,42-0,49, 0,48-0,7333) = (-0,07; -0,2533)$

$$x_1 \begin{bmatrix} 0,03^2 & ; 0,03\times 0,0667 \\ 0,03\times 0,0667 & ; 0,0667^2 \end{bmatrix} = \begin{bmatrix} 0,0009 & ; 0,0020 \\ 0,0020 & ; 0,0044 \end{bmatrix}$$

$$x_2 \begin{bmatrix} 0,04^2 & ; 0,04\times 0,1867 \\ 0,04\times 0,1867 & ; 0,1867^2 \end{bmatrix} = \begin{bmatrix} 0,0016 & ; 0,0075 \\ 0,0075 & ; 0,0349 \end{bmatrix}$$

$$x_3 \begin{bmatrix} (-0,07)^2 & ; -0,07\times(-0,2533) \\ -0,07\times(-0,2533) & ; (-0,2533)^2 \end{bmatrix} = \begin{bmatrix} 0,0049 & ; 0,0177 \\ 0,0177 & ; 0,0641 \end{bmatrix}$$

Sum of all: $\begin{bmatrix} 0,0074 & ; 0,0272 \\ 0,0272 & ; 0,1034 \end{bmatrix}$

$$\Sigma_N = \frac{1}{2}\begin{bmatrix} 0,0074 & ; 0,0272 \\ 0,0272 & ; 0,1034 \end{bmatrix} = \begin{bmatrix} 0,0037 & ; 0,0136 \\ 0,0136 & ; 0,0517 \end{bmatrix}$$

$$P(y_1, y_2 \mid \mu, \sigma^2) = \frac{1}{2\pi\sqrt{|\Sigma|}} e^{\left(-\frac{1}{2}|y-\mu|^T \Sigma^{-1} |y-\mu|\right)}$$

$$\Sigma^{-1} = \frac{1}{det(\Sigma)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad det(\Sigma) = a\times d - b\times c$$

$$det(\Sigma_{1,2P}) = 0,008634\times 0,019633 - (-0,006283)^2 = 0,0001301$$

$$\Sigma^{-1}_{1,2P} = \frac{1}{0,0001301}\begin{bmatrix} 0,019633 & 0,006283 \\ 0,006283 & 0,008634 \end{bmatrix} = \begin{bmatrix} 150,907 & 48,294 \\ 48,299 & 66,364 \end{bmatrix}$$

$$det(\Sigma_{1,2N}) = 0,0037\times 0,0517 - 0,0136^2 = 0,000006452$$

$$\Sigma^{-1}_{1,2N} = \frac{1}{0,000006452}\begin{bmatrix} 0,0517 & -0,0136 \\ -0,0136 & 0,0037 \end{bmatrix} = \begin{bmatrix} 3012,395 & -2107,840 \\ -2107,840 & 573,456 \end{bmatrix}$$

$$P(y_1, y_2 \mid \mu_P, \sigma^2 P) = \frac{1}{2\pi\sqrt{0{,}0001301}} e^{\left(\frac{-[y_1-\mu_P,\, y_2-\mu_P]^T \times \Sigma_P^{-1} \times [y_1-\mu_P,\, y_2-\mu_P]}{2}\right)}$$

$$P(y_1, y_2 \mid \mu_N, \sigma^2 N) = \frac{1}{2\pi\sqrt{0{,}000006452}} e^{\left(\frac{-[y_1-\mu_N,\, y_2-\mu_N]^T \times \Sigma_N^{-1} \times [y_1-\mu_N,\, y_2-\mu_N]}{2}\right)}$$

y_3, y_4:

$P(y_3=0, y_4=0 \mid y_6=P) = 2/3$
$P(y_3=0, y_4=0 \mid y_6=N) = 1/3$

$P(y_3=1, y_4=0 \mid y_6=P) = 1/3$
$P(y_3=1, y_4=0 \mid y_6=N) = 0$

$P(y_3=0, y_4=1 \mid y_6=P) = 0$
$P(y_3=0, y_4=1 \mid y_6=N) = 2/3$

y_5:

$P(y_6=0 \mid y_6=P) = 0$
$P(y_5=0 \mid y_6=N) = 1/3$

$P(y_5=1 \mid y_6=P) = 1$
$P(y_5=1 \mid y_6=N) = 2/3$

Uma vez que os 3 sets são independentes:

$P(x \mid P) = P(y_1, y_2 \mid P) \times P(y_3, y_4 \mid P) \times P(y_5 \mid P)$
$P(x \mid N) = P(y_1, y_2 \mid N) \times P(y_3, y_4 \mid N) \times P(y_5 \mid N)$

Posteriores:

$P(P \mid x) \propto P(y_1, y_2 \mid P) \times P(y_3, y_4 \mid P) \times P(y_5 \mid P) \times P(P)$
$P(N \mid x) \propto P(y_1, y_2 \mid N) \times P(y_3, y_4 \mid N) \times P(y_5 \mid N) \times P(N)$

b) $x_7 \to (0{,}45; 0{,}80; 0; 0; 1; N)$

$$P(P \mid 0{,}45; 0{,}80; 0; 0; 1) = P(0{,}45; 0{,}80 \mid P) \times P(0;0 \mid P) \times P(1 \mid P) \times P(P) = *_P$$

$$P(0{,}45; 0{,}80 \mid P) = \frac{1}{2\pi \sqrt{0{,}0001301}} e^{\left(-\frac{1}{2}[0{,}45-0{,}5167; 0{,}80-0{,}4233] \cdot \Sigma_p^{-1} \cdot \begin{bmatrix} 0{,}45-0{,}5167 \\ 0{,}80-0{,}4233 \end{bmatrix}\right)} =$$

$$= \frac{1}{2\pi \times 0{,}0114} \times e^{\left(-\frac{1}{2}[-0{,}067; 0{,}377] \cdot \begin{bmatrix} 150{,}207 & 48{,}294 \\ 48{,}294 & 64{,}364 \end{bmatrix} \cdot \begin{bmatrix} -0{,}067 \\ 0{,}377 \end{bmatrix}\right)} =$$

$$= \frac{1}{2\pi \times 0{,}0114} \times e^{\left(-\frac{1}{2} \times 7{,}670\right)} \approx 0{,}3016$$

$$*_P = 0{,}3016 \times \frac{2}{3} \times 1 \times \frac{1}{2} = 0{,}1005$$

$$P(N \mid 0{,}45; 0{,}80; 1; 0; 1) = P(0{,}45; 0{,}80 \mid N) \times P(0;0 \mid N) \times P(1 \mid N) \times P(N) = *_N$$

$$P(0{,}45; 0{,}80 \mid N) = \frac{1}{2\pi \times \sqrt{0{,}000006452}} e^{\left(-\frac{1}{2}[0{,}45-0{,}49; 0{,}80-0{,}7333] \cdot \Sigma_n^{-1} \cdot \begin{bmatrix} 0{,}45-0{,}49 \\ 0{,}80-0{,}7333 \end{bmatrix}\right)} =$$

$$= \frac{1}{2\pi \times 0{,}00254} e^{\left(-\frac{1}{2}[-0{,}04; 0{,}0667] \begin{bmatrix} 8012{,}895 & -2707{,}840 \\ -2707{,}840 & 573{,}456 \end{bmatrix} \begin{bmatrix} -0{,}04 \\ 0{,}0667 \end{bmatrix}\right)} =$$

$$= \frac{1}{2\pi \times 0{,}00254} e^{\left(-\frac{1}{2} \times 26{,}620\right)} \approx 0{,}000104$$

$$*_N = 0{,}000104 \times \frac{1}{3} \times \frac{2}{3} \times \frac{1}{2} = 0{,}0000116$$

$0{,}1005 > 0{,}0000116$, portanto $x_7 \to P$

$\ast_8 \rightarrow (0{,}50;\ 0{,}70;\ 0;\ 1;\ 1;\ N)$

$P(P \mid 0{,}50;\ 0{,}70;\ 0;\ 1;\ 1) = P(0{,}50;\ 0{,}70 \mid P) \times P(0{,}1 \mid P) \times P(1 \mid P) = 0$

como $P(0{,}1 \mid P) = 0$, então o resultado final também será 0.

$P(N \mid 0{,}50;\ 0{,}70;\ 0;\ 1;\ 1) = P(0{,}50;\ 0{,}70 \mid N) \times P(0{,}1 \mid N) \times P(1 \mid N) = \ast_N$

$$P(0{,}50;\ 0{,}70 \mid N) = \frac{1}{2\pi\sqrt{0{,}000006457}}\, e^{\left(-\frac{1}{2}[0{,}50-0{,}49;\ 0{,}70-0{,}7333] \times \Sigma_N^{-1} \times \begin{bmatrix} 0{,}50-0{,}49 \\ 0{,}70-0{,}7333 \end{bmatrix}\right)} =$$

$$= \frac{1}{2\pi \times 0{,}00254}\, e^{\left(-\frac{1}{2}[0{,}01;\ -0{,}0333]\begin{bmatrix} 8072{,}895 & -2707{,}840 \\ -2707{,}840 & 573{,}456 \end{bmatrix}\begin{bmatrix} 0{,}01 \\ -0{,}0333 \end{bmatrix}\right)} =$$

$$= \frac{1}{2\pi \times 0{,}00254}\, e^{\left(-\frac{1}{2}[2{,}847]\right)} \approx 15{,}138$$

$\ast_N = 15{,}138 \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{2} = 3{,}364$

$0 < 3{,}364$, logo, $\ast_8 \rightarrow N$

2

Data set

$x_1 = (0,1,1) \to N$
$x_2 = (0,0,0) \to N$
$x_3 = (0,1,1) \to N$
$x_4 = (1,0,1) \to P$
$x_5 = (0,0,1) \to P$
$x_6 = (0,0,1) \to P$
$x_7 = (0,0,1) \to P$
$x_8 = (0,1,1) \to N$

$x_1 = (0,1,1)$

Neighbours:

Distance $0 \to x_3(N), x_8(N)$ — $\leq 0 \to 2$ points
Distance $1 \to x_5(P), x_6(P), x_7(P)$ — $\leq 1 \to 5$ points
Distance $2 \to x_2(N), x_4(P)$ — not exactly 3, $K=2$

$x_3(N)$ & $x_8(N) \to x_1 : N$

$x_2 = (0,0,0)$

Neighbours:

Distance $0 \to \emptyset$ — $\leq 0 \to 0$ points
Distance $1 \to x_5(P), x_6(P), x_7(P)$ — $\leq 1 \to 3$ points
Distance $2 \to x_1(N), x_3(N), x_4(P), x_8(N)$ — exactly 3, $K=3$

$x_5(P)$ & $x_6(P)$ & $x_7(P) \to x_2 : P$

$x_3 = (0,1,1)$

Neighbours

Distance $0 \to x_1(N), x_8(N)$ — $\leq 0 \to 2$ points
Distance $1 \to x_5(P), x_6(P), x_7(P)$ — $\leq 1 \to 5$ points
Distance $2 \to x_2(N), x_4(P)$ — not exactly 3, $K=2$

$x_1(N)$ & $x_8(N) \to x_3 : N$

$x_4 = (1,0,1)$

Neighbours

Distance $0 \to \emptyset$ — $\leq 0 \to 0$ points
Distance $1 \to x_5(P), x_6(P), x_7(P)$ — $\leq 1 \to 3$ points
Distance $2 \to x_1(N), x_2(N), x_3(N), x_8(N)$ — exactly 3, $K=3$

$x_5(P)$ & $x_6(P)$ & $x_7(P) \to x_4 : P$

$X_5\ (0,0,1)$

Neighbours:

Distance 0: $X_6(P), X_7(P)$ — $\leq 0 \to 2$ points

Distance 1: $X_1(N), X_2(N), X_3(N), X_4(P), X_8(N)$ — $\leq 1 \to 7$ points

not exactly 3, $K=2$

$X_6(P)\ \&\ X_7(P) \to X_5: P$

$X_6\ (0,0,1)$

Neighbours:

Distance 0: $X_5(P), X_7(P)$ — $\leq 0 \to 2$ points

Distance 1: $X_1(N), X_2(N), X_3(N), X_4(P), X_8(N)$ — $\leq 1 \to 7$ points

not exactly 3, $K=2$

$X_5(P)\ \&\ X_7(P) \to X_6: P$

$X_7\ (0,0,1)$

Neighbours:

Distance 0: $X_5(P), X_6(P)$ — $\leq 0 \to 2$ points

Distance 1: $X_1(N), X_2(N), X_3(N), X_4(P), X_8(N)$ — $\leq 1 \to 7$ points

not exactly 3, $K=2$

$X_5(P)\ \&\ X_6(P) \to X_7: P$

$X_8\ (0,1,1)$

Neighbours:

Distance 0: $X_1(N), X_3(N)$ — $\leq 0 \to 2$ points

Distance 1: $X_5(P), X_6(P), X_7(P)$ — $\leq 1 \to 5$ points

Distance 2: $X_2(N), X_4(P)$ — not exactly 3, $K=2$

$X_1(N)\ \&\ X_3(N) \to X_8: N$

D	Predicted	True	Correct	D	Predicted	True	Correct
1	N	N	✓	5	P	P	✓
2	P	N	X	6	P	P	✓
3	N	N	✓	7	P	P	✓
4	P	P	✓	8	N	N	✓

Accuracy $= 7/8 = 87{,}5\%$

3.

a) Pelo Teorema de Bayes, para qualquer X

$P(\Theta=0|X=x)=\frac{P(X=x|\Theta=0)P(\Theta=0)}{P(X=x)}$ $\quad P(\Theta=1|X=x)=\frac{P(X=x|\Theta=1)\cdot P(\Theta=1)}{P(X=x)}$

Como o denominador é igual e $P(X=x|\Theta=0)=P(X=x|\Theta=1)$, então basta comparar $P(\Theta=0)$ com $P(\Theta=1)$, os priors, sendo independente de X.

$P(\Theta=0)=p \quad P(\Theta=1)=1-p$

Como $p>\frac{1}{2}$, então $p>1-p \Leftrightarrow P(\Theta=0|X=x)>P(\Theta=1|X=x)$

Se o classificador escolhe sempre 0 e $P(\Theta=0)=p$, então a chance de erro é dada por $E_{Bayes}=1-p$

b)

$2\times P(\Theta=0|X=x)\times P(\Theta=1|X=x)$

$$P(\Theta=0|X=x)=\frac{P(X=x|\Theta=0)P(\Theta=0)}{P(X=x)}=\frac{P(X=x|\Theta=0)P(\Theta=0)}{P(X=x|\Theta=0)P(\Theta=0)+P(X=x|\Theta=1)P(\Theta=1)}=*$$

* como $P(X=x|\Theta=0)=P(X=x|\Theta=1)$, trocamos ambos para $c(x)$ $=\frac{c(x)p}{c(x)p+c(x)(1-p)}=\frac{p}{p+(1-p)}=\frac{p}{1}=p$

Se $P(\Theta=0|X=x)=p$, então $P(\Theta=1|X=x)=1-p$

$E_{1NN}=2\times P(\Theta=0|X=x)\times P(\Theta=1|X=x)=2p\times(1-p)$

c)

Recorrendo às conclusões das alíneas anteriores:

$E_{Bayes}=1-p \Leftrightarrow p=1-E_{Bayes}$

$E_{1NN}=2p\times(1-p)=2(1-E_{Bayes})(1-1+E_{Bayes})=2\times E_{Bayes}\times(1-E_{Bayes})$