Gabor overlaps

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This note computes the overlap between two Gabor functions. Gabors are defined as a sinusoid modulated by a gaussian envelope:

$$g(\boldsymbol{x}|\boldsymbol{\mu},\sigma^2,\boldsymbol{k},\phi) = \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{\mu}|^2}{2\sigma^2}\right)\cos\left(\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{\mu})+\phi\right) \tag{1}$$

We often want to compute the overlap between two Gabor functions,

$$G_{mn} = \int d\boldsymbol{x}\, g_m(\boldsymbol{x})g_n(\boldsymbol{x}) \tag{2}$$

where each Gabor has its own parameters $g_m(\boldsymbol{x}) = g(\boldsymbol{x}|\boldsymbol{\mu}_m,\sigma_m^2,\boldsymbol{k}_m,\phi_m)$. For example, one Gabor might be a linear receptive field and the other might be a visual image.

To compute this integral, it is convenient to use complex exponentials instead of sines and cosines, $\cos t = \Re[e^{it}] = \frac{1}{2}(e^{it}+e^{-it})$. The real part of a product of complex exponentials is

$$\begin{aligned}
\Re[uv] &= \frac{1}{4}(u+\bar{u})(v+\bar{v}) && (3)\\
&= \frac{1}{4}(uv+\bar{u}v+u\bar{v}+\bar{u}\bar{v}) && (4)\\
&= \frac{1}{4}((uv+\overline{uv})+(u\bar{v}+\overline{u\bar{v}})) && (5)\\
&= \frac{1}{2}(\Re[uv]+\Re[u\bar{v}]) && (6)
\end{aligned}$$

This means that to compute the overlap (2), we can define a complex Gabor

$$h(\boldsymbol{x}|\boldsymbol{\mu},\sigma^2,\boldsymbol{k},\phi) = \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{\mu}|^2}{2\sigma^2}+i\left(\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{\mu})+\phi\right)\right) \tag{7}$$

and then compute the complex overlaps

$$H_{mn} = \int d\boldsymbol{x}\, h_m(\boldsymbol{x})h_n(\boldsymbol{x}) \tag{8}$$

Using this overlap with (6), we can compute

$$G_{mn} = \tfrac{1}{2}(\Re[H_{mn}]+\Re[H_{m\bar{n}}]) \tag{9}$$

where $H_{m\bar{n}} = \int d\boldsymbol{x}\, h_m(\boldsymbol{x})\bar{h}_n(\boldsymbol{x})$, and the only difference between h and $\bar{h}$ is that $\boldsymbol{k} \to -\boldsymbol{k}$ and $\phi \to -\phi$. Thus once we have one general H_{mn}, we can transform it to compute the target integral G_{mn}.

To compute H_{mn}, notice that the product of two complex gabors is yet another complex gabor,

$$h_m(\boldsymbol{x})h_n(\boldsymbol{x}) = \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{\mu}_m|^2}{2\sigma_m^2} - \frac{|\boldsymbol{x}-\boldsymbol{\mu}_n|^2}{2\sigma_n^2} + i\left(\boldsymbol{k}_m\cdot(\boldsymbol{x}-\boldsymbol{\mu}_m) + \phi_m + \boldsymbol{k}_n\cdot(\boldsymbol{x}-\boldsymbol{\mu}_n) + \phi_n\right)\right) \quad (10)$$

$$= \exp\left(-\frac{1}{2}a|\boldsymbol{x}|^2 + \boldsymbol{b}\cdot\boldsymbol{x} + c\right) \quad (11)$$

where we have defined

$$a = \frac{1}{\sigma_m^2} + \frac{1}{\sigma_n^2} \quad (12)$$

$$\boldsymbol{b} = \frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2} + i\boldsymbol{k}_m + i\boldsymbol{k}_n \quad (13)$$

$$c = -\frac{|\boldsymbol{\mu}_m|^2}{2\sigma_m^2} - \frac{|\boldsymbol{\mu}_n|^2}{2\sigma_n^2} + i\phi_m - i\boldsymbol{k}_m\cdot\boldsymbol{\mu}_m + i\phi_n - i\boldsymbol{k}_n\cdot\boldsymbol{\mu}_n \quad (14)$$

We can complete the square in form (11),

$$h_m(\boldsymbol{x})h_n(\boldsymbol{x}) = \exp\left(-\frac{1}{2}a|\boldsymbol{x}-\boldsymbol{b}/a|^2\right)\exp\left(c + \frac{|\boldsymbol{b}|^2}{2a}\right) \quad (15)$$

and this form can be readily integrated to give

$$H_{mn} = \int d\boldsymbol{x}\, h_m(\boldsymbol{x})h_n(\boldsymbol{x}) = \frac{2\pi}{a}\exp\left(c + \frac{|\boldsymbol{b}|^2}{2a}\right) \quad (16)$$

where $|\boldsymbol{b}|^2 = \boldsymbol{b}\cdot\boldsymbol{b}$. Note that this magnitude is a *vector* magnitude, and *not* a complex magnitude $\boldsymbol{b}\cdot\bar{\boldsymbol{b}}$, so it can therefore still have an imaginary part.

Now we have to unpack the variables $a, \boldsymbol{b}, c$. The magnitude $|\boldsymbol{b}|^2$ is

$$|\boldsymbol{b}|^2 = \left|\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right|^2 - |\boldsymbol{k}_m + \boldsymbol{k}_n|^2 + 2i\left(\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right)\cdot(\boldsymbol{k}_m + \boldsymbol{k}_n) \quad (17)$$

Separating the argument of the exponential into real and imaginary parts, we find

$$c + \frac{|\boldsymbol{b}|^2}{2a} = \left[-\frac{|\boldsymbol{\mu}_m|^2}{2\sigma_m^2} - \frac{|\boldsymbol{\mu}_n|^2}{2\sigma_n^2} + \frac{1}{2}\left(\left|\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right|^2 - |\boldsymbol{k}_m + \boldsymbol{k}_n|^2\right)\left(\frac{1}{\sigma_m^2} + \frac{1}{\sigma_n^2}\right)^{-1}\right] \quad (18)$$

$$+ i\left[(\phi_m - \boldsymbol{k}_m\cdot\boldsymbol{\mu}_m) + (\phi_n - \boldsymbol{k}_n\cdot\boldsymbol{\mu}_n) + \left(\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right)\cdot(\boldsymbol{k}_m + \boldsymbol{k}_n)\left(\frac{1}{\sigma_m^2} + \frac{1}{\sigma_n^2}\right)^{-1}\right] \quad (19)$$

$$= s + it \quad (20)$$

Using the real part of the exponential involving this term, we have

$$\Re[H_{mn}] = \frac{2\pi}{a}e^s\cos t \quad (21)$$

$$= \frac{2\pi}{\sigma_m^{-2} + \sigma_n^{-2}}\exp\left[-\frac{|\boldsymbol{\mu}_m|^2}{2\sigma_m^2} - \frac{|\boldsymbol{\mu}_n|^2}{2\sigma_n^2} + \frac{1}{2}\left(\left|\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right|^2 - |\boldsymbol{k}_m + \boldsymbol{k}_n|^2\right)\left(\frac{1}{\sigma_m^2} + \frac{1}{\sigma_n^2}\right)^{-1}\right] \quad (22)$$

$$\cdot\cos\left[(\phi_m - \boldsymbol{k}_m\cdot\boldsymbol{\mu}_m) + (\phi_n - \boldsymbol{k}_n\cdot\boldsymbol{\mu}_n) + \left(\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right)\cdot(\boldsymbol{k}_m + \boldsymbol{k}_n)\left(\frac{1}{\sigma_m^2} + \frac{1}{\sigma_n^2}\right)^{-1}\right] \quad (23)$$

The same thing holds for the term $H_{m\bar{n}}$ but with $\boldsymbol{k}_n \to -\boldsymbol{k}_n$ and $\phi_n \to -\phi_n$, giving

$$G_{mn} = \tfrac{1}{2}(\Re[H_{mn}] + \Re[H_{mn}]) \tag{24}$$

$$= \frac{\pi}{\sigma_m^{-2} + \sigma_n^{-2}} \exp\left[-\frac{|\boldsymbol{\mu}_m|^2}{2\sigma_m^2} - \frac{|\boldsymbol{\mu}_n|^2}{2\sigma_n^2} + \frac{1}{2}\left|\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right|^2 \left(\frac{1}{\sigma_m^2} + \frac{1}{\sigma_n^2}\right)^{-1}\right] \tag{25}$$

$$\left(\exp\left[-\tfrac{1}{2}\frac{|\boldsymbol{k}_m + \boldsymbol{k}_n|^2}{\sigma_m^{-2} + \sigma_n^{-2}}\right] \cos\left[(\phi_m - \boldsymbol{k}_m \cdot \boldsymbol{\mu}_m) + (\phi_n - \boldsymbol{k}_n \cdot \boldsymbol{\mu}_n) + \frac{\left(\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right) \cdot (\boldsymbol{k}_m + \boldsymbol{k}_n)}{\sigma_m^{-2} + \sigma_n^{-2}}\right]\right. \tag{26}$$

$$\left. + \exp\left[-\tfrac{1}{2}\frac{|\boldsymbol{k}_m - \boldsymbol{k}_n|^2}{\sigma_m^{-2} + \sigma_n^{-2}}\right] \cos\left[(\phi_m - \boldsymbol{k}_m \cdot \boldsymbol{\mu}_m) - (\phi_n - \boldsymbol{k}_n \cdot \boldsymbol{\mu}_n) + \frac{\left(\frac{\boldsymbol{\mu}_m}{\sigma_m^2} + \frac{\boldsymbol{\mu}_n}{\sigma_n^2}\right) \cdot (\boldsymbol{k}_m - \boldsymbol{k}_n)}{\sigma_m^{-2} + \sigma_n^{-2}}\right]\right) \tag{27}$$

where changes between the two terms have been highlighted in color.

I have checked this derivation against numerical integration and the two quantities agree.