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ME40064: Systems Modelling & Simulation
ME50344: Engineering Systems Simulation

Assignment 2

Summary

In this coursework you must extend your finite element code to be able to solve the transient form of the diffusion-reaction equation:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + \lambda c + f$$

You will then use this code to model the effects of heat damage to human skin, thereby being able to estimate the level of thermal shielding that a piece of protective clothing must provide.

Your code may use either the backward Euler or Crank-Nicolson scheme for time integration, but you must state which, and in your answers to the following specific tasks, comment on any implications your choice may have had on the accuracy of your solution.

Note that you may either re-use your existing functions to calculate element matrices, i.e. that represent the analytically integrated forms, or you can re-write them to use Gaussian quadrature to evaluate the local element integrals. You should not use Matlab's in-built functions for numerical or symbolic integration. Using the Gaussian quadrature method in your code will attract extra credit.

Part 1: Software Verification

Check that your code is working correctly by solving the following transient diffusion equation, for the domain defined from x = 0 to 1, and subject to the following initial and Dirichlet boundary conditions, and compare it to the analytical solution:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}$$

$$x = [0, 1], \quad c(x, 0) = 0, \quad c(0, t) = 0, \quad c(1, t) = 1$$

In order to achieve good accuracy in your solution, use an element size of 0.1 (i.e. a 10 element mesh) and a time step, **Δ**t = 0.01, (particularly if you choose to use the Crank-Nicolson scheme). As shown in Lecture 13, this problem has the following analytical solution:

$$c(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin(n\pi x)$$

A Matlab function to compute this analytical expression, `TransientAnalyticSoln.m`, is provided on Moodle.

Create the following two figures to demonstrate that your code is correct:

a) Plot your solution c(x) vs. x, showing the solutions at t = 0.05, 0.1, 0.3, 1.0, in the format shown in Lecture 13.
b) Plot both the analytical solution and your numerical solution at x=0.8, for t = 0 to 1.0.

Extra credit is available for the following (if used, be sure to mention this in your report)

- Appropriate use of unit tests throughout your code
- Testing both backward Euler and Crank-Nicolson time stepping methods
- Gaussian quadrature
- Implementing quadratic basis functions
- Investigating errors & convergence using the L2 norm

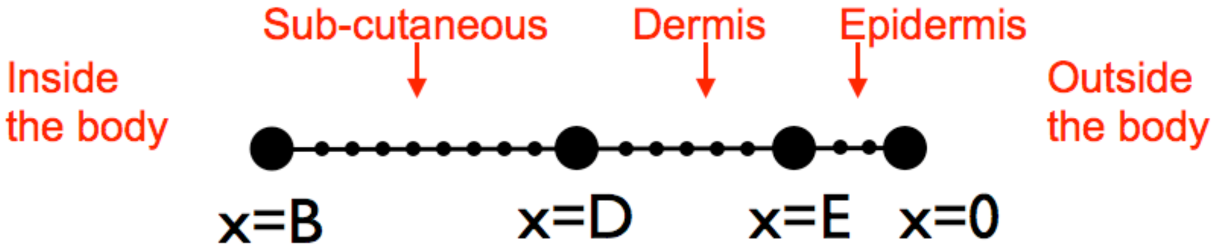
Part 2: Modelling & Simulation Results

1. Now use your code to solve the tissue burn model introduced in Lecture 14. This has a governing equation of:

$$\frac{\partial T}{\partial t} = \left(\frac{k}{\rho c}\right)\frac{\partial^2 T}{\partial x^2} - \left(\frac{G\rho_b c_b}{\rho c}\right)T + \left(\frac{G\rho_b c_b}{\rho c}\right)T_b$$

Note that you may find the backward Euler much faster than Crank-Nicolson for achieving stable results – for backward Euler a time step of 0.005 is likely to be sufficient but you should double-check this for your solutions and particular mesh resolution.

Your finite element mesh and material parameters must represent the three-layered structure of skin, as follows:



Parameter values that you should use in each layer are provided in Table 1 – with the exception that in this question we assume zero blood flow i.e. **G=0 everywhere**. The layers of the tissue are defined at the following x coordinates: **E=0.00166667, D=0.005, B=0.01.**

a) Run your code for the following initial conditions and Dirichlet boundary conditions, for a maximum of 50 seconds:

$$T(x,0) = 310.15K, \quad T(x=B,t) = 310.15K, \quad T(x=0,t) = 393.15K$$

Create a plot showing the spatial temperature distribution in the tissue for various points in time between 0 and 50 seconds. Explain the shape of the curves as they change through time based on your knowledge of the physics, initial conditions, and boundary conditions.

b) Once you have solved the temperature distribution in space and time, use this information to determine the level of tissue damage that will result. To do this, evaluate the following integral numerically at the mesh node for x=E:

$$\Gamma = \int_{t_{burn}}^{t} 2 \times 10^{98} \exp\left(-\frac{12017}{(T - 273.15)}\right) dt$$

Integrate between the time points, t_{burn}, at which the temperature T becomes greater than 317.15 Kelvin, and time, t (which will be 50 seconds for all these questions).

Note that the Matlab function, `trapz(x)`, uses the trapezium rule to integrate the vector x, assuming an interval of 1. Therefore multiply the output by the size of your timestep, **Δ**t, to obtain the final value of this integral.

If $\Gamma > 1$ at x=E, this produces a second-degree burn. State the value of Γ you have calculated at this node (don't be alarmed if it is very large or small), and hence state whether you expect a second-degree burn to occur.

2. Now use your code to determine the minimum temperature reduction (to the nearest 0.5 degree K) that must be achieved by the protective clothing at the boundary x=0, in order that second-degree burns, as measured at x=E, are not caused by t=50s. State the final Dirichlet boundary condition at x=0 that achieves this, and explain how you used your code, and any other calculations, to estimate this value. Include any data or figures that you feel are relevant in order to demonstrate your approach to estimating this.

3. Re-run your results for question 2, but now including the blood flow related reaction & source terms – use the values in Table 1 for each region of tissue. How much does this change the temperature distribution in space and time? Do you consider this to be an important effect to consider for future modelling of this problem?

4. Other than changing the geometry to a 3D representation of the body, comment briefly on the accuracy of the modelling assumptions and suggest how these could be modified in order to provide more realistic results?

Parameter	**Epidermis**	**Dermis**	**Sub-cutaneous**
k	25	40	20
G	0	0.0375	0.0375
ρ	1200	1200	1200
c	3300	3300	3300
ρ_b	-	1060	1060
c_b	-	3770	3770
T_b	-	310.15	310.15

Table 1: Parameter values for questions 2, 3 and 4. Note that some are realistic and others are not, having been chosen to allow you to solve your model in a reasonable time frame.

SUBMISSION GUIDELINES

Structure your report as a set of answers to these questions – there is no requirement to write this in a lab report format. However your report must be self-contained and therefore must not assume that the reader knows the content in this document.

- You **must** include all your Matlab source code as **text** in the Appendices. **Do not** paste your code into the document as an OLE or as an image.
- **Do not** upload zipped or compressed folders of these source files.
- Your code should use meaningful variable names and include comments, in line with good practice.
- Word limit of **2000** words (not including source code)

Submit your work using the online submission function on the unit's Moodle page.

Deadline: 4pm on Wednesday, 5th December 2018.

Marks will be awarded based on the following criteria:

- Correctness of numerical results
- Demonstration of knowledge of material presented in the course including more advanced features of the finite element method
- Clear presentation of results
- Correctness & readability of Matlab code
- Level of insight & clarity of explanation of the meaning of the results