General Relativity — Gravity

关于时空的理论

流形 度量. 取决于物质、能量的分布.

Einstein: 时空几何的结构与时空内物质和能量的分布有关 — Einstein方程

PDE 需要初值条件 但是无法给定 —— 与其它数学理论不同

时空没有局部的质量和能量，不是累加的、

测地线是不完备的.

世界观
- classical
  - Gauge theory
    - Maxwell theory
    - Yang-Mills theory → connection
  - Gravity → GR → metric
- quantum (暂不考虑)

差别在数学对象

three formalisms

① Dynamical equations ② Lagrangian formalism ③ Hamiltonian formalism

数学原理导出物理规律

physical theory check list

① Degrees of freedom: mathematical objects used to describe the physics.
   eg: GR → metric

② Dynamics (physical law): How do degrees of freedom behave

③ spacetime model.

④ symmetry

⑤ 尺度 scales

⑥ state 态/状态 Information needed to completely specify the physical system.
   量子物理不接受

⑦ observables: things physicists can measure/observable

degrees of freedom

⑧ measure (classical 不涉及)

Maths: ① quadratic form (二次型) on $\mathbb{R}^n$ (n-dim real linear space)

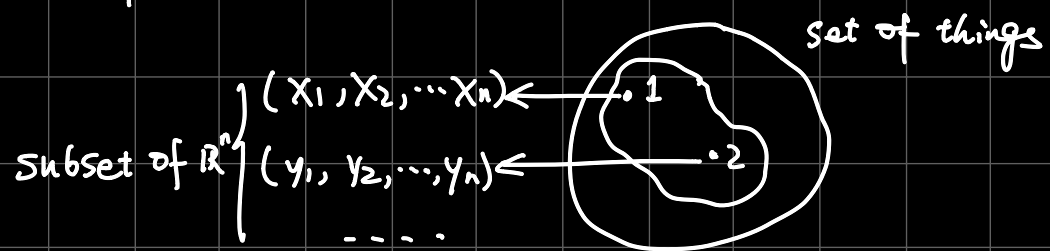
Given a basis of $\mathbb{R}^n$ a quadratic form A has a matrix representation $A=(A_{ij})$

Inertia theorem: we can find a basis that $A_{ij}=\begin{pmatrix} \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}p & & \\ & \left.\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}\right\}q & \\ & & \begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix}\end{pmatrix}$

p, q does not depend on the basis!

$p=n \Leftrightarrow$ positive definite (inner product)

② use real number to $\boxed{\text{parametrize}}$ (give name to things in sets) $\longrightarrow$ manifold as a model



① 坐标不唯一 所以可以在不同局部变换
② 局部坐标及变换，拼接
③ 为 calculus 留空间

Manifold

open subsets of M

M: topological space $\{U_\alpha\}$, $U_\alpha \subset M$.

可以是不可数无穷多

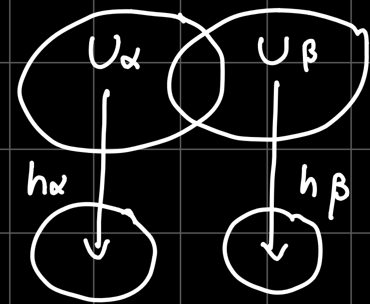
$$\bigcup_\alpha U_\alpha = M$$

$\{(U_\alpha, h_\alpha)\}$ $h_\alpha: U_\alpha \to$ open subset of $\mathbb{R}^n$

homeomorphism h_α & h_α^{-1} are continuous.

U_α: 坐标卡 chart h_α: local coordinate system.

$U_\alpha \cap U_\beta$ 有2组坐标 要求 $h_\alpha^{-1} \circ h_\beta$, $h_\beta^{-1} \circ h_\alpha$ are C^∞ (smooth)



微积分: $M \xrightarrow{f} N$ map f between M & N

$\{M, (U_\alpha, h_\alpha)\}$ $\{N, (V_\beta, g_\beta)\}$

f is $C^\infty \Leftrightarrow \forall \alpha, \beta,\ g_\beta \circ f \circ h_\alpha^{-1}$ is C^∞

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h_\alpha"'] & N \arrow[d, "g_\beta"] \\ \mathbb{R}^n \arrow[r, "1"'] & \mathbb{R}^m \end{tikzcd}$$

Two ways to describe manifolds:

1° intrinsic: directly describe M using objects on M

2° external: realize M as a subset of a higher space

eg. 2-D Sphere.

S^2: U_1, U_2 charts: $U_1 = S^2 \setminus$ northpole, $U_2 = S^2 \setminus$ southpole

隐函数定理

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$, $S^2 \subset \mathbb{R}^3$

Manifolds:

① parametrize things by sequences of real numbers — manifold as model

② Coordinates are local, non-unique

③ We can do calculus on manifold. (coordinates transformation are C^∞)

④ Intrinsic v.s. Extrinsic.

③ Groups & Symmetry

1° 集合

2° 有结构

3° 操作保持结构

4° 有很多这样的操作

先研究抽象的对称性，操作构成的集合会有什么结构，然后再研究操作如何作用于具体的集合

①逆也在这个集合

②什么都不做也保持结构

③复合.

Group (abstract symmetry)

G: set　　map: $G\times G\to G$, $(g_1,g_2)\mapsto \underbrace{g_1g_2}_{\text{multiplication}}$

properties: 1° $(g_1g_2)g_3 = g_1(g_2g_3)$

2° There's an "identity" $1\in G$, $1g=g1=g$

3° For any $g\in G$, there exists $g^{-1}\in G$ s.t. $g^{-1}g=gg^{-1}=1$

Interpretation: g: operation that preserves some fixed structure on fixed set. (symmetry)

G: set of all such operations

multiplication: composition of operations

identity: do nothing

g^{-1}: 逆操作

Examples: 1° $\mathbb{R}$: multiplication: addition

1: 0

$g\in\mathbb{R}$　　g^{-1}: $-g$

在实数轴上的平移

2° $GL(n,\mathbb{R})$: $n\times n$ real invertible matrices multiplication: matrix multiplication

保持线性空间线性结构 $\quad 1: I$

换基底 $\quad G \in GL(n,\mathbb{R}) \quad g^{-1}: G^{-1}$ (inverse matrix)

3° $O(n,\mathbb{R})$: $n\times n$ real orthogonal matrices

正交矩阵行列式±1 $\quad SO(n,\mathbb{R})$: $n\times n$ real orthogonal matrices with $\det = 1$

(half of $O(n,\mathbb{R})$)

保持标准内积

保持标准内积，保定向. → 旋转.

Groups (abstract symmetry)

Symmetries: operations on a set preserving some structures on the set
(concrete)

An Element of a group has potential to become a concrete symmetry operation.

Group action (concrete symmetry)

群作用

a group G, a set S. map: $G\times S \to S$. $g\in G, s\in S \mapsto g(s)\in S$

properties: ① $1(s)=s$

② $\underline{g_1(g_2(s))}=(g_1g_2)(s)$

操作的复合

Examples: ① G: $GL(n,\mathbb{R})$ $\quad S$: $\mathbb{R}^n$

实可逆 n阶方阵

$g\in GL(n,\mathbb{R})$ $\quad s\in\mathbb{R}^n$ $\quad s$: $n\times 1$ matrix, 列向量

$g(s) := gs$, matrix multiplication.

刻画的是S的线性结构.

The linear structure is preserved.

② $G: O(n,\mathbb{R})$ $S: \mathbb{R}^n$

n阶正交矩阵 The linear structure and standard inner product are preserved

③ $G: SO(n,\mathbb{R})$ $S: \mathbb{R}^n$

The Linear structure & standard inner product & orientation are preserved

预先选定的空间的定向.

Lie Group: G: manifold, group.

要求 ① multiplication is C^∞ map. $G\times G \to G$

② $G\to G$, $g \mapsto g^{-1}$ is C^∞

eg: $GL(n,\mathbb{R}) \supset O(n,\mathbb{R}) \supset SO(n,\mathbb{R})$

$\boxed{GL(n,\mathbb{R}) \subset \mathbb{R}^{n^2}}$ 是个开集

dim $\quad n^2 \quad \frac{n(n+1)}{2} \quad \frac{n(n+1)}{2}$

$SO(n,\mathbb{R})$ is connected.

$O(n,\mathbb{R})$ & $GL(n,\mathbb{R})$ are not connected. $\begin{cases} \det>0 \\ \det<0 \end{cases}$

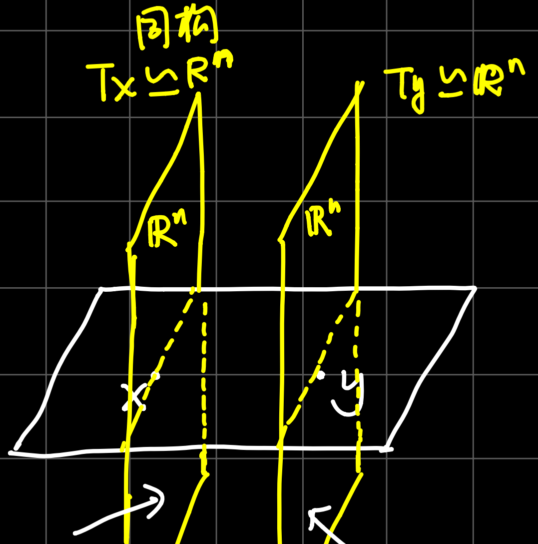
④ Standard $\boxed{\text{metric}}$ on $\mathbb{R}^n$ 1° positive-definite quadratic form.

g_{ij}: $n\times n$ matrix, 正定, 对称...

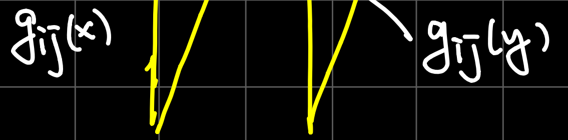
Arc length of r $\int_r ds = \int\sqrt{g_{ij}\,dx^i dx^j}$

if $g_{ij} = \mathrm{diag}(1,1,\cdots,1)$

$\int_r ds = \int\sqrt{dx^{1^2}+dx^{2^2}+\cdots+dx^{n^2}}$



2° For any $x\in\mathbb{R}^n$, there is quadratic form $g_{ij}(x)$ on tangent space of $\mathbb{R}^n$ of x.



when $g_{ij}(x) \equiv g_{ij}$, we get 1°.

Arc length of Γ $\int_\Gamma ds$.

$$ds = \sqrt{g_{ij}(x)\,dx^i dx^j}$$

Metric g on manifold M.

In local coordinates $g = g_{ij}\,dx^i dx^j$

When we do coordinate transformations, $dx^i dx^j$-part tells us how to transform g_{ij}. (we believe ds is not related to coordinates) $x \to \tilde{x} = \tilde{x}(x)$

Require $d\tilde{s}^2 = ds^2$ $\quad g_{ij}(x)\,dx^i dx^j = \tilde{g}_{ij}(\tilde{x})\,d\tilde{x}^i d\tilde{x}^j$

Intrinsic v.s. Extrinsic

eg. 1° S^2

Extrinsic: $\{x^2+y^2+z^2=1\} \subset \mathbb{R}^3$

strandard metric in $\mathbb{R}^3$ $ds^2 = dx^2+dy^2+dz^2$

Restrict to S^2: $z = \sqrt{1-(x^2+y^2)}$

$\Rightarrow ds^2 = dx^2+dy^2+d\left(\sqrt{1-(x^2+y^2)}\right)^2$

Intrinsic: spherical coordinate (θ, φ)

radius $=1$ $\quad ds^2 = \sin^2\theta\, d\varphi^2 + d\theta^2$

纬度

2° Hyperbolic plane

Intrinsic model: ① upper-half plane

$H^2 = \{(x,y) \in \mathbb{R}^2 \mid y>0\}$

$$g_{H^2} = ds^2 = \frac{dx^2+dy^2}{y^2}$$

Intrinsic Model: ② Poincare disk

what's the extrinsic model?

$D^2=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$

$g_{D^2}=ds^2=\dfrac{4dr^2}{(1-r^2)^2}$

$r^2=x^2+y^2$

From ① → ② Cayley transform:

complex number: $z=x+iy$

$H^2\to D^2$

$z\to\dfrac{z-i}{z+i}$

$g_{H^2}\to g_{D^2}$

How to feel a metric?

1° symmetry: symmetry group of a metric is called its isometry group (the group of diffeomorphisms of manifold preserving the metric)

eg. Isometry group of S^2 is $O(3,\mathbb{R})$

Extrinsic model $\begin{cases} S^2=\{x^2+y^2+z^2=1\}\in\mathbb{R}^3 \\ \text{standard metric on } \mathbb{R}^3 \text{ restricted to } S^2 \end{cases}$

dim of Isometry group of metric on n-D $\leq \dfrac{n(n+1)}{2}$

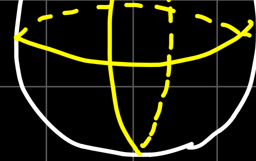
$O(3,\mathbb{R})$ acts on $\mathbb{R}^3$ preserving S^2 & standard metric

2° Geodesics (generalization of lines)

(critical point of length functional)

eg. 不一定是最短的线

不存在平行的测地线.



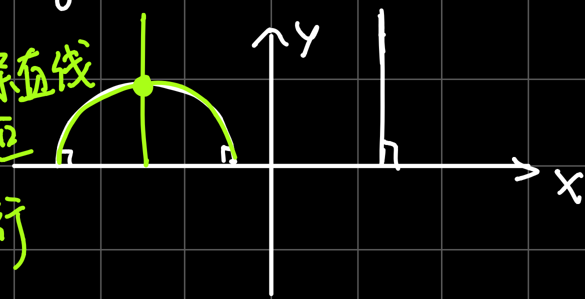
radius = radius S^2

非欧几何

Hyperbolic plane:

upper half plane model: geodesics are half circle orthogonal to x-axis or line orthogonal to x-axis.

无穷多条直线
过直线外一点
与已知直线平行



Conformal geometry: (studies properties that only depend on conformal equivalent class 共形等价类)

Manifold: M

metrics: g_1, g_2

We say they are conformally equivalent iff there is a C^∞ nonvanishing function $\Omega(x)$ on M

处处不为0 ← 不能有零点.

such $g_1 = \Omega^2 g_2$

↑

正定性

$g_2 \to g_1$ 共形变换

conformal transformation.

Lengths / distances may change,

Angles between tangent vectors do not change when we do conformal transformation.

Conformal infinity of hyperbolic plane :

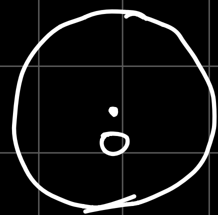
- boundary of Euclidean D^2 (conformal equivalent to hyperbolic)
- infinite for hyperbolic
- divergence for conformal vector

Hyperbolic plane, poincare disk model

$$ds^2 = \frac{4dr^2}{(1-r^2)^2} \qquad r^2 = x^2+y^2$$

$$\Omega^2 = \frac{4}{(1-r^2)^2} \quad ds^2 = \Omega^2 ds^2 \leftarrow \mathbb{R}^3 \qquad D^2 = \{x^2+y^2<1\}$$

Hyperbolic distance between O and ∂D^2



∂D^2

$r=1$

$$\int ds = \int_0^1 \frac{2}{1-r^2} dr$$

$$= \int_0^1 \frac{2}{(1-r)(1+r)} dr$$

广义积分对数发散

$\to \infty$ logarithmetically as $r \to 1$

∂D^2 is at infinity

Riemannian metric

$g_{ij}(x)dx^i dx^j$ For any x, $g_{ij}(x)$ is positive-definite

Lorentzian metric

$g_{ij}(x)dx^i dx^j$ For any x, Canonical form of $g_{ij}(x)$ is

规范型

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$