$$P(n_k)=\sum_{\{n_i\}}^{n_k} P(n_1,n_2,\cdots,n_k,\cdots)$$

keep n_k unsummed

sum over all other n_i's

Probability Reduction.

$$\langle n_{"k"}\rangle=\sum_{n_k=0}^{"\infty"} n_k P(n_k)=\sum_{n_k} n_k \sum_{\{n_i\}\backslash\{n_k\}} P(n_1,n_2,\cdots,n_k,\cdots)$$

$$P(n_k)=\sum_{\{n_i\}\backslash\{n_k\}} \underbrace{\exp(-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)+\alpha(n_1+n_2+\cdots))}_{\text{Gibbs probability}}\Big/ Z$$

partition function

$$=\frac{z_1\cdot z_2\cdot z_3\cdot z_4\cdots z_{k-1}\, z_{k+1}\cdots}{z_1\cdot z_2\cdot z_3\cdot z_4\cdots z_k\cdot z_{k+1}\cdots}\exp(-\beta n_k\epsilon_k+\alpha n_k)$$

$$=\frac{\exp(-\beta n_k\epsilon_k+\alpha n_k)}{z_k}$$

$$\langle n_{"k"}\rangle=\sum_{n_k} n_k\cdot\frac{\exp(-\beta n_k\epsilon_k+\alpha n_k)}{z_k}=\frac{1}{z_k}\sum n_k\exp(-\beta n_k\epsilon_k+\alpha n_k)$$

$$=\frac{1}{z_k}\frac{\partial}{\partial\alpha}\sum_{n_k}\exp(-\beta n_k\epsilon_k+\alpha n_k)$$

$$=\frac{1}{z_k}\frac{\partial z_k}{\partial\alpha}=\frac{\partial\ln z_k}{\partial\alpha}$$

Fermi-Dirac: $\langle n_{"k"}\rangle=\frac{\partial}{\partial\alpha}\ln(1+e^{-\beta\epsilon_k+\alpha})=\frac{e^{-\beta\epsilon_k+\alpha}}{1+e^{-\beta\epsilon_k+\alpha}}=\frac{1}{e^{\beta\epsilon_k-\alpha}+1}$

Bose-Einstein: $\langle n_{"k"}\rangle=-\frac{\partial}{\partial\alpha}\ln(1-e^{-\beta\epsilon_k+\alpha})=-\frac{-e^{-\beta\epsilon_k+\alpha}}{1-e^{-\beta\epsilon_k+\alpha}}=\frac{1}{e^{\beta\epsilon_k-\alpha}-1}$

$$\bar{n}_{"k"}^{\,FD/BE}=\frac{1}{e^{\beta\epsilon_k-\alpha}\pm1}$$ Quantum ideal gas

where's the classical limit?

4. Classical Limit → Boltzmann-Einstein Distribution

$$\bar{n}_{"k"}^{\,B}=e^{-\beta\epsilon_{"i"}}e^{\alpha}$$

normalization constant

$\bar{n}_{"k"}^{\,FD/BE}$ becomes $\bar{n}_{"k"}^{\,B}$ when $e^{\beta\epsilon_k-\alpha}\gg1$ (±1 can be dropped)

when would this happen? Quantum → Classical.

5. What is α?

$$N = n_1 + n_2 + n_3 + \cdots = \sum_{\text{“}k\text{''}} n_{\text{“}k\text{''}}$$

$$\bar{N} = \sum_{\text{“}k\text{''}}{}' \bar{n}_{\text{“}k\text{''}} = \sum_{\text{“}k\text{''}} \frac{1}{e^{\beta\epsilon_k - \alpha} + 1} = \bar{N}(\alpha) \Rightarrow \alpha \text{ is a function of } \bar{N}.$$

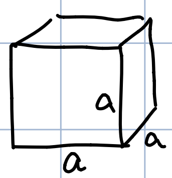
6. Classical Limit Revisited

$$e^{\beta\epsilon_k - \alpha} \gg 1 \Rightarrow \bar{n}_k^{FD/BE} \simeq \frac{1}{e^{\beta\epsilon_k - \alpha}} = e^{\alpha} e^{-\beta\epsilon_k} = \bar{n}_k^{B}$$

Meaning of $e^{\beta\epsilon_k - \alpha} \gg 1$ for all “k'', it must be true for $k=0$, the ground state.

$$e^{-\alpha} \gg e^{-\beta\epsilon_0} \simeq e^{-\beta\hbar} \simeq 1 \Rightarrow e^{-\alpha} \gg 1 \quad \text{Classical limit}$$

$$\bar{N} = \sum_{\text{“}k\text{''}} \bar{n}_k \underset{\substack{\uparrow \\ \text{classical}}}{\simeq} \sum_{\text{“}k\text{''}} e^{\alpha} e^{-\beta\epsilon_k} = e^{\alpha} \underbrace{\sum_k e^{-\beta\epsilon_k}}_{\text{one particle partition function}}$$



$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) = \epsilon\psi$$

$$\psi(x=0) = \psi(x=L) = 0$$

$$\psi = \left(\frac{2}{a}\right)^{3/2} \sin\frac{n_1\pi x}{a} \sin\frac{n_2\pi y}{a} \sin\frac{n_3\pi z}{a}$$

$$\epsilon_{\text{“}k\text{''}} = \frac{\hbar^2\pi^2}{2ma^2}(n_1^2 + n_2^2 + n_3^2) \qquad \text{“}k\text{''} \Leftrightarrow (n_1, n_2, n_3)$$

$$\bar{N} = e^{\alpha} z_1, \quad z_1 \overset{\text{g.m}}{=} \sum_{n_1,n_2,n_3=1}^{\infty} \exp\left(-\beta\frac{\hbar^2\pi^2}{2ma^2}(n_1^2+n_2^2+n_3^2)\right)$$

$$\simeq \int_0^{\infty} dn_1 \int_0^{\infty} dn_2 \int_0^{\infty} dn_3 \exp\left(-\beta\frac{\hbar^2\pi^2}{2ma^2}(n_1^2+n_2^2+n_3^2)\right)$$

$$= a^3\left(\frac{2\pi m k_B T}{h}\right)^{3/2} = V\left(\frac{2\pi m k_B T}{h}\right)^{3/2}$$

Recall: $z_1 = \frac{1}{h^3}\int d\vec{r}\, d\vec{P} \exp(-\beta P^2/2m)$ phase space, classical Hamiltonian,

$$= V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$ Gaussian Integration.

Another result: $e^{\alpha} = \bar{N}/z_1 \qquad \alpha = \mu\beta$

Go back to thermodynamics: $dF = -SdT - pdV + \mu dN$

$\mu = \frac{dF}{dN}$ $\quad \alpha = \beta \frac{dF}{dN}$. $F = -k_B T \ln Z_N$

$$
\begin{aligned}
&= -\frac{\partial \ln Z_N}{\partial N} \\
&= -\frac{\ln Z_{N+\Delta N} - \ln Z_N}{\Delta N} \\
&= -\frac{\ln Z_{N+1} - \ln Z_N}{1} \\
&= \ln \frac{Z_N}{Z_{N+1}}
\end{aligned}
$$

α is partition function difference !

$$\frac{Z_{N+1}}{Z_N} = e^{-\alpha} = \frac{Z_1}{\bar{N}}$$

$$Z_{N+1} = \frac{Z_1 Z_N}{\bar{N}} = \frac{Z_1}{N}\frac{Z_1}{N-1} Z_{N-1} = \frac{Z_1}{N}\frac{Z_1}{N-1}\frac{Z_1}{N-2} Z_{N-2} = \cdots$$

$$= \frac{Z_1^N}{N!} = Z_N$$

partition function of N particles

$$Z_N = \frac{Z_1^N}{N!}$$

$\leftarrow$ single particle partition function.

Identical or Not, label or not.

After labelling, the need of $N!$ (Gibbs Correction)