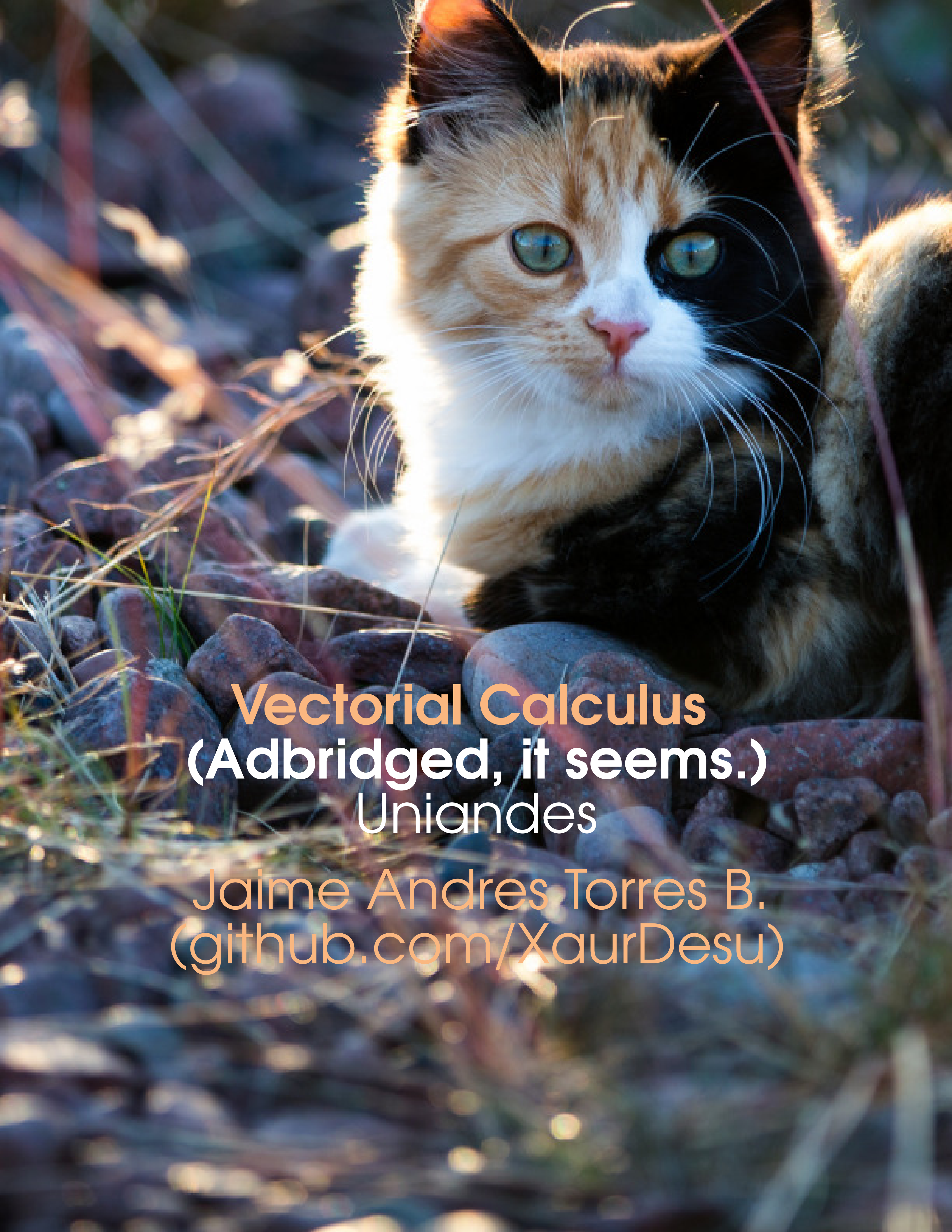

Vectorial Calculus

(Adbridged, it seems.)

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Contents

1. Introduction

Go big or go home.

Vectorial calculus is what the title says pretty much, the act of using methods proper to calculus on vectorial spaces, for the topic of this class generally referring to merely 3-dimensional ones, at the end of this book you should be able to:

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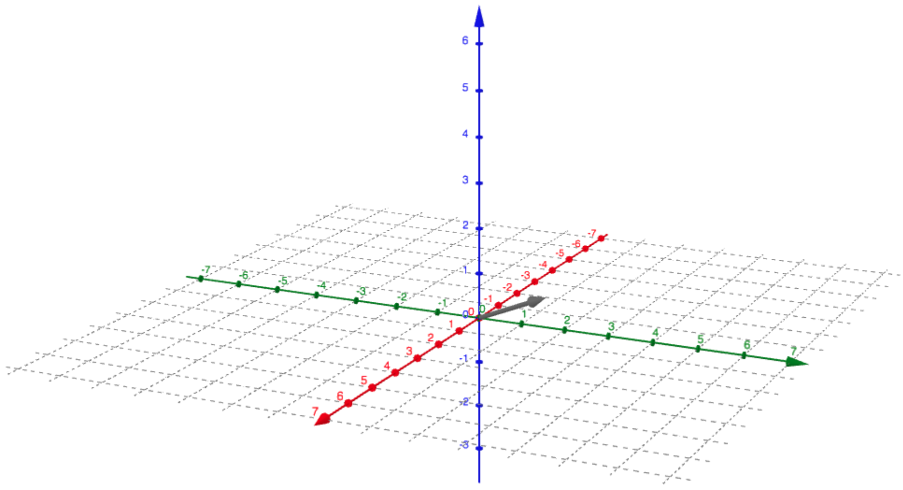
2. Linear Algebra Concepts

2.1 Vectors on a three-dimensional space.

given an $\mathbb{R}^3$ space and a point in that space $P = (a, b, c)$, we can describe a vector by either connecting the point P to another point Q, or by assuming the origin of this space (point $(0,0,0)$), this is a mathematical object with both a direction and a magnitude. The direction is given by an angle and the magnitude is given by $\sqrt{a_1^2 + a_2^2 + a_3^2}$

As an example, let's assume the vector given by $P = (1, 2, 1)$:



Vector formed by $P = (1, 2, 1)$

For this vector, we can calculate the magnitude by replacing the vectorial components by the magnitudes of the individual directions, resulting in:

Note, in this course we will be mostly only concerned with $\mathbb{R}^3$

2.1.1 Addition and Subtraction

We can take any $\vec{a}$ and $\vec{b}$ vectors on the same space and add them to each other in the form:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \tag{2.1}$$

Such form remains in the case we can do subtraction, which is expressed on the equation:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \tag{2.2}$$

This kind of operations have certain properties, shown as:

$$(\alpha + \beta)\vec{[v]} = \alpha\vec{v} + \beta\vec{v} \tag{2.3}$$

$$\vec{v} * 1 = \vec{v} \tag{2.4}$$

$$\vec{v} * \vec{0} = \vec{0} \tag{2.5}$$

$$\beta\vec{v} = \begin{pmatrix} \beta a_1 \\ \beta a_2 \\ \beta a_3 \end{pmatrix} \tag{2.6}$$

Two vectors $\vec{a}$ and $\vec{b}$ are equal if and only if:

$$\begin{cases} \vec{a} \exists \mathbb{R}^3 \\ \vec{b} \exists \mathbb{R}^3 \end{cases} \implies \begin{pmatrix} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{pmatrix} \textit{ note: this can be generalized to 'n' dimensions larger than 0} \tag{2.7}$$

in either case, $\vec{0}$ is the identity of the operation, therefore:

$$\vec{a} + \vec{0} = \vec{a} \tag{2.8}$$

2.1.2 Bases

A base in R^n can be found though n vectors on that plane, such as it would happen in R^2 with:

$$\lambda\vec{u} + \mu\vec{v} | \lambda, \mu \exists \mathbb{R} \tag{2.9}$$

this equation will form a parallelogram that can express the distorsion of space when compared to a reference system, which generally is the canonical base formed by the identity.

2.1.3 Dot product

Assume two equal-length vectors of the sort:

$$\begin{cases} \vec{a} = (a_i * n | n \exists \mathbb{R}); |\vec{a}| \exists \mathbb{R} \\ \vec{b} = (b_i * n | n \exists \mathbb{R}); |\vec{b}| \exists \mathbb{R} \end{cases} \tag{2.10}$$

in case we wanted to do obtain a scalar number, that corresponded to the sum of the internal products we could obtain:

$$A \centerdot B = ||\vec{A}||||\vec{B}|| \cos\theta \tag{2.11}$$

for cartesian vectors, we can write this as:

$$\vec{a} \centerdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \centerdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \exists \mathbb{R} \tag{2.12}$$

where θ is the angle between both vectors. We can get it by calculating

$$\cos\theta = \frac{\vec{u} x \vec{v}}{||\vec{u}|| * ||\vec{u}||}$$

If perpendicular, we can assume:

$$\vec{u} \centerdot \vec{v} = 0$$

Notable cases.

With these rules, we can infere a few interesting cases, which we'll be able to interpolate stuff with.

Implications

- $\theta < \frac{\pi}{2} \implies \cos\theta > 0$
- $\theta > \frac{\pi}{2} \implies \cos\theta < 0$
- $\theta = \frac{\pi}{2} \implies \cos\theta = 0$

Addendum: cosine values

In vectorial calculus, we'll have certain notable angles that will appear often in exercises. we can use fractions to get them approximated to numerical values, such values are listed on this table:

$\cos 0^o$	$\frac{4}{\sqrt{2}}$	1
$\cos 30^o$	$\frac{3}{\sqrt{2}}$	?
$\cos 45^o$	$\frac{2}{\sqrt{2}}$	?
$\cos 60^o$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\cos 90^o$	$\frac{0}{\sqrt{2}}$	0

Addendum 2: Triangular inequality

The triangular inequality affirms that:

$||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$

2.1.4 Cross product

A cross product is, much like the dot product, an operation that seeks to multiply the values between two vectors. it can be annotated as:

$$\vec{u}x\vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} * \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - v_1u_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \tag{2.13}$$

This is a non-commutative operation, changing the order of signs will cause the signs to invert, seen mathematically as:

$$\vec{u}x\vec{v} = -(\vec{v}x\vec{u})$$

The vector that the cross product produces is perpendicular to both evaluated vectors.

we can also use the norm of this cross product as a way to calculate the area of the paralleleipied form triangulated by $\vec{u}$ and $\vec{v}$ as

$$A = ||\vec{u}x\vec{v}||$$

This can also work for three-dimensional paralleleipied in the following formula:

$$|\vec{w}\centerdot(\vec{u}x\vec{v})| = ||\vec{w}|| * ||\vec{u}x\vec{v}|| * |\cos\vartheta| \tag{2.14}$$

we can also say, from this:

$$\vec{w}\centerdot(\vec{u}x\vec{v}) = \vec{u}\centerdot(\vec{v}x\vec{w}) = \vec{v}\centerdot(\vec{w}x\vec{u}) \tag{2.15}$$

2.1.5 Determinants

a determinant is defined as:

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.16}$$

2.2 Describing objects in a space.

2.2.1 Lines

A line is a geometrical object of the form:

$$r(t) = t\vec{v} + P, t\exists\mathbb{R} \tag{2.17}$$

generating it requires a point and a vector. Point defined by P, and vector defined by an offset 't' and a vector '$\vec{v}$'

Example *Find the equation of a line 'l' that crosses* $A = (2,1,1)$ *and* $B = (3,5,7)$

for this, we'll establish the following formula:

$$l(t) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} + t\begin{pmatrix} B_1 - A_1 \\ B_2 - A_2 \\ B_3 - A_3 \end{pmatrix} \tag{2.18}$$

Instanced, for this specific case, as:

$$l(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3-2 \\ 5-1 \\ 7-1 \end{pmatrix} \tag{2.19}$$

$$l(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} \tag{2.20}$$

Answer is equation 2.15, this can be later expanded into a parametric or simetric form of this line. But before we do that, let's try expanding the reason this works:

Example 2 *Find the equation of the line that joins points* $P = (1,2,1)$ *and* $Q = (-1,3,4)$

we can find the line that joins two points by subtracting the vectors that join them, let's take a look at the cartesian plane where we indicate 'P' and 'Q':

2.2.2 Vector Projection

A vector can be projected through the equation:

$$\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v} \tag{2.21}$$

2.2.3 Euclidian Planes

A plane is the union of all points defined by a formula of the type:

$$i_1A + i_2B + i_3C = D \tag{2.22}$$

2.3 Cylindrical and spherical coordinates.

When trying to define parts of a line in algebra, we'll usually be looking at coordinates, be them polar or cartesian. In either case, their information can be converted to the other system through the following formulas.

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan(\frac{y}{x}) \end{cases} \quad \textit{cartesian to polar} \tag{2.23}$$

$$\begin{cases} \alpha_x = \rho \cos\theta \\ \alpha_y = \rho \sin\theta \end{cases} \quad \textit{polar to cartesian} \tag{2.24}$$

Cartesian coordinates generally translate well to other dimensional spaces, such as would be the case for $\mathbb{R}^3$, however, polar coordinates as we know them usually aren't as translatable in a direct manner, and expressing them in three-dimensional spaces might be better suited to be expressed on a cylindrical or spherical condition.

Cylindrical coordinates

In the case of cylindrical coordinates, the translation is probably the most intuitive, by computing a cylinder with polar coordinates that indicate an (x,y) position, and a 'Z' variable indicating height that allows us to project the vector on a third dimension, this 'z' variable is exactly the same as it would be on a cartesian model. We can express it like such:

$\vec{v} = (\rho, \theta, Z)$

conversion to a cartesian model can be expressed as:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\cos\theta \\ \alpha_y = \rho\sin\theta \quad \textit{Cylindrical to cartesian} \\ \alpha_z = Z \end{cases} \tag{2.25}$$

Spherical coordinates

A spherical coordinate is formed by a tuple:

$$(\rho, \theta, \phi); \begin{cases} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases} \tag{2.26}$$

Where ρ is the magnitude of the vector, θ is the (x,y) coordinates, and ϕ is the (y,z) angle. this tuple can generate two vectors:

$$\begin{cases} \rho\sin\phi \\ \rho \end{cases} \tag{2.27}$$

And can be converted to a cartesian model as such:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\sin\phi\cos\theta \\ \alpha_y = \rho\sin\phi\sin\theta \quad \textit{Spherical to cartesian} \\ \alpha_z = \rho\cos\phi \end{cases} \tag{2.28}$$

Such a model works, as we might imagine, like a sphere. where we express the possible vectors through a sphere of ρ radius.

2.4 n-dimensional Euclidian Spaces

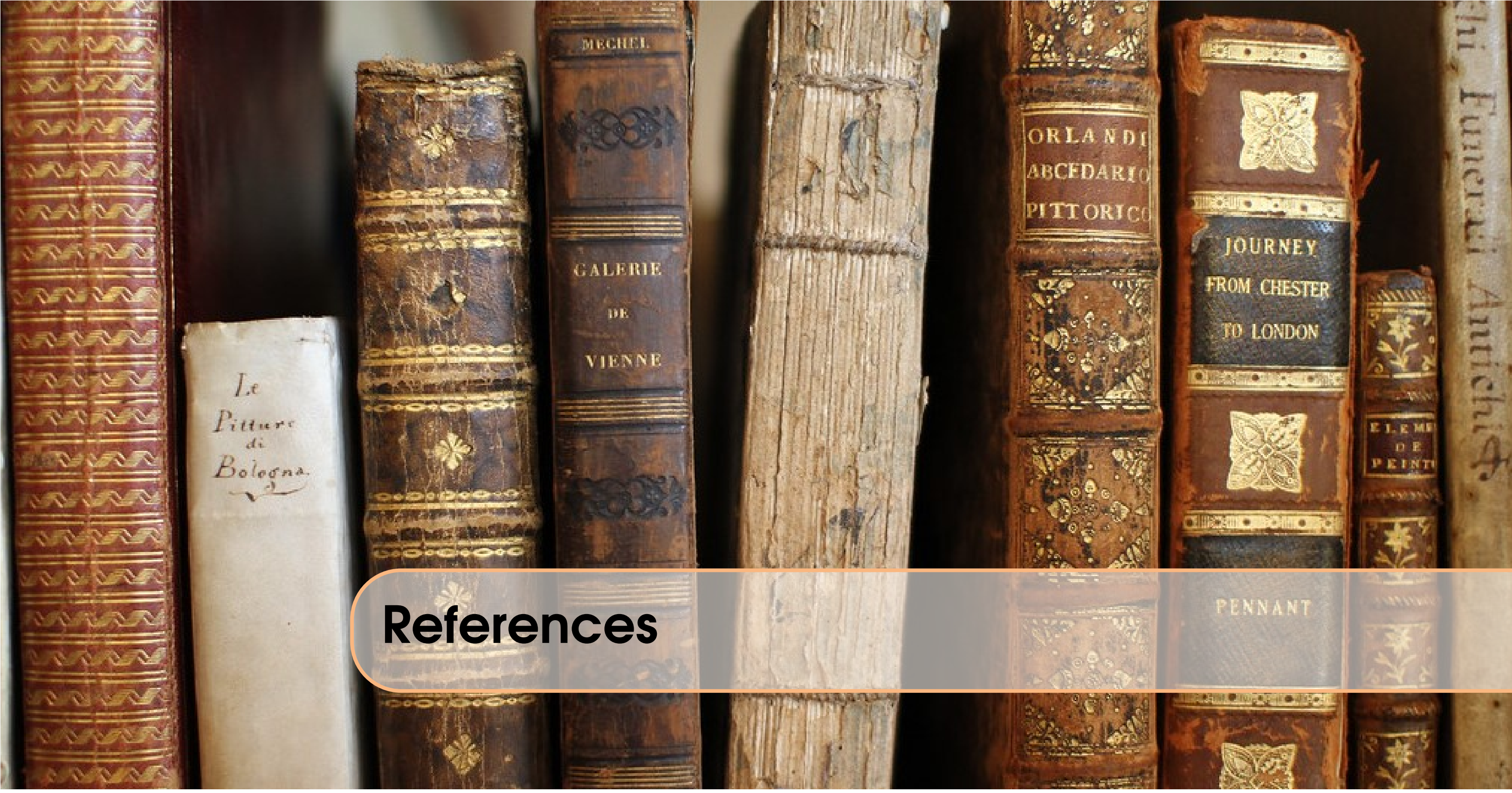
$$\mathbb{R}^n, \vec{x}\left(x_1 \ldots x_n\right); \mathbb{C}\ \textit{Operations:} \begin{cases} \vec{x} + \vec{y} \\ \alpha\vec{x} \\ \vec{x} \cdot \vec{y} \end{cases} \tag{2.29}$$

Cauchy-Schwartz Inequality

$$\text{Let: } \vec{x}, \vec{y} \exists \mathbb{R}^n \text{ then:} \tag{2.30}$$

$$|\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| \cdot ||\vec{y}|| \tag{2.31}$$

3. Functions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