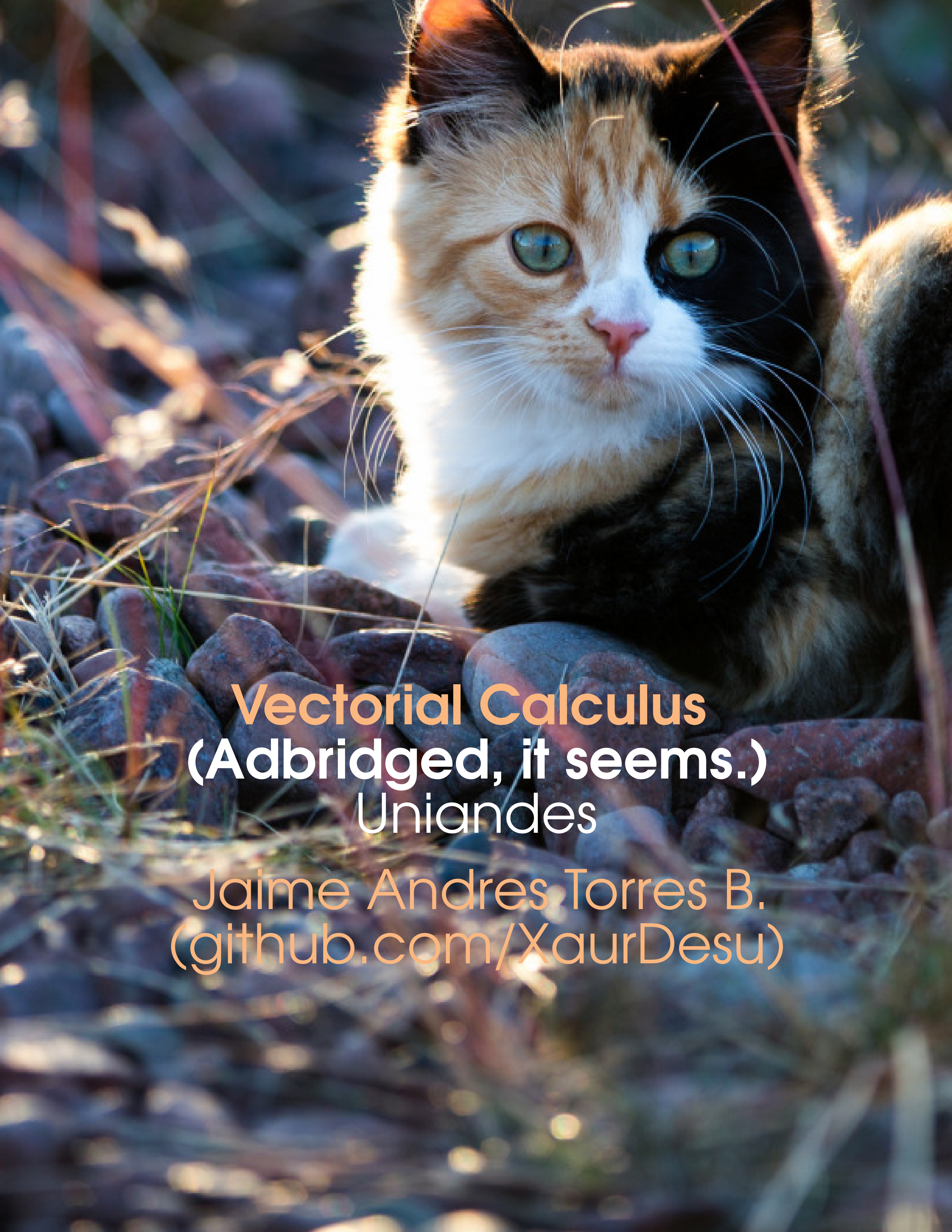

Vectorial Calculus

(Adbridged, it seems.)

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Vectorial Calculus -

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Contents

1. Introduction

Go big or go home.

Vectorial calculus is what the title says pretty much, the act of using methods proper to calculus on vectorial spaces, for the topic of this class generally referring to merely 3-dimensional ones, at the end of this book you should be able to:

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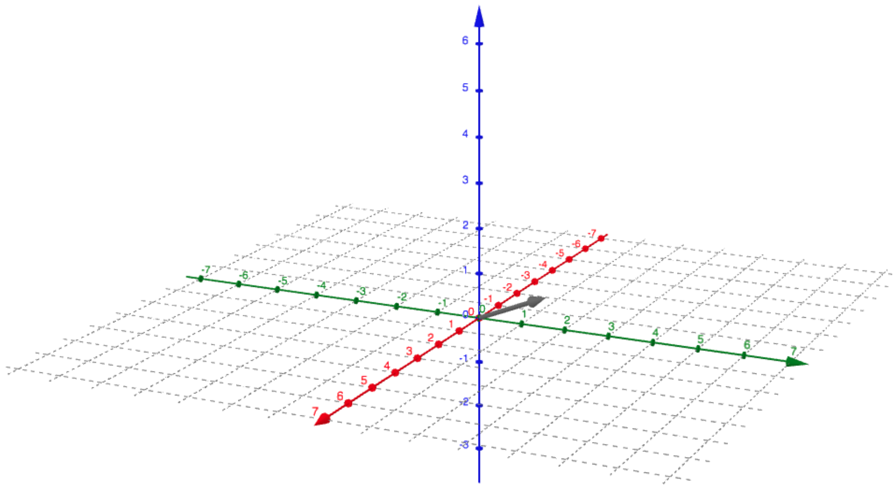
2. Linear Algebra Concepts

2.1 Vectors on a three-dimensional space.

given an $\mathbb{R}^3$ space and a point in that space $P = (a,b,c)$, we can describe a vector by either connecting the point P to another point Q, or by assuming the origin of this space (point $(0,0,0)$), this is a mathematical object with both a direction and a magnitude. The direction is given by an angle and the magnitude is given by $\sqrt{a_1^2 + a_2^2 + a_3^2}$

As an example, let's assume the vector given by $P = (1,2,1)$:



Vector formed by $P = (1,2,1)$

For this vector, we can calculate the magnitude by replacing the vectorial components by the magnitudes of the individual directions, resulting in:

Note, in this course we will be mostly only concerned with $\mathbb{R}^3$

2.1.1 Addition and Subtraction

We can take any $\vec{a}$ and $\vec{b}$ vectors on the same space and add them to each other in the form:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} \tag{2.1}$$

Such form remains in the case we can do subtraction, which is expressed on the equation:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{pmatrix} \tag{2.2}$$

This kind of operations have certain properties, shown as:

$$(\alpha+\beta)\vec[v] = \alpha\vec{v}+\beta\vec{v} \tag{2.3}$$

$$\vec{v}*1 = \vec{v} \tag{2.4}$$

$$\vec{v}*\vec{0} = \vec{0} \tag{2.5}$$

$$\beta\vec{v} = \begin{pmatrix} \beta a_1 \\ \beta a_2 \\ \beta a_3 \end{pmatrix} \tag{2.6}$$

Two vectors $\vec{a}$ and $\vec{b}$ are equal if and only if:

$$\begin{cases} \vec{a}\exists\mathbb{R}^3 \\ \vec{b}\exists\mathbb{R}^3 \end{cases} \implies \begin{pmatrix} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{pmatrix} \textit{ note: this can be generalized to 'n' dimensions larger than 0} \tag{2.7}$$

in either case, $\vec{0}$ is the identity of the operation, therefore:

$$\vec{a}+\vec{0} = \vec{a} \tag{2.8}$$

2.1.2 Bases

A base in R^n can be found though n vectors on that plane, such as it would happen in R^2 with:

$$\lambda\vec{u}+\mu\vec{v}|\lambda,\mu\exists\mathbb{R} \tag{2.9}$$

this equation will form a parallelogram that can express the distorsion of space when compared to a reference system, which generally is the canonical base formed by the identity.

2.1.3 Dot product

Assume two equal-length vectors of the sort:

$$\begin{cases} \vec{a} = (a_i*n|n\exists\mathbb{R});|\vec{a}|\exists\mathbb{R} \\ \vec{b} = (b_i*n|n\exists\mathbb{R});|\vec{b}|\exists\mathbb{R} \end{cases} \tag{2.10}$$

in case we wanted to do obtain a scalar number, that corresponded to the sum of the internal products we could obtain:

$$A \cdot B = ||\vec{A}||||\vec{B}|| \cos\theta \tag{2.11}$$

for cartesian vectors, we can write this as:

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \exists \mathbb{R} \tag{2.12}$$

where θ is the angle between both vectors. We can get it by calculating

$$\cos\theta = \frac{\vec{u} x \vec{v}}{||\vec{u}|| * ||\vec{u}||}$$

If perpendicular, we can assume:

$$\vec{u} \cdot \vec{v} = 0$$

Notable cases.

With these rules, we can infere a few interesting cases, which we'll be able to interpolate stuff with.

Implications

- $\theta < \frac{\pi}{2} \implies \cos\theta > 0$
- $\theta > \frac{\pi}{2} \implies \cos\theta < 0$
- $\theta = \frac{\pi}{2} \implies \cos\theta = 0$

Addendum: cosine values

In vectorial calculus, we'll have certain notable angles that will appear often in exercises. we can use fractions to get them approximated to numerical values, such values are listed on this table:

$\cos 0^o$	$\frac{4}{\sqrt{2}}$	1
$\cos 30^o$	$\frac{3}{\sqrt{2}}$	?
$\cos 45^o$	$\frac{2}{\sqrt{2}}$	?
$\cos 60^o$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\cos 90^o$	$\frac{0}{\sqrt{2}}$	0

Addendum 2: Triangular inequality

The triangular inequality affirms that:
$||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$

2.1.4 Cross product

A cross product is, much like the dot product, an operation that seeks to multiply the values between two vectors. it can be annotated as:

$$\vec{u}\text{x}\vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} * \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - v_1u_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \tag{2.13}$$

This is a non-commutative operation, changing the order of signs will cause the signs to invert, seen mathematically as:

$$\vec{u}x\vec{v} = -(\vec{v}x\vec{u})$$

The vector that the cross product produces is perpendicular to both evaluated vectors.

we can also use the norm of this cross product as a way to calculate the area of the paralleleipied form triangulated by $\vec{u}$ and $\vec{v}$ as

$$A = ||\vec{u}x\vec{v}||$$

This can also work for three-dimensional paralleleipied in the following formula:

$$|\vec{w} \centerdot (\vec{u}x\vec{v})| = ||\vec{w}|| * ||\vec{u}x\vec{v}|| * |\cos\vartheta| \tag{2.14}$$

we can also say, from this:

$$\vec{w} \centerdot (\vec{u}x\vec{v}) = \vec{u} \centerdot (\vec{v}x\vec{w}) = \vec{v} \centerdot (\vec{w}x\vec{u}) \tag{2.15}$$

2.1.5 Determinants

a determinant is defined as:

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.16}$$

2.2 Describing objects in a space.

2.2.1 Lines

A line is a geometrical object of the form:

$$r(t) = t\vec{v} + P, t \exists \mathbb{R} \tag{2.17}$$

generating it requires a point and a vector. Point defined by P, and vector defined by an offset 't' and a vector '$\vec{v}$'

Example *Find the equation of a line 'l' that crosses* $A = (2,1,1)$ *and* $B = (3,5,7)$

for this, we'll establish the following formula:

$$l(t) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} + t \begin{pmatrix} B_1 - A_1 \\ B_2 - A_2 \\ B_3 - A_3 \end{pmatrix} \tag{2.18}$$

Instanced, for this specific case, as:

$$l(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3-2 \\ 5-1 \\ 7-1 \end{pmatrix} \tag{2.19}$$

$$l(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} \tag{2.20}$$

Answer is equation 2.15, this can be later expanded into a parametric or simetric form of this line. But before we do that, let's try expanding the reason this works:

Example 2 *Find the equation of the line that joins points* $P = (1,2,1)$ *and* $Q = (-1,3,4)$

we can find the line that joins two points by subtracting the vectors that join them, let's take a look at the cartesian plane where we indicate 'P' and 'Q':

2.2.2 Vector Projection

A vector can be projected through the equation:

$$\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v} \tag{2.21}$$

2.2.3 Euclidian Planes

A plane is the union of all points in a 2-dimensional subset of $\mathbb{R}^{\nvDash}$ defined by a formula of the type:

$$i_1 A + i_2 B + i_3 C = D = (D \cdot ||\vec{n}||) \tag{2.22}$$

Where $\vec{n}$ is also written as:

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.23}$$

It can be determined by

- three points in $\mathbb{R}^{\nvDash}$
- Two vectors and a point in $\mathbb{R}^{\nvDash}$
- a point and the normal vector in $\mathbb{R}^{\nvDash}$

2.3 Cylindrical and spherical coordinates.

When trying to define parts of a line in algebra, we'll usually be looking at coordinates, be them polar or cartesian. In either case, their information can be converted to the other system through the following formulas.

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan(\frac{y}{x}) \end{cases} \quad \textit{cartesian to polar} \tag{2.24}$$

$$\begin{cases} \alpha_x = \rho \cos\theta \\ \alpha_y = \rho \sin\theta \end{cases} \quad \textit{polar to cartesian} \tag{2.25}$$

Cartesian coordinates generally translate well to other dimensional spaces, such as would be the case for $\mathbb{R}^3$, however, polar coordinates as we know them usually aren't as translatable in a direct manner, and expressing them in three-dimensional spaces might be better suited to be expressed on a cylindrical or spherical condition.

Cylindrical coordinates

In the case of cylindrical coordinates, the translation is probably the most intuitive, by computing a cylinder with polar coordinates that indicate an (x,y) position, and a 'Z' variable indicating height that allows us to project the vector on a third dimension, this 'z' variable is exactly the same as it would be on a cartesian model. We can express it like such:

$\vec{v} = (\rho, \theta, Z)$

conversion to a cartesian model can be expressed as:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\cos\theta \\ \alpha_y = \rho\sin\theta \quad \textit{Cylindrical to cartesian} \\ \alpha_z = Z \end{cases} \tag{2.26}$$

Spherical coordinates

A spherical coordinate is formed by a tuple:

$$(\rho, \theta, \phi); \begin{cases} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases} \tag{2.27}$$

Where ρ is the magnitude of the vector, θ is the (x,y) coordinates, and ϕ is the (y,z) angle. They must adhere to the following for it to be geometrically coherent:

$$\begin{cases} \rho > 0 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases} \tag{2.28}$$

this tuple can generate two vectors:

$$\begin{cases} \rho\sin\phi \\ \rho \end{cases} \tag{2.29}$$

And can be converted to a cartesian model as such:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\sin\phi\cos\theta \\ \alpha_y = \rho\sin\phi\sin\theta \quad \textit{Spherical to cartesian} \\ \alpha_z = \rho\cos\phi \end{cases} \tag{2.30}$$

Example

Imagine the following spherical vector:

$$\begin{cases} \rho = 2 \\ \theta = \frac{\pi}{2} \\ \phi = \frac{\pi}{4} \end{cases}$$

How do we convert it to a cartesian vector?

We'll get the vector by simply replacing the previous formulas with the values provided as it follows:

$$x = 2\sin(\frac{\pi}{4})\cos(\frac{\pi}{2}) \qquad y = 2\sin(\frac{\pi}{4})\sin(\frac{\pi}{2}) \qquad z = 2\cos(\frac{\pi}{4}) \tag{2.31}$$

We can also invert this equation and get a cartesian vector to its spherical form through the following formula:

$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\frac{y}{x} \\ \phi = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \end{cases} \quad \textit{Cartesian to Spherical} \tag{2.32}$$

Such a model works, as we might imagine, like a sphere. where we express the possible vectors through a sphere of ρ radius.

2.4 n-dimensional Euclidian Spaces

in an n-dimensional euclidian space, we can determine:

$$\mathbb{R}^n, \vec{x}\left(x_1 \ldots x_n\right); \mathbb{C} \textit{ Operations: } \begin{cases} \vec{x}+\vec{y} \\ \alpha\vec{x} \\ \vec{x}\cdot\vec{y} \end{cases} \tag{2.33}$$

2.4.1 Cauchy-Schwartz Inequality

This inequality determines that the internal product is lesser or equal to the multiplication of the norms of two vectors, written as:

$$\text{Let: } \vec{x},\vec{y}\exists\mathbb{R}^n \text{ then:} \tag{2.34}$$

$$|\vec{x}\cdot\vec{y}| \leq ||\vec{x}||\cdot||\vec{y}|| \tag{2.35}$$

and it is equal if and only if:

$$\vec{x} = \lambda\vec{y} \text{ or either } \begin{cases} \vec{x} = 0 \\ \vec{y} = 0 \end{cases} \tag{2.36}$$

2.5 Matrices

A matrix is a numerical representation of values in $\mathbb{R}^n$. They can represent planes, vectors, or even hyperplanes in $\mathbb{R}^n | n > 3$. An example in $\mathbb{R}^2$ would be:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.37}$$

Notable matrices include:

- Identity: $\begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix}$ = Id

2.5.1 Inverible Matrices

We can invert a matrix if a B_{nxn} matrix exists such as:

$$AB = BA = Id \tag{2.38}$$

we can also use the determinant to check this, as:

$$det(A) \begin{cases} = 0\text{: is Invertible} \\ \neq 0\text{: is not Invertible} \end{cases} \tag{2.39}$$

2.5.2 Matrix multiplication

We can multiplicate a matrix by another one if we define the multiplication as:

$$AxB = C_{ij} = \sum_{k=1}^{n} a_{ik} b_{jk} \tag{2.40}$$

Example

We want to multiply two matrices as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{2.41}$$

Therefore if we try doing AxB and BxA:

$$AxB = \begin{pmatrix} a & 2a+b \\ c & 2c+d \end{pmatrix} \tag{2.42}$$

$$BxA = \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix} \tag{2.43}$$

$$\tag{2.44}$$

As we can see, matrix multiplication is not commutative, but rather it is defined by the order on which A and B are written

$$Ae_j = A_j$$

3. Functions and Equations

In mathematics, even though similar, functions and equations

3.1 Geometry of functions with values in $\mathbb{R}$

We can affirm that a function is two or three dimensional if, respectively:

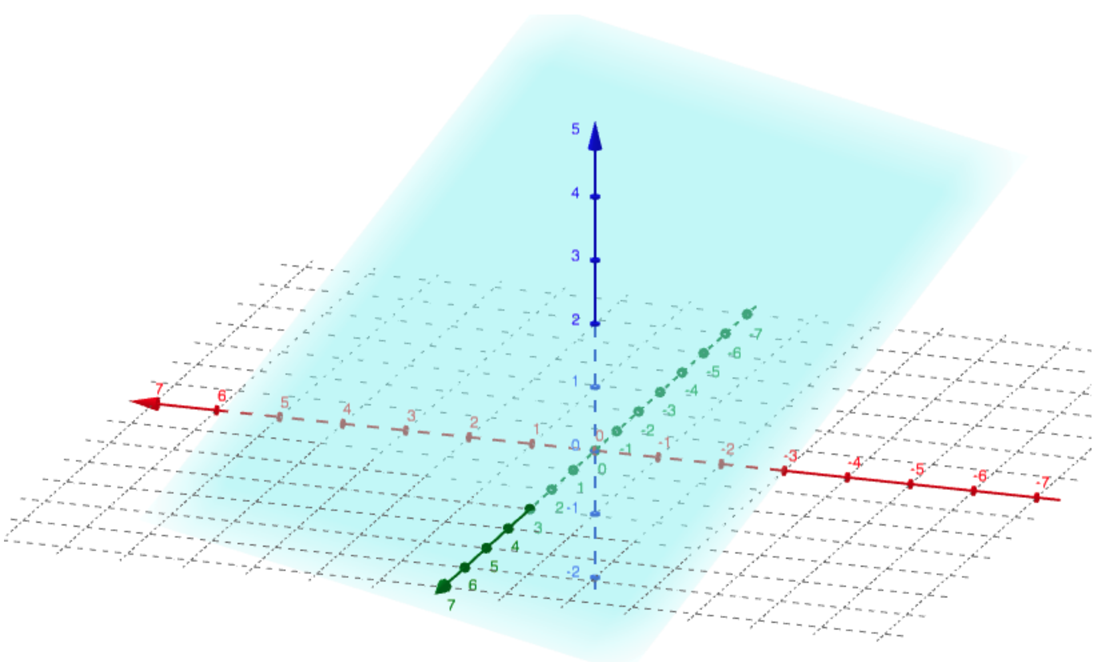
$$y = f(x) \implies \Big\{(x,y) : y = f(x), x \exists D(f) \tag{3.1}$$

$$z = f(x,y) \implies \Big\{(x,y,z) : z = f(x,y), (x,y) \exists D(f) \tag{3.2}$$

Example:

3,1,1: Graphicate the following:

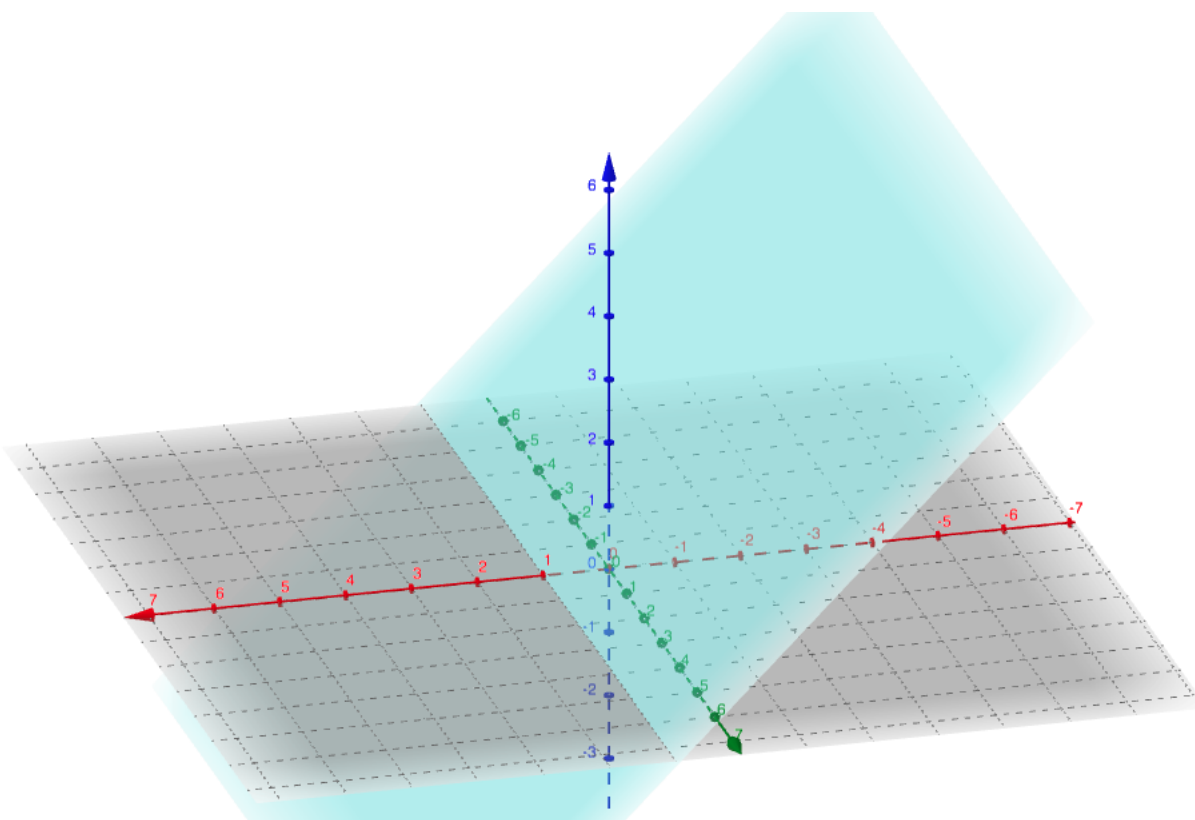
$$z = \frac{6 - x - 2y}{3}$$



As we can see, the graph makes sense because:

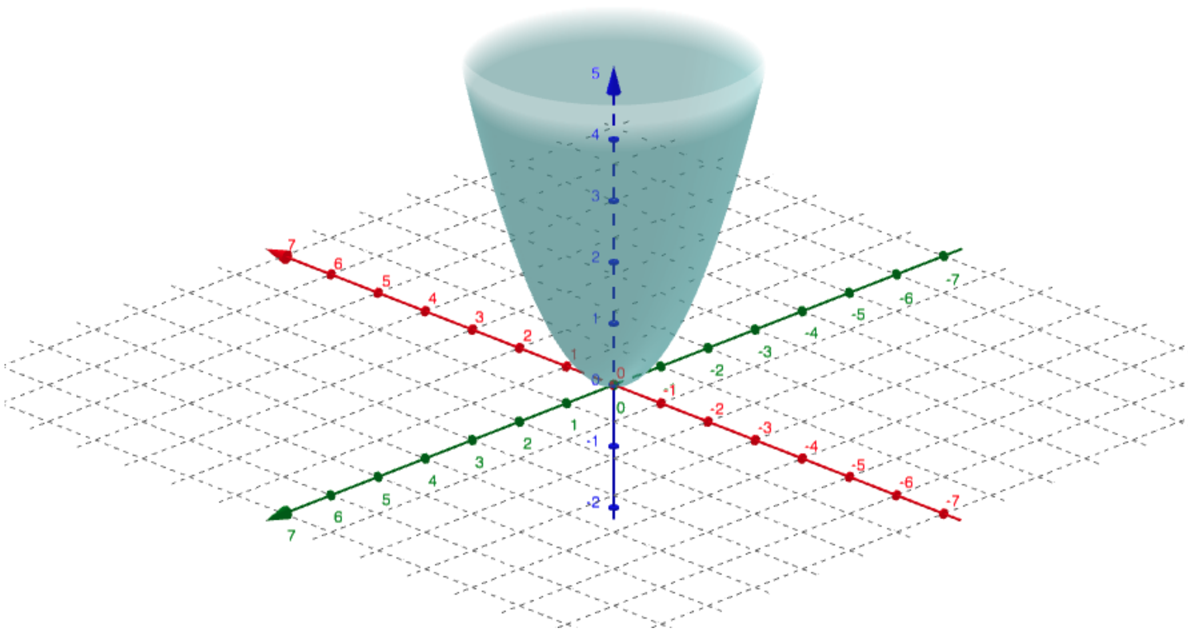
3,1,2: Graphicate the following:

$$z = 1 - x = f(x, y)$$



3,1,3: Graphicate the following:

$$z = x^2 + y^2$$



this is a three-dimensional parabola, also called more correctly as a circular paraboloid, comparable to it's two dimensional form, yet working on another dimension.

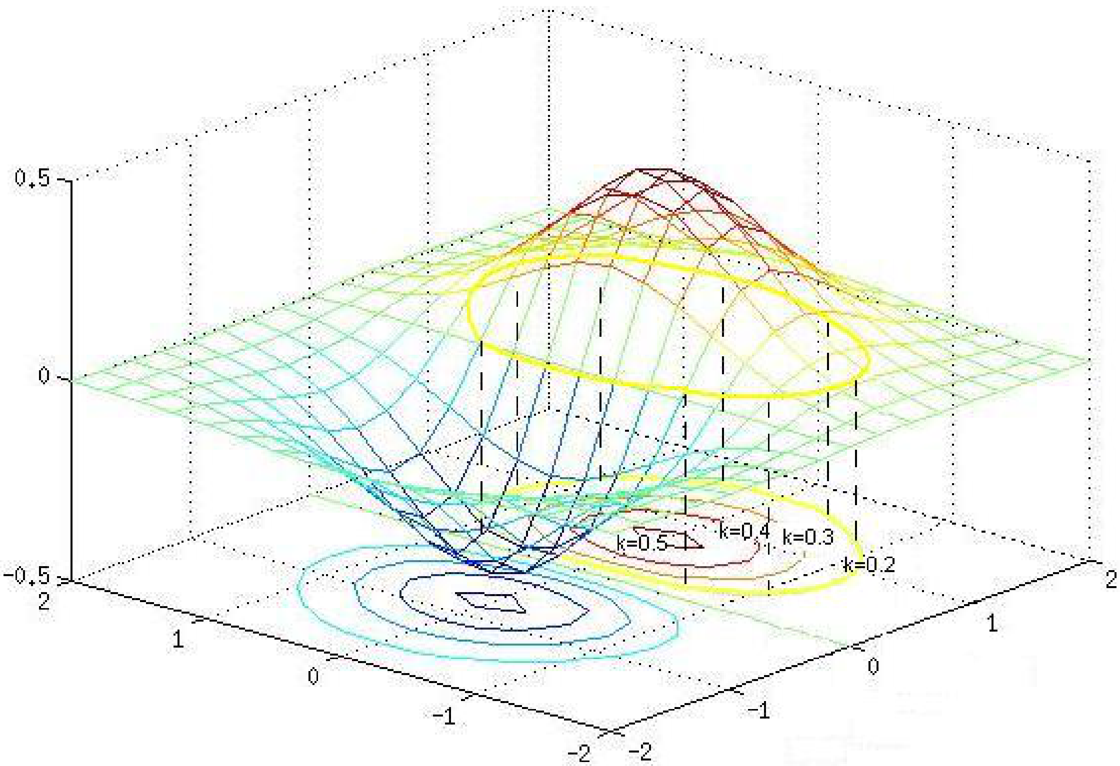
it can guarantee:

$$\begin{cases} x = 0 : z = y^2 \\ y = 0 : z = x^2 \end{cases}$$

and this equation can be deduced through this behavior. As:

$$z = z_0 : x^2 + y^2 = z$$

there are level curves, which project the curve generated by such mathematical artifacts as a two dimensional spherical object. Fields like topology or other 3-dimensional inclined math might regularly use it when measuring space. In any case, a level curve could be illustrated as such:



taken from: www.math.tamu.edu

3.2 Equations

An equation, although similar to a function such as the ones studied so far, can be distinguished by some particularities they present:

$$x^2 + y^2 + z^2 = 1$$

A few examples of objects described by equations are:

3.2.1 Elipsoids

An elipsoid is defined by the following equation:

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1 \tag{3.3}$$

3.2.2 Hyperboloid

a Hyperboloid can be seen as:

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = (\frac{z}{c})^2 + 1 \tag{3.4}$$

3.2.3 Cilinders

The general formula of a cylinder is:

$$(\frac{x}{a})^2+(\frac{y}{b})^2=1 \tag{3.5}$$

The level curves of this shape will never change when projected on a plane.

3.2.4 Parabolic

The general formula of a parabolic is:

$$y=ax^2 \tag{3.6}$$

3.2.5 sphere

$$\begin{cases} x^2+y^2+z^2=r^2 \\ z\geq 0 \end{cases} \tag{3.7}$$

3.2.6 Paraboloid

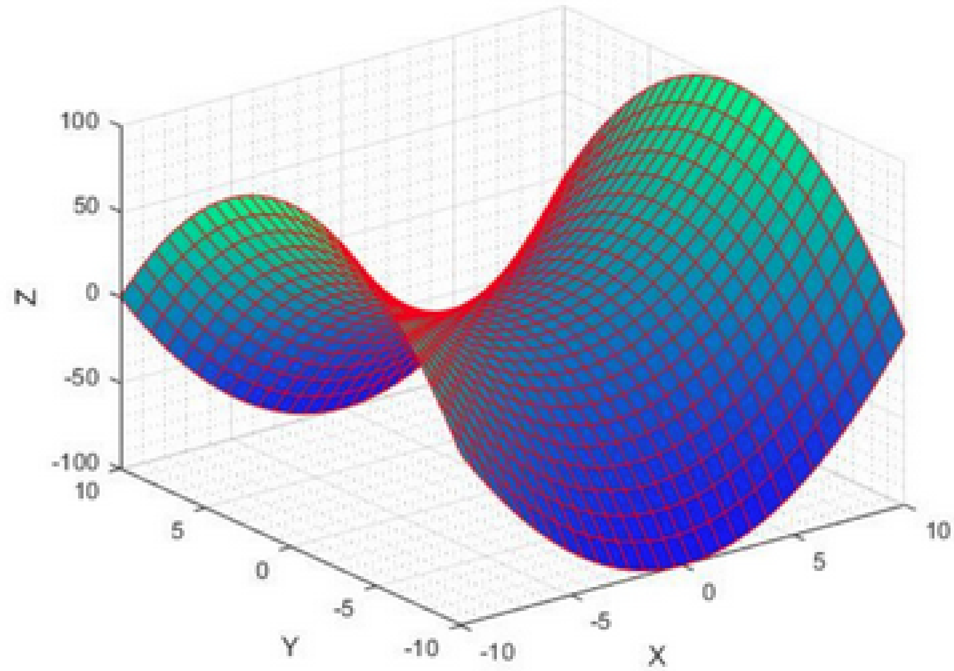
The general formula of a paraboloid is:

$$z=(\frac{x}{a})^2+(\frac{y}{b})^2 \tag{3.8}$$

This shape has a variation, called a hyperbolic paraboloid:

$$z=(\frac{x}{a})^2-(\frac{y}{b})^2 \tag{3.9}$$

It will look like this when graphicated:


100
50
0
-50
-100
Z
10
5
0
-5
-10
Y
-10
-5
0
5
10
X

4. Limits and continuity

Sets and disks

An open disk of radio 'r' and contourn $\vec{x}$ can be defined as

$$\text{Let: } \vec{x}\exists\mathbb{R}^n, r>0 \qquad D_r = \{\vec{x}\exists\mathbb{R}^n : ||\vec{x}-\vec{x_0}|| < r\} \tag{4.1}$$

and a disk that includes the internal values could be defined as

$$\text{Let: } \vec{x}\exists\mathbb{R}^n, r>0 \qquad D_r = \{\vec{x}\exists\mathbb{R}^n : ||\vec{x}-\vec{x_0}|| \leq r\} \tag{4.2}$$

Both of these are an important definition of what we'll call a **set**.
let

$$\mu \subseteq \mathbb{R}^n$$

we can say that μ is an open set if for any point $x_0\exists\mu$ there exists a r>0 such as

$$D_r(x_0) \subseteq \mu$$

Such sets have a mathematical structure called frontier points that can be defined as:

Frontier point

Let $A \subseteq \mathbb{R}^n$; a point '$x\exists\mathbb{R}^n$' is a frontier point if any $\vec{x}$ vecinity contains a point of A and at least a point outside of A

Example

Prove that '$A = (x,y)\exists\mathbb{R}^2 : y > 0$' is an open set

Given that the open set can be inferred by a disk, we can affirm:

$$\implies y > 0 \implies r = \frac{y}{2} > 0 \tag{4.3}$$

$$\text{If}: (a,b)\exists D_r(x,y) \tag{4.4}$$

$$|b-y| \leq \sqrt{(a-x)^2+(b-y)^2} < r = \frac{y}{2} \tag{4.5}$$

$$|b-y| < \frac{y}{2} \tag{4.6}$$

$$\tag{4.7}$$

4.1 Limits

A limit can be defined as

$\lim_{x\to a} f(x) = L$

Where:

$\forall\varepsilon > 0\exists\rho > 0 : |f(x)-L| < \varepsilon$ for all X, such as

$0 < |x-a| < \vartheta$

Or when said in words; "for every epsilon greater than zero that exists in a theta greater than zero, the function of value x minus L is lesser than epsilon for all X, such as the norm is greater than zero and smaller than theta".

For a limit to exist, we need three conditions to be met:

- there is a left limit
- there is a right limit
- they're equal

Multivariable limits

a limit in a multi variable plane can be defined as:

$$\lim_{(x,y)\to(a,b)} = \vec{L} \tag{4.8}$$

Meaning that we can force $|f(x,y)-L|$ to be as near to zero as posible, making (x,y) and (a,b) as close as possible without them actually ever touching each other.

Or, more formally;

$$f : \mathbb{R}^n \to \mathbb{R}^m \tag{4.9}$$

$$\text{Let: } f : A \subseteq \mathbb{R}^n \to \mathbb{R}^n \text{and} : \tag{4.10}$$

$$\text{let } \vec{x_0}\exists A \text{ or } \vec{x_0}\exists \Theta A \tag{4.11}$$

$$\lim_{x\to x_0} f(\vec{x}) = L\exists\vartheta\exists A \implies 0 < ||\vec{x}-\vec{x_0}|| < \vartheta, \implies ||f(\vec{x})-L|| < \varepsilon \tag{4.12}$$

4.1.1 Differentiation

a function in a single variable can have every single point on a curve being approximated to a line, called a tangent. This z = (x,y) line variable can be defined by partial derivatives, that can be described as:

$$z_x = \frac{D_z}{D_x} = \frac{D_f}{D_x}(x,y) = \lim_{h \to 0} = \frac{(x+h,y) - f(x,y)}{h} \tag{4.13}$$

$$\frac{d_f}{d_y}(x,y) = \lim_{h \to 0} = \frac{(x,y+h) - f(x,y)}{h} \tag{4.14}$$

Where 'h' is a real number that tends towards zero.

Example:
let:

$$f(x,y) = x^2 - y^3 - 3x^4 y \tag{4.15}$$

then, differentiate the equation.
solution:

$$\frac{d_f}{d_x} = 2xy^3 - 12x^3 y \tag{4.16}$$

$$\frac{d_f}{d_y} = 3x^2 y^2 - 3x^4 \tag{4.17}$$

When differentiating an equation on a specific variable, we take the other variables as constants. In this example, we leave 'x' untouched when differentiating on 'y' and viceversa.

4.1.2 Implicit differentiation

We can think of an implicit derivative as the process of using partial derivatives to differentiate a single, more complex equation.

Examples:

- a line that goes through $\frac{y-y_0}{x-x_0}$

For three dimensions we might imagine instead of a line being the tangent, a plane providing the same definition and mathematical role as it, defined as:

$$z = f(x,y_0) : z - z_0 = \frac{d_f}{d_x}(x,y_0)(x - x_0) \tag{4.18}$$

$$z = f(x_0,y) : z - z_0 = \frac{d_f}{d_x}(x_0,y)(y - y_0) \tag{4.19}$$

and with both this derivatives, we can define a plane as

$$z - z_0 = a(x - x_0) + b(y - y_0) \textit{Where: } \begin{cases} a = \frac{d_f}{d_y} f(x_0,y_0) \\ b = \frac{d_f}{d_x} f(x_0,y_0) \end{cases} \tag{4.20}$$

IF: $x = x_0 = 0$ and $y = y_0$

We can approximate a three-dimensional differentiation as

$$d_z = fx(x_0,y_0)(x-x_0) + f_y fx(x_0,y_0)(y-y_0) \tag{4.21}$$
$$f(x,y) \approx f(x_0,y_0) + d_z \tag{4.22}$$

note: tangent planes are a very popular test/quiz problem, you should learn how to find them for such ventures, if your professor ever mentions them in class and ESPECIALLY if they try to solve one in front of the class.

We can say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable if:

$$\Delta f = d_f + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{4.23}$$

This deffinition is a bit hard to applicate, so we'll use the differentiation criteria for looking for appliability.

Appliability Criteria: We will say it is possible to differentiate a function if every partial derivative in $\frac{d_f}{d_{xj}}$ and if they're continuous in an open set 'D'.

Example:

$$f(x,y) = \begin{cases} 0 \implies (x,y) = (0,0) \\ x^2+y^2 \implies (x,y) \neq (0,0) \end{cases} \tag{4.24}$$
$$\frac{df}{dx}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \tag{4.25}$$
$$\text{BUT THEN, that means a partial derivative for (0,0) does not exist,} \tag{4.26}$$
$$\text{ therefore this is not a continuous function.} \tag{4.27}$$
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} NE \tag{4.28}$$

4.2 Continual Functions

A function can be defined as continual if a limit can be described as:

- $$\lim_{\vec{x}\to\vec{x}_0} f(x) = f(X_0) = f(\lim_{\vec{x}\to\vec{x}_0} \vec{x})$$

This would mean that a limit is replaceable by a function where the limit of $\vec{x}$ is described according to the mathematical rules that define a function. if a function is continual and has $\frac{D_{fi}}{D_{xj}}$ then it's differentiable

4.3 Tangent Planes

We tangentially touched this topic on implicit differentiation, however, to define more formally a tangent plane,

Let:

$$f : A \subseteq \mathbb{R}^2 \to R \tag{4.29}$$

$$\vec{\Delta} f(x,y) = < \underbrace{\frac{D_f}{D_x}(x,y)}_{\text{Partial Derivative of X}} , \underbrace{\frac{D_f}{D_y}(x,y)}_{\text{Partial Derivative of y}} > \tag{4.30}$$

$$f(x_0,y_0) + D(f(\vec{x}))(\vec{x} - \vec{x_0}) \tag{4.31}$$

4.4 Derivative Properties

4.4.1 Constant multiple rule

$$\frac{\delta}{\delta_x}(cf(x,y)) = c\frac{\delta}{\delta_x}(f(x,y))$$

4.4.2 Sum rule

$$\frac{\delta}{\delta_x}(f(x,y)+g(x,y)) = \frac{\delta_f}{\delta_x}f(x,y) + \frac{\delta_g}{\delta_x}g(x,y)$$

4.4.3 Product rule

$$\frac{\delta}{\delta_x}(f(x,y) \cdot g(x,y)) = \frac{\delta_f}{\delta_x}f(x,y)g(x,y) + \frac{\delta_g}{\delta_x}g(x,y)f(x,y)$$

4.4.4 Divisor rule

$$\frac{\delta}{\delta_x}(\frac{f}{g})(x,y) = \frac{\frac{\delta_f}{\delta_x}f(x,y)g(x,y) - \frac{\delta_g}{\delta_x}g(x,y)f(x,y)}{[g(x,y)]^2}$$

4.4.5 Chain Rule

One variable

If $\vec{u}(x)$ and $\vec{x}(t)$ are differentiable, then:

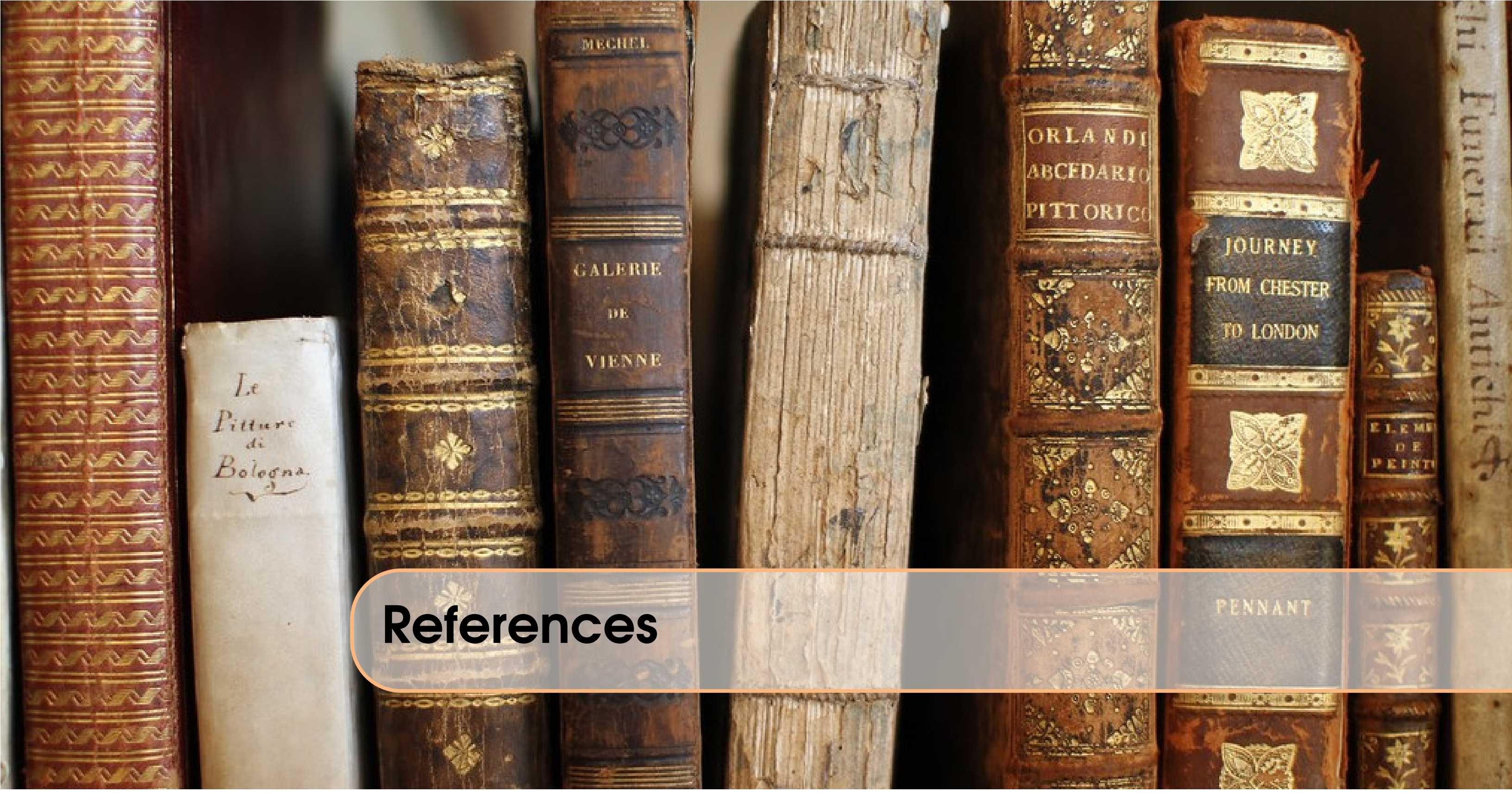
$$\frac{\delta_u}{\delta_t} = \frac{d_y}{d_x}\frac{d_x}{d_t} = u'(x|t|) + x'(t)$$

Two variables

Given $\vec{w}(x,y)$, $\vec{u}(x,y)$ and $\vec{v}(x,y)$, then $w(u(x,y),v(x,y))$ is differentable.

so, therefore:

$$y = \begin{cases} \frac{\delta w}{\delta x} = \frac{\delta w}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta w}{\delta v}\frac{\delta v}{\delta x} \\ \frac{\delta w}{\delta y} = \frac{\delta w}{\delta u}\frac{\delta u}{\delta y} + \frac{\delta w}{\delta v}\frac{\delta v}{\delta y} \end{cases} \tag{4.32}$$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References