Aprendizagem 2025/26

Homework IV

Group 21 Report
Pen&Paper

See notebook for programming
and critical analysis

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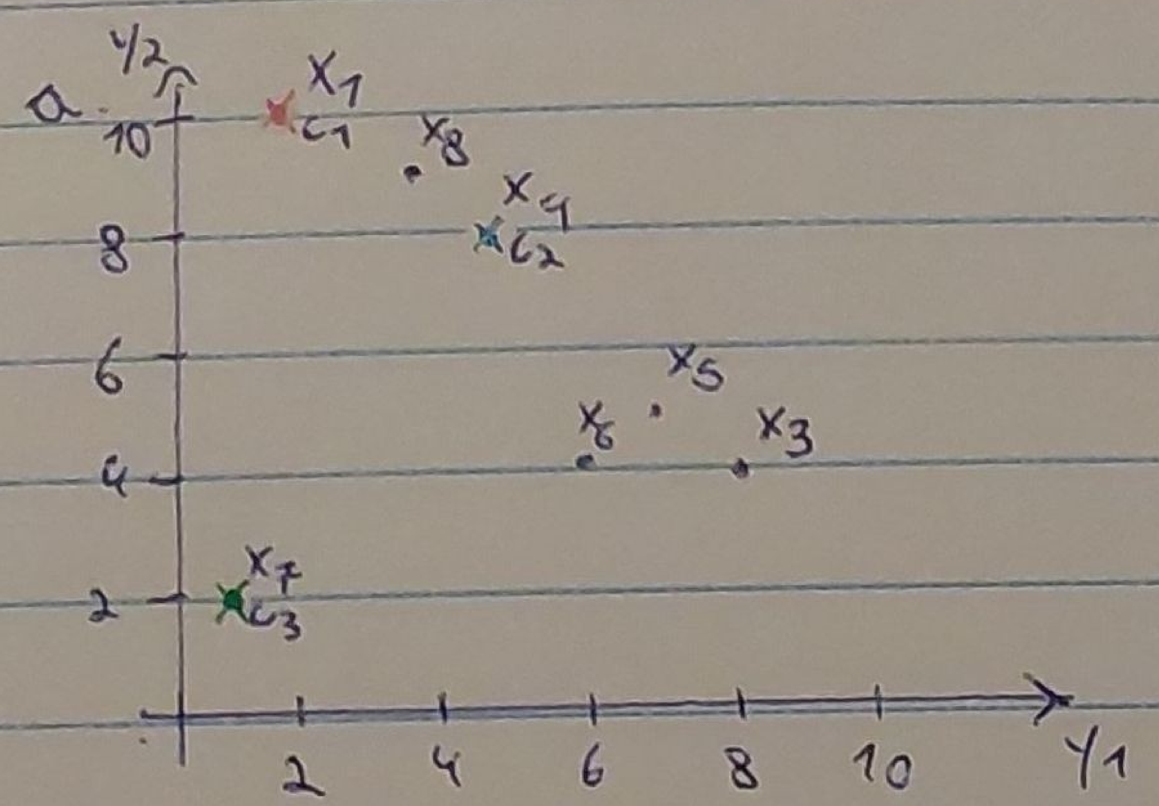
Project 4

Part A



	x	y_1	y_2
$c_1 \rightarrow$	x_1	2	10
	x_2	2	5
	x_3	8	4
$c_2 \rightarrow$	x_4	5	8
	x_5	7	5
	x_6	6	4
$c_3 \rightarrow$	x_7	1	2
	x_8	4	9

$$(d, c) = \sqrt{(X_p - x_c)^2 + (X_p - y_c)^2}$$

$x_1(2, 10)$: c_1

$x_2(2, 5)$:

$$d(x_2, c_1) = \sqrt{(2-2)^2 + (5-10)^2} = \sqrt{25}$$
$$d(x_2, c_2) = \sqrt{(2-5)^2 + (5-8)^2} = \sqrt{18}$$
$$d(x_2, c_3) = \sqrt{(2-1)^2 + (5-2)^2} = \sqrt{10}$$

$\rightarrow$ Assign To c_3

$x_3(8, 4)$:

$$d(x_3, c_1) = \sqrt{(8-2)^2 + (4-10)^2} = \sqrt{72}$$
$$d(x_3, c_2) = \sqrt{(8-5)^2 + (4-8)^2} = \sqrt{25}$$
$$d(x_3, c_3) = \sqrt{(8-1)^2 + (4-2)^2} = \sqrt{53}$$

$\rightarrow$ Assign to c_2

$x_4(5, 8)$: c_2

$x_5(7, 5)$:

$$d(x_5, c_1) = \sqrt{(7-2)^2 + (5-10)^2} = \sqrt{50}$$
$$d(x_5, c_2) = \sqrt{(7-5)^2 + (5-8)^2} = \sqrt{13}$$
$$d(x_5, c_3) = \sqrt{(7-1)^2 + (5-2)^2} = \sqrt{45}$$

$\rightarrow$ Assign To c_2

$x_6\ (6,4)$:

$d(x_6, c_1) = \sqrt{(6-2)^2+(4-10)^2} = \sqrt{52}$
$d(x_6, c_2) = \sqrt{(6-5)^2+(4-8)^2} = \sqrt{17}$
$d(x_6, c_3) = \sqrt{(6-1)^2+(4-2)^2} = \sqrt{29}$
$\rightarrow$ Assign to c_2

$x_7\ (1,2)$: c_3

$x_8\ (4,9)$:

$d(x_8, c_1) = \sqrt{(4-2)^2+(9-10)^2} = \sqrt{5}$
$d(x_8, c_2) = \sqrt{(4-5)^2+(9-8)^2} = \sqrt{2}$
$d(x_8, c_3) = \sqrt{(4-1)^2+(9-2)^2} = \sqrt{58}$
$\rightarrow$ Assign to c_2

Assign to clusters:
c_1: x_1
c_2: x_3, x_4, x_5, x_6, x_8
c_3: x_2, x_7

Calculate new clusters

$c_1' = \text{mean}\ \{x_1\} = (2,10)$

$c_2' = \text{mean}\ \{x_3, x_4, x_5, x_6, x_8\} = (6,6)$
$x = 8+5+7+6+4 = 30$
$y = 4+8+5+4+9 = 30$
$n = 5 \quad \text{mean} = (30/5, 30/5) = (6,6)$

$c_3' = \text{mean}\ \{x_2, x_7\} = (1{,}5;\ 3{,}5)$
$x = 2+1 = 3$
$y = 5+2 = 7$
$n = 2 \quad \text{mean}: (3/2, 7/2) = (1{,}5;\ 3{,}5)$

New clusters: $c_1(2,10)$, $c_2(6,6)$, $c_3(1{,}5; 3{,}5)$

Reassign points:

$x_1\ (2,10)$: C_1'

$x_2\ (2,5)$:

$d(x_2, C_1') = \sqrt{(2-2)^2 + (5-10)^2} = \sqrt{25}$
$d(x_2, C_2') = \sqrt{(2-6)^2 + (5-6)^2} = \sqrt{17}$
$d(x_2, C_3') = \sqrt{(2-1{,}5)^2 + (5-3{,}5)^2} = \sqrt{2{,}5}$
$\rightarrow$ Assign To C_3'

$x_3\ (8,4)$:

$d(x_3, C_1') = \sqrt{(8-2)^2 + (4-10)^2} = \sqrt{72}$
$d(x_3, C_2') = \sqrt{(8-6)^2 + (4-6)^2} = \sqrt{8}$
$d(x_3, C_3') = \sqrt{(8-1{,}5)^2 + (4-3{,}5)^2} = \sqrt{42{,}5}$
$\rightarrow$ Assign To C_2'

$x_4\ (5,8)$

$d(x_4, C_1') = \sqrt{(5-2)^2 + (8-10)^2} = \sqrt{13}$
$d(x_4, C_2') = \sqrt{(5-6)^2 + (8-6)^2} = \sqrt{5}$
$d(x_4, C_3') = \sqrt{(5-1{,}5)^2 + (8-3{,}5)^2} = \sqrt{32{,}5}$
$\rightarrow$ Assign To C_2'

$x_5\ (7,5)$

$d(x_5, C_1') = \sqrt{(7-2)^2 + (5-10)^2} = \sqrt{50}$
$d(x_5, C_2') = \sqrt{(7-6)^2 + (5-6)^2} = \sqrt{2}$
$d(x_5, C_3') = \sqrt{(7-1{,}5)^2 + (5-3{,}5)^2} = \sqrt{32{,}5}$
$\rightarrow$ Assign To C_2'

$x_6\ (6,4)$:

$d(x_6, C_1') = \sqrt{(6-2)^2 + (4-10)^2} = \sqrt{52}$
$d(x_6, C_2') = \sqrt{(6-6)^2 + (4-6)^2} = \sqrt{4}$
$d(x_6, C_3') = \sqrt{(6-1{,}5)^2 + (4-3{,}5)^2} = \sqrt{20{,}5}$
$\rightarrow$ Assign To C_2'

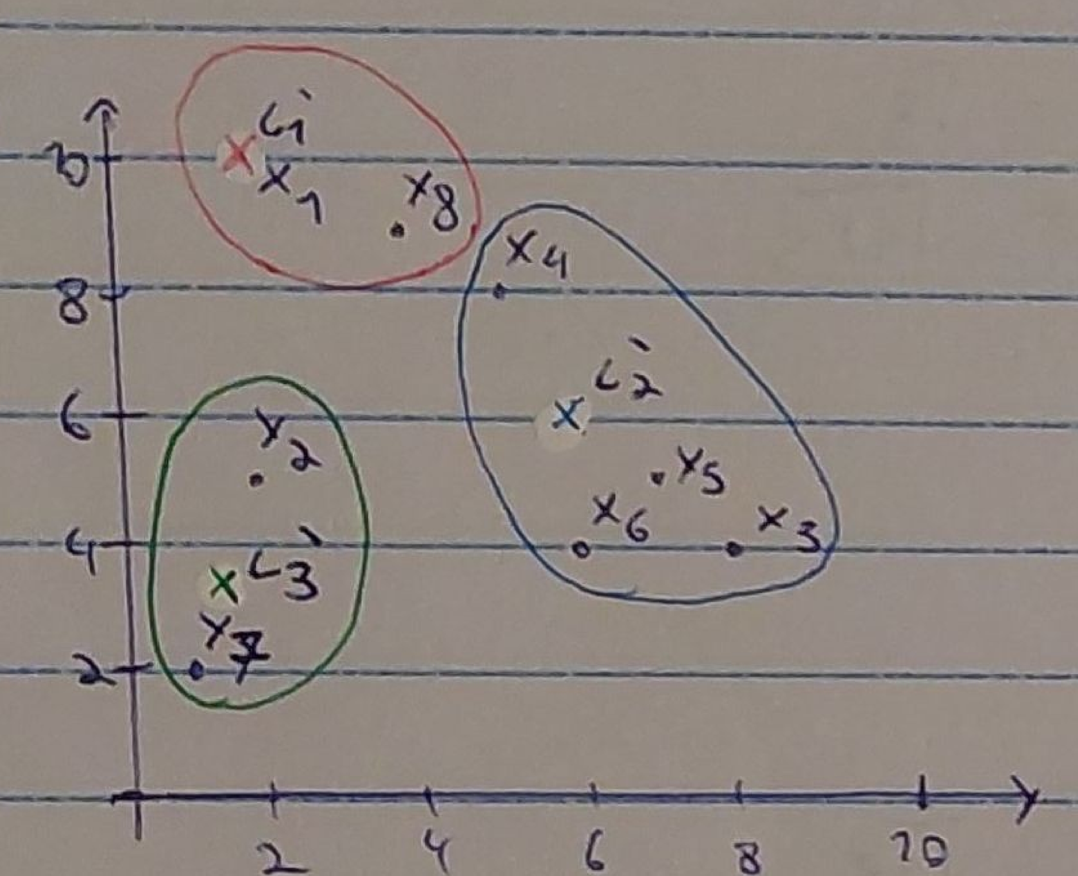
$x_7\ (1,2)$:

$d(x_7, c_1') = \sqrt{(1-2)^2 + (2-10)^2} = \sqrt{65}$
$d(x_7, c_2') = \sqrt{(1-6)^2 + (2-6)^2} = \sqrt{41}$
$d(x_7, c_3') = \sqrt{(1-1{,}5)^2 + (2-3{,}5)^2} = \sqrt{2{,}5}$
$\rightarrow$ Assign to c_3'

$x_8\ (4,9)$:

$d(x_8, c_1') = \sqrt{(4-2)^2 + (9-10)^2} \quad \sqrt{5}$
$d(x_8, c_2') = \sqrt{(4-6)^2 + (9-6)^2} \quad \sqrt{13}$
$d(x_8, c_3') = \sqrt{(4-1{,}5)^2 + (9-3{,}5)^2} = \sqrt{36{,}5}$
$\rightarrow$ Assign to c_1'

c)



d) Different centroid initializations can lead k-means to converge to different local minima. Poor choices may cause slow convergence or inaccurate clusters, while good inicalization improves stability and results. Thus, inicialization strongly affects both the speed and quality of clustering

Part B

D	y_1	y_2	y_3	y_4
x_1	5	0	1	+
x_2	0	5	0	+
x_3	1	0	-1	-

$$D = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\overline{x_1} = \frac{5+0+1}{3} = 2$$

$$\overline{x_2} = \frac{0+5+0}{3} = \frac{5}{3}$$

$$\overline{x_3} = \frac{1+0-1}{3} = 0$$

$$X - \overline{X} = \begin{bmatrix} 5-2 & 0-\frac{5}{3} & 1-0 \\ 0-2 & 5-5/3 & 0-0 \\ 1-2 & 0-5/3 & -1-0 \end{bmatrix} = \begin{bmatrix} 3 & -1,66667 & 1 \\ -2 & 3,33333 & 0 \\ -1 & -1,66667 & -1 \end{bmatrix}$$

$$Cov(x) = \frac{1}{n-1}(X-\overline{X})^T(X-\overline{X}) \quad n=3$$

$$(X-\overline{X})^T = \begin{bmatrix} 3 & -2 & 1 \\ -1,66667 & 3,33333 & -1,66667 \\ 1 & 0 & -1 \end{bmatrix}$$

$$(X-\overline{X})^T(X-\overline{X}) = \begin{bmatrix} 3 & -2 & -1 \\ -1,66667 & 3,33333 & -1,66667 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1,66667 & 1 \\ -2 & 3,33333 & 0 \\ -1 & -1,66667 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 14 & -10 & 4 \\ -10 & 16,66667 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$$Cov(x) = \frac{1}{3-1} \begin{bmatrix} 14 & -10 & 4 \\ -10 & 16,66667 & 0 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -5 & 2 \\ -5 & 8,33333 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

b)

$\det(\mathrm{cov}(x)-\lambda I)=0$

$$\begin{bmatrix} 7-\lambda & -5 & 2 \\ -5 & 8{,}33333-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix}=0$$

$\det=(7-\lambda)[(8{,}33333-\lambda)(1-\lambda)-0]-(-5)[-5(1-\lambda)-(2\times 0)]+2[-5\times 0-(8{,}33333-\lambda)\times 2]=0 \Leftrightarrow$

$(7-\lambda)[(8{,}33333-\lambda)(1-\lambda)]=(7-\lambda)(\lambda^2-9{,}33333\lambda+8{,}33333)=-\lambda^3+16{,}33333\lambda^2-73{,}66664\lambda$
$+58{,}33331$

$-(-5)[-5(1-\lambda)]=5(-5+5\lambda)=25\lambda-25$

$-2[2(8{,}33333-\lambda)]=4\lambda-33{,}33332$

$\Leftrightarrow -\lambda^3+16{,}33333\lambda^2-73{,}66664\lambda+25\lambda+4\lambda+58{,}33331-25-33{,}33332=0 \Leftrightarrow$

$\Leftrightarrow -\lambda^3+16{,}33333\lambda^2-44{,}66664\lambda+0 \Leftrightarrow$

$\Leftrightarrow \lambda(-\lambda^2+16{,}33333\lambda-44{,}66664)=0 \Leftrightarrow \rightarrow$ Quadratic formula

$\Leftrightarrow \lambda_1=12{,}86004,\ \lambda_2=3{,}47329,\ \lambda_3=0$

Vector 1

$$\begin{bmatrix} 7-12{,}86004 & -5 & 2 \\ -5 & 8{,}33333-12{,}86004 & 0 \\ 2 & 0 & 1-12{,}86004 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow$$

$\Leftrightarrow -5{,}86004V_1-5V_2+2V_3=0 \Leftrightarrow -5{,}86004V_1+5{,}52275V_1+0{,}33727V_1=0 \Leftrightarrow$

$-5V_1-4{,}52671V_2=0 \Leftrightarrow V_2=-1{,}10455V_1 \Leftrightarrow$

$2V_1-11{,}86004V_3=0 \Leftrightarrow V_3=0{,}16863V_1 \Leftrightarrow$

$\Leftrightarrow 0=0 \rightarrow V_1$ (free variable)

$V_2=-1{,}10455V_1$

$V_3=0{,}16863V_1$

$$V_1^{(n)}=\begin{bmatrix} 1 \\ -1{,}10455 \\ 0{,}16863 \end{bmatrix}$$

Normalization

$$\|V_1^{(u)}\| = \sqrt{1^2 + (-1{,}10455^2) + 0{,}16863^2} = 1{,}49949$$

$$V_1 = \frac{1}{\|V_1^{(u)}\|} V_1 = \begin{bmatrix} 0{,}66689 \\ -0{,}73667 \\ 0{,}11246 \end{bmatrix}$$

Vector 2:

$$\begin{bmatrix} 7-3{,}47329 & -5 & 2 \\ -5 & 8{,}33333-3{,}47329 & 0 \\ 2 & 0 & 1-3{,}47329 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3{,}52671 V_1 - 5V_2 + 2V_3 = 0 \\ -5 V_1 + 4{,}86004 V_2 = 0 \\ 2V_1 - 2{,}47329\, V_3 = 0 \end{cases} \Leftrightarrow \begin{cases} V_1 = 3{,}52671 V_1 - 5{,}14399 V_1 + 1{,}61728 V_1 \\ V_2 = 1{,}02880 V_1 \\ V_3 = 0{,}80864 V_1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0 = 0 \rightarrow V_1 \text{ (free variable)} \\ V_2 = 1{,}02880 V_1 \\ V_3 = 0{,}80864 V_1 \end{cases} \qquad V_2^{(u)} = \begin{bmatrix} 1 \\ 1{,}02880 \\ 0{,}80864 \end{bmatrix}$$

Normalization

$$\|V_2^{(u)}\| = \sqrt{1^2 + 1{,}02880^2 + 0{,}80864^2} = 1{,}64691$$

$$V_2 = \frac{1}{\|V_2^{(u)}\|} V_2^{(u)} = \begin{bmatrix} 0{,}60720 \\ 0{,}62469 \\ 0{,}49100 \end{bmatrix}$$

Vector 3:

$$\begin{bmatrix} 7 & -5 & 2 \\ -5 & 8{,}33333 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{cases} 7V_1 - 5V_2 + 2V_3 = 0 \\ -5V_1 + 8{,}33333 V_2 = 0 \\ 2V_1 + V_3 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 7V_1 - 3V_1 - 4V_1 = 0 \\ V_2 = 0{,}6V_1 \\ V_3 = -2V_1 \end{cases} \begin{matrix} \\ = 0 \\ = 0 \end{matrix} \Leftrightarrow \begin{cases} 0 = 0 \rightarrow V_1 \text{ (free variable)} \\ V_2 = 0{,}6 V_1 \\ V_3 = -2V_1 \end{cases} \qquad V_3^{(u)} = \begin{bmatrix} 1 \\ 0{,}6 \\ -2 \end{bmatrix}$$

Normalization:

$$\|V_3^{(u)}\| = \sqrt{1^2 + 0{,}6^2 + (-2)^2} = 2{,}31517$$

$$V_3 = \frac{1}{\|V_3^{(u)}\|} V_3^{(u)} = \begin{bmatrix} 0{,}43195 \\ 0{,}25904 \\ 0{,}86349 \end{bmatrix}$$

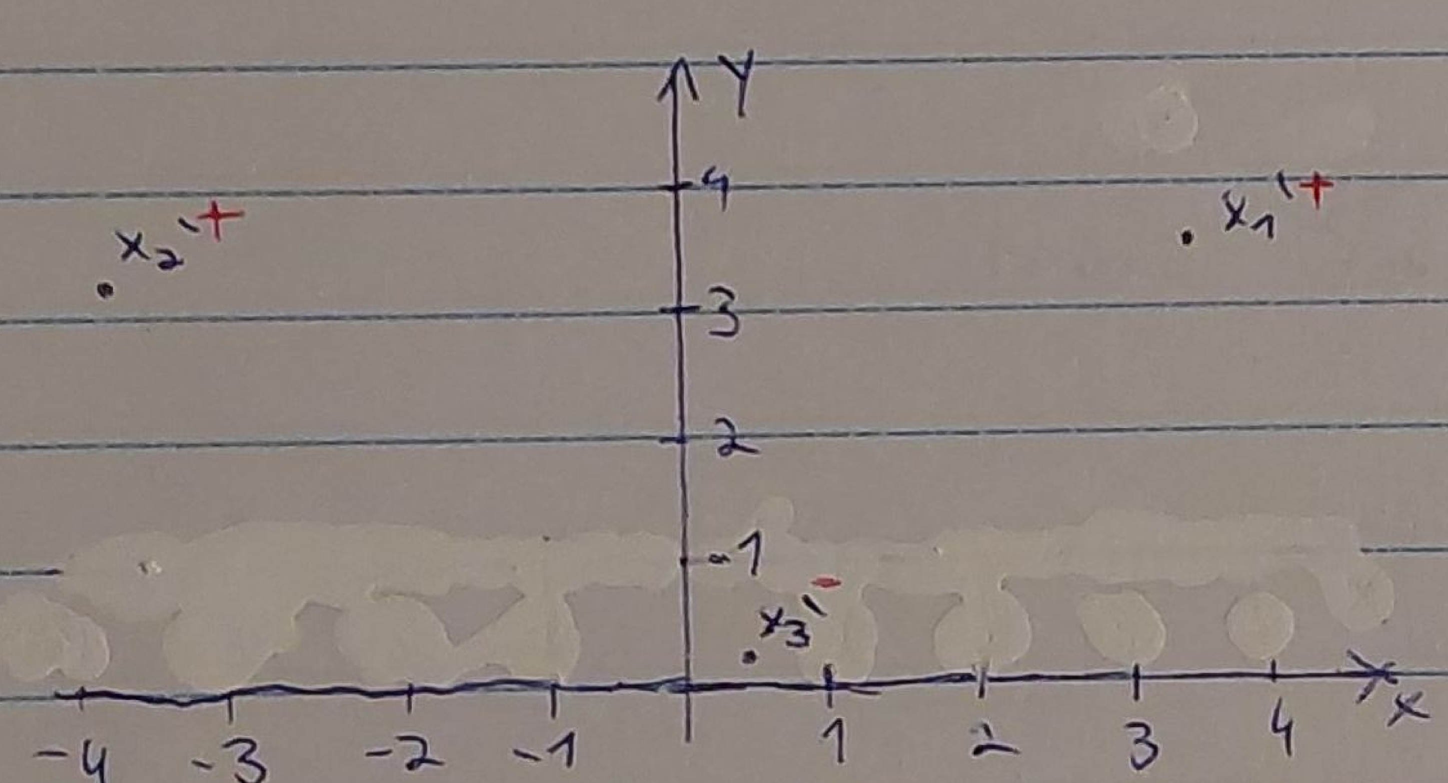
We use the two largest eigenvectores, V_1 and V_2, since they correspond to the directions with the highest variance after normalization

$$\text{Vectores} : \begin{bmatrix} 0{,}66639 & 0{,}60720 \\ -0{,}73667 & 0{,}62469 \\ 0{,}11246 & 0{,}49100 \end{bmatrix}$$

$$Z = (X)V$$

$$\begin{bmatrix} 5 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0{,}66639 & 0{,}60720 \\ -0{,}73667 & 0{,}62469 \\ 0{,}11246 & 0{,}49100 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 3{,}44691 & 3{,}52700 \\ -3{,}68335 & 3{,}12345 \\ 0{,}55443 & 0{,}11620 \end{bmatrix} \end{array} \begin{array}{l} x_1' \\ x_2' \\ x_3' \end{array}$$

c.



After projecting the data onto the PCA plane, and marking according to y_4 (+/−), we can see the two "+" points are located in the upper region, while the "−" point is seperated below. This shows that the projection plane successfully discriminates between the two classes.