Ex: 7.8

DATE / /

3-32

Q Evaluate the integrals that converge

3 $\int_{0}^{\infty} e^{-2x} dx$

This is improper integral

$\lim_{b\to\infty} \int_{0}^{b} e^{-2x} dx$

$\lim_{b\to\infty} -\frac{1}{2} e^{-2x} \Big|_{0}^{b}$

$\frac{1}{2} \lim_{b\to\infty} -e^{-2x} \Big|_{0}^{b}$

$\frac{1}{2}\left[-e^{-2b} - \left(-e^{-2(0)}\right)\right]$

$-\frac{1}{2} e^{-2(\infty)} + \frac{1}{2}(1)$

$0 + \frac{1}{2}(1)$

$\frac{1}{2}$

4 $\int_{-1}^{\infty} \frac{x}{1+x^2} dx$

$\lim_{b\to\infty} \int_{-1}^{b} \frac{x}{1+x^2} dx$

$\frac{1}{2} \lim_{b\to\infty} \ln(1+x^2) \Big|_{-1}^{b}$

$\lim_{b\to\infty} \frac{1}{2}\left[\ln(1+b^2) - \ln(1+(-1)^2)\right]$

$\frac{1}{2}\left[\ln(1+\infty^2) - \ln(2)\right]$

$= \infty$

$\therefore$ divergent integral

5 $\int_{3}^{\infty} \frac{2}{x^2-1}$

$\lim_{b\to\infty} \int_{3}^{b} \frac{2}{x^2-1}$

$2 \lim_{b\to\infty} \int_{3}^{b} \frac{1}{x^2-1}$

$-2 \lim_{b\to\infty} \int_{3}^{b} \frac{-1}{x^2-1}$

$\lim_{b\to\infty} -2\coth^{-1} x \Big|_{3}^{b}$

$\lim_{b\to\infty} -2\coth^{-1}(b) - (-2\coth^{-1}(3))$

$-2\coth^{-1}(\infty) + 2\coth^{-1}(3)$

$2\coth^{-1}(3)$

6 $\int_{0}^{\infty} xe^{-x^2} dx$

$-\frac{1}{2} \lim_{b\to\infty} \int_{0}^{b} -2xe^{-x^2} dx$

$-\frac{1}{2} \lim_{b\to\infty} e^{-x^2} \Big|_{0}^{b}$

$\lim_{b\to\infty} \frac{1}{2}\left(-e^{-b^2} - \left(-e^{-0^2}\right)\right)$

$-\frac{1}{2} e^{-\infty^2} + \frac{1}{2}$

$0 + \frac{1}{2}$

$\frac{1}{2}$

DATE / /

7 $\int_e^\infty \frac{1}{x \ln^3 x} dx$

$\lim_{b\to\infty} \int_e^b \frac{1}{x} \ln^{-3} x \, dx$

$\lim_{b\to\infty} -\frac{1}{2}\frac{1}{\ln^2 x}\Big|_e^b$

$\lim_{b\to\infty} -\frac{1}{2\ln^2(b)} - \left(-\frac{1}{2\ln^2(e)}\right)$

$-\frac{1}{2\ln^2(\infty)} + \frac{1}{2(1)}$

$\frac{1}{2}$

8 $\int_2^\infty \frac{1}{x\sqrt{\ln x}}$

$\lim_{b\to\infty} \int_2^b \frac{1}{x} (\ln x)^{-1/2}$

$2 \lim_{b\to\infty} (\ln x)^{1/2}\Big|_2^b$

$2 \lim_{b\to\infty} \sqrt{\ln x}\Big|_2^b$

$\lim_{b\to\infty} 2\sqrt{\ln b} - 2\sqrt{\ln 2}$

$2\sqrt{\ln(\infty)} - 2\sqrt{\ln 2}$

∞

$\therefore$ divergent

9 $\int_{-\infty}^0 \frac{dx}{(2x-1)^3}$

$\lim_{a\to-\infty} \int_a^0 (2x-1)^{-3} dx$

$\lim_{a\to-\infty} \frac{1}{2}\int_a^0 2(2x-1)^{-3} dx$

$\lim_{a\to-\infty} \frac{1}{2} \frac{(2x-1)^{-2}}{-2}\Big|_a^0$

$\lim_{a\to-\infty} -\frac{1}{4}(2x-1)^{-2}\Big|_a^0$

$\lim_{a\to-\infty} -\frac{1}{4}(2(0)-1)^{-2} + \frac{1}{4}(2a-1)^{-2}$

$-\frac{1}{4}(-1)^{-2} + \frac{1}{4}(2(\infty)-1)^{-2}$

$= -1/4 + \frac{1}{4}(0) = -1/4$

10 $\int_{-\infty}^3 \frac{dx}{x^2+9}$

$\lim_{a\to-\infty} \int_a^3 \frac{dx}{x^2+9}$

$\lim_{a\to-\infty} \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_a^3$

$\lim_{a\to\infty} \frac{1}{3}\left[\frac{\pi}{4} - \tan^{-1}\frac{a}{3}\right]$

apply limit

$\frac{1}{3}\left[\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right]$

$= \frac{\pi}{4}$

DATE / /

11 $\int_{-\infty}^{0} e^{3x}\,dx$

$\lim_{a\to-\infty}\int_{a}^{0} e^{3x}\,dx \quad \times 3 \times \frac{1}{3}$

$\lim_{a\to-\infty} \frac{1}{3} e^{3x}\Big|_{a}^{0}$

$\lim_{a\to-\infty}\left[\frac{1}{3}-\frac{1}{3}e^{3a}\right]$

$= \frac{1}{3} - 0$

$= 1/3$

12 $\int_{-\infty}^{0} \frac{e^{x}\,dx}{3-2e^{x}}$

$\lim_{a\to-\infty}\int_{a}^{0} \frac{e^{x}\,dx}{3-2e^{x}}$

$\lim_{a\to-\infty} -\frac{1}{2}\int_{a}^{0} \frac{-2e^{x}}{3-2e^{x}}$

$\lim_{a\to-\infty} -\frac{1}{2}\ln\left(3-2e^{x}\right)\Big|_{a}^{0}$

$\lim_{a\to-\infty} \frac{1}{2}\ln\left(3-2e^{a}\right)$

$= \frac{1}{2}\ln 3$

13 $\int_{-\infty}^{+\infty} x\,dx$

$\int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$

consider

$\lim_{a\to-\infty}\int_{a}^{0} x\,dx + \lim_{b\to\infty}\int_{0}^{b} x\,dx$

$\lim_{a\to-\infty} \frac{x^2}{2}\Big|_{a}^{0} + \lim_{b\to\infty}\frac{x^2}{2}\Big|_{0}^{b}$

$\lim_{a\to-\infty}\left(0-\frac{a^2}{2}\right) + \lim_{b\to\infty}\left(\frac{b^2}{2}-0\right)$

$-\left(\frac{-\infty^2}{2}\right) + \left(\frac{\infty^2}{2}\right)$

$= \infty$

divergent

14 $\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+2}}\,dx$

$\int_{-\infty}^{0} \frac{x}{\sqrt{x^2+2}}\,dx + \int_{0}^{\infty} \frac{x}{\sqrt{x^2+2}}\,dx$

$\lim_{a\to-\infty}\int_{a}^{0} \frac{x}{\sqrt{x^2+2}}\,dx + \lim_{b\to\infty}\int_{0}^{b} \frac{x}{\sqrt{x^2+2}}\,dx$

$\lim_{a\to-\infty} \frac{1}{2}\int_{a}^{0} \frac{2x}{\sqrt{x^2+2}}\,dx + \lim_{b\to\infty}\frac{1}{2}\int^{b} \frac{2x\,dx}{\sqrt{x^2+2}}$

$\lim_{a\to-\infty} \frac{1}{2}\frac{\sqrt{x^2+2}}{1/2}\Big|_{a}^{0} + \lim_{b\to\infty}\frac{1}{2}\frac{\sqrt{x^2+2}}{1/2}\Big|_{0}^{b}$

$\lim_{a\to-\infty}\sqrt{x^2+2}\Big|_{a}^{0} + \lim_{b\to\infty}\sqrt{x^2+2}\Big|_{0}^{b}$

$\lim_{a\to-\infty}\left(\sqrt{2}-\sqrt{a^2+2}\right) + \lim_{b\to\infty}\left(\sqrt{b^2+2}-\sqrt{2}\right)$

$\infty + \infty$

divergent

DATE / /

15 $$\int_{-\infty}^{\infty} \frac{x}{(x^2+3)^2}\,dx$$

$$\lim_{a\to-\infty}\int_a^0 \frac{x}{(x^2+3)^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{x\,dx}{(x^2+3)^2}$$

$$\lim_{a\to-\infty}\frac{1}{2}\int_a^0 \frac{2x}{(x^2+3)^2}\,dx + \lim_{b\to\infty}\frac{1}{2}\int_0^b \frac{2x}{(x^2+3)^2}$$

$$\lim_{a\to-\infty} \frac{-1}{2(x^2+3)}\Big|_a^0 + \lim_{b\to\infty} \frac{-1}{2(x^2+3)}\Big|_0^b$$

$$\lim_{a\to-\infty} -\frac{1}{2}\left(\frac{1}{3}-\frac{1}{a^2+3}\right) + \lim_{b\to\infty} -\frac{1}{2}\left(\frac{1}{b^2+3}-\frac{1}{3}\right)$$

$$-\frac{1}{6} + \frac{1}{6}$$

$$0$$

16 $$\int_{-\infty}^{\infty} \frac{e^{-1}}{1+e^{-2t}}\,dt$$

$$\lim_{a\to-\infty}\int_a^0 \frac{e^{-1}}{1+e^{-2t}}\,dx + \lim_{b\to\infty}\int_0^b \frac{e^{-1}}{1+e^{-2t}}\,dx$$

$$\lim_{a\to-\infty}\int_a^0 \frac{1}{1+e^{-t}}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+e^{-t}}\,dx$$

$$\lim_{a\to-\infty} -\tan^{-1}(e^{-t})\Big|_a^0 + \lim_{b\to\infty} -\tan^{-1}(e^{-t})\Big|_0^b$$

$$\lim_{a\to-\infty} -\left[\frac{\pi}{4} - \tan^{-1}(e^{-a})\right] + \lim_{b\to\infty} -\left[\tan^{-1}(e^{-b}) - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{2\pi}{4} = \frac{\pi}{2}$$

17 $$\int_0^4 \frac{dx}{(x-4)^2}$$

0 4 (approaching 4 from LHS)

$$\lim_{k\to4^-}\int_0^k \frac{1}{(x-4)^2}$$

$$\lim_{k\to4^-}\int_0^k (x-4)^{-2}$$

$$\lim_{k\to4^-} \frac{-1}{(x-4)}\Big|_0^k$$

$$\lim_{k\to4^-}\left[-\frac{1}{k-4} - \left(-\frac{1}{0-4}\right)\right]$$

$$= \frac{-1}{4-4} + \frac{1}{-4}$$

$$= \infty - \frac{1}{4}$$

$$= \infty$$

18 $$\int_0^8 \frac{dx}{\sqrt[3]{x}}$$

0 8 (approaching 0 from RHS)

$$\lim_{k\to0^+}\int_k^8 \frac{dx}{3\sqrt{x}}$$

$$\lim_{k\to0^+}\int_k^8 x^{3/2}\,dx$$

$$\lim_{k\to0^+} \frac{3}{2}x^{2/3}\Big|_k^8$$

$$\lim_{k\to0^+} \frac{3}{2}8^{2/3} - \frac{3}{2}k^{2/3}$$

$$= \frac{3}{2}8^{2/3} - \frac{3}{2}(0)^{2/2}$$

$$= 6$$

DATE / /

19 $\int_0^{\pi/2} \tan x \, dx$

$\lim_{k \to \frac{\pi}{2}^-} -\ln(\cos x) \Big|_0^k$

$\lim_{k \to \frac{\pi}{2}^-} -\ln(\cos k) + \ln(\cos 0)$

$= \infty$

divergent

20 $\int_0^4 \frac{dx}{\sqrt{4-x}}$

$\lim_{k \to 4^-} -\int_0^k -dx \, (4-x)^{-1/2}$

$\lim_{k \to 4^-} - \frac{\sqrt{4-x}}{1/2} \Big|_0^k$

$\lim_{k \to 4^-} -2(\sqrt{4-k} - \sqrt{4-0})$

$-2(\sqrt{4-4} - \sqrt{4-0})$

$-2(0 - 2)$

4

21 $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$\lim_{k \to 1^-} \int_0^k dx \frac{1}{\sqrt{1-x^2}}$

$\lim_{k \to 1^-} \sin^{-1} x \Big|_0^k$

$\lim_{k \to 1^-} (\sin^{-1} k - \sin^{-1}(0))$

$\sin^{-1}(1) - \sin^{-1}(0)$

$= \frac{\pi}{2}$

22 $\int_{-3}^{1} \frac{x \, dx}{\sqrt{9-x^2}}$

$\lim_{k \to -3^+} -\frac{1}{2} \int_k^1 -2x \, (9-x^2)^{-1/2} dx$

$\lim_{k \to -3^+} -\frac{1}{2} \frac{\sqrt{9-x^2}}{1/2} \Big|_k^1$

$\lim_{k \to -3^+} -\sqrt{9-x^2} \Big|_k^1$

$\lim_{k \to -3^+} -(\sqrt{9-1^2} - \sqrt{9-k^2})$

$-(\sqrt{8} - \sqrt{9-(-3)^2})$

$-(\sqrt{8} - 0)$

$-\sqrt{8}$

23 $\int_{\pi/3}^{\pi/2} \frac{\sin x}{\sqrt{1-2\cos x}} dx$

$\lim_{k \to \frac{\pi}{3}^+} \frac{1}{2} \int_k^{\pi/2} 2\sin x \, (1-2\cos x)^{-1/2} dx$

$\lim_{k \to \pi/3^+} \frac{1}{\cancel{2}} \frac{\sqrt{1-2\cos x}}{\cancel{1/2}} \Big|_k^{\pi/2}$

$\lim_{k \to \pi/3} \sqrt{1-2\cos \pi/2} - \sqrt{1-2\cos k}$

$= 1 - \sqrt{1-2\cos k}$

$= 1$

DATE / /

24 $\int_0^{\pi/4} \frac{\sec^2 x}{1-\tan x}\,dx$

$\lim_{k\to\pi/4^-} -\ln(1-\tan x)\Big|_0^k$

$\lim_{k\to\pi/4^-} -[\ln(1-\tan k) - \ln(1-\tan 0)]$

$= -\ln(1-\tan \pi/4) + \ln(1-\tan 0)$

$= \infty$

divergent

25 $\int_0^3 \frac{dx}{x-2}$

$\int_0^2 \frac{dx}{x-2} + \int_2^3 \frac{dx}{x-2}$

$\lim_{k\to 2^-} \int_0^k \frac{dx}{x-2} + \lim_{k\to 2^+} \int_k^3 \frac{dx}{x-2}$

$\lim_{k\to 2^-} \ln(x-2)\Big|_0^k + \lim_{x\to 2^+} \ln(x-2)\Big|_k^3$

$\lim_{k\to 2^-} [\ln(k-2) - \ln(0-2)] + \lim_{k\to 2^+} [\ln(3-2) - \ln(k-2)]$

$(\infty - \infty) + (0 - \infty)$

$= \infty$

divergent

26 $\int_{-2}^{2} \frac{dx}{x^2}$

$\int_{-2}^{0} \frac{dx}{x^2} + \int_0^2 \frac{dx}{x^2}$

$\lim_{k\to 0^-} \int_{-2}^{k} \frac{dx}{x^2} + \lim_{k\to 0^+} \int_k^2 \frac{dx}{x^2}$

$\lim_{k\to 0^-} -\frac{1}{x}\Big|_{-2}^{k} + \lim_{k\to 0^+} -\frac{1}{x}\Big|_k^2$

$\lim_{k\to 0^-} \left[-\frac{1}{k} + \frac{1}{-2}\right] + \lim_{k\to 0^+} \left[-\frac{1}{2} + \frac{1}{k}\right]$

$\left(-\infty - \frac{1}{2}\right) + \left(\frac{-1}{2} + \infty\right)$

∞

divergent

27 $\int_{-1}^{8} x^{-1/3}\,dx$

$\lim_{k\to 0^-} \int_{-1}^{0} x^{-1/3}\,dx + \lim_{k\to 0^+} \int_0^8 x^{-1/3}\,dx$

$\lim_{k\to 0^-} \frac{3}{2} x^{2/3}\Big|_{-1}^{k} + \lim_{k\to 0^+} \frac{3}{2} x^{2/3}\Big|_k^8$

$\lim_{k\to 0^-} \frac{3}{2}\left(k^{2/3} - (-1)^{2/3}\right) + \lim_{k\to 0^+} \frac{3}{2}\left(8^{2/3} - k^{2/3}\right)$

$\frac{3}{2}\left(0^{2/3} - 1^{2/3}\right) + \frac{3}{2}\left(8^{2/3} - 0^{2/3}\right)$

$-\frac{3}{2} + 6$

$= \frac{9}{2}$

Scanned with CamScanner

DATE / /

28 $\int_0 \frac{dx}{(x-1)^{2/3}}$

$\lim_{k\to 1^-} \int_0^k (x-1)^{-2/3}$

$\lim_{k\to 1^-} \frac{(x-1)^{1/3}}{1/3}\Big|_0^k$

$\lim_{k\to 1^-} 3(x-1)^{1/3}\Big|_0^k$

$\lim_{k\to 1^-} 3(k-1)^{1/3} - 3(0-1)^{1/3}$

$3(1-1)^{1/3} - 3(-1)^{1/3}$

$= 3$

29 $\int_0^\infty \frac{1}{(x^2)}dx$

$\lim_{k\to 0^+} \int_k^1 \frac{dx}{x^2} + \lim_{k\to\infty} \int_1^k \frac{dx}{x^2}$

$\lim_{k\to 0^-} -\frac{1}{x}\Big|_k^1 + \lim_{k\to\infty} -\frac{1}{x}\Big|_1^k$

$\lim_{k\to 0^-} \left[-\frac{1}{1} + \frac{1}{k}\right] + \lim_{k\to\infty}\left[\frac{-1}{k} + \frac{1}{1}\right]$

$(-1+\infty) + (0+1)$

$= \infty$

divergent

30 $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$

$\lim_{k\to 1^+} \int_k^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{k\to\infty} \int_2^k \frac{dx}{}$

$\lim_{k\to 1^+} \sec^{-1}x\Big|_k^2 + \lim_{k\to\infty} \sec^{-1}x\Big|_2^k$

$\lim_{k\to 1^+} (\sec^{-1}2 - \sec^{-1}k) + \lim_{k\to\infty} (\sec^{-1}k - \sec^{-1}2)$

$(\sec^{-1}2 - \sec^{-1}1) + (\sec^{-1}\infty - \sec^{-1}2)$

$(\pi/3 - 0) + (\pi/2 - \pi/3)$

$\cancel{\pi/3} + \pi/2 - \cancel{\pi/3}$

$\pi/2$

for $\sec^{-1}(\infty)$

$\sec\theta = x$

$\frac{1}{\cos\theta} = \frac{1}{0}$

$\cos\theta = 0$

$\theta = \cos^{-1}(0)$

$\theta = \pi/2$

* for $\sec^{-1}2$

$\theta = \sec^{-1}x$

$\sec\theta = x$

$\frac{1}{\cos\theta} = 2$

$\cos\theta = 1/2$

$\theta = \cos^{-1}(1/2)$

$= \pi/3$

DATE / /

31) $\int_0^1 \frac{dx}{\sqrt{x}(x+1)}$ $\quad U = \sqrt{x}$, $U^2 = x$, $2U\,dU = dx$

using U substitution

$\int_0^1 \frac{2U\,dU}{U(U^2+1)}$

$\lim_{k\to 0} \int_k^1 \frac{2\,dU}{U^2+1}$

$\lim_{k\to 0} 2 \int_k^1 \frac{dU}{U^2+1}$

$\lim_{k\to 0} 2\tan^{-1} U \Big|_k^1$

$\lim_{k\to 0} 2\tan^{-1}\sqrt{x} \Big|_k^1$

$\lim_{k\to 0} 2\tan^{-1}\sqrt{1} - 2\tan^{-1}\sqrt{k}$

$2\left[\frac{\pi}{4} - \tan^{-1}\sqrt{0}\right]$

$\frac{2\pi}{4} + 0$

$= \pi/2$

32) $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

using U substitution

$\int_0^\infty \frac{2U\,dU}{U(U^2+1)}$

$\lim_{k\to 0^+} \int_k^1 \frac{2\,dU}{U^2+1} + \lim_{k\to\infty} \int_1^\infty \frac{2\,dU}{U^2+1}$

$\lim_{k\to 0^+} 2\int_k^1 \frac{dU}{U^2+1} + \lim_{k\to\infty} 2\int_1^k \frac{dU}{U^2+1}$

$\lim_{k\to 0^+} 2\tan^{-1} U \Big|_k^1 + \lim_{k\to\infty} 2\tan^{-1} U \Big|_1^k$

$\lim_{k\to 0^+} 2\tan^{-1}\sqrt{x} \Big|_k^1 + \lim_{k\to\infty} 2\tan^{-1}\sqrt{x} \Big|_1^k$

$\lim_{k\to 0^+} 2(\tan^{-1}\sqrt{1} - \tan^{-1}\sqrt{k}) + \lim_{k\to\infty} 2(\tan^{-1}\sqrt{\infty} - \tan^{-1}\sqrt{1})$

$2\left(\frac{\pi}{4} - \tan^{-1}\sqrt{0}\right) + 2\left(\tan^{-1}\sqrt{\infty} - \frac{\pi}{4}\right)$

$2\left(\frac{\pi}{4} - 0\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$= \pi/2 + \frac{\pi}{2}$

$= \frac{2\pi}{2}$

$= \pi$