Applied Physics
NS (1001)

Vectors Topics

- Vectors and their components
- Unit Vector , adding vector by components
- Multiplying Vectors

What is a scalar?

Scalar quantities are measured with numbers and units.

length

(e.g. 16 cm)

temperature

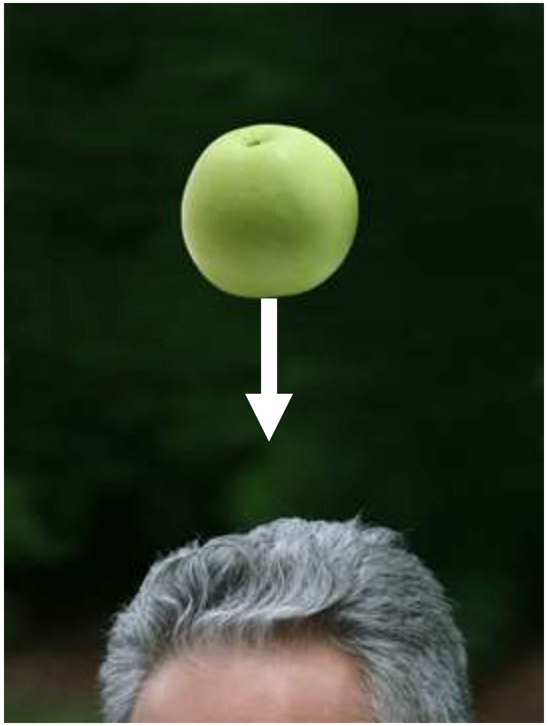
(e.g. 102 °C)

time

(e.g. 7 s)

What is a vector?

Vector quantities are measured with numbers and units, but also have a specific **direction**.



acceleration

(e.g. 30 m/s^2 upwards)

displacement

(e.g. 200 miles northwest)

force

(e.g. 2 N downwards)

Vectors

Vectors Power Graphics:
The Secret Behind Stunning Visuals: In computer graphics, vectors are used to represent points, lines, and shapes in space. Every time you see a 3D animation or play a video game, vectors are at work, determining the position and movement of objects, the lighting, and even the camera angles.

Vector Magic in Physics:
Forces and Motion: Vectors are the language of physics when it comes to describing forces, velocity, and acceleration. They allow us to break down complex physical phenomena into understandable components, like figuring out how much force is pushing an object up versus sideways.

Navigating the World:
- **GPS and Navigation**: Vectors are crucial in navigation systems like GPS. They help calculate the shortest path between two points, determine direction, and guide us to our destinations with incredible accuracy, whether on foot, by car, or even in space.

Vectors in Nature:
- **The Wind and the Waves**: Natural phenomena like wind and ocean currents are described using vectors. Meteorologists use vector fields to predict weather patterns, helping us understand how storms will move and where they will go.

Vectors

Vectors in Music:

•**Sound Waves and Acoustics**: Vectors are also used in the study of sound waves and acoustics. They help in analyzing how sound waves travel through different mediums, bounce off surfaces, and interact with each other, which is essential for designing concert halls, speakers, and even noise-canceling headphones.

Vectors in Robotics:

•**Robot Movements**: In robotics, vectors are essential for programming the movements of robots. Vectors help determine the position, orientation, and path a robot must take to perform tasks, whether it's assembling a car or performing surgery.

Vectors in Data Science:

•**Multi-Dimensional Data**: In data science, vectors are used to represent multi-dimensional data points. Techniques like machine learning rely on vector operations to classify data, recognize patterns, and make predictions.

Vectors and AI:

•**Neural Networks and Learning**: Vectors play a crucial role in artificial intelligence, especially in neural networks where data is processed as vectors. They help in understanding and interpreting complex data structures, leading to advancements in AI and machine learning.

Vectors

Vectors in Space Exploration:

•**Rocket Science**: Vectors are vital in space exploration. They help calculate trajectories, ensuring that spacecraft reach their intended destinations. The precise calculations needed to land a rover on Mars, for example, involve vector math to account for gravitational forces, velocity, and direction

Vectors and Art:

•**Vector Graphics**: In digital art and design, vector graphics allow for the creation of images that can be scaled infinitely without losing quality. Unlike pixel-based images, vector graphics are defined by mathematical equations, making them perfect for logos, fonts, and intricate designs.

Vectors in Sports:

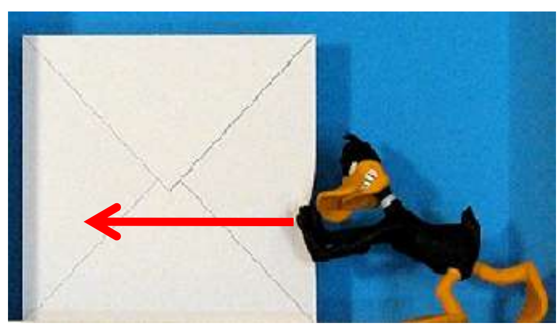
•**The Physics of Play**: In sports, vectors are used to analyze and optimize performance. Whether it's determining the perfect angle for a soccer shot, the trajectory of a basketball, or the force needed in a golf swing, vectors help athletes improve their game.

Vectors

A **vector** has magnitude as well as direction, and vectors follow certain (vector) rules of combination, which we examine in this chapter. A **vector quantity** is a quantity that has both a magnitude and a direction and thus can be represented with a vector.

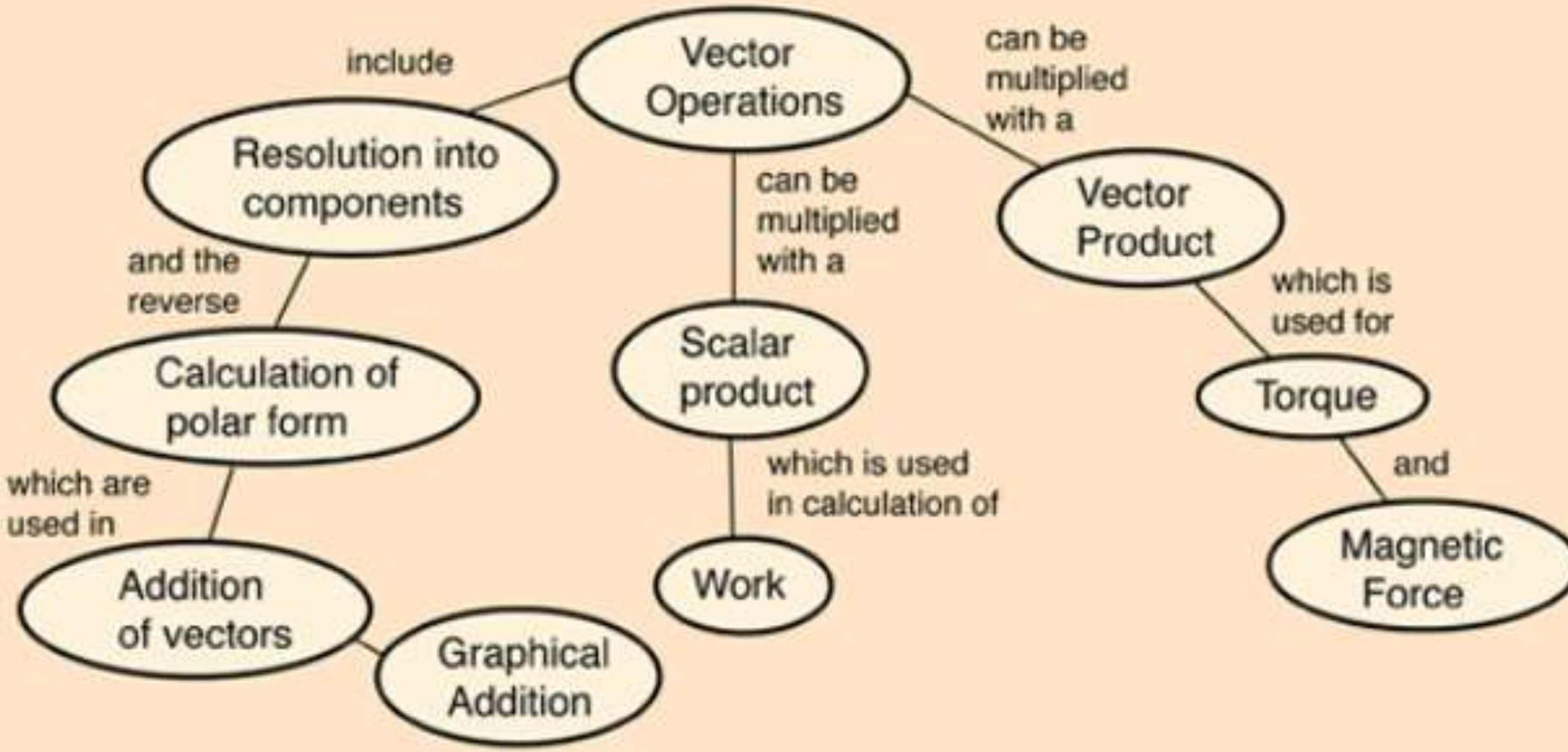
Examples of Vector Quantities:

- Displacement
- Velocity
- Acceleration
- Force



Basic Vector Operations

Both a magnitude and a direction must be specified for a vector quantity, in contrast to a scalar quantity which can be quantified with just a number. Any number of vector quantities of the same type (i.e., same units) can be combined by basic vector operations.



Vectors

The simplest vector quantity is displacement, or change of position. A vector that represents a displacement is called, reasonably, a **displacement vector.**

The displacement vector tells us nothing about the actual path that the particle takes. In Fig. 3-1*b*, for example, all three paths connecting points *A* and *B* correspond to the same displacement vector, that of Fig. 3-1*a*. Displacement vectors represent only the overall effect of the motion, not the motion itself.

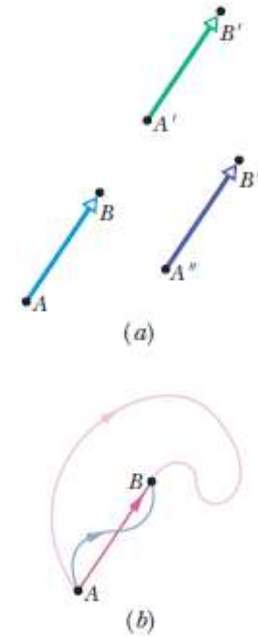
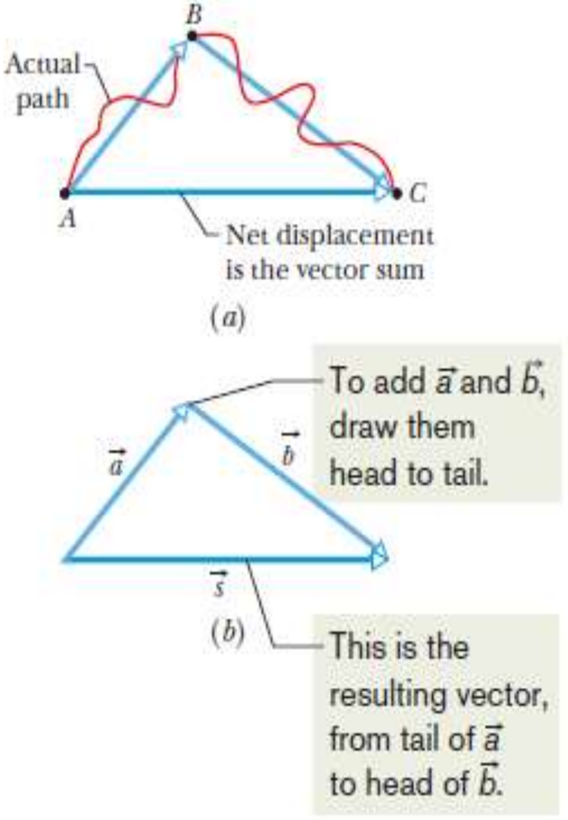


Figure 3-1 (*a*) All three arrows have the same magnitude and direction and thus represent the same displacement. (*b*) All three paths connecting the two points correspond to the same displacement vector.

Adding Vectors Geometrically

Suppose that, as in the vector diagram of Fig. 3-2*a*, a particle moves from *A* to *B* and then later from *B* to *C*.We can represent its overall displacement (no matter what its actual path) with two successive displacement vectors, *AB* and *BC*.
The *net* displacement of these two displacements is a single displacement from *A* to *C*.We call *AC* the **vector sum** (or **resultant**) of the vectors *AB* and *BC*. This sum is not the usual algebraic sum.

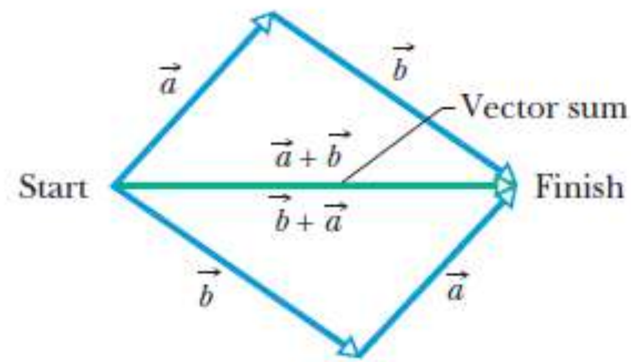


We can represent the relation among the three vectors in Fig. 3-2*b* with the *vector equation*

$$\vec{s} = \vec{a} + \vec{b},$$

Properties of Vector Addition

- Vector addition, defined in this way, has two important properties.
- 1. Commutative Law
- 2. Associative Law



$$\vec{a}+\vec{b}=\vec{b}+\vec{a} \quad \text{(commutative law).}$$

You get the same vector result for either order of adding vectors.

Figure 3-3 The two vectors $\vec{a}$ and $\vec{b}$ can be added in either order; see Eq. 3-2.

Properties of Vector Addition

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(associative law).}$$

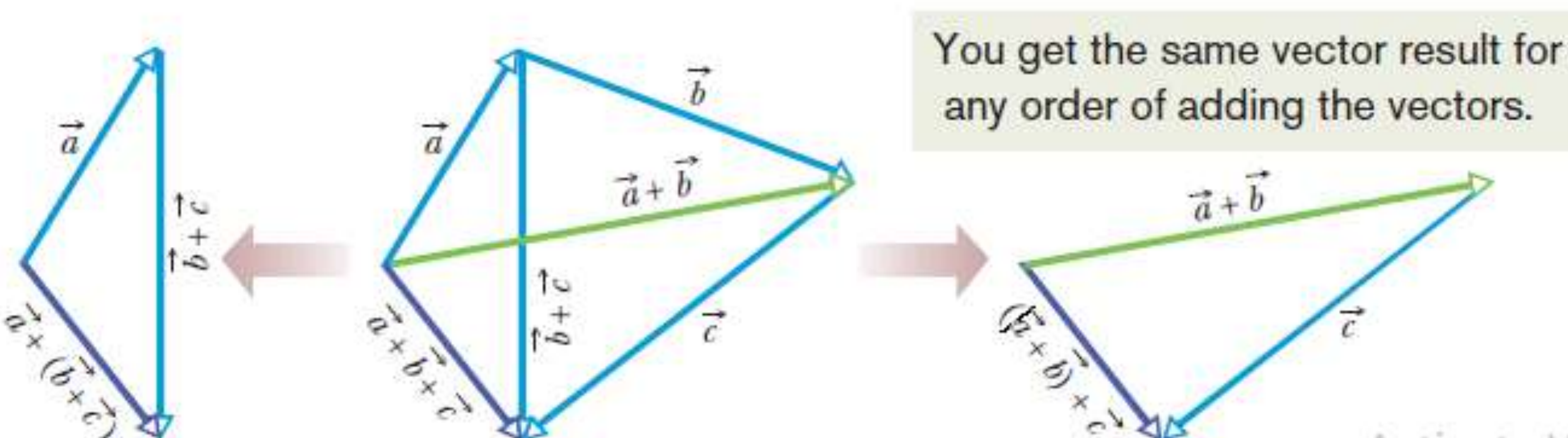


Figure 3-4 The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be grouped in any way as they are added; see Eq. 3-3.

Vector Subtraction

The vector $-\vec{b}$ is a vector with the same magnitude as $\vec{b}$ but the opposite direction (see Fig. 3-5). Adding the two vectors in Fig. 3-5 would yield

$$\vec{b} + (-\vec{b}) = 0.$$

Thus, adding $-\vec{b}$ has the effect of subtracting $\vec{b}$. We use this property to define the difference between two vectors: let $\vec{d} = \vec{a} - \vec{b}$. Then

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) \qquad \text{(vector subtraction);}$$

that is, we find the difference vector $\vec{d}$ by adding the vector $-\vec{b}$ to the vector $\vec{a}$. Figure 3-6 shows how this is done geometrically.

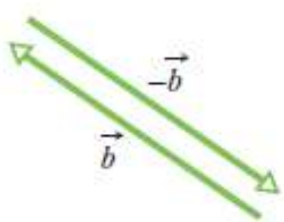


Figure 3-5 The vectors $\vec{b}$ and $-\vec{b}$ have the same magnitude and opposite directions.

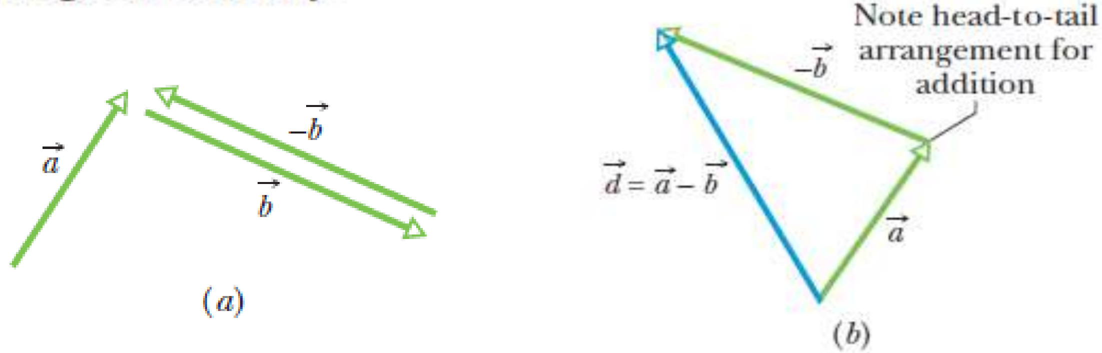
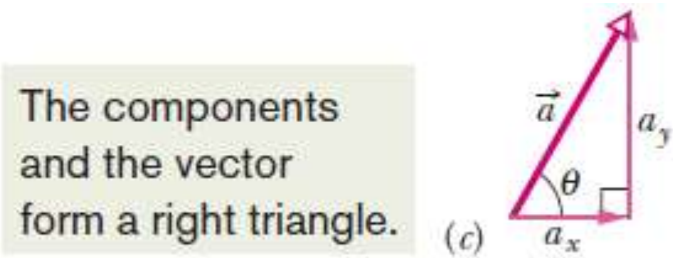


Figure 3-6 (a) Vectors $\vec{a}$, $\vec{b}$, and $-\vec{b}$. (b) To subtract vector $\vec{b}$ from vector $\vec{a}$, add vector $-\vec{b}$ to vector $\vec{a}$.

Components of Vectors

A **component** of a vector is the projection of the vector on an axis. In Fig. 3-7a, for example, a_x is the component of vector $\vec{a}$ on (or along) the x axis and a_y is the component along the y axis. To find the projection of a vector along an axis, we draw perpendicular lines from the two ends of the vector to the axis, as shown. The projection of a vector on an x axis is its x *component*, and similarly the projection on the y axis is the y *component*. The process of finding the components of a vector is called **resolving the vector.**

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta,$$



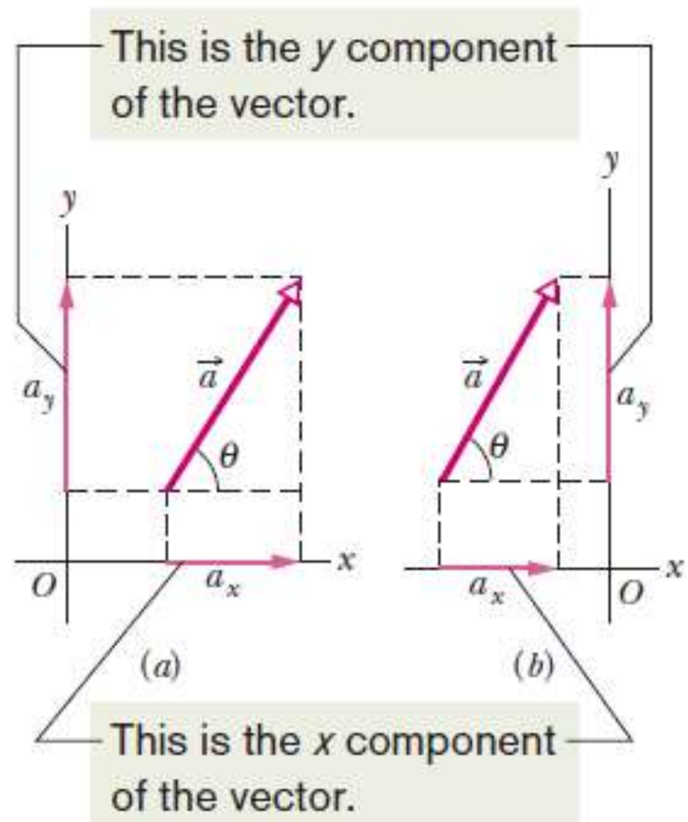


Figure 3-7 (a) The components a_x and a_y of vector $\vec{a}$. (b) The components are unchanged if the vector is shifted, as long as the magnitude and orientation are maintained. (c) The components form the legs of a right triangle whose hypotenuse is the magnitude of the vector.

Components of Vectors

Once a vector has been resolved into its components along a set of axes, the components themselves can be used in place of the vector. For example, $\vec{a}$ in Fig. 3-7a is given (completely determined) by a and θ. It can also be given by its components a_x and a_y. Both pairs of values contain the same information. If we know a vector in *component notation* (a_x and a_y) and want it in *magnitude-angle notation* (a and θ), we can use the equations

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}$$

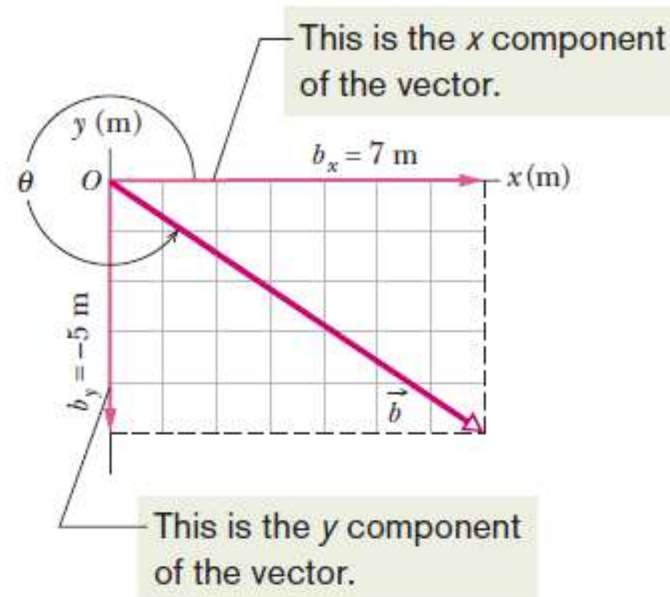
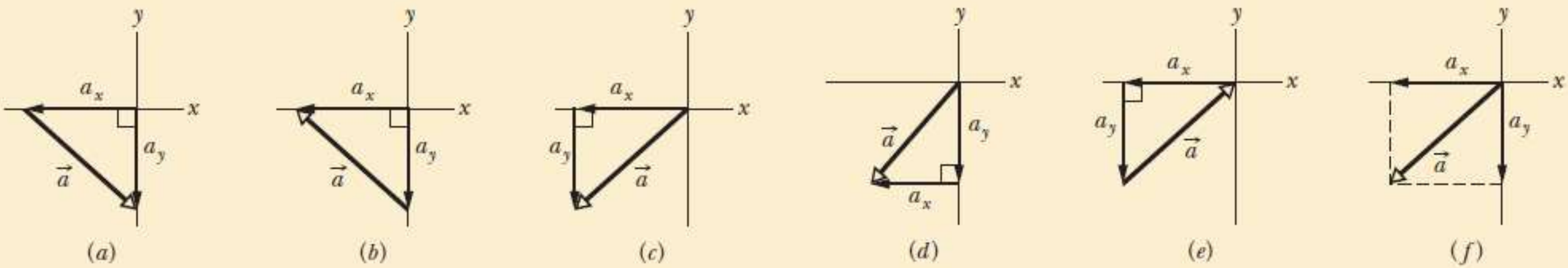


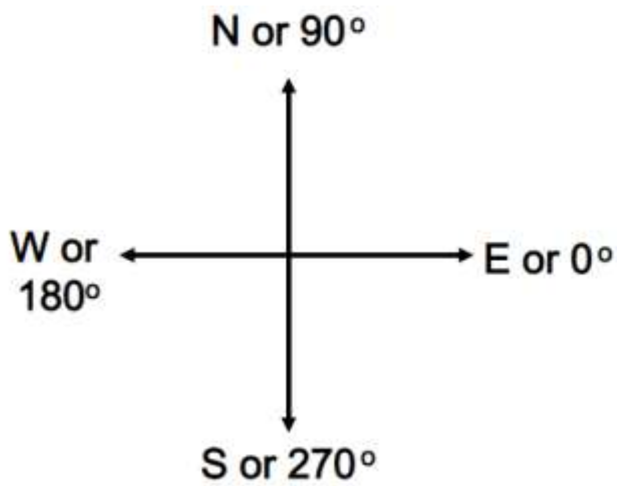
Figure 3-8 The component of $\vec{b}$ on the x axis is positive, and that on the y axis is negative.

Check points

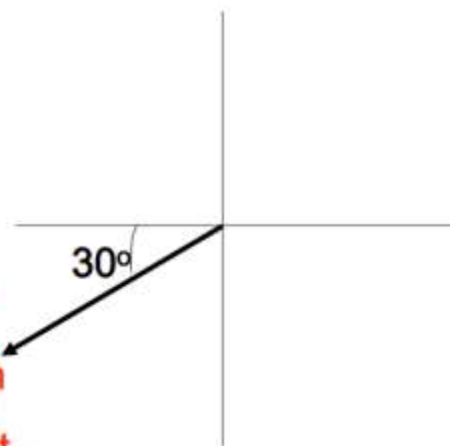
The magnitudes of displacements $\vec{a}$ and $\vec{b}$ are 3 m and 4 m, respectively, and $\vec{c} = \vec{a} + \vec{b}$. Considering various orientations of $\vec{a}$ and $\vec{b}$, what are (a) the maximum possible magnitude for $\vec{c}$ and (b) the minimum possible magnitude?

In the figure, which of the indicated methods for combining the x and y components of vector $\vec{a}$ are proper to determine that vector?





Describe the direction of the following vector in three ways?

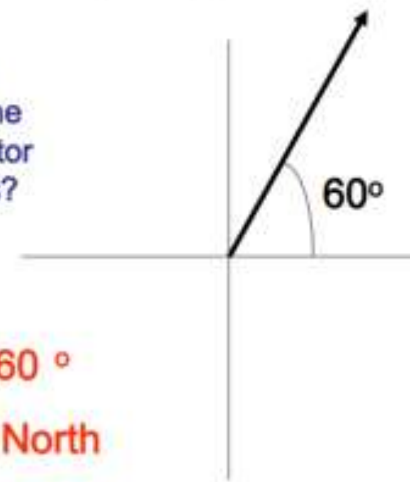


Answer: at 210 °

or West 30 ° South

or South 60 ° West

- Describe the direction of the following vector in three ways?



Answer: at 60 °

or East 60 ° North

or North 30 ° East

A small airplane leaves an airport on an overcast day and is later sighted 215 km away, in a direction making an angle of 22° east of due north. How far east and north is the airplane from the airport when sighted?

- **Solutions:**

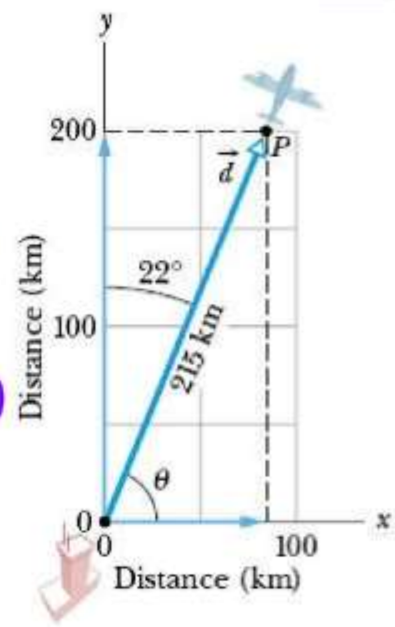
$$\theta = 90^\circ - 22^\circ = 68^\circ$$

$$d_x = d\cos\theta = (215)\cos 68^\circ$$

$$= (215)\cos 68^\circ = 81 \text{ (km)}$$

$$d_y = d\sin\theta = (215)\sin 68^\circ$$

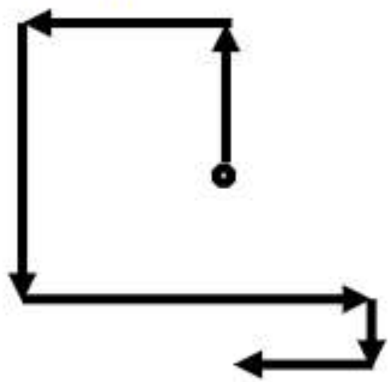
$$= 199 \text{ (km)}$$

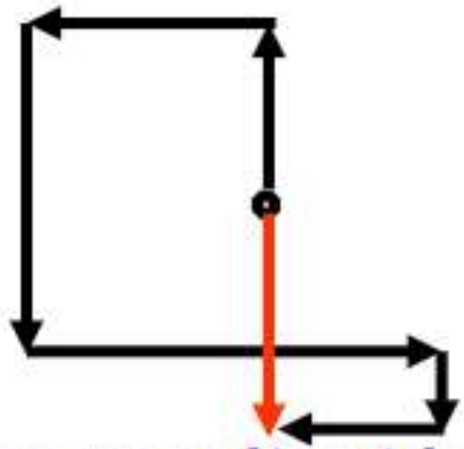


Addition of Vectors-Sample Problem

- A hiker walks 2 km to the North, 3 km to the West, 4 km to the South, 5 km to the East, 1 more km to the South, and finally 2 km to the West. How far did he end up from where he started? Hint: What is his resultant?

Shown is his path, notice all of the vectors are head to tail



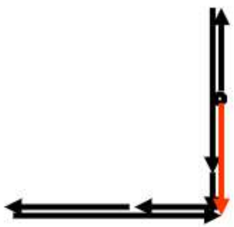


The resultant is in Red.

3 km, South

Addition of Vectors-Sample Problem

- This diagram shows the same vectors being added but in a different order, notice that the resultant is still the same.



Unit Vector

A **unit vector** is a vector that has a magnitude of exactly 1 and points in a particular direction. It lacks both dimension and unit. Its sole purpose is to point—that is, to specify a direction.

The unit vectors in the positive directions of the *x*, *y*, and *z* axes are labeled , , and , where the hat is used instead of an overhead arrow as for other vectors (Fig. 3-13).The arrangement of axes in Fig. 3-13 is said to be a **right-handed coordinate system.**

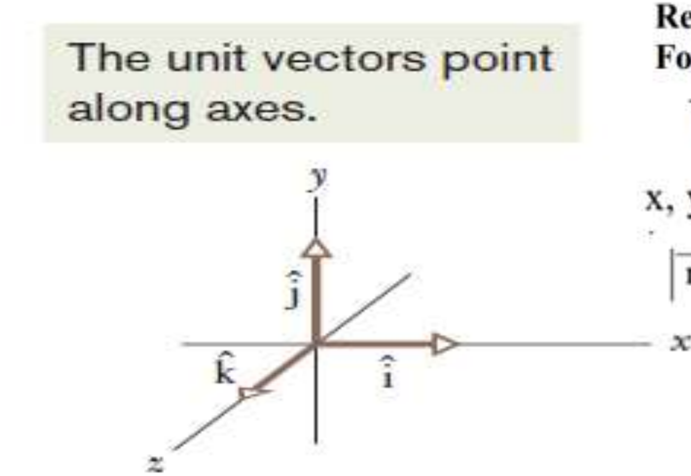


Figure 3.13 Unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ define the directions of a right-handed coordinate system.

Representation of a Vector in the Form of Unit Vectors i, j and k.

$$\overline{r} = xi + yj + zk$$

x, y and z are the magnitude

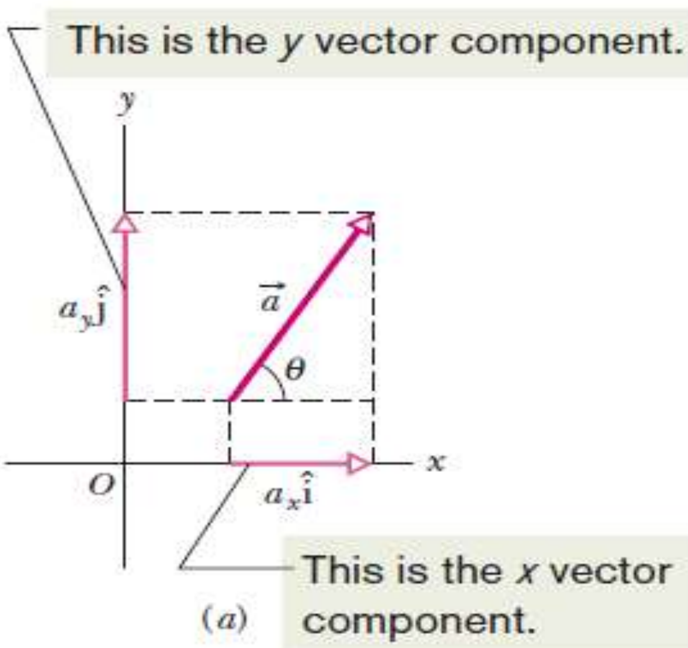
$$|\overline{r}| = \sqrt{x^2 + y^2 + z^2}$$

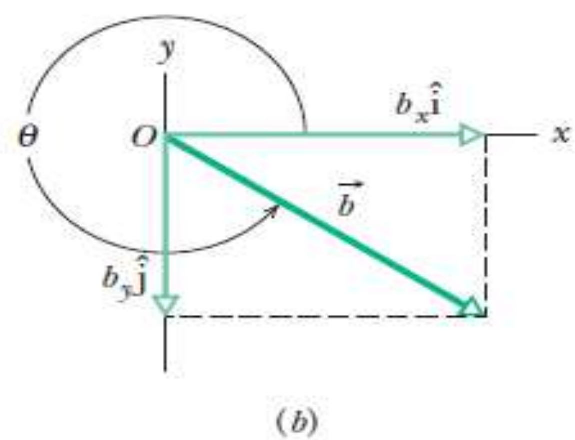
the unit vector $\hat{r} = \dfrac{\overline{r}}{|r|} = \dfrac{xi + yj + zk}{|r|}$

Unit Vector

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}}$$
$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}.$$

The quantities a_x and a_y are vectors, called the **vector components** of .The quantities a_x and a_y are scalars, called the **scalar components** of a (or, as before, simply its **components**).





ADDING VECTORS BY COMPONENTS

We can add vectors geometrically on a sketch or directly on a vector-capable calculator. A third way is to combine their components axis by axis.

To start, consider the statement

$$\vec{r} = \vec{a} + \vec{b}, \tag{3-9}$$

which says that the vector $\vec{r}$ is the same as the vector $(\vec{a} + \vec{b})$. Thus, each component of $\vec{r}$ must be the same as the corresponding component of $(\vec{a} + \vec{b})$:

$$r_x = a_x + b_x \tag{3-10}$$

$$r_y = a_y + b_y \tag{3-11}$$

$$r_z = a_z + b_z. \tag{3-12}$$

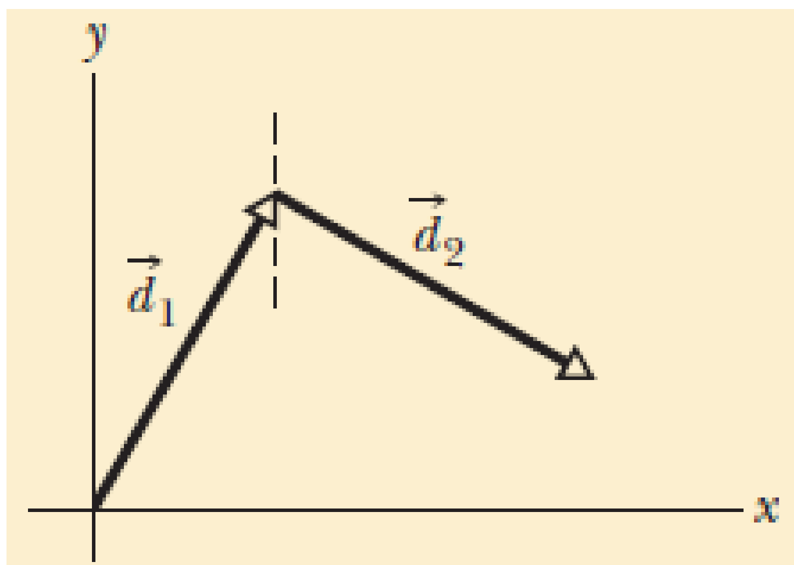
To subtract, we add (a) and (–b) by components, to

$$d_x = a_x - b_x, \quad d_y = a_y - b_y, \quad \text{and} \quad d_z = a_z - b_z,$$

$$\vec{d} = d_x\hat{\mathrm{i}} + d_y\hat{\mathrm{j}} + d_z\hat{\mathrm{k}}.$$

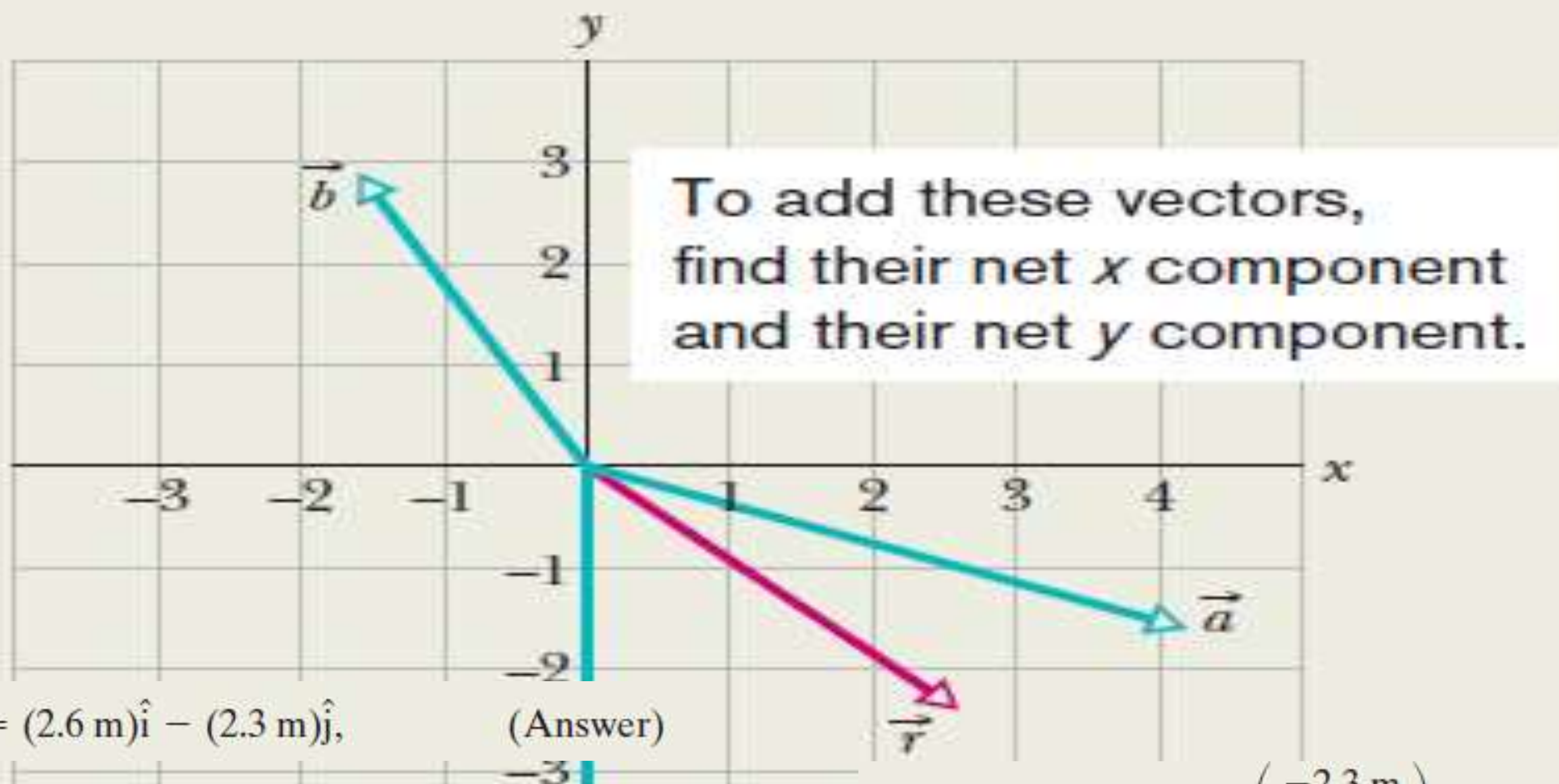
Check points

(a) In the figure here, what are the signs of the x components of $\vec{d}_1$ and $\vec{d}_2$? (b) What are the signs of the y components of $\vec{d}_1$ and $\vec{d}_2$? (c) What are the signs of the x and y components of $\vec{d}_1 + \vec{d}_2$?



$$\vec{a} = (4.2\text{ m})\hat{i} - (1.5\text{ m})\hat{j},$$
$$\vec{b} = (-1.6\text{ m})\hat{i} + (2.9\text{ m})\hat{j},$$
and
$$\vec{c} = (-3.7\text{ m})\hat{j}.$$

What is their vector sum $\vec{r}$ which is also shown?



$$\vec{r} = (2.6\text{ m})\hat{i} - (2.3\text{ m})\hat{j}, \qquad \text{(Answer)}$$

$$r = \sqrt{(2.6\text{ m})^2 + (-2.3\text{ m})^2} \approx 3.5\text{ m} \qquad \text{(Answer)}$$

$$\theta = \tan^{-1}\left(\frac{-2.3\text{ m}}{2.6\text{ m}}\right) = -41°,$$

where the minus sign means clockwise.

MULTIPLYING VECTORS

Scalar Product of two Vectors:

If $\overrightarrow{a}$ and $\overrightarrow{b}$ are non-zero vectors, and θ is the angle between them, then the scalar product of $\overrightarrow{a}$ and $\overrightarrow{b}$ is denoted by $\overrightarrow{a} \cdot \overrightarrow{b}$ and read as $\overrightarrow{a}$ dot $\overrightarrow{b}$. It is defined by the relation

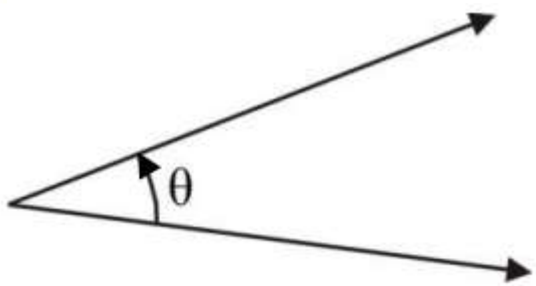


Fig.14

$$\overrightarrow{a} \cdot \overrightarrow{b} = |a|\ |b| \operatorname{Cos} \theta \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

If either $\overrightarrow{a}$ or $\overrightarrow{b}$ is the zero vector, then $\overrightarrow{a} \cdot \overrightarrow{b} = 0$

Remarks:

i. The scalar product of two vectors is also called the dot product because the "." used to indicate this kind of multiplication. Sometimes it is also called the inner product.

ii. The scalar product of two non-zero vectors is zero if and only if they are at right angles to each other. For $\overrightarrow{a} \cdot \overrightarrow{b} = 0$ implies that $\operatorname{Cos} \theta = 0$, which is the condition of perpendicularity of two vectors.

MULTIPLYING VECTORS

There are three ways in which vectors can be multiplied, but none is exactly like the usual algebraic multiplication.

Multiplying a Vector by a Scalar

If we multiply a vector $\vec{a}$ by a scalar s, we get a new vector. Its magnitude is the product of the magnitude of $\vec{a}$ and the absolute value of s. Its direction is the direction of $\vec{a}$ if s is positive but the opposite direction if s is negative. To divide $\vec{a}$ by s, we multiply $\vec{a}$ by $1/s$.

Multiplying a Vector by a Vector

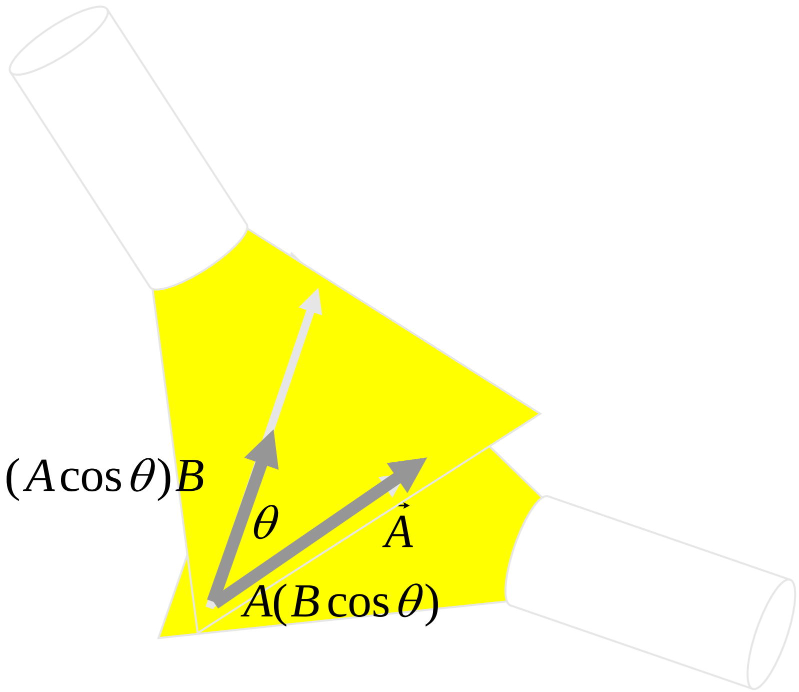
There are two ways to multiply a vector by a vector: one way produces a scalar (called the *scalar product*), and the other produces a new vector (called the *vector product*). (Students commonly confuse the two ways.)

Dot Product

- The dot product says something about how parallel two vectors are.
- The dot product (scalar product) of two vectors can be thought of as the projection of one onto the direction of the other.
- Components

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{A}\cdot\hat{i} = A\cos\theta = A_x$$



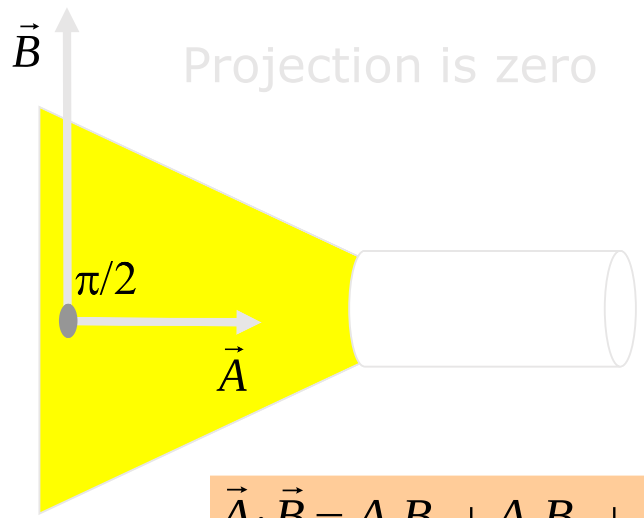
$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Projection of a Vector: Dot Product

- The dot product says something about how parallel two vectors are.
- The dot product (scalar product) of two vectors can be thought of as the projection of one onto the direction of the other.
- Components

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{A}\cdot\hat{i} = A\cos\theta = A_x$$



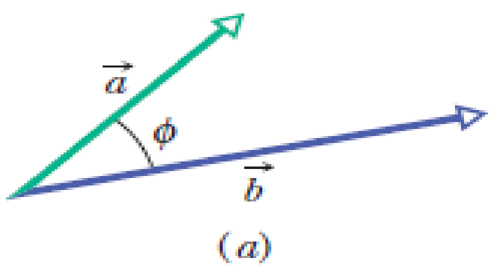
$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

The Scalar Product

The **scalar product** of the vectors $\vec{a}$ and $\vec{b}$ in Fig. 3-18*a* is written as $\vec{a} \cdot \vec{b}$ and defined to be

$$\vec{a} \cdot \vec{b} = ab \cos \phi, \tag{3-20}$$

where a is the magnitude of $\vec{a}$, b is the magnitude of $\vec{b}$, and ϕ is the angle between $\vec{a}$ and $\vec{b}$ (or, more properly, between the directions of $\vec{a}$ and $\vec{b}$). There are actually two such angles: ϕ and $360° - \phi$. Either can be used in Eq. 3-20, because their cosines are the same.



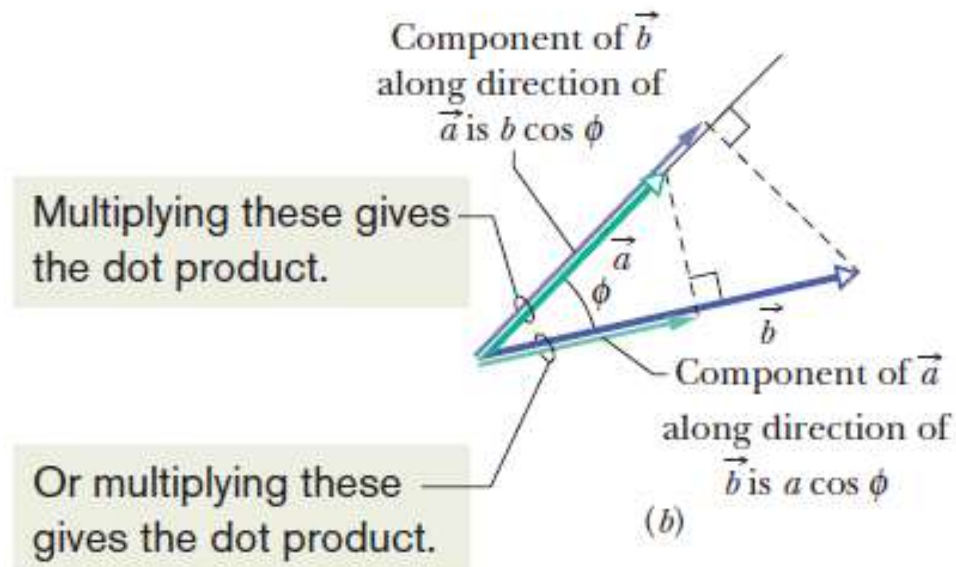


Figure 3-18 (*a*) Two vectors $\vec{a}$ and $\vec{b}$, with an angle ϕ between them. (*b*) Each vector has a component along the direction of the other vector.

The Scalar Product

Note that there are only scalars on the right side of Eq. 3-20 (including the value of $\cos\phi$). Thus $\vec{a}\cdot\vec{b}$ on the left side represents a *scalar* quantity. Because of the notation, $\vec{a}\cdot\vec{b}$ is also known as the **dot product** and is spoken as "a dot b."

A dot product can be regarded as the product of two quantities: (1) the magnitude of one of the vectors and (2) the scalar component of the second vector along the direction of the first vector. For example, in Fig. 3-18*b*, $\vec{a}$ has a scalar component $a\cos\phi$ along the direction of $\vec{b}$; note that a perpendicular dropped from the head of $\vec{a}$ onto $\vec{b}$ determines that component. Similarly, $\vec{b}$ has a scalar component $b\cos\phi$ along the direction of $\vec{a}$.

If the angle ϕ between two vectors is 0°, the component of one vector along the other is maximum, and so also is the dot product of the vectors. If, instead, ϕ is 90°, the component of one vector along the other is zero, and so is the dot product.

When two vectors are in unit-vector notation, we write their dot product as

$$\vec{a}\cdot\vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}})\cdot(b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

$$\vec{a}\cdot\vec{b} = a_xb_x + a_yb_y + a_zb_z.$$

The Vector Product

The **vector product** of $\vec{a}$ and $\vec{b}$, written $\vec{a} \times \vec{b}$, produces a third vector $\vec{c}$ whose magnitude is

$$c = ab \sin \phi, \qquad (3\text{-}24)$$

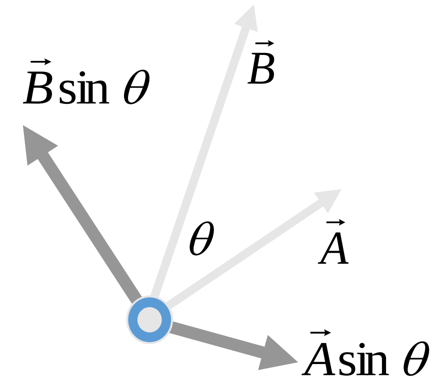
where ϕ is the *smaller* of the two angles between $\vec{a}$ and $\vec{b}$. (You must use the smaller of the two angles between the vectors because $\sin \phi$ and $\sin(360^\circ - \phi)$ differ in algebraic sign.) Because of the notation, $\vec{a} \times \vec{b}$ is also known as the **cross product,** and in speech it is "a cross b."

The direction of $\vec{c}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$. Figure 3-19*a* shows how to determine the direction of $\vec{c} = \vec{a} \times \vec{b}$ with what is known as a **right-hand rule.** Place the vectors $\vec{a}$ and $\vec{b}$ tail to tail without altering their orientations, and imagine a line that is perpendicular to their plane where they meet. Pretend to place your *right* hand around that line in such a way that your fingers would sweep $\vec{a}$ into $\vec{b}$ through the smaller angle between them. Your outstretched thumb points in the direction of $\vec{c}$.

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a} \times \vec{b} = 0$. The magnitude of $\vec{a} \times \vec{b}$, which can be written as $|\vec{a} \times \vec{b}|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

Vector Product

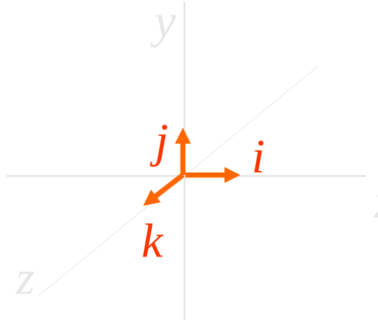
$$\vec{C} = \vec{A} \times \vec{B}$$



- The cross product of two vectors says something about how perpendicular they are.
- Magnitude:

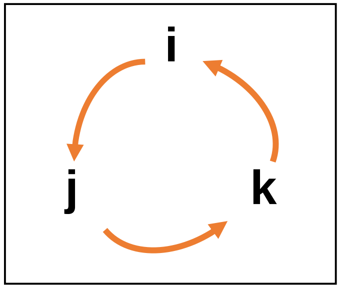
$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = AB\sin\theta$$

  - θ is smaller angle between the vectors
  - Cross product of any parallel vectors = zero
  - Cross product is maximum for perpendicular vectors
  - Cross products of Cartesian unit vectors:



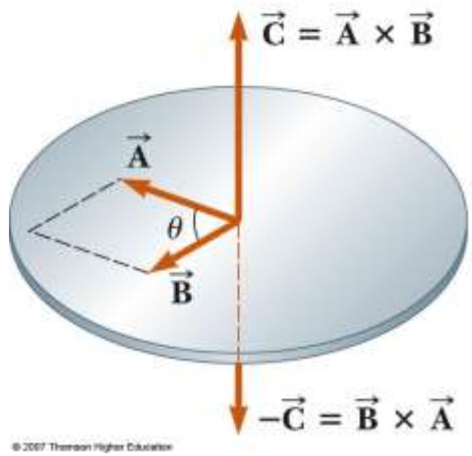
$$\hat{i} \times \hat{j} = \hat{k};\ \hat{i} \times \hat{k} = -\hat{j};\ \hat{j} \times \hat{k} = \hat{i}$$

$$\hat{i} \times \hat{i} = 0;\ \hat{j} \times \hat{j} = 0;\ \hat{k} \times \hat{k} = 0$$

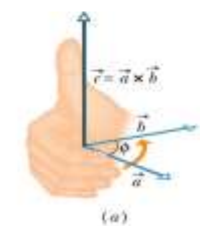


Vector Product

- Direction: C perpendicular to both A and B (right-hand rule)
  - Place A and B tail to tail
  - Right hand, not left hand
  - Four fingers are pointed along the first vector A
  - "sweep" from first vector A into second vector B through the smaller angle between them
  - Your outstretched thumb points the direction



Right-hand rule



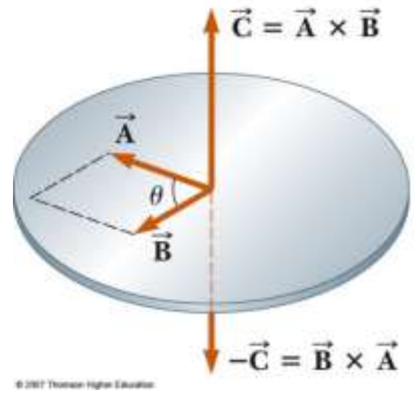
$$\vec{A} \times \vec{B} = \vec{B} \times \vec{A}?$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} \times \vec{B} = \vec{B} \times \vec{A}?$$

Vector Product

- The quantity ABsinθ is the area of the parallelogram formed by A and B
- The direction of C is perpendicular to the plane formed by A and B
- Cross product is not commutative



Right-hand rule

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

- The distributive law

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

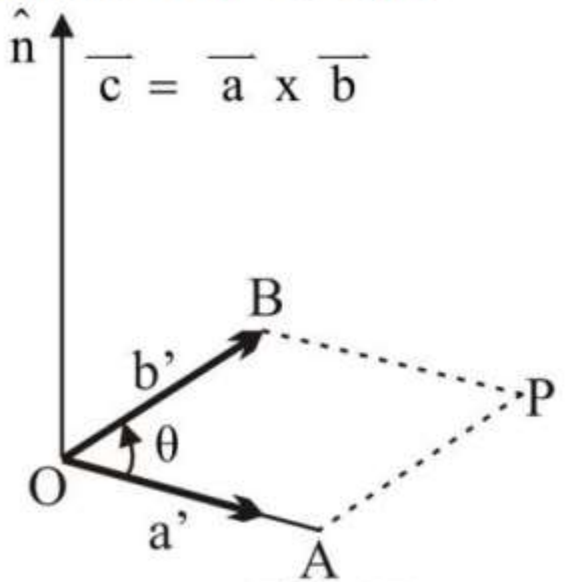
- The derivative of cross product obeys the chain rule

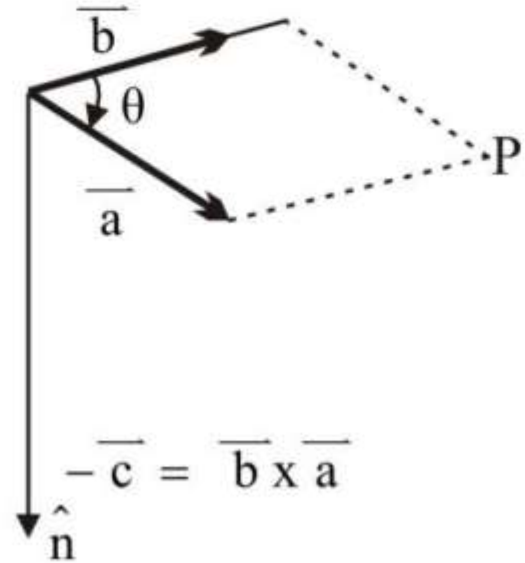
$$\frac{d}{dt}\left(\vec{A}\times\vec{B}\right) = \frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

- Calculate cross product

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

The vector product is also called the 'cross product' or 'Outer product' of the vectors.





Remarks:

If we consider $\vec{b} \times \vec{a}$, then $\vec{b} \times \vec{a}$ would be a vector which is opposite in the direction to $\vec{a} \times \vec{b}$.

Hence $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Which gives that $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ in general

Hence the vector product is not commutative.

$$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}),$$

The vector product of two non-zero vectors is zero if $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel, the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is zero. Sin $0^{o} = 0$, Hence $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} = 0$.

For a x b = 0 implies that $\sin\theta = 0$ which is the condition of parallelism of two vectors. In particular $\overrightarrow{a} \times \overrightarrow{a} = 0$. Hence for the unit vectors i, j and k,

$$i \times i = j \times j = k \times k = 0$$

If $\overrightarrow{a}$ and $\overrightarrow{b}$ are perpendicular vectors, then $\overrightarrow{a} \times \overrightarrow{b}$ is a vector whose magnitude is $|\overrightarrow{a}||\overrightarrow{b}|$ and whose direction is such that the vectors a, b, a x b form a right-handed system of three mutually perpendicular vectors. In particular i x j = (1) (1) Sin 90^{o} k (k being perpendicular to i and j) = k

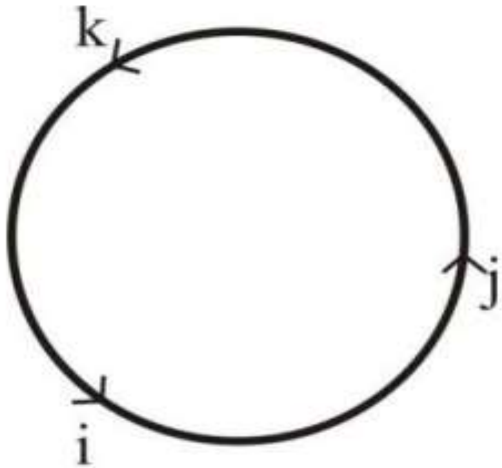


Fig. 19

Similarly j x i = −k, i x k = −j, k x j = −i

Hence the cross product of two consecutive unit vectors is the third unit vector with the plus or minus sign according as the order of the product is anti-clockwise or clockwise respectively.

Commutative property

The commutative law applies to a scalar product, so we can write

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$

the commutative law does not apply to a vector product.

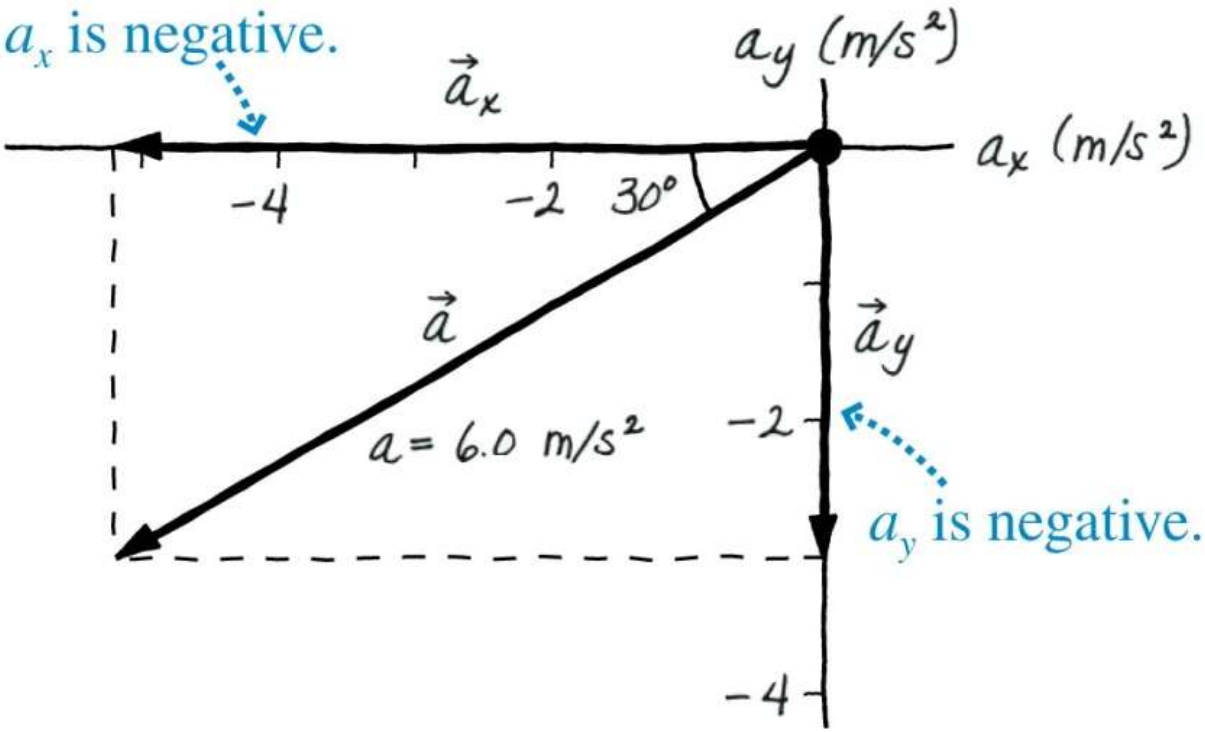
$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$

Check points

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if $\vec{C} \cdot \vec{D}$ equals (a) zero, (b) 12 units, and (c) -12 units?

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if the magnitude of the vector product $\vec{C} \times \vec{D}$ is (a) zero and (b) 12 units?

Example: Finding the Components of an Acceleration Vector



Example: Finding the Components of an Acceleration Vector

EXAMPLE 3.3 **Finding the components of an acceleration vector**

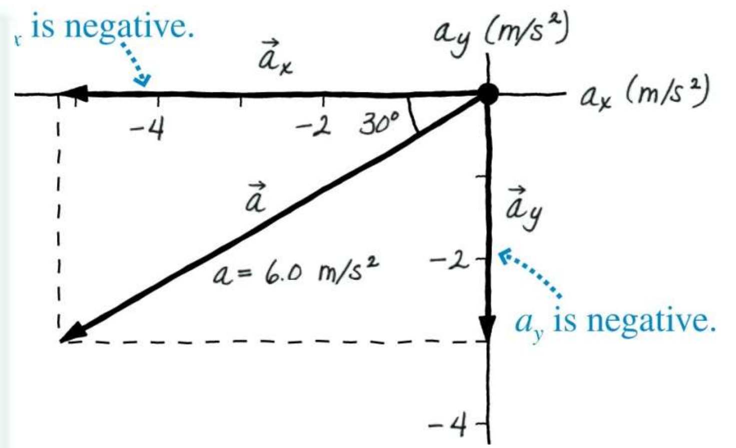
VISUALIZE It's important to *draw* vectors. The figure on the right shows the original vector $\vec{a}$ decomposed into components parallel to the axes. Notice that the axes are "acceleration axes," not *xy*-axes, because we're measuring an acceleration vector.

SOLVE The acceleration vector $\vec{a} = (6.0\ \text{m/s}^2$, 30° below the negative *x*-axis) points to the left (negative *x*-direction) and down (negative *y*-direction), so the components a_x and a_y are both negative:

$$a_x = -a\cos 30° = -(6.0\ \text{m/s}^2)\cos 30° = -5.2\ \text{m/s}^2$$

$$a_y = -a\sin 30° = -(6.0\ \text{m/s}^2)\sin 30° = -3.0\ \text{m/s}^2$$

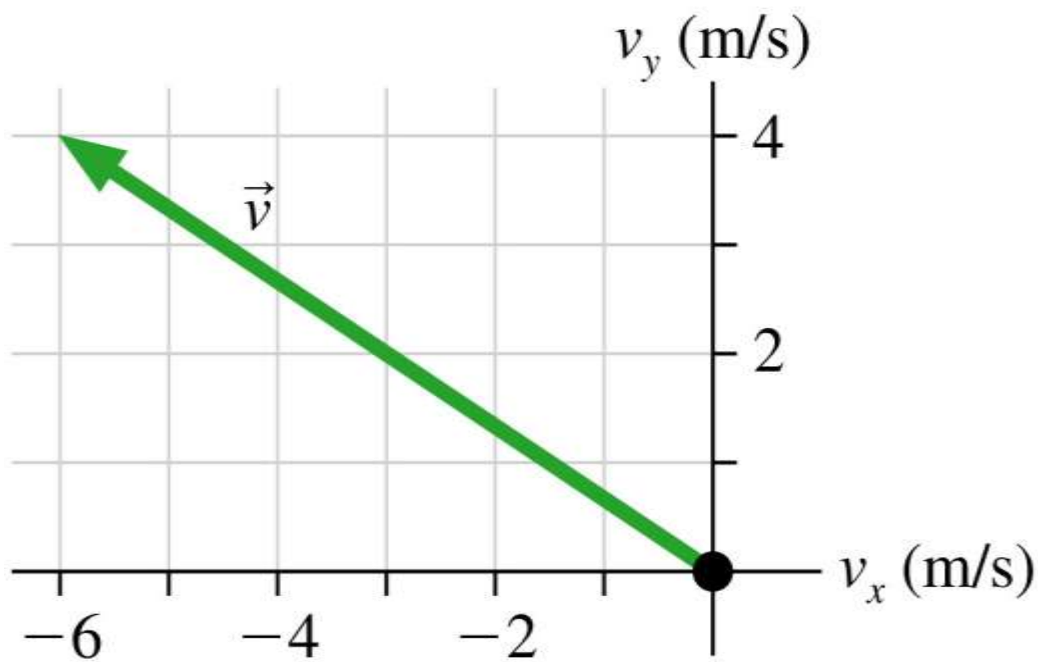
ASSESS The units of a_x and a_y are the same as the units of vector $\vec{a}$. Notice that we had to insert the minus signs manually by observing that the vector points left and down.



Example Finding the Direction of Motion

EXAMPLE 3.4 **Finding the direction of motion**

The figure below shows a car's velocity vector $\vec{v}$. Determine the car's speed and direction of motion.



Example Finding the Direction of Motion

EXAMPLE 3.4 **Finding the direction of motion**

VISUALIZE The figure on the right shows the components v_x and v_y and defines an angle θ with which we can specify the direction of motion.

SOLVE We can read the components of $\vec{v}$ directly from the axes: $v_x = -6.0$ m/s and $v_y = 4.0$ m/s. Notice that v_x is negative. This is enough information to find the car's speed v, which is the magnitude of $\vec{v}$:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-6.0\text{ m/s})^2 + (4.0\text{ m/s})^2} = 7.2\text{ m/s}$$

From trigonometry, angle θ is

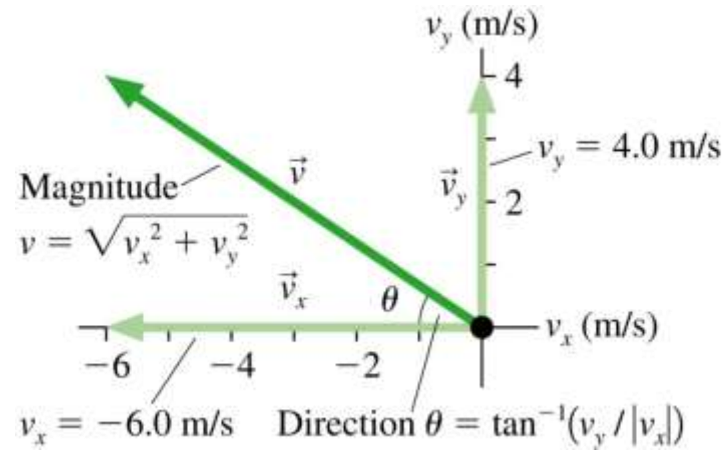
$$\theta = \tan^{-1}\left(\frac{v_y}{|v_x|}\right) = \tan^{-1}\left(\frac{4.0\text{ m/s}}{6.0\text{ m/s}}\right) = 34°$$

The absolute value signs are necessary because v_x is a negative number. The velocity vector $\vec{v}$ can be written in terms of the speed and the direction of motion as

$$\vec{v} = (7.2\text{ m/s}, 34° \text{ above the negative } x\text{-axis})$$

or, if the axes are aligned to north,

$$\vec{v} = (7.2\text{ m/s}, 34° \text{ north of west})$$

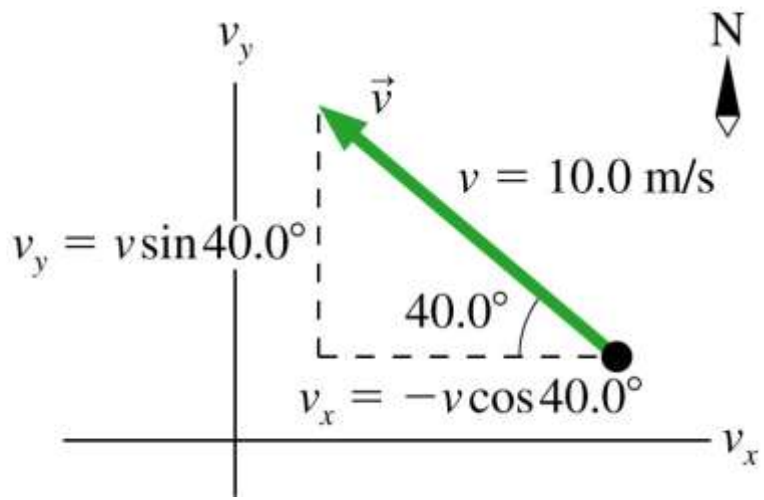


EXAMPLE 3.5 **Run rabbit run!**

A rabbit, escaping a fox, runs 40.0° north of west at 10.0 m/s. A coordinate system is established with the positive x-axis to the east and the positive y-axis to the north. Write the rabbit's velocity in terms of components and unit vectors.

EXAMPLE 3.5 **Run rabbit run!**

VISUALIZE The figure on the right shows the rabbit's velocity vector and the coordinate axes. We're showing a velocity vector, so the axes are labeled v_x and v_y rather than x and y.



SOLVE 10.0 m/s is the rabbit's *speed,* not its velocity. The velocity, which includes directional information, is

$$\vec{v} = (10.0 \text{ m/s}, 40.0° \text{ north of west})$$

Vector $\vec{v}$ points to the left and up, so the components v_x and v_y are negative and positive, respectively. The components are

$$v_x = -(10.0 \text{ m/s})\cos 40.0° = -7.66 \text{ m/s}$$

$$v_y = +(10.0 \text{ m/s})\sin 40.0° = 6.43 \text{ m/s}$$

With v_x and v_y now known, the rabbit's velocity vector is

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} = (-7.66\hat{\imath} + 6.43\hat{\jmath}) \text{ m/s}$$

Notice that we've pulled the units to the end, rather than writing them with each component.

ASSESS Notice that the minus sign for v_x was inserted manually. **Signs don't occur automatically; you have to set them after checking the vector's direction.**

Examples

What is the angle ϕ between $\vec{a} = 3.0\hat{\mathrm{i}} - 4.0\hat{\mathrm{j}}$ and $\vec{b} =$ $-2.0\hat{\mathrm{i}} + 3.0\hat{\mathrm{k}}$

$$109° \approx 110°.$$

If $\vec{a} = 3\hat{\mathrm{i}} - 4\hat{\mathrm{j}}$ and $\vec{b} = -2\hat{\mathrm{i}} + 3\hat{\mathrm{k}}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

$$-12\hat{\mathrm{i}} - 9\hat{\mathrm{j}} - 8\hat{\mathrm{k}}.$$

If $\overrightarrow{a} = 3i + 4j - k$, $\overrightarrow{b} = -2i + 3j + k$ find $\overrightarrow{a} \cdot \overrightarrow{b}$

$= 5$

Find the angle between the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, where $\overrightarrow{a} = i + 2j - k$ and $\overrightarrow{b} = -i + j - 2k$.

$$\theta = \operatorname{Cos}^{-1} \frac{1}{2} = 60^\circ$$

If $\overrightarrow{a} = 2i + 3j + 4k$ $\overrightarrow{b} = I - j + k$, Find

(i) $\overrightarrow{a} \times \overrightarrow{b}$

(ii) Sine of the angle between these vectors.

(iii) Unit vector perpendicular to each vector.

$= 7\hat{i} + 2\hat{j} - 5\hat{k}$

$\operatorname{Sin} \theta = \sqrt{\frac{26}{29}}$

$$\frac{7i + 2j - 5k}{\sqrt{78}}$$

Calculate the area of the triangle determined by the two vectors : $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = -3\hat{i} + 7\hat{j}$.

Area of the triangle determined by the two vectors $= \frac{1}{2}|\vec{A} \times \vec{B}|$

$$= \frac{1}{2}|(3\hat{i} + 4\hat{j}) \times (-3\hat{i} + 7\hat{j})|$$

$$= \frac{1}{2}|21\hat{k} - 12(-\hat{k})|$$

$$= \frac{1}{2}|33\hat{k}|$$

= 16.5 square unit

Two sides of triangle expressed as $\vec{A} = 5\hat{i} - 4\hat{j} + 3\hat{k}$ and $\vec{B} = 3\hat{i} - 2\hat{j} - \hat{k}$. Calculate area of triangle.

Area of the triangle is $\quad \text{Area} = \frac{1}{2}|\vec{A} \times \vec{B}|$

So, $\vec{A} \times \vec{B} = (5\hat{i} - 4\hat{j} + 3\hat{k}) \times (3\hat{i} - 2\hat{j} - \hat{k}) = 10\hat{i} + 14\hat{j} + 2\hat{k}$

$\Rightarrow \quad |\vec{A} \times \vec{B}| = \sqrt{10^2 + 14^2 + 2^2} = 10\sqrt{3}\ \text{m}^2$

Area of triangle $\quad \text{Area} = \frac{1}{2} \times 10\sqrt{3} = 5\sqrt{3}\ \text{m}^2$

Find the area of the parallelogram determined by the vectors : i - 3j + k and i + j + k

Let $\vec{a} = \hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

Recall the area of the parallelogram whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $|\vec{a} \times \vec{b}|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, -3, 1)$ and $(b_1, b_2, b_3) = (1, 1, 1)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(-3)(1) - (1)(1)] - \hat{j}[(1)(1) - (1)(1)] + \hat{k}[(1)(1) - (1)(-3)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[-3 - 1] - \hat{j}[1 - 1] + \hat{k}[1 + 3]$$

$$\therefore \vec{a} \times \vec{b} = -4\hat{i} + 4\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + 0^2 + 4^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{16 + 16}$$

$$\therefore |\vec{a} \times \vec{b}| = 4\sqrt{2}$$

Thus, the area of the parallelogram is $4\sqrt{2}$ square units.