MA1521 CALCULUS FOR COMPUTING[1]

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[1]This notes is exclusively for students taking MA1521 in AY2024/25 Semester 1.

Contents

Chapter 0

Real Numbers and Functions

Read Thomas' Calculus, Chapter 1.

Remark. Solution to Exercises in Lecture Notes can be downloaded at

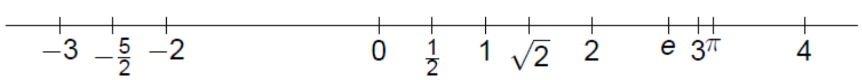
Canvas > Files > Solutions to Exercises in Lecture Notes (folder).

0.1 Numbers

The collection of all *real numbers* is denoted by $\mathbb{R}$. Thus $\mathbb{R}$ includes the integers

$$\ldots, -2, -1, 0, 1, 2, 3\ldots,$$

the *rational numbers*, p/q, where p and q are integers ($q \neq 0$), and the *irrational numbers*, like $\sqrt{2}, \pi, e$, etc.



The number line

$a \in \mathbb{R}$ means a is a member of the set $\mathbb{R}$. In other words, a is a real number. Given two real numbers a and b with $a < b$, the *closed interval* $[a, b]$ consists of all x such that $a \leq x \leq b$, and the *open interval* (a, b) consists of all x such that $a < x < b$. Similarly, we may form the half-open intervals $[a, b)$ and $(a, b]$.

0.2 Absolute Value

The *absolute value* of a number $a \in \mathbb{R}$ is written as $|a|$ and is defined as

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

For example, $|2| = 2, |-2| = 2$.

Some properties of $|x|$ are summarized as follows:

1. $|-x| = |x|$, for all $x \in \mathbb{R}$.
2. $|xy| = |x||y|$, for all $x, y \in \mathbb{R}$.
3. $-|x| \leq x \leq |x|$, for all $x \in \mathbb{R}$.
4. For a fixed $r > 0$, $|x| < r$ if and only if $x \in (-r, r)$.
5. $\sqrt{x^2} = |x|$, $x \in \mathbb{R}$.
6. (*Triangle Inequality*) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

Example 0.1. *Solve the inequality* $\dfrac{2x-1}{2x+1} < 1$.

Solution.

$$\begin{aligned}
&\frac{2x-1}{2x+1} < 1 \\
\Leftrightarrow\ & 0 < 1 - \frac{2x-1}{2x+1} \\
\Leftrightarrow\ & 0 < \frac{2x+1-2x+1}{2x+1} \\
\Leftrightarrow\ & 0 < \frac{2}{2x+1} \\
\Leftrightarrow\ & 0 < 2x+1 \\
\Leftrightarrow\ & -\frac{1}{2} < x.
\end{aligned}$$

■

Example 0.2. *Solve the inequality* $|x+1| \leq |2x-1|$.

Solution.

$$
\begin{aligned}
& |x+1| \leq |2x-1| \\
\Leftrightarrow\ & |x+1|^2 \leq |2x-1|^2 \\
\Leftrightarrow\ & x^2+2x+1 \leq 4x^2-4x+1 \\
\Leftrightarrow\ & 0 \leq 3x^2-6x \\
\Leftrightarrow\ & 0 \leq 3x(x-2) \\
\Leftrightarrow\ & x \leq 0 \text{ or } x \geq 2 \\
\Leftrightarrow\ & x \in (-\infty, 0] \cup [2, \infty).
\end{aligned}
$$

■

Exercise 0.1. *Let $r > 0$. Prove that $|x-a| < r$ if and only if $x \in (-r+a, a+r)$.*

Exercise 0.2. *Prove the triangle inequality $|x+y| \leq |x|+|y|$.*

Exercise 0.3. *Prove that for any $x, y \in \mathbb{R}$, $\big||x|-|y|\big| \leq |x-y|$.*

0.3 Functions

A *function* $f : A \longrightarrow B$ is a rule that assigns to each $a \in A$ one specific member $f(a)$ of B. Symbolically we may denote the function by $a \mapsto f(a)$. We can specify a function f by giving the rule for $f(x)$.

Example 0.3. $f(x) = x^2/(1-x)$ *assigns the number* $x^2/(1-x)$ *to each* $x \neq 1$ *in* $\mathbb{R}$.

The set A is called the *domain* of f and B is the *codomain* of f.

The *range* of f is the subset of B consisting of all the values of f. That is, the range of $f = \{f(x) \in B \mid x \in A\}$.

Given $f : A \longrightarrow \mathbb{R}$, it means that f assigns a value $f(x)$ in $\mathbb{R}$ to each $x \in A$.

Such a function is called a *real-valued function*.

For a real-valued function $f : A \longrightarrow \mathbb{R}$ defined on a subset A of $\mathbb{R}$, the *graph* of f consists of all the points $(x, f(x))$ in the xy-plane.

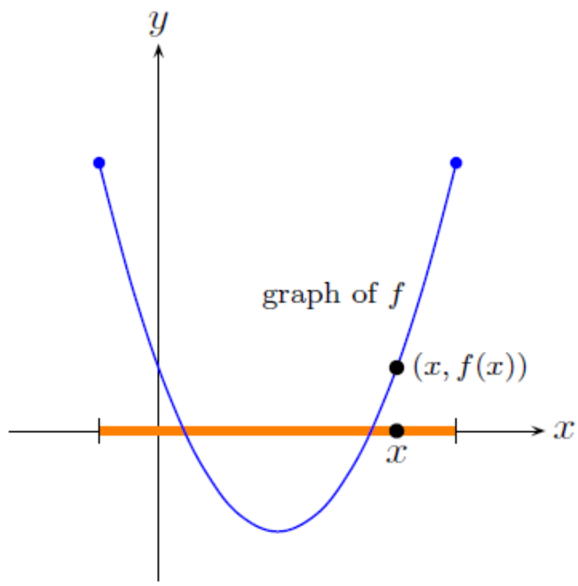
Ex 0.1 $|x-a|<r$

$-r<x-a<r$

$-r+a<x<r+a$

Ex 0.2 $(|x+y|)^2 = x^2+2xy+y^2 < |x|^2+2|x||y|+y^2 = (|x|+|y|)^2$

Ex 0.3 $||x|-|y||^2 = |x|^2-2|x||y|+|y|^2 \leq x^2-2xy+y^2 = |x-y|^2$



The graph of a function f

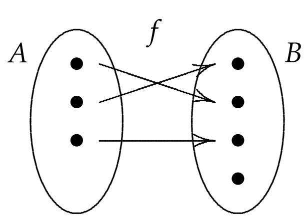
If $f : A \to B$ and $g : B \to C$, then the composite function of f and g is the function $g \circ f : A \to C$ given by $g \circ f(x) = g(f(x))$.

Example 0.4. *Let $f(x) = \frac{1}{x}$ and $g(x) = x^2 - 1$. Find $g \circ f$ and $f \circ g$.*

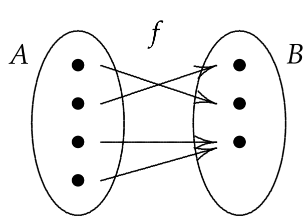
Solution. $g \circ f(x) = g(f(x)) = g(\frac{1}{x}) = (\frac{1}{x})^2 - 1 = \frac{1}{x^2} - 1$.
$f \circ g(x) = f(g(x)) = f(x^2 - 1) = \frac{1}{x^2-1}$. ■

Let $f : A \to B$. If $g : B \to A$ is a function such that $f(g(x)) = x$ for all $x \in B$ and $g(f(x)) = x$ for all $x \in A$, then g is called the inverse of f. Similarly, f is the inverse of g. The inverse function of f is usually denoted by f^{-1}.

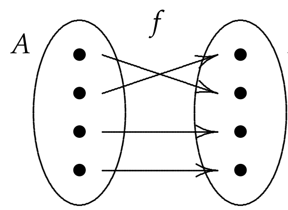
Let $f : A \to B$. f is called an *injective* function if for any $x, y \in A$, $f(x) = f(y) \Rightarrow x = y$. f is called a *surjective* function if for any $z \in B$, there is an $x \in A$ such that $f(x) = z$. f is called a *bijective* function if f is injective and surjective.



f is injective but not surjective



f is surjective but not injective



f is bijective

Exercise 0.4. *Prove that if f^{-1} exists, then f is a bijective function.*

0.4 Polynomials

A function of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, \ldots, a_n$ are constants and $a_n \neq 0$, is called a polynomial of degree n.

For example, a quadratic function $p(x) = ax^2 + bx + c$ (with $a \neq 0$) is a polynomial of degree 2.

A polynomial of degree n can be factored as a product of linear and quadratic factors.

For example, $x^4 - 1 = (x^2 + 1)(x + 1)(x - 1)$.

In general, a polynomial $p(x)$ of degree n has at most n real roots. (A root of a polynomial $p(x)$ is a point c such that $p(c) = 0$.)

For example, $x^4 - 1$ has only two real roots -1 and 1.

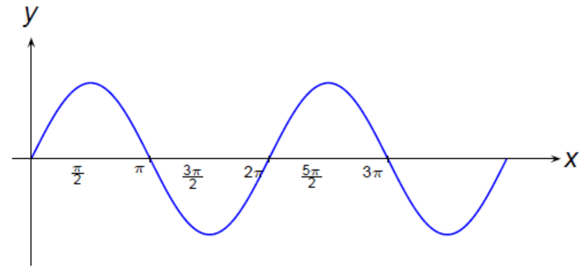
0.5 Rational Functions

A rational function is a function of the form $\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. The domain of $\dfrac{p(x)}{q(x)}$ consists of all real numbers except the roots of $q(x)$.

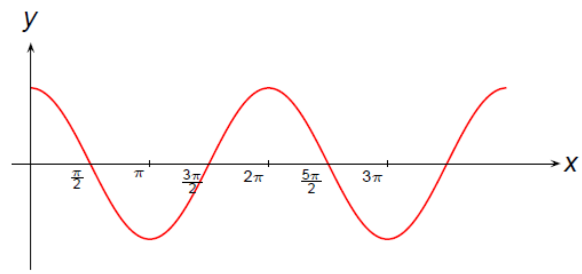
For example, the domain of $\dfrac{x^3 + 3}{x^4 - 1}$ is $\mathbb{R} \setminus \{-1, 1\}$.

0.6 Trigonometric Functions

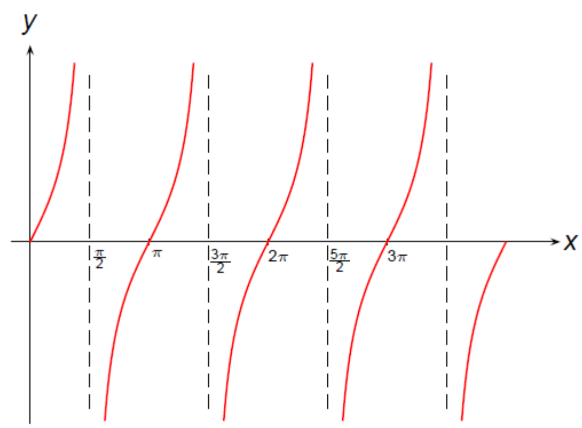
The 6 trigonometric functions are $\sin x, \cos x, \tan x, \csc x, \sec x, \cot x$. They are periodic functions of period 2π.



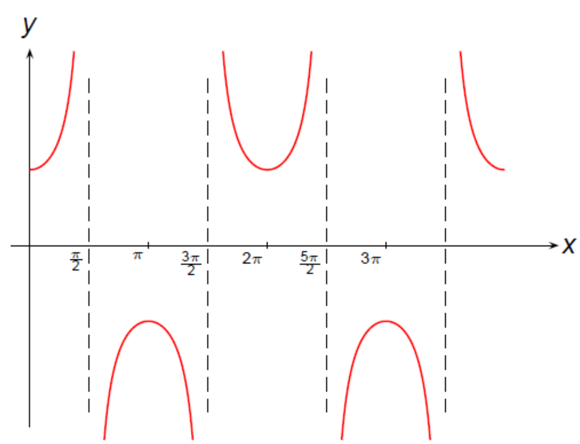
$y = \sin x$



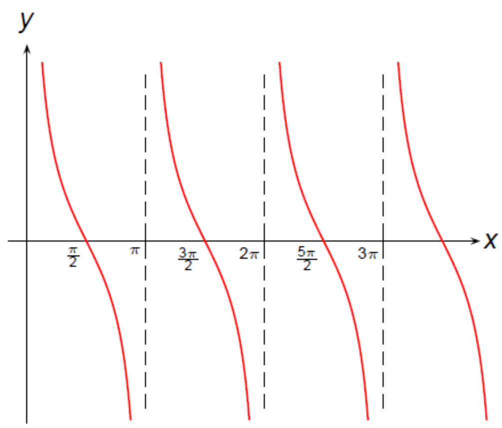
$y = \cos x$



$y = \tan x$



$y = \sec x$



$y = \cot x$

Exercise 0.5. *Sketch the graph of* $\csc(x)$.

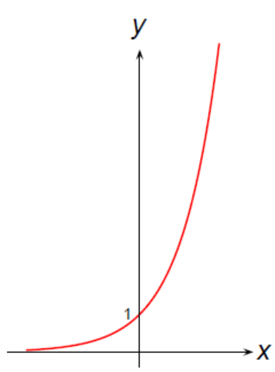
0.7 Exponential and Logarithmic Function

A function of the form $f(x) = a^x$, where $a > 0$ is called an exponential function. It's inverse function, denoted by $\log_a x$ is called the logarithmic function to the base a. ($a > 0$ and $a \neq 1$.)

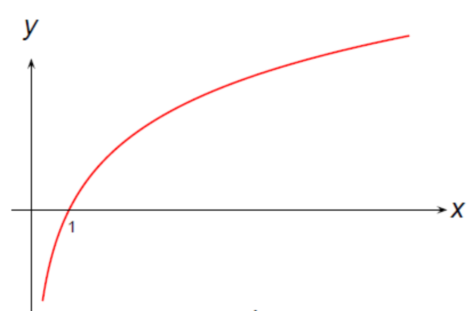
Let $e = 2.718281828459045235360287\cdots$ be the Euler number. Then the inverse of the exponential function e^x is the natural logarithm $\ln x$.

We have $e^{\ln x} = x$ for $x > 0$ and $\ln e^x = x$ for all x.

The domain of e^x is $\mathbb{R}$ and the range is the set $\mathbb{R}^+$ of all positive real numbers.



$y = e^x$



$y = \ln x$

0.8 More on the Domain and Range of a Function

If the domain of a function is not specified, then it is understood that we will take the domain to be as large as possible. This is called the *maximal domain* of the function.

In general it is not so easy to determine the range of a function. In some simple cases, basic algebraic techniques can be used to find the range of a function.

Example 0.5. *Find the maximal domain and the range of* $f(x) = \frac{1}{x-1}$.

Solution. The maximal domain of f is $\mathbb{R} \setminus \{1\}$.

Recall that the range of $f = \{f(x) \in \mathbb{R} \mid x \neq 1\}$.

To find the range of f, let $y = f(x)$. That is $y = \frac{1}{x-1}$. Solving for x, we get

$$y = \frac{1}{x-1} \implies x - 1 = \frac{1}{y} \implies x = 1 + \frac{1}{y}.$$

From this we see that if $y \neq 0$ then we may choose $x = 1 + \frac{1}{y}$ to get $f(x) = y$. Thus the range of f is $\mathbb{R} \setminus \{0\}$. ■

Example 0.6. *Find the maximal domain and range of* $f(x) = x^2 - x + 1$.

Solution. The maximal domain of f is $\mathbb{R}$.

To find the range of f, let $y = f(x)$. That is $y = x^2 - x + 1$. Solving for x (by using the quadratic formula), we get

$$y = x^2 - x + 1 \implies x^2 - x + (1 - y) = 0 \implies x = \frac{(1 \pm \sqrt{1 - 4(1-y)})}{2} \implies x = \frac{1}{2}(1 \pm \sqrt{4y - 3}).$$

From this we see that if $y \geq \frac{3}{4}$ then we may choose $x = \frac{1}{2}(1 \pm \sqrt{4y-3})$ to get $f(x) = y$. Thus the range of f is $[\frac{3}{4}, \infty)$. ■

Exercise 0.6. *Let* $f(x) = x + 5$ *and* $g(x) = x^2 - 3$. *Find the maximal domain and range of* $g(f(x))$.

Ans: Maximal domain is $\mathbb{R}$, range is $[-3, \infty)$.

Chapter 1

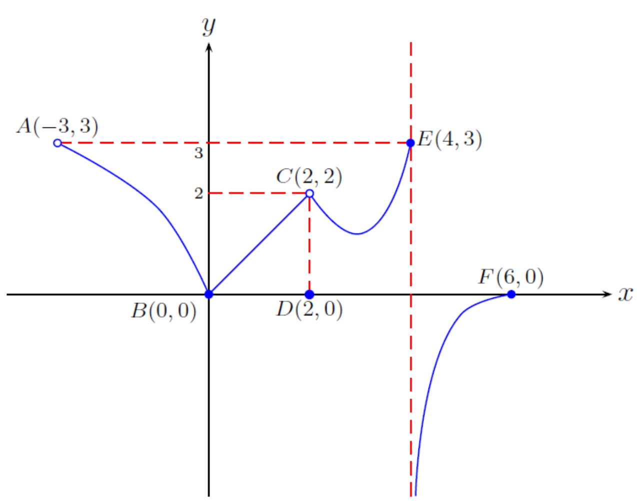
Limits and Continuity

Read Thomas' Calculus, Chapter 2.

1.1 Limits

Let f be a real-valued function defined on some interval I (e.g. (a,b), or $(a,b]$ or (a,∞)). Let c be a point in I.

- $\lim_{x\to c^-} f(x)$ is the value that $f(x)$ approaches when x approaches c from the left.
- $\lim_{x\to c^+} f(x)$ is the value that $f(x)$ approaches when x approaches c from the right.
- Let c be an interior point (i.e. not an end point). If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L \in \mathbb{R}$, we say that $\lim_{x\to c} f(x)$ exist and has value L.



c	Left Limit $\lim_{x\to c^-} f(x)$	Right Limit $\lim_{x\to c^+} f(x)$	Limit $\lim_{x\to c} f(x)$	$f(x)$
0				
2				
4				
-3				
6				

1.2 Continuity

Let f be a real-valued function defined on some interval I (e.g. (a,b), or $(a,b]$ or (a,∞)). Let c be a point in I.

Continuity at a point

Case 1 c is an interior point

- f is continuous at $x = c$ if

  (i) $\lim_{x\to c} f(x)$ exists,

  (ii) $\lim_{x\to c} f(x) = f(c)$.

Case 2 c is the left end-point

- f is continuous at $x = c$ if

  (i) $\lim_{x\to c^+} f(x)$ exists,

  (ii) $\lim_{x\to c^+} f(x) = f(c)$.

Case 3 c is the right end-point

- f is continuous at $x = c$ if

  (i) $\lim_{x\to c^-} f(x)$ exists,

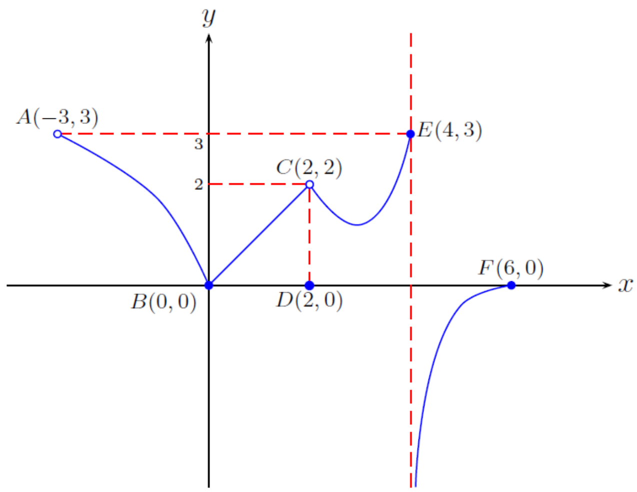
  (ii) $\lim_{x\to c^-} f(x) = f(c)$.

Continuity on an interval

- f is continuous on an interval if f is continuous at $x = c$ for all points c in I.

Remark. Roughly speaking, f is continuous at $x = c$ means that the values of f near $x = c$ all become very close to $f(c)$ when x is very close to c, so that there is no sudden jump in the values of f at $x = c$.

Example 1.1. *Find the points of discontinuity of the function f whose graph on $(-3,6]$ is given below.*



Solution.

Point of discontinuity	Reason
$x = 2$	$\lim_{x\to 2} f(x) \neq f(2)$
$x = 4$	$\lim_{x\to 4} f(x)$ does not exist

1.3 Evaluation of limits

Results (Law of limits)

The following results are true provided all the limits involved exist. The limit could be one-sided or two-sided. The number k is a constant.

1. $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
2. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$
3. $\lim_{x\to c}(f(x)g(x)) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$
4. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$
5. If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then $\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)))$.

Example 1.2. *Suppose* $\lim_{x\to 2} f(x) = 3$ *and* $\lim_{x\to 2} g(x) = 4$. *Find* $\lim_{x\to 2} \frac{f(x)g(x) + \sqrt{g(x)}}{g(x) - f(x)}$.

Solution.

$$\lim_{x\to 2} \frac{f(x)g(x) + \sqrt{g(x)}}{g(x) - f(x)} = \frac{3 \cdot 4 + \sqrt{4}}{4 - 3} = 14.$$

■

From the Laws of Limits, we have corresponding results on continuity of functions.

Results

- If f and g are continuous at $x = c$, then for any constant k and any positive constant n, each of the following functions is continuous at $x = c$.

  (i) $f \pm g$, (ii) f^n, (iii) kf, (iv) fg, (v) f/g provided $g(c) \neq 0$.

- If g is continuous at $x = c$ and f is continuous at $x = g(c)$, then the composite function $f \circ g$ is continuous at $x = c$.

  (Note $(f \circ g)(x) = f(g(x))$).

[For example, if f and g are continuous at $x = c$, then we have $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Thus

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c),$$

which implies that $f + g$ is continuous at $x = c$.]

Result. The following functions are continuous on any interval contained in their maximal domain.

1. Polynomials
2. Trigonometric Functions
3. Exponential Functions
4. Logarithmic Functions
5. A combination of any of the above on the domain it is defined.

For example, a rational function $\frac{P(x)}{Q(x)}$ is continuous at all points x where $Q(x) \neq 0$.

In particular, the rational function $\frac{1}{(x-1)(x-2)}$ is continuous on $\mathbb{R} \setminus \{1, 2\}$.

Example 1.3. *Show that the function* $\ln(x+3)$ *is continuous on the interval* $(-3,\infty)$.

Solution. Take any point $c \in (-3,\infty)$. Then $0 < c+3 < \infty$. Then the polynomial $g(x) = x+3$ is continuous at $x = c$, and the logarithmic function $f(x) = \ln x$ is continuous at $x = c+3 > 0$. Thus the composition

$$f \circ g(x) = f(g(x)) = f(x+3) = \ln(x+3)$$

is continuous at $x = c$. ■

Remark. Since $\lim_{x\to c} f(x) = f(c)$ when f is continuous at $x = c$, finding the limit at $x = c$ of any of the above functions is a matter of evaluating f at $x = c$.

Example 1.4. *Evaluate* $\lim_{x\to -2} \dfrac{x+\ln(x+3)}{\sqrt{x+6}}$.

Solution. The function $f(x) = \dfrac{x+\ln(x+3)}{\sqrt{x+6}}$ is continuous at $x = -2$, which implies that

$$\lim_{x\to -2} f(x) = f(-2).$$

Thus

$$\lim_{x\to -2} \frac{x+\ln(x+3)}{\sqrt{x+6}} = \frac{-2+\ln(-2+3)}{\sqrt{-2+6}} = -1.$$

■

Exercise 1.1. *Evaluate* $\lim_{x\to 0} \tan^3(\sin x)$.

Ans: 0.

1.4 Limits at infinity

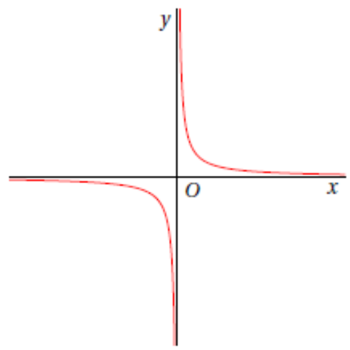
Let f be defined on $\mathbb{R}$.

- $\lim_{x\to\infty} f(x)$ is the value $f(x)$ approaches as x tends to positive infinity.
- $\lim_{x\to -\infty} f(x)$ is the value $f(x)$ approaches as x tends to negative infinity.

Graphically, if $\lim_{x\to\infty} f(x) = c \in \mathbb{R}$ or $\lim_{x\to -\infty} f(x) = c$, then the line $y = c$ is a horizontal asymptote of the graph of $f(x)$.

Example 1.5. *From the graph of* $y = \frac{1}{x}$, *one sees that* $\frac{1}{x}$ *approaches* 0 *as* x *tends to positive infinity. Thus we have* $\lim_{x\to\infty} \frac{1}{x} = 0$.

Similarly we have $\lim_{x\to-\infty} \frac{1}{x} = 0$.



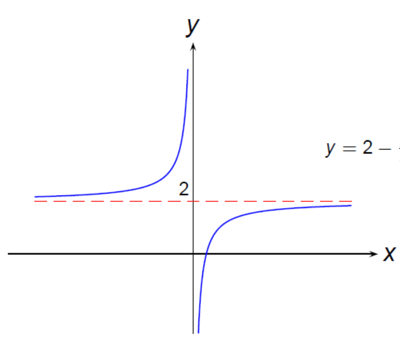
Then for any positive integer n, *we also have*

$$\lim_{x\to\infty} \frac{1}{x^n} = \left(\lim_{x\to\infty} \frac{1}{x}\right)^n = 0^n = 0, \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = \left(\lim_{x\to-\infty} \frac{1}{x}\right)^n = 0^n = 0.$$

Example 1.6. *Using the result in the previous example, we have*

$$\lim_{x\to\infty} (2 - \frac{1}{x}) = 2 - 0 = 2,$$

and thus $y = 2$ *is a horizontal asymptote of the graph of* $y = 2 - \frac{1}{x}$.



Also, $\lim_{x\to-\infty} (2 - \frac{1}{x}) = 2$.

Example 1.7. *Evaluate* $\lim_{x\to\infty}\left(\frac{3}{2x}+\sqrt{4-e^{-x}}\right)^2$.

Solution. $\lim_{x\to\infty}\left(\frac{3}{2x}+\sqrt{4-e^{-x}}\right)^2 = (0+\sqrt{4})^2 = 4.$ ■

Exercise 1.2. *Evaluate* $\lim_{x\to-\infty}\ln\left(3-2\sin\frac{4}{x}\right)$.

Ans: $\ln 3$.

1.5 More on Limits

Indeterminate forms.

(a) A limit of the form $\lim_{x\to c}\frac{f(x)}{g(x)}$ where $f(x)\to 0$ and $g(x)\to 0$ as $x\to c$ is called an indeterminate form of the type $\frac{0}{0}$.

(b) A limit of the form $\lim_{x\to c}\frac{f(x)}{g(x)}$ where $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to c$ is called an indeterminate form of the type $\frac{\infty}{\infty}$.

Replacement rule. Let I be an open interval containing the point $x=c$. Suppose $f(x)=g(x)$ for all $x\in I$, except possibly at $x=c$. Then $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)$.

Example 1.8. *Evaluate* $\lim_{x\to 6}\frac{x^2-7x+6}{36-x^2}$.

Solution. $\lim_{x\to 6}\frac{x^2-7x+6}{36-x^2} = \lim_{x\to 6}\frac{(x-1)(x-6)}{-(6+x)(x-6)} == \lim_{x\to 6}\frac{x-1}{-(6+x)} = -\frac{5}{12}.$ ■

Exercise 1.3. *Evaluate* $\lim_{x\to-3}\frac{\sqrt{x+12}-\sqrt{6-x}}{18-2x^2}$.

Ans: $\frac{1}{36}$.

Result. Limits of the form $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x.

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\pm\infty}\frac{\overbrace{Ax^{\alpha}}^{\textit{leading term}}\quad+\cdots}{\underbrace{Bx^{\beta}}_{\textit{leading term}}\quad+\cdots}=\begin{cases}0 & \text{if } \alpha<\beta\\ \frac{A}{B} & \text{if } \alpha=\beta\\ \underbrace{\infty \text{ or } -\infty}_{\textit{depends on the question}} & \text{if } \alpha>\beta\end{cases}$$

Example 1.9. *Evaluate* $\lim\limits_{x\to\infty}\dfrac{(18x^2+5x-1)(2\sqrt{x}-1)^3}{(3x-1)^4}$.

Solution. $\lim\limits_{x\to\infty}\dfrac{(18x^2+5x-1)(2\sqrt{x}-1)^3}{(3x-1)^4}=\lim\limits_{x\to\infty}\dfrac{144x^{\frac{7}{2}}+\cdots}{81x^4+\cdots}=0.$ ■

Exercise 1.4. *Evaluate* $\lim\limits_{x\to-\infty}\dfrac{(1+2x)^3}{\sqrt{16x^6+9x-1}}$.

Ans: -2.

Useful results.

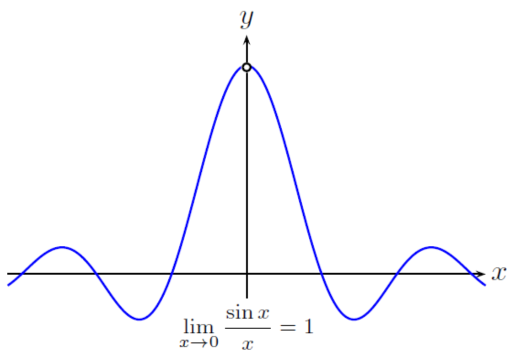
If $\lim\limits_{x\to c}g(x)=0$, then

- $\lim\limits_{x\to c}\dfrac{\sin(g(x))}{g(x)}=\lim\limits_{x\to c}\dfrac{g(x)}{\sin(g(x))}=1,$
- $\lim\limits_{x\to c}\dfrac{\tan(g(x))}{g(x)}=\lim\limits_{x\to c}\dfrac{g(x)}{\tan(g(x))}=1.$

In particular, when $c=0$ and $g(x)=x$,

- $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{x}{\sin x}=1,$
- $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=\lim\limits_{x\to 0}\dfrac{x}{\tan x}=1.$

For example, $\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}=1$, $\lim\limits_{x\to 1}\dfrac{\ln x}{\tan(\ln x)}=1$, $\lim\limits_{x\to\infty}\dfrac{\sin(e^{-x})}{e^{-x}}=1$.



Example 1.10. *Evaluate* $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{\sin(3x^2) + x\tan(2x)}$.

Solution.

$$\lim_{x\to 0} \frac{\sin^2 x}{\sin(3x^2) + x\tan(2x)} = \lim_{x\to 0} \frac{x^2 \dfrac{\sin^2 x}{x^2}}{3x^2 \dfrac{\sin(3x^2)}{3x^2} + 2x^2 \dfrac{\tan(2x)}{2x}}$$
$$= \lim_{x\to 0} \frac{\left(\dfrac{\sin x}{x}\right)^2}{3\dfrac{\sin(3x^2)}{3x^2} + 2\dfrac{\tan(2x)}{2x}} = \frac{1^2}{3+2} = \frac{1}{5}$$

■

Exercise 1.5. *Evaluate* $\displaystyle\lim_{x\to 0^+} \left(x^2 \cot(2x)\csc^2(3\sqrt{x})\right)$.

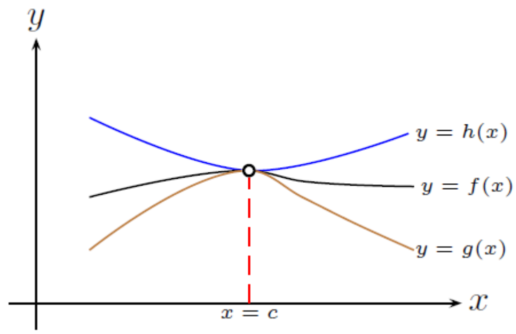
Ans: $\frac{1}{18}$.

Exercise 1.6. *Evaluate* $\displaystyle\lim_{x\to 4} \frac{\tan(\sqrt{x}-2)}{\sin(16-x^2)}$.

Ans: $-\frac{1}{32}$.

1.6 Squeeze (Sandwich) Theorem

Squeeze Theorem. Suppose $g(x) \le f(x) \le h(x)$ for all x in some open interval containing a point c, except possibly at $x = c$. If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$.



Example 1.11. *It is given that $3 - x^2 \leq f(x) \leq 1 + 2e^x$ for all x. Find $\lim\limits_{x\to 0} f(x)$.*

Solution. As $\lim\limits_{x\to 0} 3 - x^2 = 3$ and $\lim\limits_{x\to 0} 1 + 2e^x = 3$, we have by Squeeze Theorem that $\lim\limits_{x\to 0} f(x) = 3$.
■

Exercise 1.7. *Use Squeeze Theorem to show that $\lim\limits_{x\to c} |f(x)| = 0 \Rightarrow \lim\limits_{x\to c} f(x) = 0$.*

Remark The converse of the above result, namely $\lim\limits_{x\to c} f(x) = 0 \Rightarrow \lim\limits_{x\to c} |f(x)| = 0$ is true.

Hence, we have

Result. $\lim\limits_{x\to c} f(x) = 0 \Leftrightarrow \lim\limits_{x\to c} |f(x)| = 0$.

Example 1.12. *Evaluate $\lim\limits_{x\to 0}\left(x^3\cos(\dfrac{2}{\sin x})\right)$.*

Solution. Notice that for all x, we have

$$\begin{aligned} -1 \leq \cos(\frac{2}{\sin x}) \leq 1 &\implies \left|\cos(\frac{2}{\sin x})\right| \leq 1 \\ &\implies \left|x^3\cos(\frac{2}{\sin x})\right| \leq |x^3| = |x|^3 \\ &\implies -|x|^3 \leq x^3\cos(\frac{2}{\sin x}) \leq |x|^3. \end{aligned}$$

Notice that $\lim\limits_{x\to 0} |x|^3 = 0^3 = 0$ and thus also $\lim\limits_{x\to 0} -|x|^3 = 0$.

Hence by Squeeze Theorem, we have $\lim\limits_{x\to 0}\left(x^3\cos(\dfrac{2}{\sin x})\right) = 0$.

■

Exercise 1.8. *Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{\sin(2\ln x)}{\ln x} + 2x\sin(\dfrac{1}{x})\right)$.*

Ans: 2.

Ex 1.8

since $-1 \leq \sin(2\ln x) \leq 1$, then $\lim_{x\to\infty} -\frac{1}{\ln x} \leq \lim_{x\to\infty} \frac{\sin(2\ln x)}{\ln x} \leq \lim_{x\to\infty} \frac{\sin(2\ln x)}{\ln x}$

$\Rightarrow 0 \leq \lim_{x\to\infty} \frac{\sin(2\ln x)}{\ln x} \leq 0$

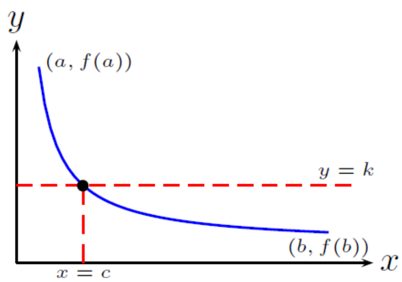
$\therefore \lim_{x\to\infty} \frac{\sin(2\ln x)}{\ln x} = 0$

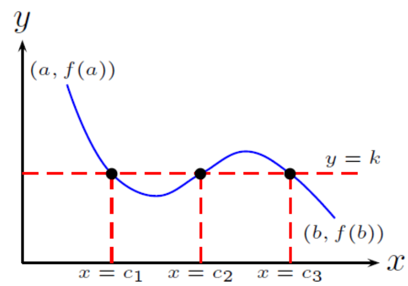
$\therefore \lim_{x\to\infty} \left(\frac{\sin(2\ln x)}{\ln x} + 2x\sin\left(\frac{1}{x}\right) \right) = 0 + \lim_{x\to\infty} 2\,\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$

$= 2$

1.7 Intermediate Value Theorem (IVT)

If a real-valued function f is continuous on $[a,b]$ and k is a number between $f(a)$ and $f(b)$, then $f(c)=k$ for some $c\in[a,b]$.





Example 1.13. *Show that the equation $x^3e^x=10$ has a solution between 1 and 1.5.*

Solution. Let $f(x)=x^3e^x$. f is continuous on $\mathbb{R}$. We have $f(1)=e=2.718$ and $f(1.5)=1.5^3e^{1.5}=15.126$. Thus $f(1)<10<f(1.5)$. By Intermediate Value Theorem, there is a solution to $f(x)=10$ between 1 and 1.5.

Exercise 1.9. *Show that the equation $10=x+2\tan(2x)$ has a solution between 3 and 4.*

1.8 Appendix: The Precise Definition of the Limit of a Function

Remark: Section 1.8 will be excluded from the assessments (quizzes and the Final Exam).

Remark: The precise definition of the limit of a function introduced in this section will not be needed in MA1521, and it will be studied in detail in MA2108 Mathematical Analysis I, where many results in calculus will be proved rigorously.

Let $f(x)$ be defined on an open interval containing the point c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x\to c} f(x)=L,$$

if, for every number $\epsilon>0$, there exists a corresponding number $\delta>0$ such that for all x,

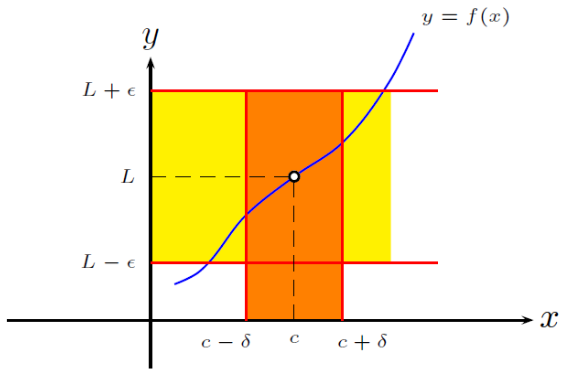
$$0<|x-c|<\delta \Rightarrow |f(x)-L|<\epsilon.$$

Ex 1.9

$\theta \to \frac{\pi}{2} \Rightarrow \tan\theta \to \infty$ $\quad$ $\exists\, \theta \in (0, \frac{\pi}{2})$ st $\tan\theta > 10$ $\quad$ $\therefore \exists\, 2x \in [2\pi, 2\pi + \theta], \theta \in (0, \frac{\pi}{2}]$ st $\tan\theta 2x = 10$

since $\pi > 3$ and $\pi + 2\tan(2\pi) = \pi < 10$, $\pi \leq x + 2\tan(2x) \leq 20 + \frac{5\pi}{4}$ $\forall x \in [\pi, \frac{5\pi}{4}]$

$\therefore \exists\, x \in (3, 4)$ st $x + 2\tan 2x = 10$



The $\epsilon - \delta$ definition of the limit of a function

Example 1.14. *Prove from definition that* $\lim_{x \to 1} 5x - 3 = 2$.

Solution. Note that $|(5x-3)-2| = 5|x-1|$. Given $\epsilon > 0$, we choose $\delta = \epsilon/5$. Then for all x, $0 < |x-1| < \delta \Rightarrow 5|x-1| < 5\delta \Rightarrow |(5x-3)-2| < \epsilon$.

■

Exercise 1.10. *Prove from definition that* $\lim_{x \to -6} \frac{x}{3} + 3 = 1$.

Chapter 2

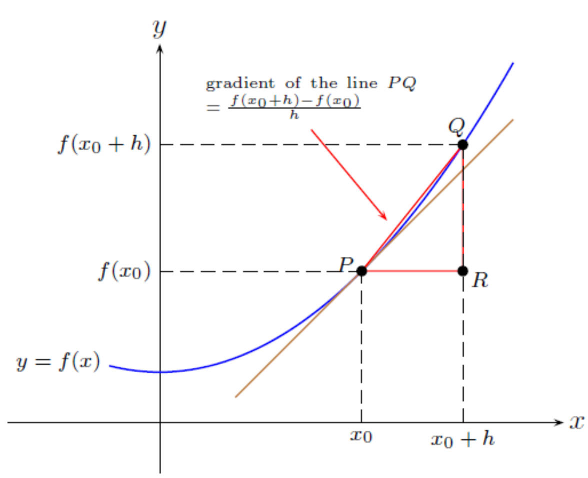
Derivatives

Read Thomas' Calculus, Chapter 3.

2.1 Differentiability

Definition 2.1. *The* **derivative** *of a function f at the point x_0, denoted by $f'(x_0)$, is given by the following limit*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$



Consider the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. From the diagram, as $h \to 0$, Q approaches P and hence, the gradient of the chord PQ, namely

$$\frac{f(x_0 + h) - f(x_0)}{h},$$

approaches the limiting value

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0).$$

When $f'(x_0)$ exists, we say that f is differentiable at $x=x_0$. Geometrically in this case, $f'(x_o)$ is equal to the slope of the tangent line to the curve $y=f(x)$ at the point $(x_0, f(x_0))$.

Remark. An alternative formula for $f'(x_0)$ is

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.$$

(This can be seen by letting $x=x_0+h$, and noting that '$h\to 0$' is the same as '$x\to x_0$'.)

Example 2.1. *Let $f(x)=x^2$. Find $f'(3)$ (that is, $x_0=3$).*

Solution.

$$\begin{aligned} f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0}\frac{(3+h)^2-3^2}{h} \\ &= \lim_{h\to 0}\frac{9+6h+h^2-9}{h} \\ &= \lim_{h\to 0}(6+h) \\ &= 6. \end{aligned}$$

Remark. Thus the slope of the tangent line to the curve $y=x^2$ at the point $(3,9)$ is equal to 6.

■

Definition 2.2. *Suppose the derivative $f'(x)$ exists for all x in an open interval I. We can then treat $f'(x)$ as a function defined on I. The process of finding the derivative of a function is called differentiation. If $y=f(x)$, we can also write $\frac{d}{dx}f(x)$, $\frac{dy}{dx}$ or $\frac{df}{dx}$ to denote $f'(x)$.*

In summary, we have

$$\frac{d}{dx}f(x)=\frac{dy}{dx}=\frac{df}{dx}=f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Example 2.2. *(a) Use the above definition of derivative to differentiate the function* $\dfrac{1}{2+\sqrt{x}}$.

(b) Find the equation of the tangent to the curve $y = \frac{1}{2+\sqrt{x}}$ *at the point* $(1, \frac{1}{3})$.

Ans: (a) $\frac{-1}{2\sqrt{x}(2+\sqrt{x})^2}$, (b) $(y - \frac{1}{3}) = -\frac{1}{18}(x-1)$.

Solution. (a)

$$
\begin{aligned}
\frac{d}{dx}\frac{1}{2+\sqrt{x}} &= \lim_{h\to 0}\frac{\dfrac{1}{2+\sqrt{x+h}} - \dfrac{1}{2+\sqrt{x}}}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\frac{(2+\sqrt{x})-(2+\sqrt{x+h})}{(2+\sqrt{x+h})(2+\sqrt{x})} \\
&= \lim_{h\to 0}\frac{1}{h}\frac{\sqrt{x}-\sqrt{x+h}}{(2+\sqrt{x+h})(2+\sqrt{x})} \\
&= \lim_{h\to 0}\frac{1}{h}\frac{\sqrt{x}-\sqrt{x+h}}{(2+\sqrt{x+h})(2+\sqrt{x})}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\
&= \lim_{h\to 0}\frac{1}{h}\frac{-h}{(2+\sqrt{x+h})(2+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0}\frac{-1}{(2+\sqrt{x+h})(2+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \\
&= \frac{-1}{2\sqrt{x}(2+\sqrt{x})^2}.
\end{aligned}
$$

(b) From part (a), $\dfrac{d}{dx}\dfrac{1}{2+\sqrt{x}} = \dfrac{-1}{2\sqrt{x}(2+\sqrt{x})^2}$.

At $x = 1$, derivative is $\dfrac{-1}{2\sqrt{1}(2+\sqrt{1})^2} = -\dfrac{1}{18}$.

Thus the equation of the tangent at $(1, \dfrac{1}{3})$ is $\dfrac{y-\dfrac{1}{3}}{x-1} = -\dfrac{1}{18}$.

∎

Differentiability implies continuity.

Theorem 2.1. *If f is differentiable at $x = x_0$, then f is continuous at $x = x_0$.*

Ex 2.2

a) $\frac{d}{dx}\left(\frac{1}{2+\sqrt{x}}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{2+\sqrt{x+h}} - \frac{1}{2+\sqrt{x}}}{h}\right) = \lim_{h\to 0}\left(\frac{1}{h}\left(\frac{2+\sqrt{x}-(2+\sqrt{x+h})}{(2+\sqrt{x})(2+\sqrt{x+h})}\right)\right) = \lim_{h\to 0}\left(\frac{\sqrt{x}-\sqrt{x+h}}{(2+\sqrt{x})(2+\sqrt{x+h})}\right)$

$= \lim_{h\to 0}\left(\frac{x-(x+h)}{(2+\sqrt{x})(2+\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}\right) = \lim_{h\to 0}\left(\frac{-1}{(2+\sqrt{x})(2+\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}\right) = -\frac{1}{2\sqrt{x}(2+\sqrt{x})^2}$

b) $y+c = mx$ @ $(1, \frac{1}{3})$,

$y+c = \frac{dy}{dx}\Big|_{x=1} x$

$= \frac{-1}{2(1)(2+\sqrt{1})^2} x = -\frac{1}{18} x$

$y+c = -\frac{1}{18} x$

$c = -\frac{1}{18} - \frac{1}{3}$

$= -\frac{7}{18}$

$\therefore y - \frac{7}{18} = \frac{1}{18} x$

Proof. It is given that $f'(x_0)$ exists. For all x near x_0, we have

$$f(x) = \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) + f(x_0).$$

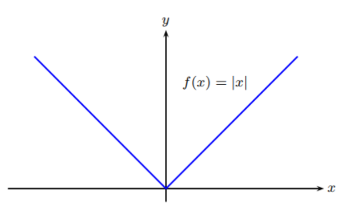
Letting $x \to x_0$, we have

$$\begin{aligned}\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) + f(x_0)\right) \\ &= f'(x_0) \cdot (x_0 - x_0) + f(x_0) \\ &= f(x_0).\end{aligned}$$

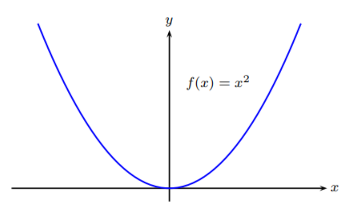
Hence f is continuous at $x = x_0$.

■

Remark. The converse of the above result is not true in general. For example the absolute value function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.



$f(x) = |x|$ is not differentiable at $x = 0$.



$f(x) = x^2$ is differentiable at $x = 0$.

Differentiability on Intervals.

Definition 2.3. *A function f is said to be differentiable on an interval I if it is differentiable at every point in I.*

Remark. If the interval has endpoints, then the limit in defining the derivative at an endpoint should be replaced by the appropriate one-sided limit.

Exercise 2.1. *Show that $f(x) = |x^2 - 2x|$ is* **not** *differentiable at $x = 2$.*

Example 2.3. *Let n be a positive integer. Show that $\frac{d}{dx}x^n = nx^{n-1}$.*

Solution.

$$\frac{d}{dx}x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Ex 2.3 $f'(2) = \lim_{h \to 0} \frac{|(2+h)^2 - 2(2+h)| - |2^2 - 2(2)|}{2+h-2} = \lim_{h \to 0} \frac{|(2+h)^2 - (4+2h)| - 0}{h}$

$= \lim_{h \to 0} \frac{|h^2+2h|}{h}$

$= \lim_{h \to 0} \frac{|h||h+2|}{h}$

Since $\lim_{h \to 0^-} \frac{|h||h+2|}{h} < 0$ and $\lim_{h \to 0^+} \frac{|h||h+2|}{h} > 0$,

$\lim_{h \to 0^-} \frac{|h||h+2|}{h} \neq \lim_{h \to 0^+} \frac{|h||h+2|}{h}$

$$= \lim_{h \to 0} \frac{(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n) - x^n}{h}$$

(using binomial theorem)

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + \tfrac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1} = nx^{n-1}.$$

■

Example 2.4. *Show that* $\frac{d}{dx}\sin x = \cos x$.

Solution.

$$\frac{d}{dx}\sin x = \lim_{h \to 0}\frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0}\frac{2\cos\frac{2x+h}{2}\sin\frac{h}{2}}{h}$$

(using $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$)

$$= \lim_{h \to 0}\cos(x + \frac{h}{2}) \cdot \lim_{h \to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= (\cos x) \cdot 1 = \cos x.$$

■

2.2 Standard Derivatives & Differentiation Rules

Table 1

Function	**Derivative**
x^n	nx^{n-1}
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\cot^{-1}(x)$	$-\frac{1}{1+x^2}$
$\sec^{-1}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert > 1$
$\csc^{-1}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert > 1$

Table 2

Function	Derivative
$(g(x))^n$	$ng'(x)(g(x))^{n-1}$
$\cos(g(x))$	$-g'(x)\sin(g(x))$
$\sin(g(x))$	$g'(x)\cos(g(x))$
$\tan(g(x))$	$g'(x)\sec^2(g(x))$
$\sec(g(x))$	$g'(x)\sec(g(x))\tan(g(x))$
$\csc(g(x))$	$-g'(x)\csc(g(x))\cot(g(x))$
$\cot(g(x))$	$-g'(x)\csc^2(g(x))$
$e^{g(x)}$	$g'(x)e^{g(x)}$
$\ln(g(x))$	$\frac{g'(x)}{g(x)}$
$\sin^{-1}(g(x))$	$\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\cos^{-1}(g(x))$	$-\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\tan^{-1}(g(x))$	$\frac{g'(x)}{1+g(x)^2}$
$\cot^{-1}(g(x))$	$-\frac{g'(x)}{1+g(x)^2}$
$\sec^{-1}(g(x))$	$\frac{g'(x)}{\lvert g(x)\rvert\sqrt{g(x)^2-1}},\ \lvert g(x)\rvert > 1$
$\csc^{-1}(g(x))$	$-\frac{g'(x)}{\lvert g(x)\rvert\sqrt{g(x)^2-1}},\ \lvert g(x)\rvert > 1$

Note that when $g(x) = x$, the formulae in Table 2 reduce to those in Table 1.

The formulae in Table 2 follow readily from those in Table 1 and the Chain Rule (which we will see very soon).

Rules of Differentiation

Let u and v be differentiable functions of x, and let c be a constant.

Constant Rule	$\frac{d}{dx}(c) = 0$
Constant Multiple Rule	$\frac{d}{dx}(cu) = c\frac{du}{dx}$
Sum Rule	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
Product Rule	$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$
Quotient Rule	$\frac{d}{dx}(\frac{u}{v}) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Example 2.5. *Differentiate $x^3 + x^2 \tan x$ with respect to x.*

Solution. By the sum rule and product rule,

$$\begin{aligned}\frac{d}{dx}(x^3 + x^2 \tan x) &= \frac{d}{dx}x^3 + \frac{d}{dx}(x^2 \tan x) \\ &= 3x^2 + \frac{d}{dx}(x^2) \cdot \tan x + x^2 \cdot \frac{d}{dx} \tan x \\ &= 3x^2 + 2x \tan x + x^2 \sec^2 x.\end{aligned}$$

■

Example 2.6. *Differentiate $\frac{x^2}{\ln x}$ with respect to x.*

Solution. By the quotient rule,

$$\begin{aligned}\frac{d}{dx}\frac{x^2}{\ln x} &= \frac{(\frac{d}{dx}x^2) \cdot \ln x - x^2 \cdot \frac{d}{dx}\ln x}{(\ln x)^2} \\ &= \frac{2x \ln x - x^2 \cdot \frac{1}{x}}{(\ln x)^2} \\ &= \frac{2x \ln x - x}{(\ln x)^2}.\end{aligned}$$

■

Let $f(u)$ be differentiable at $u = g(x)$, and let g be a differentiable function of x.

Chain Rule	$\dfrac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Write $y = (f(g(x))$ and $u = g(x)$ as functions of x. Then the Chain Rule can be abbreviated as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

with the understanding that $\dfrac{dy}{du}$ is evaluated at $u = g(x)$, so that all three expressions are regarded as functions of x.

Example 2.7. *Differentiate* $\sin(x^3 + x + 2)$ *with respect to* x.

Solution. Let $u = x^3 + x + 2$, so that

$$y = \sin(x^3 + x + 2) = \sin u, \quad \text{and} \quad u = x^3 + x + 2.$$

Then by the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \cos u \cdot (3x^2 + 1) \\ &= (\cos(x^3 + x + 2)) \cdot (3x^2 + 1) \\ \Longrightarrow \quad \frac{d}{dx} \sin(x^3 + x + 2) &= (3x^2 + 1)\cos(x^3 + x + 2). \end{aligned}$$

■

Example 2.8. *Use the Chain Rule to derive the formula* $\dfrac{d}{dx}(g(x))^n = ng'(x)(g(x))^{n-1}$ *(in Table 2) from a corresponding formula in Table 1.*

Solution. Let $u = g(x)$, so that

$$y = (g(x))^n = u^n, \quad \text{and} \quad u = g(x).$$

Then by the Chain Rule and the first formula in Table 1,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= nu^{n-1} \cdot g'(x) \\ &= n(g(x))^{n-1} \cdot g'(x) \\ \Longrightarrow \quad \frac{d}{dx}(g(x))^n &= ng'(x)(g(x))^{n-1}. \end{aligned}$$

■

Example 2.9. *Differentiate* $\ln\left(\dfrac{x^2}{(6x-7)^2}\right)$ *with respect to* x.

Ans: $\frac{2}{x} - \frac{12}{6x-7}$.

Solution. First we have

$$\ln\left(\frac{x^2}{(6x-7)^2}\right) = 2\ln\left(\frac{x}{(6x-7)}\right) = 2\ln(x) - 2\ln(6x-7).$$

Thus $\quad \dfrac{d}{dx}\ln\left(\dfrac{x^2}{(6x-7)^2}\right) = \dfrac{2}{x} - 2\cdot\dfrac{1}{6x-7}\cdot 6 = \dfrac{2}{x} - \dfrac{12}{6x-7}.$

■

Exercise 2.2. *Differentiate with respect to* x.
(a) $(x+1)^2\tan^{-1}(\sqrt{x})$
(b) $\dfrac{\sin^{-1}(2x)}{\sqrt{1-4x^2}}$

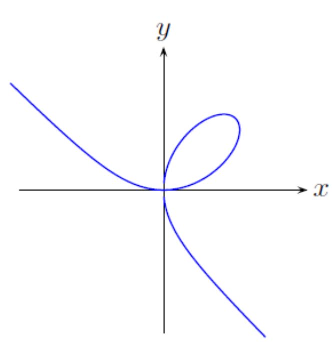
Ans: (a) $2(x+1)\tan^{-1}(\sqrt{x}) + \dfrac{x+1}{2\sqrt{x}}$, (b) $\dfrac{2}{1-4x^2} + \dfrac{4x\sin^{-1}(2x)}{(1-4x^2)^{\frac{3}{2}}}$.

2.3 Implicit Differentiation

The diagram shows the curve given implicitly by the equation

$$x^3 + y^3 - 9xy = 0$$

known as the *folium of Descartes*.



Folium of Descartes $x^3 + y^3 = 9xy$

Ex 2.2

a) $$\frac{d}{dx}\left[(x+1)^2\tan^{-1}(\sqrt{x})\right] = \tan^{-1}(\sqrt{x})\frac{d}{dx}(x+1)^2) + (x+1)^2\frac{d}{dx}(\tan^{-1}(\sqrt{x}))$$

$$= 2(x+1)\tan^{-1}(\sqrt{x}) + (x+1)^2\frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}(\sqrt{x})$$

$$= 2(x+1)\tan^{-1}(\sqrt{x}) + (x+1)^2\left[\frac{\frac{1}{2}(x^{-\frac{1}{2}})}{x+1}\right]$$

$$= 2(x+1)\tan^{-1}(\sqrt{x}) + \frac{x+1}{2\sqrt{x}}$$

b) $$\frac{d}{dx}\left(\frac{\sin^{-1}(2x)}{\sqrt{1-4x^2}}\right) = \frac{\sqrt{1-4x^2}\frac{d}{dx}\sin^{-1}(2x) - \sin^{-1}(2x)\frac{d}{dx}(\sqrt{1-4x^2})}{(\sqrt{1-4x^2})^2}$$

$$= \frac{\sqrt{1-4x^2}\left(\frac{1}{\sqrt{1-(2x)^2}}\right)\frac{d}{dx}(2x)}{(\sqrt{1-4x^2})^2} - \frac{\sin^{-1}(2x)\left[\frac{1}{2}\sqrt{1-4x^2}^{\,-1}\right]\frac{d}{dx}(4x^2)}{(\sqrt{1-4x^2})^2}$$

$$= \frac{2}{1-4x^2} - \frac{4x\sin^{-1}(2x)}{(1-4x^2)^{\frac{3}{2}}}$$

Suppose we wish to find the gradient of the curve at the point $(2,4)$. For this example, getting an explicit expression for y in terms of x is challenging. It turns out that it is possible to find $\frac{dy}{dx}$ by a method known as *implicit differentiation*. This consists of differentiating both sides of the given equation with respect to x and solving the resulting equation for $\frac{dy}{dx}$. When differentiating a function in y with respect to x, we need to use the Chain Rule as follows:

$$\frac{d}{dx}g(y) = g'(y)\frac{dy}{dx}.$$

Example 2.10. *Consider the curve $x^3 + y^3 - 9xy = 0$. Find $\frac{dy}{dx}$.*

Ans: $\frac{dy}{dx} = -\frac{3x^2-9y}{3y^2-9x}$.

Solution. Differentiating both sides of the equation with respect to x, we have

$$3x^2 + 3y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0.$$

Solving for $\dfrac{dy}{dx}$, we obtain

$$\begin{aligned}(3y^2 - 9x)\frac{dy}{dx} &= -3x^2 + 9y \\ \implies \frac{dy}{dx} &= -\frac{3x^2 - 9y}{3y^2 - 9x}.\end{aligned}$$

■

Example 2.11. *Find $\frac{dy}{dx}$ for points on the curve $x^3e^y + \cos(xy) = 2024$.*

Solution. Differentiating both sides of the equation with respect to x, we have

$$\overbrace{3x^2e^y + x^3e^y\frac{dy}{dx}}^{\text{Apply product rule to } x^3e^y} - \sin(xy)\left(x\frac{dy}{dx} + y\right) = 0.$$

Solving for $\dfrac{dy}{dx}$, we obtain

$$\begin{aligned}(x^3e^y - x\sin(xy))\frac{dy}{dx} &= -3x^2e^y + y\sin(xy) \\ \implies \frac{dy}{dx} &= \frac{-3x^2e^y + y\sin(xy)}{x^3e^y - x\sin(xy)}.\end{aligned}$$

■

2.4 Derivatives of Inverse Functions

Recall that

(i) a bijective function (or one-one function) f has an inverse f^{-1} defined on the range of f.

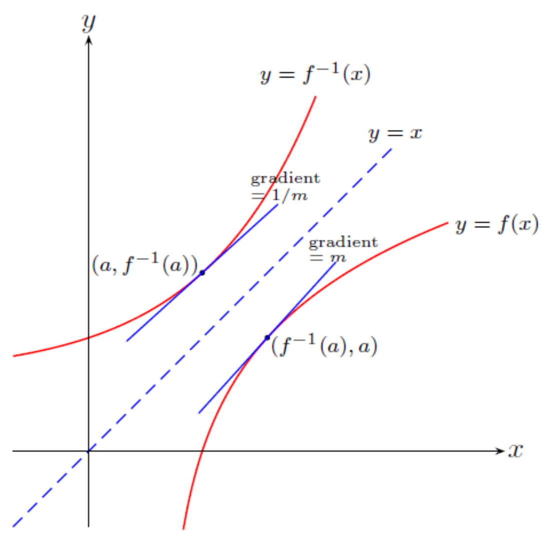
(ii) increasing or decreasing functions are bijective.

Theorem 2.2. *Let f be bijective and differentiable on an open interval I. Then*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Let $b = f^{-1}(a)$, so that $a = f(b)$. At the point $(a, f^{-1}(a)) = (a, b)$ on the curve $y = f^{-1}(x)$, the gradient of the tangent is

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(b)}.$$



Proof. Let $b = f^{-1}(a)$, so that $a = f(b)$. As f^{-1} is the inverse function of f, we have $f^{-1}(f(x)) = x$ for all x. By chain rule,

$$\begin{aligned}
&(f^{-1})'(f(x)) \cdot f'(x) = 1 \\
\Rightarrow\ &(f^{-1})'(f(b)) \cdot f'(b) = 1 \\
\Rightarrow\ &(f^{-1})'(a) \cdot f'(b) = 1 \\
\Rightarrow\ &(f^{-1})'(a) = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}.
\end{aligned}$$

■

Remark. Using $a = f(b)$ and letting b vary over I, one obtains the formula

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

relating the derivative functions of f^{-1} and f. This formula is often abbreviated as follows: Let $y = y(x)$ be a function. Suppose y has an inverse function $x = x(y)$. Then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

with the understanding that $\frac{dx}{dy}$ is evaluated at $y = y(x)$, and both expressions are regarded as functions of x.

This can also be obtained directly as follows: Note that

$$x = x(y(x))$$

for all x. Differentiating both sides (and applying the abbreviated Chain Rule on the right hand side), we get

$$\frac{dx}{dx} = \frac{d}{dx}(x(y(x)) \implies 1 = \frac{dx}{dy} \cdot \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{\frac{dy}{dx}}$$

with the understanding that $\frac{dx}{dy}$ is evaluated at $y = y(x)$. ■

Example 2.12. *Show that* $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$.

(Recall that $\sin^{-1}(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *for all* $x \in -[1,1]$.*)*

Solution. Let $y = \sin^{-1}(x)$, so that $x = \sin y$. Then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \implies \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\frac{d}{dy}(\sin y)} = \frac{1}{\cos y}.$$

Note that

$$y = \sin^{-1}(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \implies \cos y \geq 0 \implies \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-x^2}.$$

Thus $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$.

■

Exercise 2.3. *Let* $f(x) = x^5 - 2e^{-2x} + 3$.

(a) Show that f *is bijective by showing that* f *is increasing on* $\mathbb{R}$.

(b) Find the gradient of the tangent to the curve $y = f^{-1}(x)$ *at the point* $(1,0)$.

(c) Let $g(x) = \dfrac{1}{2x^{-1} + 3f^{-1}(x)}$. *Find the value of* $g'(1)$.

Ans: (b) $\frac{1}{4}$, (c) $\frac{5}{16}$.

2.5 Higher-order Derivatives

Given a differentiable function f, we can find the derivative of its derivative function f' with respect to x to get a new function, denoted by f'' or $f^{(2)}$, called the *second derivative* of f provided that f' is differentiable. That is,

$$f''(x) = \frac{d}{dx} f'(x).$$

Notations. Let $y = f(x)$.

$$f^{(2)}(x) = f''(x) = \frac{d^2y}{dx^2} = y'' = D^2 f(x).$$

In general, we can define the nth order derivative of f for any positive integer n provided the derivative exists. For example, the third derivative of $y = f(x)$ is defined by

$$f^{(3)}(x) = f'''(x) = \frac{d}{dx} f^{(2)}(x) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right)\right).$$

We shall denote the nth order derivative of f by

$$f^{(n)}(x) = \frac{d^ny}{dx^n} = D^n y = D^n f(x).$$

Example 2.13. *Let* n *be a positive integer. For an non-negative integer* k, *find*

$$\frac{d^k}{dx^k} x^n.$$

Solution.

$k = 1$: $\qquad \dfrac{d}{dx} x^n = nx^{n-1}.$

Ex 2.3

a) $f'(x) = 5x^4 + 4e^{-2x}$

b) $(f^{-1})'(1) = \frac{1}{f'(f(1))} = \frac{1}{5(0)^4 + 4e^{-2(0)}} = \frac{1}{4}$

c) $\frac{d}{dx}\left(\frac{1}{2x^{-1} + 3f^{-1}(x)}\right) = -\frac{\frac{d}{dx}\left(2x^{-1} + 3f^{-1}(x)\right)}{(2x^{-1} + 3f^{-1}(x))^2}$

$= -\frac{-2x^{-2} + 3(f^{-1})'(x)}{(2x^{-1} + 3f^{-1}(x))^2}$

$g'(1) = -\frac{-2(1)^{-2} + 3(f^{-1})'(1)}{(2(1)^{-1} + 3f^{-1}(1))^2}$

$= -\frac{-2 + 3(\frac{1}{4})}{2^2}$

$= -\frac{-8+3}{16}$

$= \frac{5}{16}$

$k = 2$: $\qquad \dfrac{d^2}{dx^2}x^n = n(n-1)x^{n-2}.$

$k = 3$: $\qquad \dfrac{d^3}{dx^3}x^n = n(n-1)(n-2)x^{n-3}.$

$\vdots$

$k \le n$: $\qquad \dfrac{d^k}{dx^k}x^n = n(n-1)(n-2)\cdots(n-k+1)x^{n-k}.$

In particular, when $k = n$, we have $\dfrac{d^n}{dx^n}x^n = n(n-1)(n-2)\cdots(1) = n!$.

$k = n+1$: $\qquad \dfrac{d^{n+1}}{dx^{n+1}}x^n = \dfrac{d}{dx}(\dfrac{d^n}{dx^n}x^n) = \dfrac{d}{dx}(n!) = 0.$

Similarly, we have $\dfrac{d^k}{dx^k}x^n = 0$ for all $k \ge n+1$.

2.6 Parametric Equations

A curve defined by the parametric equations

$$x = f(t) \text{ and } y = g(t), \ (t \text{ is the parameter})$$

is differentiable at a point where $t = t_0$ if both f and g are differentiable at $t = t_0$. Usually we also assume $f'(t_0) \ne 0$ or $g'(t_0) \ne 0$.

By chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \implies \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{g'(t)}{f'(t)},$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt} = \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{f'(t)} = \frac{g''(t)f'(t) - g'(t)f''(t)}{f'(t)^3}.$$

Examples of parametric curves

- **Ellipses**

$$x = a\cos t + x_0 \text{ and } y = b\sin t + y_0,$$

where $a > 0, b > 0, x_0$ and y_0 are fixed constants and $0 \le t < 2\pi$.

- **Circles**
$$x = r\cos t + x_0 \text{ and } y = r\sin t + y_0,$$
where $r > 0, x_0$ and y_0 are fixed constants and $0 \le t < 2\pi$.

- **Hyperbolas**
$$x = a\sec t + x_0 \text{ and } y = b\tan t + y_0,$$
or
$$x = a\tan t + x_0 \text{ and } y = b\sec t + y_0,$$
where $a > 0, b > 0, x_0$ and y_0 are fixed constants and $-\pi \le t \le \pi$, $t \ne -\frac{\pi}{2}, \frac{\pi}{2}$.

Example 2.14. *For the parametric curve given by*
$$x = 2t - t^2, \; y = t - t^3,$$
find the point(s) on the curve at which the tangent is parallel to the line $2y = x + 2024$.

Ans: $(0,0)$ and $(\frac{5}{9}, \frac{8}{27})$.

Solution. We are given $x = 2t - t^2, \; y = t - t^3$.

First $\quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1-3t^2}{2-2t}.$

The gradient of the line $2y = x + 2024$ is $\dfrac{1}{2}$.

So we look for the value(s) of t such that $\dfrac{dy}{dx} = \dfrac{1}{2}$. Thus
$$\frac{1-3t^2}{2-2t} = \frac{1}{2} \iff 1 - 3t^2 = 1 - t \iff t(3t-1) = 0 \iff t = 0, \frac{1}{3}.$$
When $t = 0$, $(x, y) = (2 \cdot 0 - 0^2, 0 - 0^3) = (0, 0)$.

When $t = \dfrac{1}{3}$, $(x, y) = (2 \cdot \dfrac{1}{3} - (\dfrac{1}{3})^2, \dfrac{1}{3} - (\dfrac{1}{3})^3) = (\dfrac{5}{9}, \dfrac{8}{27})$

The corresponding points on the curve are $(0,0)$ and $(\dfrac{5}{9}, \dfrac{8}{27})$.

■

2.7 Miscellaneous examples

Functions of the form $f(x)^{g(x)}$.

The derivative of $y = f(x)^{g(x)}$ can be found by first finding the derivative of $\ln y$ and then solving for $\frac{dy}{dx}$.

Example 2.15. *Differentiate with respect to x.*

(a) $(x^2-1)^{4\tan x}$,

(b) $5^{x\ln x}$.

Ans: (a) $(x^2-1)^{4\tan x}\left(4\sec^2 x\ln(x^2-1)+\dfrac{8x\tan x}{x^2-1}\right)$,

(b) $5^{x\ln x}\ln 5(1+\ln x)$.

Solution. (a) Let $y=(x^2-1)^{4\tan x}$, so that

$$\ln y = \ln(x^2-1)^{4\tan x} = 4\tan x\cdot\ln(x^2-1).$$

Differentiating both sides with respect to x, we have

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= (4\tan x)'\ln(x^2-1)+\frac{(x^2-1)'}{(x^2-1)}(4\tan x)\\
&= 4\sec^2 x\ln(x^2-1)+\frac{2x}{x^2-1}\cdot 4\tan x\\
&= 4\sec^2 x\ln(x^2-1)+\frac{8x\tan x}{x^2-1}\\
\Longrightarrow\quad \frac{dy}{dx} &= y\cdot\left(4\sec^2 x\ln(x^2-1)+\frac{8x\tan x}{x^2-1}\right)\\
&= (x^2-1)^{4\tan x}\left(4\sec^2 x\ln(x^2-1)+\frac{8x\tan x}{x^2-1}\right).
\end{aligned}$$

(b) Let $y=5^{x\ln x}$. Then

$$\ln y = \ln(5^{x\ln x}) = x\ln x\cdot\ln 5.$$

Differentiating both sides with respect to x, we have

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \ln 5\cdot(x\ln x)'\\
&= \ln 5\cdot\left(x\cdot\frac{1}{x}+1\cdot\ln x\right)\\
&= \ln 5\cdot(1+\ln x)\\
\Longrightarrow\quad \frac{dy}{dx} &= y\cdot\ln 5\cdot(1+\ln x)\\
\Longrightarrow\quad \frac{d}{dx}(5^{x\ln x}) &= 5^{x\ln x}\ln 5\,(1+\ln x).
\end{aligned}$$

■

Change of base formula

$$\log_a x = \frac{\ln x}{\ln a}, a > 0 \text{ and } a \neq 1.$$

Example 2.16. *Differentiate* $\log_{(1+x^2)} \sqrt{x}$ *with respect to* x.

Ans: $\dfrac{1}{2(\ln(1+x^2))^2}\left(\dfrac{\ln(1+x^2)}{x} - \dfrac{2x\ln x}{1+x^2}\right)$.

Solution.

$$\begin{aligned}
&\frac{d}{dx}\log_{(1+x^2)}\sqrt{x} \\
= \ & \frac{d}{dx}\frac{\ln\sqrt{x}}{\ln(1+x^2)} = \frac{d}{dx}\frac{\ln x}{2\ln(1+x^2)} \\
= \ & \frac{1}{2}\cdot\frac{(\ln x)'\ln(1+x^2) - \ln x\cdot(\ln(1+x^2))'}{(\ln(1+x^2))^2} \\
= \ & \frac{1}{2}\cdot\frac{\frac{1}{x}\ln(1+x^2) - \ln x\cdot\frac{2x}{1+x^2}}{(\ln(1+x^2))^2} \\
= \ & \frac{1}{2(\ln(1+x^2))^2}\left(\frac{\ln(1+x^2)}{x} - \frac{2x\ln x}{1+x^2}\right).
\end{aligned}$$

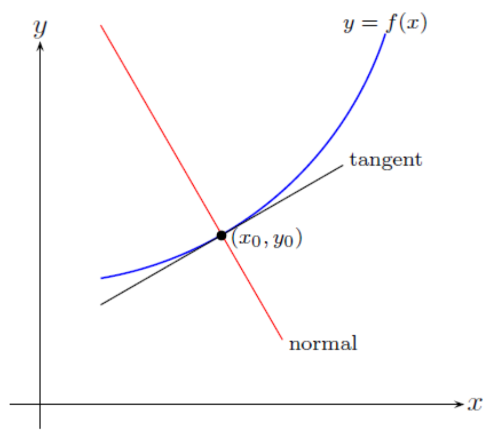
■

Chapter 3

Applications of Differentiation

Read Thomas' Calculus, Chapter 4.

3.1 Tangents and Normals



- The tangent at the point $(x_0, f(x_0))$ on the graph of a differentiable function f has equation
$$y - f(x_0) = m(x - x_0).$$
- The normal at the point $(x_0, f(x_0))$ on the graph of a differentiable function f has equation
$$y - f(x_0) = -\frac{1}{m}(x - x_0),$$
where $m = f'(x_0)$.

Example 3.1. *The curve C has equation $x^2 + y^2 + 3xy = 5$.*

(a) Find the equations of the tangent and normal at the point $(1, 1)$.

(b) Find (if any) the equations of the tangents that are parallel to the axes.

Ans: (a) tangent: $y = -x + 2$, normal: $y = x$.

Solution. (a) Differentiating the equation $x^2 + y^2 + 3xy = 5$ with respect to x, we have

$$2x + 2yy' + 3y + 3xy' = 0.$$

Thus $\quad y' = -\dfrac{2x + 3y}{3x + 2y}$.

At $(1,1)$, we have $y' = -\dfrac{5}{5} = -1$. Thus the gradient of the tangent at $(1,1)$ is -1 and the gradient of the normal is 1.

Therefore, the equation of tangent is $y - 1 = -(x - 1)$. That is $y = -x + 2$.

The equation of normal is $y - 1 = x - 1$. That is $y = x$.

(b) Recall that $y' = -\dfrac{2x + 3y}{3x + 2y}$.

If a tangent is parallel to the x-axis, then its gradient is 0. Thus we shall find the points on the curve such that $y' = 0$. That is $-\dfrac{2x + 3y}{3x + 2y} = 0 \iff 2x + 3y = 0 \iff y = -\dfrac{2x}{3}$.

Substituting this into the equation of the curve $x^2 + y^2 + 3xy = 5$, we obtain

$$x^2 + (-\frac{2x}{3})^2 + 3x(-\frac{2x}{3}) = 5.$$

That is $-\dfrac{5x^2}{9} = 5$, which has no solution.

■

If a curve is defined parametrically by

$$x = x(t) \text{ and } y = y(t),$$

then

- the equation of the tangent at the point where $t = t_0$ is
$$y - y(t_0) = m(x - x(t_0)),$$
- the equation of the normal at the point where $t = t_0$ is
$$y - y(t_0) = -\frac{1}{m}(x - x(t_0)),$$
  where m is the value of $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$ at $t = t_0$.

Example 3.2. *A curve is defined by*

$$x = t^2 - t \text{ and } y = (t+1)^2.$$

The tangent at the point A on the curve passes through $(-1,0)$ and $(1,8)$. Find the equation of the normal at the point A.

Ans: $y = -\frac{x}{4} + 4$.

Solution. The gradient of the line through $(-1,0)$ and $(1,8)$ is $\dfrac{8-0}{1-(-1)} = 4$.

Also $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{2t+2}{2t-1}$.

Solving $\dfrac{dy}{dx} = 4$, that is $\dfrac{2t+2}{2t-1} = 4$, gives $t = 1$.

Thus the point A on the curve has coordinates $(1^2 - 1, (1+1)^2) = (0,4)$.

The equation of the normal through A is

$$\frac{y-4}{x-0} = -\frac{1}{4} \quad \text{or equivalently,} \quad y = -\frac{x}{4} + 4.$$

■

3.2 Increasing and Decreasing Functions

Definition 3.1. *(a) The function f is* increasing *on an interval I if $f(x_2) > f(x_1)$ for $x_1, x_2 \in I$ with $x_2 > x_1$.*

(b) The function f is decreasing *on an interval I if $f(x_2) < f(x_1)$ for $x_1, x_2 \in I$ with $x_2 > x_1$.*

Theorem 3.1. *Let f be differentiable on (a,b) and continuous on $[a,b]$.*

(a) f is increasing on $[a,b]$ if $f'(x) > 0$ for all x in (a,b).

(b) f is decreasing on $[a,b]$ if $f'(x) < 0$ for all x in (a,b).

Thus if $f'(x) > 0 (< 0)$ on (a,b) except possibly at a finite number of points at which $f'(x) = 0$, then f is increasing (decreasing) on $[a,b]$.

Example 3.3. *Show that the function $f(x) = 3x^3 - 3e^{-x} - \frac{4}{x}$ is bijective (one-one) on the interval $(0,\infty)$ to its range $\mathbb{R}$.*

Solution. $f'(x) = 9x^2 + 3e^{-x} + \dfrac{4}{x^2} > 0$ for all $x \in (0,\infty)$.

Thus f is increasing on $(0,\infty)$. It follows that f is injective on $(0,\infty)$.

Note that $\lim_{x\to 0^+} 3x^3 - 3e^{-x} - \dfrac{4}{x} = -\infty$ and $\lim_{x\to+\infty} 3x^3 - 3e^{-x} - \dfrac{4}{x} = +\infty$.

As f is continuous on $(0,\infty)$, the range of f is $\mathbb{R}$.

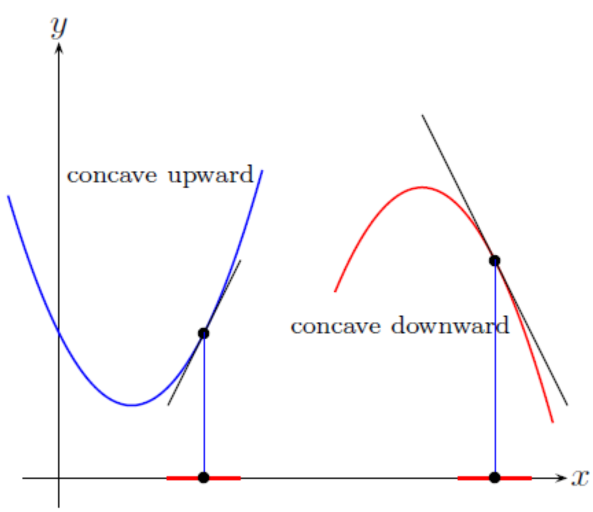
Thus $f(x)$ is bijective from $(0,\infty)$ onto its range $\mathbb{R}$.

■

3.3 Concave Upward and Concave Downward Functions

Definition 3.2. *(a) The graph of f is concave upward (downward) at $(c, f(c))$ if $f'(c)$ exists and there is an open interval I containing c such that for all $x \neq c$ in I, the point $(x, f(x))$ on the graph of f is above (below) the tangent line to the graph of f at $x = c$.*

(b) The graph of f is concave upward (downward) on (a,b) if it is concave upward (downward) at every point in (a,b).

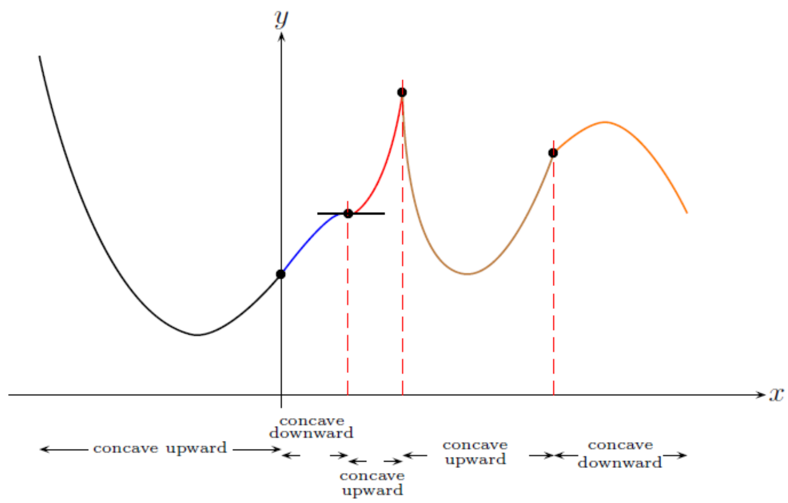


The following theorem gives a test for concavity.

Theorem 3.2. *Let f be differentiable on (a,b). Let $c \in (a,b)$.*

(a) If $f''(c) > 0$, then the graph of f is concave upward at $(c, f(c))$.

(b) If $f''(c) < 0$, then the graph of f is concave downward at $(c, f(c))$.



Note that the converse of the theorem is not true. For example, if $f(x) = x^4$, then the graph of f is concave upward at $(0,0)$, but $f''(0) = 0$.

Definition 3.3. *A point $(c, f(c))$ where the graph of a function f has a tangent line and where the concavity changes is called a point of inflection.*

Point of inflection: change of concavity

Theorem 3.3. *Let f be differentiable on (a,b). Let $c \in (a,b)$. If $(c, f(c))$ is a point of inflection of the graph of f and $f''(c)$ exists, then $f''(c) = 0$.*

Example 3.4. *Determine the intervals on which the function*

$$f(x) = -2x^3 + 15x^2 - 24x + 7$$

is

(i) increasing, (ii) decreasing, and its graph is (iii) concave upward (iv) concave downward. (v) Find the point(s) of inflection of the graph of f.

Ans: (i) $[1,4]$, (ii) $(-\infty,1] \cup [4,\infty)$, (iii) $(-\infty, \frac{5}{2})$, (iv) $(\frac{5}{2}, \infty)$, (v) $(\frac{5}{2}, \frac{19}{2})$.

Solution. $f'(x) = -6x^2 + 30x - 24 = -6(x-1)(x-4)$.

x	$x<1$	$x=1$	$1<x<4$	$x=4$	$4<x$
$f'(x)$	$-$	0	$+$	0	$-$

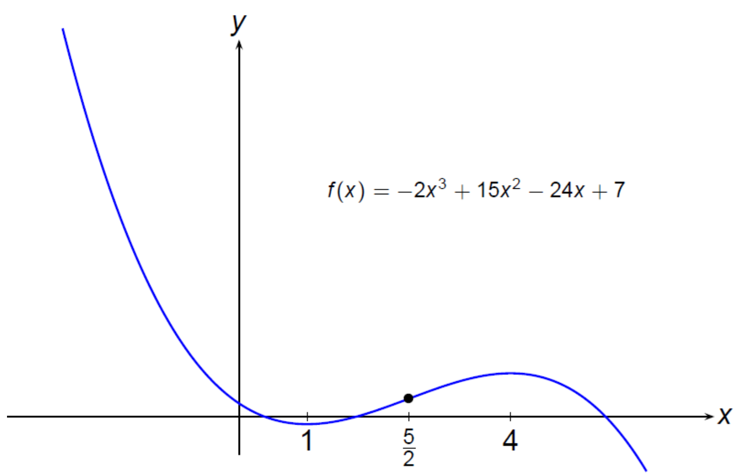
Thus f is increasing on $[1,4]$, and decreasing on $(-\infty,1]\cup[4,\infty)$,

We have $f''(x) = -12x+30 = -12(x-\frac{5}{2})$.

x	$x<\frac{5}{2}$	$x=\frac{5}{2}$	$\frac{5}{2}<x$
$f''(x)$	$+$	0	$-$

Thus the graph of f is concave upward on $(-\infty,\frac{5}{2})$ and concave downward on $(\frac{5}{2},\infty)$.

There is a change of concavity of the graph of f at $(\frac{5}{2},\frac{19}{2})$. Thus $(\frac{5}{2},\frac{19}{2})$ is a point of inflection of the graph of f.



Summarizing, (i) increasing on $[1,4]$, (ii) decreasing on $(-\infty,1]$, $[4,\infty)$, (iii) concave upward on $(-\infty,\frac{5}{2})$, (iv) concave downward $(\frac{5}{2},\infty)$, (v) inflection point at $x=\frac{5}{2}$.

3.4 Related Rates

Let $y = f(x)$ and let x and y be functions of a third variable t that represents, for example, time. By the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Example 3.5. *Water is flowing at a rate of $100\ cm^3$ per second into an inverted conical flask of height 16cm and base radius 4cm. At the instant when the height of water level is 12cm, the water level is rising at the rate of 3cm per sec. Calculate the rate at which water is leaking from the flask.*

Ans: 15.18 cm^3 per second.

Solution. By similar triangles, $\frac{r}{h} = \frac{4}{16} \Rightarrow r = \frac{h}{4}$. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{h}{4})^2 h.$$

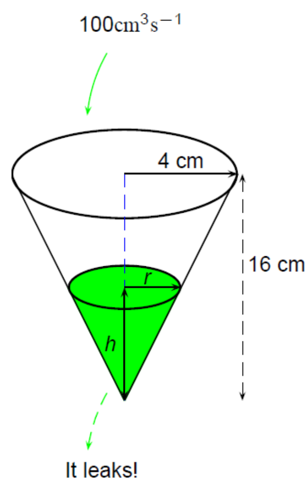
Thus $V = \frac{\pi}{48}h^3$. Differentiating with respect to t,

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \frac{\pi}{48}\cdot 3h^2\cdot\frac{dh}{dt}.$$

At $h = 12$, $\frac{dh}{dt} = 3$ cm per second. Thus

$\frac{dV}{dt} = \frac{\pi}{48}\times 3\times(12)^2\times 3 = 27\pi = 84.82$ cm^3 per second.

Therefore, leaking rate is $100 - 27\pi = 15.18$ cm^3 per second.



∎

Exercise 3.1. *A particle is moving horizontally in the x–y plane along the line $y = 5$ in such a way that its distance from the origin $(0,0)$ is increasing at a rate of 1 unit per sec. Calculate the rate at which the particle is moving horizontally at the instant when it is 13 units from $(0,0)$.*

Ans: $\frac{13}{12}$ unit per sec.

3.5 Maximum and Minimum Values

Definition 3.4. *(Absolute Extrema) A function f has an*

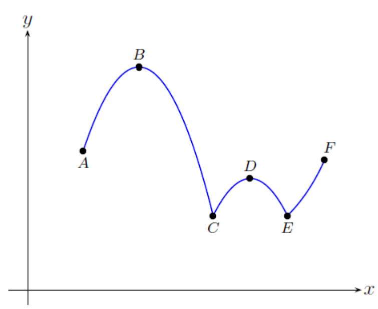
(a) absolute/global maximum at $x = c$ if $f(x) \leq f(c)$ for all x in the domain of f. (Such point $x = c$ is called an absolute/global maximum point of f, and the value $f(c)$ is called the absolute/global maximum value of f.)

(b) absolute/global minimum at $x = c$ if $f(x) \geq f(c)$ for all x in the domain of f. (Such point $x = c$ is called an absolute/global minimum point of f, and the value $f(c)$ is called the absolute/global minimum value of f.)

Definition 3.5. *(Local Extrema) A function f defined on some interval I has a*

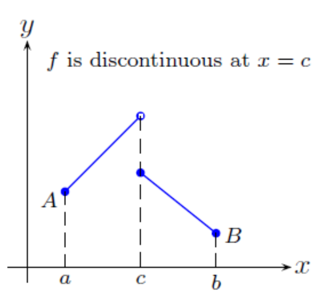
(a) local/relative maximum at $x = c$ if $f(x) \leq f(c)$ for x in some open interval containing $x = c$. (Such point $x = c$ is called a local/relative maximum point of f, and the value $f(c)$ is called a local/relative maximum value of f.)

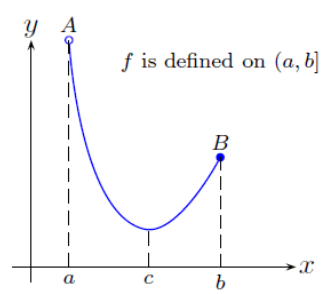
(b) local/relative minimum at $x = c$ if $f(x) \geq f(c)$ for x in some open interval containing $x = c$. (Such point $x = c$ is called a local/relative minimum point of f, and the value $f(c)$ is called a local/relative minimum value of f.)



Theorem 3.4. *(Extreme Value Theorem) If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum at some points in $[a, b]$.*

Question. What if f is not continuous or if the domain is not a closed interval of the form $[a, b]$?





In such cases, the conclusion of the Extreme Value Theorem may not hold. (In the two diagrams above, none of the two functions has an absolute maximum on its domain.)

Theorem 3.5. *If f is differentiable on an open interval containing $x = c$ and f has a local extremum (i.e., local maximum or local minimum) at $x = c$, then $f'(c) = 0$.*

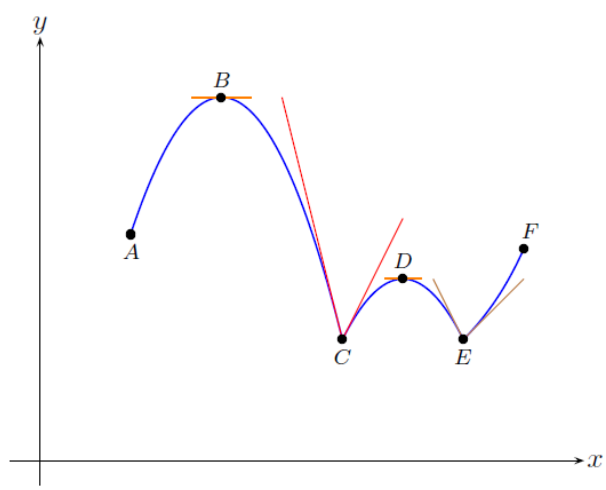
Definition 3.6. *(Critical Point) A number c in the domain of a function f is a* **critical point** *of f if the following 2 conditions hold:*

(i) it is not an end-point,

(ii) either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 3.6. *If f has a local minimum/maximum at $x = c$, then c is a critical point of f.*

However, the converse of this result is not true in general.



From the above results, we conclude that an absolute extremum occurs either at the end point or at a critical point.

Hence, to find absolute extrema of a continuous function f defined on $[a, b]$, we

(1) find the values of f at all critical points of f on (a, b),

(2) find the values of $f(a)$ and $f(b)$.

The largest and smallest values from steps (1) and (2) are the absolute maximum value and absolute minimum value of f respectively.

Remark. If the function f is continuous on $[a, b]$ and is considered as defined on $(a, b]$ and the largest (smallest) value of f obtained from steps 1 and 2 occurs only at $x = a$, then f has no absolute maximum (minimum) on $(a, b]$.

Example 3.6. *Find the absolute maximum and minimum values of*

$$g(x) = \frac{x}{x^2+1}$$

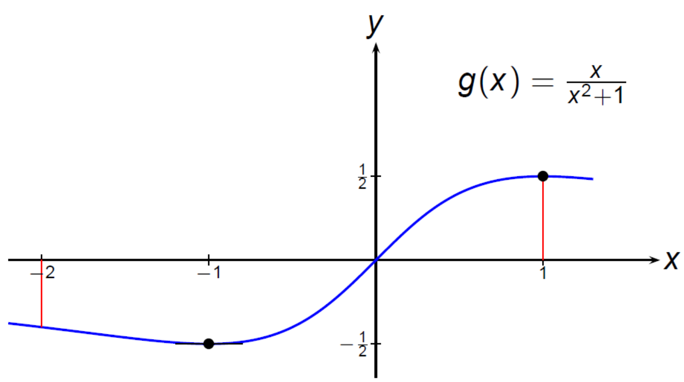
on (a) $[-2, 1]$, *(b)* $(-2, 1)$, *(c)* $(-1, 1]$.

Ans: (a) absolute maximum value $= \frac{1}{2}$, absolute minimum value $= -\frac{1}{2}$, (b) no absolute maximum value, absolute minimum value $= -\frac{1}{2}$, (c) absolute maximum value $= \frac{1}{2}$, no absolute minimum value.

Solution. $g'(x) = \dfrac{1 \cdot (x^2+1) - x(2x)}{x^2+1} = \dfrac{1-x^2}{x^2+1} = \dfrac{(1-x)(1+x)}{x^2+1}$.

$g'(x) = 0 \Leftrightarrow x = -1, 1$. Thus g has a critical point at $x = -1, 1$.

x	-2	-1	1
$g(x)$	$-\frac{2}{5}$	$-\frac{1}{2}$	$\frac{1}{2}$



(a) on $[-2, 1]$: absolute maximum value $= \dfrac{1}{2}$, absolute minimum value $= -\dfrac{1}{2}$,

(b) on $(-2, 1)$: no absolute maximum value, absolute minimum value $= -\dfrac{1}{2}$,

(c) on $(-1, 1]$: absolute maximum value $= \dfrac{1}{2}$, no absolute minimum value.

■

Exercise 3.2. *Find the absolute maximum and minimum values of*

$$h(x) = x^{5/3} - x^{2/3}$$

on (a) $[-1, 8]$, *(b)* $(-1, 1)$.

Ans: (a) absolute maximum value = 28, absolute minimum value = −2.

Exercise 3.3. *Let* $f(x) = \begin{cases} |x| & -5 \le x < 2 \\ x^2 - 6x + 10 & 2 \le x \le 4 \end{cases}$. *Find*

(a) the critical points of f,

(b) the absolute maximum and minimum values of f.

Ans: (a) critical points at $x = 0, 2, 3,$ (b) absolute maximum value = 5, absolute minimum = 0.

Theorem 3.7. *(First Derivative Test for Absolute Extrema) Let f be differentiable on an open interval containing a critical point c except possibly at c and f is continuous at c.*

(1) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then f has an absolute maximum at c.

(2) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then f has an absolute minimum at c.

Example 3.7. *Find the absolute maximum value of the function*

$$h(x) = 6x^{1/2} - x^{3/2}$$

on the interval $(1, \infty)$.

Ans: absolute maximum value $= 4\sqrt{2}$.

Solution. $h(x) = 6x^{\frac{1}{2}} - x^{\frac{3}{2}}$. Then $h'(x) = 3x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{3(2-x)}{2x^{\frac{1}{2}}}$.

Thus h has 1 critical point in the interval $(1, \infty)$ given by $x = 2$.

When $1 < x < 2$, $h'(x) > 0$. When $x > 2$, $h'(x) < 0$.

Thus by the first derivative test for absolute extrema, h has an absolute maximum on the interval $(1, \infty)$ given by $x = 2$, and the maximum value of h is $h(2) = 6 \cdot 2^{\frac{1}{2}} - 2^{\frac{3}{2}} = 6\sqrt{2} - 2\sqrt{2} = 4\sqrt{2}$.

■

Theorem 3.8. *(First Derivative Test for Local Extrema) Let f be differentiable on an open interval containing a critical point c except possibly at c and f is continuous at c.*

(1) If f' changes from positive to negative at $x = c$, then f has a local maximum at c.

(2) If f' changes from negative to positive at $x = c$, then f has a local minimum at c.

(3) If f' does not change sign at $x = c$, then f has no local extremum at c.

Example 3.8. *Let $f(x) = 3x^4 - 8x^3$.*

(a) Find the critical points of f.

(b) Determine whether a local maximum or local minimum or neither occurs at each of these points.

Ans: (a) critical point at $x = 0, 2$, (b) f has a local minimum at $x = 2$.

Solution. It is given that $f(x) = 3x^4 - 8x^3$. Then $f'(x) = 12x^3 - 24x^2 = 12x^2(x-2)$.

Thus $f'(x) = 0 \iff 12x^2(x-2) = 0 \iff x = 0, 2$.

Hence f has critical points at $x = 0, 2$.

x	$x < 0$	0	$0 < x < 2$	2	$2 < x$
$f'(x)$	$-$	0	$-$	0	$+$

By the first derivative test (for local extrema), f has no local extremum at $x = 0$, and f has a local minimum at $x = 2$.

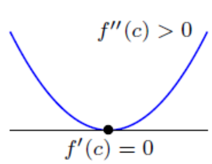
■

Theorem 3.9. *(Second Derivative Test) Let f be a twice differentiable function defined in an open interval containing c.*

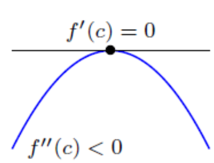
(1) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(2) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(3) No conclusion can be drawn if $f''(c) = 0$.



a local minimum at $x = c$



a local maximum at $x = c$

Remark. The functions x^4, $-x^4$, x^3, has a local min, max, neither a max nor a min at $x = 0$, respectively, but have 0 second derivative at $x = 0$.

Example 3.9. *Let $f(x) = (x-4)e^x$.*

(a) Find the critical points of f.

(b) Determine whether a local maximum or local minimum or neither occurs at each of these points.

Ans: (a) critical point at $x = 3$, (b) f has a local minimum at $x = 3$.

Solution. It is given that $f(x) = (x-4)e^x$. Then $f'(x) = 1 \cdot e^x + (x-4)e^x = (x-3)e^x$.

Thus $f'(x) = 0 \iff (x-3)e^x = 0 \iff x = 3$. Hence f has a critical point at $x = 3$.

Then $f''(x) = 1 \cdot e^x + (x-3)e^x = (x-2)e^x$. Thus $f''(3) = (3-2)e^3 = e^3 > 0$.

By the second derivative test, f has a local minimum at $x = 3$.

■

Exercise 3.4. *Let* $f(x) = \begin{cases} 2x - x^2, & 0 \le x < 2 \\ (x-2)^2, & x \ge 2 \end{cases}$.

(a) Find the critical points of f.

(b) Determine whether a local maximum or local minimum or neither occurs at each of these points.

Ans: (a) critical points at $x = 1$ and $x = 2$, (b) f has a local maximum at $x = 1$ and a local minimum at $x = 2$.

3.6 Applied Maximum and Minimum Problems

Example 3.10. *The number of viewers of a YouTube video at time t is given by*

$$V(t) = \frac{10^6 t}{(t^2 + 400)^{\frac{3}{2}}}, \quad t > 0.$$

Find the value of t when the number of viewers V is at its peak. Justify your answer.

Ans: $t = 10\sqrt{2}$.

Solution. $V(t) = \dfrac{10^6 t}{(t^2+400)^{\frac{3}{2}}}$, $t > 0$.

The domain of V is given to be $t > 0$. To find the time t when the number of viewers is at its peak, it is the same as to find the time t at which the function V has an absolute maximum.

$$\begin{aligned}\frac{dV}{dt} &= \frac{10^6[(t^2+400)^{\frac{3}{2}}\cdot 1 - t\cdot\frac{3}{2}(t^2+400)^{\frac{1}{2}}2t]}{(t^2+400)^3}\\ &= \frac{10^6(t^2+400)^{\frac{1}{2}}[(t^2+400)-3t^2]}{(t^2+400)^3}\\ &= \frac{10^6(400-2t^2)}{(t^2+400)^{\frac{5}{2}}} = \frac{10^6\cdot 2(200-t^2)}{(t^2+400)^{\frac{5}{2}}} = \frac{10^6\cdot 2(10\sqrt{2}+t)(10\sqrt{2}-t)}{(t^2+400)^{\frac{5}{2}}}.\end{aligned}$$

Thus $\dfrac{dV}{dt} = 0 \iff t = \pm 10\sqrt{2}$.

Since $t > 0$, the only critical point of $V(t)$ is at $t = 10\sqrt{2}$.

When $t < 10\sqrt{2}$, $\dfrac{dV}{dt} > 0$. When $t > 10\sqrt{2}$, $\dfrac{dV}{dt} < 0$.

Thus V has an absolute maximum at $t = 10\sqrt{2}$.

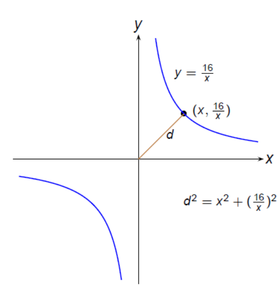
■

Exercise 3.5. *Determine the coordinates of the points on the curve $y = \frac{16}{x}$ that are closest to the origin. With the aid of a graph, justify that your answer indeed gives the shortest distance.*

Ans: $(4,4)$, $(-4,-4)$.

Solution Let d be the distance from $(x, \dfrac{16}{x})$ to the origin.

By Pythagoras' theorem, $d^2 = x^2 + (\dfrac{16}{x})^2$.



Note that d attains a minimum value L at (x_0, y_0) if and only if d^2 attains a minimum value L^2 at (x_0, y_0).

Thus it suffices to minimize d^2. Let $D = d^2$. we have

$$D = x^2 + (\frac{16}{x})^2.$$

Note that $x \neq 0$.

Hence,

$$\frac{dD}{dx} = 2x - \frac{2 \cdot 16^2}{x^3} = \frac{2(x^4 - 4^4)}{x^3}.$$

$$\frac{dD}{dx} = 0 \iff \frac{2(x^4 - 4^4)}{x^3} = 0 \iff x^4 = 4^4 \iff x = \pm 4.$$

When $0 < x < 4$, $\frac{dD}{dx} < 0$. When $4 < x$, $\frac{dD}{dx} > 0$.

Thus D attains the absolute minimum at $x = 4$ on $(0, \infty)$.

Similarly, when $x < -4$, $\frac{dD}{dx} < 0$. When $-4 < x < 0$, $\frac{dD}{dx} > 0$.

Thus D attains the absolute minimum at $x = -4$ on $(-\infty, 0)$.

The 2 points are $(4, 4)$ and $(-4, -4)$.

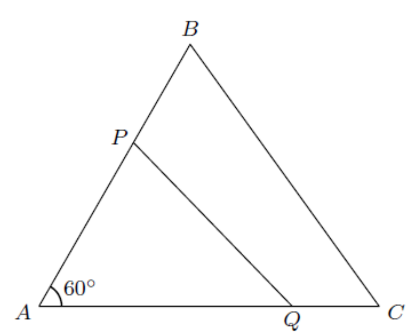
■

Exercise 3.6. *A triangular plot ABC in which* $\angle A = 60°$ *and AB is* 80 *metres long is to be divided into two plots of equal areas by a fence built along the line PQ. The fence costs* \$10 *per metre. Let the lengths of AC and AP be* $10b$ *and* $10x$ *respectively.*

(a) Show that the length of PQ is $10z$, *where* $z^2 = x^2 + \frac{16b^2}{x^2} - 4b$.

(b) Show that $4 \leq x \leq 8$.

(c) Determine, to the nearest dollar, the minimum cost of fencing, and the corresponding value of x, *when (i)* $b = 9$, *(ii)* $b = 25$.

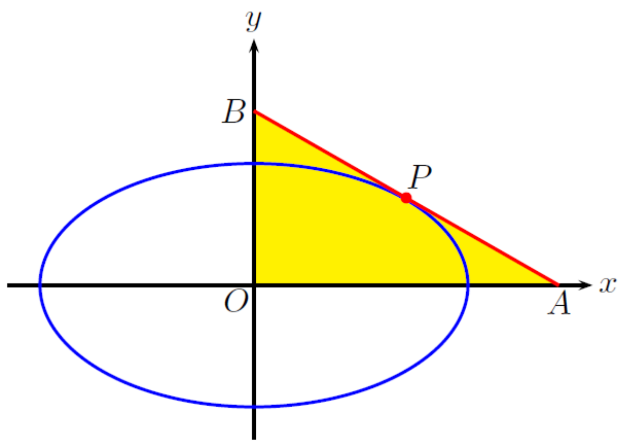


Ans: (a) \$600, *(b)* \$1096.59.

Exercise 3.7. *A bus carries* 60 *passengers each day from a train station to a shopping mall. It costs* \$1.50 *per passenger to ride the bus. Research reveals that* 4 *more (fewer) people would ride the bus for each* 5 *cents decrease (increase) in bus fare. Determine the bus fare (to the nearest cent) per passenger that will maximise revenue of the bus operator.*

Ans: \$1.12.

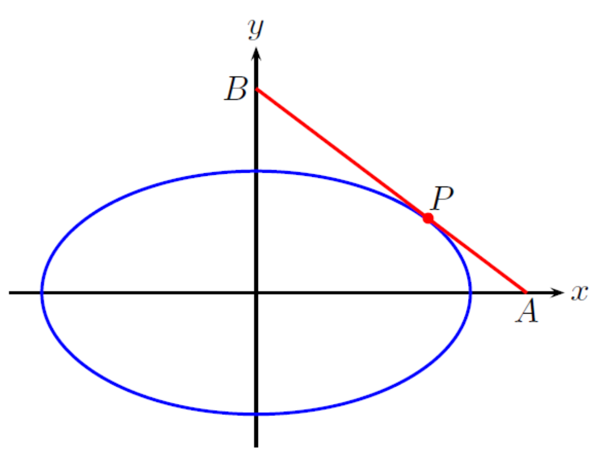
Exercise 3.8. *Find the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the first quadrant such that the triangle bounded by the axes of the ellipse and the tangent to the ellipse at the point P has the least area.*



$\triangle OAB$ has the least area.

Ans: $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$.

Exercise 3.9. *Find the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the first quadrant such that the line segment tangent to the ellipse at the point P with endpoints on the axes of the ellipse has the least length.*



AB has the shortest length.

Ans: $\left(\frac{a^{\frac{3}{2}}}{(a+b)^{\frac{1}{2}}}, \frac{b^{\frac{3}{2}}}{(a+b)^{\frac{1}{2}}}\right)$.

3.7 L'Hôpital's Rule

Theorem 3.10. *Let f and g be differentiable at all points in some open interval containing $x = c$ (except possibly at c). If $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ or $\lim_{x\to c} f(x) = \infty = \lim_{x\to c} g(x)$, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists or equals ∞ or $-\infty$.

The result also holds

(1) for limits at infinity, i.e. $c = \pm\infty$.

(2) for one-sided limits.

Sketch of Proof. Here we give a proof in the special case when $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$, and the functions f, g, f' and g' are continuous at $x = c$, and $g'(c) \neq 0$. Then

$$f(c) = \lim_{x\to c} f(x) = 0 \quad \text{and} \quad g(c) = \lim_{x\to c} g(x) = 0.$$

Thus

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{x\to c} \frac{f(x)-0}{g(x)-0} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} \\
&= \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} \\
&= \frac{f'(c)}{g'(c)} \quad \text{(by definition of } f'(c) \text{ and } g'(c) \text{ and noting that } g'(c) \neq 0) \\
&= \frac{\lim_{x\to c} f'(x)}{\lim_{x\to c} g'(x)} \quad \text{(by continuity of } f', g' \text{ at } x = c) \\
&= \lim_{x\to c} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

■

Example 3.11. *Evaluate*

(a) $\lim_{x\to\infty} \dfrac{3x^2 + x\ln x}{x^2 + 2\ln x}$

(b) $\lim_{x\to 0}\left(2\csc 2x - \dfrac{1}{x}\right)$

(c) $\lim_{x\to 0^+} x\ln x$

Ans: (a) 3, (b) 0, (c) 0.

Solution. (a) We apply L'Hôpital's Rule twice.

$$\lim_{x\to\infty} \frac{3x^2 + x\ln x}{x^2 + 2\ln x}$$

$$= \lim_{x\to\infty} \frac{6x + \ln x + 1}{2x + \dfrac{2}{x}} \quad \text{(by L'Hôpital's Rule)}$$

$$= \lim_{x\to\infty} \frac{6 + \dfrac{1}{x} + 0}{2 - \dfrac{2}{x^2}} \quad \text{(by L'Hôpital's Rule)}$$

$$= \frac{6+0}{2-0} = 3.$$

(b) We apply L'Hôpital's Rule twice.

$$\lim_{x\to 0}\left(2\csc 2x - \frac{1}{x}\right)$$

$$= \lim_{x\to 0} \frac{2x - \sin(2x)}{x\sin(2x)}$$

$$= \lim_{x\to 0} \frac{2 - 2\cos(2x)}{\sin(2x) + 2x\cos(2x)} \quad \text{(by L'Hôpital's Rule)}$$

$$= \lim_{x\to 0} \frac{0 + 4\sin(2x)}{2\cos(2x) + (2\cos(2x) - 4x\sin(2x))} \quad \text{(by L'Hôpital's Rule)}$$

$$= \frac{0}{2+2-0} = 0.$$

(c) Exercise.

■

There are 3 indeterminate forms of the following types:

(1) 0^0,   (2) ∞^0,   (3) 1^∞.

Example 3.12.

0^0: $\lim\limits_{x\to0^+} x^x = \lim\limits_{x\to0^+} e^{\ln(x^x)} = \lim\limits_{x\to0^+} e^{x\ln x} = e^{\lim\limits_{x\to0^+} x\ln x} = e^{\lim\limits_{x\to0^+} \frac{\ln x}{x^{-1}}} = e^{\lim\limits_{x\to0^+} \frac{x^{-1}}{-x^{-2}}} = e^{\lim\limits_{x\to0^+} -x} = e^0 = 1.$

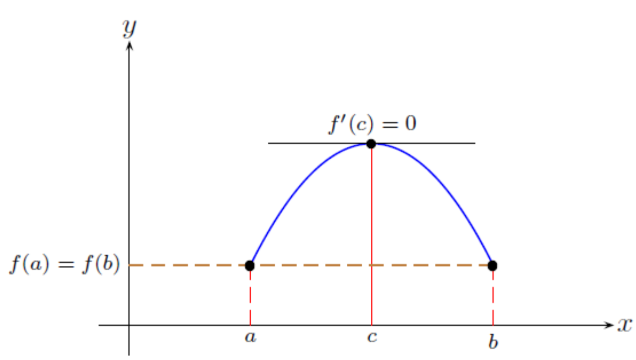
∞^0: $\lim\limits_{x\to0^+} (\frac{2}{x})^x = \lim\limits_{x\to0^+} \frac{2^x}{x^x} = \frac{2^0}{1} = 1.$

1^∞: $\lim\limits_{x\to0^+} (1+x)^{\frac{1}{x}} = \lim\limits_{x\to0^+} e^{\ln((1+x)^{\frac{1}{x}})} = \lim\limits_{x\to0^+} e^{\frac{\ln(1+x)}{x}} = e^{\lim\limits_{x\to0^+} \frac{\ln(1+x)}{x}} = e^{\lim\limits_{x\to0^+} \frac{\frac{1}{1+x}}{1}} = e^{\frac{\frac{1}{1+0}}{1}} = e.$

Remark. *In the last example, we have* $\lim\limits_{x\to0^+} (1+x)^{\frac{1}{x}} = e$ *(note that the value of the limit is not* 1 *or* ∞*). When* x *is positive and gets closer to* 0*, the expression* $1+x$ *becomes smaller and gets closer to* 1*, which tends to make the expression* $(1+x)^{\frac{1}{x}}$ *smaller. On the other hand, the exponent* $\frac{1}{x}$ *gets larger and tends to make the expression* $(1+x)^{\frac{1}{x}}$ *larger. As it turns out, the combined effect of these two opposing forces is that* $\lim\limits_{x\to0^+} (1+x)^{\frac{1}{x}} = e \approx 2.718$*, which is somewhere between* 1 *and* ∞*.*

3.8 Rolle's Theorem and Mean Value Theorem

Theorem 3.11. *(Rolle's Theorem) Let* f *be continuous on* $[a,b]$ *and differentiable on* (a,b)*. If* $f(a) = f(b)$*, then there is at least one number* c *in* (a,b) *such that* $f'(c) = 0$*.*



Example 3.13. *Use Rolle's Theorem to prove that the equation*

$$\ln x + 2x = 3$$

has at most one positive solution.

Solution. Let $f(x) = \ln x + 2x$. Note that f is a differentiable function defined for $x > 0$. Also $f'(x) = \dfrac{1}{x} + 2$. Suppose $f(x) = 3$ has two positive solutions a and b with $0 < a < b$. That is $f(a) = f(b) = 3$. By Rolle's Theorem applied to f on the interval $[a,b]$, there exists $c \in (a,b)$ such that $f'(c) = \dfrac{1}{c} + 2 = 0$, which is not true as $c > a > 0$. This contradiction shows that $f(x) = 3$ cannot have two positive solutions. In other words, $\ln x + 2x = 3$ has at most one positive solution.

Remark. Note that

$$f(1) = \ln(1) + 2 = 2 \quad \text{and} \quad f(e) = \ln e + 2e = 1 + 2e > 3.$$

Thus we have

$$f(1) < 3 < f(e).$$

Also f is differentiable and thus continuous on $[1,e]$. Thus by Intermediate Value Theorem (see Theorem 1.7 of Lecture Notes), there exists a number $c \in [1,3]$ such that $f(c) = 3$.

Since we have also proved that the equation $f(x) = 3$ has at most one positive solution, we knows that the equation $f(x) = 3$ has exactly one positive solution.

■

Exercise 3.10. *Use Rolle's Theorem to prove that the equation*

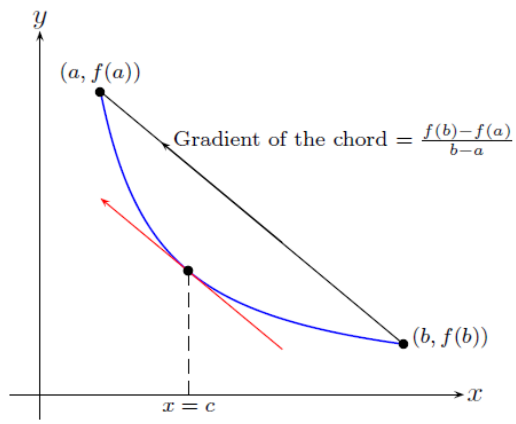
$$2e^x + x^2 + 3x = 0$$

has at most two real solutions.

Rolle's Theorem can be used to prove the following result.

Theorem 3.12. *(Mean Value Theorem) Let f be continuous on $[a,b]$ and differentiable on (a,b). Then, there is at least one number c in (a,b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Theorem 3.13. *Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) > 0(< 0)$ for all $x \in (a,b)$, then f is increasing (decreasing) on $[a,b]$.*

Proof. Let's suppose $f'(x) > 0$ for all $x \in (a,b)$. Let $x_1, x_2 \in [a,b]$ and $x_1 < x_2$. Thus f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) as f is continuous on $[a,b]$ and differentiable on (a,b). By mean value theorem applied to f on $[x_1, x_2]$, there is a number c in (x_1, x_2) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

Since $f'(c) > 0$ and $x_2 > x_1$, we have $f(x_2) > f(x_1)$. That is f is increasing on $[a,b]$.

■

Exercise 3.11. *Use the Mean Value Theorem to prove that for all real numbers x and y,*

$$|\cos x - \cos y| \leq |x - y|.$$

Exercise 3.12. *Use the Mean Value Theorem to show that if $f'(x) = 0$ for all x in (a,b), then f is a constant function on (a,b).*

Exercise 3.13. *Use the Mean Value Theorem to prove that there exists a real number $\theta \in (\frac{\pi}{5}, \frac{\pi}{4})$ such that*

$$\sin\frac{\pi}{5} = \sin\frac{\pi}{4} - \frac{\pi}{20}\cos\theta.$$

Deduce that $\sin\frac{\pi}{5} < \frac{\sqrt{2}}{40}(20 - \pi)$.

Chapter 4

Integrals

Read Thomas' Calculus, Chapter 5 and 8.

4.1 Antiderivatives

Definition 4.1. *F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.*

Theorem 4.1. *(1) If F is an antiderivative of f on an interval I, then so is $F + C$ for any constant C. Furthermore, any antiderivative of f on I is of the form $F + C$ for some constant C. This can be expressed as*

$$\int f(x)\,dx = F(x) + C.$$

$\int f(x)\,dx$ is called an indefinite integral.

(2) Let α and β be any constants. Then

$$\int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx.$$

Example 4.1. *An anti-derivative of x^n is $\frac{1}{n+1}x^{n+1}$, where $n \neq -1$ as $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = x^n$. Thus*

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

Example 4.2. *Find all the anti-derivatives of $2x - \cos 3x$.*

Solution. $\int 2x - \cos 3x\,dx = x^2 - \frac{1}{3}\sin 3x + C$.

4.2 Standard Integrals

1. $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C \quad (n \neq -1)$
2. $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + C$
3. $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$
4. $\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$
5. $\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$
6. $\int \tan(ax+b) \, dx = \frac{1}{a} \ln|\sec(ax+b)| + C$
7. $\int \sec(ax+b) \, dx = \frac{1}{a} \ln|\sec(ax+b) + \tan(ax+b)| + C$
8. $\int \csc(ax+b) \, dx = -\frac{1}{a} \ln|\csc(ax+b) + \cot(ax+b)| + C$
9. $\int \cot(ax+b) \, dx = -\frac{1}{a} \ln|\csc(ax+b)| + C$
10. $\int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$
11. $\int \csc^2(ax+b) \, dx = -\frac{1}{a} \cot(ax+b) + C$
12. $\int \sec(ax+b)\tan(ax+b) \, dx = \frac{1}{a} \sec(ax+b) + C$
13. $\int \csc(ax+b)\cot(ax+b) \, dx = -\frac{1}{a} \csc(ax+b) + C$
14. $\int \frac{1}{a^2+(x+b)^2} \, dx = \frac{1}{a} \tan^{-1}(\frac{x+b}{a}) + C$
15. $\int \frac{1}{\sqrt{a^2-(x+b)^2}} \, dx = \sin^{-1}(\frac{x+b}{a}) + C$
16. $\int \frac{-1}{\sqrt{a^2-(x+b)^2}} \, dx = \cos^{-1}(\frac{x+b}{a}) + C$
17. $\int \frac{1}{a^2-(x+b)^2} \, dx = \frac{1}{2a} \ln\left|\frac{x+b+a}{x+b-a}\right| + C$
18. $\int \frac{1}{(x+b)^2-a^2} \, dx = \frac{1}{2a} \ln\left|\frac{x+b-a}{x+b+a}\right| + C$
19. $\int \frac{1}{\sqrt{(x+b)^2+a^2}} \, dx = \ln\left|(x+b) + \sqrt{(x+b)^2+a^2}\right| + C$
20. $\int \frac{1}{\sqrt{(x+b)^2-a^2}} \, dx = \ln\left|(x+b) + \sqrt{(x+b)^2-a^2}\right| + C$
21. $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$
22. $\int \sqrt{x^2-a^2} \, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2-a^2}| + C$

Example 4.3. *Let $a \neq 0$. Show that $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$, for $x \neq -\frac{b}{a}$.*

Solution. We have to show $\frac{d}{dx}(\frac{1}{a}\ln|ax+b|) = \frac{1}{ax+b}$.

For $x > -\frac{b}{a}$, we have $|ax+b| = ax+b$. Thus

$$\frac{d}{dx}(\frac{1}{a}\ln|ax+b|) = \frac{d}{dx}(\frac{1}{a}\ln(ax+b)) = \frac{1}{ax+b}.$$

For $x < -\frac{b}{a}$, we have $|ax+b| = -(ax+b)$. Thus

$$\frac{d}{dx}(\frac{1}{a}\ln|ax+b|) = \frac{d}{dx}(\frac{1}{a}\ln(-(ax+b))) = \frac{1}{a}\cdot\frac{-a}{-(ax+b)} = \frac{1}{ax+b}.$$

∎

Example 4.4. *Find*

(a) $\int \frac{1}{\sqrt{x^2-4x+29}}\,dx$

(b) $\int \frac{1}{\sqrt{3+6x-9x^2}}\,dx$

Solution. (a) $\int \frac{1}{\sqrt{x^2-4x+29}}\,dx = \int \frac{1}{\sqrt{(x-2)^2+5^2}}\,dx$

$$= \ln\left|(x-2)+\sqrt{(x-2)^2+5^2}\right| + C$$
$$= \ln\left|(x-2)+\sqrt{x^2-4x+29}\right| + C.$$

(b) $\int \frac{1}{\sqrt{3+6x-9x^2}}\,dx = \int \frac{1}{\sqrt{2^2-(3x-1)^2}}\,dx$

$$= \frac{1}{3}\int \frac{1}{\sqrt{(\frac{2}{3})^2-(x-\frac{1}{3})^2}}\,dx$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{x-\frac{1}{3}}{\frac{2}{3}}\right) + C$$
$$= \frac{1}{3}\sin^{-1}(\frac{3x-1}{2}) + C.$$

∎

Exercise 4.1. *Find*

(a) $\int \left(\frac{3x-1}{2x+1}\right)^2 dx$

(b) $\int \frac{(2e^{2x-1}-e^{-x})^2}{e^{x+1}}\,dx$

Ans: (a) $\frac{9x}{4} - \frac{15}{4}\ln|2x+1| - \frac{25}{8(2x+1)} + C$, (b) $\frac{4}{3}e^{3x-3} - 4e^{-2}x - \frac{1}{3}e^{-3x-1} + C$.

Trigonometric Identities Useful for Integration

1. $\sec^2 x - 1 = \tan^2 x$
2. $\csc^2 x - 1 = \cot^2 x$
3. $\sin A \cos A = \frac{1}{2}\sin 2A$
4. $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$
5. $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$
6. $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$
7. $\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$
8. $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$
9. $\sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$

Example 4.5. *Find* $\int \cos\frac{x}{6}\sin\frac{x}{3}\,dx$

Solution.

$$\begin{aligned}
\int \cos\tfrac{x}{6}\sin\tfrac{x}{3}\,dx &= \int \tfrac{1}{2}(\sin\tfrac{x}{2} - \sin(-\tfrac{x}{6}))\,dx \\
&= \tfrac{1}{2}\int \sin\tfrac{x}{2} + \sin\tfrac{x}{6}\,dx \\
&= \tfrac{1}{2}(-2\cos\tfrac{x}{2} - 6\cos\tfrac{x}{6}) + C \\
&= -\cos\tfrac{x}{2} - 3\cos\tfrac{x}{6} + C.
\end{aligned}$$

■

Example 4.6. *Show that* $\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$.

Solution. First we have

$$\begin{aligned}
\cos^4 x &= (\tfrac{1}{2}(1+\cos 2x))^2 = \tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x) \\
&= \tfrac{1}{4}(1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)) \\
&= \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x.
\end{aligned}$$

Thus $\int \cos^4 x\, dx = \int \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\, dx$
$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

■

Exercise 4.2. *Find* $\int \left(\frac{\sin 4x}{1+\cos 4x}\right)^2 dx$

Ans: $\frac{1}{2}\tan 2x - x + C$.

4.3 Partial Fractions

Let $P(x)$ and $Q(x)$ be two polynomials. Suppose $Q(x)$ is a product of linear or quadratic factors with real coefficients. Then, the rational function $\frac{P(x)}{Q(x)}$ can be expressed as a sum of simple fractions whose denominators are factors of $Q(x)$.

Factors of $Q(x)$	Partial fractions
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^2$	$\dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2}$
$ax^2+bx+c,\ b^2-4ac<0$	$\dfrac{Ax+B}{ax^2+bx+c}$

The rational function $\frac{P(x)}{Q(x)}$ is said to be a proper fraction if the degree of $P(x)$ is smaller that the degree of $Q(x)$. Otherwise, it is called an improper fraction. If $\frac{P(x)}{Q(x)}$ is an improper fraction, one can perform long division to write it as $A(x) + \frac{B(x)}{Q(x)}$, where $\frac{B(x)}{Q(x)}$ is a proper fraction.

Examples

(1) $\dfrac{2x+4}{x^2-9} = \dfrac{2x+4}{(x-3)(x+3)} = \dfrac{\frac{5}{3}}{x-3} + \dfrac{\frac{1}{3}}{x+3}$.

(2) $\dfrac{3x^2+x+4}{x^2+x-2} = 3 + \dfrac{-2x+10}{x^2+x-2} = 3 - \dfrac{\frac{14}{3}}{x+2} + \dfrac{\frac{8}{3}}{x-1}$.
(2) is an example of an improper fraction.

Example 4.7. *Find* $\displaystyle\int \frac{3x^2+x+4}{x^2+x-2}\, dx$.

Solution.

$$\int \frac{3x^2+x+4}{x^2+x-2}\,dx = \int 3 - \frac{\frac{14}{3}}{x+2} + \frac{\frac{8}{3}}{x-1}\,dx = 3x - \frac{14}{3}\ln|x+2| + \frac{8}{3}\ln|x-1| + C.$$

■

4.4 Integration by Substitution

Theorem 4.2. *Let $u = g(x)$ be a differentiable function whose range is some interval I and let f be continuous on I. Then,*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example 4.8. *Find* $\displaystyle\int \frac{e^{3x}}{\sqrt{2e^{3x}+4}}\,dx.$

Solution. Let $u = 2e^{3x}+4$. Then $\dfrac{du}{dx} = 6e^{3x}$, so that $du = 6e^{3x}dx$. Then

$$\int \frac{e^{3x}}{\sqrt{2e^{3x}+4}}\,dx = \int \frac{1}{6\sqrt{u}}\,du = \frac{1}{3}\sqrt{u} + C = \frac{1}{3}\sqrt{2e^{3x}+4} + C.$$

■

Exercise 4.3. *Find* $\displaystyle\int \frac{(3-\tan 4x)^5}{\cos^2 4x}\,dx.$

Ans: $-\frac{1}{24}(3-\tan 4x)^6 + C$.

Exercise 4.4. *Find* $\displaystyle\int \frac{8}{x\sqrt{\ln x}}\,dx.$

Ans: $16\sqrt{\ln x} + C$.

Interchanging the roles of u and x in Theorem 4.2 and then renaming u and x by x and t respectively, we get

Theorem 4.3. *Let $x = g(t)$ be a differentiable function whose range is some interval I and let f be continuous on I. Then*

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt.$$

Example 4.9. *Find* $\int \frac{\ln x}{x}\,dx.$

Solution. Let $x = e^t$. Then $\frac{dx}{dt} = e^t$, so that $dx = e^t dt$. Then

$$\int \frac{\ln x}{x}\,dx = \int \frac{\ln(e^t)}{e^t}\,e^t\,dt = \int t\,dt = \frac{t^2}{2} + C = \frac{(\ln x)^2}{2} + C.$$

■

Trigonometric Substitution

Expression	Substitution	Identity involved
$\sqrt{a^2-(x+b)^2}$	$x+b = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+(x+b)^2}$	$x+b = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{(x+b)^2-a^2}$	$x+b = a\sec\theta,\ 0 < \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example 4.10. *Find* $\int \frac{\sqrt{25-4x^2}}{x^2}\,dx.$

Solution. Let $x = \frac{5}{2}\sin\theta$. That is $\theta = \sin^{-1}\frac{2x}{5}$. Then $2x = 5\sin\theta$, so that $\sqrt{25-4x^2} = \sqrt{25-25\sin^2\theta} = 5\cos\theta$. Also $\frac{dx}{d\theta} = \frac{5}{2}\cos\theta$, so that $dx = \frac{5}{2}\cos\theta d\theta$. Then

$$\begin{aligned}\int \frac{\sqrt{25-4x^2}}{x^2}\,dx &= \int \frac{5\cos\theta}{(\frac{5}{2}\sin\theta)^2}\cdot\frac{5}{2}\cos\theta\,d\theta\\ &= \int 2\cot^2\theta\,d\theta\\ &= 2\int \csc^2\theta - 1\,d\theta\\ &= -2\cot\theta - 2\theta + C\\ &= -\frac{1}{x}\sqrt{25-4x^2} - 2\sin^{-1}(\frac{2x}{5}) + C.\end{aligned}$$

■

Example 4.11. *Find* $\int \frac{1}{x\sqrt{9x^2+1}}\,dx.$

Solution. Let $x = \dfrac{1}{3}\tan\theta$. That is $\theta = \tan^{-1}(3x)$. Then $\sqrt{1+9x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$. Also, $\dfrac{dx}{d\theta} = \dfrac{1}{3}\sec^2\theta$, so that $dx = \dfrac{1}{3}\sec^2\theta d\theta$. Then

$$\begin{aligned}
\int \frac{1}{x\sqrt{9x^2+1}}\,dx &= \int \frac{1}{\frac{1}{3}\tan\theta\sec\theta}\cdot\frac{1}{3}\sec^2\theta\,d\theta \\
&= \int \csc\theta\,d\theta \\
&= -\ln|\csc\theta+\cot\theta| + C \\
&= -\ln\left|\frac{\sqrt{9x^2+1}}{3x}+\frac{1}{3x}\right| + C \\
&= -\ln\left|\frac{\sqrt{9x^2+1}+1}{3x}\right| + C.
\end{aligned}$$

■

Exercise 4.5. *Find* $\displaystyle\int \sqrt{6x-x^2}\,dx$.

Ans: $\frac{9}{2}\sin^{-1}(\frac{x-3}{3}) + \frac{1}{2}(x-3)\sqrt{6x-x^2} + C$

4.5 Integration by Parts

Recall the product rule of differentiation:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides of the above equation with respect to x gives

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx,$$

or

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

This suggests the following way of performing integration by parts:

$$\int f'(x)g(x)\,dx = \overbrace{f(x)}^{\substack{\text{integrate}\\ f'}} \cdot \overbrace{g(x)}^{\text{keep } g} - \int \overbrace{f(x)}^{\text{keep } f}\ \overbrace{g'(x)}^{\substack{\text{differentiate}\\ g}}\ dx.$$

Example 4.12. *Find* $\int x \sin 3x\,dx$.

Solution.

$$\begin{aligned}
\int x\sin 3x\,dx &= \int (\sin 3x)\cdot x, dx \\
&= \overbrace{\frac{-\cos 3x}{3}}^{\text{integrate } \sin 3x} \overbrace{x}^{\text{keep } x} - \int \overbrace{\frac{-\cos 3x}{3}}^{\text{keep integral of } \sin 3x} \quad \overbrace{1}^{\text{differentiate } x}\ dx \\
&= \frac{-x\cos 3x}{3} + \frac{\sin 3x}{9} + C.
\end{aligned}$$

■

Example 4.13. *Find* $\int x \ln x\,dx$.

Solution. $$\int x\ln x\,dx = \overbrace{\frac{x^2}{2}}^{\text{integrate } x} \overbrace{\ln x}^{\text{keep } \ln x} - \int \overbrace{\frac{x^2}{2}}^{\substack{\text{keep integral of}\\ x}} \quad \overbrace{\frac{1}{x}}^{\text{differentiate } \ln x}\ dx$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

■

Basic rules to determine which function to integrate and which function to differentiate.

The choice of integration follows the reverse order of the following:

Types of functions	**Examples**	**Remark**
Logarithmic Function	$\ln(ax+b)$ or its higher powers	differentiate it
Inverse Trigonometric Functions	$\sin^{-1}(ax+b)$, $\cos^{-1}(ax+b)$, $\tan^{-1}(ax+b)$	differentiate it
Algebraic Functions	Power functions x^a, polynomials	differentiate it
Trigonometric Functions	$\sin(ax+b)$, $\cos(ax+b)$, $\tan(ax+b)$, $\csc(ax+b)$, $\sec(ax+b)$, $\cot(ax+b)$, or a combinations of these	differentiate it integrate it
Exponential Functions	e^{ax+b}	integrate it

Exercise 4.6. *Find* $\displaystyle\int (2x+1)\ln(2x-3)\,dx.$

Ans: $(x^2+x)\ln(2x-3) - \frac{x^2}{2} - \frac{5x}{2} - \frac{15}{4}\ln(2x-3) + C.$

Exercise 4.7. *Find* $\displaystyle\int x\tan^{-1}(2x)\,dx.$

Ans: $(\frac{1}{8} + \frac{x^2}{2})\tan^{-1} 2x - \frac{x}{4} + C.$

Exercise 4.8. *Find* $\displaystyle\int \frac{\sin 2x}{e^{2x}}\,dx.$

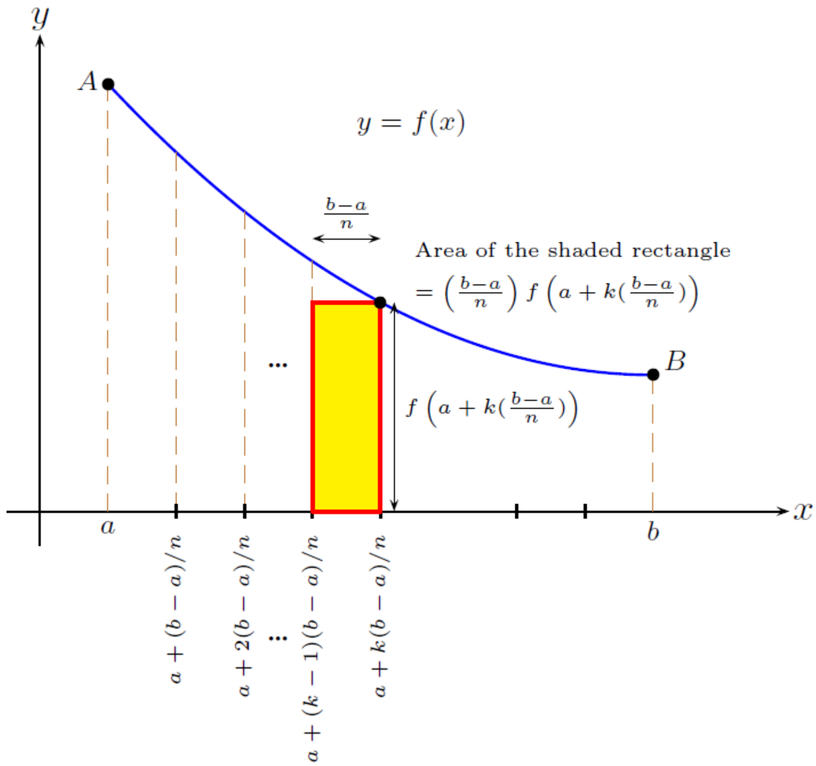
Ans: $-\frac{1}{4}e^{-2x}(\sin 2x + \cos 2x) + C.$

Exercise 4.9. *Find* $\displaystyle\int \sec^3 x\,dx.$

Ans: $\frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) + C.$

4.6 Riemann Sums and Definite Integrals

Let f be continuous on $[a,b]$.



The following limit exists and is known as the *definite integral* of f from $x = a$ to $x = b$.

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k(\frac{b-a}{n})\right)\right\}.$$

It is denoted by

$$\int_a^b f(x)\,dx.$$

Geometrically the definite integral gives the area of the region under the graph of f from $x = a$ to $x = b$ (at least in the case when $f(x) \geq 0$ and $a < b$). The numbers a and b are respectively called the *lower* and *upper limits* of the definite integral. The function $f(x)$ is called the *integrand* of the definite integral.

The finite series $\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k(\frac{b-a}{n})\right)$ is known as a *Riemann sum* of f.

Summing up, we have the following definition of $\int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k(\frac{b-a}{n})\right)\right\}.$$

Approximation. For sufficiently large n,

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n}\left(\frac{b-a}{n}\right) f\left(a+k(\frac{b-a}{n})\right).$$

Example 4.14. *Use Riemann sum to compute* $\int_0^3 x^2\,dx$.

The summation formula $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ *is needed to compute the sum.*

Ans: 9.

Solution. Here $f(x) = x^2$, $a = 0$, $b = 3$. Thus for each n, the corresponding Riemann sum is given by

$$\sum_{k=1}^{n} \frac{3-0}{n} \cdot f(0+k\cdot\frac{3-0}{n}) = \sum_{k=1}^{n} \frac{3}{n}(\frac{3k}{n})^2.$$

Using the given summation formula, we get

$$\sum_{k=1}^{n} \frac{3}{n}(\frac{3k}{n})^2 = \frac{27}{n^3}\sum_{k=1}^{n} k^2 = \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{9(n+1)(2n+1)}{2n^2}.$$

Thus

$$\int_0^3 x^2\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{3}{n}(\frac{3k}{n})^2 = \lim_{n\to\infty}\frac{9(n+1)(2n+1)}{2n^2}$$
$$= \lim_{n\to\infty}\frac{9(1+\frac{1}{n})(2+\frac{1}{n})}{2} = \frac{9(1+0)(2+0)}{2} = 9.$$

4.7 Fundamental Theorem of Calculus (FTC)

The method of using Riemann sums to evaluate definite integrals is tedious. The following result, known as the First Fundamental Theorem of Calculus (FTC 1), provides us with a simpler way of calculating definite integrals when the anti-derivatives of the integrand can be found.

Theorem 4.4 (FTC 1)**.** *Let f be continuous on $[a,b]$ and let F be an anti-derivative of f. Then,*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

That is,

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Notation: Often we denote $\Big[F(x)\Big]_a^b = F(b) - F(a)$. Then the FTC 1 takes the form

$$\int_a^b F'(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Remark. We will see later that FTC 1 follows from Theorem 4.5 (FTC 2).

Example 4.15. *Evaluate* $\displaystyle\int_1^e \frac{(\ln x)^{\frac{1}{3}}}{x}\,dx.$

Ans: $\frac{3}{4}$.

Solution: We need to find an anti-derivative of $f(x) = \frac{(\ln x)^{\frac{1}{3}}}{x}$. For this, we make the substitution $u = \ln x$ to get

$$\int \frac{(\ln x)^{\frac{1}{3}}}{x}\,dx = \int (\ln x)^{\frac{1}{3}}\,d(\ln x) = \frac{3}{4}(\ln x)^{\frac{4}{3}} + C.$$

Pick $C = 0$, and we get an $F(x) = \dfrac{3}{4}(\ln x)^{\frac{4}{3}}$. Then by FTC 1, we have

$$\int_1^e \frac{(\ln x)^{\frac{1}{3}}}{x}\,dx = \left[\frac{3}{4}(\ln x)^{\frac{4}{3}}\right]_1^e = \frac{3}{4}(\ln e)^{\frac{4}{3}} - \frac{3}{4}(\ln 1)^{\frac{4}{3}} = \frac{3}{4} - 0 = \frac{3}{4}.$$

Remark. We often present our solution more compactly as follows:

$$\begin{aligned}\int_1^e \frac{(\ln x)^{\frac{1}{3}}}{x}\,dx &= \int_1^e (\ln x)^{\frac{1}{3}}\,d(\ln x)\\ &= \left[\frac{3}{4}(\ln x)^{\frac{4}{3}}\right]_1^e = \frac{3}{4}(\ln e)^{\frac{4}{3}} - \frac{3}{4}(\ln 1)^{\frac{4}{3}} = \frac{3}{4} - 0 = \frac{3}{4}.\end{aligned}$$

■

Example 4.16. *Evaluate* $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx.$

Ans: $\frac{\pi}{2}-1$.

Solution. Using integration by parts, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\cos x\,dx &= \Big[x\sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx \\ &= (\frac{\pi}{2}-0) + \Big[\cos x\Big]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} + (0-1) \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

■

Example 4.17. *Use a Riemann sum to show that* $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3k+n} = \frac{1}{3}\ln 4$.

Solution. Recall that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k(\frac{b-a}{n})\right) = \int_a^b f(x)\,dx.$$

To express $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3k+n}$ as a Riemann sum, we need to identify the function $f(x)$ and the interval $[a,b]$. First we have

$$\sum_{k=1}^{n}\frac{1}{3k+n} = \sum_{k=1}^{n}\frac{1}{k(\frac{3}{n})+1}\frac{1}{n} = \frac{1}{3}\sum_{k=1}^{n}\frac{1}{1+k(\frac{4-1}{n})}\frac{4-1}{n}.$$

From this we see that $a=1$, $b=4$ and $f(x)=\frac{1}{x}$. Therefore,

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3k+n} &= \lim_{n\to\infty}\frac{1}{3}\sum_{k=1}^{n}\frac{1}{1+k(\frac{4-1}{n})}\frac{4-1}{n} \\ &= \frac{1}{3}\int_1^4\frac{1}{x}\,dx = \frac{1}{3}\Big[\ln x\Big]_1^4 = \frac{1}{3}\ln 4.\end{aligned}$$

Next we have the Second Fundamental Theorem of Calculus (FTC 2) as follows:

Theorem 4.5 (FTC 2)**.** *Let f be continuous on $[a,b]$. The function g defined by*

$$g(x) = \int_a^x f(t)\,dt,\quad a\le x\le b$$

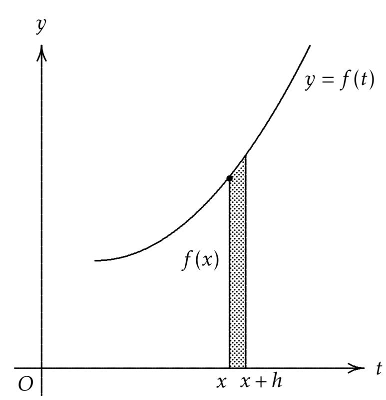
is continuous and differentiable on (a,b), and $g'(x) = f(x)$. That is

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Remark. FTC 2 essentially says that the area function $\int_a^x f(t)\,dt$ is an anti-derivative of (the integrand) f.

Sketch of Proof of Theorem 4.5 (FTC 2). For small h,

$$\begin{aligned}
g(x+h)-g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\
&= \int_x^{x+h} f(t)\,dt \\
&= \text{area under the graph of } f \text{ over } [x,x+h] \\
&\approx f(x)\cdot h.
\end{aligned}$$



Area under the graph of f over $[x,x+h] \approx f(x)\cdot h$

Thus

$$\frac{g(x+h)-g(x)}{h} \approx f(x).$$

Letting $h \to 0$, we get

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x).$$

■

As mentioned earlier, one can deduce FTC 1 from FTC 2 as follows:

Proof of Theorem 4.4 (FTC 1). Let F be an antiderivative of a continuous function f over an interval I, and let a,b be in I. By Theorem 4.5 (FTC 2), $\displaystyle\int_a^x f(t)\,dt$ is also an antiderivative of f. Thus by Theorem 4.1, we have

$$F(x) = \int_a^x f(t)\,dt + C$$

for some constant C. Then

$$F(b) - F(a) = \Big(\int_a^b f(t)\,dt + C\Big) - \Big(\int_a^a f(t)\,dt + C\Big) = \int_a^b f(t)\,dt.$$

■

Remark. If $g(x)$ is differentiable, then using the Chain Rule, one has

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x).$$

Proof. We let

$$F(u) = \int_a^u f(t)\,dt \quad \text{and} \quad u = g(x).$$

Thus

$$F(g(x)) = \int_a^{g(x)} f(t)\,dt.$$

By Theorem 4.5 (FTC 2), one has

$$\frac{dF}{du} = f(u).$$

Thus,

$$\begin{aligned}\frac{d}{dx}\int_a^{g(x)} f(t)\,dt &= \frac{d}{dx}F(g(x)) \\ &= \frac{dF}{du}\cdot\frac{du}{dx} \quad \text{(by Chain Rule)} \\ &= f(g(x))g'(x).\end{aligned}$$

■

Example 4.18. *Find* $\dfrac{d}{dx}\displaystyle\int_{-2}^{\sin x} \sqrt{1+t^6}\,dt.$

Ans: $\cos x\sqrt{1+\sin^6 x}$.

Solution. We apply the formula

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x)$$

(with $f(t) = \sqrt{1+t^6}$ and $g(x) = \sin x$). Note that $g'(x) = \cos x$. Thus

$$\frac{d}{dx}\int_{-2}^{\sin x} \sqrt{1+t^6}\,dt = \sqrt{1+(\sin x)^6}\cdot\cos x = \cos x\sqrt{1+\sin^6 x}.$$

■

Properties of Definite Integrals

Let $c \in [a,b]$ and $\alpha, \beta \in \mathbb{R}$.

1. $\int_a^b \alpha\,dx = \alpha(b-a)$
2. $\int_c^c f(x)\,dx = 0$
3. $\int_a^b (\alpha f(x) + \beta g(x))\,dx = \int_a^b \alpha f(x)\,dx + \int_a^b \beta g(x)\,dx$
4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
6. $\int_a^b f(x)\,dx \geq 0$ if $f(x) \geq 0$ for $a \leq x \leq b$
7. $\int_a^b f(x)\,dx \leq 0$ if $f(x) \leq 0$ for $a \leq x \leq b$
8. $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ if $f(x) \geq g(x)$ for $a \leq x \leq b$
9. $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$ if $f(x) \leq g(x)$ for $a \leq x \leq b$
10. $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ if $m \leq f(x) \leq M$ for $a \leq x \leq b$
11. $\int_{-a}^a f(x)\,dx = 0$ if f is an odd function defined on $[-a,a]$
12. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if f is an even function defined on $[-a,a]$

4.8 Miscellaneous Examples

Example 4.19. *(Integrals of the type* $\int \frac{px+q}{ax^2+bx+c}\,dx$*)*

Evaluate $\int_{-2}^{-1} \frac{3x+7}{x^2+4x+5}\,dx$.

Solution.

$$\int_{-2}^{-1} \frac{3x+7}{x^2+4x+5}\,dx$$

$$= \int_{-2}^{-1} \frac{3}{2}\frac{2x+4}{x^2+4x+5} + \frac{1}{x^2+4x+5}\,dx$$

$$= \int_{-2}^{-1} \frac{3}{2}\frac{2x+4}{x^2+4x+5} + \frac{1}{1+(x+2)^2}\,dx$$

$$= \left[\frac{3}{2}\ln|x^2+4x+5|\right]_{-2}^{-1} + \left[\tan^{-1}(x+2)\right]_{-2}^{-1}$$

$$= \frac{3}{2}(\ln 2 - \ln 1) + \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{3}{2}\ln 2 + \frac{\pi}{4}.$$

■

Example 4.20. *(Integrals of the type* $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$*)*

Show that $\int_1^2 \frac{2x+3}{\sqrt{4x-x^2}}\,dx = 2\sqrt{3} + \frac{7\pi}{6} - 4.$

Solution:

$$\int_1^2 \frac{2x+3}{\sqrt{4x-x^2}}\,dx = \int_1^2 \frac{-(4-2x)}{\sqrt{4x-x^2}} + \frac{7}{\sqrt{4x-x^2}}\,dx$$

$$= \int_1^2 \frac{-(4-2x)}{\sqrt{4x-x^2}} + \frac{7}{\sqrt{2^2-(x-2)^2}}\,dx$$

$$= \left[-2\sqrt{4x-x^2}\right]_1^2 + \left[7\sin^{-1}(\frac{x-2}{2})\right]_1^2$$

$$= -2(2-\sqrt{3}) + 7(\sin^{-1}0 - \sin^{-1}(-\frac{1}{2}))$$

$$= -2(2-\sqrt{3}) - 7(-\frac{\pi}{6})$$

$$= 2\sqrt{3} - 4 + \frac{7\pi}{6}.$$

■

Exercise 4.10. *Using the identity* $1+\cos x+\sin x = 2\cos^2\dfrac{x}{2}(1+\tan\dfrac{x}{2})$, *evaluate*

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx.$$

Ans: $\ln 2$.

Exercise 4.11. *Using the substitution* $x = \frac{\pi}{2} - y$, *show that*

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x+\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\cos x+\sin x}\,dx.$$

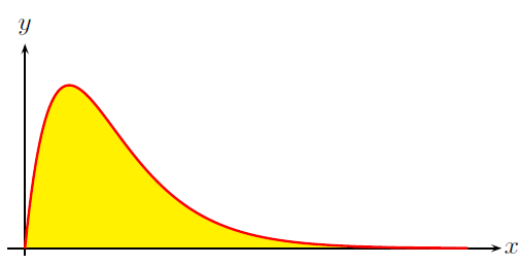
Exercise 4.12. *Using the results in exercise* 4.10 *and* 4.11 *and the identity* $\sin^2 x+\cos^2 x = 1$, *show that*

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x+\sin x}\,dx = \frac{\ln 2}{2}.$$

4.9 Improper Integrals

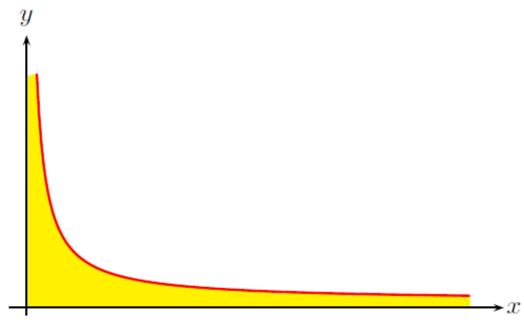
The definite integrals that we have studied so far have the following characteristics: (i) the domain of integration is a finite closed interval $[a,b]$, (ii) the function or the integrand has finite values on the domain of integration. It is possible that we would encounter problems that do not meet these conditions.

The integral for the area under the curve $y = xe^{-x}$ from $x = 0$ to $x = \infty$ has domain of integration which is infinite.



The graph of $y = xe^{-x}$.

The integral for the area under the curve $y = \frac{1}{\sqrt{x}}$ from $x = 0$ to $x = 10$ requires us to integrate the function $\frac{1}{\sqrt{x}}$ whose value at $x = 0$ is infinite.



The graph of $y = \frac{1}{\sqrt{x}}$.

In either case, the integrals are called *improper integrals* and are calculated as limits.

Definition 4.2. *Integrals with infinite limits of integration are improper integrals of* **Type I**.

1. If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, then we say that the improper integral diverges.

Example 4.21. *Evaluate the Type I improper integral*

$$\int_1^\infty \frac{\ln x}{x^2}\,dx.$$

Solution. We first use integration by parts to compute the indefinite integral.

$$\int \ln x \cdot \frac{1}{x^2}\,dx = \ln x \cdot \frac{-1}{x} - \int \frac{1}{x}\cdot\frac{-1}{x}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C = -\frac{(1+\ln x)}{x} + C.$$

Thus

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{(1+\ln x)}{x}\right]_1^b = \lim_{b\to\infty} 1 - \frac{1+\ln b}{b} = 1.$$

Here $\lim_{b\to\infty}\frac{\ln b}{b} = 0$ by L'Hôpital's rule. ■

Example 4.22. *Evaluate the Type I improper integral*

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

Solution. We need to evaluate $\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$ and $\int_0^\infty \frac{1}{1+x^2}\,dx$.

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty}\int_b^0 \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty}\left[\tan^{-1}x\right]_b^0 = \lim_{b\to-\infty}(0-\tan^{-1}b) = -(-\frac{\pi}{2}) = \frac{\pi}{2}.$$

Similarly,

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Therefore,

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

■

Definition 4.3. *Integrals of functions that become infinite at a point within the interval of integration are improper integrals of* **Type II**.

1. If $f(x)$ is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c with $a<c<b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Example 4.23. *Evaluate the Type II improper integral*

$$\int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx.$$

Solution.

$$\int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = \lim_{c\to 1^-}\int_0^c \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = \lim_{c\to 1^-}\left[3(x-1)^{\frac{1}{3}}\right]_0^c = \lim_{c\to 1^-} 3(c-1)^{\frac{1}{3}} + 3 = 3.$$

■

Exercise 4.13. *Evaluate the Type I improper integral*

$$\int_0^\infty \frac{\tan^{-1} x}{1+x^2}\,dx.$$

Ans: $\frac{\pi^2}{8}$.

Exercise 4.14. *Evaluate the Type II improper integral*

$$\int_0^1 \frac{1}{2\sqrt{x}(1+x)}\,dx.$$

Ans: $\frac{\pi}{4}$.

Exercise 4.15. *Evaluate the Type I improper integral*

$$\int_{-\infty}^\infty \frac{1}{e^{-x}+e^{x}}\,dx.$$

Ans: $\frac{\pi}{2}$.

Chapter 5

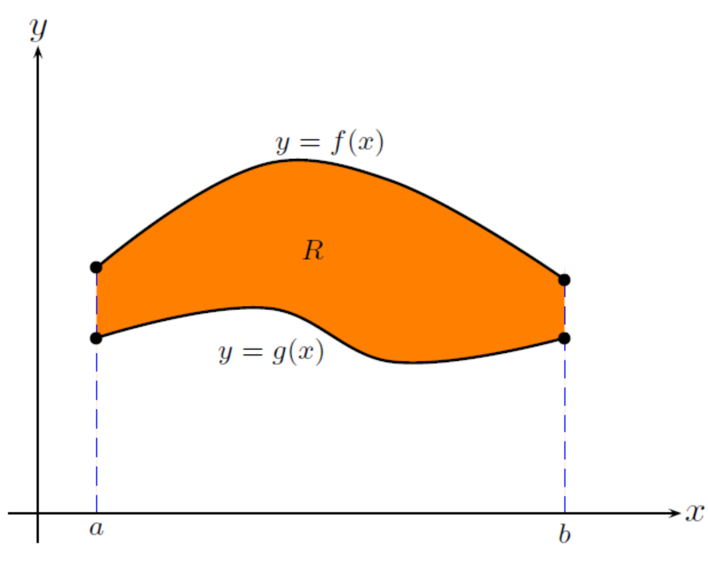
Applications of Integration

Read Thomas' Calculus, Chapter 6.

5.1 Area Between Curves

Theorem 5.1. *Let f and g be continuous on $[a,b]$ with $f(x) \geq g(x)$ for all $a \leq x \leq b$. The area of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$ is given by*

$$A = \int_a^b (f(x) - g(x))\,dx.$$



In particular, if $g(x) = 0$, we obtain the following result.

Theorem 5.2. *Let f be continuous on $[a,b]$ with $f(x) \geq 0$ for all $a \leq x \leq b$. The area of the region bounded by the curve $y = f(x)$, the x-axis $y = 0$, and the lines $x = a$ and $x = b$ is given by*

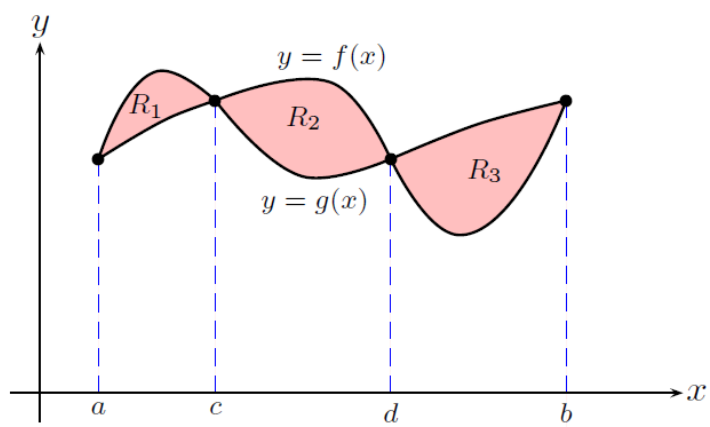
$$A = \int_a^b f(x)\,dx.$$

In general we have the following result.

Theorem 5.3. *Let f and g be continuous on $[a,b]$ (not necessarily with $f(x) \geq g(x)$ for all $a \leq x \leq b$). The area of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$ is given by*

$$A = \int_a^b |f(x) - g(x)|\,dx.$$

To evaluate the above integral, we split it into two or more integrals, each corresponding to the region where either $f(x) - g(x) \geq 0$ or $f(x) - g(x) \leq 0$.



In particular, if $g(x) = 0$, then we obtain the following result.

Theorem 5.4. *Let f be continuous on $[a,b]$. The area of the region bounded by the curve $y = f(x)$, and the lines $x = a$ and $x = b$ is given by*

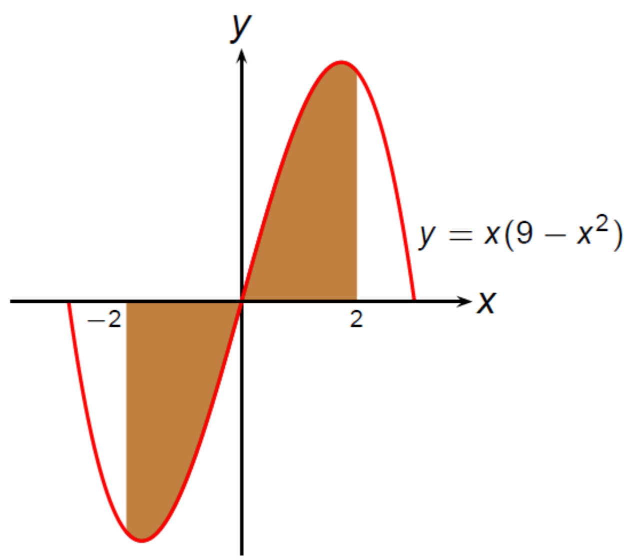
$$A = \int_a^b |f(x)|\,dx.$$

To evaluate the above integral, we split it into two or more integrals, each corresponding to the region where either $f(x) \geq 0$ or $f(x) \leq 0$.

Example 5.1. *Find the area of the region bounded the curve $y = x(9-x^2)$, $(-2 \leq x \leq 2)$, the x-axis, the line $x = -2$ and the line $x = 2$.*

Solution. Let $f(x) = x(9-x^2)$. Notice that $f(x) \geq 0$ for $0 \leq x \leq 2$ and $f(x) \leq 0$ for $-2 \leq x \leq 0$.

$$\begin{aligned}
\text{Area} &= \int_{-2}^{2} |x(9-x^2)|\,dx \\
&= \int_{-2}^{0} -x(9-x^2)\,dx + \int_{0}^{2} x(9-x^2)\,dx \\
&= \left[-\frac{9x^2}{2} + \frac{x^4}{4}\right]_{-2}^{0} + \left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_{0}^{2} \\
&= 0 - (-18+4) + (18-4) - 0 = 28.
\end{aligned}$$



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Example 5.2. *Find the area of the region bounded by the curves $y = e^{2x} - 2$, $(x \geq 0)$, and $y = 10 - e^x$, $(x \geq 0)$, and*

(a) the x-axis,

(b) the y-axis.

Ans: (a) $-\frac{7}{2} - 12\ln 3 + 10\ln 10 + \ln 2$,
(b) $12\ln 3 - 6$.

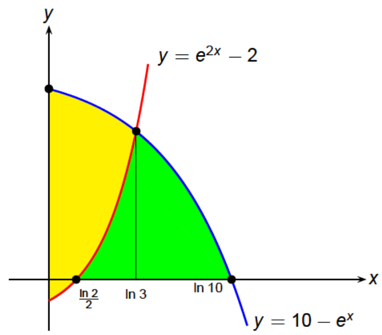
Solution.

1. $e^{2x} - 2 = 0 \Leftrightarrow x = \dfrac{\ln 2}{2}$.

2. $e^{2x} - 2 = 10 - e^x \iff e^{2x} + e^x - 12 = 0$

$\iff (e^x + 4)(e^x - 3) = 0 \Leftrightarrow x = \ln 3.$

3. $10 - e^x = 0 \iff x = \ln 10.$



(a) Green Area $= \displaystyle\int_{\frac{\ln 2}{2}}^{\ln 3} (e^{2x} - 2)\,dx + \int_{\ln 3}^{\ln 10} (10 - e^x)\,dx$

$$= \left[\frac{1}{2}e^{2x} - 2x\right]_{\frac{\ln 2}{2}}^{\ln 3} + \left[10x - e^x\right]_{\ln 3}^{\ln 10}$$

$$= (\frac{9}{2} - 2\ln 3) - (1 - \ln 2) + (10\ln 10 - 10) - (10\ln 3 - 3)$$

$$= -\frac{7}{2} - 12\ln 3 + 10\ln 10 + \ln 2.$$

(b) Yellow Area $= \displaystyle\int_0^{\ln 3} (10 - e^x) - (e^{2x} - 2)\,dx$

$$= \int_0^{\ln 3} 12 - e^x - e^{2x}\,dx$$

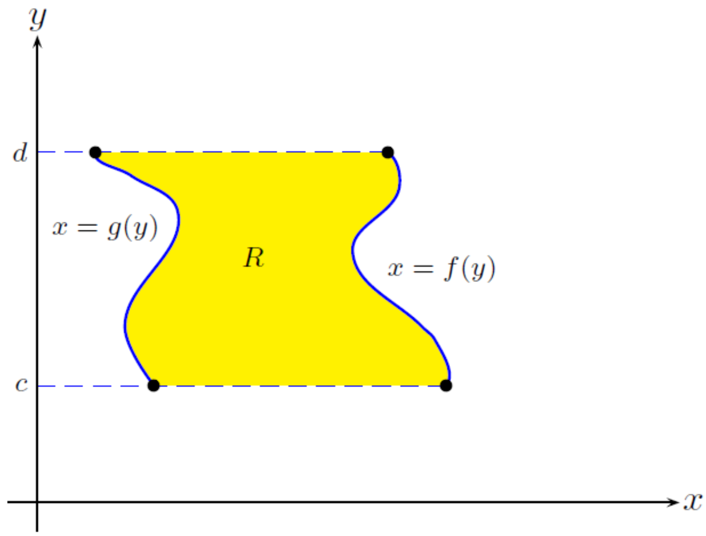
$$= \left[12x - e^x - \frac{1}{2}e^{2x}\right]_0^{\ln 3}$$

$$= (12\ln 3 - 3 - \frac{9}{2}) - (-1 - \frac{1}{2})$$

$$= 12\ln 3 - 6.$$

Theorem 5.5. *Let f and g be continuous on $[c,d]$ with $f(y) \geq g(y)$ for all $c \leq y \leq d$. The area of the region bounded by the curves $x = f(y)$, $x = g(y)$, and the lines $y = c$ and $y = d$ is given by*

$$A = \int_c^d (f(y) - g(y))\, dy.$$



In general, we have the following result.

Theorem 5.6. *Let f and g be continuous on $[c,d]$ (not necessarily with $f(y) \geq g(y)$ for all $c \leq y \leq d$). The area of the region bounded by the curves $x = f(y), x = g(y)$, and the lines $y = c$ and $y = d$ is given by*

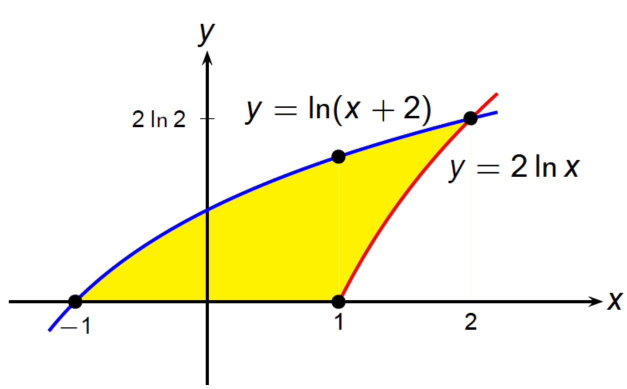
$$A = \int_c^d |f(y) - g(y)|\, dy.$$

To evaluate the above integral, we split it into two or more integrals, each corresponding to the region where either $f(y) - g(y) \geq 0$ or $f(y) - g(y) \leq 0$.

Example 5.3. *Find the area of the region bounded the curve $y = \ln(x+2)$, $y = 2\ln x$, the x-axis.*

Ans: $4\ln 2 - 1$.

Solution.

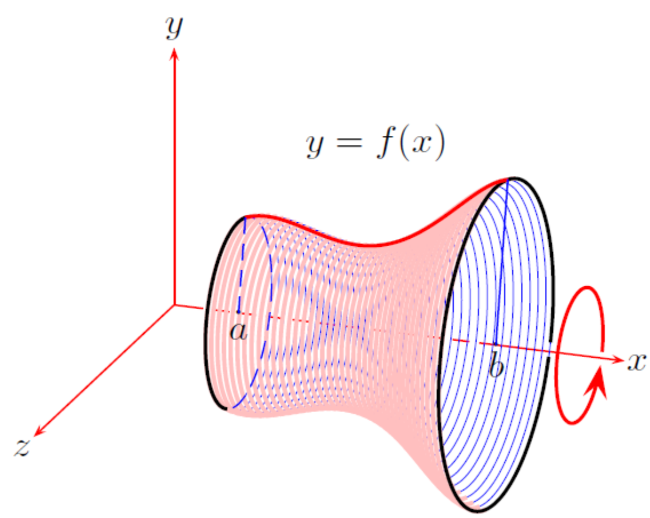


Solving $y = \ln(x+2)$ and $y = 2\ln x$ gives $(x,y) = (2, \ln 2)$. Also $y = \ln(x+2) \Leftrightarrow x = e^y - 2$ and $y = 2\ln x \Leftrightarrow x = e^{\frac{y}{2}}$.

Thus the area is $A = \int_0^{2\ln 2} e^{\frac{y}{2}} - e^y + 2\,dy = \left[2e^{\frac{y}{2}} - e^y + 2y\right]_0^{2\ln 2} = 4\ln 2 - 1$.

■

5.2 Volume of Solid of Revolution by Disk Method

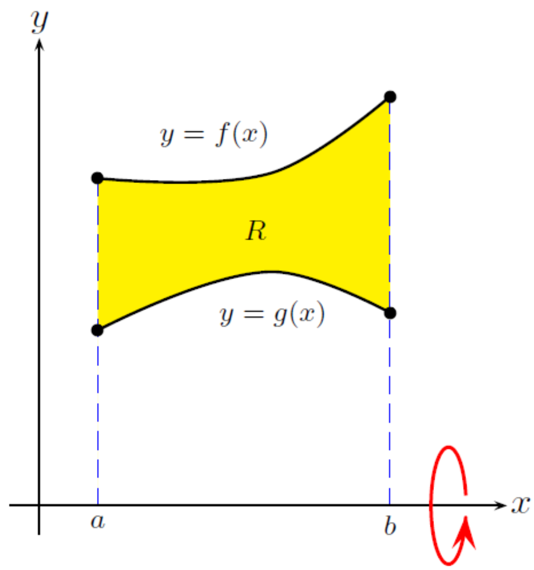


Theorem 5.7. *When the plane region bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$ is revolved completely about the x-axis, the volume of the solid formed is*

$$V = \pi \int_a^b f(x)^2\,dx.$$

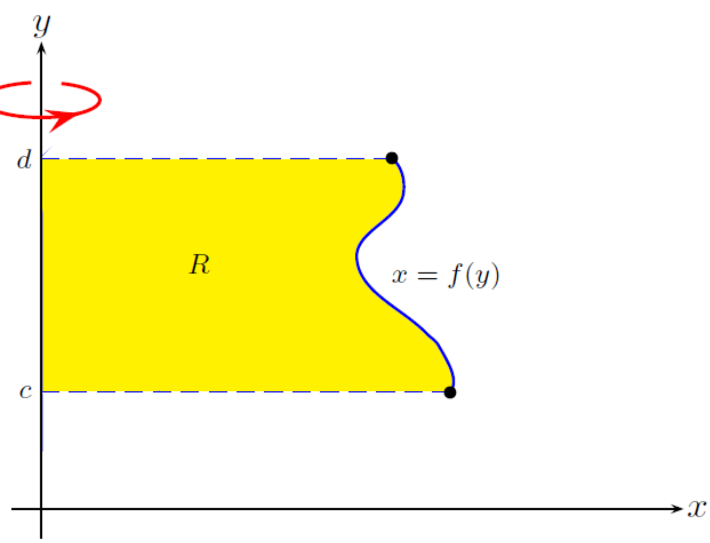
The above formula is known as the disk method.

Remark. Informally we can visualize the above formula as follows: Think of the expression $\pi(f(x)^2\,dx$ as the volume of a thin vertical circular disk of radius $f(x)$ and thickness dx (so that its volume is $\pi \cdot (\text{radius})^2 \cdot \text{thickness} = \pi(f(x)^2\,dx)$. Also we think of the definite integral sign $\int_a^b$ as the process of 'taking the limit of a Riemann sum'. Then the formula essentially says that the volume V of solid of revolution is equal to the limit of the Riemann sum (i.e. the definite integral $\int_a^b$) of the volumes of the thin circular disks (given by $\pi(f(x)^2\,dx$)



Theorem 5.8. *Let f and g be continuous on $[a,b]$ with $f(x) \geq g(x) \geq 0$ for all $a \leq x \leq b$. When the region bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ is revolved completely about the x-axis, the volume of the solid formed is*

$$V = \pi \int_a^b f(x)^2\,dx - \pi \int_a^b g(x)^2\,dx.$$

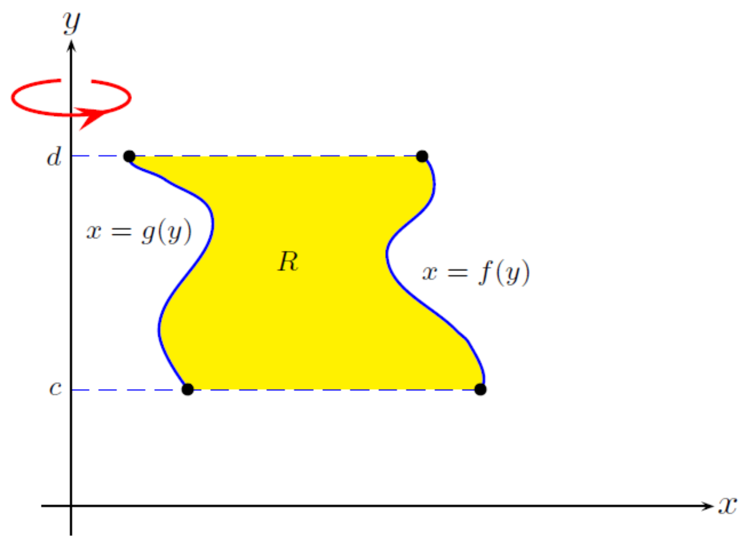


Theorem 5.9. *Let f be continuous on $[c,d]$. When the plane region bounded by the curve $x = f(y)$ and the lines $y = c$ and $y = d$ is revolved completely about the y-axis, the volume of*

the solid formed is

$$V = \pi \int_a^b f(y)^2\, dy.$$

Remark. Informally we can visualize the above formula as follows: The volume V of the solid of revolution is equal to the limit of a Riemann sum (i.e. $\int_c^d$) of volumes of thin horizontal circular disks (given by $\pi \cdot (f(y))^2 \cdot dy$).



Theorem 5.10. *Let f and g be continuous on $[c,d]$ with $f(y) \geq g(y) \geq 0$ for all $c \leq y \leq d$. When the region bounded by the curves $x = f(y)$ and $x = g(y)$ for $c \leq y \leq d$ is revolved completely about the y-axis, the volume of the solid formed is*

$$V = \pi \int_c^d f(y)^2\, dy - \pi \int_c^d g(y)^2\, dy.$$

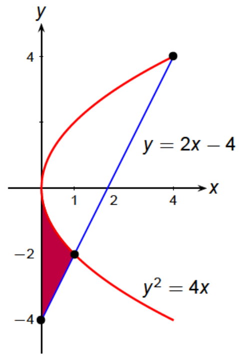
Example 5.4. *Find the volume of the solid generated by rotating completely the region bounded by the curve $y^2 = 4x$, the line $y = 2x - 4$ and the y-axis about the*

(a) x-axis,

(b) y-axis.

Ans: (a) $\frac{22\pi}{3}$, (b) $\frac{16\pi}{15}$.

Solution. The two curves intersect at $(1,-2), (4,4)$. The required region is the one shaded in purple in the figure.



(a) (Rotating about the x-axis) Using the disk method, the volume is

$$V = \pi \int_0^1 (2x-4)^2 - (-\sqrt{4x})^2 \, dx$$
$$= 4\pi \int_0^1 x^2 - 5x + 4 \, dx = 4\pi \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x \right]_0^1 = \frac{22\pi}{3}.$$

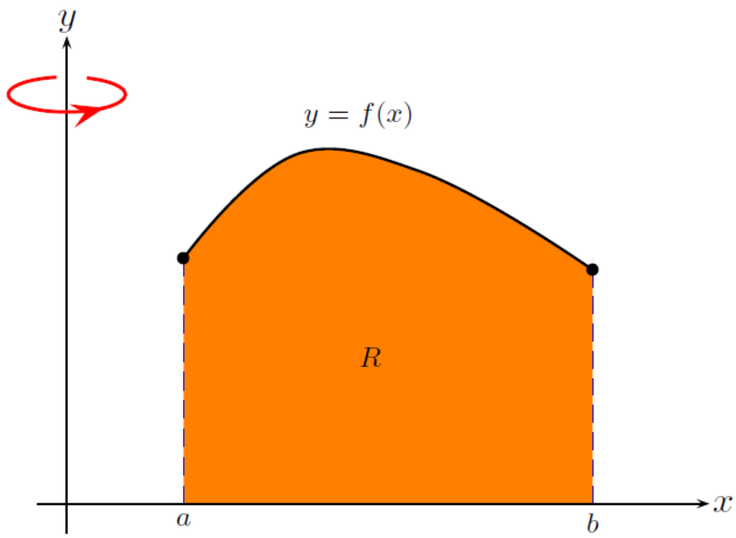
(b) (Rotating about the y-axis) Using the disk method, the volume is

$$V = \pi \int_{-2}^{0} (\frac{1}{4}y^2)^2 \, dy + \pi \int_{-4}^{-2} (\frac{1}{2}(y+4))^2 \, dy$$
$$= \pi \left[\frac{y^5}{80} \right]_{-2}^{0} + \pi \left[\frac{(y+4)^3}{12} \right]_{-4}^{-2} = \pi(\frac{2}{5} + \frac{2}{3}) = \frac{16\pi}{15}.$$

■

5.3 Cylindrical Shell Method

Consider the solid of revolution obtained by revolving the region R about the y-axis. If we apply the disk method, it is necessary to express the equation $y = f(x)$ in the form $x = g(y)$. Quite often, it is difficult or even impossible to do so.

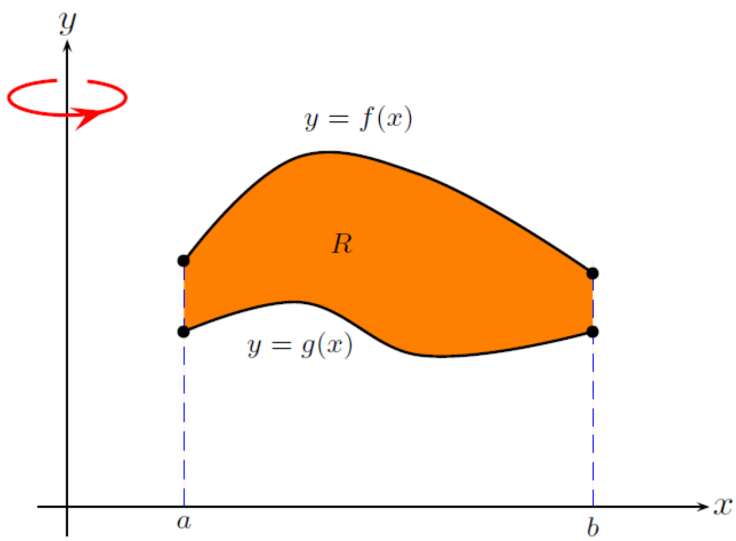


The following result, known as the method of cylindrical shell, provides a solution to this problem.

Theorem 5.11. *When the plane region bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$, where $0 \le a < b$, is revolved completely about the y-axis, the volume of the solid formed is*

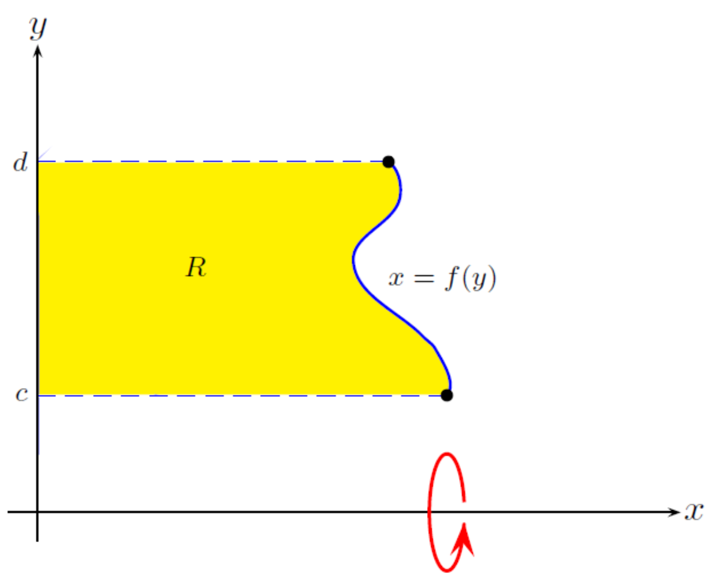
$$V = 2\pi \int_a^b x|f(x)|\,dx.$$

Remark. Informally we can visualize the above formula as follows: Think of the expression $2\pi x \cdot |f(x)| \cdot dx$ as the volume of a thin vertical cylindrical shell of base radius x and of height $|f(x)|$ and of thickness dx (so that its volume is given by (surface area of the cylinder)$\cdot$ thickness $= (2\pi x \cdot |f(x)|) \cdot dx$). Then the formula says that the volume V of the solid of revolution is equal to the limit of a Riemann sum (i.e. $\displaystyle\int_a^b$) of volumes of thin vertical cylindrical shells (given by $2\pi x \cdot |f(x)| \cdot dx$).



Theorem 5.12. *When the plane region bounded by the curve $y = f(x), y = g(x)$ and the lines $x = a$ and $x = b$, where $0 \leq a < b$, is revolved completely about the y-axis, the volume of the solid formed is*

$$V = 2\pi \int_a^b x|f(x) - g(x)|\, dx.$$

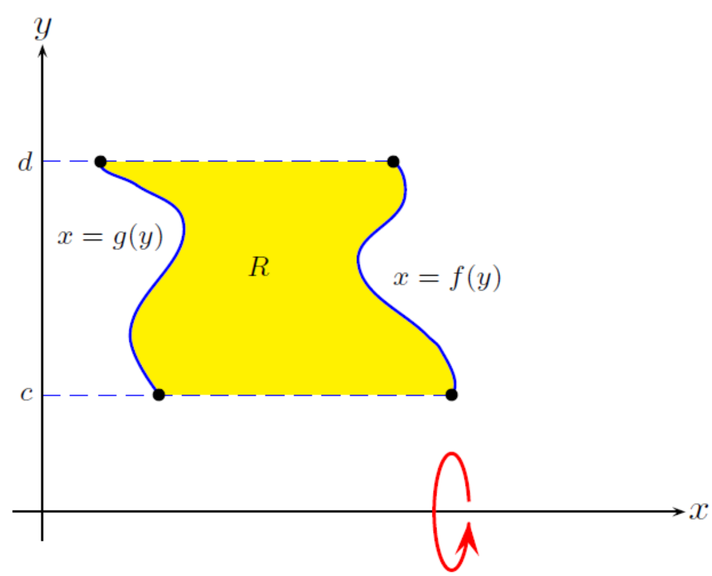


Theorem 5.13. *When the plane region bounded by the curve $x = f(y)$ and the lines $y = c$ and $y = d$, where $0 \leq c < d$, is revolved completely about the x-axis, the volume of the solid formed*

is

$$V = 2\pi \int_c^d y|f(y)|\,dy.$$

Remark. Informally we can visualize the above formula as follows: The volume V of the solid of revolution is equal to the limit of a Riemann sum (i.e. $\displaystyle\int_c^d$) of volumes of thin horizontal cylindrical shells (given by $2\pi y \cdot |f(y)| \cdot dy$).



Theorem 5.14. *When the plane region bounded by the curve $x = f(y), x = g(y)$ and the lines $y = c$ and $y = d$, where $0 \le c < d$, is revolved completely about the x-axis, the volume of the solid formed is*

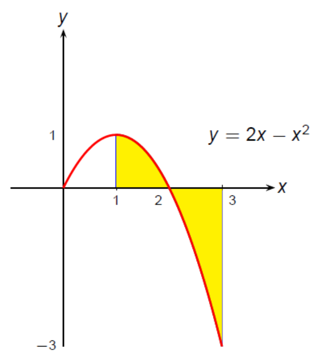
$$V = 2\pi \int_c^d y|f(y) - g(y)|\,dy.$$

Example 5.5. *The regions bounded by the curve $y = 2x - x^2$ $(1 \le x \le 3)$, the line $x = 1$, the line $x = 3$ and the x-axis is revolved completely about the y-axis. Calculate the volume of the solid generated.*

Ans: 9π.

Solution. We use the shell method. The volume is given by

$$
\begin{aligned}
V &= 2\pi \int_1^3 x|f(x)|\,dx = 2\pi \int_1^2 x|2x - x^2|\,dx \\
&= 2\pi\left(\int_1^2 x(2x - x^2)\,dx + \int_2^3 x(-2x + x^2)\,dx\right) \\
&= 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 + 2\pi\left[\frac{x^4}{4} - \frac{2x^3}{3}\right]_2^3 = 2\pi(\frac{11}{12} + \frac{43}{12}) = 9\pi.
\end{aligned}
$$



Example 5.6. *Sketch the curve whose equation is $y = \ln(2x-1)$ for $x > \frac{1}{2}$.*

The region bounded by this curve, the axes and the line $y = \ln 3$ is rotated completely about the x-axis. Calculate the volume of the solid generated.

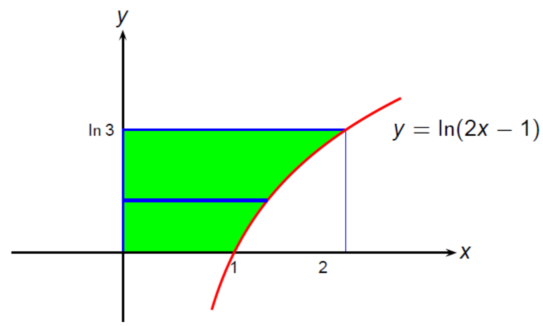
Ans: $\pi(3\ln 3 + \frac{1}{2}(\ln 3)^2 - 2)$.

Solution. We use the shell method. First

$$
y = \ln(2x-1) \Leftrightarrow x = \frac{1}{2}(e^y + 1).
$$

The volume is given by

$$
\begin{aligned}
V &= 2\pi \int_0^{\ln 3} y\left|f(y)\right|dy = 2\pi \int_0^{\ln 3} y\left|\frac{1}{2}(e^y + 1)\right|dy \\
&= 2\pi \int_0^{\ln 3} y\frac{1}{2}(e^y + 1)\,dy = \pi \int_0^{\ln 3} ye^y + y\,dy \\
&= \pi\left[ye^y - e^y + \frac{y^2}{2}\right]_0^{\ln 3} = \pi(3\ln 3 + \frac{1}{2}(\ln 3)^2 - 2).
\end{aligned}
$$



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5.4 Arc Length of a curve

Let f be continuous on $[a,b]$. Using Riemann sums, we can prove:

The length of the curve $y = f(x), a \le x \le b$ is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Remark. To visualize the above expression, we can think of a curve as consisting of many small slanted line segments, and each slanted line segment is the hypotenuse of a right-angled triangle of base length dx and of height dy. Then by Pythagoras' Theorem, the length of a slanted line segment is given by $\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2} \cdot dx = \sqrt{1 + f'(x)^2}\, dx$. Then the length of the curve is the limit of a Riemann sum (i.e. $\displaystyle\int_a^b$) of lengths of small slanted line segments given by $\sqrt{1 + f'(x)^2}\, dx$. ($ds \equiv \sqrt{1 + f'(x)^2}\, dx$ is called the arc length differential).

Example 5.7. *Calculate the length of the curve*

$$y = \frac{x^4 + 3}{6x}, \ 1 \le x \le 2.$$

Solution. Given $y = \frac{x^4+3}{6x} = \frac{1}{6}(x^3 + \frac{3}{x})$, we have

$$y' = \frac{1}{6}(3x^2 - \frac{3}{x^2}) = \frac{1}{2}(x^2 - \frac{1}{x^2}).$$

Thus $1 + y'^2 = 1 + \frac{1}{4}(x^2 - \frac{1}{x^2})^2 = \left(\frac{1}{2}(x^2 + \frac{1}{x^2})\right)^2$ so that $\sqrt{1 + y'^2} = \frac{1}{2}(x^2 + \frac{1}{x^2})$.

The arclength is $\int_1^2 \sqrt{1 + y'^2}\, dx = \int_1^2 \frac{1}{2}(x^2 + \frac{1}{x^2})\, dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_1^2 = \frac{17}{12}$.

■

Similarly, if the curve is given as the graph of a function of y, we have the following formula.

The length of the curve $x = g(y), p \le y \le q$ is

$$\int_p^q \sqrt{1 + g'(y)^2}\, dy.$$

Exercise 5.1. *Calculate the length of the curve*

$$y = 1 + \left(\frac{3x}{2}\right)^{\frac{2}{3}},\ 0 \le x \le \frac{16}{3}.$$

Ans: $\frac{2}{3}(5^{\frac{3}{2}} - 1)$.

Exercise 5.2. *Sketch the curves $y = 2 - e^{x-1}$ and $y = 4x^2 - 3$ for $x \ge 0$ on a single diagram. It is given that the two curves meet at a point where $x = 1$. Calculate the area of the region bounded by the two curves and the y-axis.*

Ans: $\frac{8}{3} + \frac{1}{e}$.

Exercise 5.3. *Let $I_n = \int_0^1 (2x+1)^n e^{-x}\, dx$, where n is a non-negative integer.*
(a) Show that for $n \ge 1$,

$$I_n = (1 - \frac{3^n}{e}) + 2nI_{n-1}.$$

(b) The region bounded by the curve $y = (2x+1)^2 e^{-x}$, the axes and the line $x = 1$ is rotated completely about the y-axis. Use the result in (a) to find the value of the solid generated.

Ans: (b) $\pi(66 - \frac{172}{e})$.

Exercise 5.4. *Consider the region R bounded by $y = 2x^2$, the line $x = 2$ and the x-axis. For $0 < p < 2$, the vertical line $x = p$ divides R into two parts R_1 and R_2, where R_1 denotes the part on the right of $x = p$ and R_2 denotes the part on the left of $x = p$. Let V_1 be the volume of the solid generated by revolving R_1 about the x-axis, and V_2 be the volume of the solid generated by revolving R_2 about the y-axis. Find R_1 and R_2 in terms of p. Find also the value of p that maximizes the total volume given by $V = V_1 + V_2$.*

Ans: $p = 1$ gives the maximum V.

Chapter 6

Sequences and Series

Read Thomas' Calculus, Chapter 9.

6.1 Sequences

An infinite sequence of numbers is an infinite ordered list of numbers

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

For each n, the nth number in the list is called the nth term of the sequence.

We usually denote a sequence by $\{a_n\}_{n=1}^{\infty}$ (or simply $\{a_n\}$ when the reference to n is clear). Formally,

Definition 6.1. *An infinite sequence of numbers is a function whose domain is the set of positive integers.*

In this formal definition, a_n is the value of the function evaluated at n.

Example 6.1. *The sequence of arithmetic progression is given by $\{a+(n-1)d\}_{n=1}^{\infty}$, where a is the first term of the sequence and d is called the common difference.*

The function that defines this sequence is: $f(n) = a+(n-1)d$.

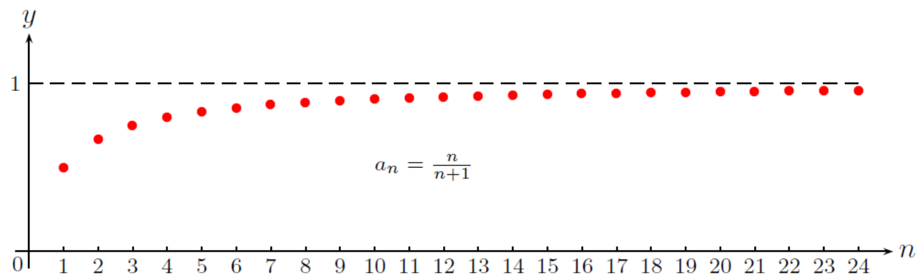
The sequence of geometric progression is given by $\{ar^{n-1}\}_{n=1}^{\infty}$, where a is the first term of the sequence and r is known as the common ratio of the geometric progression.

The function that defines this sequence is: $f(n) = ar^{n-1}$.

For example, $1, 3, 5, 7, 9, \ldots$, is an arithmetic progression (with $a = 1$, $d = 2$).

$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots$, *is a geometric progression (with $a = 1$, $r = \frac{1}{2}$).*

We now consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{n}{n+1}$, and determine what happens to a_n when n is getting large. The following graph shows how the terms approach 1.



We say that the sequence $\{\frac{n}{n+1}\}$ approaches 1 as n increases and write

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

We now state formally the meaning of a limit of a sequence.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

Then $\lim\limits_{n\to\infty} a_n$ is the value a_n approaches as n tends to positive infinity.

If $\lim\limits_{n\to\infty} a_n$ exists as a real (finite) number L, then we say that the sequence $\{a_n\}$ converges (or more detailedly, $\{a_n\}$ converges to L). Sometimes we simply write $a_n \to L$.

We say that the sequence $\{a_n\}$ diverges if $\lim\limits_{n\to\infty} a_n$ does not exist as a real (finite) number.

Example 6.2. *(i)* *As we have seen, we have* $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$.

Thus the sequence $\left\{\dfrac{n}{n+1}\right\}$ *converges (to 1), and we also write* $\dfrac{n}{n+1} \to 1$.

(ii) Note that the sequence

$$-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \cdots$$

does not converge to any real number. Thus the sequence $\{(-1)^n\}$ *diverges.*

(iii) Clearly we have $\lim\limits_{n\to\infty} 2n = \infty$. *But* ∞ *is not a real number, so the sequence* $\{2n\}$ *diverges.*

6.2 Finding the Limit of a Sequence

This following theorem gives a shortcut to evaluate the limit of some sequences using the limit of functions.

Theorem 6.1. *Let $f(x)$ be a function, and $\{a_n\}$ be a sequence such that $f(n) = a_n$ for all n. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.*

Example 6.3. *Find the following limits:*

(i) $\lim_{n\to\infty} \frac{1}{n}$. *(ii)* $\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution. (i) Consider the function $f(x) = \frac{1}{x}$. Note that $\frac{1}{n} = f(n)$ for all n.

Since $\lim_{x\to\infty} \frac{1}{x} = 0$, it follows from Theorem 6.1 that $\lim_{n\to\infty} \frac{1}{n} = 0$.
(ii) By L'Hôpital's rule, we have

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Thus we also have $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

∎

6.3 Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then we have

- $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$.
- $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.
- $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $\lim_{n\to\infty} b_n \neq 0$.

Example 6.4. *From the previous example, we know that*

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

It follows that we have

$$\begin{aligned}\lim_{n\to\infty} \frac{2n^2 + 3\ln n}{n^2} &= \lim_{n\to\infty} (2 + \frac{3\ln n}{n^2}) \\ &= 2 + 3\lim_{n\to\infty} \frac{\ln n}{n} \cdot \frac{1}{n} \\ &= 2 + 3 \cdot 0 \cdot 0 = 2.\end{aligned}$$

Theorem 6.2. *(Squeeze Theorem for Sequence) If $a_n \le b_n \le c_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Example 6.5. *Show that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Solution. Note that $-|a_n| \le a_n \le |a_n|$ for all n,
and it is given that $\lim_{n\to\infty} |a_n| = 0$, so that we also have $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = -0 = 0$. Thus it follows from Squeeze Theorem that $\lim_{n\to\infty} a_n = 0$.

■

Example 6.6. *Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.*

Solution. Since $0 \le \frac{n!}{n^n} = \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots \times \frac{2}{n} \times \frac{1}{n} \le \frac{1}{n}$ for all n and $\lim_{n\to\infty} \frac{1}{n} = 0$, the result follows from Squeeze Theorem.

■

6.4 Series

An expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

is called an *infinite series* or simply a *series*. To compute the value (called the *sum*) of this infinite series, we construct a new sequence $\{S_n\}$ defined by

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

$\{S_n\}$ is called the sequence of *partial sums* of the given series. Then the sum of the infinite series is defined as the limit of the sequence $\{S_n\}$. In other words, we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* if the sequence $\{S_n\}$ converges (which means that $\lim_{n\to\infty} S_n$ exists as a real number). Thus the series $\sum_{n=1}^{\infty} a_n$ is convergent means that it has a finite sum.

We say that the series $\sum_{n=1}^{\infty} a_n$ *diverges* if the sequence $\{S_n\}$ diverges (which means that $\lim_{n\to\infty} S_n$ does not exist as a real number). Thus the series $\sum_{n=1}^{\infty} a_n$ is divergent means that it does not have a finite sum.

Example 6.7. *The constant sequence $\{1\}$ is a convergent sequence, since $\lim_{n\to\infty} 1 = 1$, which is a real number.*

Now consider the series $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$.

For each n, the nth partial sum is given by $S_n = \sum_{i=1}^{n} 1 = n$. Thus, we have $\sum_{n=1}^{\infty} 1 = \lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$.

But ∞ is not a real number, and thus the series $\sum_{n=1}^{\infty} 1$ is divergent.

Example 6.8. *(a) An important example of an infinite series is the geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1}, \quad (a \neq 0).$$

The geometric series is convergent with its sum equal to $\dfrac{a}{1-r}$ when $|r| < 1$, and it is divergent when $|r| \geq 1$.

(b) Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

(c) Find the sum of the series $\sum_{n=1}^{\infty} x^n$, *for* $|x| < 1$.

Solution. (a) Note that for each n,

$$S_n = \sum_{i=1}^{n} ar^{i-1} = \begin{cases} \dfrac{a(1-r^n)}{1-r} & \text{if } r \neq 1, \\ an & \text{if } r = 1. \end{cases}$$

Note that $a \neq 0$. Then $\lim_{n\to\infty} an = \begin{cases} \infty & \text{if } a > 0, \\ -\infty & \text{if } a < 0. \end{cases}$

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1, \\ 0 & \text{if } -1 < r < 1, \\ \text{does not exist} & \text{if } r \leq -1. \end{cases}$$

Then it follows readily that

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} S_n = \begin{cases} \infty & \text{if } a > 0 \text{ and } r \geq 1, \\ -\infty & \text{if } a < 0 \text{ and } r \geq 1, \\ \dfrac{a}{1-r} & \text{if } -1 < r < 1, \\ \text{does not exist} & \text{if } r \leq -1. \end{cases}$$

In summary, $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent (with its sum $= \frac{a}{1-r}$) when $|r| < 1$, and $\sum_{n=1}^{\infty} ar^{n-1}$ is divergent when $|r| \geq 1$.

(b) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = 3 \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ is a geometric series with $r = \frac{4}{3} > 1$, and thus it is divergent.

(c) $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} x \cdot x^{n-1}$ is a geometric series with $a = x$ and $r = x$. When $|x| < 1$, it follows from (a) that the series is convergent and $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$.

∎

Example 6.9. *Show that the series* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *is convergent, and find its sum.*

Solution. For each n,

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$
$$= (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n} - \frac{1}{n+1})$$
$$= 1 - \frac{1}{1+n}.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{1+n}\right) = 1$, which is a real number. Thus the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

■

Theorem 6.3. *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, so are the series $\sum_{n=1}^{\infty} ca_n$ (where c is a constant) and $\sum_{n=1}^{\infty} (a_n + b_n)$. Moreover,*

(a) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$, and

(b) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

Example 6.10. *From Example 6.9, we know that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. It follows that*

$$\sum_{n=1}^{\infty} (\frac{2}{3^n} + \frac{4}{n(n+1)}) = \sum_{n=1}^{\infty} \frac{2}{3^n} + 4\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{\frac{2}{3}}{1-\frac{1}{3}} + 4 \cdot 1 = 1 + 4 = 5.$$

(Note that $\sum_{n=1}^{\infty} \frac{2}{3^n}$ is a geometric series with $a = \frac{2}{3}$ and $r = \frac{1}{3}$.)

■

Lemma 6.4. *If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Suppose $\sum_{n=1}^{\infty} a_n = L$. Let $S_n = a_1 + \cdots + a_n$. Then $\lim_{n\to\infty} S_n = L$. Note that $a_n = S_n - S_{n-1}$ for all $n \geq 2$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = L - L = 0.$$

■

Theorem 6.5. *(The n^{th} Term Test for Divergence)*

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

This is also known simply as the n^{th} term test.

Example 6.11. *Is the series $\sum_{n=1}^{\infty} \frac{n^2}{7n^2+3}$ convergent or divergent?*

Solution. $\lim_{n\to\infty} \frac{n^2}{7n^2+3} = \lim_{n\to\infty} \frac{1}{7+\frac{3}{n^2}} = \frac{1}{7+0} = \frac{1}{7} \neq 0$. Thus by the n^{th} term test, the series $\sum_{n=1}^{\infty} \frac{n^2}{7n^2+3}$ is divergent.

■

Warning. The n^{th} term test is inconclusive if $\lim_{n\to\infty} a_n = 0$.

(To see this, consider the two series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Note that both series satisfy the condition $\lim_{n\to\infty} a_n = 0$. But we will later show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.)

■

For series of nonnegative terms (i.e., each $a_n \geq 0$), we have the following fundamental result:

Theorem 6.6. *A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above (i.e., there exists a constant K such that $S_n < K$ for all n.)*

Remark. Basically, this theorem means that if each $a_n \geq 0$, then the sum of the series $\sum_{n=1}^{\infty} a_n$ is either a finite number or ∞.

Example 6.12. *The series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is convergent.*

Solution. Let $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Then for each n,

$$\begin{aligned}
S_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \\
&\leq 1 + \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-2)\times(n-1)} + \frac{1}{(n-1)\times n} \\
&= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-2} - \frac{1}{n-1}) + (\frac{1}{n-1} - \frac{1}{n}) \\
&= 2 - \frac{1}{n} < 2.
\end{aligned}$$

Thus the sequence of partial sums $\{S_n\}$ is bounded above (by 2). Therefore by the above theorem, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. ∎

In general, it is difficult to determine if a given series in convergent. In the next few sections, we will discuss some methods that will enable us to test the convergence of certain series.

6.5 Integral Test

Example 6.13. *The harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

Solution. Observe that

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{\geq \frac{8}{16} = \frac{1}{2}} + \cdots$$

From this bracketing of the terms, one can see that the partial sums $\{S_n\}$ satisfy

$$S_2 \geq \frac{1}{2}, \ \ S_4 \geq 2\cdot\frac{1}{2}, \ \ S_8 \geq 3\cdot\frac{1}{2}, \ \ S_{16} \geq 4\cdot\frac{1}{2}, \cdots,$$

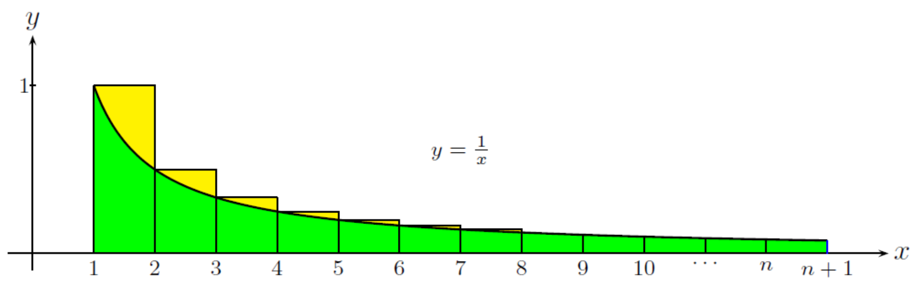
and more generally,

$$S_{2^k} \geq \frac{k}{2} \quad \text{for all } k.$$

So the sequence of partial sums is not bounded from above. By the above theorem, the harmonic series diverges.

■

For $n = 1, 2, \ldots$, the rectangle with height $\frac{1}{n}$ erected on the unit interval $[n, n+1]$ has area $\frac{1}{n}$. The harmonic series may be viewed as the sum of the areas of these rectangles. Graphically we see that the union of all these rectangles contains the green region R bounded by the graph of $y = \frac{1}{x}$ and the x-axis for $x \geq 1$. However, the region R has an infinite area as the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges. This implies the harmonic series diverges.



Theorem 6.7. *(Integral Test) Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent. In other words:*

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 6.14. *Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.*

Solution. Clearly, $\frac{1}{x^2+1}$ is a decreasing function for $x > 0$.
$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} [\tan^{-1} x]_1^b = \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. Thus the improper integral $\int_1^\infty \frac{1}{x^2+1}\,dx$ converges. By the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges. (But note that $\frac{\pi}{4}$ is *not* the sum of the series.)

■

Theorem 6.8. *(The p-series) The p-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if and only if* $p > 1$.

Proof. Let $p > 1$. $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \lim_{b\to\infty} \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right) = \frac{1}{p-1}$. Thus the series converges by the integral test.

If $p = 1$, then the series is the harmonic series which is divergent.

Let $0 < p < 1$. Then $1 - p > 0$. $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \frac{1}{1-p}\left(b^{1-p} - 1\right) = \infty$. Thus the series diverges by the integral test.

Let $p \leq 0$. Then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$. By Theorem 6.5 (the nth term test), the series diverges.

∎

Example 6.15. *The series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *is divergent, since it is a p-series with* $p = \frac{1}{2} \leq 1$.

The series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ *is convergent, since it is a p-series with* $p = \frac{3}{2} > 1$.

6.6 The Comparison Test

Theorem 6.9. *(Comparison Test)*

Suppose $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with nonnegative terms such that* $0 \leq a_n \leq b_n$ *for all* n.

(i) If $\sum_{n=1}^{\infty} b_n$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

(ii) If $\sum_{n=1}^{\infty} a_n$ *is divergent, then* $\sum_{n=1}^{\infty} b_n$ *is divergent.*

Remark. Roughly speaking, (i) means that if the bigger series has a finite sum, then the smaller series also has a finite sum.

Also, (ii) means that if the sum of the smaller series is ∞, then the sum of the bigger series is also ∞.

Remark. When applying the Comparison Test, we often compare a given series with an appropriate p-series or geometric series.

Example 6.16. *Determine the convergence of the series* $\displaystyle\sum_{n=1}^{\infty} \frac{7}{2n^2+4n+3}$.

Solution. Note that $0 \le \dfrac{7}{2n^2+4n+3} \le \dfrac{7}{n^2}$ for all $n \ge 1$. We know $\displaystyle\sum_{n=1}^{\infty} \frac{7}{n^2} = 7\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as it is a p-series with $p = 2 > 1$. Thus by the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{7}{2n^2+4n+3}$ converges.

■

Example 6.17. *Determine the convergence of the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

Solution. Note that for all $n \ge 1$,

$$2^n - 1 \ge \frac{1}{2}\cdot 2^n = 2^{n-1} \implies 0 \le \frac{1}{2^n-1} \le \frac{1}{2^{n-1}}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent, since it is a geometric series with $|r| = \dfrac{1}{2} < 1$. Thus by the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent.

■

6.7 The Ratio Test and Root Test

Theorem 6.10. *(The Ratio Test) Suppose* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is a series such that* $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ *(L is a finite number or ∞).*

(i) If $0 \le L < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ *is absolutely convergent. That is* $\displaystyle\sum_{n=1}^{\infty} |a_n|$ *is convergent.*

(ii) If $L > 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $L = 1$, then the ratio test is inconclusive.

Proof. (Optional Reading Exercise) See Section 6.13. ■

Remark. Theorem 6.13 (later) will tell us: $\sum_{n=1}^{\infty} |a_n|$ is convergent $\implies \sum_{n=1}^{\infty} a_n$ is convergent .

Thus in (i) (when $0 \le L < 1$), the series $\sum_{n=1}^{\infty} a_n$ itself is also convergent.

Remark. For a geometric series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ar^{n-1}$, one has

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{ar^n}{ar^{n-1}} \right| = \lim_{n\to\infty} |r| = |r|.$$

Thus roughly speaking, the Ratio Test says that if a series resembles a geometric series, then its convergence/divergence property also resembles that of the geometric series (in the cases of (i) and (ii)).

Remark. (iii) can be illustrated by the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$, as both series have $L = 1$.

Example 6.18. *Test the series*

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

for absolute convergence.

Solution.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \frac{1}{3} \left(\frac{n+1}{n} \right)^3 = \lim_{n\to\infty} \frac{1}{3} \left(1 + \frac{1}{n} \right)^3 = \frac{1}{3} < 1.$$

Thus by the ratio test, $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ converges absolutely.

■

Example 6.19. *Test the series*

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

for absolute convergence.

Solution.

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right| = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{\lim_{n\to\infty} (1+\frac{1}{n})^n}.$$

By L'Hôpital's rule,

$$\begin{aligned} \lim_{x\to\infty} (1+\frac{1}{x})^x &= e^{\lim_{x\to\infty} \ln(1+\frac{1}{x})^x} = e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})} = e^{\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}} \\ &= e^{\lim_{x\to\infty} \frac{\frac{1}{(1+\frac{1}{x})}\cdot(-\frac{1}{x^2})}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}} = e^{\frac{1}{1+0}} = e. \end{aligned}$$

Thus we have $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$, and it follows that $L = \frac{1}{e} < 1$.

Hence by the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is absolutely convergent.

■

Exercise 6.1. *Test for convergence of the following series.*

(a) $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$ *(b)* $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ *(c)* $\sum_{n=1}^{\infty} \frac{4^n(n!)^2}{(2n)!}$

Ans. (a) convergent, (b) divergent, (c) divergent.

Our next test resembles the Ratio Test closely:

Theorem 6.11. *(The Root Test) Suppose* $\sum_{n=1}^{\infty} a_n$ *is a series such that* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ *(L is a finite number or ∞).*

(i) If $0 \le L < 1$, then $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

(ii) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $L = 1$, then the root test is inconclusive.

Remark. For a geometric series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ar^{n-1}$, one has

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|ar^{n-1}|} = \lim_{n\to\infty} |a|^{\frac{1}{n}} |r|^{\frac{n-1}{n}} = |a|^0 \cdot |r|^1 = 1 \cdot |r| = |r|.$$

As such, the conclusions in (i) and (ii) in the Root Test are consistent with what we know about convergence/divergence property of a geometric series.

Remark. Again (iii) can be illustrated by the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$, as both series have $L = 1$.

Example 6.20. *Test the series*

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$$

for convergence.

Solution.

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1.$$

By the root test, $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent.

■

Example 6.21. *Test the series*

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{3n+1}}$$

for convergence.

Solution. $\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^n}{3^{3n+1}}\right|} = \lim_{n\to\infty} \frac{n}{3^{3+\frac{1}{n}}} = \infty$. By the root test, $\sum_{n=1}^{\infty} \frac{n^n}{3^{3n+1}}$ is divergent.

■

Exercise 6.2. *Test for convergence of the following series.*

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ *(b)* $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ *(c)* $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Ans. (a) convergent, (b) divergent, (c) convergent.

6.8 Alternating Series

An *alternating series* is a series whose terms are alternatively positive and negative.

Example 6.22. *(i) An example of an alternating series is given by*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots,$$

which is known as the alternating harmonic series. The name reflects the fact that $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ *is the harmonic series.*

(ii) Another example of an alternating series is given by

$$\sum_{n=1}^{\infty}(-1)^{n}n=-1+2-3+4-5+6-\cdots$$

Remark. The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. Nonetheless, it turns out that the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is convergent.

This can be seen as follows:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}&=(\frac{1}{1}-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+\cdots\\&=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots\\&=\sum_{n=1}^{\infty}\frac{1}{(2n-1)\cdot 2n}.\end{aligned}$$

For each $n\geq 1$, we have

$$2n-1\geq n \implies 0\leq\frac{1}{(2n-1)\cdot 2n}\leq\frac{1}{2n^2}.$$

The series $\sum_{n=1}^{\infty}\frac{1}{2n^2}=\frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, since it is a p-series with $p=2>1$. Thus by the Comparison Test, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)\cdot 2n}$ is convergent. Thus the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is also convergent.

The above conclusion also follows from the following result:

Theorem 6.12. *(The Alternating Series Test) If b_n is a sequence of positive numbers such that*

(i) *b_n is decreasing (that is, $b_n \geq b_{n+1}$ for all n), and*

(ii) $\lim_{n\to\infty} b_n = 0$,

then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots \quad \text{and}$$
$$\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

are convergent.

Example 6.23. *Use the Alternating Series Test to show that the series* $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ *is convergent.*

Proof. $\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n}$ is an alternating series with each $b_n = \frac{1}{n} > 0$. For each $n \geq 1$, we have $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0$, and thus the sequence $\left\{\frac{1}{n}\right\}$ is decreasing. Also we have $\lim_{n\to\infty}\frac{1}{n} = 0$. Thus by the alternating series test, the series $\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent.

Example 6.24. *Show that the series* $\sum_{n=1}^{\infty}\frac{(-1)^{n}n^2}{n^3+1}$ *is convergent.*

Solution. $\sum_{n=1}^{\infty}\frac{(-1)^{n}n^2}{n^3+1}$ is an alternating series with each $b_n = \frac{n^2}{n^3+1} > 0$. For each $n \geq 1$, we have, by direct calculation,

$$\begin{aligned} b_n - b_{n+1} = \frac{n^2}{n^3+1} - \frac{(n+1)^2}{(n+1)^3+1} &= \frac{n^2((n+1)^3+1) - (n+1)^2(n^3+1)}{(n^3+1)((n+1)^3+1)} \\ &= \frac{n^4 + 2n^3 + n^2 - 2n - 1}{(n^3+1)((n+1)^3+1)} \\ &= \frac{n^4 + 2n^2(n-1) + (n^2-1)}{(n^3+1)((n+1)^3+1)} \geq 0. \end{aligned}$$

Thus the sequence $\left\{\frac{n^2}{n^3+1}\right\}$ is decreasing.

Also, $\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = \frac{0}{1+0} = 0.$

Thus by the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$ is convergent.

■

6.9 Absolute Convergence

Given a series $\sum_{n=1}^{\infty} a_n$, we can construct a new series $\sum_{n=1}^{\infty} |a_n|$, whose terms are the absolute values of the terms of the original series.

Theorem 6.13. *If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*

Proof. Note that $0 \le (a_n + |a_n|) \le 2|a_n|$ for all n. Since $\sum_{n=1}^{\infty} 2|a_n|$ converges, we have by comparison test, $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges too. Therefore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

converges.

■

Definition 6.2. *A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

The series $\sum_{n=1}^{\infty} a_n$ is said to be conditionally convergent if it is convergent but not absolutely convergent.

Theorem 6.13 states that every absolutely convergent series is convergent.

Example 6.25. *Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.*

Solution. The series $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=0}^{\infty}\frac{1}{n^2}$ is convergent, since it is a p-series with $p = 2 > 1$.

Therefore, $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^2}$ converges absolutely.

Example 6.26. *Show that the series* $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ *is conditionally convergent.*

Solution. From Example $\boxed{6.23}$, we know, by the alternating series test, that $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent.

Also, $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=0}^{\infty}\frac{1}{n}$ is the harmonic series which is divergent.

Therefore, $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n}$ converges conditionally.

■

Example 6.27. *Show that the series* $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ *is absolutely convergent.*

6.10 Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

where x is a variable, and the c_n's are constants called *coefficients* of the series. For each fixed x, the power series is a series of numbers that we can test for convergence or divergence.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots,$$

is called a *power series centred at a* or a *power series about a*.

Note that the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges at $x = a$.

Example 6.28. *For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?*

Solution. If $x \neq 0$, then $\lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty} (n+1)|x| = \infty$. By ratio test, $\sum_{n=0}^{\infty} n!x^n$ diverges. Therefore, $\sum_{n=0}^{\infty} n!x^n$ converges if and only if $x = 0$.

■

Example 6.29. *For what values of x is the series $\sum_{n=0}^{\infty} \frac{(x-7)^n}{n}$ convergent?*

Solution. $\lim_{n\to\infty} \left|\frac{\frac{(x-7)^{n+1}}{n+1}}{\frac{(x-7)^n}{n}}\right| = \lim_{n\to\infty} \frac{n}{n+1}|x-7| = |x-7|$. By ratio test, we have the following conclusions (i) and (ii).

(i) If $|x-7| < 1$, then $\sum_{n=0}^{\infty} \frac{(x-7)^n}{n}$ is absolutely convergent.

(ii) If $|x-7| > 1$, then $\sum_{n=0}^{\infty} \frac{(x-7)^n}{n}$ is divergent.

(iii) If $x = 6$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, which is an alternating series. By the alternating series test, it is convergent.

(iv) If $x = 8$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{n}$, which is the harmonic series which is divergent.

Summarizing, the series $\sum_{n=0}^{\infty} \frac{(x-7)^n}{n}$ is convergent if and only if $x \in [6, 8)$.

■

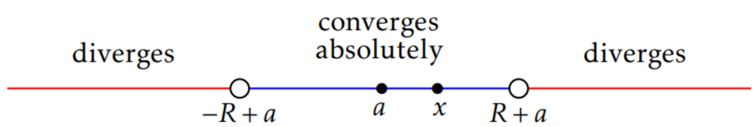
Theorem 6.14. *For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following possibilities holds:*

(i) The series converges at $x = a$ only.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) is called the radius of convergence of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The interval of convergence of a power series is the interval consisting of all values of x for which the series converges.



The interval of convergence can be $(a-R, a+R), [a-R, a+R), (a-R, a+R], [a-R, a+R]$.

In some cases, we can compute R by the following method.

Theorem 6.15. *Consider the power series* $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, *where* $c_n \neq 0$ *for all* n. *If* $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$ *or* $\displaystyle\lim_{n\to\infty}\sqrt[n]{|c_n|} = L$, *where* L *is a real number or* ∞, *then* $R = \frac{1}{L}$.

By convention, if $L = 0$, *then* $R = \infty$, *and if* $L = \infty$, *then* $R = 0$.

Proof. Suppose $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$.

Thus $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right||x-a| = L|x-a|$.

By ratio test, the series converges absolutely for $L|x-a| < 1$, that is $|x-a| < \frac{1}{L}$; and the series diverges for $L|x-a| > 1$, that is $|x-a| > \frac{1}{L}$. Therefore the radius of convergence is $\frac{1}{L}$.

The second case where $\displaystyle\lim_{n\to\infty}\sqrt[n]{|c_n|} = L$ follows similarly by root test.

■

Example 6.30. *Find the radius of convergence and interval of convergence of the power series* $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$.

Solution. $\displaystyle\lim_{n\to\infty}\left|\frac{\frac{(-3)^{n+1}}{\sqrt{n+2}}}{\frac{(-3)^n}{\sqrt{n+1}}}\right| = \lim_{n\to\infty} 3\sqrt{\frac{n+1}{n+2}} = 3$. Thus the radius of convergence is $\frac{1}{3}$.

When $x = \frac{1}{3}$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is an alternating series. By the alternating series test, it is convergent.

When $x = -\frac{1}{3}$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is divergent by integral test.

Consequently, the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

■

Example 6.31. *Find the radius of convergence and interval of convergence of the power series* $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

Solution. $\lim_{n\to\infty} \left| \frac{\frac{n+1}{3^{n+2}}}{\frac{n}{3^{n+1}}} \right| = \lim_{n\to\infty} \frac{n+1}{3n} = \frac{1}{3}$. Thus the radius of convergence is 3. The centre of the power series is at $x = -2$. Next we consider the two endpoints $x = -2 \pm 3$, that is, $x = -5, 1$.

When $x = -5$, the series becomes $\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}}$, that is $\sum_{n=0}^{\infty} \frac{n(-1)^n}{3}$, which is divergent by the nth term test .

When $x = 1$, the series becomes $\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}}$, that is $\sum_{n=0}^{\infty} \frac{n}{3}$, which is also divergent by the nth term test .

Consequently, the interval of convergence is $(-5, 1)$.

■

Example 6.32. *Find the radius of convergence and interval of convergence of the power series* $\sum_{n=1}^{\infty} \frac{(4x-3)^{2n+1}}{4^n n^{4/3}}$.

Solution. Note that we cannot apply Theorem 6.15 directly since all the coefficients of the even powers of x are zero. Let $u_n = \frac{(4x-3)^{2n+1}}{4^n n^{4/3}}$.

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(4x-3)^{2n+3}}{4^{n+1}(n+1)^{4/3}} \cdot \frac{4^n n^{4/3}}{(4x-3)^{2n+1}} \right|$$
$$= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{\frac{4}{3}} \frac{|4x-3|^2}{4} = \frac{|4x-3|^2}{4}.$$

By ratio test, the power series converges absolutely for $\frac{|4x-3|^2}{4} < 1 \Leftrightarrow \frac{|4x-3|}{2} < 1 \Leftrightarrow |x - \frac{3}{4}| < \frac{1}{2}$, and diverges for $|x - \frac{3}{4}| > \frac{1}{2}$.

Therefore, the radius of convergence is $\frac{1}{2}$.

At $x = \frac{5}{4}$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+1}}{4^n n^{4/3}} = \sum_{n=1}^{\infty} \frac{2}{n^{4/3}}$, which is convergent since it is a p-series with $p = \frac{4}{3} > 1$.

At $x = \frac{1}{4}$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n+1}}{4^n n^{4/3}} = \sum_{n=1}^{\infty} \frac{-2}{n^{4/3}}$, which is convergent since it is a p-series with $p = \frac{4}{3} > 1$.

Therefore, the interval of convergence is $[\frac{1}{4}, \frac{5}{4}]$.

■

6.11 Power Series Representation

Recall that for $|x| < 1$, the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

$\displaystyle\sum_{n=0}^{\infty} x^n$ is called a power series representation of the function $\frac{1}{1-x}$ about $x = 0$.

Example 6.33. *(i) Find a power series representation of* $\dfrac{1}{1+x}$ *about* $x = 0$.

(ii) Find a power series representation of $\dfrac{x^3}{x+2}$ *about* $x = 0$.

Solution. (i) We make use of the above geometric series.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + x^4 + \cdots,$$

which is valid for $|-x| < 1$. That is,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{for } |x| < 1.$$

(ii) Using the power series representation in (i), we get

$$\frac{x^3}{x+2} = \frac{x^3}{2} \cdot \frac{1}{1+\frac{x}{2}} = \frac{x^3}{2} \cdot \sum_{n=0}^{\infty} (-1)^n \Big(\frac{x}{2}\Big)^n \quad \text{for } |\frac{x}{2}| < 1.$$

That is,

$$\frac{x^3}{x+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \quad \text{for } |x| < 2.$$

■

Example 6.34. *Find a power series representation of* $\frac{1}{x^2+3x+2}$ *about* $x = 0$.

Solution.

$$\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{1+x} - \frac{1}{2}\frac{1}{1+\frac{x}{2}} = \sum_{n=0}^{\infty}(-1)^n x^n - \frac{1}{2}\cdot\sum_{n=0}^{\infty}(-1)^n(\frac{x}{2})^n,$$

which is valid when both $|x| < 1$ and $\left|\frac{x}{2}\right| < 1$. Note that

$$|x| < 1 \text{ and } \left|\frac{x}{2}\right| < 1 \iff |x| < 1 \text{ and } |x| < 2 \iff |x| < 1.$$

Thus

$$\frac{1}{x^2+3x+2} = \sum_{n=0}^{\infty}(-1)^n(1-\frac{1}{2^{n+1}})x^n \quad \text{for } |x| < 1.$$

■

Theorem 6.16. *If the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* f *defined by*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $|x-a| < R$ *and*

(i) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$, *for* $|x-a| < R$.

(ii) $\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$, *for* $|x-a| < R$.

Example 6.35. *Find a power series representation of* $\ln(1-x)$ *and its radius of convergence.*

Solution. For $|x| < 1$, $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Thus by Theorem 6.16 (ii),

$$\int \frac{1}{1-x}\,dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$
$$\implies -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C, \quad \text{for } |x| < 1.$$

When $x = 0$, we have $0 = 0 + C$ so that $C = 0$. Therefore,

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{n+1}, \quad \text{for } |x| < 1.$$

The radius of convergence is 1 by ratio test.

■

Example 6.36. *Find a power series representation of* $\tan^{-1} x$.

Solution. For $|x| < 1$, $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$. Thus by Theorem 6.16 (ii),

$$\int \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$
$$\implies \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C, \quad \text{for } |x| < 1.$$

When $x = 0$, we have $0 = 0 + C$ so that $C = 0$. Therefore,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \text{for } |x| < 1.$$

■

6.12 Taylor and Maclaurin Series

By repeated use of Theorem 6.16 (i), we deduce the following.

Theorem 6.17. *If f has a power series representation at $x = a$, that is*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R, \textit{ for some } R > 0,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

If f has a power series representation at $x = a$, then it is unique and has the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is called the Taylor series of f at $x = a$.

The Maclaurin series of f is the special case of Taylor series when $a = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Exercise 6.3. *Assume that each of the functions* e^x, $\sin x$ *and* $\cos x$ *has a Maclaurin series representation. Find the Maclaurin series and its radius of convergence for (a)* e^x, $(b) \sin x$, *(c)* $\cos x$.

Ans: (a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, (b) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, (c) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.
All three Maclaurin series converge for all x in $\mathbb{R}$.

Solution. (a) Let's work out the Maclaurin series of e^x. Since $\frac{d^n e^x}{dx^n} = e^x$ for all $n \geq 0$. Thus $\frac{d^n e^x}{dx^n}(0) = 1$ for all $n \geq 0$.

Therefore, the Maclaurin series of e^x is

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Since $\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0$, the radius of convergence is ∞ and so the interval of convergence is $\mathbb{R}$.

Assuming e^x has a Maclaurin series representation, we thus have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

(b), (c): Exercise.

■

6.13 Appendix: The Precise Definition of the Limit of a Sequence and some Proofs

Remark: Section 6.13 will be excluded from the assessments (quizzes and the Final Exam).

Remark: The precise definition of the limit of a sequence and the proofs given in this section will will be studied in detail in MA2108 Mathematical Analysis I.

Definition 6.3. *The sequence $\{a_n\}$ converges to the number L if for every positive number ϵ, there corresponds an integer N such that for all n,*

$$n > N \text{ implies that } |a_n - L| < \epsilon.$$

If no such number exists, we say that $\{a_n\}$ diverges. If $\{a_n\}$ converges to L, we write

$$\lim_{n\to\infty} a_n = L$$

or simply $a_n \to L$ and call L the limit of the sequence.

We write $\lim\limits_{n\to\infty} a_n = \infty$ if a_n is arbitrarily large by taking n sufficiently large. Formally, it means for every $M > 0$, there exists a number N such that

$$n > N \Rightarrow a_n > M.$$

Example 6.37. *Show that* $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0.$

Solution. Given $\epsilon > 0$, choose a positive integer N such that $\frac{1}{N} < \epsilon$. Then for any integer $n > N$, we have $|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$. This verifies the definition of limit. Therefore we have shown that $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

■

Now we give the proofs of the Ratio Test (Theorem 6.10) and the Alternating Series Test (Theorem 6.12).

Proof of Theorem 6.10 (Ratio Test). Notation and setting as in Theorem 6.10. Suppose $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$. Choose r such that $L < r < 1$. Then there exists a positive integer N such that

$$n > N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < r.$$

Then $|a_n| \le |a_{N+1}| r^{n-N-1}$ for all $n > N$. Since the geometric series $\sum |a_{N+1}| r^{n-N-1}$ is convergent, by comparison test, $\sum |a_n|$ is convergent.

Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$. Choose r such that $1 < r < L$. Then there exists a positive integer N such that

$$n > N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > r.$$

Then $|a_n| \ge |a_{N+1}| r^{n-N-1}$ for all $n > N$. Since $\lim_{n\to\infty} |a_{N+1}| r^{n-N-1} = \infty$ as $r > 1$, we have $\lim_{n\to\infty} |a_n| = \infty$.

Thus $\lim_{n\to\infty} a_n \neq 0$. Therefore, $\sum_{n=1}^{\infty} a_n$ is divergent by the test for divergence (Theorem 6.5).

■

Proof of Theorem 6.12 (Alternating Series Test). Notation and setting as in Theorem 6.12.

For simplicity, we only consider the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$. Let $S_n = \sum_{i=1}^{n} (-1)^{i-1} b_i$. Then $S_{2n} = (b_1 - b_2) + \cdots + (b_{2n-1} - b_{2n}) = b_1 - (b_2 - b_3) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$. The first equality shows that $S_{2n} \ge 0$ and $S_{2(n+1)} \ge S_{2n}$, The second equality shows that $S_{2n} \le b_1$ for all n. The sequence $\{S_{2n}\}$ is non-decreasing and bounded above, so it have a limit, say

$$\lim_{n\to\infty} S_{2n} = L.$$

As $S_{2n+1} = S_{2n} + b_{2n+1}$, we have

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1} = L + 0 = L.$$

Lastly, for any term S_n with $n \ge 2$, there exists a positive integer m such that $S_{2m} \le S_n \le S_{2m+1}$. [For example, $S_2 \le S_2 \le S_3, S_2 \le S_3 \le S_3, S_4 \le S_4 \le S_5, S_4 \le S_5 \le S_5$, etc.] Also, as n tends to infinity, m tends to infinity. Thus

$$\lim_{m\to\infty} S_{2m} \le \lim_{n\to\infty} S_n \le \lim_{m\to\infty} S_{2m+1}.$$

Since $\lim_{m\to\infty} S_{2m} = \lim_{m\to\infty} S_{2m+1} = L$, we have by Squeeze Theorem that $\lim_{m\to\infty} S_n = L$. This means the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is convergent.

■

Chapter 7

Vectors and Geometry of Space

Read Thomas' Calculus, Chapter 11.

In this chapter,

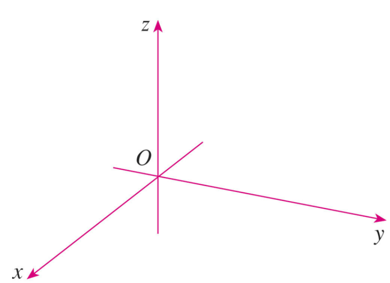
• we introduce the coordinate systems for three-dimensional space $\mathbb{R}^3$. This provides the setting for our study of calculus of functions of two and three variables. The simplest geometric notion is the distance between two points in space.

• we define vectors geometrically followed by the study their algebraic properties. We emphasize the power of algebraic manipulation of vectors. In particular, we look at dot product and cross product of vectors and their applications.

• we define lines and planes in $\mathbb{R}^3$.

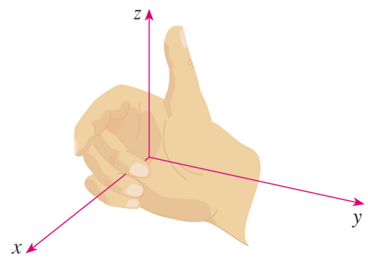
7.1 The 3D-Coordinate System

We set up the 3D coordinate system by fixing a point O in space (called the origin) and take three lines passing through O that are perpendicular to each other. These lines are labeled as x-axis, y-axis and z-axis respectively.

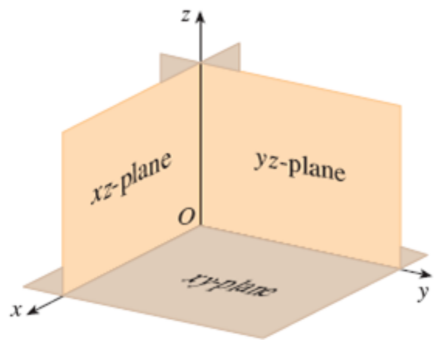


A point P in space can be represented by an ordered triple (a, b, c) where a , b and c are projections of the point P onto the x-, y- and z-axis respectively. The three dimensional space is also called the xyz-space.

The direction of the z-axis is determined by the right-hand rule:

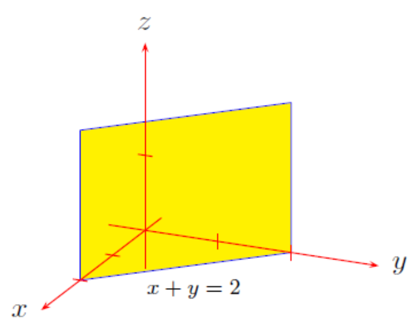


Any two of the axes determine a plane:



Example 7.1. *Describe and sketch the surface in* $\mathbb{R}^3$ *represented by the equation* $x + y = 2$.

Solution. Note that $x+y=2$ represents a line on the xy-plane. However, in $\mathbb{R}^3$, it represents the plane containing all points whose x- and y-coordinate sum to 2. This is a vertical plane.



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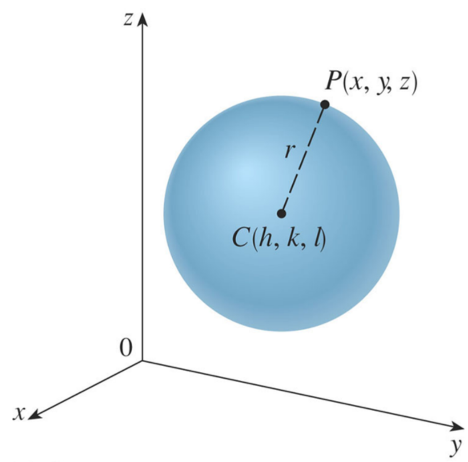
Theorem 7.1 (Distance Formula)**.**
The distance $|P_1P_2|$ *between the points* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *is*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Consequently, we have the following equation of a sphere:

Theorem 7.2 (Equation of Sphere)**.**
An equation of a sphere with center $C(h,k,l)$ and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2.$$



7.2 Vectors

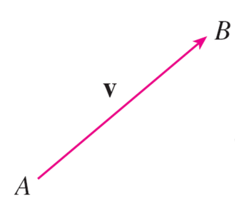
It can be complicated (messy) to describe an object in space directly using the coordinates x, y and z. It turns out to be easier by using **vectors**.

A vector is often represented by an arrow.

• The length of the arrow represents the magnitude of the vector.

• The arrow points in the direction of the vector.

For instance, suppose a particle moves along a line segment from point A to point B.



The vector $\mathbf{v}$ has initial point A (the tail) and terminal point B (the tip). We indicate this by writing $\mathbf{v}=\overrightarrow{AB}$. Call this the **displacement vector** of a particle from A to B.

We denote a vector by either:

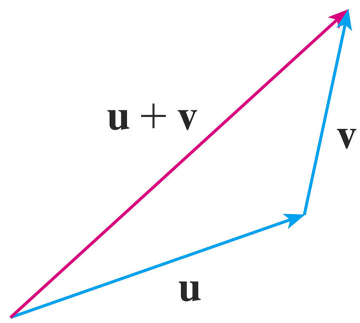
• Printing a letter in boldface $\mathbf{v}$, or

• Putting an arrow above the letter $\vec{v}$.

The zero vector, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

Note that a vector does not depend on its initial point.

Definition 7.1 (Adding Vectors – The Triangle Law).
Let $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors. Then their sum* $\mathbf{u}+\mathbf{v}$ *is the vector from the initial point of* $\mathbf{u}$ *to the terminal point of* $\mathbf{v}$ *when we position the vectors so that the initial point of* $\mathbf{v}$ *coincide with the terminal point of* $\mathbf{u}$.



Definition 7.2 (Scalar Multiplication).
Let $c \in \mathbb{R}$ *and* $\mathbf{u}$ *be a vector. The scalar multiple* $c\mathbf{u}$ *is the vector whose length is* $|c|$ *times the length of* $\mathbf{u}$ *and whose direction is the same as* $\mathbf{u}$ *if* $c > 0$ *and is opposite to* $\mathbf{u}$ *if* $c < 0$. *If* $c = 0$ *or* $\mathbf{u} = \mathbf{0}$, *then* $c\mathbf{u} = \mathbf{0}$.

Notice two nonzero vectors are parallel if they are scalar multiple of each other. By the difference $\mathbf{u}-\mathbf{v}$, we mean

$$\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v}).$$

To treat vectors systematically (algebraically), we place the initial point of a vector $\mathbf{u}$ at the origin O. In doing so, the terminal point has the coordinate (u_1, u_2) or (u_1, u_2, u_3) for some $u_1, u_2, u_3 \in \mathbb{R}$, depending on whether $\mathbf{u}$ is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Denote $\mathbf{u}$ by

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle.$$

u_1, u_2, u_3 are called the **components** of $\mathbf{u}$.

$\langle u_1, u_2, u_3 \rangle$ is also called the **position vector** of the point (u_1, u_2, u_3).

In other words, position vectors are vectors whose initial point is the origin.

Any vector can be represented by a position vector.

What is so nice about position vectors? The main advantage of representing vectors using position vectors is that we can simplify calculations by algebraic manipulations!

• To add position vectors, we can just add their corresponding components. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then

$$\mathbf{a}+\mathbf{b} = \langle a_1+b_1, a_2+b_2, a_3+b_3 \rangle.$$

• To multiply $\mathbf{a}$ by a scalar c, we multiply each component by that scalar.

$$c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

Theorem 7.3.
Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ *representing* $\overrightarrow{AB}$ *is*

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Proof.

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\
&= \overrightarrow{OB} - \overrightarrow{OA} \\
&= \langle x_2, y_2, z_2 \rangle - \langle x_1, y_1, z_1 \rangle \\
&= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.
\end{aligned}$$

■

Theorem 7.4 (Properties of Vectors)**.**
Suppose $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are vectors, and* $c, d \in \mathbb{R}$ *are scalars. Then*

(1) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	(2) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
(3) $\mathbf{a} + \mathbf{0} = \mathbf{a}$	(4) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
(5) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	(6) $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
(7) $(cd)\mathbf{a} = c(d\mathbf{a})$	(8) $1\mathbf{a} = \mathbf{a}$

These properties are readily verified geometrically or algebraically.

Three vectors in $\mathbb{V}_3$ play a special role. They are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle, \mathbf{k} = \langle 0, 0, 1 \rangle.$$

These vectors are called the **standard basis vectors**. They have length 1 and point in the direction of the positive x-, y- and z-axes respectively. Thus, any vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ can be written as

$$\begin{aligned}
\mathbf{a} &= \langle a_1, a_2, a_3 \rangle \\
&= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \\
&= a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}.
\end{aligned}$$

The **length** of a vector $\mathbf{u}$ is the length of any of its representation, and is denoted by $\|\mathbf{u}\|$ (in the textbook, the notation $|\mathbf{u}|$ is used). Using the distance formula, we have the following

The length of the vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

A **unit vector** is a vector whose length is 1. For example, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors.

Theorem 7.5.
If $\mathbf{a} \neq \mathbf{0}$*, then a unit vector in the same direction as* $\mathbf{a}$ *is given by*

$$\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|.$$

Notice $1/\|\mathbf{a}\|$ is a positive scalar, so $\mathbf{u}$ is in the same direction as $\mathbf{a}$. Now,

$$\|\mathbf{u}\| = \left\|\frac{\mathbf{a}}{\|\mathbf{a}\|}\right\| = \frac{1}{\|\mathbf{a}\|}\|\mathbf{a}\| = 1.$$

So $\mathbf{u}$ is the unit vector in the same direction as $\mathbf{a}$. ■

7.3 The Dot Product

The **dot product** of two vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is defined to be

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

The dot product satisfies the following properties

Theorem 7.6 (Properties of Dot Product)**.**
For vectors $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *and any scalar* d*,*

(i) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ *(commutativity)*

(ii) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ *(distributive law)*

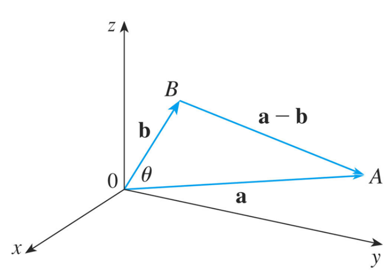
(iii) $(d\mathbf{a})\cdot\mathbf{b} = d(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(d\mathbf{b})$

(iv) $\mathbf{0}\cdot\mathbf{a} = 0$

(v) $\mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2$.

Notice $\mathbf{a}\cdot\mathbf{b} = 0$ does not imply that $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

For two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{V}_3$, we define the **angle** θ between them to be the smaller angle between $\mathbf{a}$ and $\mathbf{b}$, formed by placing their initial points at the origin.



- $\mathbf{a}$ and $\mathbf{b}$ have the same direction iff $\theta = 0$.
- $\mathbf{a}$ and $\mathbf{b}$ have opposite direction iff $\theta = \pi$.
- $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (perpendicular) iff $\theta = \frac{\pi}{2}$.

Theorem 7.7.
Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta.$$

Proof. Recall that the Law of Cosines says that

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\| \|\mathbf{b}\| \cos\theta.$$

$$\begin{aligned}
\|\mathbf{a} - \mathbf{b}\|^2 &= \|\langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle\|^2 \\
&= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\
&= (a_1^2 - 2a_1 b_1 + b_1^2) + (a_2^2 - 2a_2 b_2 + b_2^2) \\
&\quad + (a_3^2 - 2a_3 b_3 + b_3^2) \\
&= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b}.
\end{aligned}$$

Rearranging,

$$\begin{aligned}
2\mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 \\
&= 2\|\mathbf{a}\| \|\mathbf{b}\| \cos\theta,
\end{aligned}$$

the last equality follows from the Law of Cosines.
So

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta.$$

■

Example 7.2. *Find the angle between the vectors $\mathbf{a} = \langle 2, 1, -3 \rangle$ and $\mathbf{b} = \langle 1, 5, 6 \rangle$.*

Solution. We have

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{-11}{\sqrt{14}\sqrt{62}}.$$

It follows that

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{14}\sqrt{62}} \right) \approx 1.953\text{radian}.$$

■

Theorem 7.8.
Two vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Proof. If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a}$ and $\mathbf{b}$ are orthogonal as $\mathbf{0}$ is considered to be orthogonal to every vector.

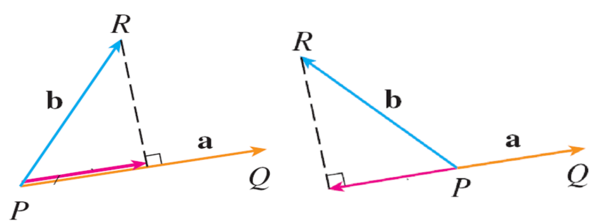
We may assume that $\mathbf{a}$ and $\mathbf{b}$ are nonzero. Then

$$\|\mathbf{a}\| \|\mathbf{b}\| \cos\theta = \mathbf{a} \cdot \mathbf{b} = 0$$

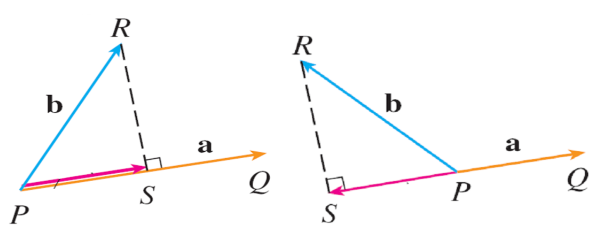
if and only if $\cos\theta = 0$, if and only if $\theta = \frac{\pi}{2}$, which is equivalent to saying that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. ■

7.4 Projections

The figure shows two vectors $\mathbf{a}$ and $\mathbf{b}$ with the same initial point representing $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.



Let S be the foot of the perpendicular line from R to the line containing $\overrightarrow{PQ}$.



The vector $\overrightarrow{PS}$ is called the **vector projection** of $\mathbf{b}$ **onto** $\mathbf{a}$, denoted by

$$\text{proj}_{\mathbf{a}}\mathbf{b}.$$

The **scalar projection** of $\mathbf{b}$ **onto** $\mathbf{a}$ (also called the **component of** $\mathbf{b}$ **along** $\mathbf{a}$) is defined to be the signed magnitude of the vector projection, and is denoted by

$$\text{comp}_{\mathbf{a}}\mathbf{b}.$$

Notice

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}.$$

This value is negative if $\frac{\pi}{2} < \theta \leq \pi$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Therefore,

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{a}}\mathbf{b} \times \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}.$$

Example 7.3. *Let* $\mathbf{a} = \langle -2, 3, 1\rangle$ *and* $\mathbf{b} = \langle 1, 1, 2\rangle$. *Find the scalar projection and vector projection of* $\mathbf{b}$ *onto* $\mathbf{a}$.

Solution. Notice $\|\mathbf{a}\| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$. So

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

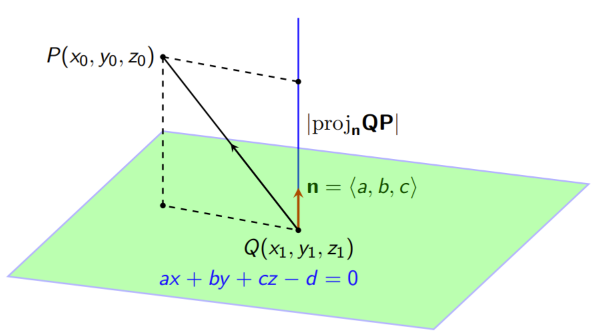
It follows that

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{3}{14}\mathbf{a} = \langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\rangle.$$

■

Theorem 7.9 (Distance from a point to a plane)**.**
The (shortest) distance from a point $P(x_0, y_0, z_0)$ *to the plane* $ax + by + cz = d$ *is given by*

$$\frac{\left|ax_0 + by_0 + cz_0 - d\right|}{\sqrt{a^2 + b^2 + c^2}}.$$



Proof. A normal vector to the plane is $\mathbf{n} = \langle a, b, c\rangle$. (See Section 7.7.) Pick any point $Q(x_1, y_1, z_1)$

on the plane so that $ax_1 + by_1 + cz_1 = d$. Then the shortest distance from P to the plane is

$$\begin{aligned}
\left\|\text{proj}_{\mathbf{n}} \overrightarrow{QP}\right\| &= \left|\text{comp}_{\mathbf{n}} \overrightarrow{QP}\right| \\
&= \left|\frac{\overrightarrow{QP} \cdot \mathbf{n}}{\|\mathbf{n}\|}\right| \\
&= \frac{\left|\langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle \cdot \langle a, b, c\rangle\right|}{\|\mathbf{n}\|} \\
&= \frac{\left|ax_0 + by_0 + cz_0 - d\right|}{\sqrt{a^2 + b^2 + c^2}}.
\end{aligned}$$

■

Example 7.4. *Find the distance from the point $(2, -3, 4)$ to the plane $x + 2y + 3z = 13$.*

Solution. We have $(x_0, y_0, z_0) = (2, -3, 4)$ and $a = 1, b = 2, c = 3, d = 13$. Using the formula, the distance is

$$\frac{|2(1) + (-3)(2) + 4(3) - 13|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{5}{\sqrt{14}}.$$

■

7.5 The Cross Product

We now define a second type of product of vectors, called the cross product or vector product. While the dot product is a scalar, the cross product is a vector.

For two vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, define the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ to be

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \\
&= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.
\end{aligned}$$

To compute $\mathbf{a} \times \mathbf{b}$, we must write the components of $\mathbf{a}$ in the second row of the determinant, and the components of $\mathbf{b}$ in the third row. The order is important!

One of the most important properties of the cross product is the following theorem.

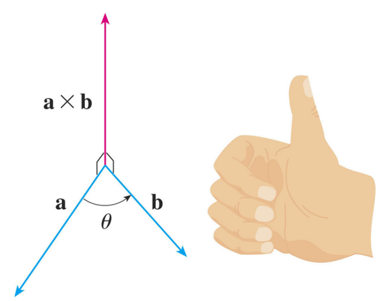
Theorem 7.10.
The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Proof. To show $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= (a_2b_3 - a_3b_2)a_1 - (a_1b_3 - a_3b_1)a_2 + (a_1b_2 - a_2b_1)a_3 \\ &= 0.\end{aligned}$$

A similar computation shows that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$. ■

The vector $\mathbf{a}\times\mathbf{b}$ points in a direction perpendicular to $\mathbf{a}$ and $\mathbf{b}$. This can be given by the right-hand rule as follows:



What is the geometric meaning of the length $\|\mathbf{a}\times\mathbf{b}\|$? This is given by the following theorem.

Theorem 7.11.
If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

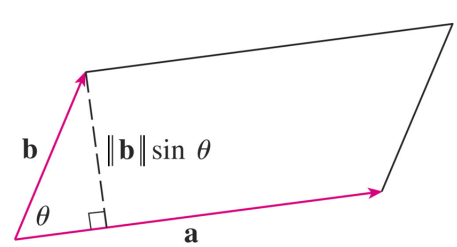
$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta.$$

We can use cross product

- to find the area of a parallelogram
- to find the distance from a point to a line in $\mathbb{R}^3$.

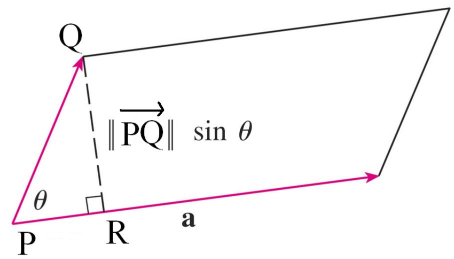
If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point then they determine a parallelogram with base $\|\mathbf{a}\|$, altitude $\|\mathbf{b}\|\sin\theta$. Therefore, the area of the parallelogram is given by

$$\|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta = \|\mathbf{a}\times\mathbf{b}\|.$$



The distance from Q to the line through P and R is

$$\left\|\overrightarrow{PQ}\right\|\sin\theta = \frac{\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|}{\left\|\overrightarrow{PR}\right\|}.$$



Some of the usual laws of algebra hold for cross products.

Theorem 7.12.
If $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are vectors and* d *is a scalar, then*

(i) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

(ii) $(d\mathbf{a})\times\mathbf{b} = d(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(d\mathbf{b})$

(iii) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$

(iv) $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$

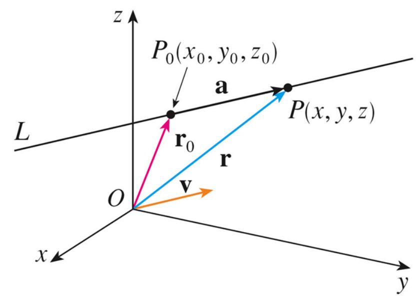
7.6 Lines

How do we write down an equation of a line in space? To do it, we must describe the behavior of a general point on the line. We shall see that vectors can help us to achieve this goal with minimal effort.

So let $P(x,y,z)$ denote an arbitrary point on the line L.

Let $\mathbf{r}$ and $\mathbf{r}_0$ denote the position vectors of P and P_0 respectively, where P_0 is a point on L which we have fixed. Our aim is to describe $\mathbf{r}$, the position vector of an arbitrary point on L.

Let $\mathbf{v}$ be a vector parallel to L, so $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar t since $\overrightarrow{P_0P}$ and $\mathbf{v}$ are parallel.



Then

$$\mathbf{r} = \mathbf{r}_0 + \overrightarrow{P_0P},$$

so that

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a **vector equation** of L.

Each **parameter** t gives the position vector $\mathbf{r}$ of a point on L. As t varies, the line is traced out by the tip of the vector $\mathbf{r}$.

We can write the vector equation in the component form:

$$\mathbf{v} = \langle a, b, c \rangle, \quad \mathbf{r_0} = \langle x_0, y_0, z_0 \rangle, \quad \mathbf{r} = \langle x, y, z \rangle.$$

Two vectors are equal if and only if the corresponding components are equal. Therefore, we have

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle.$$

Theorem 7.13 (Parametric Equation of Line)**.**

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Usually the parameter t (in the parametric equation of line) takes values on the entire $\mathbb{R}$ or an interval I.

The numbers a, b and c are called **direction numbers** of the line L.

The vector equation and parametric equations of a line are not unique.

If we change the point $\mathbf{r}_0$ or the parameter t or choose a different parallel vector $\mathbf{v}$, then the equations change. Therefore, direction numbers are not unique.

Example 7.5. *Find an equation of the line passing through $P(1, 2, -1)$ and $Q(5, -3, 4)$.*

Solution. A vector parallel to the line is

$$\overrightarrow{PQ} = \langle 5-1, -3-2, 4-(-1)\rangle = \langle 4, -5, 5\rangle.$$

Pick a point on the line, say $(1, 2, -1)$. Then the parametric equations for the line are

$$x = 1 + 4t, y = 2 - 5t, z = -1 + 5t.$$

■

Let L_1 and L_2 be two lines in $\mathbb{R}^3$, with parallel vectors **a** and **b**, respectively, and let θ be the angle between **a** and **b**.

• The lines L_1 and L_2 are parallel whenever **a** and **b** are parallel.

• If L_1 and L_2 intersect then θ is an angle between L_1 and L_2. Notice $\pi - \theta$ is also an angle between the lines.

In 2-D, two lines are either parallel or intersect. This is not true in 3-D. Non-parallel, non-intersecting lines are called **skew** lines.

Example 7.6. *Show that the lines*

$$L_1 : x - 2 = -t, y - 1 = 2t, z - 5 = 2t,$$

$$L_2 : x - 1 = s, y - 2 = -s, z - 1 = 3s,$$

are skew.

Solution. The lines are not parallel since a vector parallel to L_1 is $\mathbf{a} = \langle -1, 2, 2\rangle$ and a vector parallel to L_2 is $\mathbf{b} = \langle 1, -1, 3\rangle$. Since **a** is not a scalar multiple of **b**, these vectors are not parallel.

Assume for a contradiction that L_1 and L_2 intersect. Then there must exist a choice of the parameter t and s such that the values of x, y and z are the same. In particular, for the x-coordinate,

$$2 - t = 1 + s,$$

so that $s = 1 - t$.

On the other hand, the y-coordinate must satisfy

$$y = 1 + 2t = 2 - s.$$

Substituting $s = 1 - t$ into the last equation, we have $t = 0$ and so $s = 1$.

Now, the z-coordinate must satisfy

$$z = 5 + 2t = 5,$$

$$z = 1 + 3s = 4,$$

which is absurd!

Hence our assumption that L_1 and L_2 intersects was wrong. So the lines are skew, as desired.

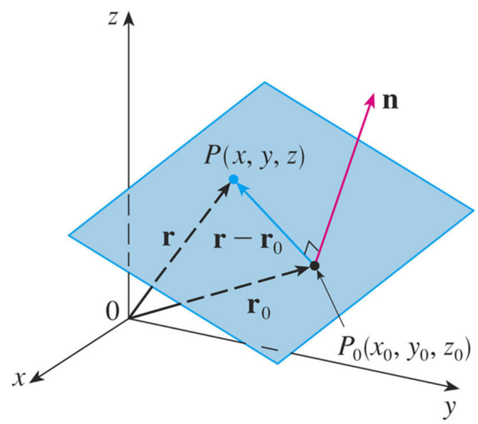
■

7.7 Planes

To get an equation for the plane, we need to describe an arbitrary point $P(x, y, z)$ on the plane. Again, we use vectors to help us do that.

Let $\mathbf{r}$ and $\mathbf{r}_0$ denote the position vectors of P and P_0 respectively, where P_0 is a fixed point on the given plane.

Then $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$.



The normal vector $\mathbf{n}$ (which is orthogonal to the plane) is always orthogonal to $\mathbf{r} - \mathbf{r}_0$. Therefore we have

Theorem 7.14 (Vector Equation of Plane)**.**

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be written as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0.$$

To obtain a scalar equation for the plane, write the vectors in component form and equate corresponding components:

$$\mathbf{n} = \langle a, b, c \rangle, \quad \mathbf{r} = \langle x, y, z \rangle, \quad \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle.$$

Then $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$ becomes

$$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle$$

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

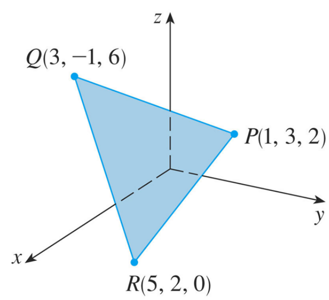
Theorem 7.15 (Linear Equation of Plane)**.**

$$ax + by + cz + d = 0,$$

where

$$d = -(ax_0 + by_0 + cz_0).$$

Example 7.7. *Find an equation of the plane that passes through the points* $P(1,3,2)$*,* $Q(3,-1,6)$*,* $R(5,2,0)$*.*



Solution. First, we need a vector **n** orthogonal to the plane. This can be given by

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

Notice

$$\overrightarrow{PQ} = \langle 2, -4, 4\rangle, \overrightarrow{PR} = \langle 4, -1, -2\rangle.$$

So

$$\begin{aligned} \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} \\ &= 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}. \end{aligned}$$

With the point $P(1,3,2)$ and the normal vector **n**, an equation of the plane is:

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

or after simplifications,

$$6x + 10y + 7z = 50.$$

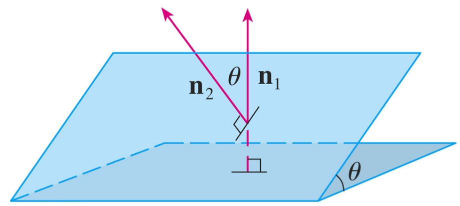
■

Two planes are parallel if their normal vectors are parallel.

If two planes are not parallel, then

• They intersect in a straight line.

• The angle between the two planes is defined as the **acute** angle between their normal vectors



Example 7.8. *(a) Find the angle between the planes $x + 2y + z = 3$ and $x - 4y + 3z = 5$.*

(b) Find the line of intersection of these two planes.

Solution. (a) The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 2, 1 \rangle, \mathbf{n}_2 = \langle 1, -4, 3 \rangle.$$

So, if θ is the angle between them, then

$$\begin{aligned} \theta &= \cos^{-1} \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \\ &= \cos^{-1} \frac{1(1) + 2(-4) + 1(3)}{\sqrt{1+4+1}\sqrt{1+16+9}} \\ &= \cos^{-1} \frac{-4}{\sqrt{156}} \approx 108.7^\circ \end{aligned}$$

Therefore, the angle between the planes is 71.3°.

(b) Solving both equations for x,

$$x = 3 - 2y - z \quad \text{and} \quad x = 5 + 4y - 3z.$$

Setting them to be equal gives

$$3 - 2y - z = 5 + 4y - 3z.$$

Solving for z gives

$$z = 3y + 1.$$

Substituting this into the first equation,

$$x = 3 - 2y - (3y + 1) = -5y + 2.$$

Let $y = t$ be the parameter, we obtain a parametric equation for the line of intersection

$$x = -5t + 2, \quad y = t, \quad z = 3t + 1.$$

■

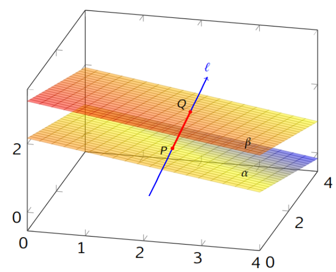
Example 7.9. *Consider the planes* $\alpha : 3x + 4y + 12z = 26$ *and* $\beta : 3x + 4y + 12z = 39$.

(a) Find the distance between α *and* β.

(b) Let ℓ *be the line with parametric equations*

$$x = 2 + 3t, \;\; y = 2 + 4t, \;\; z = 1 + 12t.$$

Find the intersection point Q *of* ℓ *with* β.



Ans: (a) 1, (b) $(\frac{29}{13}, \frac{30}{13}, \frac{25}{13})$.

Solution. (a) Note that $P(2,2,1)$ is a point on α, and the two planes α and β are parallel. Then the distance between α and β is given by

$$\frac{|3(2) + 4(2) + 12(1) - 39|}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{|-13|}{13} = 1.$$

(b) Substituting the parametric equations of ℓ into the equation of β, we have

$3(2 + 3t) + 4(2 + 4t) + 12(1 + 12t) = 39 \Leftrightarrow 26 + 169t = 39 \Leftrightarrow t = \frac{1}{13}$.

Therefore, the intersection point Q is

$$(2 + \frac{3}{13}, 2 + \frac{4}{13}, 1 + \frac{12}{13}) = (\frac{29}{13}, \frac{30}{13}, \frac{25}{13}).$$

Note that **PQ** is perpendicular to α and β, and $||\mathbf{PQ}|| = 1$.

■

Exercise 7.1. *Let* **a** *and* **b** *be vectors in* $\mathbb{R}^3$. *Prove that* $(\mathbf{a} \cdot \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$.

Exercise 7.2. *Find the distance from the point* $P(3,3,3)$ *to the line* $\ell : \; x = t, y = \frac{t}{2}, z = t$.

Ans. $\sqrt{2}$.

Exercise 7.3. *Find the point on the surface* $z = x^2 + y^2 + 10$ *that is nearest to the plane* $x + 2y - z = 0$.

Ans. $(\frac{1}{2}, 1, \frac{45}{4})$.

Chapter 8

Functions of Several Variables

Read Thomas' Calculus, Chapter 13.

8.1 Vector Functions of One Variable

Recall the vector equation of line:
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}.$$
We have seen that the tip of the vector $\mathbf{r}(t)$ traces out a line as t varies.

We can rewrite the above as follows:

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle.$$

Notice that each component of $\mathbf{r}(t)$ is a scalar function of t.

In general, a vector-valued function is

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$$

where $f(t)$, $g(t)$ and $h(t)$ are scalar functions of t.

Formally,

A **vector-valued function** $\mathbf{r}(t)$ is a mapping from its domain $D \subseteq \mathbb{R}$ to its range $R \subseteq \mathbb{V}_3$, so that for each $t \in D$, $\mathbf{r}(t) = \mathbf{v}$ for exactly one vector $\mathbf{v} \in \mathbb{V}_3$.

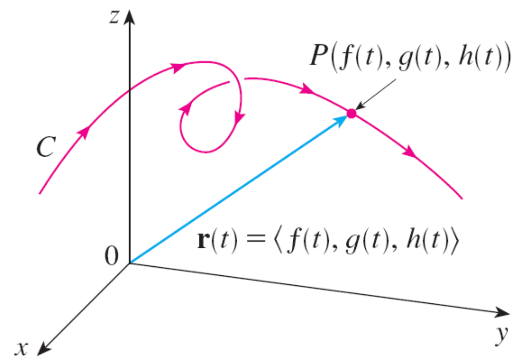
We write a vector-valued function as

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

or

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$$

for some scalar function f, g and h (called the **component functions** of $\mathbf{r}$).



Suppose $\mathbf{r}(t)$ traces out the curve C, we say that $\mathbf{r}(t)$ is a **parametrization** of C.

A curve C can have more than one parametrizations.

For example, both

$$\mathbf{r}(t) = \langle t, t^2 \rangle, \quad t \in \mathbb{R}$$

$$\mathbf{r}(t) = \langle t^3, t^6 \rangle, \quad t \in \mathbb{R}$$

parameterize the parabola $f(x) = x^2$ on the xy-plane.

Example 8.1. *Sketch the curve traced out by the vector-valued function* $\mathbf{r}(t) = \sin t\mathbf{i} - 3\cos t\mathbf{j} + 2t\mathbf{k}$.

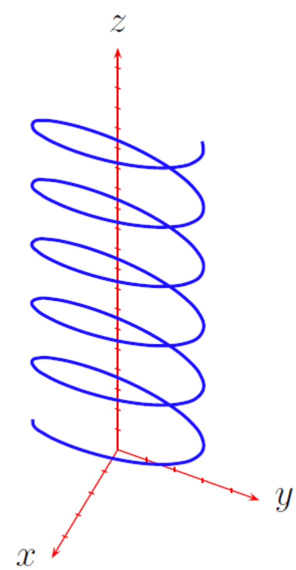
Solution. There is a relationship between x and y here:

$$x^2 + \left(\frac{y}{3}\right)^2 = \sin^2 t + \cos^2 t = 1$$

which is the equation of an ellipse in 2-D. In 3-D, since the equation does not involve z, it becomes the equation of an elliptic cylinder whose axis is the z-axis.

The curve will wind its way up the cylinder anticlockwise as t increases.

We call this curve an **elliptical helix**.



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8.2 Calculus of Vector Functions

To extend differentiation and integration to vector-valued functions, just think

'component-wise'!!!

Definition 8.1. *The* **derivative** $\mathbf{r}'(t)$ *of the vector-valued function* $\mathbf{r}(t)$ *is defined by*

$$\mathbf{r}'(t) = \lim_{\triangle t \to 0} \frac{\mathbf{r}(t + \triangle t) - \mathbf{r}(t)}{\triangle t}$$

for any values of t for which the limit exists.

When the limit exists for $t = a$, we say that $\mathbf{r}$ is **differentiable** at $t = a$.

The derivative of a vector-valued function can be found directly from the derivatives of the components.

Theorem 8.1 (Derivative of Vector-valued Function)**.**
Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and suppose that the components f, g and h are all differentiable at $t = a$. Then $\mathbf{r}$ is differentiable at $t = a$ and its derivative is given by

$$\mathbf{r}'(a) = \langle f'(a), g'(a), h'(a) \rangle.$$

Example 8.2. *Let* $\mathbf{r}(t) = \langle 3t^2 + 2t + 1, \sin(\pi t), e^{2t} \rangle$*. Find* $\mathbf{r}'(t)$*, and compute* $\mathbf{r}'(1)$*.*

Solution.

$$\begin{aligned}\mathbf{r}'(t) &= \langle \frac{d}{dt}(3t^2 + 2t + 1), \frac{d}{dt}\sin(\pi t), \frac{d}{dt}e^{2t} \rangle \\ &= \langle 6t + 2, \pi\cos(\pi t), 2e^{2t} \rangle.\end{aligned}$$

Hence

$$\mathbf{r}'(1) = \langle 6 + 2, \pi\cos\pi, 2e^2 \rangle = \langle 8, -\pi, 2e^2 \rangle.$$

■

Theorem 8.2 (Derivative Rules)**.**
Suppose $\mathbf{r}(t)$ *and* $\mathbf{s}(t)$ *are differentiable vector-valued functions,* $f(t)$ *is a differentiable scalar function and* c *is a scalar constant. Then*

(i) $\frac{d}{dt}(\mathbf{r}(t) + \mathbf{s}(t)) = \mathbf{r}'(t) + \mathbf{s}'(t)$

(ii) $\frac{d}{dt}(c\mathbf{r}(t)) = c\mathbf{r}'(t)$

(iii) $\frac{d}{dt}f(t)\mathbf{r}(t) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$

(iv) $\frac{d}{dt}\mathbf{r}(t) \cdot \mathbf{s}(t) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$

(v) $\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{s}(t)) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$.

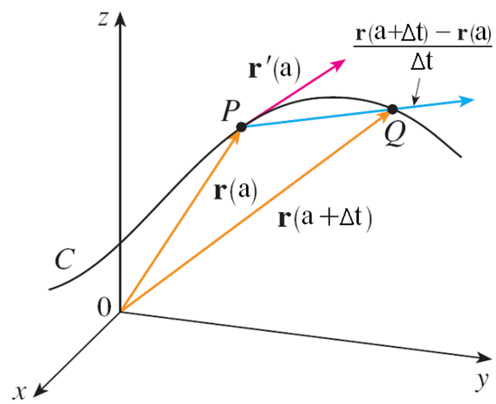
8.3 Tangent Vector and Tangent Line to a Curve

Recall that one interpretation of the derivative of a scalar function is that the value of the derivative at a point gives the slope of the tangent line to the curve at that point.

There is a similar interpretation for the derivative of vector-valued functions.

Recall

$$\mathbf{r}'(a) = \lim_{\triangle t \to 0} \frac{\mathbf{r}(a + \triangle t) - \mathbf{r}(a)}{\triangle t}.$$



The tangent vector

Notice that for $\triangle t > 0$, the vector $\frac{\mathbf{r}(a+\triangle t)-\mathbf{r}(a)}{\triangle t}$ points in the same direction as $\mathbf{r}(a+\triangle t)-\mathbf{r}(a)$.

As $\triangle t \to 0$, $\frac{\mathbf{r}(a+\triangle t)-\mathbf{r}(a)}{\triangle t}$ approaches $\mathbf{r}'(a)$.

This is a vector tangent to the curve at $\mathbf{r}(a)$. We call $\mathbf{r}'(a)$ a **tangent vector** to the curve at $t = a$.

Example 8.3. *Find the tangent line L to the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ at $(0,1,\pi/2)$.*

Solution. At point $(0,1,\pi/2)$, $t = \pi/2$. Since $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$, a direction of the tangent line L is

$$\mathbf{r}'(\pi/2) = \langle -\sin(\pi/2), \cos(\pi/2), 1\rangle = \langle -1, 0, 1\rangle$$

So a parametric equation of L is

$$x = 0 + (-1)t, \quad y = 1 + (0)t, \quad z = \pi/2 + (1)t,$$

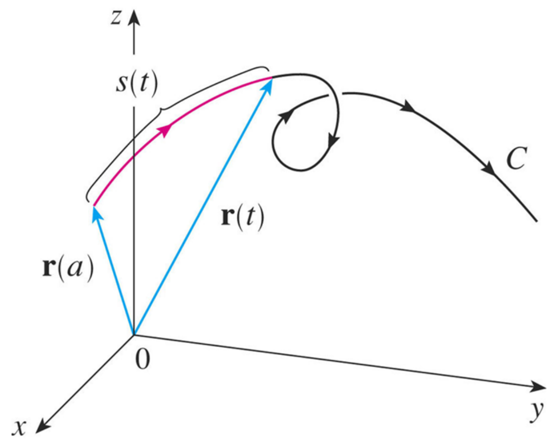
that is

$$x = -t, \quad y = 1, \quad z = \pi/2 + t.$$

■

8.4 Arc Length of a Space Curve

Suppose that a smooth curve C is traced out by the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, f', g, g', h, h' are all continuous for $t \in [a,b]$, and the curve is traversed exactly once as t increases from a to b. Then the arc length of C is given by the following result.



Theorem 8.3 (Arc Length Formula)**.**
Let C be the curve given by

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle, \quad a \le t \le b$$

where f', g' and h' are continuous. If C is traversed exactly once as t increases from a to b, then its length is

$$\begin{aligned} s &= \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt \\ &= \int_a^b \left\|\mathbf{r}'(t)\right\| dt. \end{aligned}$$

Example 8.4. *Find the arclength of the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle 2t, \ln t, t^2\rangle$ for $1 \le t \le e$.*

Solution.

$$\begin{aligned} s &= \int_1^e \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\,dt \\ &= \int_1^e \sqrt{\frac{1 + 4t^2 + 4t^4}{t^2}}\,dt \\ &= \int_1^e \sqrt{\frac{(1 + 2t^2)^2}{t^2}}\,dt \\ &= \int_1^e \frac{1 + 2t^2}{t}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt \\ &= \left(\ln|t| + t^2\right)\Big|_1^e \\ &= (\ln e + e^2) - (\ln 1 + 1) = e^2. \end{aligned}$$

■

So far we have seen functions of one variable, i.e. the domain is a subset of $\mathbb{R}$

function	Domain D	Range R
(scalar) $f(t)$	$D \subseteq \mathbb{R}$	$R \subseteq \mathbb{R}$
(vector) $\mathbf{r}(t)$	$D \subseteq \mathbb{R}$	$R \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$

However, in the real world, physical quantities often depend on two or more variables.

8.5 Functions of Two Variables

Definition 8.2.
A function f of two variables is a rule that assigns to each **ordered pair** *of real numbers (x, y) in a set $D \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ a* **unique** *real number denoted by $f(x, y)$.*

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be:

the set of all pairs (x, y) for which the given expression is a well-defined real number.

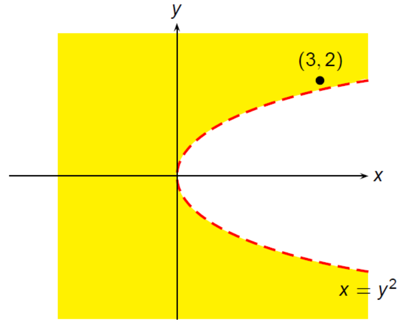
Example 8.5. *Find the domain of*

$$f(x, y) = x \ln(y^2 - x).$$

Solution. $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$.

So the domain of f is

$$D = \{(x, y) : x < y^2\}.$$



Domain of the function $f(x, y) = x \ln(y^2 - x)$

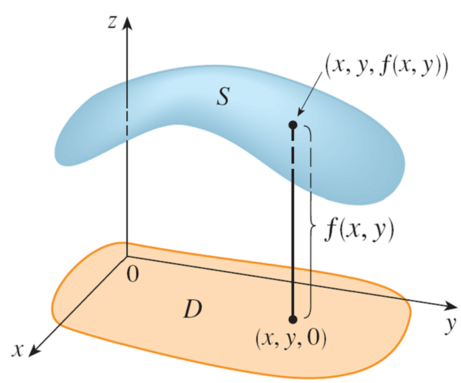
■

One way of visualizing $z = f(x,y)$ is to draw its graph.

If f is a function of two variables with domain D, then the **graph** of f is the set of all points $(x,y,z) \in \mathbb{R}^3$ such that $z = f(x,y)$ and $(x,y) \in D$.

The graph of a function f of two variables is also called the **surface** S with equation $z = f(x,y)$.

We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane.



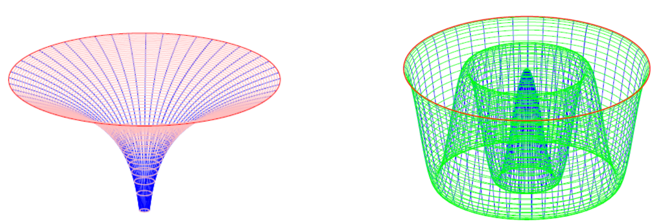
Graphing functions of more than one variable is not simple!

For most functions of two variables, to identify the surface, we must

(1) take hints from the expressions $z = f(x,y)$

(2) think through the traces and piece together the clues

Example 8.6. *Match the functions $f(x,y) = \ln(x^2+y^2)$ and $g(x,y) = \cos(x^2+y^2)$ to the surfaces shown below:*



Solution. Notice both functions contain the expression x^2+y^2. This is significant: given any value r and any point (x,y) on the circle $x^2+y^2 = r^2$, the height of the surface at the point (x,y) is a constant. That is, the surface has circular cross sections parallel; to the xy-plane.

However, both surfaces shown have this property, so we cannot yet tell which surface is matched by which function.

Notice the cosine of any angle lies between 1 and -1. So the second graph is $g(x,y)$.

An important property of $f(x,y)$ is that the logarithm tends to $-\infty$ as its argument x^2+y^2 approaches 0. This appears in the first graph.

■

Another way to visualize functions of several variables is to use the contour plot which provide the same information condensed into a 2-D picture.

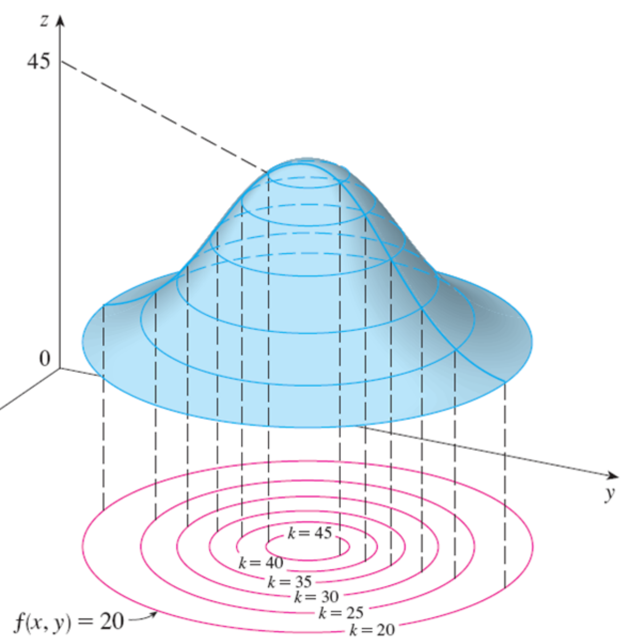
Definition 8.3 (Level Curve)**.**
A **level curve** *of $f(x,y)$ is the two-dimensional graph of the equation $f(x,y)=k$ for some constant k.*

Definition 8.4 (Contour Plot)**.**
A **contour plot** *of $f(x,y)$ is a graph of numerous level curves $f(x,y)=k$, for representative values of k.*

To sketch contour plots, we use values of k that are **equally spaced**. The surface is:

• steep where the level curves are close together.

• flatter where the level curves are farther apart.



Example 8.7. *Sketch some level curves of $h(x,y)=4x^2+y^2$.*

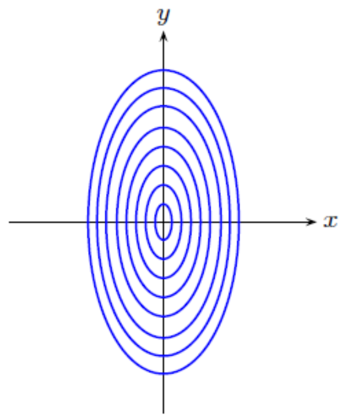
Solution. If $k<0$, then $4x^2+y^2=k$ has no solution, so there is no level curves for $k<0$.

If $k=0$, then there is only one solution $(0,0)$, so the level curve is a single point $(0,0)$.

If $k > 0$, then $4x^2 + y^2 = k$ is an ellipse:

$$\frac{x^2}{(\sqrt{k}/2)^2} + \frac{y^2}{\sqrt{k}^2} = 1.$$

Thus, larger k gives rise to an ellipse with longer major and minor axes.



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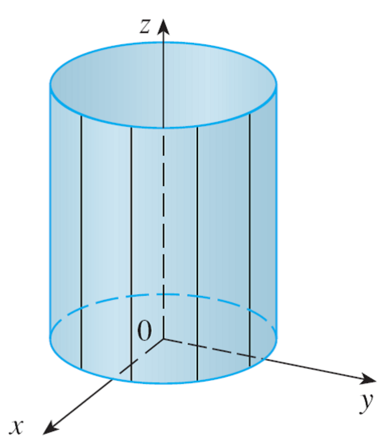
8.6 Cylinders and Quadric Surfaces

We have seen that the graph of functions of two variables are surfaces in space $\mathbb{R}^3$. To appreciate the calculus of functions of two variables, we need more examples of surfaces in space other than planes $(ax + by + cz = d)$and spheres $((x-a)^2 + (y-b)^2 + (z-c)^2 = d^2)$.

It is not easy to draw surfaces in $\mathbb{R}^3$. Our goal here is to identify and sketch some special type surfaces, namely the **cylinders** and some **quadric surfaces**.

8.6.1 Cylinders

When we mention the word cylinder, we probably think of the following object



Actually, the term cylinder is used to refer to a surface more general that the one we saw.

Definition 8.5.
A surface is a cylinder if there is a plane P such that all the planes parallel to P intersect the surface in the same curve (when viewed in 2-dimension).

Example 8.8. *Show that the surface given by*

$$y^2 + z^2 = 1$$

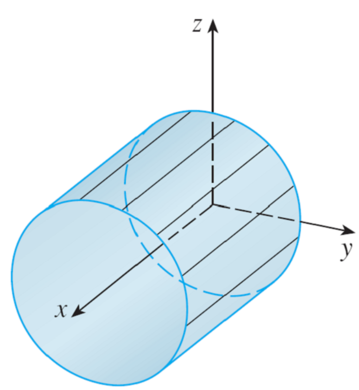
is a cylinder.

Solution. Notice x is missing in the equation. When $x = 0$, $y^2 + z^2 = 1$ is a circle with radius 1 in the yz-plane, which is the intersection of the surface and the yz-plane.

Generally, $x = k$ represent a plane parallel to the yz-plane, and the intersection the surface and this plane is always the circle $y^2 + z^2 = 1$.

Therefore, the surface $y^2 + z^2 = 1$ is a cylinder.

In fact, any equation in x, y and z where one of the variable is missing is a cylinder.



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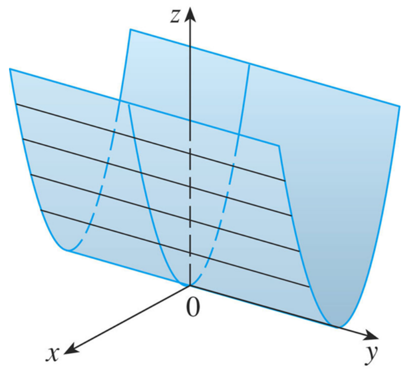
Example 8.9. *Sketch the graph of the surface $z = x^2$.*

Solution. Notice that the equation of the graph, $z = x^2$, does not involve y.

This means that any vertical plane with equation $y = k$ (parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$. So the surface $z = x^2$ is a cylinder.



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8.6.2 Quadric Surface

Definition 8.6 (Quadric Surface)**.**
A **quadric surface** *is the graph of a second-degree equation in three variables x, y and z:*

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where A, B, ..., J are constants.

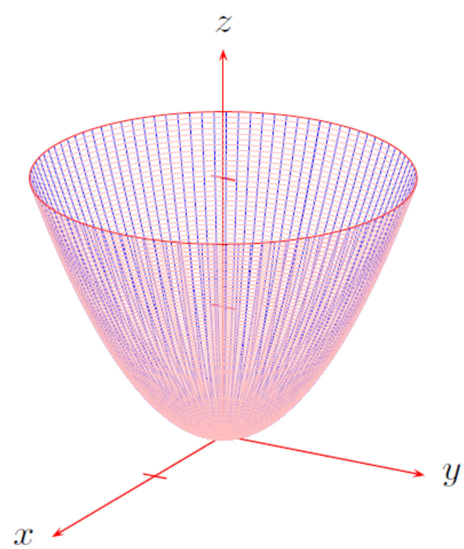
There are six basic quadric surfaces. But we shall focus on two of them:

- Elliptic paraboloid
- Ellipsoid

Definition 8.7 (Elliptic paraboloid – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

The graph of the elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ when $c > 0$.



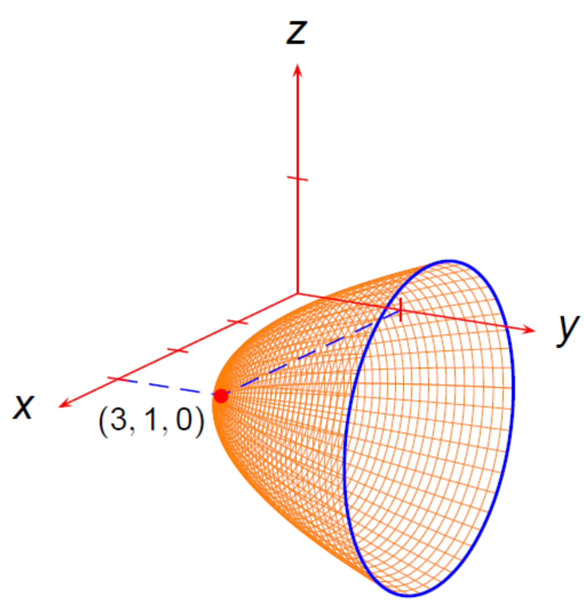
Example 8.10. *Identify and sketch the surface*

$$x^2 + 2z^2 - 6x - y + 10 = 0.$$

Solution. By completing squares, we rewrite the equation as

$$(y-1) = (x-3)^2 + \frac{z^2}{1/2}$$

It represents an elliptic paraboloid. However, it has been shifted so that its vertex is the point $(3,1,0)$, and is symmetric about the line which is parallel to the y-axis and passes through $(3,1,0)$.



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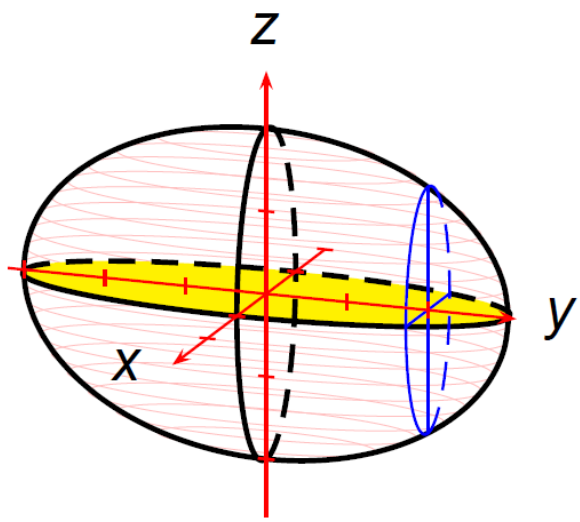
Definition 8.8 (Ellipsoid)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

If $a = b = c$, then the ellipsoid is a sphere.



Example 8.11. *Sketch the quadric surface with equation*

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$$

Solution. The surface intersects the xy-plane ($z = 0$) in the ellipse

$$x^2 + \frac{y^2}{9} = 1.$$

In general, the surface intersects the plane $z = k$ in the ellipse

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4},$$

provided $1 - \frac{k^2}{4} > 0$, that is $-2 < k < 2$.

The surface also intersects every plane $x = k$ (which is parallel to the yz-plane ($x = 0$)) in the ellipse

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2,$$

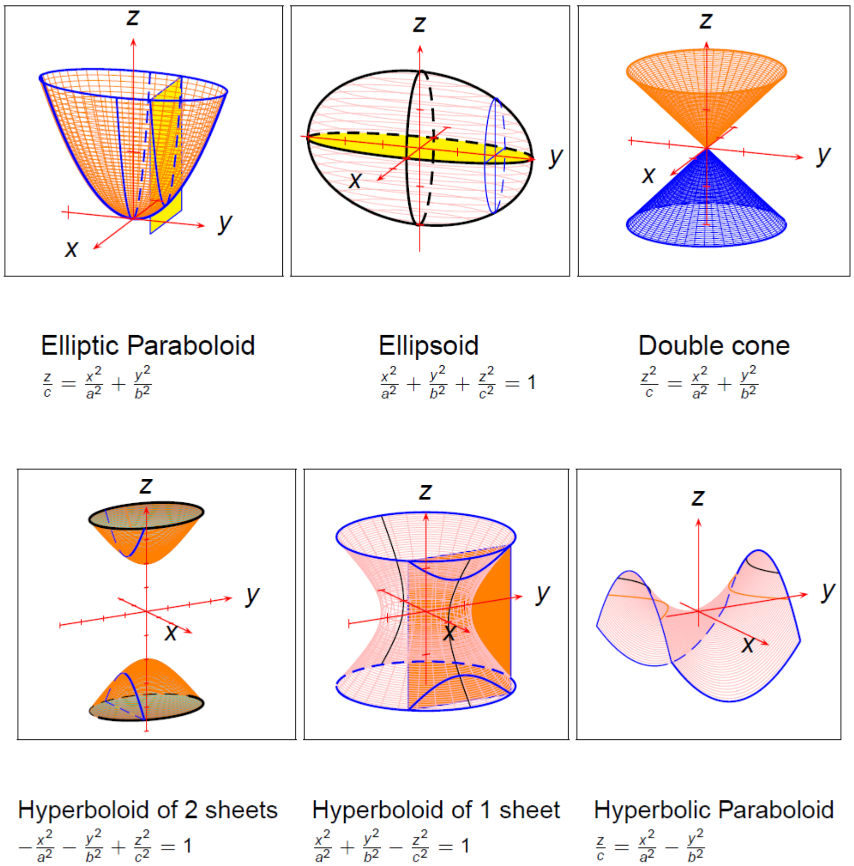
provided $1 - k^2 > 0$, that is $-1 < k < 1$.

The surface also intersects every planes $y = k$ which is parallel to the xz-plane ($y = 0$) in ellipse

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9},$$

provided $1 - \frac{k^2}{9} > 0$, that is $-3 < k < 3$.

■



8.7 Functions of Three Variables

Definition 8.9.
A function f of three variables is a rule that assigns to each **ordered triple** *of real numbers (x, y, z) in a set $D \subseteq \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ a* **unique** *real number denoted by $f(x, y, z)$.*

Unlike functions of two variables, it is very difficult to visualize a function f of three variables by its graph.

That would lie in a four-dimensional space!!!

However, we do gain some insight into f by examining its level surfaces (counterparts of level curves in two-variable case).

Definition 8.10 (Level Surface).
A **level surface** *of $f(x,y,z)$ is the three-dimensional graph of the equation $f(x,y,z) = k$ for some constant k.*

If the point (x,y,z) moves along a level surface, the value of $f(x,y,z)$ remains fixed.

Example 8.12. *Find the level surfaces of the function*

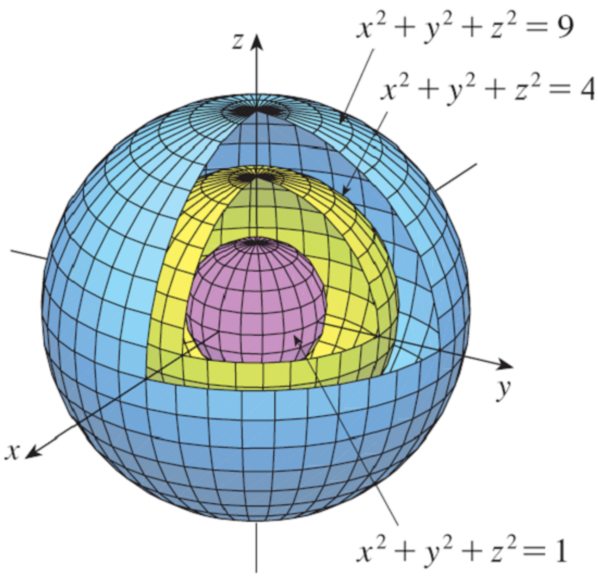
$$f(x,y,z) = x^2 + y^2 + z^2.$$

Solution. The level surfaces are:

$$x^2 + y^2 + z^2 = k$$

where $k \geq 0$.

These form a family of concentric spheres with radius $\sqrt{k}$.



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8.8 Partial Derivatives

Recall that for a function f of a single variable, we define the derivative function as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for any values of x for which the limit exists.

At a particular point $x = a$, we interpret $f'(a)$ as the instantaneous rate of change of f with respect to x at that point.

We want to generalize the notion of derivative to functions of more than one variable.

The idea is to 'vary' one variable and keep other variable(s) fixed.

Definition 8.11 (Partial Derivative)**.**
If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by:

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h},$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$

Example 8.13. *Let $f(x,y) = x^2y$. Then*

$$f_x(x,y) = \lim_{h\to 0} \frac{(x+h)^2y - x^2y}{h} = \lim_{h\to 0} \frac{(x^2y+2xhy+h^2y)-x^2y}{h} = \lim_{h\to 0}(2xy+hy) = 2xy.$$

$$f_y(x,y) = \lim_{h\to 0} \frac{x^2(y+h)-x^2y}{h} \lim_{h\to 0} x^2 = x^2.$$

There are many alternative notations for partial derivatives:

Instead of f_x, we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the first variable x) or

$$\frac{\partial f}{\partial x}.$$

To compute the partial derivative f_x, one may simply do the following: Treat (temporarily) the other variable y in $f(x,y)$ as a constant, and differentiate $f(x,y)$ with respect to the variable x.

Similarly, to compute f_y, one may simply do this: Treat (temporarily) the other variable x in $f(x,y)$ as a constant, and differentiate $f(x,y)$ with respect to the variable y.

Example 8.14. *For $f(x,y) = e^{xy} + \frac{x}{y}$, compute f_x and f_y. Find also $\frac{\partial f}{\partial x}(2,1)$ and $\frac{\partial f}{\partial y}(2,1)$.*

Solution. Treating y as a constant, we have

$$f_x(x,y) \quad = \quad ye^{xy} + \frac{1}{y}.$$

Treating x as a constant, we have

$$f_y(x,y) \quad = \quad xe^{xy} - \frac{x}{y^2}.$$

Hence at $(x, y) = (2, 1)$, we have

$$\frac{\partial f}{\partial x}(2,1) = f_x(2,1) = 1 \cdot e^{2\cdot 1} + \frac{1}{1} = e^2 + 1,$$
$$\frac{\partial f}{\partial y}(2,1) = f_y(2,1) = 2 \cdot e^{2\cdot 1} - \frac{2}{1^2} = 2e^2 - 2.$$

■

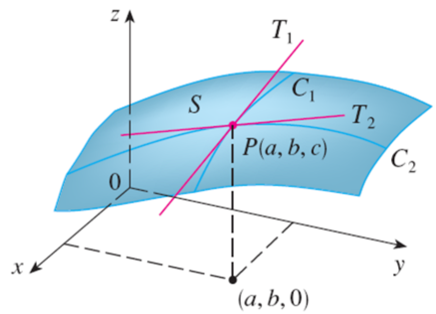
To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f).

If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.

By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. That is, C_1 is the trace of S in the plane $y = b$.

Likewise, the vertical plane $x = a$ intersects S in a curve C_2.

Both the curves C_1 and C_2 pass through P.



• The curve C_1 is the graph of the function $g(x) = f(x, b)$. So, the slope of its tangent T_1 at P is: $g'(a) = f_x(a, b)$.

• The curve C_2 is the graph of the function $h(x) = f(a, y)$. So, the slope of its tangent T_2 at P is: $h'(b) = f_y(a, b)$.

Thus, the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as:

The slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

For functions of more than two variables, such as $w = f(x, y, z)$, we can similarly define

$$f_x, f_y, f_z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \quad \text{or} \quad \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}.$$

Example 8.15. *Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation* $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution. Take partial derivative with respect to x on both sides:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6yx\frac{\partial z}{\partial x} = 0.$$

Solving for $\frac{\partial z}{\partial x}$, we have

$$(3z^2 + 6yx)\frac{\partial z}{\partial x} = -(3x^2 + 6yz) \implies \frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Similarly, we have $\quad \dfrac{\partial z}{\partial y} = -\dfrac{y^2 + 2xz}{z^2 + 2xy}.$

■

8.9 Higher Order Partial Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.

So, we can consider their partial derivatives

$$(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y.$$

These are called the second partial derivatives of f.

If $z = f(x,y)$, we use the following notation:

$$\begin{array}{ccccccccccc}
(f_x)_x & = & f_{xx} & = & f_{11} & = & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & = & \frac{\partial^2 f}{\partial x^2} & = & \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y & = & f_{xy} & = & f_{12} & = & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) & = & \frac{\partial^2 f}{\partial y \partial x} & = & \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x & = & f_{yx} & = & f_{21} & = & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) & = & \frac{\partial^2 f}{\partial x \partial y} & = & \frac{\partial^2 z}{\partial x \partial y} \\
(f_y)_y & = & f_{yy} & = & f_{22} & = & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) & = & \frac{\partial^2 f}{\partial y^2} & = & \frac{\partial^2 z}{\partial y^2}
\end{array}$$

Thus, the notation f_{xy} means that we first differentiate with respect to x and then with respect to y.

In computing f_{yx} , the order is reversed.

Example 8.16. *Find all second-order partial derivatives of $f(x,y) = x^2y - y^3 + \ln x$.*

Solution. First, we compute the first-order derivatives:

$$f_x = 2xy + \frac{1}{x},$$

$$f_y = x^2 - 3y^2.$$

Then we have

$$f_{xx} = \frac{\partial}{\partial x}\left(2xy + \frac{1}{x}\right) = 2y - \frac{1}{x^2},$$

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}\left(2xy + \frac{1}{x}\right) &=& \quad 2x, \\ f_{yx} &= \frac{\partial}{\partial x}\left(x^2 - 3y^2\right) &=& \quad 2x, \\ f_{yy} &= \frac{\partial}{\partial y}\left(x^2 - 3y^2\right) &=& \quad -6y. \end{aligned}$$

■

Notice $f_{xy} = f_{yx}$ in the preceding example. This is not a coincidence.

It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most (not all) functions that one meets in practice.

The following theorem, discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Theorem 8.4 (Clairaut's Theorem)**.**
Suppose f is defined on a disk D that contains (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Partial derivatives of order 3 and higher can also be defined. For example, $f_{xyy} = (f_{xy})_y$.

Using Clairaut's Theorem, it can be shown that

$$f_{xyy} = f_{yxy} = f_{yyx}$$

if these functions are continuous.

8.10 Tangent Planes

Recall that we use derivative $f'(a)$ to get the tangent line to the curve $y = f(x)$ at $x = a$:

$$y = f(a) + f'(a)(x-a).$$

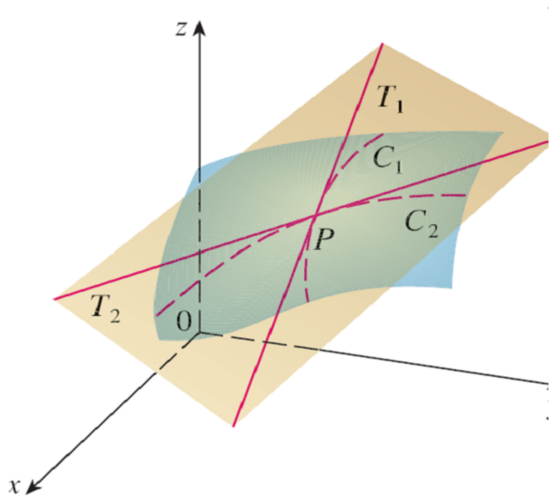
In the same spirit, we shall use partial derivatives to obtain the **tangent plane** to the surface $z = f(x,y)$ at a given point.

Consider the surface S which is the graph of $z = f(x,y)$. Suppose f has continuous first partial derivatives.

Let $P(a,b,c)$ be a point on S. Notice $c = f(a,b)$.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = b$ and $x = a$ with the surface S. Notice P lies on both C_1 and C_2.

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.



Then, the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

How to find an equation for the tangent plane?

Recall that any plane passing through $P(a,b,c)$ has an equation of the form

$$\mathbf{n} \cdot \langle x-a, y-b, z-c \rangle = 0$$

where $\mathbf{n}$ is a vector normal to the plane.

Notice the tangent line T_1 lies on the plane $y = b$. Along T_1 at $x = a$, a change of 1 unit in x corresponds to a change of $f_x(a,b)$ in z (here we require f_x to be continuous). The value of y does not change along the line. A vector with the same direction as T_1 is

$$\langle 1, 0, f_x(a,b) \rangle.$$

Similarly, a vector with the same direction as T_2 is

$$\langle 0, 1, f_y(a,b) \rangle.$$

We have found two vectors parallel to the tangent plane:

$$\langle 1, 0, f_x(a,b) \rangle, \langle 0, 1, f_y(a,b) \rangle.$$

A vector normal to the plane is given by the cross product:

$$\langle 0, 1, f_y(a,b)\rangle \times \langle 1, 0, f_x(a,b)\rangle = \langle f_x(a,b), f_y(a,b), -1\rangle.$$

Theorem 8.5 (Equation of Tangent Plane)**.**
Suppose $f(x,y)$ has continuous first partial derivatives at (a,b). A normal vector to the tangent plane at $(a,b,f(a,b))$ to the surface $z = f(x,y)$ is

$$\langle f_x(a,b), f_y(a,b), -1\rangle.$$

Further, an equation of the tangent plane is given by

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$$

or

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Example 8.17. *Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.*

Solution. Notice

$$f_x(x,y) = 4x, f_x(1,1) = 4,$$

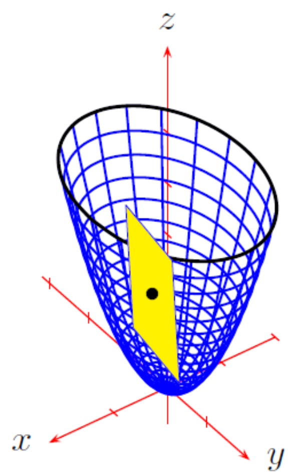
$$f_y(x,y) = 2y, f_y(1,1) = 2.$$

The equation of the plane is

$$z = f(1,1) + 4(x-1) + 2(y-1),$$

$$z = 4x + 2y - 3.$$

The figure shows the elliptic paraboloid and its tangent plane at $(1,1,3)$ that we found in the preceding example



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8.11 Differentiability and Chain Rule

For single-variable function $f(x)$, we say that f is differentiable at a if and only if $f'(a)$ exists. For two-variable function $f(x,y)$, it is tempting to say that f is differentiable at (a,b) if $f_x(a,b)$ and $f_y(a,b)$ exist. However, such definition would fail to capture the true nature of 'differentiability'!

Definition 8.12.
Informally, we say that f is **differentiable** *at (a,b) if the tangent plane at (a,b) is a good approximation to f at points close to (a,b).*

The above definition is not precise since we do not define what do we mean by 'a good approximation'. Do not worry! All the functions we encounter in this course will be differentiable at points in its domain. Recall that the Chain Rule for functions of a single variable gives the following rule for differentiating a composite function.

If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t, and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

We shall extend the chain rule to functions of several variables. This takes several slightly different forms, depending on the number of independent variables.

The first version deals with a function $z = f(x,y)$ where $x = g(t)$ and $y = h(t)$ are both functions of a single variable t:

$$z = f(g(t), h(t)).$$

Theorem 8.6 (The Chain Rule - Case 1).
Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then, z is a differentiable function of t, and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example 8.18. *For $z = f(x,y) = x^2 e^y$, $x = g(t) = t^2 - 1$ and $y = h(t) = \sin t$, find the derivative $\frac{dz}{dt}$.*

Solution. First, compute the partial derivatives:

$$\frac{\partial z}{\partial x} = 2xe^y, \quad \frac{\partial z}{\partial y} = x^2 e^y.$$

Next, compute the derivatives:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = \cos t.$$

Therefore, using the chain rule,

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= 2xe^y(2t) + x^2 e^y \cos t \\
&= 2(t^2-1)e^{\sin t}(2t) + (t^2-1)^2 e^{\sin t}\cos t.
\end{aligned}$$

■

Notice, in the preceding example, you could have first substituted for x and y and then compute the derivative of

$$f(g(t),h(t)) = (t^2-1)^2 e^{\sin t}$$

using the usual rules of differentiation for functions of a single variable.

We can easily extend The Chain Rule to the case of a function $f(x,y)$ where x and y now are both functions of two independent variables s and t, $x = g(s,t)$ and $y = h(s,t)$.

Then, z is indirectly a function of s and t:

$$z = f(g(s,t),h(s,t)).$$

We wish to find

$$\frac{\partial z}{\partial s}, \quad \frac{\partial z}{\partial t}.$$

Recall that, in computing $\frac{\partial z}{\partial t}$, we hold s fixed and compute the ordinary derivative of z with respect to t. (This is the situation in The Chain Rule - Case 1)

Similarly, in computing $\frac{\partial z}{\partial s}$, we hold t fixed and compute the ordinary derivative of z with respect to s. (This is the situation in The Chain Rule - Case 1)

We have the following (for free!)

Theorem 8.7 (The Chain Rule - Case 2)**.**
Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then,

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Case 2 of the Chain Rule contains three types of variables:

- s and t are independent variables.
- x and y are called intermediate variables.
- z is the dependent variable.

Example 8.19. *If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.*

Solution. Applying Case 2 of Chain Rule,

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t). \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\ &= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t). \end{aligned}$$

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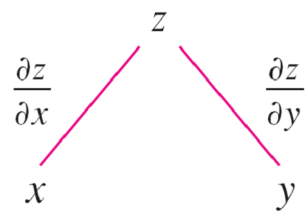
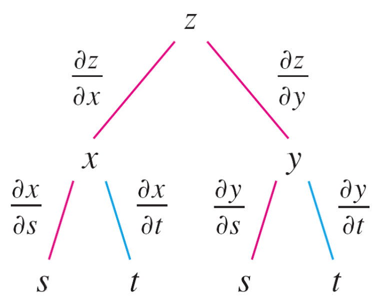
To remember the Chain Rule, it's helpful to draw a tree diagram, as follows:

We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y.



Then, we draw branches from x and y to the independent variables s and t. On each branch, we write the corresponding partial derivative.



To find $\frac{\partial z}{\partial s}$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

Similarly, we find $\frac{\partial z}{\partial t}$ by using the paths from z to t.

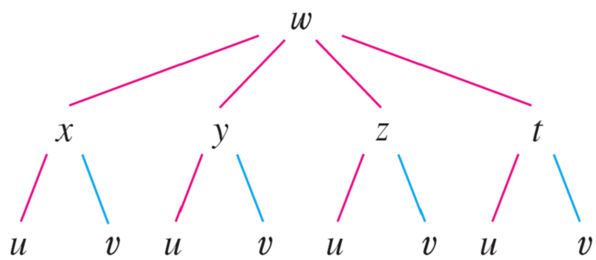
Theorem 8.8 (The Chain Rule - General Version)**.**
Suppose that u is a differentiable function of n variables $x_1, \ldots, x_n$, and each x_j is a differentiable function of m variables $t_1, \ldots, t_m$. Then u is a function of $t_1, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, \ldots, m$.

Example 8.20. *Write out the Chain Rule for the case where $w = f(x,y,z,t)$ and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $t = t(u,v)$.*

Solution. The figure shows the tree diagram.



With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}.$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}.$$

■

Example 8.21. *If $w = f(x^2 - y^2, y^2 - x^2)$ and f is differentiable, show that*

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0.$$

Solution. Introduce intermediate variables:

$$u = x^2 - y^2, \quad v = y^2 - x^2.$$

Using Chain Rule,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u}(2x) + \frac{\partial w}{\partial v}(-2x)$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}(-2y) + \frac{\partial w}{\partial v}(2y)$$

Therefore

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y}$$

$$= \left(\frac{\partial w}{\partial u}(2xy) + \frac{\partial w}{\partial v}(-2xy)\right) + \left(\frac{\partial w}{\partial u}(-2xy) + \frac{\partial w}{\partial v}(2xy)\right) = 0.$$

■

8.12 Implicit Differentiation

Consider a surface defined by an equation

$$F(x,y,z)=0$$

where $F(x,y,z)$ is differentiable.

Suppose z is implicitly defined as a function of independent variables x and y, that is, for every choice of x and y, there is a unique z such that $F(x,y,z)=0$.

Suppose we are interested in $\frac{\partial z}{\partial x}$.

If we can solve the above equation for z, say $z=f(x,y)$, then we can compute $\frac{\partial z}{\partial x}$ directly. But life is complicated enough that we may not be able to solve for z. Using Chain Rule to differentiate the equation $F(x,y,z)=0$ with respect to x:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0.$$

But

$$\frac{\partial x}{\partial x}=1,\quad \frac{\partial y}{\partial x}=0$$

since x and y are independent variables. Therefore, if $\frac{\partial F}{\partial z}\neq 0$, then

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{F_x}{F_z}.$$

Theorem 8.9 (Implicit Differentiation: Two Independent Variables)**.**
Suppose the equation $F(x,y,z)=0$, where F is differentiable, defines z implicitly as a differentiable function of x and y. Then,

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)},\quad \frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

provided $F_z(x,y,z)\neq 0$.

Example 8.22. *Find $\frac{\partial z}{\partial x}$ if*

$$x^3+y^3+z^3+6xyz=1.$$

Solution. Let $F(x,y,z)=x^3+y^3+z^3+6xyz-1$. Then

$$F_x=3x^2+6yz,\quad F_z=3z^2+6xy.$$

Therefore, by the Implicit Differentiation Theorem,

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+6yz}{3z^2+6xy}.$$

■

8.13 Increments and Differentials

Definition 8.13.
Let $z = f(x,y)$. Suppose $\triangle x$ and $\triangle y$ are increments in the independent variable x and y respectively.

Then the **increment** *in z is defined by*

$$\triangle z = f(x + \triangle x, y + \triangle y) - f(x,y).$$

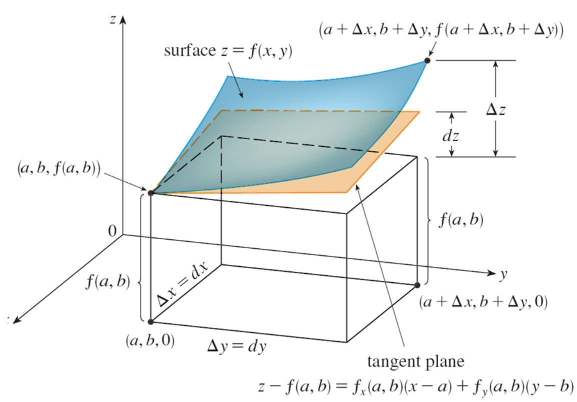
Definition 8.14.
Let $z = f(x,y)$. Suppose $\triangle x$ and $\triangle y$ are increments in the independent variable x and y respectively.

Then the **differentials** *of the independent variables x and y are*

$$dx = \triangle x, \quad dy = \triangle y.$$

The **differential** *(or* **total differential***) of the dependent variable z is*

$$dz = f_x(x,y)dx + f_y(x,y)dy.$$



Notice that

• the increment $\triangle z$ is the change in z as (x,y) changes from (a,b) to $(a + \triangle x, b + \triangle b)$.

• the differential dz is the change in the tangent plane as (x,y) changes from (a,b) to $(a + \triangle x, b + \triangle b)$.

Example 8.23. *Let $z = 2x^2 - xy$. Find $\triangle z$. Use this result to find the change of z if (x,y) changes from $(1,1)$ to $(0.98, 1.03)$.*

Solution.

$$\begin{aligned}\triangle z &= f(x+\triangle x, y+\triangle y) - f(x,y) \\ &= \Big(2(x+\triangle x)^2 - (x+\triangle x)(y+\triangle y)\Big) - (2x^2 - xy) \\ &= (4x-y)\triangle x - x\triangle y + 2(\triangle x)^2 - \triangle x \triangle y.\end{aligned}$$

As (x,y) changes from $(1,1)$ to $(0.98, 1.03)$, we have $\triangle x = 0.98 - 1 = -0.02$ and $\triangle y = 1.03 - 1 = 0.03$. Substituting these values into the expression of $\triangle z$ above, we obtain

$$\triangle z = -0.0886.$$

■

From the previous example, it seems quite complicated to calculate $\triangle z$. Is there a way to approximate $\triangle z$?

It turns out that dz gives a good approximation of $\triangle z$ **provided** $\triangle x$ and $\triangle y$ are small and $f(x,y)$ is differentiable.

Theorem 8.10.
Suppose f is differentiable at (a,b). Let $\triangle x$ and $\triangle y$ be small increments in x and y respectively from (a,b). Then

$$\triangle z \approx dz = f_x(a,b)\,dx + f_y(a,b)\,dy = f_x(a,b)\triangle x + f_y(a,b)\triangle y.$$

Example 8.24. *The base radius and height of a circular cone are measured as $10cm$ and $25cm$ respectively, with a possible error in measurement of as much as $0.1cm$ in each. Use differential to estimate the maximum error in the calculated volume of the cone.*

Solution. The volume of the cone is $V = \pi r^2 h/3$. So

$$dV = V_r dr + V_h dh = \frac{2\pi rh}{3} dr + \frac{\pi r^2}{3} dh.$$

Since each error is at most $0.1cm$, we can take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$ to give

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi.$$

The maximum error required is $20\pi cm^3$. ■

8.14 Directional Derivatives and the Gradient Vector

Imagine you are hiking in the Grand Canyon. Let's think of your altitude at the point given by longitude x and latitude y as a function $f(x,y)$.

Facing east (in the direction of positive x-axis), the slope is given by the partial derivative $\frac{\partial f}{\partial x}$.

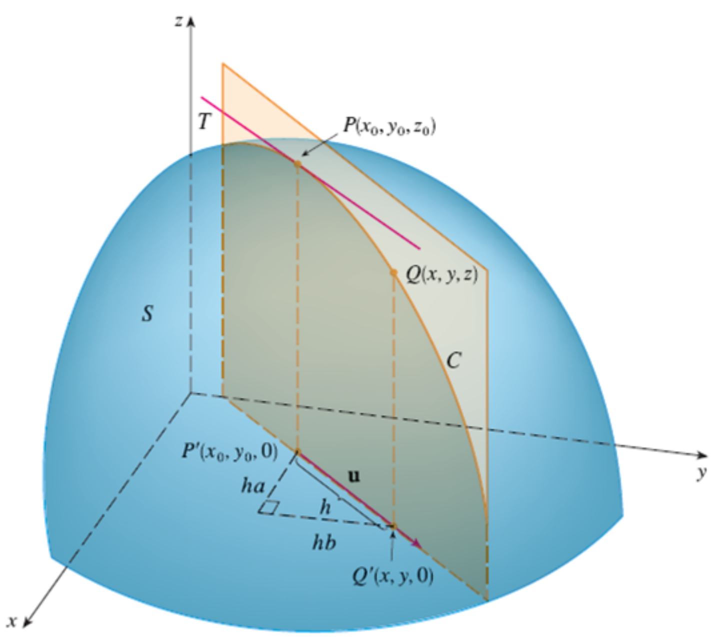
Facing north (in the direction of positive y-axis), the slope is given by the partial derivative $\frac{\partial f}{\partial y}$.

How to compute the slope when you are facing any given direction, say north-east?

Definition 8.15 (Directional Derivative)**.**
The **directional derivative** *of $f(x,y)$ at (x_0,y_0) in the direction of unit vector $\mathbf{u} = \langle a,b\rangle$ is*

$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha,y_0+hb)-f(x_0,y_0)}{h}$$

provided this limit exists.



By looking at the figure above, we can think of the directional derivative $D_{\mathbf{u}}f(x_0,y_0)$ as the slope to the point $P(x_0,y_0,z_0)$ on the surface in the direction given by $\mathbf{u}$.

Notice

- if $\mathbf{u} = \mathbf{i} = \langle 1,0\rangle$ then

$$D_{\mathbf{i}}f = f_x.$$

• if $\mathbf{u} = \mathbf{j} = \langle 0, 1\rangle$ then
$$D_{\mathbf{j}} f = f_y.$$

In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

In practice, we do not usually compute the directional derivative using the definition. Instead, we compute it using the dot product of the vector consisting of partial derivatives and the unit direction vector $\mathbf{u}$.

Theorem 8.11 (Computing Directional Derivative)**.**
If $f(x,y)$ is a differentiable function, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and
$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b.$$
We can rewrite it in terms of vectors:
$$D_{\mathbf{u}} f(x,y) = \langle f_x, f_y\rangle \cdot \langle a, b\rangle = \langle f_x, f_y\rangle \cdot \mathbf{u}.$$

Consider the vector $\langle f_x, f_y\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

It turns out that this vector has much significance. So we give it a special name.

Definition 8.16 (Gradient)**.**
The **gradient** *of $f(x,y)$ is the vector-valued function*
$$\nabla f(x,y) = \langle f_x, f_y\rangle = f_x\mathbf{i} + f_y\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$
provided both partial derivatives exist.

∇f is read 'del f'.

With this notation, we have
$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

Example 8.25. *Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.*

Solution. First compute the gradient vector at $(2,-1)$:
$$\nabla f(x,y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$
$$\nabla f(2,-1) = -4\mathbf{i} + 8\mathbf{j}.$$

Notice $\mathbf{v}$ is NOT a unit vector, since

$$\|\mathbf{v}\| = \sqrt{2^2 + 5^2} = \sqrt{29}.$$

The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}.$$

Therefore

$$\begin{aligned} D_{\mathbf{u}} f(2,-1) &= \nabla f(2,-1) \cdot \mathbf{u} \\ &= \langle -4, 8 \rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle \\ &= \frac{32}{\sqrt{29}}. \end{aligned}$$

■

For functions of three variables, we can define directional derivative in a similar manner.

Definition 8.17 (3-D Directional Derivative)**.**
The **directional derivative** *of $f(x,y,z)$ at (x_0, y_0, z_0) in the direction of unit vector $\mathbf{u} = \langle a, b, c \rangle$ is*

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

provided this limit exists.

Just as with functions of two variables, we have

Theorem 8.12 (Computing 3-D Directional Derivative)**.**

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u}$$

where

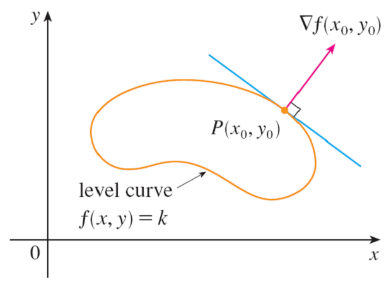
$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is the gradient vector.

What is so significant about ∇f?

Theorem 8.13 (Level Curve vs ∇f).
Suppose $f(x,y)$ is differentiable function of x and y at (x_0,y_0).

Suppose $\nabla f(x_0,y_0) \neq \mathbf{0}$. Then $\nabla f(x_0,y_0)$ is perpendicular/normal to the level curve $f(x,y) = k$ at the point (x_0,y_0) where $f(x_0,y_0) = k$.



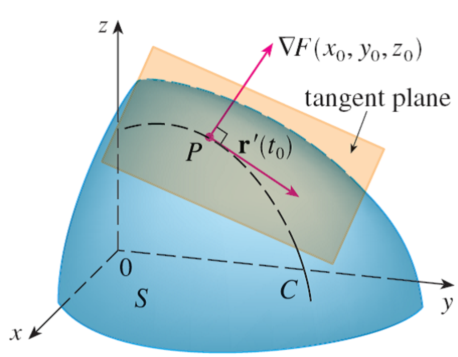
Using a similar argument, we can prove that this phenomenon also holds for level surfaces $F(x,y,z) = k$.

Theorem 8.14 (Level Surface vs ∇f).
Suppose $F(x,y,z)$ is differentiable function of x, y and z at (x_0,y_0,z_0). Suppose S is the level surface $F(x,y,z) = k$ containing (x_0,y_0,z_0). Let C be any curve that lies on S and passes through (x_0,y_0,z_0). Let $\mathbf{r}(t)$ be a parametric equation of C such that $\mathbf{r}(t_0) = \langle x_0,y_0,z_0\rangle$.

Suppose $\nabla F(x_0,y_0,z_0) \neq \mathbf{0}$. Then

$$\nabla F(x_0,y_0,z_0) \cdot \mathbf{r}'(t_0) = 0,$$

That is, the $\nabla F(x_0,y_0,z_0)$ is perpendicular/normal to tangent vector $\mathbf{r}'(t_0)$ to **any** *curve C on the surface S that passes through (x_0,y_0,z_0).*



$F(x_0,y_0,z_0) = k$, $\nabla F(x_0,y_0,z_0) \cdot \mathbf{r}'(t_0) = 0$.

Consequently, the **tangent plane to the level surface** $F(x,y,z) = k$ at (x_0,y_0,z_0) is given by the equation

Theorem 8.15 (Tangent Plane to Level Surface)**.**

$$\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or equivalently,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Example 8.26. *Find the equations of the tangent plane and normal line at the point* $(-2, 1, -3)$ *to the ellipsoid*

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Solution. The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}.$$

Therefore,

$$F_x(x, y, z) = \frac{x}{2}, \;\; F_y(x, y, z) = 2y, \;\; F_z = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1, \;\; F_y(-2, 1, -3) = 2, \;\; F_z(-2, 1, -3) = -\frac{2}{3}.$$

The equation of the tangent plane at $(-2, 1, -3)$ is

$$\nabla F(-2, 1, -3) \cdot \langle x - (-2), y - 1, z - (-3) \rangle$$

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0,$$

which simplifies to

$$3x - 6y + 2z + 18 = 0.$$

The normal vector to the plane is $\langle 3, -6, 2 \rangle$. So the parametric equations of the normal line are

$$x = -2 + 3t, \;\; y = 1 - 6t, \;\; z = -3 + 2t, \;\; t \in \mathbb{R}.$$

■

Remark. Theorem 8.5 is a special case of Theorem 8.15, and one may deduce Theorem 8.5 from Theorem 8.15 as follows:

Suppose S is the surface defined by $z = f(x,y)$, and $(a,b,f(a,b))$ is a point on S. Now consider the function $F(x,y,z) = f(x,y) - z$. Notice that

$$z = f(x,y) \iff f(x,y) - z = 0 \iff F(x,y,z) = 0.$$

Thus we may regard S as the level surface $F(x,y,z) = 0$. Note also that at a point (x_0,y_0,z_0), we have

$$F_x(x_0,y_0,z_0) = f_x(x_0,y_0), \quad F_y(x_0,y_0,z_0) = f_y(x_0,y_0), \quad F_z(x_0,y_0,z_0) = -1.$$

Then from Theorem 8.15 (with $(z_0,y_0,z_0) = (a,b,f(a,b))$, the equation of the tangent plane at $(a,b,f(a,b))$ to S (as the level surface $F(x,y,z) = 0$) is given by

$$F_x(a,b,f(a,b))(x-a) + F_y(a,b,f(a,b))(y-b) + F_z(a,b,f(a,b))(z - f(a,b)) = 0$$

or equivalently,

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0,$$

which is the same equation as given in Theorem 8.5.

■

Let's return to the directional derivative $D_{\mathbf{u}} f(x,y,z)$ and ask some questions.

We know that $D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$ is a scalar function of x, y and z (because it is a dot product of two vectors). Geometrically, we think of $D_{\mathbf{u}} f(x_0,y_0,z_0)$ as the rate of change of f at (x_0,y_0,z_0) in the direction of $\mathbf{u}$.

Question: at a given point (x_0,y_0,z_0), in which direction does f change the fastest? In other words, what is the maximum rate of change of f at (x_0,y_0,z_0)?

The answer lies in ∇f!

Let θ be the angle between ∇f and $\mathbf{u}$. Then

$$\begin{aligned} D_{\mathbf{u}} f &= \nabla f \cdot \mathbf{u} \\ &= \|\nabla f\| \|\mathbf{u}\| \cos\theta \\ &= \|\nabla f\| \cos\theta \quad \text{since } \mathbf{u} \text{ is a unit vector} \end{aligned}$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta.$$

The maximum value of $\cos\theta$ is 1 and this happens when $\theta = 0$.

So the maximum value of $D_{\mathbf{u}} f$ is $\|\nabla f\|$ and it occurs when $\theta = 0$, i.e. $\mathbf{u}$ points in the direction of ∇f.

The minimum value of $\cos\theta$ is -1 and this happens when $\theta = \pi$.

So the minimum value of $D_{\mathbf{u}} f$ is $-\|\nabla f\|$ and it occurs when $\theta = \pi$, i.e. $\mathbf{u}$ points in the direction of $-\nabla f$.

Theorem 8.16 (Maximizing Rate of Increase/Decrease of f)**.**
Suppose f is a differentiable function of two or three variables. Let P denote a given point. Assume $\nabla f(P) \neq \mathbf{0}$. Let $\mathbf{u}$ be a unit vector making an angle θ with ∇f. Then

$$D_{\mathbf{u}} f(P) = \|\nabla f(P)\| \cos\theta.$$

Moreover,

- *$\nabla f(P)$ points in the direction of maximum rate of increase of f at P (maximum value of $D_{\mathbf{u}} f(P)$ is $\|\nabla f(P)\|$)*
- *$-\nabla f(P)$ points in the direction of maximum rate of decrease of f at P (minimum value of $D_{\mathbf{u}} f(P)$ is $-\|\nabla f(P)\|$)*

Example 8.27. *Let $f(x,y) = xe^y$. In what direction does f have the maximum rate of change at the point $P(2,0)$? What is this maximum rate of change?*

Solution. Note that

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle.$$

f increases fastest in the direction of the gradient vector

$$\nabla f(2,0) = \langle 1, 2 \rangle.$$

The maximum rate of change is

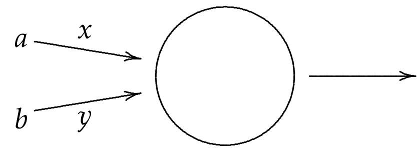
$$\|\nabla f(2,0)\| = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

■

Example 8.28. *In a toy model of a neural network with only one artificial neuron, there are two inputs a and b to the neuron, and they are attached with (variable) weights of x and y respectively. In a (repeated) training of the neuron, the values of the inputs are fixed at $(a,b) = (2,3)$. The activation function of the neuron is given by $\varphi(s) = \dfrac{1}{1+e^{-s}}$ near $s = -1$, so that for (x,y) near $(4,5)$, the (actual) output of the neuron is given by $\varphi(2x+3y-24) = \dfrac{1}{1+e^{-(2x+3y-24)}}$ (here the term -24 is called a 'bias') (note also that $2\cdot4+3\cdot5-24 = -1$). With the input fixed at $(a,b) = (2,3)$ and the corresponding target output of the neuron set at $\dfrac{1}{2}$, the cost function is given by*

$$C(x,y) = \left(\frac{1}{2} - \frac{1}{1+e^{-(2x+3y-24)}}\right)^2 \quad \textit{for } (x,y) \textit{ near } (4,5),$$

which is a measure of the discrepancy between the actual output and the target output. Find the direction in which the cost function C will have maximum rate of decrease when the weights are at $(x,y) = (4,5)$.



Remark. *Such information is useful in training the neuron, which involves modifying the weights (x,y) efficiently so that for the same input $(a,b) = (2,3)$ (but with the weights modified from $(x,y) = (4,5)$), the new output of the neuron will become closer to the target output.*

Solution. Given that

$$C(x,y) = \left(\frac{1}{2} - \frac{1}{1+e^{-(2x+3y-24)}}\right)^2 \quad \text{for } (x,y) \text{ near } (4,5).$$

We have, for (x,y) near $(4,5)$,

$$\begin{aligned}
C_x &= 2\cdot\left(\frac{1}{2} - \frac{1}{1+e^{-(2x+3y-24)}}\right)\cdot\frac{(-1)(-1)}{(1+e^{-(2x+3y-24)})^2}\cdot e^{-(2x+3y-24)}\cdot(-2)\\
&= -\frac{4e^{-(2x+3y-24)}}{(1+e^{-(2x+3y-24)})^2}\cdot\left(\frac{1}{2} - \frac{1}{1+e^{-(2x+3y-24)}}\right),\\
C_y &= 2\cdot\left(\frac{1}{2} - \frac{1}{1+e^{-(2x+3y-24)}}\right)\cdot\frac{(-1)(-1)}{(1+e^{-(2x+3y-24)})^2}\cdot e^{-(2x+3y-24)}\cdot(-3)\\
&= -\frac{6e^{-(2x+3y-24)}}{(1+e^{-(2x+3y-24)})^2}\cdot\left(\frac{1}{2} - \frac{1}{1+e^{-(2x+3y-24)}}\right).
\end{aligned}$$

A direct computation gives

$$C_x(4,5) = \frac{-2e(e-1)}{(1+e)^3}, \quad C_y(4,5) = \frac{-3e(e-1)}{(1+e)^3}.$$

At the point $(x,y) = (4,5)$, the cost function C decreases fastest in the direction of the gradient vector

$$-\nabla C(4,5) = -\langle C_x(4,5), C_y(4,5)\rangle = -\left\langle \frac{-2e(e-1)}{(1+e)^3}, \frac{-3e(e-1)}{(1+e)^3}\right\rangle = \frac{e(e-1)}{(1+e)^3}\langle 2,3\rangle.$$

Thus the required direction is given by the unit vector

$$\mathbf{u} = \frac{-\nabla C(4,5)}{\|-\nabla C(4,5)\|} = \frac{\langle 2,3\rangle}{\|\langle 2,3\rangle\|} = \frac{\langle 2,3\rangle}{\sqrt{13}}.$$

■

8.15 Extrema of Functions of Two Variables

In the real world, we always seek to optimize our resources.

Given our constraints (our time, ability, finance, health, family background and what not), getting an A in MA1521 itself can be seen as an optimization problem.

Similar to the study of extrema of functions of one variable, two key concepts are the local maximum/minimum and the absolute maximum/minimum.

Definition 8.18 (Local and Absolute Maximum).
Let $f(x,y): D \to \mathbb{R}$. Then

- *f has a* **local maximum** *at (a,b) if $f(x,y) \le f(a,b)$ for all points in some disk with center (a,b). Such point (a,b) is called a* **local maximum point** *of f, and the number $f(a,b)$ is called a* **local maximum value** *of f.*
- *f has an* **absolute maximum** *at (a,b) if $f(x,y) \le f(a,b)$ for all points in the domain D. Such point (a,b) is called an* **absolute maximum point** *of f, and the number $f(a,b)$ is called the* **absolute maximum value** *of f.*

Definition 8.19 (Local and Absolute Minimum).
Let $f(x,y): D \to \mathbb{R}$. Then

- *f has a* **local minimum** *at (a,b) if $f(x,y) \ge f(a,b)$ for all points in some disk with center (a,b). Such point (a,b) is called a* **local minimum point** *of f, and the number $f(a,b)$ is called a* **local minimum value** *of f.*
- *f has an* **absolute minimum** *at (a,b) if $f(x,y) \ge f(a,b)$ for all points in the domain D. Such point (a,b) is called an* **absolute minimum point** *of f, and the number $f(a,b)$ is called the* **absolute minimum value** *of f.*

8.15.1 Local Extrema

A key observation that will be used repeatedly when finding local extrema of functions is the following.

Theorem 8.17.
If f has a local maximum or minimum at (a,b) and the first-order derivatives of f exist there, then

$$f_x(a,b) = f_y(a,b) = 0.$$

Proof. Let $g(x) = f(x,b)$. Then g is a function of a single variable x. If f has a local maximum/minimum at $(x,y) = (a,b)$ then g has a local maximim/minimim at $x = a$. So $g'(a) = 0$.

But $g'(a) = f_x(a,b)$. So $f_x(a,b) = 0$.

Similarly, $f_y(a,b) = 0$.

■

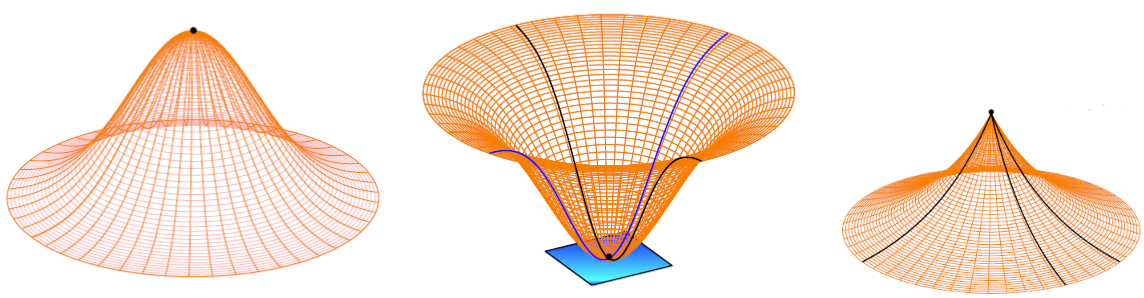
There is a geometric interpretation of the preceding theorem:

If f has a tangent plane at a local maximum/minimum (a,b), then the tangent plane has equation

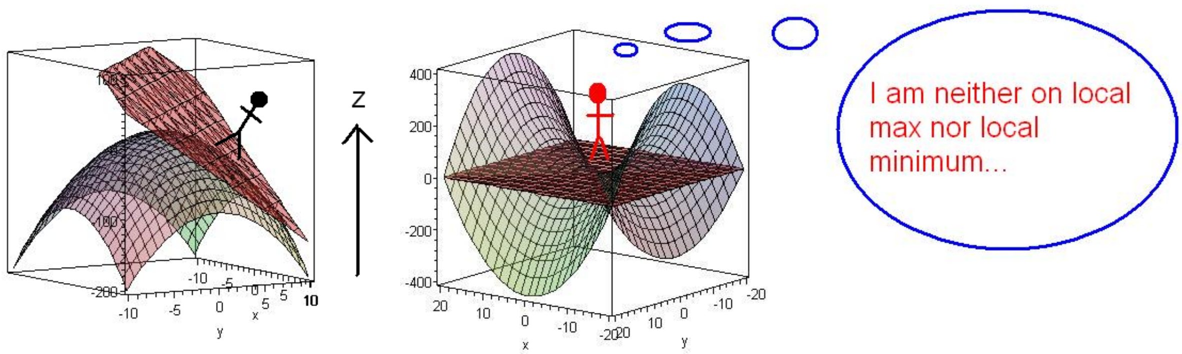
$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = f(a,b),$$

that is the tangent plane is a horizontal plane parallel to the xy-plane.

If I stand on a local maximum/minimum then ......



In the following, I am definitely NOT standing on local maximum/minimum ......



Definition 8.20 (Critical or Stationary Point).
Let $f(x,y): D \to \mathbb{R}$. Then a point (a,b) is called a **critical point** *of f if*

- $f_x(a,b) = 0$ *and* $f_y(a,b) = 0$, *OR*
- *one of the partial derivatives does not exist.*

Clearly,

$$(a,b) \text{ local maximum/minimum point} \quad \Longrightarrow \quad (a,b) \text{ critical point.}$$

However, the converse IS NOT TRUE!

Example 8.29. *Find the extreme (maximum/minimum) values of $f(x,y) = y^2 - x^2$.*

Solution. Extreme values can only occur at critical points. Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0,0)$.

We still have to check whether $f(0,0)$ is a maximum/minimum value.

Note that $f(0,0) = 0$.

If $y = 0$ and $x \neq 0$, then

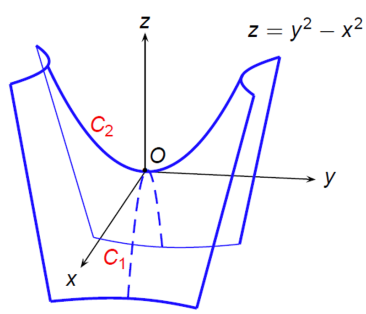
$$f(x,y) = -x^2 < 0 = f(0,0).$$

If $x = 0$ and $y \neq 0$, then

$$f(x,y) = y^2 > 0 = f(0,0).$$

Therefore, $f(0,0)$ cannot be an extreme value for f. So f has no extreme values.

■



In the preceding example, we see that a critical point needs not be an extreme point. But the behavior of the critical point in the preceding example is interesting.

If you look at the graph of $f(x,y) = y^2 - x^2$, you will see that $f(0,0) = 0$ is a maximum in the direction of the x-axis but a minimum in the direction of the y-axis.

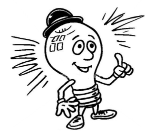
This motivates the following definition.

Definition 8.21 (Saddle Point)**.**
Let $f(x,y) : D \to \mathbb{R}$. Then

A point (a,b) is called a **saddle point** *of f if*

• *it is a critical point of f, AND*

• *every open disk centered at (a,b) contains points $(x,y) \in D$ for which $f(x,y) < f(a,b)$ and points $(x,y) \in D$ for which $f(x,y) > f(a,b)$.*



Suppose you are standing on a surface and you are standing upright (parallel to the z-axis). Moreover, when you begin walking, some directions take you uphill while other directions take you downhill.

Then you are standing at a saddle point!

We cannot rely on our visualization of 3D-graphs to locate extreme points.

Luckily, we have the **second derivative test** to determine whether a given critical point is local maximum/minimum, saddle point or neither.

Theorem 8.18 (Second Derivative Test)**.**
Suppose $f(x,y)$ has continuous second-order partial derivatives on some open disk centered at (a,b). Suppose $f_x(a,b) = f_y(a,b) = 0$ (that is (a,b) is a critical point). Define the **discriminant** *D for the point (a,b) by*

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2.$$

(a) *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*

(b) *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*

(c) *If $D < 0$, then (a,b) is a saddle point of f.*

(d) *If $D = 0$, then no conclusion can be drawn.*

Example 8.30. *Locate and classify all critical points for $f(x,y) = x^3 - 2y^2 - 2y^4 + 3x^2y$.*

Solution. We have

$$f_x = 3x^2 + 6xy, \quad f_y = -4y - 8y^3 + 3x^2.$$

Step 1. Locate critical points.

Let's solve the system

$$\begin{aligned} 3x^2 + 6xy &= 0 \quad (1) \\ -4y - 8y^3 + 3x^2 &= 0 \quad (2) \end{aligned}$$

If $x = 0$, then by (2) we have $-4y - 8y^3 = 0 \Leftrightarrow -4y(1+2y^2) = 0 \Leftrightarrow y = 0$. Thus we obtain one solution $(0,0)$.

If $x \neq 0$, then by (1) we have $y = -\frac{x}{2}$. Substituting this into (2), we have
$-4(-\frac{x}{2}) - 8(-\frac{x}{2})^3 + 3x^2 = 0 \Leftrightarrow 2x + x^3 + 3x^2 = 0 \Leftrightarrow x(x+2)(x+1) = 0 \Leftrightarrow x = -1, -2$. Note that $x \neq 0$.

Using $y = -\frac{x}{2}$, we obtain the two solutions $(-1, \frac{1}{2}), (-2, 1)$.

Therefore, the critical points are $(0,0), (-1, \frac{1}{2}), (-2, 1)$.

Step 2. Classify (if possible) these critical points using Second Derivative Test.

We need

$$f_{xx} = 6x + 6y, \quad f_{xy} = 6x, \quad f_{yy} = -4 - 24y^2.$$

At the critical point $(-2, 1)$, we have

$$D(-2,1) = f_{xx}(-2,1)f_{yy}(-2,1) - [f_{xy}(-2,1)]^2 = (-6)\cdot(-28) - (-12)^2 = 24 > 0.$$

Note also that $f_{xx}(-2,1) = -6 > 0$. Thus by the Second Derivative Test, we know that f has a local maximum point at $(-2, 1)$.

We can make similar computations at the other two critical points $(0,0)$ and $(-1, \frac{1}{2})$, and the result is tabulated as follows:

critical point	D	f_{xx}	2nd Derivative Test's Conclusion
$(0,0)$	0		inconclusive
$(-1, \frac{1}{2})$	$-6 < 0$		saddle point
$(-2, 1)$	$24 > 0$	$-6 < 0$	local maximum

We need a different analysis to deal with the critical point $(0,0)$.

Notice in the plane $y = 0$, $f(x,y) = f(x,0) = x^3$. We know from Calculus that this curve has an inflection point at $x = 0$. So there is no local extremum at this point.

Moreover, when we start walking from $(0,0)$ in positive direction of x, we will be walking uphill; in the negative direction of x, we will be walking downhill.

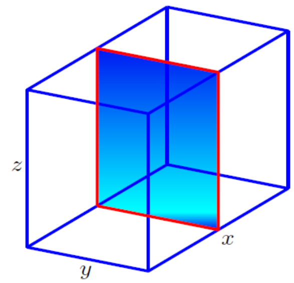
So $(0,0)$ is a saddle point.

■

Exercise 8.1. *Find the critical points of $f(x,y) = -3xe^{-x^2-y^2}$ and classify them.*

Ans. Local minimum at $\frac{1}{\sqrt{2}}$, Local maximum at $-\frac{1}{\sqrt{2}}$.

Exercise 8.2. *A delivery company only accepts rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 270 cm. Find the dimensions of an acceptable box of largest volume.*



The girth is $2y + 2z$.

Ans. 90cm $\times$ 45cm $\times$ 45cm.

Chapter 9

Double Integrals

Read Thomas' Calculus, Chapter 14.

9.1 Riemann Sum

Having studied derivatives for functions of several variables and their applications, we now turn to introducing the idea of integral for functions of several variables. It turns out that these ideas are useful in many practical problems.

Recall that in Calculus of Single Variable, our attempt to find the area under a curve led to the definition of a definite integral.

We now seek to find volume under a surface and in the process we arrive at the definition of a double integral.

We start by reviewing how we arrive at the definite integral of functions of a single variable:

Step 1. Suppose $f(x)$ is defined for $a \le x \le b$. We divide the interval $[a, b]$ into n subintervals of equal size $\triangle x = \frac{b-a}{n}$.

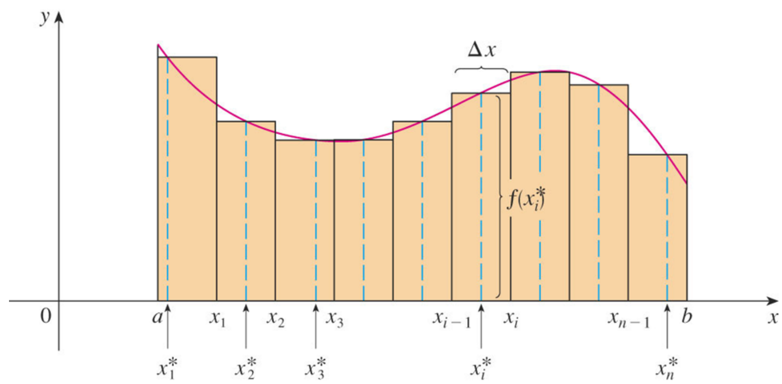
Step 2. We choose sample points x_i^* from these subintervals and form the **Riemann Sum**

$$\sum_{i=1}^{n} f(x_i^*)\triangle x.$$

Step 3. Take the limit of such sum as $n \to \infty$ to obtain the definite integral of f from a to b:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x.$$

In the special case where $f(x) \ge 0$, the integral $\int_a^b f(x)dx$ represents the area under the curve $f(x)$ from a to b.

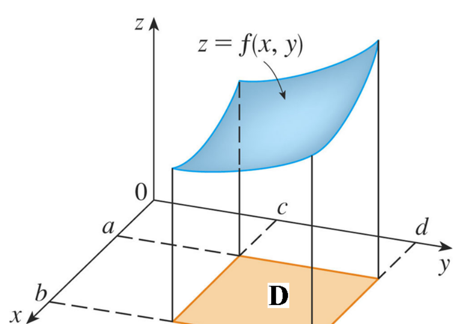


9.2 Volume and Double Integral

Suppose $f(x,y)$ is a function of two variables defined on a closed rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, c \le y \le d\}.$$

Suppose $f(x,y) \ge 0$. The graph of f is a surface with $z = f(x,y)$ above the region R.
Let S be the solid that lies above R and under the graph of f.
How can we find the volume of S?



We can estimate the volume of S as follows:

Step 1. divide the rectangle R into subrectangles. We do this by

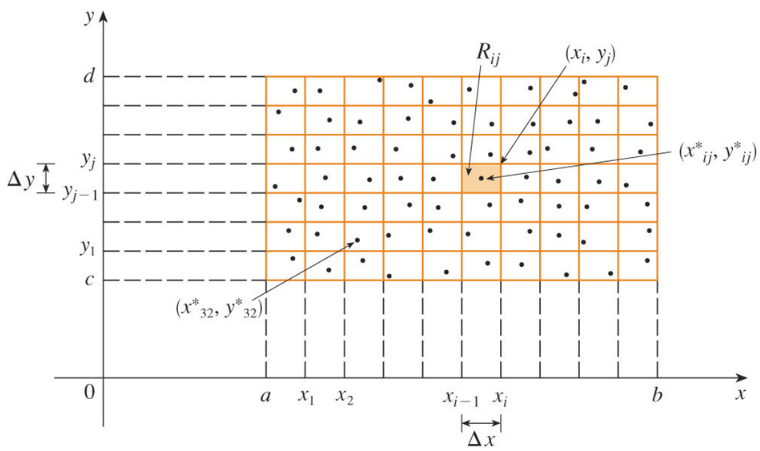
• dividing the interval $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal length $\triangle x = \frac{b-a}{m}$, and

• dividing the interval $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of equal length $\triangle y = \frac{d-c}{n}$.

Form subrectangles

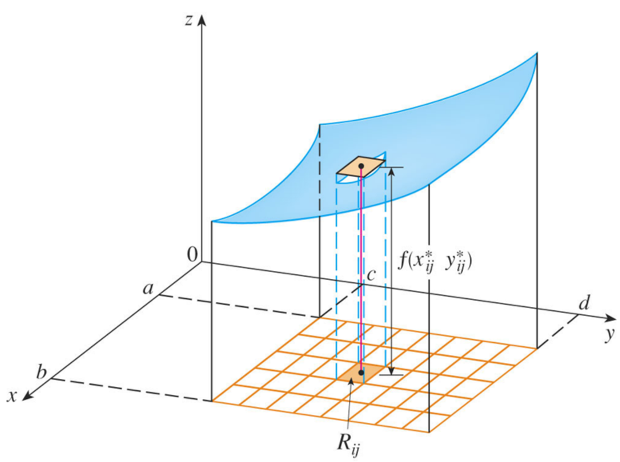
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad \text{for all } 1 \le i \le m, 1 \le j \le n.$$

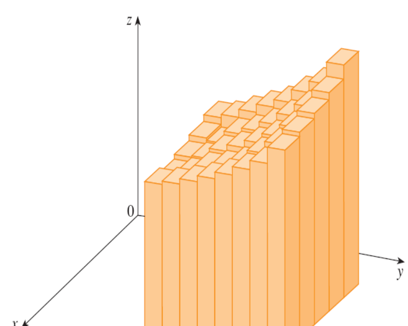
Each of these subrectangles has area $\triangle A = \triangle x \triangle y$.



Step 2. Choose a sample point (x^*_{ij}, y^*_{ij}) in each R_{ij}. Then approximate the part of S lies above R_{ij} by a thin rectangle box with base R_{ij} and height $f(x^*_{ij}, y^*_{ij})$. The volume of this box is given by

$$f(x^*_{ij}, y^*_{ij})\triangle A.$$



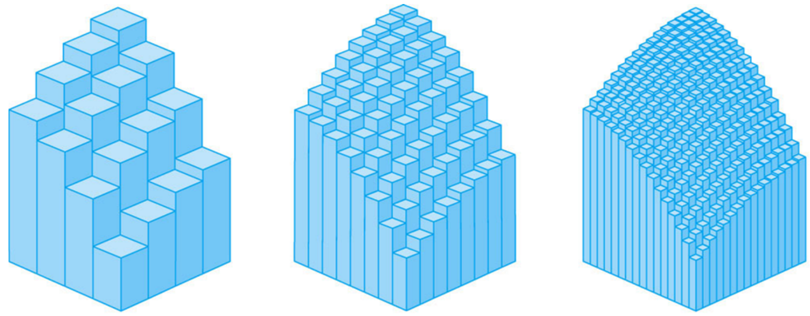


It follows that by adding the volumes of all these thin boxes, we get an approximation of the total volume of S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\triangle A.$$

Our intuition tells us that the approximation becomes better as $m, n \to \infty$. So we would expect

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\triangle A.$$



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

We make the following definition:

Definition 9.1 (Double Integral)**.**
The **double integral** *of f over the rectangle R is*

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\triangle A$$

provided the limit exists and is the same for any choice of the sample points (x_{ij}^*, y_{ij}^*) in R_{ij}, for $1 \le i \le m$, $1 \le j \le n$.

When this happens, we say that f is **integrable** *over R.*

Remark. It can be shown that all continuous functions are integrable.

By comparing our definition of integral and volume, we have

Theorem 9.1 (Volume as a Double Integral)**.**
If $f(x,y) \ge 0$, the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA.$$

Some properties of double integral:

Assuming all the integrals exist, we have

1. $\displaystyle\iint_R (f(x,y)+g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$

2. $\displaystyle\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA.$

3. If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

9.3 Iterated Double Integral

Recall that it is usually difficult to evaluate a single integral directly from definition, but the Fundamental Theorem of Calculus provides a much easier method:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

For double integral, it is even more difficult to compute it from first principles. We now see how to express a double integral as an **iterated integral** which can be evaluated by calculating two single integrals.

Suppose $f(x,y)$ is integrable over the rectangle $R = [a,b] \times [c,d]$.

We use the notation $\int_c^d f(x,y)\,dy$ to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from c to d.

This procedure is called **partial integration with respect to** y. Notice the similarity to partial differentiation.

So $\int_c^d f(x,y)\,dy$ is a function of x, as it depends on the value of x: set

$$A(x) = \int_c^d f(x,y)\,dy.$$

We now integrate $A(x)$ from a to b:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The integral on the right-hand side is called an **iterated integral**.

Usually, we omit the brackets:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

Definition 9.2 (Iterated Integral)**.**

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

means we first integrate with respect to y from c to d (keeping x fixed) and then with respect to x from a to b.

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

means we first integrate with respect to x from a to b (keeping y fixed) and then with respect to y from c to d.

Example 9.1. *Evaluate the iterated integral*

$$\int_1^2 \int_0^3 x^2 y\,dx\,dy.$$

Solution. We first integrate with respect to x and then with respect to y:

$$\begin{aligned}
\int_1^2 \int_0^3 x^2 y\,dx\,dy &= \int_1^2 \left[\int_0^3 x^2 y\,dx\right] dy \\
&= \int_1^2 \left[\frac{x^3 y}{3}\right]_0^3 dy \\
&= \int_1^2 9y\,dy \\
&= \left[\frac{9y^2}{2}\right]_1^2 \\
&= \frac{27}{2}.
\end{aligned}$$

■

Example 9.2. *Evaluate the iterated integral*

$$\int_0^3 \int_1^2 x^2 y\,dy\,dx.$$

Solution. We first integrate with respect to y and then with respect to x:

$$\begin{aligned}\int_0^3\int_1^2 x^2y\,dy\,dx &= \int_0^3\left[\int_1^2 x^2y\,dy\right]dx\\ &= \int_0^3\left[\frac{x^2y^2}{2}\right]_1^2 dx\\ &= \int_0^3 \frac{3}{2}x^2\,dx\\ &= \left[\frac{x^3}{2}\right]_0^3\\ &= \frac{27}{2}.\end{aligned}$$

■

Notice in both of the preceding examples, we obtained the same answer.

It seems that the order of integration (with respect to x or y first) does not matter. This is similar to Clairaut's Theorem for mixed partial derivatives.

Indeed, if f is continuous on R, this is always true. Moreover, the (iterated) integral is equal to the corresponding double integral.

The following theorem gives a practical way for evaluating a double integral by expressing it as an iterated integral (in either order):

Theorem 9.2 (Fubini's Theorem)**.**
If f is continuous on the rectangle $R = [a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 9.3. *Evaluate*

$$\iint_R y\sin(xy)\,dA$$

where $R = [1,2]\times[0,\pi]$.

Solution 1. Using Fubini's Theorem, let's integrate first with respect to x.

$$
\begin{aligned}
\iint_R y\sin(xy)\,dA &= \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy \\
&= \int_0^{\pi} [-\cos(xy)]_1^2\,dy \\
&= \int_0^{\pi} (-\cos 2y + \cos y)\,dy \\
&= \left[-\frac{1}{2}\sin 2y + \sin y\right]_0^{\pi} = 0.
\end{aligned}
$$

Solution 2. By Fubini's Theorem, we should get the same answer if we first integrate with respect to y:

$$\iint_R y\sin(xy)\,dA = \int_1^2\int_0^{\pi} y\sin(xy)\,dy\,dx.$$

We first need to compute

$$\int_0^{\pi} y\sin(xy)\,dy.$$

Using integration by parts,

$$
\begin{aligned}
\int_0^{\pi} y\sin(xy)\,dy &= \left[-\frac{y\cos(xy)}{x}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos(xy)}{x}\,dy \\
&= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\,[\sin xy]_0^{\pi} \\
&= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}.
\end{aligned}
$$

Now, integrating the first term by parts, we have

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

So

$$\int\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}.$$

Hence

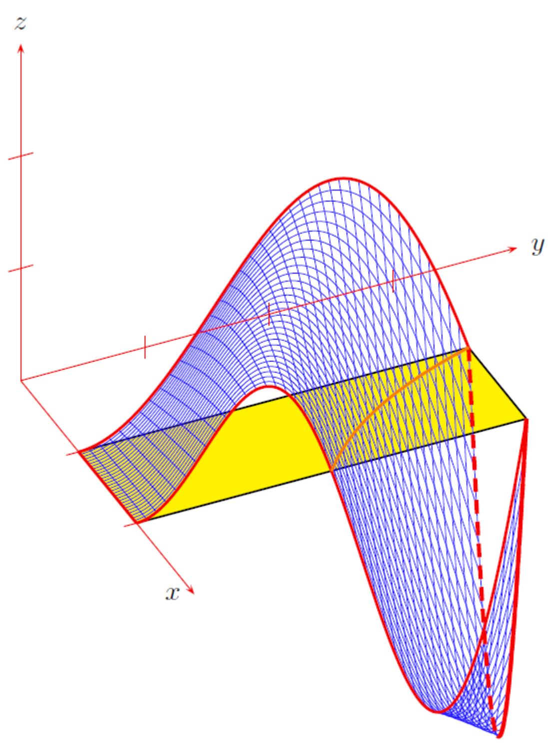
$$\int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx = \left[-\frac{\sin \pi x}{x}\right]_1^2$$
$$= -\frac{\sin 2\pi}{2} + \sin \pi = 0.$$

■

Now, we make some observation about the last example.

• Though both solutions give the same answer, the first solution is much easier than the second one. Therefore, when we evaluate double integrals, it is wise to choose the right order of integration that yields simpler calculations.

• Consider the surface $z = y\sin(xy)$.



This function takes both positive and negative values on $R = [1,2]\times[0,\pi]$.

For such a function, $\iint_R f(x,y)\,dA$ is a difference of volumes: $V_1 - V_2$ where V_1 is the volume above R and below the graph of f and V_2 is the volume below R and above the graph.

The fact that the integral is 0 in the preceding example means that these two volumes V_1 and V_2 are equal.

9.4 A Special Case

Sometimes $f(x,y)$ can be factored as the product of a function of x only and a function of y only. That is

$$f(x,y) = g(x)h(y).$$

Then Fubini's Theorem gives

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_c^d \int_a^b g(x)h(y)\,dx\,dy \\ &= \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy\end{aligned}$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_c^d \left[h(y)\int_a^b g(x)\,dx\right] dy \\ &= \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)\end{aligned}$$

since $\int_a^b g(x)\,dx$ is a constant.
To summarize:

Theorem 9.3 (A Special Case)**.**

$$\iint_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

where $R = [a,b]\times[c,d]$.

Example 9.4. *Evaluate*

$$\iint_R \frac{2x}{y}\,dA$$

where $R = [3,4]\times[1,2]$.

Solution: By Theorem 9.3,

$$\iint_R \frac{2x}{y}\,dA = \iint_R 2x\cdot\frac{1}{y}\,dA = \left(\int_3^4 2x\,dx\right)\cdot\left(\int_1^2 \frac{1}{y}\,dy\right) = \left[x^2\right]_3^4\cdot\left[\ln|y|\right]_1^2 = (4^2-3^2)\cdot(\ln 2-\ln 1) = 7\ln 2.$$

■

9.5 Double Integral over General Region

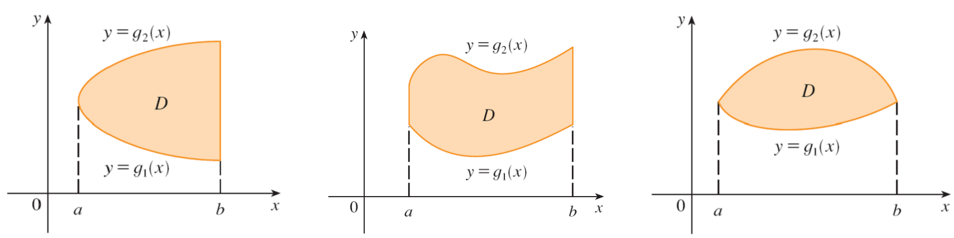
So far we have defined double integrals over domains which are rectangles. In this section, we shall define double integrals over domains which are more general than rectangles. In particular, they are regions which are bounded between two continuous curves. They are called **Type I** and **Type II** regions respectively.

Definition 9.3. *Type I Region*
A plane region D is said to be of **Type I** *if it lies between the graphs of two continuous functions of x, that is,*

$$D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where $g_1(x)$ and $g_2(x)$ are continuous on $[a,b]$.

Some examples of Type I region:

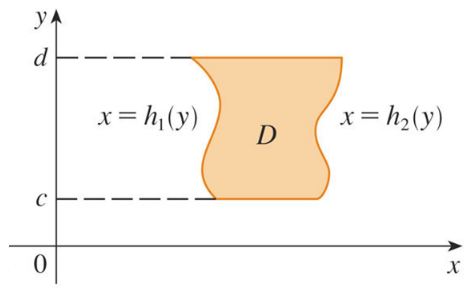


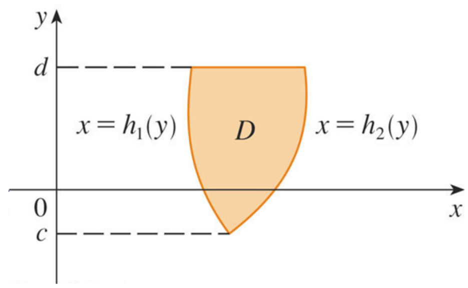
Definition 9.4. *Type II Region*
A plane region D is said to be of **Type II** *if it lies between the graphs of two continuous functions of y, that is,*

$$D = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where $h_1(y)$ and $h_2(y)$ are continuous on $[c,d]$.

Some examples of Type II region:





How do we compute the integral of $f(x,y)$ over Type I region D?

Theorem 9.4. *Double Integral over Type I Domain*
If f is continuous on a Type I domain D such that

$$D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Observe that the expression on the right-hand side of

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

is an iterated integral similar to the ones we have for rectangle region, except that in the inner integral, we regard x as being constant not only in $f(x,y)$ but also in the limits of the integration, $g_1(x)$ and $g_2(x)$.

Similarly, we have

Theorem 9.5. *Double Integral over Type II Domain*
If f is continuous on a Type II domain D such that

$$D = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

then
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Example 9.5. *Evaluate* $\iint_D (x+2y)\,dA$ *where* D *is the region bounded by the parabolas* $y = 2x^2$ *and* $y = 1 + x^2$.

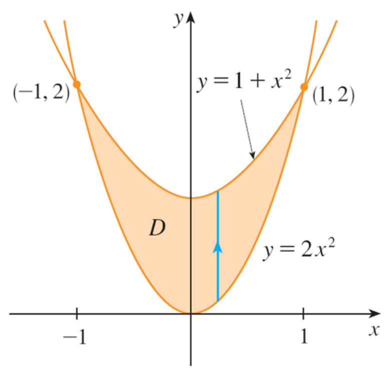
Solution.

Step 1. Identify the region.

Notice the parabolas intersect when $2x^2 = 1 + x^2$, that is, $x = \pm 1$.

We note that D is a Type I region:

$$D = \{(x,y) : -1 \le x \le 1, 2x^2 \le y \le 1 + x^2\}.$$



Step 2. Set up the iterated integral.

Therefore,
$$\iint_D (x+2y)\,dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx.$$

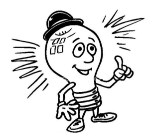
Step 3. Evaluate the inner integral.

$$\begin{aligned}
\int_{2x^2}^{1+x^2} (x+2y)\,dy &= \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} \\
&= x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2 \\
&= -3x^4 - x^3 + 2x^2 + x + 1.
\end{aligned}$$

Step 4. Complete the computation.

$$
\begin{aligned}
\iint_D (x+2y)\,dA &= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx \\
&= \left[-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{1} \\
&= \frac{32}{15}.
\end{aligned}
$$

■



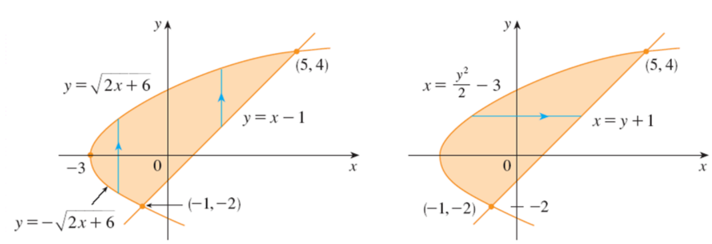
When we set up a double integral, it is helpful to draw a diagram.

For Type I region, it is helpful to draw a **vertical** arrow which starts at the lower boundary $y = g_1(x)$ and ends at the upper boundary $y = g_2(x)$. This corresponds to the inner integral.

For Type II region, it is helpful to draw a **horizontal** arrow which starts at the left boundary $x = h_1(x)$ and ends at the right boundary $x = h_2(x)$. This corresponds to the inner integral.

Example 9.6. *Evaluate* $\iint_D xy\,dA$ *where* D *is the region bounded by the line* $y = x - 1$ *and the parabola* $y^2 = 2x + 6$.

Solution. The region D can be of Type I or II:

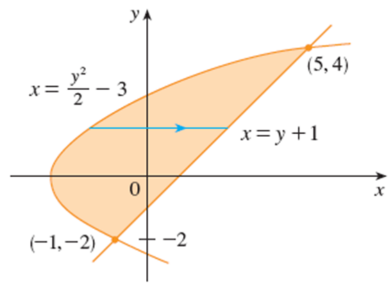


(a) D as a type I region (b) D as a type II region

But we prefer D as Type II because as a Type I region, the lower boundary of D is more complicated, in particular, it consists of two parts: one for $-3 \le x \le -1$ and another for $-1 \le x \le 5$.

Therefore, we let

$$D = \{(x, y) : -2 \le y \le 4, \frac{1}{2}y^2 - 3 \le x \le y + 1\}.$$



D as a type II region

So

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy.$$

Lets first compute the inner integral:

$$\begin{aligned}
\int_{\frac{y^2}{2}-3}^{y+1} xy\,dx &= \left[y\cdot\frac{x^2}{2}\right]_{x=\frac{y^2}{2}-3}^{x=y+1}\\
&= \frac{1}{2}\left(y(y+1)^2 - y(\frac{y^2}{2}-3+1)^2\right)\\
&= \frac{1}{2}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint_D xy\,dA &= \int_{-2}^{4}\frac{1}{2}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)dy\\
&= \frac{1}{2}\left[-\frac{y^6}{24}+y^4+2\frac{y^3}{3}-4y^2\right]_{-2}^{4}\\
&= 36.
\end{aligned}$$

■

Example 9.7. *Find the volume of the tetrahedron T bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$ and $z=0$.*

Solution. For question like this, it is wise to draw two diagrams:

• one for the solid (tetrahedron) T,

• another for the domain D.

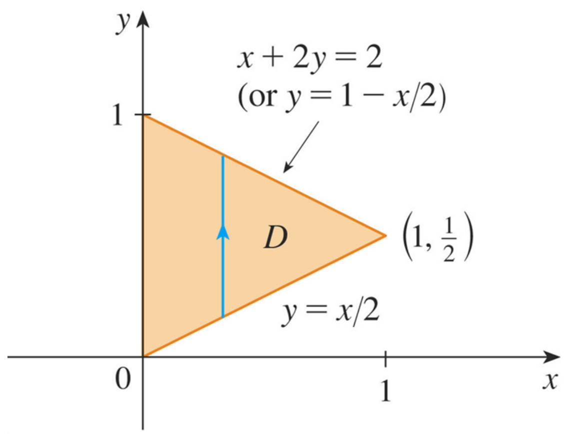
How do we start drawing?

There is no general rule, it depends on the problem in the question.

Notice the plane $x+2y+z=2$ intersects the xy-plane in the line $x+2y=2$. (Set $z=0$ in the equation of the plane).

Together with the restrictions that the solid is bounded by $z=0$ (above the xy-plane), $x=0$ (the yz-plane) and the plane $x=2y$, we see that T lies above the region D in the xy-plane bounded by the lines:

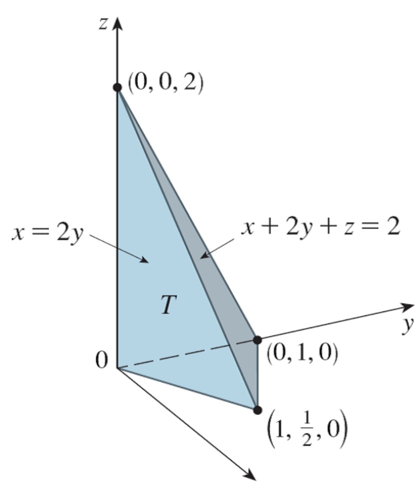
- $x=2y$,
- $x+2y=2$ (the intersection of the plane $x+2y+z=2$ and the plane $z=0$),
- $x=0$.



Notice that $(1, \frac{1}{2}, 0)$ and $(0,1,0)$ are two points on the plane $x+2y+z=2$.

There is another point on this plane: $(0,0,2)$.

We can now draw the tetrahedron T as follows:



So the required volume V lies under the graph $z=2-x-2y$ and above

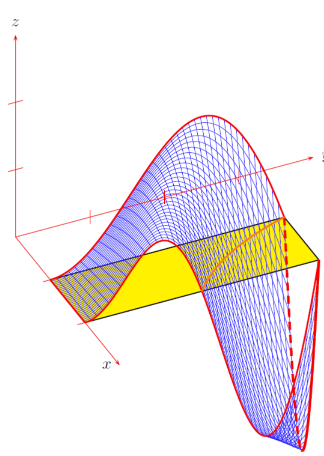
$$D = \{(x,y) : 0 \le x \le 1, \frac{x}{2} \le y \le 1 - \frac{x}{2}\}.$$

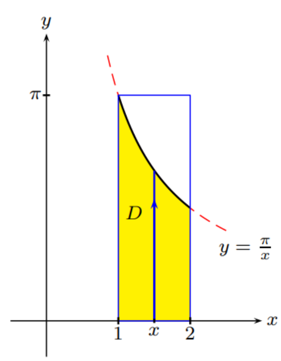
Therefore

$$\begin{aligned}
V &= \iint_D (2 - x - 2y)\, dA \\
&= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx \\
&= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx \\
&= \int_0^1 (x^2 - 2x + 1)\, dx \\
&= \left[\frac{x^3}{3} - x^2 + x\right]_0^1 \\
&= \frac{1}{3}.
\end{aligned}$$

■

Example 9.8. *Find the volume of the solid above the xy-plane and bounded by the graph of* $z = y\sin(xy)$ *and xy-plane for* $1 \le x \le 2$ *and* $0 \le y \le \pi$.





Ans. $\pi/2$.

Solution. First, for the rectangle $[1,2] \times [0,\pi]$ in the xy-plane,

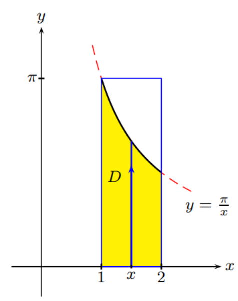
$$\begin{aligned}
y\sin(xy) = 0 &\iff y = 0 \text{ or } \sin(xy) = 0 \\
&\iff y = 0 \text{ or } xy = \pi \text{ or } (x,y) = (2,\pi).
\end{aligned}$$

Therefore, the surface $z = y\sin(xy)$ intersects the interior of the rectangle $[1,2]\times[0,\pi]$ in the curve $y = \dfrac{\pi}{x}$. Denote the region bounded by the curve $y = \dfrac{\pi}{x}$ and the x-axis from $x = 1$ to $x = 2$ by D. Note that D is a type I region given by

$$D = \{(x,y) : 1 \le x \le 2,\ \ 0 \le y \le \frac{\pi}{x}\}.$$

Note also that for points $(x,y) \ne (2,\pi)$ in the rectangle $[1,2]\times[0,\pi]$ (so that $0 \le xy < 2\pi$), we have

$$z = y\sin(xy) \ge 0 \iff \sin(xy) \ge 0 \iff 0 \le xy \le \pi \iff 0 \le y \le \frac{y}{x} \iff (x,y) \text{ is in } D.$$



Thus the required volume V is given by

$$\begin{aligned}
V = \iint_D y\sin(xy)\,dA &= \int_1^2 \int_0^{\frac{\pi}{x}} y\sin(xy)\,dy\,dx \\
&= \int_1^2 \left[-\frac{y\cos(xy)}{x} + \frac{\sin xy}{x^2}\right]_{y=0}^{y=\frac{\pi}{x}} dx \\
&= \int_1^2 -\frac{\pi\cos(\pi)}{x^2} + \frac{\sin(\pi)}{x^2}\,dx \\
&= \int_1^2 \frac{\pi}{x^2}\,dx \\
&= \left[-\frac{\pi}{x}\right]_1^2 = \frac{\pi}{2}.
\end{aligned}$$

■

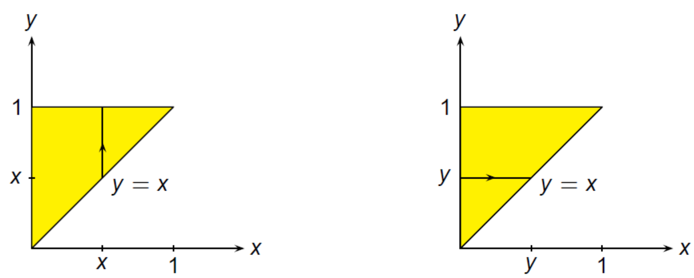
Remark. (Interchanging the order of integration) When computing an iterated integral over a domain which is of both type I and type II, it is sometimes easier to compute the integral by interchanging the order of integration.

Example 9.9. Evaluate the iterated integral $I = \int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$ by interchanging the order of integration.

[Remark. It is difficult to compute the inner integral $\int_x^1 \sin(y^2)\,dy$*.]*

Ans: $\frac{1}{2}(1-\cos 1)$.

Solution. The region of integration D is the triangular region bounded by the lines $y = x, x = 0$ and $y = 1$. Notice that D is a domain of both type I and type II.



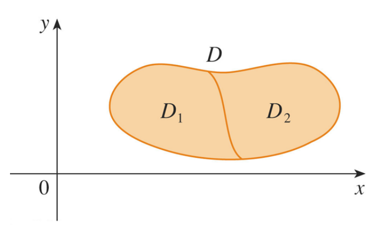
$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \int_D \sin(y^2)\,dA = \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$
$$= \int_0^1 \left[x\sin(y^2)\right]_{x=0}^{x=y} dy = \int_0^1 y\sin(y^2)\,dy$$
$$= \left[-\frac{1}{2}\cos(y^2)\right]_0^1 = \frac{1}{2}(1-\cos 1).$$

9.6 Decomposing Domain into Smaller Domains

Double integrals are additive with respect to the domain: if D is the union of domains $D_1, \ldots, D_n$ that do not overlap except possibly on boundary curves, then

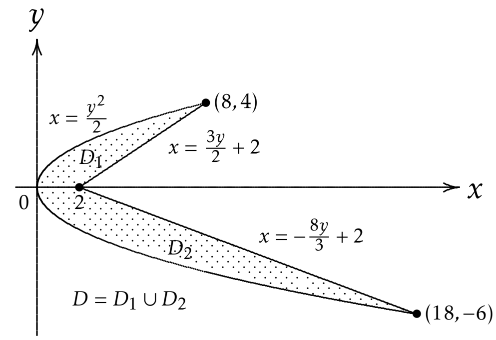
Theorem 9.6. *Additivity With Respect to Domain*

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \cdots + \iint_{D_n} f(x,y)\,dA.$$



Additivity may be used to evaluate double integrals over domain D which is neither of Type I nor II but can be decomposed into finitely many domains of Type I or II.

Example 9.10. *Suppose we want to compute the double integral* $\iint_D xy\ dA$, *where* D *is the shaded region bounded by the curve* $y^2 = 2x$, *and the lines* $2x - 3y - 4 = 0$ *and* $3x + 8y - 6 = 0$ *as shown below.*



Notice that the region D *is not a domain of type* I *or type* II. *Nonetheless,* D *is a union of two domains of type* II *as follows:*

$$
\begin{aligned}
D &= D_1 \cup D_2, \quad \textit{where} \\
D_1 &= \{(x,y) : 0 \le y \le 4, \frac{y^2}{2} \le x \le \frac{3y}{2} + 2\}, \\
D_2 &= \{(x,y) : -6 \le y \le 0, \frac{y^2}{2} \le x \le -\frac{8y}{3} + 2\}.
\end{aligned}
$$

Then by Theorem 9.6, we have

$$
\begin{aligned}
\iint_D xy\ dA &= \iint_{D_1} xy\ dA + \iint_{D_2} xy\ dA \\
&= \int_0^4 \int_{\frac{y^2}{2}}^{\frac{3y}{2}+2} xy\ dxdy + \int_{-6}^0 \int_{\frac{y^2}{2}}^{\frac{8y}{3}+2} xy\ dxdy,
\end{aligned}
$$

and then the two iterated itegrals in the last line can be computed readily.

■

9.7 Properties of Double Integral

The following properties for double integral over D follow from the corresponding properties for double integrals over a rectangle region R:

Theorem 9.7.

$$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

Theorem 9.8.

$$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$$

Theorem 9.9.
*If $f(x,y) \geq g(x,y)$ for **all** $(x,y) \in D$ then*

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA.$$

Example 9.11. *(i) Let D be a domain in $\mathbb{R}^2$. Then Theorem 9.7 and Theorem 9.8, we have*

$$\iint_D (x^2+2xy)\,dA = \iint_D x^2\,dA + 2\int_D xy\,dA.$$

(ii) Show that

$$\iint_D (x^2+y^2)\,dA \geq \int_D 2xy\,dA.$$

Proof of (ii). *For all (x,y) in D, we have*

$$(x-y)^2 \geq 0 \implies x^2-2xy+y^2 \geq 0 \implies x^2+y^2 \geq 2xy.$$

Thus by Theorem 9.9, we have $\iint_D (x^2+y^2)\,dA \geq \iint_D 2xy\,dA.$

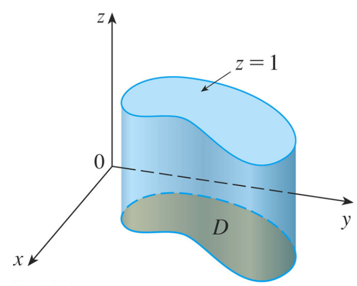
■

9.8 An Application – Finding Area

We can use double integral to compute area of a region D on the plane:

Theorem 9.10. *Area of plane region*
Let $f(x,y) = 1$ over a given region D. Then the area of D is

$$A(D) = \iint_D 1\, dA.$$



Proof. By considering the constant function $f(x,y) = 1$ for all (x,y) in D, we see that $\iint_D 1\, dA$ is the volume of the solid which is a cylinder whose base is $A(D)$ and height 1.

Another way of computing the volume of a cylinder is

$$\text{area of base} \;\times\; \text{height}$$

which is

$$A(D) \cdot 1$$

in this case. So

$$A(D) = \iint_D 1\, dA,$$

as required.

∎

9.9 Double Integrals in Polar Coordinates

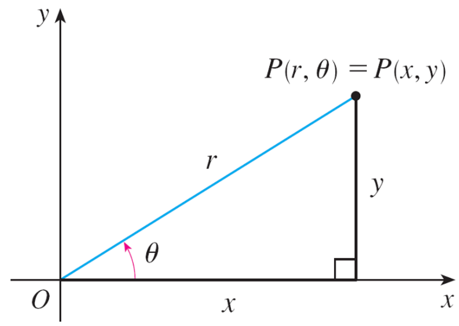
We have learned how to evaluate double integral over D where D is of the following type:

• rectangle;

• region of Type I or Type II.

Sometimes, the region D is not so easily described in terms of x and y coordinates.

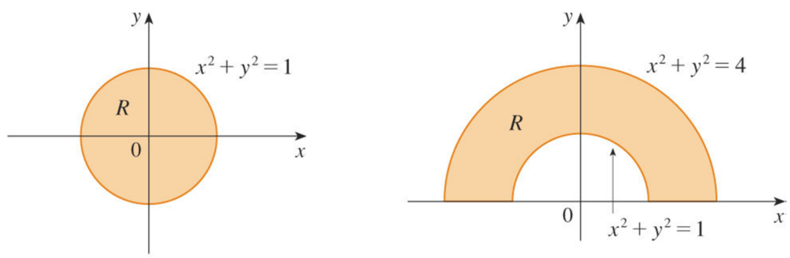
Sometimes, such regions can be conveniently described using polar coordinates (r,θ). The following figure which shows the relationship between polar coordinates and the rectangle coordinates:



Instead of using (x,y), we note that any point on the xy-plane can be represented by an ordered pair (r,θ) where

- r is the distance from the origin to the point
- θ is the angle from the positive x-axis to the straight line joining the origin and the point.

For example consider the following region given in terms of its polar coordinates:



(a) $R = \{(r,\theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ (b) $R = \{(r,\theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

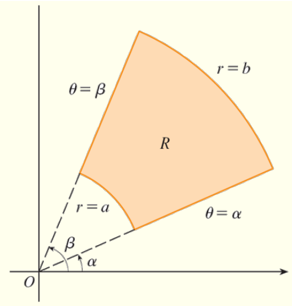
Polar coordinates (r,θ) of a point are related to the rectangle coordinate (x,y) by the equations

Theorem 9.11. *Polar Coordinates Versus Rectangle Coordinates*

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

Definition 9.5 (Polar Rectangle)**.**
A **polar rectangle** *is a region*

$$R = \{(r,\theta) : a \leq r \leq b, \ \ \alpha \leq \theta \leq \beta\}.$$



How do we compute

$$\iint_R f(x,y)\,dA$$

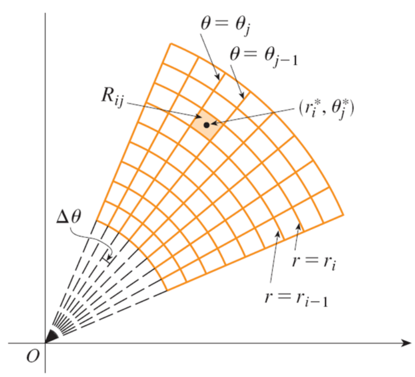
where R is a polar rectangle?

Recall that for $\iint_R f(x,y)\,dA$ over the usual rectangle R, we can think of $dA = dxdy$ as the area of the 'little rectangle' $\triangle A = \triangle x \triangle y$:

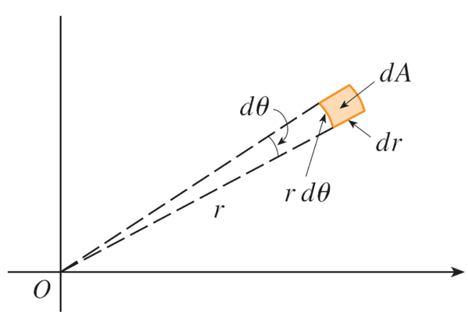
To compute $\iint_R f(x,y)\,dA$ over the **polar rectangle** R given by

$$R = \{(r,\theta) : a \le r \le b, \ \ \alpha \le \theta \le \beta\}.$$

we partition R as follows:



Then we can think of $dA = r\,dr\,d\theta$ as the area of the 'little polar rectangle' $\triangle A \approx \triangle r \cdot r\triangle\theta$:



Note that the arc of the polar rectangle is $r\,\triangle\theta$ which depends on r.

Theorem 9.12. *Change to Polar Coordinates in Double Integral*
If f is continuous on a polar rectangle R given by

$$R = \{(r,\theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$$

where $0 \le \beta - \alpha \le 2\pi$, then

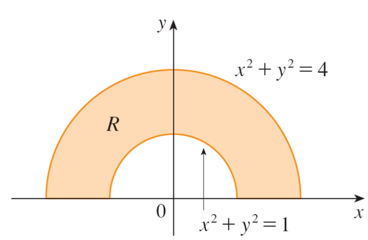
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

The formula says that we convert from rectangle to polar coordinates in a double integral by:

- writing $x = r\cos\theta$, $y = r\sin\theta$
- using the appropriate limits of integration for r and θ
- replacing dA by $r\ dr\ d\theta$ (**do not forget the additional** r in $r\ dr\ d\theta$)

Example 9.12. *Evaluate $\iint_R (3x+4y^2)\,dA$ where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.*

Solution. The region R is shown below:



$R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

So

$$R = \{(r,\theta) : 1 \le r \le 2, 0 \le \theta \le \pi\}.$$

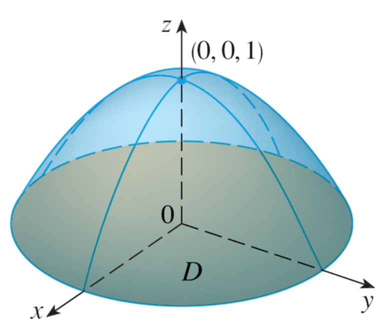
Changing to polar coordinates for the double integral, we have

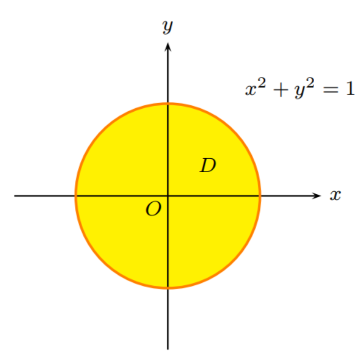
$$
\begin{aligned}
\iint_R (3x+4y^2)\,dA &= \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta) r\,dr\,d\theta \\
&= \int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta \\
&= \int_0^{\pi} \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_{r=1}^{r=2} d\theta \\
&= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta \\
&= \int_0^{\pi} \left[7\cos\theta + \frac{15}{2}(1-\cos 2\theta)\right] d\theta \\
&= \left[7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\right]_0^{\pi} \\
&= \frac{15\pi}{2}.
\end{aligned}
$$

■

Example 9.13. *Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.*

Solution. Notice the plane and the paraboloid intersect in the circle $x^2 + y^2 = 1$.
So the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \le 1$.





In polar coordinates, D is given by

$$D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}.$$

Since $1 - x^2 - y^2 = 1 - r^2$, we have

$$
\begin{aligned}
\text{Volume} &= \iint_D (1-x^2-y^2)\,dA \\
&= \int_0^{2\pi}\int_0^1 (1-r^2) r\,dr\,d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (r-r^3)\,dr\right) \\
&= 2\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 \\
&= \frac{\pi}{2}.
\end{aligned}
$$

■

Exercise 9.1. *Let R be the circular region bounded by the circle $x^2+(y-1)^2=1$. It is known that*

$$\iint_R \frac{dA}{(1+2x^2+2y^2)^2} = \frac{\pi}{a},$$

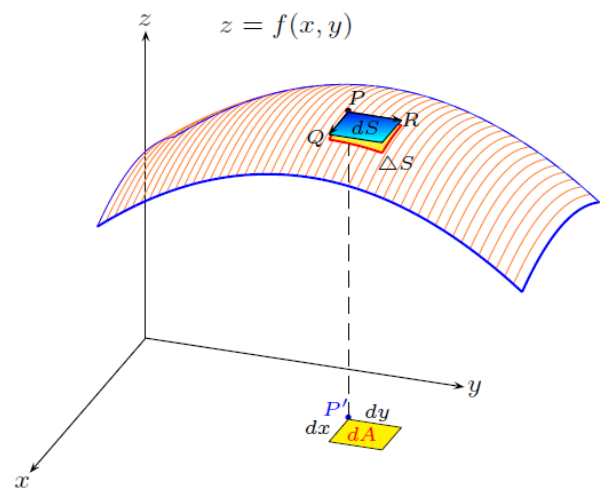
where a is a positive integer. Determine the value of a.

[Hint: Use polar coordinates and evaluate the resulting integral by means of the substitution $t=\tan\theta$].

Ans. 6.

9.10 Surface Area

Let f be a differentiable function of 2 variables defined on a domain D. We wish to find the surface area of the graph of f over D. It is simply equal to $\iint_D dS$, where dS is the differential of the surface area of the graph of f. Therefore we need to express dS in terms of the differential dA of the domain. To do so, take any point $P'(x,y)$ in D and let P be the corresponding point on the graph of f. Consider an increment dx along the x-direction and an increment dy along the y-direction at the point P'. Thus $dA=|dxdy|$. These increments sweep out an increment of surface area on the surface at P. The differential dS of this area at P is given by the corresponding area on the tangent plane to the surface at P.



Let $\overrightarrow{\boldsymbol{PQ}}$ be the vector on the tangent plane at P with x-component dx, and $\overrightarrow{\boldsymbol{PR}}$ the vector with y-component dy. Thus, $\overrightarrow{\boldsymbol{PQ}} = \langle dx, 0, f_x(x,y)dx\rangle$ and $\overrightarrow{\boldsymbol{PR}} = \langle 0, dy, f_y(x,y)dy\rangle$. The area of the parallelogram spanned by $\overrightarrow{\boldsymbol{PQ}}$ and $\overrightarrow{\boldsymbol{PR}}$ is the magnitude of the cross product $\overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}}$.

$$\overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = \langle -f_x, -f_y, 1\rangle dxdy.$$

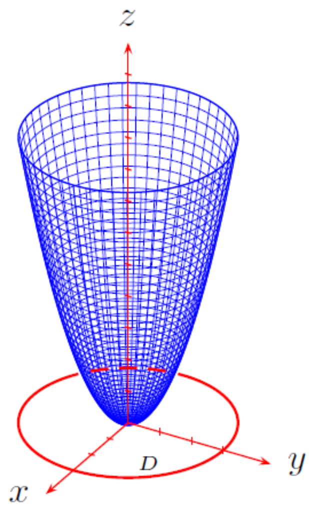
Therefore, $dS = |\langle -f_x, -f_y, 1\rangle dxdy| = \sqrt{f_x^2 + f_y^2 + 1}\, dA$. Consequently,

$$\text{Surface area} = \iint_D dS = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

Example 9.14. *Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.*

Solution. The paraboloid lies above the circular disk

$$D = \{(r, \theta) \mid 0 \le \theta \le 2\pi, 0 \le r \le 3\}.$$



The paraboloid is defined by $z = f(x,y)$, where $f(x,y) = x^2+y^2$. Thus we have $f_x = 2x$, $f_y = 2y$. Then

$$
\begin{aligned}
\text{Surface area} &= \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA \\
&= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA \\
&= \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \qquad \text{(change to polar coordinates)} \\
&= 2\pi \left[\tfrac{1}{12}(1 + 4r^2)^{\frac{3}{2}}\right]_0^3 \\
&= \frac{\pi}{6}(37\sqrt{37} - 1).
\end{aligned}
$$

■

Exercise 9.2. *The surface area of the portion on the cylinder*

$$y^2 + z^2 = 1$$

bounded by the planes $y = x + 2$ and $y = x - 2$ is equal to πa. Determine the value of a.

Ans. 8.

Exercise 9.3. *Let $y = f(x)$ be a curve on the xy-plane, where $f'(x)$ is continuous and $f(x) \geq 0$ for all $x \in [a,b]$. The curve $y = f(x)$, $x \in [a,b]$, situated on the xy-plane in $\mathbb{R}^3$, is rotated about the x-axis through 360° to generate a surface S. Let D be the Type I region on the xy-plane bounded between the curves $y = f(x)$ and $y = -f(x)$ for x from a to b. That is*

$$D = \{(x,y) \mid -f(x) \leq y \leq f(x), a \leq x \leq b\}.$$

Show that the function $p(x,y)$ defined on D whose graph is the portion of S above the xy-plane is given by $p(x,y) = \sqrt{f(x)^2 - y^2}$. Hence, or otherwise, show that the surface area of S is given by

$$\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$

Exercise 9.4. *The curve $y = \sin x$, $x \in [0, \frac{\pi}{2}]$, situated on the xy-plane in $\mathbb{R}^3$, is rotated about the x-axis through $360°$ to generate a surface S. Find the surface area of S.*

Ans. $\pi[\sqrt{2} + \ln(1 + \sqrt{2})]$.

Chapter 10

Ordinary Differential Equations

Read Thomas' Calculus, Chapter 16.

10.1 First Order Ordinary Differential Equations

Let y be a function of x. An equation involving x, y and at least one derivative of y is called an ordinary differential equation (ODE). The order of an ODE is the order of the highest derivative that occurs in the equation. We consider only first order ordinary differential equations.

Separable ODE

A separable first order ODE is of the form

$$\frac{dy}{dx} = f(x)g(y).$$

Separating the variables,

$$\frac{1}{g(y)}dy = f(x)dx.$$

Integrating both sides,

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C.$$

Example 10.1. *Solve* $y' = (1+y^2)e^x$.

Solution. First we note that the differential equation $\frac{dy}{dx} = (1+y^2)e^x$ is separable. We separate the variables to obtain $\frac{1}{1+y^2}dy = e^x dx$. Thus $\int \frac{1}{1+y^2}dy = \int e^x dx$. That is $\tan^{-1} y = e^x + C$, or $y = \tan(e^x + C)$.

■

Example 10.2. *Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with 2 mg at certain time, say $t = 0$, what can be said about the amount available at a later time?*

Solution. Let y be the amount of substance in mg at time t in years. Then $\frac{dy}{dt} = -ky$, $y(0) = 2$, where k is a positive constant. Thus $\frac{dy}{y} = -kdt$. Integrating, $\int \frac{dy}{y} = \int -kdt$. That is $\ln|y| = -kt + C$, or equivalently, $|y| = e^{-kt+C} = e^C e^{-kt}$. Therefore, $y = e^C e^{-kt}$ or $y = -e^C e^{-kt}$. In other words, $y = Ae^{-kt}$, where A is a constant. As $y(0) = 2$, we have $2 = Ae^{-k\times 0} = A$. Consequently, $y = 2e^{-kt}$.

■

Remark. How to find k? The value of k depends on the radioactive substance. Usually we can calculate k by looking up the half-life of the substance in a chemistry table.

For example, the half-life of the substance is T years. From the above solution, we know $y = Ae^{-kt}$. Thus $\frac{A}{2} = Ae^{-kT}$. That is $-\ln 2 = -kT$. From this we obtain $k = \frac{\ln 2}{T}$.

In the report 'Stemming the tide 2020: The reality of the Fukushima radioactive water crisis', Greenpeace claimed that the contaminated water contained "dangerous levels of carbon-14", a radioactive substance that has the "potential to damage human DNA".

Carbon-14 is unstable and has a half-life of 5730 ± 40 years.

Example 10.3. *A copper ball is heated to $100°$ C. At time $t = 0$, it is placed in water which is maintained at $30°$ C. At the end of 3 mins, the temperature of the ball is reduced to $70°$ C. Find the time at which the temperature of the ball is $31°$ C.*

[Physical information: Experiments show that the rate of change of the temperature T of the ball with respect to time t is proportional to the difference between T and the temperature of the surrounding medium.

Also heat flows so rapidly in copper that at any time the temperature is practically the same at all points of the ball.]

Solution. Let T be the temperature of the ball at time t. Then $\frac{dT}{dt} = k(T - 30)$, $T(0) = 100, T(3) = 70$. Thus $\int \frac{dT}{T-30} = \int kdt$. That is $\ln|T - 30| = kt + C$, or equivalently, $T - 30 = Ae^{kt}$. $T(0) = 100 \Rightarrow 100 - 30 = Ae^{k\times 0} \Rightarrow A = 70$. Therefore, $T = 30 + 70e^{kt}$.

$T(3) = 70 \Rightarrow 70 = 30 + 70e^{3k} \Rightarrow 4 = 7e^{3k} \Rightarrow k = \frac{1}{3}\ln\frac{4}{7} = \frac{1}{3}(\ln 4 - \ln 7)$. Therefore, $T = 30 + 70e^{\frac{t}{3}(\ln 4 - \ln 7)}$. Then

$$\begin{aligned} T = 31 &\implies 31 = 30 + 70e^{\frac{t}{3}(\ln 4 - \ln 7)} \\ &\implies \frac{t}{3}(\ln 4 - \ln 7) = \ln\frac{1}{70} = -\ln 70 \\ &\implies t = \frac{3\ln 70}{\ln 7 - \ln 4} = 22.78\,\text{min}. \end{aligned}$$

■

Example 10.4. *A skydiver together with his equipment has a combined weight of m kg. After he jumps and the parachute opens at time $t = 0$, he falls freely and is descending with velocity v m/s at the moment when the time is t s. The air resistance against his descending motion is known to be bv^2 N, where b is a positive constant, and v is his velocity at that moment. Show that the skydiver eventually approaches a terminal speed of $k \equiv \sqrt{\frac{mg}{b}}$ m/s, where $g = 9.81$ m/s^2 is the acceleration due to gravity.*

Solution. By Newton's second law, we have $m\dfrac{dv}{dt} = mg - bv^2$.

Note that $k \equiv \sqrt{\dfrac{mg}{b}} \implies mg = bk^2$, and the equation can thus be rewritten as

$$m\frac{dv}{dt} = bk^2 - bv^2 \implies \frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2).$$

Separating the variables, we get

$$\begin{aligned} &\frac{dv}{v^2 - k^2} = -\frac{b}{m}dt \\ \implies &\frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right)dv = -\frac{b}{m}dt \\ \implies &\left(\frac{1}{v-k} - \frac{1}{v+k}\right)dv = -\frac{2kb}{m}dt. \end{aligned}$$

Integrating, we get

$$\begin{aligned}
\int (\frac{1}{v-k} - \frac{1}{v+k}) dv &= \int -\frac{2kb}{m} dt \\
\Longrightarrow \ln|v-k| - \ln|v+k| &= -\frac{2kb}{m}t + C \\
\Longrightarrow \ln\left|\frac{v-k}{v+k}\right| &= -\frac{2kb}{m}t + C \\
\Longrightarrow \left|\frac{v-k}{v+k}\right| &= e^C \cdot e^{-\frac{2kb}{m}t} \\
\Longrightarrow \frac{v-k}{v+k} &= Ae^{-\frac{2kb}{m}t},
\end{aligned}$$

where A is a constant ($= e^C$ or $-e^C$). Solving for v, we obtain

$$v = \left(\frac{1 + Ae^{-\frac{2kb}{m}t}}{1 - Ae^{-\frac{2kb}{m}t}} \right) k.$$

From this, we see that

$$\lim_{t\to\infty} v = \frac{1+0}{1-0} \cdot k = k.$$

■

Exercise 10.1. *Solve the differential equation*

$$\frac{dy}{dx} = xe^{3x-2y}.$$

Ans: $\frac{1}{2}e^{2y} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$.

Exercise 10.2. *A curve C that passes through the point $(2,1)$ is such that at any point (x,y) on the curve,*

$$x^2 \frac{dy}{dx} = y(x^3 + 4).$$

Find the equation of the curve.

Ans: $y = e^{\frac{x^2}{2} - \frac{4}{x}}$.

10.2 Reduction to Separable Form

Certain first order differential equations are not separable, but they can be made separable by a simple change of variables.

This holds for equations of the form

$$y' = g(\frac{y}{x}),$$

where g is any function of $\frac{y}{x}$. Let $v = \dfrac{y}{x}$. Then $y = vx$ and $y' = v + xv'$. Then the equation $y' = g(\dfrac{y}{x})$ can be written as $v + xv' = g(v)$ or equivalently $v' = \dfrac{g(v) - v}{x}$, which is separable. We can now solve for v, and then solve for y.

Example 10.5. *Solve* $2xyy' - y^2 + x^2 = 0$.

Solution. We may rewrite the equation as

$$y' = \frac{y^2 - x^2}{2xy} \quad \text{or equivalently} \quad y' = \frac{-1 + (\frac{y}{x})^2}{2(\frac{y}{x})},$$

where the right hand side is a function of $\dfrac{y}{x}$. Let $v = \dfrac{y}{x}$, so that $y' = (xv)' = v + xv'$. Then the equation can be written as

$$v + xv' = \frac{-1 + v^2}{2v} \iff x\frac{dv}{dx} = \frac{-1 + v^2}{2v} - v = -\frac{1 + v^2}{2v}.$$

Separating the variables, we get

$$\frac{2v\,dv}{1 + v^2} = -\frac{dx}{x}.$$

Integrating, we get

$$\begin{aligned} \int \frac{2v\,dv}{1 + v^2} &= \int -\frac{dx}{x} \\ \implies \ln|1 + v^2| &= -\ln|x| + C \\ \implies \ln|x(1 + v^2)| &= C \\ \implies |x(1 + v^2)| &= e^C \\ \implies x(1 + v^2) &= A, \end{aligned}$$

where A is a constant ($= e^C$ or $-e^C$). Therefore, we have

$$x(1 + \frac{y^2}{x^2}) = A, \text{ or equivalently, } x^2 + y^2 = Ax.$$

■

A differential equation of the form $y' = f(ax+by)$, where f is continuous and $b \neq 0$, can be solved by setting $u = ax+by$. (If $b = 0$, then the equation itself is separable.)

Example 10.6. *Solve* $(2x-4y+5)y' + x - 2y + 3 = 0.$ *qquad* *(*)*

Solution. Note that the equation can be rewritten as

$$y' = \frac{-(x-2y+3)}{2x-4y+5} = \frac{-(x-2y)-3}{2(x-2y)+5},$$

where the right hand side is a function of $x-2y$.

Let $u = x-2y$. Then $u' = 1-2y'$, and thus $y' = \dfrac{1-u'}{2}$. Thus the equation becomes

$$\frac{1-u'}{2} = \frac{-u-3}{2u+5},$$

which gives

$$u' = \frac{4u+11}{2u+5} \iff \frac{du}{dx} = \frac{4u+11}{2u+5}.$$

Separating the variables, we get

$$\frac{2u+5}{4u+11}du = dx,$$

which gives

$$\frac{\frac{1}{2}(4u+11)-\frac{1}{2}}{4u+11}du = dx \iff (1-\frac{1}{4u+11})du = 2dx.$$

Integrating, we get

$$\begin{aligned}
&\int (1-\frac{1}{4u+11})du = \int 2dx \\
\Longrightarrow\ & u - \frac{1}{4}\ln|4u+11| = 2x + C_1 \\
\Longrightarrow\ & 4u - \ln|4u+11| = 8x + 4C_1 \\
\Longrightarrow\ & 4x - 8y - \ln|4x-8y+11| = 8x + 4C_1 \quad (\text{since } u = x-2y) \\
\Longrightarrow\ & 4x + 8y + \ln|4x-8y+11| + C = 0, \quad \text{where } C = 4C_1.
\end{aligned}$$

In the calculations above, the two expressions

$$2u+5 = 2x-4y+5 \quad \text{and} \quad 4u+11 = 4x-8y+11$$

have appeared in the denominators, and thus the equations

$$2x-4y+5 = 0 \quad \text{and} \quad 4x-8y+11 = 0$$

are possible solutions to the DE (*) that we may have missed out in our calculations.

Case 1: $2x-4y+5=0$. Then $y=\dfrac{2x+5}{4}$ and $y'=\dfrac{1}{2}$. Substituting this into the left hand side of (*), we get

$$(2x-4y+5)y'+x-2y+3=0\cdot y'+x-2\cdot\frac{2x+5}{4}+3=-\frac{1}{2}\neq 0.$$

Thus $2x-4y+5=0$ does not satisfy (*).

Case 2: $4x-8y+11=0$. Then $y=\dfrac{4x+11}{8}$ and $y'=\dfrac{1}{2}$. Substituting this into the left hand side of (*), we get

$$(2x-4y+5)y'+x-2y+3=(2x-4\cdot\frac{4x+11}{8}+5)\cdot\frac{1}{2}+x-2\cdot\frac{4x+11}{8}+3=(-\frac{11}{2}+5)\cdot\frac{1}{2}+x-x-\frac{11}{4}+3=0.$$

Thus $4x-8y+11=0$ satisfies (*).

Thus the solutions to (*) are

$$4x+8y+\ln|4x-8y+11|+C=0 \quad\text{and}\quad 4x-8y+11=0.$$

■

Exercise 10.3. *Solve the initial value problem* $y'=\frac{y}{x}+\frac{2x^3\cos(x^2)}{y}$, $y(\sqrt{\pi})=0$.

Ans: $y=\pm x\sqrt{2\sin(x^2)}$.

Solution: Note that we may rewrite the differential equation as

$$y'=\frac{y}{x}+\frac{2x^2\cos(x^2)}{\frac{y}{x}},$$

where the right hand side is a function of $\dfrac{y}{x}$ and x. Let $v=\dfrac{y}{x}$. Then $y=xv$ and $y'=v+xv'$. Thus the equation becomes

$$\begin{aligned} v+xv' &= v+\frac{2x^2\cos(x^2)}{v} \\ \Longrightarrow v' &= \frac{2x\cos(x^2)}{v} \\ \Longrightarrow \frac{dv}{dx} &= \frac{2x\cos(x^2)}{v}. \end{aligned}$$

Separating the variables, we get

$$v\,dv=2x\cos(x^2)\,dx.$$

Integrating, we get

$$\begin{aligned}
\int v\,dv &= \int 2x\cos(x^2)\,dx \\
\implies \frac{1}{2}v^2 &= \sin(x^2)+C \\
\implies \frac{1}{2}\cdot\frac{y^2}{x^2} &= \sin(x^2)+C \\
\implies y^2 &= 2x^2(\sin(x^2)+C).
\end{aligned}$$

Now

$$y(\sqrt{\pi})=0 \implies 0=2\pi(\sin\pi+C) \implies C=0.$$

Consequently, the solution is $y^2=2x^2\sin(x^2)$, or $y=\pm x\sqrt{2\sin(x^2)}$.

■

Exercise 10.4. *Solve* $(x+2y-1)+3(x+2y)y'=0$.

Ans: $x+3y+C=3\ln|x+2y+2|$, $x+2y+2=0$.

10.3 Linear First Order ODE

A linear first order ODE is of the form

$$\frac{dy}{dx}+P(x)y=Q(x),$$

where $P(x),Q(x)$ are continuous functions. Note that the above ODE is separable if $P(x)$ is identically equal to $Q(x)$. This is the *standard form* of a linear first order ODE.

Let $I(x)=e^{\int P(x)\,dx}$. We call $I(x)$ an integrating factor. Multiplying both sides of the above ODE by $I(x)$, we get

$$\frac{dy}{dx}e^{\int P(x)\,dx}+P(x)e^{\int P(x)\,dx}y=Q(x)e^{\int P(x)\,dx}.$$

But

$$\frac{dy}{dx}e^{\int P(x)\,dx}+P(x)e^{\int P(x)\,dx}y=\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right),$$

which can be shown by applying the product rule and the Fundamental Theorem of Calculus. Hence,

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right)=Q(x)e^{\int P(x)\,dx}.$$

Thus we have shown that

$$\frac{d}{dx}(y \cdot I(x)) = Q(x) \cdot I(x).$$

Integrating both sides gives

$$y \cdot I(x) = \int Q(x) \cdot I(x)\, dx$$

from which the solution for y can be obtained.

Example 10.7. *Solve* $xy' - 3y = x^2, x > 0$.

Solution. First we rewrite the DE in the standard form $y' - \frac{3}{x}y = x$. An integrating factor is $e^{\int -\frac{3}{x}\,dx} = e^{-3\ln x} = \frac{1}{x^3}$. Multiplying the DE (the standard form) by this integrating factor, we have $(y' - \dfrac{3}{x}y) \cdot \dfrac{1}{x^3} = x \cdot \dfrac{1}{x^3}$, which gives $(\frac{y}{x^3})' = \frac{1}{x^2}$. Integrating, we have $\frac{y}{x^3} = -\frac{1}{x} + C$. That is $y = -x^2 + Cx^3$.

■

Example 10.8. *Solve* $y' - y = e^{2x}$.

Solution. An integrating factor is $e^{\int -1\,dx} = e^{-x}$. Multiplying the DE by this integrating factor, we have $(y' - y) \cdot e^{-x} = e^{2x} \cdot e^{-x}$, which gives $(ye^{-x})' = e^x$. Integrating, we obtain $ye^{-x} = e^x + C$. That is $y = e^{2x} + Ce^x$. ■

Exercise 10.5. *Solve the differential equation*

$$(x+1)^2 \frac{dy}{dx} - (x+1)y = 2,\ x > -1.$$

Ans: $y = -\frac{1}{x+1} + C(x+1)$.

Exercise 10.6. *Solve the differential equation*

$$\frac{dy}{dx} = \frac{4 + y\sin x}{\cos x},\ -\frac{\pi}{2} < x < \frac{\pi}{2},$$

given that $y = 6$ *when* $x = 0$.

Ans: $y = \frac{4x+6}{\cos x}$.

Exercise 10.7. *An object of mass m dropped from rest in a medium that offers a resistance proportional to the magnitude of the instantaneous velocity of the object. Let* $x(t)$ *be the displacement of the object measured vertically downward at time t so that* $x(0) = 0$. *Show that*

$$x(t) = \frac{mg}{k}t + \frac{m^2 g}{k^2}(e^{-\frac{k}{m}t} - 1),$$

where k is the proportional (positive) constant of the force of resistance of the medium.

[Set up the DE for the velocity first: $m\frac{dv}{dt} = mg - kv$.]

10.4 The Bernoulli Equation.

An ODE in the form
$$y' + p(x)y = q(x)y^n,$$
where $n \neq 0, 1$, is called the *Bernoulli equation*. The functions $p(x)$ and $q(x)$ are continuous functions on an interval J.

Let $u = y^{1-n}$. Substituting into the Bernoulli equation we get
$$u' + (1-n)p(x)u = (1-n)q(x).$$
This is a first order linear ODE.

Remark. (i) When $n = 0$ or 1, the Bernoulli equation itself is a first order linear ODE.

(ii) When $n > 0$, the constant zero function $y(x) = 0$ is automatically a solution of the Bernoulli equation.

Example 10.9. *Solve* $y' + y = x^2y^2$.

Solution. This is a Bernoulli equation with $n = 2$. Let $z = y^{1-2} = y^{-1}$. Then $z' = -y^{-2}y'$, so that $y' = -y^2z'$. Thus the given Bernoulli equation can be written as
$$-y^2z' + y = x^2y^2 \Leftrightarrow z' - y^{-1} = -x^2 \Leftrightarrow z' - z = -x^2.$$
This is a first order linear equation. Multiplying by the integrating factor $e^{\int -dx} = e^{-x}$, we have
$$(z' - z)e^{-x} = -x^2e^{-x},$$
which gives
$$(ze^{-x})' = -x^2e^{-x}.$$
Integrating, we get $ze^{-x} = \int -x^2e^{-x}\,dx$.

Using integration by parts, we have
$$\int -x^2e^{-x}\,dx = x^2e^{-x} + 2xe^{-x} + 2e^{-x} + C.$$
Thus $\quad z = e^x(x^2e^{-x} + 2xe^{-x} + 2e^{-x} + C) = x^2 + 2x + 2 + Ce^x$.

Therefore, $\dfrac{1}{y} = x^2 + 2x + 2 + Ce^x$, and thus $y = \dfrac{1}{x^2 + 2x + 2 + Ce^x}$.

Since $n = 2 > 0$, $y = 0$ is also a solution.

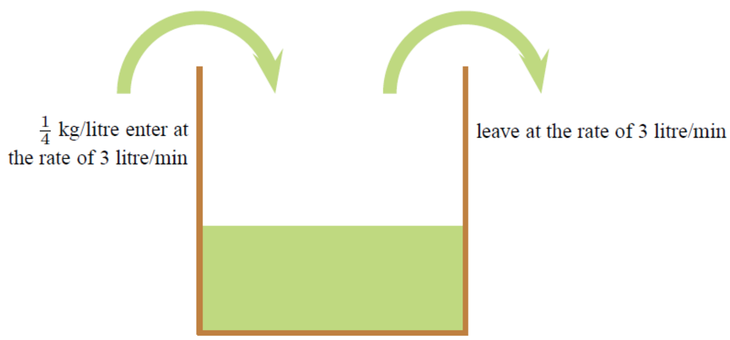
■

Exercise 10.8. Solve $xy' + y = x^4y^3$.

Ans: $\frac{1}{y^2} = -x^4 + cx^2$, or $y = 0$.

10.5 Applications of ODE

Example 10.10. *At time $t = 0$, a tank contains 20 kg of salt dissolved in 100 litres of water. Assume that water containing $\frac{1}{4}$ kg of salt per litre is entering the tank at the rate of 3 litre per min, and the well-stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t.*



At $t = 0$, there is 100 litre of water with 20 kg salt dissolved in it.
At time t, there remains 100 litre of water with Q kg salt dissolved in it.

Solution. First note that the volume of the solution remains constant which is 100 litres. Let Q be the amount of salt in kg at time t. The concentration of salt in the solution is $Q/100$ kg per litre. Suppose at time $t + dt$, the amount of salt is $Q + dQ$. Then

$$dQ = \text{salt input} - \text{salt output} = 3 \times \frac{1}{4} \times dt - 3 \times \frac{Q}{100} \times dt.$$

Thus

$$\frac{dQ}{dt} = \frac{3}{4} - \frac{3Q}{100}.$$

That is

$$\frac{dQ}{dt} = -\frac{3}{100}(Q - 25).$$

The general solution to this first order linear DE is $Q = 25 + Ce^{-\frac{3t}{100}}$. Since $Q(0) = 20$, we have $20 = 25 + C$ so that $C = -5$. Consequently, $Q = 25 - 5e^{-\frac{3t}{100}}$.

Note that $\lim\limits_{t\to\infty} Q(t) = 25$. Thus after sufficiently long time, the salt concentration will approach 25 kg per 100 litres.

■

Example 10.11. *A body was found at a crime scene. You are a member of the CSI team and you arrived at the crime scene at 8AM. Immediately upon arrival, you took the temperature of the victim and found that it was 26°C. At 9AM, you took the temperature of the victim again and found that it was 24°C. You estimate that the victim's temperature was 37°C just before death* *and that the temperature at the crime scene stayed approximately constant at 21°C. What is your estimate on the time of death?*

Remark: Newton's law of cooling states that the rate of cooling of an object is proportional to the difference in temperature between the object and its surroundings.

Solution. Set time $t = 0$ at 8AM, where t is measured in hours. Let T be the temperature of the body at time t. By Newton's law of cooling, we have $\frac{dT}{dt} = k(T - 21)$, where k is a constant. The general solution is $T = 21 + Ae^{kt}$. As $T(0) = 26$, we have $26 = 21 + A$ so that $A = 5$. Therefore, $T = 21 + 5e^{kt}$. At 9AM, that is 1 hour later, $T(1) = 24$. Thus $24 = 21 + 5e^{k}$ so that $k = \ln(\frac{3}{5})$. Hence

$$T = 21 + 5e^{t\cdot\ln(\frac{3}{5})} = 21 + 5e^{\ln((\frac{3}{5})^t)} = 21 + 5\left(\frac{3}{5}\right)^t.$$

If τ is the time of death, then $T(\tau) = 37$. Therefore,

$$37 = 21 + 5\left(\frac{3}{5}\right)^{\tau}.$$

That is $\frac{16}{5} = \left(\frac{3}{5}\right)^{\tau} \Leftrightarrow \ln(\frac{16}{5}) = \tau\ln(\frac{3}{5}) \Leftrightarrow \tau = \ln(\frac{16}{5})/\ln(\frac{3}{5}) = -2.277$ hours, (or equivalently negative 2 hour 17 mins). Thus time of death is about 5 : 43AM.

■

Exercise 10.9. *The Jurong lake has a volume of* $700000\,\mathrm{m}^3$*. At time* $t = 0$*, the government starts a water cleaning process so that only fresh clean water flows into the lake. After* 5 *years, it is found that the pollution in the lake is reduced by* 50%*. If fresh water flows into the lake at a rate of* r *cubic metres per year and lake water flows out to the sea at the same rate, what is the value of* r *correct to the nearest thousands?*

Ans: 97000.

Exercise 10.10. *Newton's law of cooling states that the rate of cooling of an object is proportional to the difference in temperature between the object and its surroundings. If an object is kept in an environment whose temperature is kept constant at* 15°C *and the object takes* 20 *minutes to cool from* 95°C *to* 55°C*, determine how much longer it will take for the object to cool down to* 25°C*.*

Ans: 40 mins more.

Exercise 10.11. *In a chemical reaction, the rate at which the mass,* m *(in grams) of a chemical compound at time* t *(in seconds) is proportional to* $m^2 - 9m + 18$*,* $(0 < m < 3)$*. Initially* $(t = 0)$*, we assume* $m = 0$*. After* 1 *second, the mass of the chemical compound has increased to* 2g*. Write down a differential equation in* m *and* t*, and show that*

$$\frac{m-6}{m-3} = 2^{t+1}.$$

Find the exact mass of the chemical compound after 2 *seconds.*

Ans: 18/7 grams.

Chapter 11

More on ODE

Read Thomas' Calculus, Chapter 16.

Remark: Chapter 11 will be excluded from the Final Exam.

11.1 Euler's Method

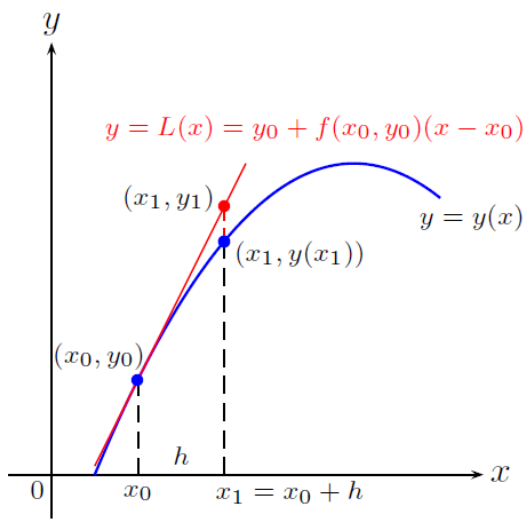
Remark: Section 11.1 will be excluded from the Final Exam.

Not all the first order ODE's can be solved explicitly in closed form. In that case, we have to rely on numerical solutions. In this section, we introduce Euler's method which is a numerical method in approximating a first order ODE. Given a differential equation $\frac{dy}{dx} = f(x,y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) \text{ or } L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

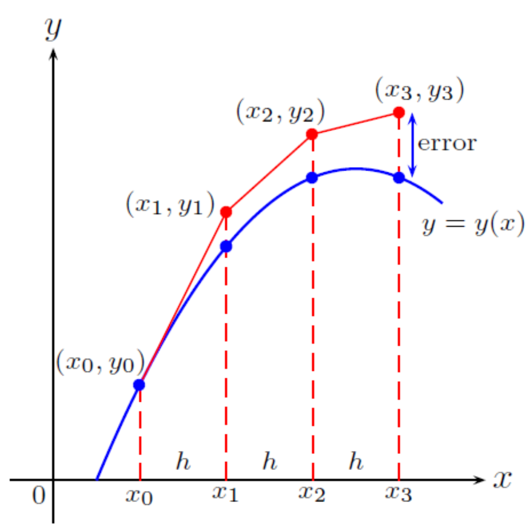
The linear function $L(x)$ gives a good approximation to the solution in a short interval about x_0. The idea of Euler's method is to put together a sequence of such linearizations in a successive manner to approximate the solution curve over a longer interval.

First we know that the point (x_0, y_0) lies on the solution curve. Consider a small increment from x_0 to $x_1 \equiv x_0 + h$. The graph of $L(x)$ is the tangent line with slope $f(x_0, y_0)$ to the solution curve $y = y(x)$ at the point (x_0, y_0). So if h is small, $y_1 \equiv L(x_1)$ is a good approximation to $y(x_1)$. In other word, the point (x_1, y_1) is close to the solution curve $y = y(x)$.



The linearization $L(x)$ at $x = x_0$.

The first step approximates $y(x_1)$ with $y_1 = L(x_1)$.



The red polygonal curve is Euler's approximation.

Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve through (x_1, y_1), we take a second step. Setting $x_2 = x_1 + h$, we use the linearization of the solution curve through (x_1, y_1) to calculate $y_2 = y_1 + f(x_1, y_1)h$.

This gives the next approximation (x_2, y_2) to the value along the solution curve $y = y(x)$. Continuing in this way, we take a third step from the point (x_2, y_2) with slope $f(x_2, y_2)$ to obtain the third approximation $y_3 = y_2 + f(x_2, y_2)h$, and so on.

In other words, we are building an approximation to one of the solution by following the direction of the slope field of the differential equation.

The following steps summarize Euler's method. Suppose we wish to approximate the solution over the interval $[a,b]$. Choose an integer n as the number of steps. Let $h = \frac{b-a}{n}$. Let

$$\begin{aligned} x_0 &= a \\ x_1 &= x_0 + h \\ x_2 &= x_1 + h \\ &\vdots \\ b = x_n &= x_{n-1} + h. \end{aligned}$$

Then calculate the approximations to the solution as follows.

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)h \\ y_2 &= y_1 + f(x_1, y_1)h \\ &\vdots \\ y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})h. \end{aligned}$$

The polygonal curve joining the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ successively is an approximation to the solution curve of the DE $y' = f(x,y)$ through the point (x_0, y_0).

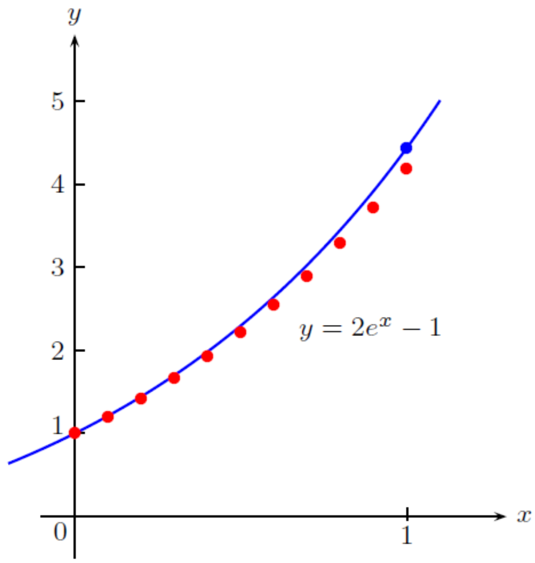
Example 11.1. *Use Euler's method to solve*

$$y' = 1 + y, y(0) = 1,$$

on the interval $[0,1]$ *starting at* $x_0 = 0$ *by taking* $h = 0.1$. *Find the approximate value of* $y(1)$ *and compare it with the exact value.*

Solution. Taking $n = 10$ and $h = 0.1$, the result is tabulated in the following table. The exact solution to the DE is $y = 2e^x - 1$. Thus the exact value at $x = 1$ is $y(1) = 2e - 1 = 4.4366$. The approximate value is 4.1875.

Euler solution of $y' = 1 + y, y(0) = 1$, $h = 0.1$			
x	y(Euler)	y(Exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.221	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1	4.1875	4.4366	0.2491



The graph of $y = 2e^x - 1$ and its Euler's approximation

Exercise 11.1. *Use Euler's method to calculate the first three approximations to the initial value problem:*

$$y' = 2xy + 2y,\ y(0) = 3,$$

by taking $h = 0.2$.

Ans: $y_1 = 4.2, y_2 = 6.216, y_3 = 9.697$.

Remark. In the film Hidden Figures, Katherine Goble resorts to Euler's method in calculating the re-entry of astronaut John Glenn from Earth orbit.

11.2 2nd Order Linear Equations with Constant Coefficients

Remark: Section 11.2 will be excluded from the Final Exam.

Let us begin with second order homogenous linear equation with constant coefficients

$$y'' + ay' + by = 0, \tag{11.1}$$

where a and b are real constants. We look for a solution of the form $y = e^{\lambda x}$. Plugging into (11.1) we find that, $e^{\lambda x}$ is a solution of (11.1) if and only if

$$\lambda^2 + a\lambda + b = 0. \tag{11.2}$$

(11.2) is called the *auxiliary equation* or *characteristic equation* of (11.1). The roots of (11.2) are called *characteristic values* (or eigenvalues):

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}),$$
$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}).$$

1. If $a^2 - 4b > 0$, (11.2) has two distinct real roots λ_1, λ_2, and the general solutions of (11.1) is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

2. If $a^2 - 4b = 0$, (11.2) has one real root λ (we may say that (11.2) has two equal roots $\lambda_1 = \lambda_2$). The general solution of (11.1) is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}.$$

3. If $a^2 - 4b < 0$, (11.2) has a pair of complex conjugate roots

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta.$$

The general solution of (11.1) is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example 11.2. Solve $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$.

Solution. The characteristic values are $1, -2$. Thus the solution is

$$y = e^x + 3e^{-2x}.$$

■

Example 11.3. Solve $y'' - 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 1$.

Solution. The characteristic values are 2 with multiplicity 2. Thus the solution is

$$y = (3 - 5x)e^{2x}.$$

■

Example 11.4. Solve $y'' - 2y' + 10y = 0$.

Solution. The characteristic values are $1 + 3i$, $1 - 3i$. Thus the solution is

$$y = e^x(c_1 \cos 3x + c_2 \sin 3x).$$

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11.3 Method of Undetermined Coefficients

Remark: Section 11.3 will be excluded from the Final Exam.

Consider the equation $y'' + ay' + by = f(x)$, where a and b are real constants. To solve this non-homogeneous linear DE, we look for a particular solution y_p of $y'' + ay' + by = f(x)$.

Then the general solution is the sum of the general solution y_c of the associated homogeneous linear DE: $y'' + ay' + by = 0$ and this particular solution y_p. That is

$$y = y_c + y_p.$$

Case 1. $f(x) = P_n(x)e^{\alpha x}$, where $P_n(x)$ is a polynomial of degree $n \geq 0$.
We look for a particular solution in the form

$$y = Q(x)e^{\alpha x},$$

where $Q(x)$ is a polynomial. Plugging it into $y'' + ay' + by = f(x)$ we find

$$Q'' + (2\alpha + a)Q' + (\alpha^2 + a\alpha + b)Q = P_n(x). \tag{11.3}$$

Subcase 1.1. If $\alpha^2 + a\alpha + b \neq 0$, namely, α is not a root of the characteristic equation, we choose $Q = R_n$, a polynomial of degree n, and

$$y = R_n(x)e^{\alpha x}.$$

The coefficients of R_n can be determined by comparing the terms of same power in the two sides of (11.3). Note that in this case both sides of (11.3) are polynomials of degree n.

Subcase 1.2. If $\alpha^2 + a\alpha + b = 0$ but $2\alpha + a \neq 0$, namely, α is a simple root of the characteristic equation, then (11.3) is reduced to

$$Q'' + (2\alpha + a)Q' = P_n. \tag{11.4}$$

We choose Q to be a polynomial of degree $n+1$. Since the constant term of Q does not appear in (11.4), we may choose $Q(x) = xR_n(x)$, where $R_n(x)$ is a polynomial of degree n.

$$y = xR_n(x)e^{\alpha x}.$$

Subcase 1.3 If $\alpha^2 + a\alpha + b = 0$ and $2\alpha + a = 0$, namely, α is a root of the characteristic equation with multiplicity 2, then (11.3) is reduced to

$$Q'' = P_n. \tag{11.5}$$

We choose $Q(x) = x^2R_n(x)$, where $R_n(x)$ is a polynomial of degree n.

$$y = x^2R_n(x)e^{\alpha x}.$$

Example 11.5. Find the general solution of $y'' - y' - 2y = 4x^2$.

Solution. The homogeneous equation has $\lambda^2 - \lambda - 2 = 0$ as its characteristic equation with roots $\lambda = 2, -1$.

Therefore the general solution of the associated homogeneous equation is $y = c_1 e^{2x} + c_2 e^{-x}$.

Note that $4x^2 = 4x^2 e^{0x}$ and 0 is not a root of the characteristic equation. We can try a particular solution of the form

$$y_p = A + Bx + Cx^2.$$

Substituting this into the equation, we have

$$2C - (B + 2Cx) - 2(A + Bx + Cx^2) = 4x^2.$$

Equating coefficients, we have

$$\begin{array}{rcl} 2C - B - 2A & = & 0 \\ -2C - 2B & = & 0 \\ -2C & = & 4 \end{array}$$

Thus $A = -3, B = 2, C = -2$, and $y = -3 + 2x - 2x^2$.

The general solution is

$$y = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2.$$

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Example 11.6. Solve $y'' - 2y' + y = xe^x$.

Solution. The general solution of the associated homogeneous DE is $C_1 e^x + C_2 xe^x$.

Here $\alpha = 1$ is a double root of the characteristic equation $\lambda^2 - 2\lambda + 1 = 0$. Therefore, we try a particular solution of the form $y = x^2(A + Bx)e^x$.

We have $y' = (Bx^3 + (A + 3B)x^2 + 2Ax)e^x$ and $y'' = (Bx^3 + (A + 6B)x^2 + (4A + 6B)x + 2A)e^x$.

Substituting these into the DE, we have $(2A + 6Bx)e^x = xe^x$. Thus $A = 0$ and $B = \frac{1}{6}$.

Consequently, the general solution is $y = C_1 e^x + C_2 xe^x + \frac{1}{6}x^3 e^x$.

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Case 2. $f(x) = P_n(x)e^{\alpha x}\cos(\beta x)$ or $f(x) = P_n(x)e^{\alpha x}\sin(\beta x)$, where $P_n(x)$ is a polynomial of degree $n \geq 0$.

We first look for a solution of

$$y'' + ay' + by = P_n(x)e^{(\alpha + i\beta)x}. \tag{11.6}$$

Using the method in Case 1 we obtain a complex-valued solution

$$z(x) = u(x) + iv(x),$$

where $u(x) = \mathfrak{Re}(z(x))$, $v(x) = \mathfrak{Im}(z(x))$. Substituting $z(x) = u(x) + iv(x)$ into (11.6) and taking the real and imaginary parts, we can show that $u(x) = \mathfrak{Re}(z(x))$ is a solution of

$$y'' + ay' + by = P_n(x)e^{\alpha x}\cos(\beta x), \tag{11.7}$$

and $v(x) = \mathfrak{Im}(z(x))$ is a solution of

$$y'' + ay' + by = P_n(x)e^{\alpha x}\sin(\beta x). \tag{11.8}$$

Example 11.7. Solve $y'' - 2y' + 2y = e^x \cos x$.

Solution. The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$ with roots $1 + i$ and $1 - i$.

The general solution of the associated homogeneous DE is

$$y = c_1 e^x \cos x + c_2 e^x \sin x.$$

Now consider the DE $y'' - 2y' + 2y = e^{(1+i)x}$. Let's find a particular solution.

Since $(1+i)$ is a root of the characteristic equation, we should try a particular solution of the form $y = Axe^{(1+i)x}$.

Thus $y' = (A + A(1+i)x)e^{(1+i)x}, y'' = (2A(1+i) + A(1+i)^2 x)e^{(1+i)x}$.

Therefore,

$$\begin{aligned} &y'' - 2y' + 2 \\ &= \left[(2A(1+i) + A(1+i)^2 x) - 2(A + A(1+i)x) + 2Ax\right]e^{(1+i)x} \\ &= 2Aie^{(1+i)x}. \end{aligned}$$

From this, $1 = 2Ai$ or $A = -\frac{i}{2}$.

Thus a particular solution is given by $y = -\frac{i}{2}xe^{(1+i)x}$, or equivalently, $y = \frac{1}{2}xe^x \sin x - \frac{i}{2}xe^x \cos x$.

Taking the real part, $y_p = \frac{1}{2}xe^x \sin x$ is a particular solution of the given DE.

Consequently, the general solution is

$$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{2}xe^x \sin x.$$

■

Remark. Alternatively to solve (11.7) or (11.8), one can try a solution of the form

$$Q_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)$$

if $\alpha + i\beta$ is not a root of $\lambda^2 + a\lambda + b = 0$, and

$$xQ_n(x)e^{\alpha x}\cos(\beta x) + xR_n(x)e^{\alpha x}\sin(\beta x)$$

if $\alpha + i\beta$ is a root of $\lambda^2 + a\lambda + b = 0$, where Q_n and R_n are polynomials of degree n.

11.4 Appendix: Malthus Model of Population

Remark: Section 11.4 will be excluded from the Final Exam.

The total population $N(t)$ of a country or a colony is clearly a function of time. $N(t)$ *though should be integer valued and is great than* 0, *is considered as a continuous and in fact differentiable function of time, especially its value is usually very huge.*

Given the population now, can one predict the future population?

Suppose B is a function giving the *"per capita birth rate"* in a given society, i.e. B is the number of babies born per second, divided by the total population N of the country at that moment. Note that B could be small in a big country and large in a small country - it depends on whether there is a strong social pressure on people to get married and have kids. Now B could depend on time (people might gradually come to realise that large families are no fun, etc..) and it could depend on N. But suppose you don't believe these things. Instead suppose people will always have as many kids as they can, no matter what. Then B is a constant. Thus

$$\textit{number of babies born in time interval } dt = BN\,dt.$$

Similarly, let D be the death rate per capita. Again it could be a function of t (in case the society has better medicine, fewer smokers etc) or N (overcrowding leads to famine or disease). But if we assume that it is constant, then

$$\textit{number of deaths in time interval } dt = DN\,dt.$$

So the change in N, denoted by dN within the time interval dt is

$$dN = \textit{number of births} - \textit{number of deaths},$$

provided there is no emigration or immigration. Thus

$$dN = (B-D)N\,dt.$$

That is

$$\frac{dN}{dt} = (B-D)N = kN, \tag{1}$$

where $k = B - D$.

This model of society was put forward by Thomas Malthus in 1798. Clearly Malthus was assuming a socially static society in which human reproductive behaviour never changes with time or overcrowding, poverty etc.. What does Malthus' model predict?

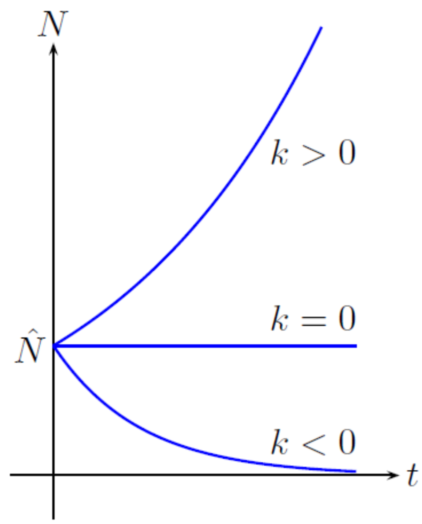
Suppose that the population now is $\hat{N}$ and let $t = 0$ now.

From (1), $\frac{dN}{dt} = kN \Rightarrow \int \frac{dN}{N} = k\int dt = kt + C \Rightarrow \ln(N) = kt + C \Rightarrow N(t) = Ae^{kt}$.

Since $\hat{N} = N(0) = A$, we get

$$N(t) = \hat{N}e^{kt}. \tag{2}$$

(1) is the logistic equation and (2) is the solution of the standard Malthus's model.



Graphs of $N(t)$ for different values of k.

The population collapses if $k < 0$ (more deaths than births per capita), remain stable if (and only if) $k = 0$, and it explodes if $k > 0$ (more births than deaths). Malthus observed that the population of Europe was increasing, so he predicted a catastrophic population explosion; since the food supply could not be expanded so fast, this would be disastrous.

In fact, this didn't happen in Europe. So Malthus' model is not quite correct; as many millions went to the US, and many millions died in wars.

Malthus' model can be improved. Note that Malthus' model is interesting because it shows that static behaviour patterns can lead to disaster. But precisely because the term e^{kt} grows so quickly, Malthus' assumptions must eventually go wrong - obviously there is a limit to the possible population. Eventually, if we don't control B, then D will have to increase. So we have to assume that D is a function of N.

Clearly, D must be an increasing function of N, but which function? The simplest possible choice is

$$D = sN, \tag{3}$$

where s is a constant.

Now we want to solve

$$\frac{dN}{dt} = BN - sN^2, \ \ N(0) = \hat{N}.$$

Rewrite the equation as

$$\frac{dN}{dt} - BN = -sN^2.$$

This is a Bernoulli equation.

Let $z = N^{1-2} = \frac{1}{N}$. Then $\frac{dz}{dt} = -\frac{1}{N^2}\frac{dN}{dt}$. Thus $-\frac{N^2 dz}{dt} - BN = -sN^2$. That is $\frac{dz}{dt} + Bz = s$ which is a linear equation in z. An integrating factor is e^{Bt}. Thus

$\frac{dz}{dt} + Bz = s \Leftrightarrow \frac{d(ze^{Bt})}{dt} = se^{Bt} \Leftrightarrow ze^{Bt} = \frac{s}{B}e^{Bt} + C \Leftrightarrow z = \frac{s}{B} + Ce^{-Bt}$. That is $\frac{1}{N} = \frac{s}{B} + Ce^{-Bt}$.

Let $N_\infty = \frac{B}{s}$ which is the *carrying capacity*. Then $\frac{1}{N} = \frac{1}{N_\infty} + Ce^{-Bt}$.

Now $N(0) = \hat{N} \Rightarrow \frac{1}{\hat{N}} = \frac{1}{N_\infty} + C \Rightarrow C = \frac{1}{\hat{N}} - \frac{1}{N_\infty}$.

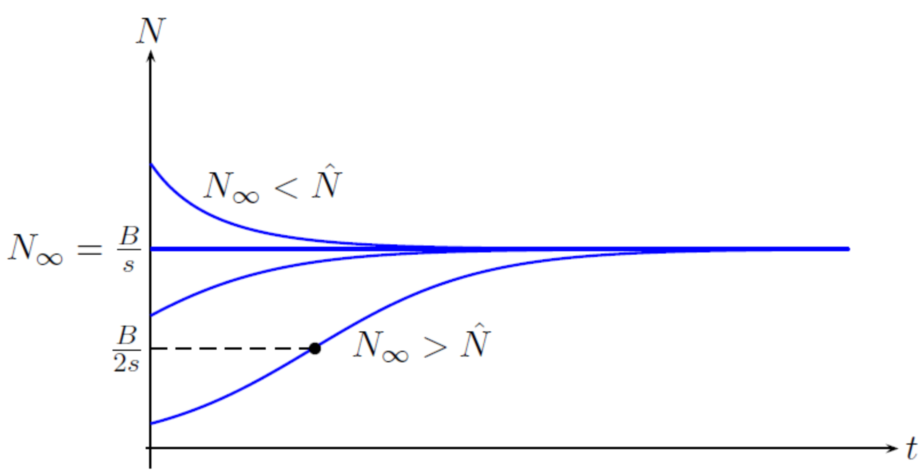
Thus

$$\frac{1}{N} = \frac{1}{N_\infty} + \left(\frac{1}{\hat{N}} - \frac{1}{N_\infty}\right)e^{-Bt}.$$

Rearranging,

$$N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right)e^{-Bt}}. \tag{4}$$

Note that $\lim\limits_{t\to\infty} N = N_\infty$, as $B > 0$. Also $N(t)$ is increasing if $N_\infty > \hat{N}$, and $N(t)$ is decreasing if $N_\infty < \hat{N}$. Thus $\frac{dN}{dt} \neq 0$ if $N_\infty \neq \hat{N}$. (4) is the solution of the improved Malthus' model.



$N_\infty = \frac{B}{s}$ is the carrying capacity. Point of inflection at $N = \frac{B}{2s}$.

Example 11.8. *The growth of rabbits in your rabbit farm followed a logistic population model with a birth rate per capita of 10 rabbits per rabbit per year. You observed that their number had approached to a logistic equilibrium population of 2500 rabbits. One day your friend Dr. Good visited your farm and suggested that you try to mix some of his latest invention of Vitamin X into your rabbit feed to boost the reproduction rate. You followed his suggestion and after a long period of time, observed that the rabbit population had reached a new logistic equilibrium of 3000 rabbits. If the new rabbit birth rate per capita after Vitamin X was introduced was B rabbits per rabbit per year, what is the value of B?*

Solution. We have $\frac{10}{s} = 2500$ so that $s = \frac{1}{250}$. Thus $\frac{B}{s} = 3000 \Rightarrow B = \frac{3000}{250} = 12$.

$\blacksquare$

Exercise 11.2. *Suppose $N_\infty > 2\hat{N}$. Show that there is a point of inflection on the graph of N at $t > 0$.*