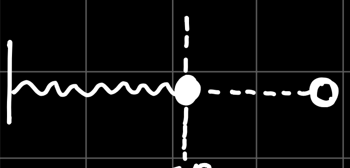
Outline

- motivation why QFT?
  - QFT ≠ PT (perturbation Theory)
  - EFT (effective field theory)
- spin-0, spin-$\frac{1}{2}$, spin-1 Fields (唯一能量子化的场)

  引力的自旋为2，无法量子化
- Renormalization
- Scattering Theory, Perturbation Theory
- QED & Feynman Diagram & Path Integral

Warm-Up: A Review of QM not in a QM course

- Harmonic Oscillator (1-D) 有精确解，可以用来做近似



Classic: $L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2$

广义坐标

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$$

$$H = \frac{p^2}{2} + \frac{q^2}{2}\omega^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \Rightarrow \ddot{q} + \omega^2 q = 0$$

$$\Rightarrow q = a e^{-i\omega t} + a^* e^{i\omega t} + \text{B.C.}$$

p, q 都是数, $[p, q] = 0$

Quantum: $\hbar \equiv 1$

$$q \to \hat{q}, \quad p \to \hat{p}, \quad [\hat{p}, \hat{q}] = i$$

正则量子化

canonic Quantization

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 \Rightarrow \ddot{q} + \omega^2 q = 0$$

$$\Rightarrow \hat{q} = \sqrt{\frac{1}{2\omega}}\,(\underbrace{e^{i\omega t}\hat{a}^\dagger}_{q^-} + \underbrace{e^{-i\omega t}\hat{a}}_{q^+}) + \text{B.C.}$$

$$\hat{p} = \frac{\partial q}{\partial t} = \sqrt{\frac{1}{2\omega}}\, i\omega\,(e^{i\omega t}\hat{a}^\dagger - e^{-i\omega t}\hat{a})$$

$$\Rightarrow \hat{a} = \sqrt{\frac{\omega}{2}}\left(q - \frac{p}{i\omega}\right)e^{-i\omega t}$$

$$\hat{a}^\dagger = \sqrt{\frac{\omega}{2}}\left(q + \frac{p}{i\omega}\right)e^{i\omega t}$$

$$\Rightarrow [\hat{a}, \hat{a}^\dagger] = 1$$

产生湮灭算符就是系数

transmition amplitude

at $t=t_i$, $X=X_i$ at $t=t_f$?

Newton: $\bullet - - - - - - - \bullet$

$t=t_i$, $X=X_i$ $\qquad$ $t=t_f$

QM: Anywhere

就是振幅的模方

Q: find the amplitude for when $t=t_f$, particle at x_f.

$$\langle x_f | e^{-iH(t_f-t_i)} | x_i \rangle$$

$$= \sum_{n,m} \langle x_f | n \rangle \langle n | e^{-iHT} | m \rangle \langle m | x_i \rangle$$

$$= \sum_n \psi_n^*(x_f)\, \psi_n(x_i)\, e^{-iE_nT}$$

$$\xrightarrow[\text{if } x_f=x_i]{} \int dx \sum_n |\psi_n|^2 e^{-iE_nT} = \sum_n e^{-iE_nT} \equiv K$$

$$\xrightarrow[\text{wick rotation}]{T\to -iT} \sum_n e^{-E_nT} \xrightarrow{T\to\infty} e^{-E_0T} + e^{-E_1T} + \cdots$$

解析延拓

算基态 $E_0 = \lim\limits_{T\to\infty} -\dfrac{\ln K}{E_0}$

$$H = \frac{p^2}{2} + \frac{q^2}{2}\omega^2 = (a^\dagger a + \tfrac{1}{2})$$

time-independent

$$\Rightarrow [H, a] = -a\omega \quad [H, a^\dagger] = a^\dagger\omega$$

Assume $|n\rangle$ is the eigenstate of H:

$$H|n\rangle = E_n|n\rangle$$

$$\Rightarrow H(a|n\rangle) = aH|n\rangle - \omega a|n\rangle = (E_n - \omega)a|n\rangle$$

$\Rightarrow$ $a|n\rangle$ is also the eigenstate of H. the eigenvalue is $E_n - \omega$

Similarly: $H(a^\dagger|n\rangle) = (E_n+\omega)a^\dagger|n\rangle$

$\Rightarrow$ $a^\dagger|n\rangle$ is the eigenstate of H, the eigenvalue is $E_n+\omega$.

$$H\hat{a}^N|n\rangle = (E_n - N\omega)\hat{a}^N|n\rangle \quad \cancel{N\to\infty}$$

$\Rightarrow$ 存在基态 $|0\rangle$

$\rightarrow$ Fock Space

引入 $|0\rangle$，有产生湮灭算符就确定了系统

$$|n\rangle = \frac{a^{\dagger n}}{(n!)^{1/2}}|0\rangle$$

归一化系数.

解波函数: $\psi_n(q) = \langle q | n \rangle$ q表象

$$\langle q | a | 0 \rangle = 0$$

$$\langle q | q - \frac{p}{i\omega} | 0 \rangle = 0$$

$$\Rightarrow \left(q + \frac{1}{\omega}\frac{\partial}{\partial q}\right)\varphi_0(q) = 0$$

$$\Rightarrow \frac{d}{dq}\varphi_0 = -\omega q \varphi_0$$

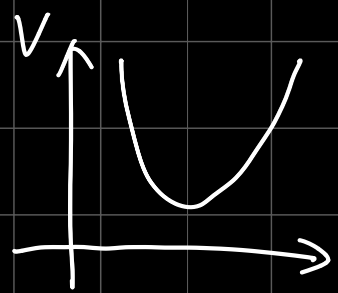
$$\Rightarrow \frac{d\varphi_0}{\varphi_0} = -\omega q\, dq$$

$$\Rightarrow \ln \varphi_0 = -\frac{1}{2}\omega q^2 + C$$

$$\Rightarrow \underline{\varphi_0 = C e^{-\frac{1}{2}\omega q^2}}$$ 基态波函数

激发态 $\varphi_n = \langle q | \frac{a^{\dagger n}}{\sqrt{n!}} | 0 \rangle$

$$= \frac{C}{\sqrt{n!}} \left[\sqrt{\frac{\omega}{2}} \left(q - \frac{1}{\omega}\frac{\partial}{\partial q} \right) \right]^n e^{-\frac{1}{2}\omega q^2}$$



$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{1}{4!}\lambda q^4 \qquad \lambda \to 0 \quad \text{perturbation}$$

量子力学和量子场论的微扰做的越精确越远离

Ground Energy. interaction picture 相互作用绘景

S eq. $i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle = (H_0 + V)|\psi\rangle$ 　　已知 $H_0|n\rangle = E_n^{(0)}|n\rangle$

$\uparrow$ Schrödiger picture

introduce $|\psi\rangle_I \equiv e^{iH_0 t}|\psi, t\rangle$

看 $|\psi\rangle_I$ 随时间的演化

$$\Rightarrow -i\frac{\partial}{\partial t}|\psi\rangle_I = H_0 e^{iH_0 t}|\psi(t)\rangle + e^{iH_0 t}(H_0+V)|\psi(t)\rangle$$

$$= e^{iH_0 t} V e^{-iH_0 t} e^{iH_0 t}|\psi(t)\rangle \equiv V_I(t)|\psi\rangle_I$$

$$\Rightarrow |\psi_I(t)\rangle = e^{-i\int dt\, V_I(t)}|\psi_I(0)\rangle$$

$$= \underline{\left[1 - i\int dt\, V_I(t) + \frac{(-i)^2}{2}\int dt\, V_I(t)\int dt'\, V_I(t') + \cdots \right]}|\psi_I\rangle$$

阶表

transition amplitude: $\langle x_f | x_i(t)\rangle_{t=t_f-t_i}$

$$\langle x_f | e^{-iH_0 t} | x_i(t)\rangle_I \big|_{t=T}$$

$$= \langle x_f | e^{-iH_0 T} e^{-i\int dt\, V_I(t)} | x_i\rangle$$

$$= \sum_{n,l,m} \langle x_f|n\rangle\langle n|e^{-iH_0 T}|m\rangle\langle m|\exp[-i\int dt\, V_I]|l\rangle\langle l|x_i\rangle$$

$$= \sum_{n,l} \underline{\varphi_n^*(x_f)} e^{-iE_n T}\langle n|\exp[-i\int dt\, V_I]|l\rangle \underline{\varphi_l(x_i)}$$

$$\xrightarrow{\text{积分}} \sum_n e^{-iE_n^{(0)}T}\langle n|\exp[-i\int dt\, V_I]|n\rangle$$ 　　积分 $\int_{nl}$

$$\xrightarrow{T\to -iT} \sum_n e^{-E_n^{(0)}T}\langle n|\exp[-i\int dt\, V_I]|n\rangle$$ 像配分函数 　　假设能量都大于0

$$\xrightarrow{T\to\infty} e^{-E_0^{(0)}T}\langle n|\exp[-i\int dt\, V_I]|n\rangle$$

$E_0^{(0)}=\frac{1}{2}\omega$ $\quad = e^{-E_0 T}$

LO (leading order) in λ, λ^0. $\langle 0|1|0\rangle = 1 \quad E_0 = E_0^{(0)} = \frac{1}{2}\omega$

NLO (第一阶) in λ, λ, $\langle 0|(-i)\int_0^T dt\, V_I|0\rangle$

$$\xrightarrow[T\to -iT]{t\to -i\tau} \langle 0|(-i)\int_0^{-iT} d(-i\tau)\, V_I(-i\tau)|0\rangle$$

$$= -\int_0^T \langle 0|V_I|0\rangle\, d\tau$$

$$\langle 0|V_I|0\rangle = \langle 0|e^{iH_0 t} V e^{-iH_0 t}|0\rangle$$
$$= e^{iE_0^{(0)}t}\langle 0|V|0\rangle e^{-iE_0^{(0)}t}$$
$$= \langle 0|\frac{1}{4!}\lambda q^4|0\rangle$$
$$= \frac{\lambda}{4!}\langle 0|\left[\sqrt{\frac{1}{2\omega}}\left(e^{\omega\tau}a^\dagger + e^{-\omega\tau}a\right)\right]^4|0\rangle$$
$$= \frac{\lambda}{4!}\frac{1}{4\omega^2}\langle 0|e^{-\omega\tau}a\begin{Bmatrix} a e^{-\omega\tau}\, a^\dagger e^{\omega\tau} \\ a^\dagger e^{\omega\tau}\, a e^{-\omega\tau}\end{Bmatrix} a^\dagger e^{\omega\tau}|0\rangle$$
$$= \frac{\lambda}{4!}\frac{1}{4\omega^2}(1+2) = \frac{\lambda}{4!}\frac{3}{4\omega^2} \text{ 不含时}$$

$$e^{-E_0^{(0)}T}\left(1-\frac{\lambda}{4!}\frac{3T}{4\omega^2}\right) = e^{-E_0 T}$$

$$\Rightarrow -E_0^{(0)}T + \ln\left(1-\frac{\lambda}{4!}\frac{3T}{4\omega^2}\right) = -E_0 T$$

$$\Rightarrow -E_0^{(0)}T - \frac{\lambda}{4!}\frac{3T}{4\omega^2} = -E_0 T$$

$$\Rightarrow E_0 = E_0^{(0)} + \frac{\lambda}{4!}\frac{3}{4\omega^2}$$

NNLO 第二阶. λ^2 $\quad \sum_m \int_0^T d\tau \langle 0|V_I(\tau)|m\rangle \int_0^\tau d\tau' \langle m|V_I|0\rangle$

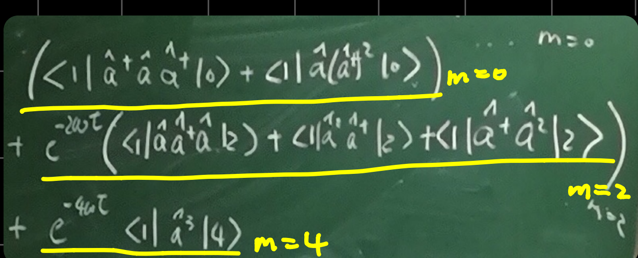
$$= \sum_m \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \int_0^T d\tau' \int_0^\tau d\tau\, \langle 0|e^{H_0\tau} V e^{-H_0\tau}|m\rangle\langle m|e^{H_0\tau'} V e^{-H_0\tau'}|0\rangle$$

$$= \sum_m \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \int_0^T d\tau' \int_0^\tau d\tau\, \langle 0|V|m\rangle\langle m|V|0\rangle e^{-E_m^{(0)}(\tau-\tau')} e^{+E_0^{(0)}(\tau-\tau')}$$

$$= \sum_m \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \int_0^T d\tau' \int_0^\tau d\tau\, \langle 0|V|m\rangle\langle m|V|0\rangle e^{-m\omega(\tau-\tau')}$$

$$= \left(\frac{9}{2}T^2 + \frac{72T}{4\omega} + \frac{24}{8\omega}T\right)\left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2$$

$$\langle 0|V|m\rangle = \langle 0|\left(e^{\omega\tau}a^\dagger + e^{-\omega\tau}a\right)|m\rangle$$
$$= \langle 0|e^{-\omega\tau}a\left(e^{\omega\tau}a^\dagger + e^{-\omega\tau}a\right)^3|m\rangle$$
$$=$$



$\langle 0|V|m\rangle\langle m|V|0\rangle =$ 三行各自平方相加

$$= (1+2)^2 + (6\sqrt{2})^2 e^{-4\omega(\tau-\tau')}$$
$$+ 4!\, e^{-8\omega(\tau-\tau')}$$
$$= 9 + 72 e^{-4\omega(\tau-\tau')} + 24 e^{-8\omega(\tau-\tau')}$$

$$-E_0 T = -E_0^{(0)}T + \ln\left(1 - \frac{\lambda}{4!}\frac{3T}{4\omega^2} + \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2\left(\frac{9}{2}T^2 + \frac{21}{\omega}T\right)\right)$$

$$= -E_0^{(0)}T - \frac{\lambda}{4!}\frac{3T}{4w^2} - \frac{1}{2}\left(\frac{\lambda}{4!}\frac{3T}{4w^2}\right)^2 + \left(\frac{\lambda}{4!}\frac{1}{4w^2}\right)^2\left(\frac{9}{2}T^2 + \frac{21}{\omega}T\right)$$

$$\Rightarrow E_0 = E_0^{(0)} + \frac{\lambda}{4!}\frac{3}{4w^2} - \left(\frac{\lambda}{4!}\frac{1}{4w^2}\right)^2\frac{21}{\omega} + \cdots\cdots + \lambda^n$$

修正交错

$$\Delta E^{(n)} \propto (-1)^{n+1}\frac{3^{n+\frac{1}{2}}}{2\pi^{3/2}}\Gamma(n+\tfrac{1}{2})\lambda^n$$ — Benden & T.T. Wu 1969

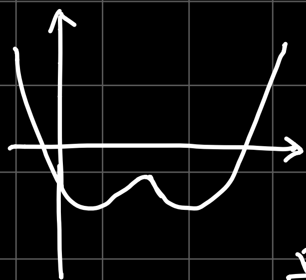
渐近自由

微扰论 asymptotic series 渐近序列 not convergent

—Borel summation

What if $H = \frac{1}{2}p^2 + \frac{1}{2}q^2 - \frac{\lambda}{4!}q^4$?

微扰论能不能用？



$\xrightarrow{PT}$ degenerate

$\xrightarrow{real}$ non-degenerate

基态简并？

量子隧穿破坏简并

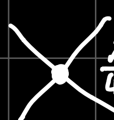
$$E^{(1)} - E^{(0)} = e^{-A/\lambda}$$

微扰论不可能算出这样的结果

—路径积分

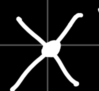
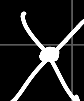
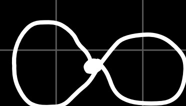
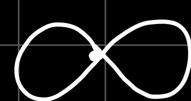
Feynman Diagram

$$3^2 + 72e^{-4w(\tau-\tau')} + 4!\,e^{-8\omega(\tau-\tau')}$$

$\frac{\lambda}{4!}q^4 \equiv$  $\frac{\lambda}{4!}$

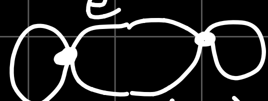
τ —— τ' $\equiv e^{-2w(\tau-\tau')}$

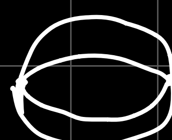
disconnected

① $\langle 0|$()$|0\rangle\langle 0|$()$|0\rangle \longrightarrow$  $\times$ 

bubble

$$\left(\frac{\lambda}{4!}\right)^2(3)^2$$

② 

③  $e^{-8w(\tau-\tau')}\left(\frac{\lambda}{4!}\right)^2\cdot 24$

$$e^{-4w(\tau-\tau')}\left(\frac{\lambda}{4!}\right)^2\cdot N^{=72}$$

所有的可能性

—维谐振子的量子化场

研究编时乘积

$$T[q(t_1), q(t_2)] = q(t_1)q(t_2)\theta(t_1-t_2) + q(t_2)q(t_1)\theta(t_2-t_1)$$

$$= \frac{1}{2\omega}\left[aa e^{-i\omega(t_1+t_2)} + \underline{\underline{aa^+ e^{-i\omega(t_1-t_2)}}} + a^+a e^{+i\omega(t_1-t_2)} + a^+a^+ e^{i\omega(t_1+t_2)}\right]\theta(t_1-t_2) + (1\leftrightarrow 2)$$

$$= \frac{1}{2\omega}\left(\cdots + a^+a e^{-i\omega(t_1-t_2)} + \cdots\right)\theta(t_1-t_2) + \frac{1}{2\omega}e^{-i\omega(t_2-t_1)}\theta(t_1-t_2) + \frac{1}{2\omega}(\cdots)\theta(t_2-t_1) + e^{i\omega(t_1-t_2)}\frac{1}{2\omega}\theta$$

$$= \frac{1}{2\omega}\left(\cdots + a^+a e^{-i\omega(t_1-t_2)} + \cdots\right) + \frac{1}{2\omega}e^{-i\omega(t_2-t_1)}\theta(t_1-t_2) + \frac{1}{2\omega}e^{i\omega(t_1-t_2)}\theta(t_2-t_1)$$

定义是 a^+在左边

$$= :q(t_1)q(t_2): + \boxed{e^{-i\omega|t_1-t_2|}\frac{1}{2\omega}} = \Delta F(t_1-t_2)$$

费曼传播子（一个数）

$$= :q(t_1)q(t_2): + \underbrace{q(t_1)q(t_2)}_{\text{contraction}}$$

$$\underbrace{q(t_1)q(t_2)} = \begin{cases} [q^+(t_1), q^-(t_2)], & t_1 > t_2 \\ [q^+(t_2), q^-(t_1)], & t_2 > t_1 \end{cases}$$

Wick's Theorem: $T[q(t_1), \cdots q(t_n)] = :q(t_1)q(t_2)\cdots q(t_n):$

$+ :\text{All possible contractions}:$

1° $T[q(t_1)] = q(t_1) = :q(t_1): + 0$

2° $T[q(t_1), q(t_2)] = :q(t_1)q(t_2): + \underbrace{q(t_1)q(t_2)}$

3° $T[q(t_1), q(t_2), q(t_3)]$ 不妨假设 $t_1 > t_2,\ t_1 > t_3$

$$= q(t_1)T[q(t_2), q(t_3)]$$

$$= q(t_1):q(t_2)q(t_3): + q(t_1):\underbrace{q(t_2)q(t_3)}:$$

对于未来说可以不要

$$= q(t_1):q(t_2)q(t_3): + :q(t_1)\underbrace{q(t_2)q(t_3)}:$$

$$= :q_1q_2q_3: + :\underbrace{q_1q_2}q_3: + :\underbrace{q_1q_2q_3}: + :q_1\underbrace{q_2q_3}:$$

$$q(t_1):q(t_2)q(t_3):$$

$$= (q_1^+ + q_1^-):q_2q_3:$$

$a \qquad a^\dagger \qquad a^\dagger a^\dagger \cdots aa \cdots$

$$= :q_1^- q_2 q_3: + q_1^+ :q_2 q_3:$$

$$= :q_1^- q_2 q_3: + :q_1^+ q_2 q_3: + :[q_1^+, q_2^-] q_3: + :q_2 [q_1^+, q_3^-]:$$

$$= :q_1^- q_2 q_3: + :q_1^+ q_2 q_3: + :\underbrace{q_1 q_2} q_3: + :\underbrace{q_1 q_2 q_3}:$$

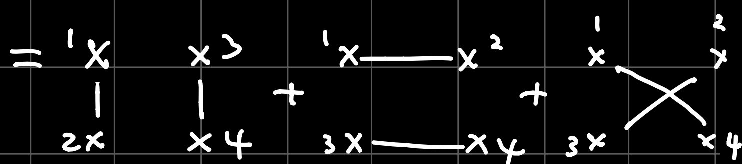
$$= :q_1 q_2 q_3: + :\underbrace{q_1 q_2} q_3: + :\underbrace{q_1 q_2 q_3}:$$

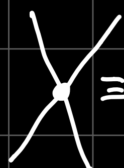
$$\langle 0 | T[q(t_1), q(t_2), \cdots, q(t_n)] | 0 \rangle = :\text{All possible full contractions}:$$

有余项就不行

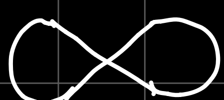
Examples: 1° $\langle 0 | T[q_1, q_2, q_3, q_4] | 0 \rangle = \underbrace{q_1 q_2} \underbrace{q_3 q_4} + \underbrace{q_1 q_2 q_3} q_4 + \underbrace{q_1 q_2 q_3 q_4}$

$= \Delta F(t_1 - t_2) \Delta F(t_3 - t_4) + \Delta F(t_1 - t_3) \Delta F(t_2 - t_4) + \Delta F(t_1 - t_4) \Delta F(t_2 - t_3)$

$=$ 

2°  $\equiv \int_0^T d\tau \left(-\frac{\lambda}{4!}\right)$

$\equiv \Delta F = \frac{1}{2\omega} e^{-|t_1 - t_2| \omega}$

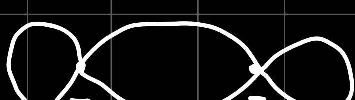


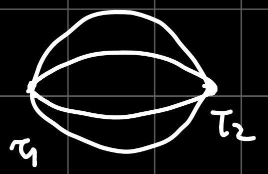
$$\left(-\frac{\lambda}{4!}\right) \int_0^T dt \; \Delta F^2(0) \cdot 3$$

3种可能

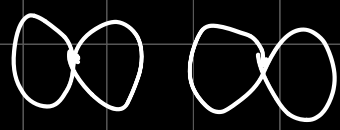
$$= -\frac{\lambda}{4!} 3 \left(\frac{1}{2\omega}\right)^2 T$$

3° $\frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \int_0^T d\tau_1 d\tau_2 \langle 0 | T\{V(\tau_1) V(\tau_2)\} | 0 \rangle$


$\frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \int_0^T d\tau_1 d\tau_2 \, \Delta F^2(0) \left(\frac{1}{2\omega}\right)^2 e^{-2\omega|\tau_1 - \tau_2|} \cdot 72$



$$\frac{1}{2}\left(-\frac{\lambda}{4!}\right)^2\int_0^T d\tau_1 d\tau_2 \left(\frac{1}{2w}\right)^4 e^{-4w(\tau_1-\tau_2)}\cdot 4!$$



$$\frac{1}{2!}\left[\left(-\frac{\lambda}{4!}\right)3\left(\frac{1}{2w}\right)^2 T\right]^2$$

欧拉-Lagrangian方程 $\ddot{q}+w^2q=0 \longrightarrow q_0$

Green's function $\ddot{q}+w^2q=\delta(t) \longrightarrow G$

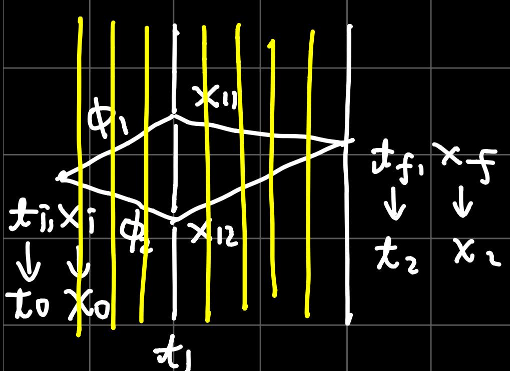
$$\ddot{q}+w^2q=F(t) \Rightarrow q=q_0+\int G(t-t')F(t')\,dt'$$

$\ddot{q}+w^2q=\delta(t)$ Fourier Transform $\Rightarrow G_F = i\left(\frac{1}{2w}\right)^2 e^{-iw|t|}$

$= i\Delta_F(t)$

Feynman Path Integral

Double slits experiment



$$\langle x_2, t_2 | x_{11}, t_1\rangle\langle x_{11}, t_1 | x_0, t_0\rangle$$
$$+\langle x_2, t_2 | x_{12}, t_1\rangle\langle x_{12}, t_1 | x_0, t_0\rangle$$
$$= |\phi_1+\phi_2|^2$$ 双缝，一个板

$$\xrightarrow{\text{缝为}n} \sum_{i=1}^{n}\langle x_2, t_2 | x_{1i}, t_1\rangle\langle x_{1i}, t_1 | x_0, t_0\rangle$$

$$\xrightarrow{n\to\infty} \propto \int dx_1 \langle x_2, t_2 | x_1, t_1\rangle\langle x_1, t_1 | x_0, t_0\rangle$$

$$\xrightarrow{\text{板为}N} \propto \int dx_1 dx_2\cdots dx_{N-1}\prod_{i=0}^{N-1}\langle x_{i+1}, t_{i+1} | x_i, t_i\rangle$$

$$\boxed{\langle x_{i+1}, t_{i+1} | x_i, t_i\rangle = \langle x_{i+1}| e^{-i\hat{H}(t_{i+1}-t_i)} | x_i\rangle}$$

$$= \int dx_1 dx_2\cdots dx_{N-1}\prod_{i=0}^{N-1}\langle x_{i+1}| e^{-i\hat{H}(t_{i+1}-t_i)} | x_i\rangle$$

$$\underset{t_0}{|}\overset{\varepsilon}{\;}|\overset{\varepsilon}{\;}|\overset{\varepsilon}{\;}|\;\cdots\;|\overset{\varepsilon}{\;}\underset{t_N}{|}\longrightarrow$$

$$\varepsilon = \frac{t_N - t_0}{N}$$

$$\langle x_{i+1}| e^{-iH(t_{i+1}-t_i)} |x_i\rangle$$

$$= \int dp \, \langle x_{i+1}|p\rangle\langle p| e^{-i(p^2/2m+V)\varepsilon} |x_i\rangle$$

$$= \frac{1}{\sqrt{2\pi}}\int dp \, e^{ipx_{i+1}} e^{-i\varepsilon(p^2/2m+V)} \langle p|x_i\rangle$$

$$= \frac{1}{2\pi}\int dp \, e^{ip(x_{i+1}-x_i)} e^{-i\varepsilon p^2/2m} e^{-i\varepsilon V}$$

$$= \frac{1}{2\pi}\int dp \, e^{ip\varepsilon\frac{x_{i+1}-x_i}{\varepsilon}} e^{-i\varepsilon p^2/2m} e^{-i\varepsilon V}$$

$$= \frac{1}{2\pi}\int dp \, e^{ip\varepsilon\dot{x}} e^{-i\varepsilon p^2/2m} e^{-i\varepsilon V(x_i)}$$

$$\overset{m=1}{=} \frac{1}{2\pi}\int dp \, e^{-\frac{i\varepsilon}{2}(p^2-2p\dot{x}_i+\dot{x}_i^2)} e^{\frac{i\varepsilon}{2}\dot{x}_i^2} e^{-i\varepsilon V(x_i)}$$

$$= \frac{1}{2\pi}\int dp \, e^{-\frac{i\varepsilon}{2}(p-\dot{x}_i)^2} e^{\frac{i\varepsilon}{2}(\dot{x}_i^2-2V(x_i))}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi}{i\varepsilon}}\, e^{\frac{i\varepsilon}{2}(\dot{x}_i^2-2V(x_i))}$$

$$= \frac{1}{\sqrt{2\pi i\varepsilon}}\, e^{i\varepsilon L(\dot{x}_i, x_i)}$$

$$t_{i+1}-t_i = \frac{t_N - t_0}{N}$$

$$= (2\pi i\varepsilon)^{-\frac{N}{2}} \int d^{N-1}x \; e^{\sum_{i=1}^{\frac{N}{2}} i\varepsilon L(\dot{x}_i, x_i)}$$

$$\xrightarrow{N\to\infty} = (2\pi i\varepsilon)^{-\frac{N}{2}} \int d^{N-1}x \; e^{i\int_{t_0}^{t_N} dt\, L(\dot{x}_i, x_i)}$$

$$= \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \left[\frac{1}{2\pi i\varepsilon}\right]^{\frac{N}{2}} \int d^{N-1}x \; e^{iS}$$

$$\equiv N\int D[x]\, e^{iS}$$ not well-defined 振荡

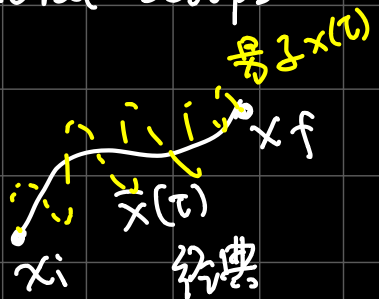
$$\xrightarrow{t=-i\tau} = N\int D[x]\, e^{-S_E}$$ 可以进行微扰和非微扰计算

S_E的定义？ $L = L(\dot{x}, x) = \frac{\dot{x}^2}{2} - V(x)$

$$\xrightarrow{t=-i\tau} -\frac{\dot{x}^2}{2} - V(x) = -\Big(\underbrace{\frac{\dot{x}^2}{2} + V(x)}_{L_E}\Big)$$

$$i\int dt\, L = -\int d\tau\, L_E = -S_E$$

General Setups.



$$\frac{\delta S}{\delta \bar{x}} = 0 \quad \bar{x} \text{ 满足 Euler-Lagrange 方程}$$

$$\Rightarrow \frac{d^2\bar{x}}{d\tau^2} - V'(\bar{x}) = 0$$

量子 $x(\tau) = \bar{x}(\tau) + \underline{\delta x}$

不是变分，只是差量 $\delta x(t_i) = \delta x(t_f) = 0$

$$S_E = \int d\tau \left(\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V\right)$$

$$= \int d\tau \left(\underline{\frac{1}{2}\left(\frac{d\bar{x}}{d\tau}\right)^2} + \frac{1}{2}\left(\frac{d\delta x}{d\tau}\right)^2 + \frac{d\bar{x}}{d\tau}\frac{dx}{d\tau} + \underline{V(\bar{x})} + V'(\bar{x})\delta x + \frac{1}{2}V''(\bar{x})\delta x^2 + \cdots\right)$$

$$= \int d\tau \left[L_E(\bar{x}, \dot{\bar{x}}) + \delta x\left(-\frac{d^2\bar{x}}{d\tau^2} + V'(x)\right) + \Delta\right]$$

$$= \int d\tau \left[L_E(\bar{x}, \dot{\bar{x}}) + \Delta\right]$$

$$= \int d\tau \left[L_E(\bar{x}, \dot{\bar{x}}) + \frac{1}{2}\delta x\left[-\frac{d^2}{d\tau^2} + V''\right]\delta x\right]$$

$$= S_E(\bar{x}) + \underline{\frac{1}{2}\int_{t_i}^{t_f} d\tau\, \delta x\left[-\frac{d^2}{d\tau^2} + V''(\bar{x})\right]\delta x} + O(\delta^3 x)$$

类比：设 δx 是任一波函数，$\delta x = \sum c_i \delta x_i$

δx_i 正交，$\int_{t_i}^{t_f} \delta x_i \delta x_j d\tau = \delta_{ij}$

δx_i 是本征态：$\left[-\frac{d^2}{d\tau^2} + V''(\bar{x})\right]\delta x = \epsilon_i \delta x_i$

eigenvalue

$$= \frac{1}{2}\int_{t_i}^{t_f} \sum c_i \delta x_i \; H \; \sum c_j \delta x_j$$

$$= \frac{1}{2}\sum |c_i|^2 \epsilon_i$$

$$= S_E(\bar{x}) + \frac{1}{2}\sum |c_i|^2 \epsilon_i$$

$$P[x(\tau)] = D[\bar{x}+\delta x(\tau)] = D[\delta x] = D[\Sigma C_i \delta x_i]$$

$$\langle\ \rangle = N\int_{-\infty}^{+\infty} \prod_i dC_i\, e^{-S_E(\bar{x})} e^{-\frac{1}{2}\Sigma C_i^2 \epsilon_i}$$

$$\boxed{= N e^{-S_E(\bar{x})} \prod_i \epsilon_i^{-\frac{1}{2}} = N e^{-S_E(x)} \operatorname{Det}\left[-\frac{d^2}{d\tau^2}+V''\right]^{-\frac{1}{2}}}$$

Saddle approximation

Remark: $N\int D[x(\tau)] \equiv \lim_{N\to\infty} \left(\frac{1}{2\pi i\hbar\epsilon}\right)^N \int \prod_{j=1}^{N} dx_j \qquad \epsilon = \frac{T}{N}$

Harmonic Oscillator

$$\left(-\frac{d^2}{d\tau^2}+w^2\right)\delta x_n = \epsilon_n \delta x_n$$

$$\epsilon_n = \frac{n^2\pi^2}{T^2}+w^2$$

$$\langle x|x\rangle_{t_i\, t_f} = N e^{-S_E(\bar{x})} \prod_n \left(\frac{n^2\pi^2}{T^2}+w^2\right)^{-\frac{1}{2}}$$

$$= \underline{\underline{N}} e^{-S_E(\bar{x})} \underline{\underline{\left[\prod_n \left(\frac{n^2\pi^2}{T^2}\right)^{-\frac{1}{2}}\right]}} \left[\prod_n \left(1+\frac{T^2w^2}{n^2\pi^2}\right)^{-\frac{1}{2}}\right]$$

自由

$$= e^{-S_E(\bar{x})} \left[N \prod_n \left(\frac{n^2\pi^2}{T^2}\right)^{-\frac{1}{2}}\right] \left[\prod_n \left(1+\frac{T^2w^2}{n^2\pi^2}\right)^{-\frac{1}{2}}\right]$$

$$= e^{-S_E(\bar{x})} \frac{1}{\sqrt{2\pi T}} \prod_n \left(1+\frac{T^2w^2}{n^2\pi^2}\right)^{-\frac{1}{2}} = e^{-S_E(\bar{x})} \frac{1}{\sqrt{2\pi T}} \left[\frac{\sinh(Tw)}{Tw}\right]^{-\frac{1}{2}}$$

$$= e^{-S_E(\bar{x})} \left(\frac{w}{\pi}\right)^{\frac{1}{2}} e^{-\frac{wT}{2}} \left[1+\frac{1}{2}e^{-2wT}+\cdots\right]$$

谐振子 $|\psi_0(0)|^2$ $\qquad \Rightarrow E_0 = \frac{1}{2}w$

$$|\psi_0(0)|^2 = \left(\frac{w}{\pi}\right)^{\frac{1}{2}}$$

$\bar{x}$ satisfies $\left.\frac{\delta S_E}{\delta x}\right|_{\bar{x}} = 0 \Rightarrow$ E-L方程 $-\ddot{\bar{x}}+w^2\bar{x}=0$

$$\Rightarrow \bar{x} = Ae^{-w\tau}+Be^{w\tau}$$

$$\begin{cases} A+B = x \\ Ae^{-wt}+Be^{-wt} = x \end{cases}$$ 初末态在同一位置

$$\Rightarrow A.B$$

$$\Rightarrow S_E=\int_0^T L_E=\int_0^T \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2+V(x)$$

$$=\omega x^2\tanh\left(\frac{\omega T}{2}\right)$$

$$=e^{-(\omega x^2\tanh\frac{\omega T}{2})}\frac{1}{\sqrt{2\pi T}}\left[\frac{\sinh\omega T}{\omega T}\right]^{-\frac{1}{2}}=\sum_n|\psi_n(x)|^2\exp(-E_nT).$$

$$=\sqrt{\frac{\omega}{\pi}}\exp(-z)\boxed{\exp\left(-\frac{\omega}{2}T\right)}\left(1+2z\exp(-\omega T)+\frac{1}{2}(-1-4z+4z^2)\exp(-2\omega T)+\dots\right]$$

$$z=\omega x^2$$

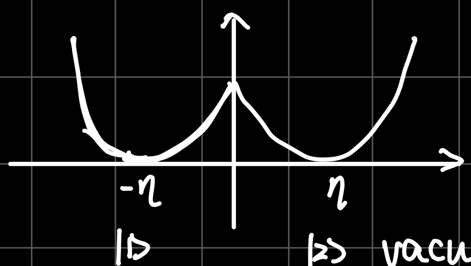
$$=\boxed{\exp(-E_0T)}\sum|\psi_n(x)|^2\exp(-(E_n-E_0)T)$$

$\Rightarrow E_n-E_0=n\omega \qquad E_n=(n+\frac{1}{2})\omega$

$|\psi_n(x)|^2=\sqrt{\frac{\omega}{\pi}}\exp(-z)(\boxed{\text{与}n\text{有关}})$

check: $\psi_n(x)=\frac{1}{\sqrt{2^n n!}}\left(\frac{\omega}{\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{z}{2}\right)H_n(z)$.

Double Well Potential 求系统的基态能.



$V(q)=\frac{\lambda}{4!}(q^2-\eta^2)^2$

在$\pm\eta$附近大约是谐振子 $\omega^2=\frac{\lambda\eta^2}{6}$

Perturbation theory doesn't work here! —— degeneracy.

$H|1\rangle=E|1\rangle+\Delta|2\rangle$

$H|2\rangle=E|2\rangle+\Delta|1\rangle$

$H=\begin{pmatrix}E & \Delta\\ \Delta & E\end{pmatrix}$ eigenvalue: $\lambda_+=E+\Delta \qquad \frac{1}{\sqrt{2}}(|1\rangle-|2\rangle)$

$\lambda_-=E-\Delta. \qquad \frac{1}{\sqrt{2}}(|1\rangle+|2\rangle)$

E基本就是$\frac{1}{2}\omega$ 基态能不再简并

$$\langle x_f|e^{-ET}|x_i\rangle=Ne^{-\bar{S}_E}\det[-\partial_t^2+V''(\bar{q})]^{-\frac{1}{2}}$$

① Solve for $e^{-\bar{S}_E}=e^{-S_E(\bar{q})}$:

$$L_E=\frac{1}{2}\dot{q}^2+\frac{\lambda}{4!}(q^2-\eta^2)^2\xrightarrow{E-L\text{方程}}\ddot{q}+\omega^2q-\frac{\lambda}{3!}q^3=0$$

(没法解)

用能量守恒计 $\bar{q}$

$$\frac{1}{2}\dot{q}^2 + V_E = 0 \Rightarrow \frac{1}{2}\dot{q}^2 - \frac{\lambda}{4!}(q^2-\eta^2)^2 = 0$$

$$\Rightarrow \dot{q} = \sqrt{\frac{\lambda}{12}}\sqrt{(q^2-\eta^2)^2}$$

$$\Rightarrow \pm\int_{\tau_c=0}^{\tau} dq\,(q^2-\eta^2)^{-1} = \sqrt{\frac{\lambda}{12}}\int_{\tau_c}^{\tau} d\tau$$

$$\Rightarrow \pm\frac{1}{\eta}\tanh\frac{\bar{q}}{\eta} = \sqrt{\frac{\lambda}{12}}(\tau-\tau_c)$$