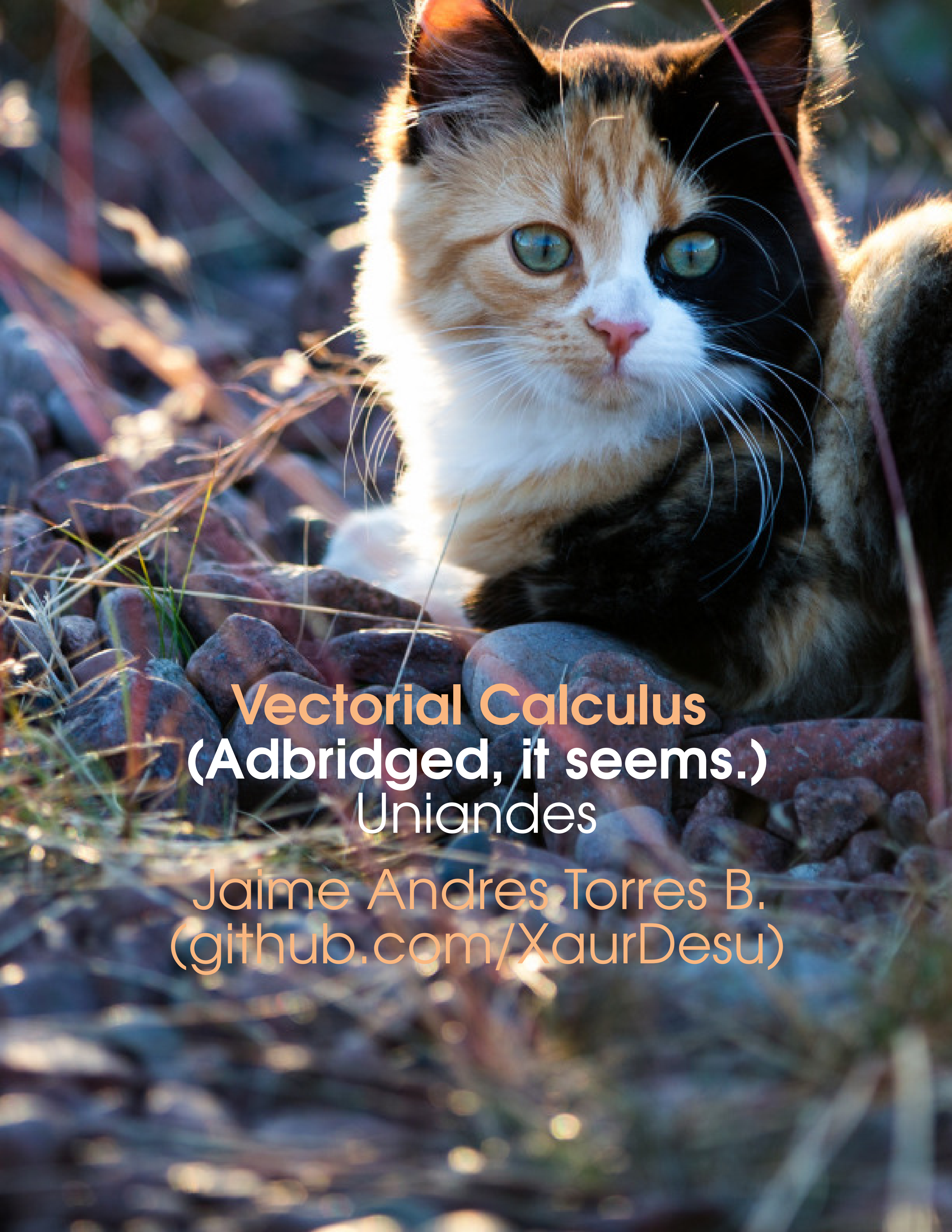

# Vectorial Calculus

**(Adbridged, it seems.)**

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**Vectorial Calculus -**

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# Contents

# 1. Introduction

*Go big or go home.*

Vectorial calculus is what the title says pretty much, the act of using methods proper to calculus on vectorial spaces, for the topic of this class generally referring to merely 3-dimensional ones, at the end of this book you should be able to:

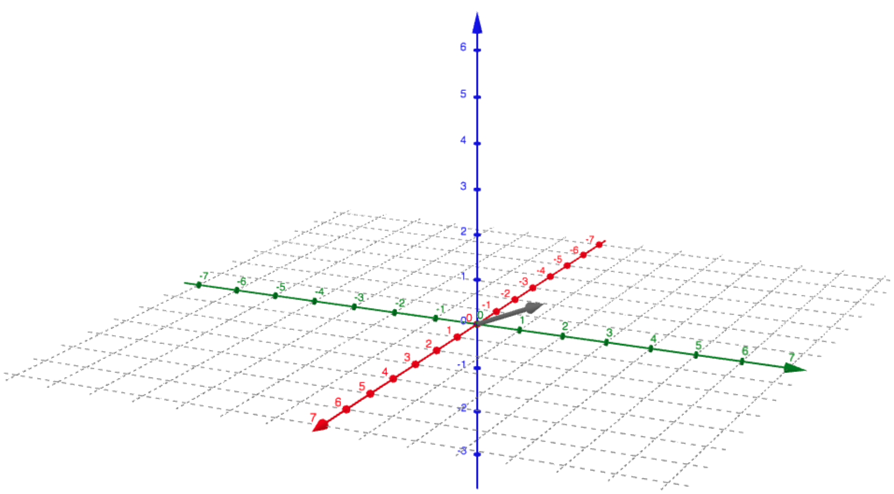
•

# 2. Linear Algebra Concepts

## 2.1 Vectors on a three-dimensional space.

given an $\mathbb{R}^3$ space and a point in that space $P = (a, b, c)$, we can describe a vector by either connecting the point $P$ to another point $Q$, or by assuming the origin of this space (point $(0,0,0)$), this is a mathematical object with both a direction and a magnitude. The direction is given by an angle and the magnitude is given by $\sqrt{a_1^2 + a_2^2 + a_3^2}$

As an example, let's assume the vector given by $P = (1, 2, 1)$:



*Vector formed by $P = (1, 2, 1)$*

For this vector, we can calculate the magnitude by replacing the vectorial components by the magnitudes of the individual directions, resulting in:

*Note, in this course we will be mostly only concerned with $\mathbb{R}^3$*

### 2.1.1 Addition and Subtraction

We can take any $\vec{a}$ and $\vec{b}$ vectors on the same space and add them to each other in the form:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} \tag{2.1}$$

Such form remains in the case we can do subtraction, which is expressed on the equation:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{pmatrix} \tag{2.2}$$

This kind of operations have certain properties, shown as:

$$(\alpha+\beta)\vec{[}v] = \alpha\vec{v}+\beta\vec{v} \tag{2.3}$$

$$\vec{v}*1 = \vec{v} \tag{2.4}$$

$$\vec{v}*\vec{0} = \vec{0} \tag{2.5}$$

$$\beta\vec{v} = \begin{pmatrix} \beta a_1 \\ \beta a_2 \\ \beta a_3 \end{pmatrix} \tag{2.6}$$

Two vectors $\vec{a}$ and $\vec{b}$ are equal if and only if:

$$\begin{cases} \vec{a}\exists\mathbb{R}^3 \\ \vec{b}\exists\mathbb{R}^3 \end{cases} \implies \begin{pmatrix} a_1=b_1 \\ a_2=b_2 \\ a_3=b_3 \end{pmatrix} \textit{ note: this can be generalized to 'n' dimensions larger than 0} \tag{2.7}$$

in either case, $\vec{0}$ is the identity of the operation, therefore:

$$\vec{a}+\vec{0} = \vec{a} \tag{2.8}$$

### 2.1.2 Bases

A base in $R^n$ can be found though $n$ vectors on that plane, such as it would happen in $R^2$ with:

$$\lambda\vec{u}+\mu\vec{v}|\lambda,\mu\exists\mathbb{R} \tag{2.9}$$

this equation will form a parallelogram that can express the distorsion of space when compared to a reference system, which generally is the canonical base formed by the identity.

### 2.1.3 Dot product

Assume two equal-length vectors of the sort:

$$\begin{cases} \vec{a} = (a_i*n|n\exists\mathbb{R}); |\vec{a}|\exists\mathbb{R} \\ \vec{b} = (b_i*n|n\exists\mathbb{R}); |\vec{b}|\exists\mathbb{R} \end{cases} \tag{2.10}$$

in case we wanted to do obtain a scalar number, that corresponded to the sum of the internal products we could obtain:

$$A \cdot B = ||\vec{A}||||\vec{B}||\cos\theta \tag{2.11}$$

for cartesian vectors, we can write this as:

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \exists \mathbb{R} \tag{2.12}$$

where $\theta$ is the angle between both vectors. We can get it by calculating

$$\cos\theta = \frac{\vec{u}x\vec{v}}{||\vec{u}|| * ||\vec{u}||}$$

If perpendicular, we can assume:

$$\vec{u} \cdot \vec{v} = 0$$

#### Notable cases.

With these rules, we can infere a few interesting cases, which we'll be able to interpolate stuff with.

#### Implications

- $\theta < \frac{\pi}{2} \implies \cos\theta > 0$
- $\theta > \frac{\pi}{2} \implies \cos\theta < 0$
- $\theta = \frac{\pi}{2} \implies \cos\theta = 0$

#### Addendum: cosine values

In vectorial calculus, we'll have certain notable angles that will appear often in exercises. we can use fractions to get them approximated to numerical values, such values are listed on this table:

| | | |
|---|---|---|
| $\cos 0^o$ | $\frac{4}{\sqrt{2}}$ | 1 |
| $\cos 30^o$ | $\frac{3}{\sqrt{2}}$ | ? |
| $\cos 45^o$ | $\frac{2}{\sqrt{2}}$ | ? |
| $\cos 60^o$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ |
| $\cos 90^o$ | $\frac{0}{\sqrt{2}}$ | 0 |

#### Addendum 2: Triangular inequality

The triangular inequality affirms that:

$||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$

### 2.1.4 Cross product

A cross product is, much like the dot product, an operation that seeks to multiply the values between two vectors. it can be annotated as:

$$\vec{u}\mathrm{x}\vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} * \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - v_1u_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \tag{2.13}$$

This is a non-commutative operation, changing the order of signs will cause the signs to invert, seen mathematically as:

$$\vec{u}x\vec{v} = -(\vec{v}x\vec{u})$$

The vector that the cross product produces is perpendicular to both evaluated vectors.

we can also use the norm of this cross product as a way to calculate the area of the paralleleipied form triangulated by $\vec{u}$ and $\vec{v}$ as

$$A = ||\vec{u}x\vec{v}||$$

This can also work for three-dimensional paralleleipied in the following formula:

$$|\vec{w}\,\mathbf{.}\,(\vec{u}x\vec{v})| = ||\vec{w}|| * ||\vec{u}x\vec{v}|| * |\cos\vartheta| \tag{2.14}$$

we can also say, from this:

$$\vec{w}\,\mathbf{.}\,(\vec{u}x\vec{v}) = \vec{u}\,\mathbf{.}\,(\vec{v}x\vec{w}) = \vec{v}\,\mathbf{.}\,(\vec{w}x\vec{u}) \tag{2.15}$$

### 2.1.5 Determinants

a determinant is defined as:

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.16}$$

## 2.2 Describing objects in a space.

### 2.2.1 Lines

A line is a geometrical object of the form:

$$r(t) = t\vec{v} + P, t\exists\mathbb{R} \tag{2.17}$$

generating it requires a point and a vector. Point defined by P, and vector defined by an offset 't' and a vector '$\vec{v}$'

**Example** *Find the equation of a line 'l' that crosses* $A = (2,1,1)$ *and* $B = (3,5,7)$

for this, we'll establish the following formula:

$$l(t) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} + t\begin{pmatrix} B_1 - A_1 \\ B_2 - A_2 \\ B_3 - A_3 \end{pmatrix} \tag{2.18}$$

Instanced, for this specific case, as:

$$l(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3-2 \\ 5-1 \\ 7-1 \end{pmatrix} \tag{2.19}$$

$$l(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} \tag{2.20}$$

Answer is equation 2.15, this can be later expanded into a parametric or simetric form of this line. But before we do that, let's try expanding the reason this works:

**Example 2** *Find the equation of the line that joins points* $P = (1,2,1)$ *and* $Q = (-1,3,4)$

we can find the line that joins two points by subtracting the vectors that join them, let's take a look at the cartesian plane where we indicate 'P' and 'Q':

### 2.2.2 Vector Projection

A vector can be projected through the equation:

$$\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v} \tag{2.21}$$

### 2.2.3 Euclidian Planes

A plane is the union of all points in a 2-dimensional subset of $\mathbb{R}^{\not\models}$ defined by a formula of the type:

$$i_1A + i_2B + i_3C = D = (D \cdot ||\vec{n}||) \tag{2.22}$$

Where $\vec{n}$ is also written as:

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.23}$$

It can be determined by

- three points in $\mathbb{R}^{\not\models}$
- Two vectors and a point in $\mathbb{R}^{\not\models}$
- a point and the normal vector in $\mathbb{R}^{\not\models}$

## 2.3 Cylindrical and spherical coordinates.

When trying to define parts of a line in algebra, we'll usually be looking at coordinates, be them polar or cartesian. In either case, their information can be converted to the other system through the following formulas.

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan(\frac{y}{x}) \end{cases} \quad \textit{cartesian to polar} \tag{2.24}$$

$$\begin{cases} \alpha_x = \rho \cos \theta \\ \alpha_y = \rho \sin \theta \end{cases} \quad \textit{polar to cartesian} \tag{2.25}$$

Cartesian coordinates generally translate well to other dimensional spaces, such as would be the case for $\mathbb{R}^3$, however, polar coordinates as we know them usually aren't as translatable in a direct manner, and expressing them in three-dimensional spaces might be better suited to be expressed on a cylindrical or spherical condition.

### 2.3.1 Cylindrical coordinates

In the case of cylindrical coordinates, the translation is probably the most intuitive, by computing a cylinder with polar coordinates that indicate an (x,y) position, and a 'Z' variable indicating height that allows us to project the vector on a third dimension, this 'z' variable is exactly the same as it would be on a cartesian model. We can express it like such:

$\vec{v} = (\rho, \theta, Z)$

conversion to a cartesian model can be expressed as:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\cos\theta \\ \alpha_y = \rho\sin\theta \quad \textit{Cylindrical to cartesian} \\ \alpha_z = Z \end{cases} \tag{2.26}$$

$$\vec{(}\alpha) = \begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan(\frac{y}{x}) \quad \textit{Cartesian to Cylindrical} \\ Z = Z \end{cases} \tag{2.27}$$

### 2.3.2 Spherical coordinates

A spherical coordinate is formed by a tuple:

$$(\rho, \theta, \phi); \begin{cases} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases} \tag{2.28}$$

Where $\rho$ is the magnitude of the vector, $\theta$ is the (x,y) coordinates, and $\phi$ is the (y,z) angle. They must adhere to the following for it to be geometrically coherent:

$$\begin{cases} \rho > 0 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases} \tag{2.29}$$

this tuple can generate two vectors:

$$\begin{cases} \rho\sin\phi \\ \rho \end{cases} \tag{2.30}$$

And can be converted to a cartesian model as such:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho \sin\phi \cos\theta \\ \alpha_y = \rho \sin\phi \sin\theta \\ \alpha_z = \rho \cos\phi \end{cases} \quad \textit{Spherical to cartesian} \tag{2.31}$$

$$\vec{(}\alpha) = \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan(\frac{y}{x}) \\ \phi = \arccos(\frac{z}{\rho}) \end{cases} \quad \textit{Cartesian to Spherical} \tag{2.32}$$

**Example**
Imagine the following spherical vector:

$$\begin{cases} \rho = 2 \\ \theta = \frac{\pi}{2} \\ \phi = \frac{\pi}{4} \end{cases}$$

*How do we convert it to a cartesian vector?*

We'll get the vector by simply replacing the previous formulas with the values provided as it follows:

$$x = 2\sin(\frac{\pi}{4})\cos(\frac{\pi}{2}) \qquad y = 2\sin(\frac{\pi}{4})\sin(\frac{\pi}{2}) \qquad z = 2\cos(\frac{\pi}{4}) \tag{2.33}$$

We can also invert this equation and get a cartesian vector to its spherical form through the following formula:

$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\frac{y}{x} \\ \phi = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \end{cases} \quad \textit{Cartesian to Spherical} \tag{2.34}$$

Such a model works, as we might imagine, like a sphere. where we express the possible vectors through a sphere of $\rho$ radius.

## 2.4 n-dimensional Euclidian Spaces

in an n-dimensional euclidian space, we can determine:

$$\mathbb{R}^n, \vec{x}\left(x_1 \ldots x_n\right); \mathbb{C} \textit{ Operations: } \begin{cases} \vec{x}+\vec{y} \\ \alpha\vec{x} \\ \vec{x}\,\boldsymbol{.}\,\vec{y} \end{cases} \tag{2.35}$$

### 2.4.1 Cauchy-Schwartz Inequality

This inequality determines that the internal product is lesser or equal to the multiplication of the norms of two vectors, written as:

$$\text{Let: } \vec{x},\vec{y} \exists \mathbb{R}^n \text{ then:} \tag{2.36}$$
$$|\vec{x}\,.\,\vec{y}| \leq ||\vec{x}||\,.\,||\vec{y}|| \tag{2.37}$$

and it is equal if and only if:

$$\vec{x} = \lambda\vec{y} \text{ or either } \begin{cases} \vec{x} = 0 \\ \vec{y} = 0 \end{cases} \tag{2.38}$$

## 2.5 Matrices

A matrix is a numerical representation of values in $\mathbb{R}^n$. They can represent planes, vectors, or even hyperplanes in $\mathbb{R}^n|n > 3$. An example in $\mathbb{R}^2$ would be:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.39}$$

Notable matrices include:

- Identity: $\begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix}$ = Id

### 2.5.1 Inverible Matrices

We can invert a matrix if a $B_{nxn}$ matrix exists such as:

$$AB = BA = Id \tag{2.40}$$

we can also use the determinant to check this, as:

$$det(A) \begin{cases} = 0\text{: is Invertible} \\ \neq 0\text{: is not Invertible} \end{cases} \tag{2.41}$$

### 2.5.2 Matrix multiplication

We can multiplicate a matrix by another one if we define the multiplication as:

$$AxB = C_{ij} = \sum_{k=1}^{n} a_{ik} b_{jk} \tag{2.42}$$

**Example**

We want to multiply two matrices as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{2.43}$$

Therefore if we try doing AxB and BxA:

$$AxB = \begin{pmatrix} a & 2a+b \\ c & 2c+d \end{pmatrix} \tag{2.44}$$

$$BxA = \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix} \tag{2.45}$$

$$\tag{2.46}$$

As we can see, matrix multiplication is not commutative, but rather it is defined by the order on which A and B are written

$$Ae_j = A_j$$

# 3. Functions and Equations

In mathematics, even though similar, functions and equations

## 3.1 Geometry of functions with values in $\mathbb{R}$

We can affirm that a function is two or three dimensional if, respectively:

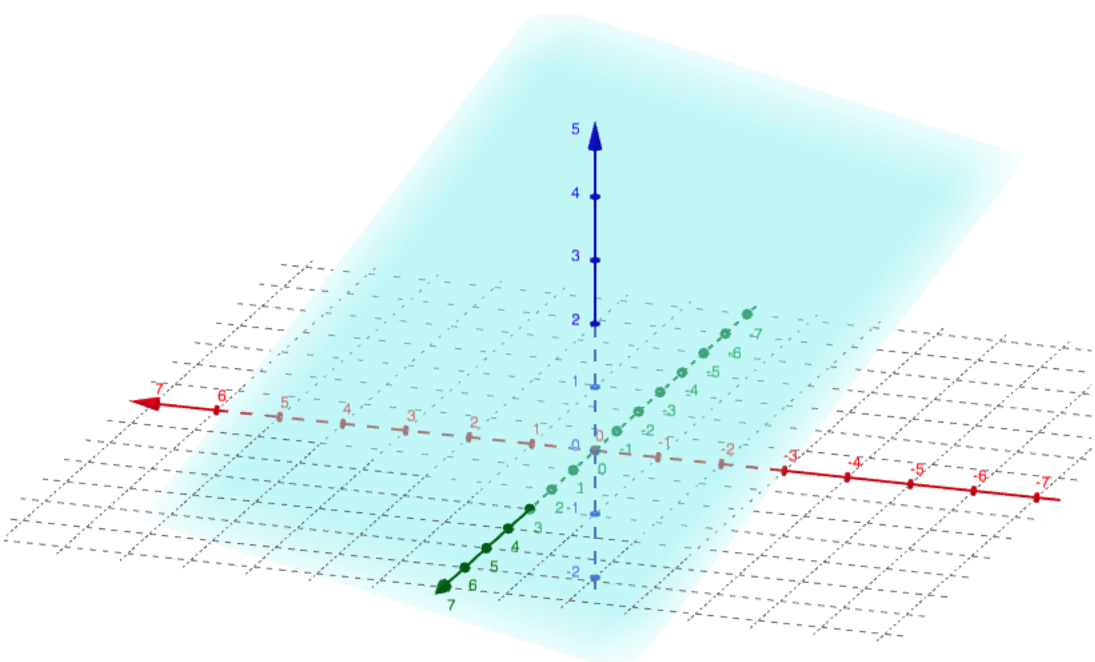
$$y = f(x) \implies \Big\{(x,y) : y = f(x), x \exists D(f) \tag{3.1}$$

$$z = f(x,y) \implies \Big\{(x,y,z) : z = f(x,y), (x,y) \exists D(f) \tag{3.2}$$

**Example:**

*3,1,1: Graphicate the following:*

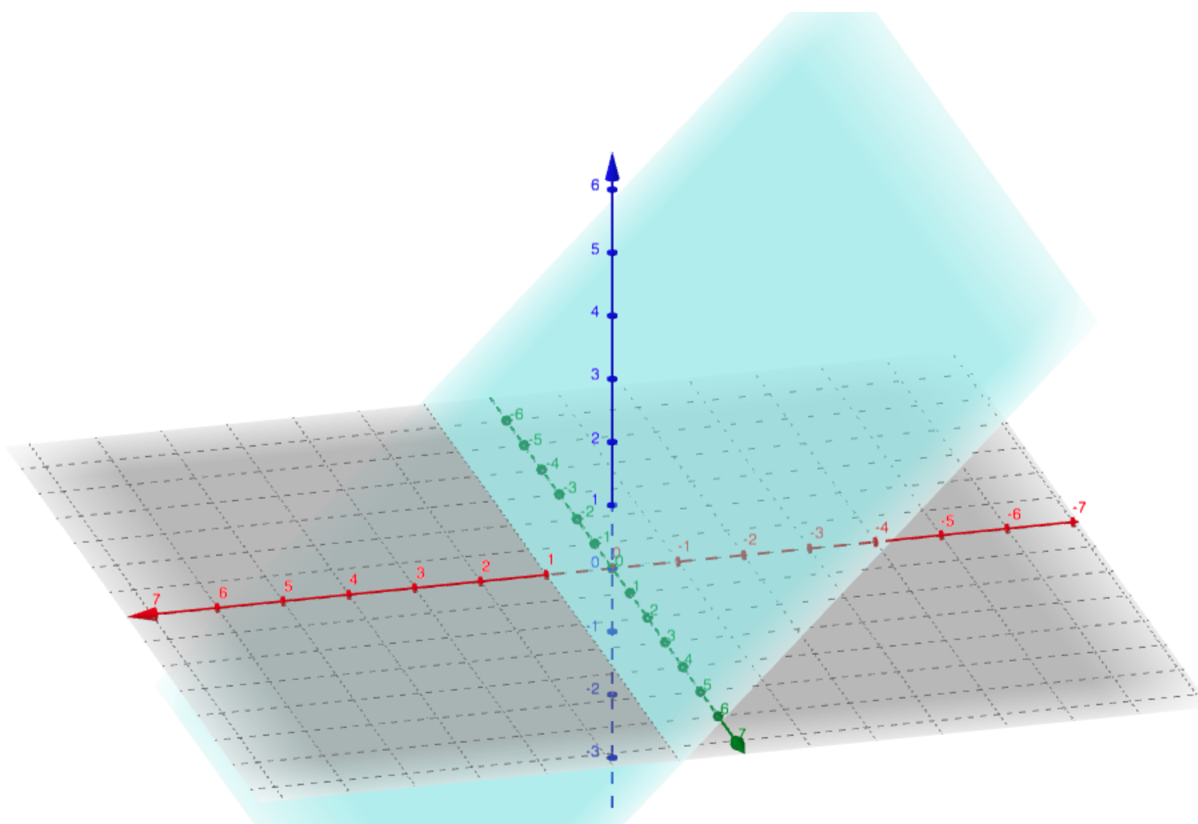
$$z = \frac{6 - x - 2y}{3}$$



As we can see, the graph makes sense because:

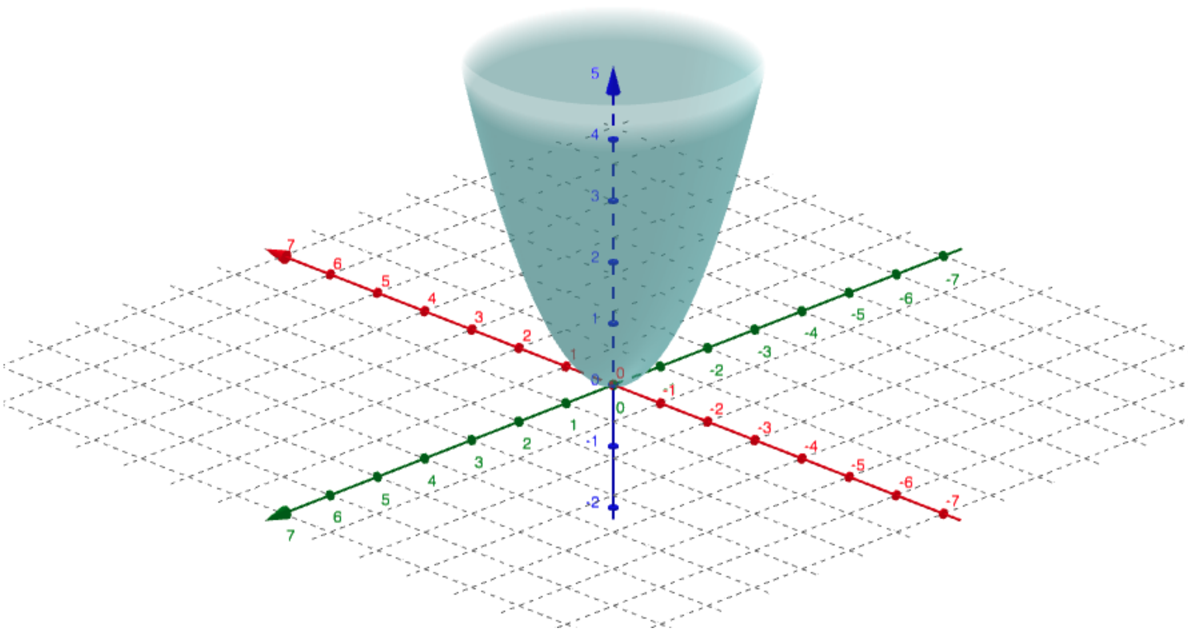
*3,1,2: Graphicate the following:*

$$z = 1 - x = f(x, y)$$



*3,1,3: Graphicate the following:*

$$z = x^2 + y^2$$




this is a three-dimensional parabola, also called more correctly as a circular paraboloid, comparable to it's two dimensional form, yet working on another dimension.

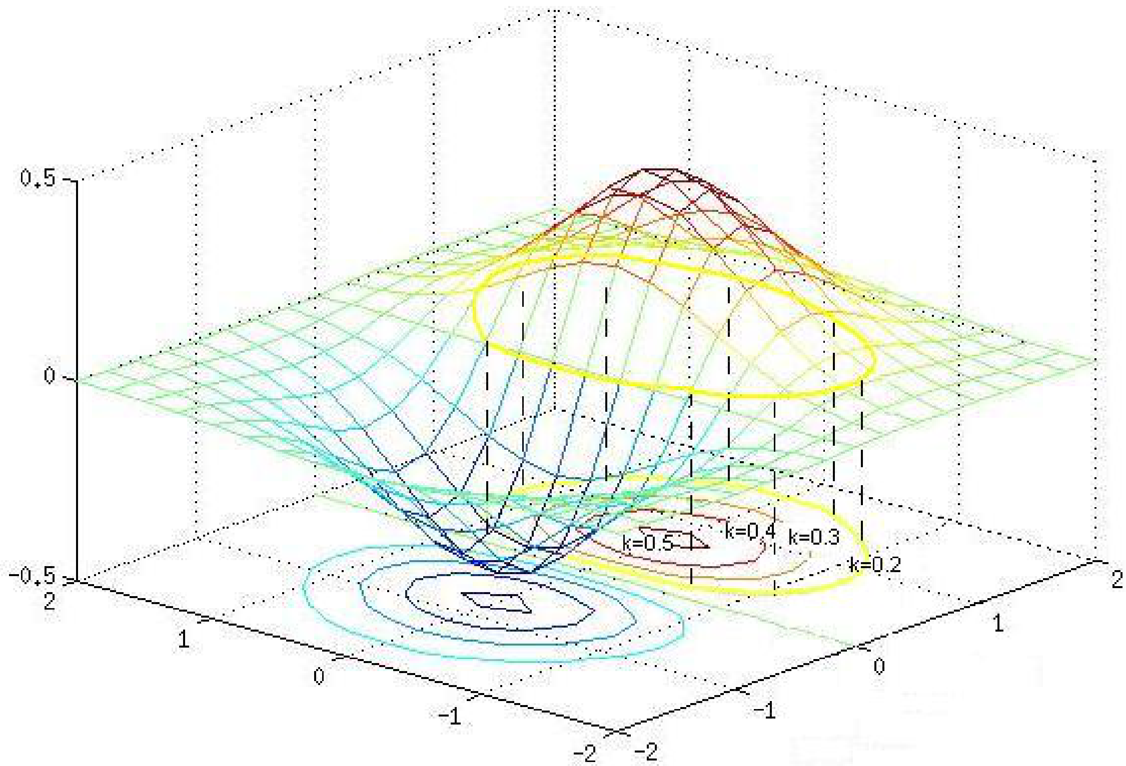
it can guarantee:

$$\begin{cases} x = 0 : z = y^2 \\ y = 0 : z = x^2 \end{cases}$$

and this equation can be deduced through this behavior. As:

$$z = z_0 : x^2 + y^2 = z$$

there are level curves, which project the curve generated by such mathematical artifacts as a two dimensional spherical object. Fields like topology or other 3-dimensional inclined math might regularly use it when measuring space. In any case, a level curve could be illustrated as such:




*taken from: www.math.tamu.edu*

## 3.2 Equations

An equation, although similar to a function such as the ones studied so far, can be distinguished by some particularities they present:

$$x^2+y^2+z^2=1$$

A few examples of objects described by equations are:

### 3.2.1 Planes

A plane is a 2-dimensional shape that can be expressed as:

$$Ax+By+Cz=Ax_0+By_0+Cz_0 \tag{3.3}$$

There are different ways of getting the values regarded in this shape, as seen in 2.2.3 of these class notes, and we will mostly do it by passing different geometrical expressions to a point and a vector normal to the plane, and replacing the vector on A,B,C, and the point in $x_0, y_0, z_0$

#### Example

Find a plane normal to: $\vec{n}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ That passes through: $P=\begin{pmatrix}1\\1\\1\end{pmatrix}$

We replace fot this as following:

$$1x+2y+3z = 1*1+2*1+3*1 \implies 1x+2y+3z = 6 \tag{3.4}$$

z = f(x,y) can also be a way to define a plane through a function, however, this is a bit limiting, as z is not free. However, tangent planes can be found in a fairly easy way

### 3.2.2 Elipsoids

An elipsoid is defined by the following equation:

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1 \tag{3.5}$$

### 3.2.3 Hyperboloid

a Hyperboloid can be seen as:

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = (\frac{z}{c})^2 + 1 \tag{3.6}$$

### 3.2.4 Cilinders

The general formula of a cylinder is:

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1 \tag{3.7}$$

The level curves of this shape will never change when projected on a plane.

### 3.2.5 Parabolic

The general formula of a parabolic is:

$$y = ax^2 \tag{3.8}$$

### 3.2.6 Spheres

$$\begin{cases} x^2+y^2+z^2 = r^2 \\ z \geq 0 \end{cases} \tag{3.9}$$

### 3.2.7 Paraboloid

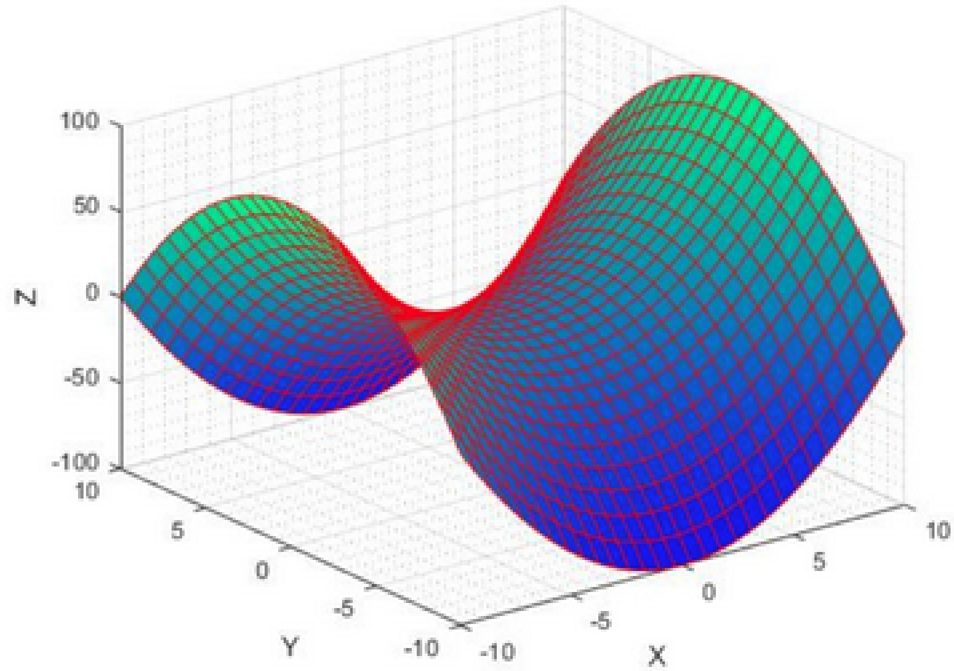
The general formula of a paraboloid is:

$$z = (\frac{x}{a})^2 + (\frac{y}{b})^2 \tag{3.10}$$

This shape has a variation, called a hyperbolic paraboloid:

$$z = (\frac{x}{a})^2 - (\frac{y}{b})^2 \tag{3.11}$$

It will look like this when graphicated:




## 3.3 Examples

**Calculate the tangent plane of $ze^z xy^2 = 0$ in the point $(\frac{\pi}{2}, 1, 0)$**

*Solution*

$$ze^z xy^2 = 0 \tag{3.12}$$

$$f(x,y) = \cos x \sin y e^z \tag{3.13}$$

$$\begin{cases} \frac{\partial_F}{\partial_x} = -\sin x \cos y e^z \\ \frac{\partial_F}{\partial_y} = -\cos x \cos y e^z \\ \frac{\partial_F}{\partial_z} = \cos x \cos y e^z \end{cases} \tag{3.14}$$

$$\frac{\partial_F}{\partial_x} = -\sin\frac{\pi}{2}\cos(1)e^0 = -\cos(1) \tag{3.15}$$

$$\frac{\partial_F}{\partial_y} = -\cos\frac{\pi}{2}\sin 1 e^0 = 0 \tag{3.16}$$

$$\frac{\partial_F}{\partial_z} = 0 \tag{3.17}$$

$$-\cos(1)(x - \frac{\pi}{2}) + 0(y-1) + 0(z-0) = 0 \tag{3.18}$$

$$-\cos(1)(x - \frac{\pi}{2}) = 0 \tag{3.19}$$

$$x - \frac{\pi}{2} = 0 \tag{3.20}$$

$$x = \frac{\pi}{2} \tag{3.21}$$

# 4. Limits and continuity

### Sets and disks

An open disk of radio 'r' and contourn $\vec{x}$ can be defined as

$$\text{Let: } \vec{x} \exists \mathbb{R}^n, r > 0 \qquad D_r = \{\vec{x} \exists \mathbb{R}^n : ||\vec{x} - \vec{x_0}|| < r\} \tag{4.1}$$

and a disk that includes the internal values could be defined as

$$\text{Let: } \vec{x} \exists \mathbb{R}^n, r > 0 \qquad D_r = \{\vec{x} \exists \mathbb{R}^n : ||\vec{x} - \vec{x_0}|| \leq r\} \tag{4.2}$$

Both of these are an important definition of what we'll call a **set**.
let

$$\mu \subseteq \mathbb{R}^n$$

we can say that $\mu$ is an open set if for any point $x_0 \exists \mu$ there exists a r>0 such as

$$D_r(x_0) \subseteq \mu$$

Such sets have a mathematical structure called frontier points that can be defined as:

### Frontier point

Let $A \subseteq \mathbb{R}^n$; a point '$x \exists \mathbb{R}^n$' is a frontier point if any $\vec{x}$ vecinity contains a point of A and at least a point outside of A

### Example

Prove that '$A = (x, y) \exists \mathbb{R}^2 : y > 0$' is an open set

Given that the open set can be inferred by a disk, we can affirm:

$$\implies y > 0 \implies r = \frac{y}{2} > 0 \tag{4.3}$$

$$\text{If : } (a,b)\exists D_r(x,y) \tag{4.4}$$

$$|b-y| \leq \sqrt{(a-x)^2+(b-y)^2} < r = \frac{y}{2} \tag{4.5}$$

$$|b-y| < \frac{y}{2} \tag{4.6}$$

$$\tag{4.7}$$

## 4.1 Limits

A limit can be defined as

$\lim_{x \to a} f(x) = L$

Where:

$\forall \varepsilon > 0 \exists \rho > 0 : |f(x) - L| < \varepsilon$ for all X, such as

$0 < |x-a| < \vartheta$

Or when said in words; "for every epsilon greater than zero that exists in a theta greater than zero, the function of value x minus L is lesser than epsilon for all X, such as the norm is greater than zero and smaller than theta".

For a limit to exist, we need three conditions to be met:

- there is a left limit
- there is a right limit
- they're equal

#### Multivariable limits

a limit in a multi variable plane can be defined as:

$$\lim_{(x,y)\to(a,b)} = \vec{L} \tag{4.8}$$

Meaning that we can force $|f(x,y) - L|$ to be as near to zero as posible, making (x,y) and (a,b) as close as possible without them actually ever touching each other.

Or, more formally;

$$f : \mathbb{R}^n \to \mathbb{R}^m \tag{4.9}$$

$$\text{Let: } f : A \subseteq \mathbb{R}^n \to \mathbb{R}^n \text{and :} \tag{4.10}$$

$$\text{let } \vec{x}_0 \exists A \text{ or } \vec{x}_0 \exists \Theta A \tag{4.11}$$

$$\lim_{x \to x_0} f(\vec{x}) = L \exists \vartheta \exists A \implies 0 < ||\vec{x} - \vec{x}_0|| < \vartheta, \implies ||f(\vec{x}) - L|| < \varepsilon \tag{4.12}$$

### 4.1.1 Differentiation

a function in a single variable can have every single point on a curve being approximated to a line, called a tangent. This z = (x,y) line variable can be defined by partial derivatives, that can be described as:

$$z_x = \frac{D_z}{D_x} = \frac{D_f}{D_x}(x,y) = \lim_{h\to 0} = \frac{(x+h,y)-f(x,y)}{h} \quad (4.13)$$

$$\frac{d_f}{d_y}(x,y) = \lim_{h\to 0} = \frac{(x,y+h)-f(x,y)}{h} \quad (4.14)$$

Where 'h' is a real number that tends towards zero.

**Example:**

*let:*

$$f(x,y) = x^2 - y^3 - 3x^4 y \quad (4.15)$$

*then, differentiate the equation.*

**solution:**

$$\frac{d_f}{d_x} = 2xy^3 - 12x^3 y \quad (4.16)$$

$$\frac{d_f}{d_y} = 3x^2y^2 - 3x^4 \quad (4.17)$$

When differentiating an equation on a specific variable, we take the other variables as constants. In this example, we leave 'x' untouched when differentiating on 'y' and viceversa.

### 4.1.2 Implicit differentiation

We can think of an implicit derivative as the process of using partial derivatives to differentiate a single, more complex equation.

**Examples:**

- a line that goes through $\frac{y-y_0}{x-x_0}$

For three dimensions we might imagine instead of a line being the tangent, a plane providing the same definition and mathematical role as it, defined as:

$$z = f(x,y_0) : z - z_0 = \frac{d_f}{d_x}(x,y_0)(x-x_0) \quad (4.18)$$

$$z = f(x_0,y) : z - z_0 = \frac{d_f}{d_x}(x_0,y)(y-y_0) \quad (4.19)$$

and with both this derivatives, we can define a plane as

$$z - z_0 = a(x-x_0) + b(y-y_0) \textit{Where: } \begin{cases} a = \frac{d_f}{d_y} f(x_0,y_0) \\ b = \frac{d_f}{d_x} f(x_0,y_0) \end{cases} \quad (4.20)$$

IF: $x = x_0 = 0$ and $y = y_0$

We can approximate a three-dimensional differentiation as

$$d_z = fx(x_0, y_0)(x - x_0) + f_y fx(x_0, y_0)(y - y_0) \tag{4.21}$$
$$f(x,y) \approx f(x_0, y_0) + d_z \tag{4.22}$$

*note: tangent planes are a very popular test/quiz problem, you should learn how to find them for such ventures, if your professor ever mentions them in class and ESPECIALLY if they try to solve one in front of the class.*

We can say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable if:

$$\Delta f = d_f + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{4.23}$$

This deffinition is a bit hard to applicate, so we'll use the differentiation criteria for looking for appliability.

*Appliability Criteria:* We will say it is possible to differentiate a function if every partial derivative in $\frac{d_f}{d_{xj}}$ and if they're continuous in an open set 'D'.

**Example:**

$$f(x,y) = \begin{cases} 0 \implies (x,y) = (0,0) \\ x^2 + y^2 \implies (x,y) \neq (0,0) \end{cases} \tag{4.24}$$
$$\frac{df}{dx}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0 \tag{4.25}$$
$$\text{BUT THEN, that means a partial derivative for (0,0) does not exist,} \tag{4.26}$$
$$\text{therefore this is not a continuous function.} \tag{4.27}$$
$$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} NE \tag{4.28}$$

### 4.1.3 Examples

- Solve:

$$\lim_{(x,y) \to (0,0)} \frac{\cos -1 - \frac{x^2}{2}}{x^4 + y^4} \tag{4.29}$$

This limit **DOES NOT EXIST**:
because going by paths we can assume:

$$< x = 0 >$$

Would imply:

$$\lim_{y \to 0} \frac{\cos 0 - 1 - \frac{0^2}{2}}{0^4 + y^4} \tag{4.30}$$
$$\lim_{y \to 0} \frac{0}{y^4} = \lim_{y \to 0} 0 = 0 \tag{4.31}$$

$$< y = 0 >$$

Would imply:

$$\lim_{x\to 0} \frac{\cos x - 1 - \frac{x^2}{2}}{x^4} \implies \frac{0}{0} \tag{4.32}$$

<l'Hopital because of indetermination> (4.33)

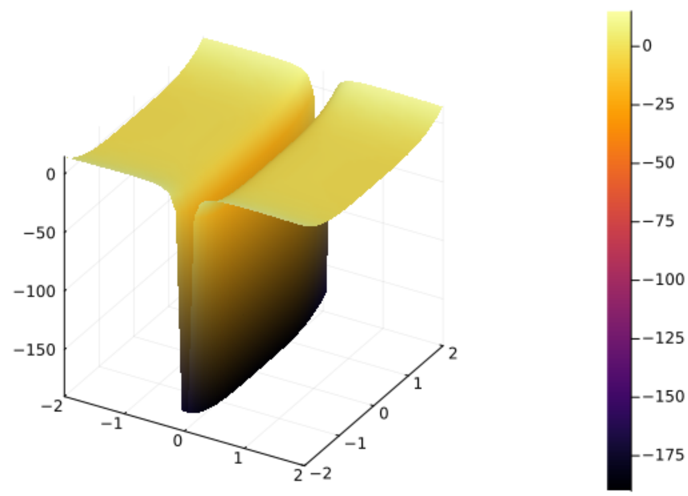
$$\lim_{x\to 0} \frac{-\sin x - x}{4x^3} \tag{4.34}$$

<l'Hopital because of indetermination... again.> (4.35)

$$\lim_{x\to 0} \frac{-\cos x - 1}{12x^2} \tag{4.36}$$

$$\ldots \tag{4.37}$$

...and it will never stop differentiating until a $\mathbb{R}$ value is divided by 0, which isn't possible without adding multiple things to our framework of reference, maybe imaginary numbers or something.
we can graphicate such a function as:




Would imply:

## 4.2 Continual Functions

A function can be defined as continual if a limit can be described as:

- $$\lim_{\vec{x}\to\vec{x}_0} f(x) = f(X_0) = f(\lim_{\vec{x}\to\vec{x}_0} \vec{x})$$

This would mean that a limit is replaceable by a function where the limit of $\vec{x}$ is described according to the mathematical rules that define a function. if a function is continual and has $\frac{D_{fi}}{D_{xj}}$ then it's differentiable

## 4.3 Tangent Planes

We tangentially touched this topic on implicit differentiation, however, to define more formally a tangent plane,

Let:

$$f : A \subseteq \mathbb{R}^2 \to R \tag{4.38}$$

$$\vec{\Delta} f(x,y) =< \underbrace{\frac{D_f}{D_x}(x,y)}_{\text{Partial Derivative of X}} , \underbrace{\frac{D_f}{D_y}(x,y)}_{\text{Partial Derivative of y}} > \tag{4.39}$$

$$f(x_0,y_0) + D(f(\vec{x}))(\vec{x} - \vec{x_0}) \tag{4.40}$$

## 4.4 Derivative Properties

### 4.4.1 Constant multiple rule

$$\frac{\delta}{\delta_x}(cf(x,y)) = c\frac{\delta}{\delta_x}(f(x,y))$$

### 4.4.2 Sum rule

$$\frac{\delta}{\delta_x}(f(x,y)+g(x,y)) = \frac{\delta_f}{\delta_x}f(x,y) + \frac{\delta_g}{\delta_x}g(x,y)$$

### 4.4.3 Product rule

$$\frac{\delta}{\delta_x}(f(x,y) \cdot g(x,y)) = \frac{\delta_f}{\delta_x}f(x,y)g(x,y) + \frac{\delta_g}{\delta_x}g(x,y)f(x,y)$$

### 4.4.4 Divisor rule

$$\frac{\delta}{\delta_x}(\frac{f}{g})(x,y) = \frac{\frac{\delta_f}{\delta_x}f(x,y)g(x,y) - \frac{\delta_g}{\delta_x}g(x,y)f(x,y)}{[g(x,y)]^2}$$

### 4.4.5 Chain Rule

#### One variable

If $\vec{u}(x)$ and $\vec{x}(t)$ are differentiable, then:

$$\frac{\delta_u}{\delta_t} = \frac{d_y}{d_x}\frac{d_x}{d_t} = u'(x|t|) + x'(t)$$

#### Two variables

Given $\vec{w}(x,y)$, $\vec{u}(x,y)$ and $\vec{v}(x,y)$, then $w(u(x,y),v(x,y))$ is differentable.

so, therefore:

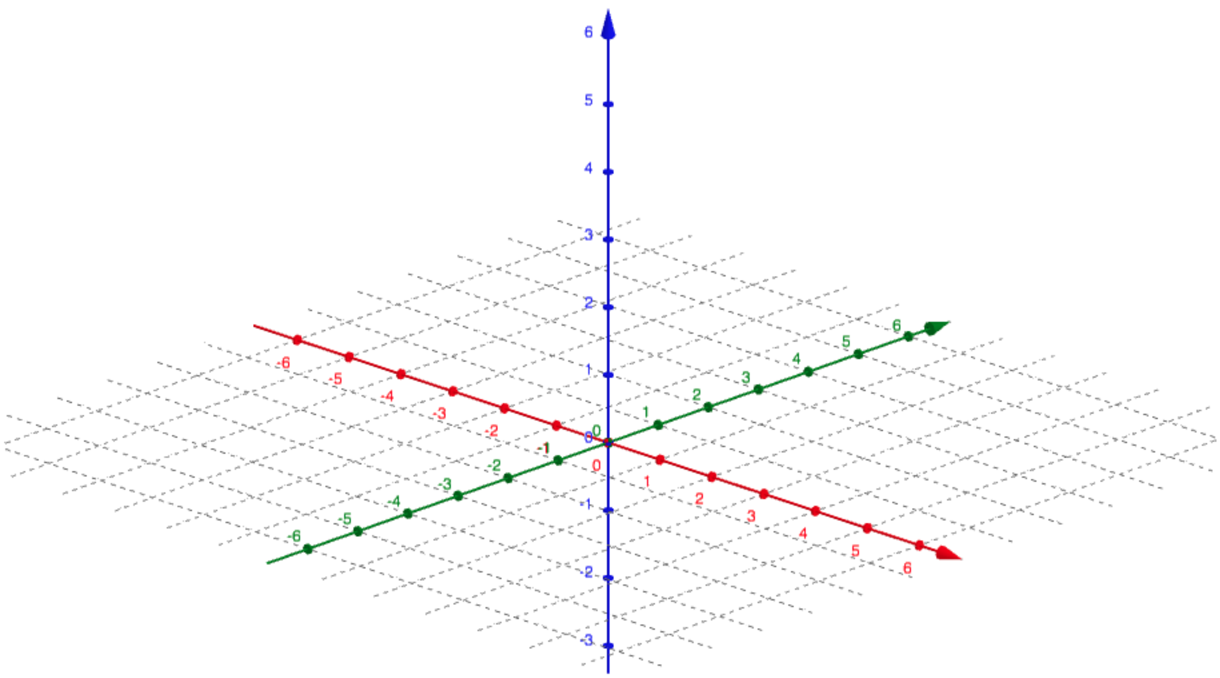
$$y = \begin{cases} \frac{\delta w}{\delta x} = \frac{\delta w}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta w}{\delta v}\frac{\delta v}{\delta x} \\ \frac{\delta w}{\delta y} = \frac{\delta w}{\delta u}\frac{\delta u}{\delta y} + \frac{\delta w}{\delta v}\frac{\delta v}{\delta y} \end{cases} \tag{4.41}$$

In general, we can define this rule as

$$D(fog) = Df(g(\vec{x_0}))Dg(\vec{x_0}) \tag{4.42}$$

## 4.5 Gradients and directional derivatives.

assume a 3D plane as such:




Def: a directional derivative is given by:

$$D_{\vec{u}}f(x_0,y_0) = \frac{d}{dt}f(x_0+tu_1,y_0+tu_2);t=0 \tag{4.43}$$

And we say that if a derivative exists, then $||\vec{u}|| = 1$; f depends on x and y, and x and y both depend on t, this can be seen mathematically as:

$$\frac{df}{dx}f(x_0+tu_1,y_0+tu_2)\underbrace{U_1}_{\frac{dx}{dt}}+\frac{df}{dy}(x_0+tu_1,y_0+tu_2)\underbrace{U_2}_{\frac{dy}{dt}} \tag{4.44}$$

from this we can observe

$$D_{\vec{u}}\ f(x_0,y_0) = \vec{\Delta}f(x_0,y_0)\,.\,\vec{u} = ||\vec{\Delta}f(x_0,y_0)||\,.\,||\vec{u}||\,.\cos\theta \tag{4.45}$$

So, given this set of conditions:

$$\begin{cases} D_{\vec{u}}f(x_0,y_0) \text{ is maximal if } \theta \text{ is } 0 \\ D_{\vec{u}}f(x_0,y_0) \text{ is minimal if } \theta \text{ is } \pi \end{cases} \tag{4.46}$$

We can infer then, regarding our direction:

$$\vec{u} = \frac{\vec{\Delta}f(x_0,y_0)}{||\vec{\Delta}f(x_0,y_0)||} \tag{4.47}$$

will make the function grow faster, and:

$$\vec{u} = -\frac{\vec{\Delta}f(x_0,y_0)}{||\vec{\Delta}f(x_0,y_0)||} \tag{4.48}$$

will make it decrease faster.

### 4.5.1 implicit equations

An implicit surface equation given by $F(x,y,z) = C$ we can define, as a curve:

$$\vec{r}t = < x(t), y(t), z(t) > \tag{4.49}$$

$$\vec{r}t' = < x'(t), y'(t), z'(t) > \tag{4.50}$$

And from there we can assume that this curve is a part of S if:

$$F(x(t), y(y), z(t)) = C; \forall t \tag{4.51}$$

Given this, F depends on x,y, and z and those three variables depend on t, allowing us to affirm:

$$F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} \tag{4.52}$$

$$\vec{\Delta}f \cdot \vec{r}t' = 0 \tag{4.53}$$

geometrically this means the gradient is perpendicular to the tangent and therefore, the tangent plane. to S in $(x_0,y_0,z_0)$, which can be expressed vectorially (and therefore, more usefully) as:

$$\vec{\Delta}F(x_0,y_0,z_0) \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} = 0 \tag{4.54}$$

The line normal to S in $(x_0,y_0,z_0)$ can be defined as:

$$\vec{n}(t) = \vec{x_0} + t\vec{\Delta}F(x_0,y_0,z_0) \tag{4.55}$$

## 4.6 Iterated Partial derivatives:

#### addendum

Remember:

- $f \exists C^o$ means a function is continuous
- $f \exists C^1$ means f' is continuous
- $f \exists C^k$ means the k derivative is continuous

### 4.6.1 Iterated derivatives

let:

$$f(x,y) = x^2y^3 - 3x^4y \tag{4.56}$$

then:

$$f_x(x,y) = 2xy^3 - 12x^3y \tag{4.57}$$
$$f_y(x,y) = 3x^2y^2 - 3x^4 \tag{4.58}$$
$$f_{xx} = 2y^3 - 36x^2y \tag{4.59}$$
$$f_{xy} = 6xy^2 - 12x^3 \tag{4.60}$$
$$f_{yx} = 6xy^2 - 12x^3 \tag{4.61}$$
$$f_(yy) = 6x^2y \tag{4.62}$$

As we can see, equations 4,51 and 4,52 are equivalent, this is because when we do such chained differentiations, we can guarantee:

$$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x} \tag{4.63}$$

### 4.6.2 Clairaut's Theorem

If $f(xy), f(yx)$ are continuous, then $f_{xy} = f_{yx}$

#### Proof

Okay, this might get a bit technical, but here we go:

As written by Kiril Datchev from Purdue[, ] (give his stuff a read, it's pretty cool), we can take as a given:

By definition:

- $$\partial_x \partial_y f(a,b) = \lim_{h \to 0} \frac{\partial_y f(a+h,b) - \partial_y f(a,b)}{h} \tag{4.64}$$

So from this, we can deduce:

(4.65)

(4.66)

## 4.7 Examples

**Find the gradient in P=(0,3) if** $f(x,y) = 2ye^{xy} + y\cos x$
We can use what we know of partial derivatives to affirm:

$$\vec{\Delta} f(x,y) =< 2y^2e^{xy} - y\sin x; 2e^{xy} + 2xe^{xy} + \cos x >; \tag{4.67}$$

<replace for P>; (4.68)

$$\vec{\Delta} f(0,3) =< 2*3^2e^{0*3} - 3\sin 0\ ;\ 2e^{0*3} + 2*0*e^{0*3} + \cos(0) >; \tag{4.69}$$

$$\vec{\Delta} f(0,3) =< 18,3 > \tag{4.70}$$

**Find the derivative in** $f(x,y) = \frac{x}{y}$ **in the point (6,-2) and the direction of** $\vec{v} =< 1,3 >$
This can be said as:

$$\vec{\Delta}(x,y) =< \frac{1}{y}, -\frac{x}{y^2} >; \tag{4.71}$$

$$\vec{\Delta}(6,-2) =< -\frac{1}{2}, -\frac{3}{2} > \tag{4.72}$$

Given this initial state, we can then use the following declarations on this problem:

$$||\vec{v}|| = \sqrt{10}; \tag{4.73}$$

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{||\vec{v}||}\vec{v} \tag{4.74}$$

And therefore, conclude:

$$D_{\vec{u}} f(6,-2) = \frac{1}{\sqrt{10}}(\frac{1}{2} - \frac{9}{2}) = \frac{4}{\sqrt{10}} = -\sqrt{\frac{8}{5}} \tag{4.75}$$

**Find the change rate in** $h(x,y,z) = \cos(xy) + e^{yz} + \ln(xz)$ **in the point (1, 0, 0.5) moving towards** $P_1 = (2,2,\frac{5}{2})$
Much as in our first example, we can use partial derivatives to get a delta for this problem. The change rate of a function **Is the same as a derivative, for a derivative is interested in how much a function changes**; we shall then affirm:

$$\vec{\Delta} h(x,y,z) =< -\delta\sin(x,y) + \frac{1}{x}, -x\sin y + ze^{yz}, ye^{yz} + \frac{1}{z} > \tag{4.76}$$

**Find the tangent plane and normal line of:**
$x^2 + y^2 = -4 + 2xy + x - 3y + z^2$ in $P = (1,2,\sqrt{10})$
We can reweite this as:

$$F(x,y,z) = -4 \tag{4.77}$$

$$\vec{\Delta} F(x,y,z) =< 2x - 2y - 1; 2y - 2x + 3, -2z > \tag{4.78}$$

$$\vec{\Delta} F(1,2,\sqrt{10}) =< -3,5,-2\sqrt{10} > \tag{4.79}$$

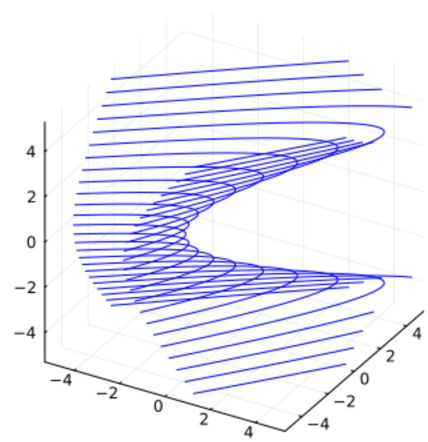
From here, we can define the plane as:

$$-3(x-1)+5(y-2)-2\sqrt{10}(z-\sqrt{10})=0 \tag{4.80}$$
$$-3x+5y-2\sqrt{10}z=-13 \tag{4.81}$$
$$3x-5y+2\sqrt{10}z=13 \tag{4.82}$$

And the line can be defined therefore as:

$$\vec{n}(t)=\begin{pmatrix}1\\2\\\sqrt{10}\end{pmatrix}+t\begin{pmatrix}-3\\5\\-2\sqrt{10}\end{pmatrix} \tag{4.83}$$

Just as a curiosity, we can write the curve as this:




## 4.8 important expressions

- Tangent Planes:

$$\frac{\delta F}{\delta x}(x_0,y_0,z_0)(x-x_0)+\frac{\delta F}{\delta y}(x_0,y_0,z_0)(y-y_0)+\frac{\delta F}{\delta z}(x_0,y_0,z_0)(z-z_0)=0 \tag{4.84}$$

- Directional Derivative:

$$\frac{d}{dt}f(x_0+tu,y_0+tu_z)|0 \tag{4.85}$$


# 5. Superior order derivatives

## 5.1 Taylor's Theorem

let

$$f : \mathbb{R} \to \mathbb{R}$$

Then:

$$F(x) = F(a) + F'(a)(x-a) + \frac{F''(a)}{\alpha!}(x-a)^2 + \cdots + \frac{F^{(k)}(x-a)^k}{k!}(x-a)^k + error \tag{5.1}$$

this can also be written as:

$$\Delta F(t_0) = F(t_0 + \Delta t) - F(t_0) \tag{5.2}$$

Generally, we can assume that given a theorem from $\mathbb{R}^n \to \mathbb{R}^n$

$$\vec{x_0} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}, \vec{h} = \begin{pmatrix} h_1 \\ . \\ . \\ . \\ h_n \end{pmatrix} \text{ then:} \tag{5.3}$$

$$f(\vec{x_0} + \vec{h}) = f(\vec{x_0}) + \sum_1^n \frac{\partial f}{\partial x_i}(\vec{x_0}) + \frac{1}{2!}\sum_{i,j=1}^n h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}\vec{x_0} + \frac{1}{3!}\sum_{i,j,k=1}^n h_i h_j h_k \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\vec{x_0} + \ldots \tag{5.4}$$

...That being said, we can mostly just stay with two. Triple derivatives are sure as hell weird.

## 5.2 Function extremes with real values.

Given a set 'D' that is bound if we can find a value R > 0 such as:

$$D \subseteq x^2 + y^2 \leq R^2$$

and:

$$D \subseteq \mathbb{R}^2 \to \mathbb{R}$$

We can express this as:

$$f : D \subseteq \mathbb{R}^2 \to \mathbb{R} \tag{5.5}$$

$$p\exists D\ is: \begin{cases} \max\ if\ f(p) \geq f(x)\forall x\exists D \\ \min\ if\ f(p) \leq f(x) \end{cases} \tag{5.6}$$

And we can also localize this to a specific region of the set, as a local minimum or maximum, think of it as p being a subset of a set, and then write it in the same way as you write above:

**Theorem:**
'f' has a local maximum or a minimum in $(x_0, y_0)$ , if an interior point of D and the partial derivatives of 'f' exist in $(x_0, y_0)$ , then:

$$f_x = (x_0, y_0) = 0 \wedge f_y(x_0, y_0) = 0 \tag{5.7}$$

### 5.2.1 Critical Points

p is a critical point of an f function if all partial derivatives are zero in such a point.

given a continuous function $f : D \to \mathbb{R}$ that's closed and bound, then f will achieve its minimum and maximum values.

**Example**

$$f(x,y) = x^2 + y^2 = \begin{cases} f_x = 2x = 0 \\ f_y = 2y = 0 \end{cases} \quad \min(0,0) \tag{5.8}$$

We can see similar behavior in a hyperbolic paraboloid, but here, it wont be an absolute minimum for both points.

**Example**

$$f(x,y) = x^2 - y^2 = \begin{cases} f_x = -2x = 0 \\ f_y = -2y = 0 \end{cases} \tag{5.9}$$

Every critical point is a candidate to be an absolute maximum or minimum, but won't nescessarily be. in any case, partial derivatives are a good tool for finding them through the critical points of a function.

If f has continuous second-order continuous derivatives, then:

$$\vec{D}^2_{\vec{u}} f(x_0, y_0) = \vec{D}_{\vec{u}} \vec{D}_{\vec{u}} f(x_0, y_0) \tag{5.10}$$

Is a directional derivative of (x,y) in u , we can calculate it as:

$$D_{\vec{u}}f(x,y) = \vec{\nabla}f \cdot \vec{u} =< fx, fy > \cdot < h,k > \tag{5.11}$$

$$\vec{\nabla}(hf_x + kf_y) \cdot \vec{u} =< hf_{xx} + kf_{yx}, hf_{xy} + kf_{yy} > \tag{5.12}$$

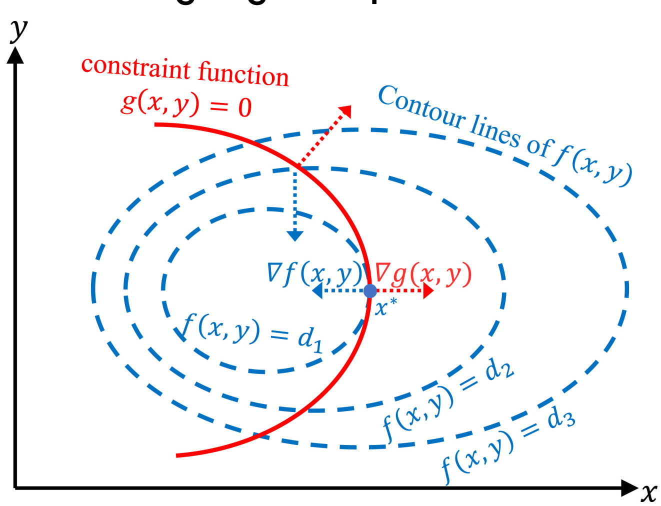
By developing this further, we'll arrive to a discriminant matrix of the sort:

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = det \tag{5.13}$$

from whatever this gives us, we can say:

$$D = det \implies \begin{cases} \text{if: } D > 0 \wedge f_{xx} > 0 \implies (x_0, y_0) = \text{min local} \\ \text{if: } D > 0 \wedge f_{xx} < 0 \implies (x_0, y_0) = \text{max local} \end{cases} \tag{5.14}$$

## 5.3 Restricted extremes and Lagrange multiplicators




Given a curve 'C' in the plane x,y in the plane: g(x,y) = k, given that g has $\partial'$ and $\vec{\Delta}g \neq 0$, finding minimums and maximums of f(x,y) alongside 'C. f has continuous $\partial'$.

If we assume f to have a maximum or minimum in $(x_0, y_0)$, then:

$$D_{\vec{T}}f(x_0,y_0) = 0 \tag{5.15}$$

$$\underbrace{\vec{\Delta}f(x_0,y_0)}_{\exists\mathbb{R}^2} \cdot \underbrace{\vec{T}}_{\exists\mathbb{R}} = 0 \tag{5.16}$$

If we assume f(x,y,z) having an extreme point $(x_0, y_0, z_0)$ on a surface S: $g(x,y,z) = k$ then for a plane P:

$$D_{\vec{u}}f(x_0,y_0,z_0) = 0 \forall \vec{u} \exists P \tag{5.17}$$

$$\vec{\nabla}f(x_0,y_0,z_0) = 0 \forall \vec{u} \tag{5.18}$$

$$\tag{5.19}$$

## 5.4 Implicit function theorem

Let a function 'F(x,y) = 0' that is the implicit equation of a curve in xy; then:

**Example**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{5.20}$$

$$y = \pm\sqrt{(1 - \frac{x^2}{a^2})b^2} = \pm\sqrt{\frac{b^2}{a^2}(a^2 - x^2)}; x\exists[-a, a] \tag{5.21}$$

This is an example of a situation where fiven F(x,y)=0 and $(x_0, y_0)$, expressed together as $F(x_0, y_0) = 0$ we would like to know if we gan obtain

### 5.4.1 Theorem 1

Let:

- a F(x,y) defined and continuous in a rectangle centered on our point $(x_0, y_0)$; then $D = [x_0 - \triangle, x_0 + \triangle; y_0 - \triangle, y_0 + \triangle]$
- $F(x_0, y_0) = 0$
- $y \to F(x, y)$ increases (or reduces) in a strictly monotonous fashion.

Then:

- in a vecinity of the point, (*) can determine as a function of $x : y = f(x)$
- $f(x_0) = y_0$
- f is continuous

### 5.4.2 Theorem 2

We'll assume, besides that we assumed on our first theorem:

- $F_y = (x_0, y_0) \neq 0$
- $\partial_x, \partial_y$ exist and are continuous.

Then, including what we concluded on the first iteration, we can conclude that:

- in a vecinity of the point, (*) can determine as a function of $x : y = f(x)$
- $f(x_0) = y_0$
- f is continuous
- f'() exists and is continuous

let's remember a function is differentiable when:

$$F_x(x,y)\Delta x + F_y(x,y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \varepsilon_1, \varepsilon_2 \to 0 \tag{5.22}$$

$$= \Delta x(F_x(x,y) + \varepsilon_1) + \Delta y(F_y(x,y) + \varepsilon_2) \tag{5.23}$$

$$\frac{\Delta y}{\Delta x} = -\frac{F_x + \varepsilon_1}{F_y = \varepsilon_2}, \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{F_x(x,y)}{F_y(x,y) \neq 0} \tag{5.24}$$

$$f'(x) \tag{5.25}$$

### 5.4.3 Theorem 3

We assume:

- $F(x_1, \ldots, x_n, y)$ is defined and continuous over $D = [\ldots]$

- $f(x_1^0,\dots,x_n^0,y_0)=0$

- $f_y \neq 0$
- $f_{x1},\dots,f_{xn},f_y$ exist and are continuous

then:

- in a vecinity $(x_1^0,\dots,x_n^0,y_0),(**)$ determines as a function

### 5.4.4 General Theorem

In general, we'll say that $F_1(x_1\dots,x_n,y_1,\dots,y_n)=0=(***)$

then $\underbrace{(***)}_{determines}$ $y_1,\dots,y_n$ as functions of $x_1,\dots,x_n$ if for every $(x_1,\dots,x_n)\exists(a,b,\dots;\dots)=z$ the system has a single solution $y_1,\dots,y_n$

Instead of $F_y(x_0,y_0)\neq 0$ we can say that a Jacobian can be determined as a Hessian of the type:

$$J=\frac{D(F_1,\dots,F_n)}{D(y_1,\dots,y_n}=\begin{vmatrix}\frac{\partial F_1}{\partial y_1} & \dots & \frac{\partial F_1}{\partial y_n}\\ \dots & \dots & \dots\\ \frac{\partial F_n}{y_1} & \dots & \frac{\partial F_n}{y_n}\end{vmatrix} \tag{5.26}$$

Put under the logic of the previous statements, we would say:

- $F_i$ is defined and continuous
- The point $(x_1^0,\dots,y_n^0;y_1^0,\dots,y_n^0)$ satisfies (***)
- $J(x_1^0,\ dots,y_n^0)\neq 0$
- $\partial'$ of $F_i$ are continuous in all variables

Then:

- in a $(x_1^0,\dots,y_n^0;y_1^0,\dots,y_n^0)$ vecinity it can be determined by (***)
- $f_j(x_1^0,\dots,x_n^0)=y_j^0$
- $f_j$ is continuous
- Partial derivatives of $f_j$ are continuous

**Demonstration**

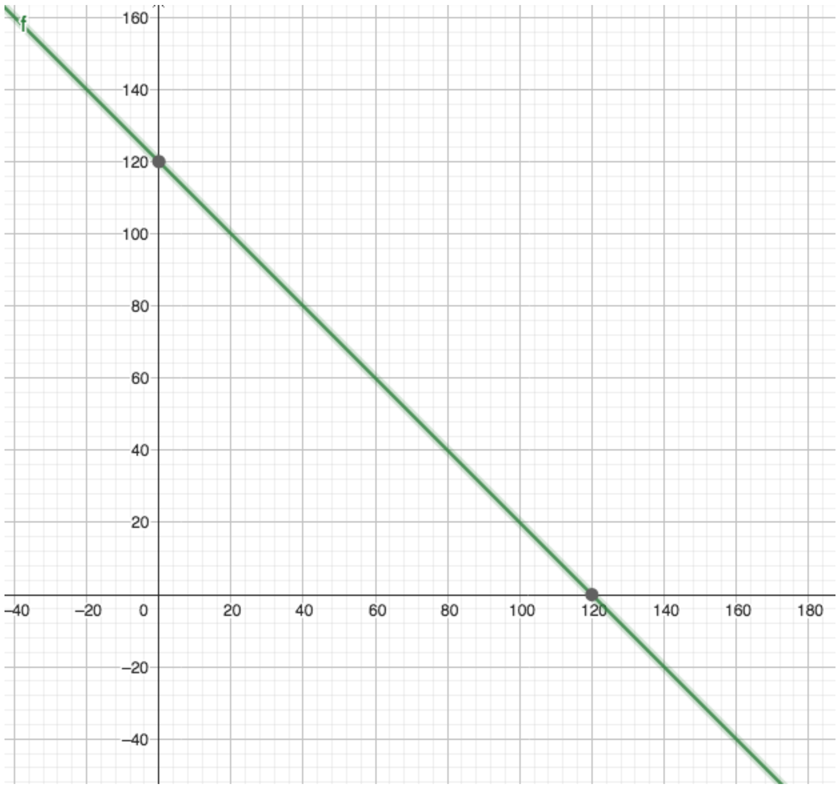
$$m = 1 \; complies \; Theorem \; 3; \tag{5.27}$$
$$\text{Theorem } 3m-1 \implies \text{Theorem } 3m \tag{5.28}$$
$$J(X_1^0, \ldots y_1^0) \neq 0 \tag{5.29}$$
$$< induct \; \frac{\partial F_m}{\partial y_m}(x_1^0 \ldots x_n^0) \neq 0 > \tag{5.30}$$
$$< Theorem \; 3 = T_3 > \tag{5.31}$$
$$T_3 \implies F_m(x_1, \ldots y_m) = 0; D^* \subseteq D \tag{5.32}$$
$$y_m = f(n)(x,y) \tag{5.33}$$
$$y_m = \phi(x_1 \ldots x_n, y_1 \ldots y_n) \tag{5.34}$$
$$\implies F_m(x_1 \ldots y_{m-1}; \phi(x_1 \ldots x_n, y_1 \ldots y_n)) = 0, \tag{5.35}$$
$$< \phi \; continuous \; ; \partial' \; exist > \tag{5.36}$$
$$J* = \frac{d(\phi_1, \ldots \phi_m - 1)}{d(y_1, \ldots, y_{m-1})} = \begin{vmatrix} \frac{\partial \phi_1}{\partial y_0} & \ldots & \frac{\partial \phi_1}{\partial y_{m-1}} \\ \ldots & \ldots & \ldots \\ \frac{\partial \phi_1}{\partial y_{m-1}} & \ldots & \frac{\partial \phi_{m-1}}{\partial y_{m-1}} \end{vmatrix} \tag{5.37}$$
$$< Chain \; Rule > \tag{5.38}$$
$$F_m(x_1, \ldots, y_m - 1, \phi(x_1, \ldots, y_m - 1)) = 0 \tag{5.39}$$
$$J = J^* \cdot \frac{\partial F_m}{\partial y_m} \neq 0 \implies J \exists T_3 \tag{5.40}$$

## 5.5 Examples

**Given 3 non-negative numbers with a 120 overall sum, find the maximal value of x,y,z**

$$x + y + z = 120 \implies z = 120 - x - y \tag{5.41}$$
$$f(x,y) = xy(120 - x - y) \tag{5.42}$$
$$\{x, y | x, y \exists \mathbb{R}^2 \geq 0; x \leq 120 - y; y \leq x - 120\} \tag{5.43}$$
$$\text{<Implies y = -x + 120 >} \tag{5.44}$$

we can graph it as:




from this we can say, by calculating the area:

$$f(x,y) = 120xy - x^2y - xy^2 \tag{5.45}$$

And now we differentiate:

$$f_x = 120y - 2xy - y^2 \tag{5.46}$$
$$f_y = 120x - 2xy - y^2 \tag{5.47}$$
$$\text{<Minimals>} \tag{5.48}$$
$$120 - 2x - y = 0 \implies x = 0 \vee y = 0 \tag{5.49}$$

and from linear algebra, we can do:

$$\begin{pmatrix} -2 & -1| - 120 \\ -1 & -2| - 120 \end{pmatrix} \tag{5.50}$$

after solving this we can say:

$$x = 40 \implies y = 40 \implies z = 40 \tag{5.51}$$
$$f(40,40,40) = (40^3) \tag{5.52}$$

**Find critical points of** $z = x^2 + y^2 + 3xy$

$$\frac{\partial_z}{\partial_x} = 2x + 3y = 0 \tag{5.53}$$
$$\frac{\partial_z}{\partial_y} = 2y + 3x = 0 \tag{5.54}$$
$$\begin{cases} 2x + 3y = 0 \\ 2y + 3x = 0 \end{cases} \implies y = -\frac{2x}{3} \tag{5.55}$$
$$2(-\frac{2x}{3}) + 3x = 0 \tag{5.56}$$
$$-\frac{4}{3}x + 3x = 0 \tag{5.57}$$
$$\frac{5}{3}x = 0 \tag{5.58}$$
$$x = 0 \tag{5.59}$$
$$y = 0 \tag{5.60}$$
$$\textit{(0,0) is the only critical point} \tag{5.61}$$

Now we might imagine:

$$D(\nabla f) = \underbrace{H}_{\text{Hessian}} f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

And from that:

$$\frac{\partial^2 f}{\partial x^2} = 2 \tag{5.62}$$
$$\frac{\partial^2 f}{\partial x \partial y} = 3 \tag{5.63}$$
$$\frac{\partial^2 f}{\partial y \partial x} = 3 \tag{5.64}$$
$$\frac{\partial^2 f}{\partial y^2} = 2 \tag{5.65}$$
$$\implies H = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \tag{5.66}$$
$$\Delta = det\ Hf(0,0) \tag{5.67}$$
$$\Delta = -5 \tag{5.68}$$
$$\text{(Inflection point)} \tag{5.69}$$

**Find the absolute maximum and minimums of $f(x,y) = x^2 + 2xy + 3y^2$ in a closed triangle with vertices in $(-1,1),(2,1),(-1,2)$**

$$f_x = 2x + 2y = 0 \tag{5.70}$$
$$f_y = 2x + 6y = 0 \tag{5.71}$$
$$< therefore > \tag{5.72}$$
$$y = 1 \tag{5.73}$$
$$0.x - 4y + 0 \implies y = 0; x = 0; f(0,0) = 0 \tag{5.74}$$
$$y = 1; f(x,1) = x^2 + 2x + 3 \tag{5.75}$$
$$x = -1 \tag{5.76}$$
$$g'(x) = 2x + 2 = 0 \implies x = -1 \implies f(-1,1) = 1 - 2 + 3 = 2 \tag{5.77}$$
$$g(y) = f(-1,y) = 1 - 2y + 3y^2 \tag{5.78}$$
$$g'(y) = 6y - 2 + 0 \implies y = \frac{1}{3} \implies (-1,\frac{1}{3}), f(-1,\frac{1}{3}) = \frac{2}{3} \tag{5.79}$$
$$y = x - 1 \tag{5.80}$$
$$\tag{5.81}$$

**Find the points over the surface $xyz = 8$ that are the closest to the origin (0,0,0)**

*pictured: the surface to be analyzed*

$$f(x,y) = \sqrt{x^2 + y^2 + \frac{6y}{x^2y^2}} \tag{5.82}$$
$$g(x,y) = x^2 + y^2 + \frac{6y}{x^2y^2} \tag{5.83}$$
$$\begin{cases} g_x = 2x - \frac{128}{x^3y^2} \\ g_y = 2y - \frac{128}{x^2y^3} \end{cases} \implies 2x^4y^4 = 128 <=> x^4 = \frac{64}{y^2} <=> x^2 = \pm\frac{8}{y} \tag{5.84}$$
$$< Case\ 1 :> \ x^2 = \frac{8}{y} \tag{5.85}$$
$$2y^4x^2 = 128 \implies 2y^4\frac{8}{y} \implies y^3 = 8 \implies y = 2 \implies x^2 = 4 \implies x \pm 2 \implies (2,2) \wedge (-2,2) \tag{5.86}$$
$$< Case\ 2 :> \ x^2 = -\frac{8}{y} \tag{5.87}$$
$$\ldots y^3 = -8 \implies y = -2 \implies x^2 = 4 \implies x \pm 2 \implies (2,-2) \wedge (-2,-2) \tag{5.88}$$
$$f(\pm 2, \pm 2) = \sqrt{12} \tag{5.89}$$

The closer points to the origin are (2,2,2), (-2,2,-2), (2,-2,-2), (-2,-2,2)

**Find the extremes of $f(x,y) = xy$ alongside the elipse $\underbrace{4x^2 + y^2 = 4}_{g(x,y)}$**

Note: extreme values are different from 0, and xy is different from 0

$$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y) \tag{5.90}$$

$$< x,y >= \lambda < 8x, 2y ><=> \begin{cases} y = \lambda 8x : I \\ x = \lambda 2y : II \\ 4x^2 + y^2 = 4 : III \end{cases} \tag{5.91}$$

$$\frac{I}{II} : \frac{y}{x} = \frac{4x}{y} <=> y^2 = 4x^2 \tag{5.92}$$

$$4x^2 + 4x^2 = 4 \tag{5.93}$$

$$8x^2 = 4 \tag{5.94}$$

$$x^2 = \frac{1}{2} \tag{5.95}$$

$$x \pm \frac{1}{\sqrt{2}} \tag{5.96}$$

$$y^2 = 2, y \pm \sqrt{2} \tag{5.97}$$

**Find the volume of a rectangular box defined by three planes and a vertex in (0,0,0), under** $3x+2y+z=6$

$$f(x,y,z) = xyz \tag{5.98}$$

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) \tag{5.99}$$

$$< yz, xz, xy >= \lambda < 3,2,1 > \tag{5.100}$$

$$\begin{cases} yz = 3\lambda \\ xz + 2\lambda \\ xy = \lambda \\ 3x + 2y + z = 6 \end{cases} \tag{5.101}$$

$$\frac{I}{II} <=> \frac{y}{x} = \frac{3}{2} <=> 2y = 3x \tag{5.102}$$

$$\frac{II}{III} <=> \frac{7}{y} = 2 <=> z = 2y \tag{5.103}$$

$$3x + 3x + 3x = 6 \tag{5.104}$$

$$9x = 6 \implies x = \frac{2}{3} \implies y = 1 \tag{5.105}$$

$$\implies z = 2 \tag{5.106}$$

**Calculate critical points of the following function:** $f(x,y) = x^2y - xy^2 - x + y$

$$\Delta f(x,y) = (\overbrace{2xy - y^2 - 1}^{\frac{\partial f}{\partial x}}, \overbrace{x^2 - 2xy + 1}^{\frac{\partial f}{\partial y}}) \tag{5.107}$$

$$\Delta f(x,y) = (0,0) <=> f_x = 0, f_y = 0 \tag{5.108}$$

$$\begin{cases} 2xy - y^2 - 1 = 0 \\ x^2 - 2xy + 1 = 0 \end{cases} \tag{5.109}$$

$$2xy - y^2 - 1 = 0 \tag{5.110}$$

$$2xy = y^2 + 1 \tag{5.111}$$

$$x^2 - (y^2 + 1) + 1 = 0 \tag{5.112}$$

$$x^2 = y^2 \tag{5.113}$$

$$|x| = |y| \tag{5.114}$$

$$y = \pm x \implies \begin{cases} C1 = y = x \\ C2 = y = -x \end{cases} \tag{5.115}$$

$$< C1 > \tag{5.116}$$

$$y = x \tag{5.117}$$

$$2xy - y^2 - 1 = 0 \tag{5.118}$$

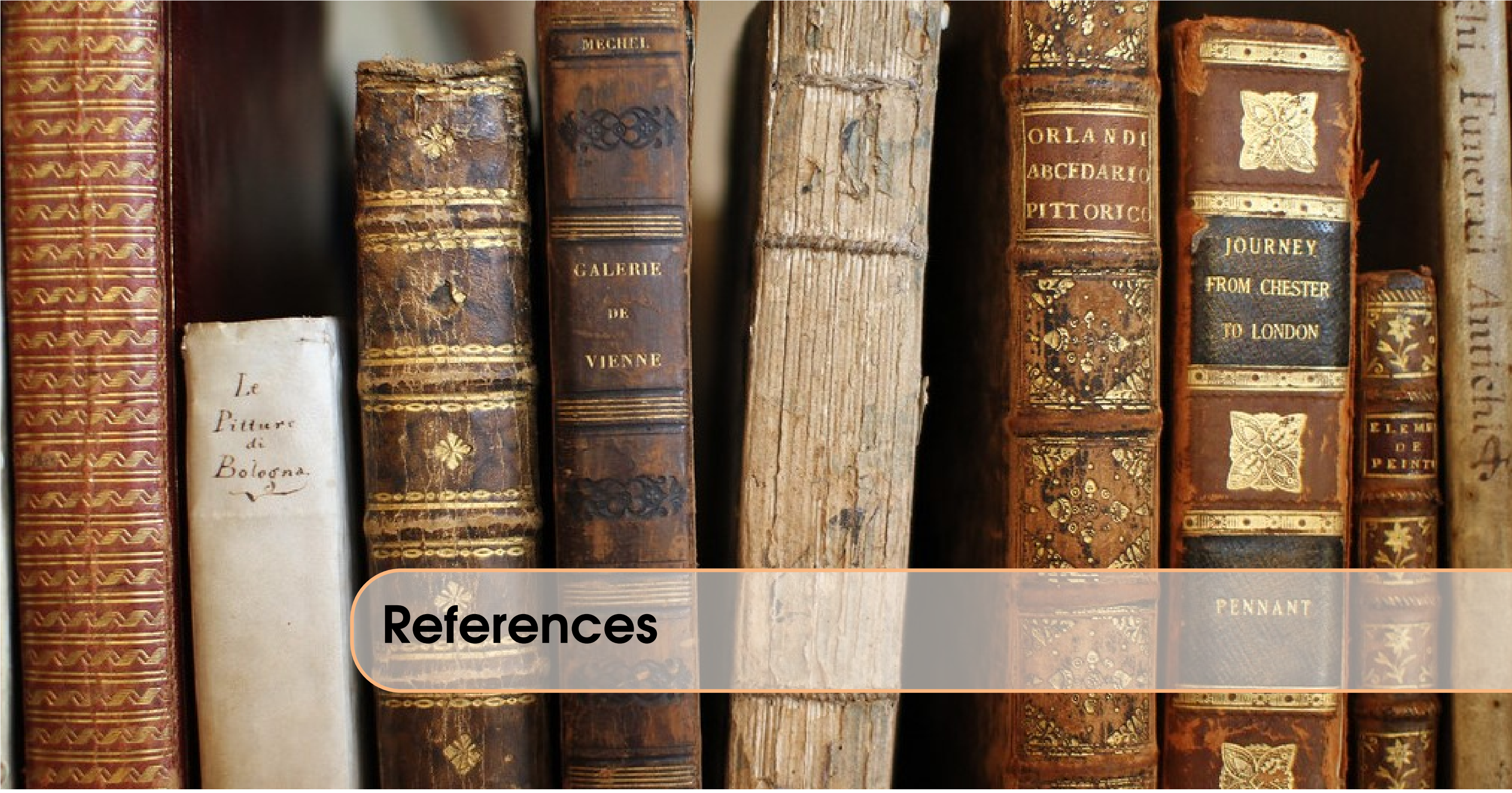
$$2x^2 - x^2 - 1 = 0 \tag{5.119}$$

$$x^2 - 1 = 0 \tag{5.120}$$

$$x^2 = 1x = \pm 1 \tag{5.121}$$

$$Hf = \begin{vmatrix} 2y & 2x - 2y \\ 2x - 2y & -2x \end{vmatrix} \tag{5.122}$$

$$a\Delta = -4xy - (2x - 2y)^2 \tag{5.123}$$


Le Pitture di Bologna
MECHEL
GALERIE DE VIENNE
ORLANDI ABCEDARIO PITTORICO
JOURNEY FROM CHESTER TO LONDON
PENNANT

# References