1. Basics of Entanglement Entropy

(1) Definition of EE

$$\rho_{tot} = |\Psi\rangle\langle\Psi|$$

von Neumann entropy of the total system $S_{tot} = -\mathrm{tr}\,\rho_{tot} \log \rho_{tot} = 0$

$\longrightarrow$ divide the system into 2 subsystems A & B. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

$\rho_A = \mathrm{tr}_B\, \rho_{tot}$. 部分求迹

$S_A = -\mathrm{tr}_A\, \rho_A \log \rho_A$

考虑 static system. 于是不用考虑时间项

(2) properties of EE (zero temperature)

1° B is the complement of A $\Rightarrow S_A = S_B \Rightarrow$ EE isn't an extensive quantity (violated at finite temperature)

2° A is divided into 2 submanifolds $\Rightarrow S_{A_1} + S_{A_2} \geq S_A$ subadditivity

3°. $\forall$ 3 subsystems A, B, C don't intersect each other, $\Rightarrow$

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C}$$ strong subadditivity inequality.

A more strong version: $S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$.

(3) EE in QFTs and Area law

consider $QFT_{d+1} = \mathbb{R} \times N$

define: a subsystem $A \subset N$ at $t = t_0$ (d-Dim)

A's complement: B

∂A divides the manifold N into 2 submanifolds A & B

$\Rightarrow S_A = -\mathrm{tr}_A\, \rho_A \log \rho_A$ where $\rho_A = \mathrm{tr}_B\, \rho_{tot}$ geometric entropy

(因为依赖于A的geometry)

EE provides us with a convenient way to measure how closely entangled a given wave function $|\Psi\rangle$ is.

EE is divergent in a continuum theory → introduce "a"

$$\Rightarrow \quad S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading terms.}$$

Area law

γ: depends on the system.

planck unit

最初来源于 numerical computations & checked in many later arguments.

1-D时有问题，但是本文只考虑 $d \geq 2$

Bekenstein – Hawking entropy (BH) of Black holes

$$S_{BH} = \frac{\text{Area of Horizon}}{4 G_N}$$

Anology: S_A: the entropy for an observer accessible to A not B

B → the inside of the black hole horizon

A → outside of the horizon (observer at A)

An original motivation of the EE was its similarity to the BH entropy.

2. RT formula

proposal: $$\boxed{S_A = \frac{\text{Area of } \gamma_A}{4 G_N^{(d+2)}}}$$

γ_A: the d-Dim static minimal surface in AdS_{d+2} (boundary: ∂A)

$G_N^{(d+2)}$: d+2 -Dim Newton constant

→ suggests that the minimal surface γ_A plays the role of a holographic screen for an observer who is only accessible to A.

RT formula

distinguish new topological phases & characterise critical points. AdS/CFT & RT formula relates quantum entanglement to spacetime geometry (1406.1471). thus EE becomes an important tool to study quantum gravity & holography

3. Introduction to entangement contour

$f_A: A \to \mathbb{R} \quad f_A(i) \geq 0 \quad \sum f_A(i) = S_A$ → describes the

① distribution of contribution to Entanglement from each point of A. 不依赖于 partition.

① positivity ② Normalization ② characterize the spatial structure of the entanglement in A.

③ Symmetry: if T is a symmetry of ρ^A, i.e. $T\rho^A T^\dagger = \rho^A$. and T exchanges site i with site j, then $f_A(i) = f_A(j)$

④ Invariance under local unitary transformations: the contribution from region X to the entanglement between A & B is not affected by a redefinition of the sites or changes of basis within region X.

⑤ upper bound: $f_A(x) \le S(\Omega_A)$ cannot uniquely determine the EC function

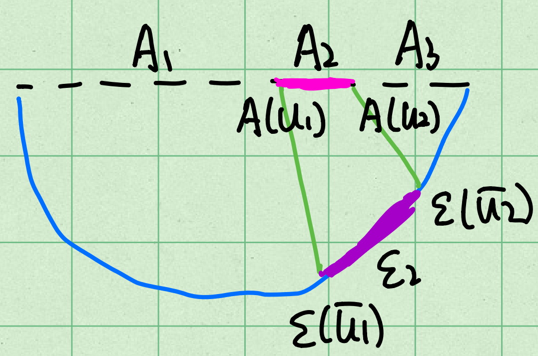
And the fundamental relationship between EC and the reduced density matrix isnot clear.

4. Qiang Wen's proposal.

1803.05524 Qiang Wen Fine structure in holographic entanglement and EC

(Explore the fine correspondence between the points on the boundary interval A and the points on the according RT surface $\mathcal{E}_A$ with the guidance of modular flows)

A point on the boundary interval is related to a unique point on the RT surface.



$$\bar{u}_0 = \frac{2 l_u^2 u_0}{4u_0^2 + l_u^2}$$

$$S_A(A_2) = \int_{A_2} f_A(x)\,dx$$

proposal (natural): Length ($\mathcal{E}_2$) captures the contribution from A_2 to the EE S_A.

知道了点的对应关系 ⇒ 知道了任意一段 A_2 对应的 $\mathcal{E}_2$ ⇒ length($\mathcal{E}_2$) $\xrightarrow{RT}$ $S_A(A_2)$ ⇒ 知道了 $f_A(x)$.

通过一种特殊情形得到的 $f_A(x)=\frac{1}{4G}\frac{4\ell}{\ell^2-4x^2}$

$$S_A(A_2)=\frac{c}{6}\log\frac{(\ell_1+\ell_2)(\ell_2+\ell_3)}{\ell_1\ell_3}$$ 不知道咋算的

$$=\int_{A_2} f_A(x)dx$$

满足线性组合的关系

$$\boxed{S_A(A_2)=\frac{1}{2}(S_{A_1\cup A_2}+S_{A_2\cup A_3}-S_{A_1}-S_{A_3})}$$ partial EE

propose that the above simple combination gives the contour function of entanglement entropy in general 1+1 dimensional theories.

5. Proof of many properties. (for $S_A(A_2)$) and comments.

(1) Additivity

$$\underset{A_1}{\rule{1cm}{0.4pt}}\,\underset{A_2^a}{\rule{1cm}{0.4pt}}\,\underset{A_2^b}{\rule{1cm}{0.4pt}}\,\underset{A_3}{\rule{1cm}{0.4pt}}$$

$A_2=A_2^a\cup A_2^b$

$$\left.\begin{aligned}S_A(A_2^a)&=\frac{1}{2}(\cancel{S_{A_1\cup A_2^a}}+S_{A_2\cup A_3}-S_{A_1}-\cancel{S_{A_2^b\cup A_3}})\\ S_A(A_2^b)&=\frac{1}{2}(S_{A_1\cup A_2}+\cancel{S_{A_2^b\cup A_3}}-\cancel{S_{A_1\cup A_2^a}}-S_{A_3})\end{aligned}\right\}\oplus$$

$$S_A(A_2^a)+S_A(A_2^b)=\frac{1}{2}(S_{A_1A_2}+S_{A_2A_3}-S_{A_1}-S_{A_3})$$

$$\Rightarrow S_A(A_2)=S_A(A_2^a)+S_A(A_2^b)$$

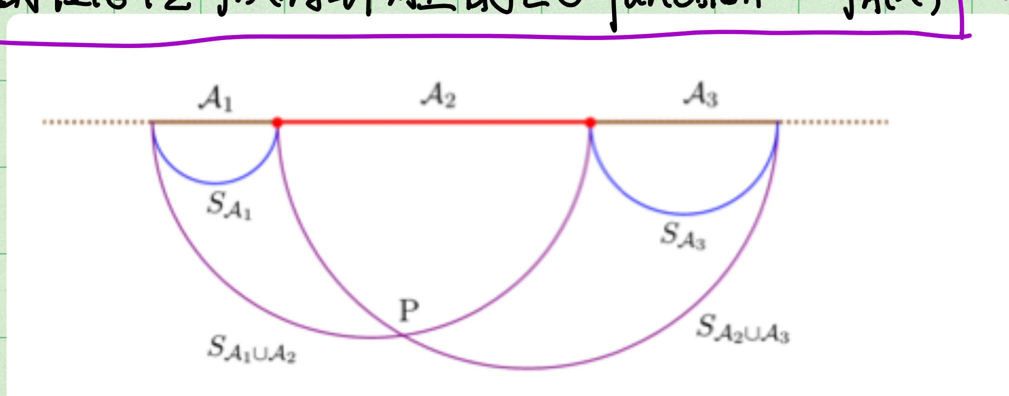
(2) Normalization

$$S_A(A_2)=\int_{A_2} f_A(x)dx \xrightarrow{A_2\to A} S_A=S_A(A_2)|_{A_2\to A}\quad \text{i.e. } S_A=\int_A f_A(x)dx$$

comment:

由 A_2, A_2^a 和 A_2^b 的任意性可以得到 A 上的 EC function $f_A(x)$ (1)(2)

(3) positivity.



$S_{A_1 \cup A_2} + S_{A_2 A_3} - S_{A_1} - S_{A_3} > 0$ (From the figure) strong subadditivity

i.e. $S_A(A_2) > 0$

Something wrong with the proof?

comment: $f_A(x) \geq 0$

(4) Invariance under local unitary transformations:

the EE is invariant under local transformations that only act on the subregion, by the proposal we know that $S_A(A_2)$ is invariant under local transformations which act only inside A_1, A_2 and A_3 respectively.

stronger than ④
A_1, A_2, A_3 respectively
v.s. A_2

(5) upper bound

$$S_A(A_2) = \frac{1}{2}(S_{A_1A_2} + S_{A_2A_3} - S_{A_1} - S_{A_3})$$

usually $S_{A_1} + S_{A_2} \geq S_{A_1A_2}$, $S_{A_2} + S_{A_3} \geq S_{A_2A_3}$

$$\Rightarrow S_A(A_2) \leq \frac{1}{2}(S_{A_1} + S_{A_2} + S_{A_2} + S_{A_3} - S_{A_1} - S_{A_3}) = S_{A_2}.$$

i.e. the contribution from A_2 to the total EE S_A shouldn't be larger than S_{A_2}. $S_A(A_2)$ doesn't count the entanglement between A_2 and other subsets inside A.

(6) Lower bound

the monogamy of mutual info.

$$S_{A_1} + S_{A_2} + S_{A_3} + S_A \leq S_{A_1A_2} + S_{A_2A_3} + S_{A_1A_3}$$

$$\Rightarrow S_A(A_2) = \frac{1}{2}(S_{A_1A_2} + S_{A_2A_3} - S_{A_1} - S_{A_3}) \geq \frac{1}{2}(S_{A_2} + S_A - S_{A_1A_3}) \geq 0$$

Araki-Lieb triangle inequality:
$S_A \geq |S_{A_1A_3} - S_{A_2}|$

(7) Symmetry

$$A_1, A_2, A_3 \xrightarrow{T} A_1', A_2', A_3'$$

ρ_A is invariant $\Rightarrow$ EE are invariant, i.e. $S_{A_i} = S_{A_i'}$

T_A is invariant ∴ EE are invariant i.e. $S_{A_i} = S_{A'_i}$

$S_{A_iA_j} = S_{A'_iA'_j}$

$\Rightarrow S_A(A_2) = S_A(A'_2) \qquad A'_2 = TA_2$

Don't understand.

6. Evidences for the EC proposal

① ansatz (拟设)

$$S_A(A_2) = \underline{C_1} S_{A_1} + \underline{C_2} S_{A_2} + \underline{C_3} S_{A_3} + \underline{C_{12}} S_{A_1A_2} + \underline{C_{23}} S_{A_2A_3} + \underline{C_{123}} S_A$$

coefficients are constants, don't depend on A_2.

additivity & limit $\Rightarrow$ proposal

② modular flow