Notes of Quantum Field Theory

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Contents

Part I

Foundations

Chapter 1

Relativistic Quantum Mechanics

1.1 Quantum Lorentz Transformation

According to Einstein's principle of relativity, if the coordinates in one inertial frame are x^μ, and the coordinates in another inertial frame are $\tilde{x}^\mu$, then they must satisfy (in flat space-time)

$$\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu \tag{1.1}$$

or equivalently

$$\eta_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} = \eta_{\rho\sigma} \tag{1.2}$$

where $\eta_{\mu\nu}$ is called the Minkowski metric with only diagonal components

$$\eta_{00} = -1, \ \eta_{11} = \eta_{22} = \eta_{33} = 1 \tag{1.3}$$

and the transformation in eq.(1.2) should be linear

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1.4}$$

where a^μ is a constant and $\Lambda^\mu{}_\nu$ must satisfy

$$\eta_{\mu\nu} \Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma = \eta_{\rho\sigma} \tag{1.5}$$

If we have two coordinate transformations $T(\Lambda, a) : x \to x'$ and $T(\bar{\Lambda}, \bar{a}) : x' \to x''$, then the transformation from x to x'' writes

$$x''^\mu = \bar{\Lambda}^\mu{}_\nu x'^\nu + \bar{a}^\mu = \bar{\Lambda}^\mu{}_\nu (\Lambda^\nu{}_\sigma x^\sigma + a^\nu) + \bar{a}^\mu = \bar{\Lambda}^\mu{}_\nu \Lambda^\nu{}_\sigma x^\sigma + \bar{\Lambda}^\mu{}_\nu a^\nu + \bar{a}^\mu \tag{1.6}$$

so

$$T(\bar{\Lambda}, \bar{a}) T(\Lambda, a) = T(\bar{\Lambda}\Lambda, \bar{\Lambda} a + \bar{a}) \tag{1.7}$$

Thus, let the right hand side of eq.(1.7) be the identity transformation $\Lambda(\mathbb{1}, 0)$, we obtain the inverse of transformation $T(\Lambda, a)$

$$T^{-1}(\Lambda, a) = T(\Lambda^{-1}, -\Lambda^{-1}a)^1 \tag{1.8}$$

Hence all the Lorentz transformations with binary operation shown in eq.(1.7) and identity element $T(\mathbb{1}, 0)$ form a group, which is known as the **inhomogeneous Lorentz group**, or **Poincaré group**. For those transformation with $a = 0$, they form a subgroup called **homogeneous Lorentz group**. Any homogeneous Lorentz group satisfies

$$\det(\Lambda) = \pm 1, \quad \Lambda^0{}_0 \leqslant 1 \text{ or } \Lambda^0{}_0 \geqslant -1 \tag{1.9}$$

The subgroup with $\det(\Lambda) > 1$ and $\Lambda^0{}_0 \leqslant 1$ is called **proper orthochronous Lorentz group**.

1.2 Connected Lie Group

The group of transformations $T(\theta)$ is called a **connected Lie group** if it is described by a set of finite parameters θ^a and each element of the group is connected to the identity by a path.The multiplication for a connected Lie group has the following form

$$T(\theta)T(\bar{\theta}) = T(f(\theta, \bar{\theta})) \tag{1.10}$$

These transformations must be represented on the Hilbert space by unitary operators $U(T(\theta))$, which can be expressed by a power seriesin the neighbourhood of the identity

$$U(T(\theta)) = \mathbb{1} + \mathrm{i}\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \cdots \tag{1.11}$$

where t_a, t_{bc} are operators independent of θ. Suppose that the $U(T(\theta))$ form an ordinary representation of this group

$$U(T(\theta))U(T(\bar{\theta})) = U(T(f(\theta, \bar{\theta}))) \tag{1.12}$$

and the expansion of $f(\theta, \bar{\theta})$ takes the form of

$$f^a(\theta, \bar{\theta}) = \theta + \theta^a + f^a{}_{bc}\theta^b\bar{\theta}^c + \cdots \tag{1.13}$$

So due to eq.(1.11) and (1.12) we have

$$\begin{aligned}
&(\mathbb{1} + \mathrm{i}\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \cdots) \times (\mathbb{1} + \mathrm{i}\bar{\theta}^a t_a + \frac{1}{2}\bar{\theta}^b\bar{\theta}^c t_{bc} + \cdots) \\
&= \mathbb{1} + \mathrm{i}(\theta + \theta^a + f^a{}_{bc}\theta^b\bar{\theta}^c + \cdots)t_a \\
&+ \frac{1}{2}(\theta + \theta^b + f^b{}_{de}\theta^d\bar{\theta}^e + \cdots)(\theta + \theta^c + f^c{}_{de}\theta^d\bar{\theta}^e + \cdots)t_{bc} + \cdots
\end{aligned} \tag{1.14}$$

[1]Notice that $(\det\{\Lambda\})^2 = 1$ so there must exist an inverse of Λ.

From the $\theta\bar{\theta}$ terms we obtain

$$t_{bc} = -t_b t_c - \mathrm{i} f^a{}_{bc} t_a \tag{1.15}$$

Hence if the structure of the group is given, then the terms of $U(T(\theta))$ can be calculated. Since t_{bc} is the second-order derivative of $f(\theta, \bar{\theta})$, it must be symmetric in b and c. Therefore we have

$$[t_b, t_c] = \mathrm{i} C^a{}_{bc} t_a = \mathrm{i}(-f^a{}_{bc} + f^a{}_{cb}) t_a \tag{1.16}$$

Such commutation relationship is called a **Lie algebra**.

Specially, it the function $f(\theta, \bar{\theta})$ simply takes the form

$$f^a(\theta, \bar{\theta}) = \theta^a + \bar{\theta}^a \tag{1.17}$$

or equivalently

$$[t_b, t_c] = 0 \tag{1.18}$$

Such group is called **Abelian group**. Under such condition, the operator can be calculated by

$$U(T(\theta)) = \lim_{N \to \infty} [U(T(\theta/N))]^N = \lim_{N \to \infty} (\mathbb{1} + \mathrm{i}\frac{\theta^a t_a}{N})^N = \exp\{\mathrm{i}\theta^a t_a\} \tag{1.19}$$

If θ is taken to be infinitesimal, then such operator can be approximated by

$$U(T(\theta)) = \mathbb{1} + \mathrm{i}\theta^a t_a \tag{1.20}$$

This result shows the case for formalism in infinitesimal translation or rotation. The groups for either translations or rotations are Abelian groups since two different transformations are commutable.

1.3 Poincaré Algebra

From the above section we know that for Poincaré group, the identity element is $\mathbb{1} = \delta^\mu_\nu, a = 0$. We now consider the transformations with infinitesimal change from the identity transformation

$$T: \ \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \ a^\mu = \epsilon^\mu \tag{1.21}$$

where ω and ϵ are both infinitesimal. So the Lorentz condition eq.(1.5) reads

$$\eta_{\mu\nu} = \eta_{\rho\sigma}(\delta^\rho{}_\mu + \omega^\rho{}_\mu)(\delta^\sigma{}_\nu + \omega^\sigma{}_\nu) = \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} + O(\omega^2)^2 \tag{1.22}$$

So ω must be anti-symmetric matrices with 6 independent components. Since ϵ has 4 independent components, the total independent components for a inhomogeneous Lorentz transformation is 10-corresponding to 10 independent symmetries: time

[2] $\omega_{\mu\nu} = \eta_{\rho\nu}\omega^\rho{}_\mu$.

translational symmetry (t), 3 spatial translation symmetry (x, y, z), 3 rotational symmetry (along x, y, z) and 3 Lorentz transformation symmetry (x, y, z).

Due to eq.(1.11), for an infinitesimal Lorentz transformation, U can be expanded by

$$U(\mathbb{1}+\omega,\epsilon)=\mathbb{1}+\frac{1}{2}\mathrm{i}\omega_{\rho\sigma}J^{\rho\sigma}-\mathrm{i}\epsilon_\rho P^\rho+\cdots \tag{1.23}$$

where $J^{\rho\sigma}$ and P^ρ are Hermitian operators independent of ω and ϵ.

In order to introduce the Lie algebra of Poincaré group, we examine the product

$$U(\Lambda,a)U(\mathbb{1}+\omega,\epsilon)U^{-1}(\Lambda,a) \tag{1.24}$$

From eqs.(1.8) and (1.23) and equating the coefficients of ω and ϵ we obtain

$$U(\Lambda,a)J^{\rho\sigma}U^{-1}(\Lambda,a)=\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma(J^{\mu\nu}-a^\mu P^\nu+a^\nu P^\mu) \tag{1.25}$$

$$U(\Lambda,a)P^\rho U^{-1}(\Lambda,a)=\Lambda_\mu{}^\rho P^\mu \tag{1.26}$$

For homogeneous Lorentz transformation, $a^\mu=0$, so eq. (1.27) becomes

$$U(\Lambda,a)J^{\rho\sigma}U^{-1}(\Lambda,a)=\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma J^{\mu\nu} \tag{1.27}$$

For infinitesimal transformation $\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\omega^\mu{}_\nu$, by expanding $U(\Lambda,a)$ and $U^{-1}(\Lambda,a)$ and using eqs(1.27) and (1.26) we have

$$\mathrm{i}[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-\epsilon_\mu P^\mu,J^{\rho\sigma}]=\omega_\mu{}^\rho J^{\mu\sigma}+\omega_\nu{}^\sigma J^{\rho\nu}-\epsilon^\rho P^\sigma+\epsilon^\sigma P^\rho \tag{1.28}$$

$$\mathrm{i}[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-\epsilon_\mu P^\mu,P^\rho]=\omega_\mu{}^\rho P^\mu \tag{1.29}$$

Equating coefficients of ω and ϵ we have

$$\mathrm{i}[J^{\mu\nu},J^{\rho\sigma}]=\eta^{\nu\rho}J^{\mu\sigma}-\eta^{\mu\rho}J^{\sigma\nu}-\eta^{\sigma\mu}J^{\rho\nu}+\eta^{\sigma\nu}J^{\rho\mu}\text{[3]} \tag{1.30}$$

$$\mathrm{i}[P^\mu,J^{\rho\sigma}]=\eta^{\mu\rho}P^\sigma-\eta^{\mu\sigma}P^\rho \tag{1.31}$$

$$[P^\mu,P^\rho]=0 \tag{1.32}$$

This is the Lie algebra of Poincaré group. In another formation, we define the **momentum vector**

$$\boldsymbol{P}=\{P^1,P^2,P^3\} \tag{1.33}$$

and the **angular momentum vector**

$$\boldsymbol{J}=\{J^{23},J^{31},J^{12}\} \tag{1.34}$$

and the **boost vector**

$$\boldsymbol{K}=\{J^{01},J^{02},J^{03}\} \tag{1.35}$$

and finally the energy operator $P^0=H$. The commutation relationship between these vectors can also be given

$$[J_i,J_j]=\mathrm{i}\epsilon_{ijk}J_k \tag{1.36}$$

[3]The factor 1/2 is cancelled by using $[J^{\mu\nu},J^{\rho\sigma}]=-[J^{\rho\sigma},J^{\mu\nu}]$.

$$[J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k \tag{1.37}$$

$$[K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k \tag{1.38}$$

$$[J_i, P_j] = \mathrm{i}\epsilon_{ijk}P_k \tag{1.39}$$

$$[K_i, P_j] = -\mathrm{i}H\delta_{ij} \tag{1.40}$$

$$[J_i, H] = [P_i, H] = [H, H] = 0 \tag{1.41}$$

$$[K_i, H] = -\mathrm{i}P_i \tag{1.42}$$

As mentioned above, pure translations (both spatial- and time-) or pure rotations form an Abelian group. Thus the unitary operator can be calculated by eq.(1.19). For pure finite translations $T(\mathbb{1}, a)$, we have

$$U(1, a) = \exp\{-\mathrm{i}P^\mu a_\mu\} \tag{1.43}$$

Likewise for pure finite rotation $T(\boldsymbol{R_\theta}, 0)$, we have

$$U(\boldsymbol{R_\theta}, 0) = \exp\{\mathrm{i}\boldsymbol{J}\cdot\boldsymbol{\theta}\} \tag{1.44}$$

1.4 One-Particle State

We now consider the classification of on-particle states according to the transformation under the inhomogeneous Lorentz group. First of all, the eigenstates of four-momentum satisfy

$$\hat{P}^\mu \left|p, \sigma\right\rangle = p^\mu \left|p, \sigma\right\rangle \tag{1.45}$$

Here σ labels a series of discrete quantum numbers. From eq.(1.43) we have

$$U(\mathbb{1}, a)\left|p, a\right\rangle = \mathrm{e}^{-\mathrm{i}pa}\left|p, a\right\rangle \tag{1.46}$$

We now consider how these states transform under the homogeneous transformations. For an arbitrary homogeneous Lorentz transformation $U(\Lambda, 0)$, we have

$$P_\mu U(\Lambda)\left|p, a\right\rangle = U(\Lambda)(U^{-1}(\Lambda)P^\mu U(\Lambda))\left|p, \sigma\right\rangle = U(\Lambda)(\Lambda_\rho^{-1\mu}p^\rho)\left|p, \sigma\right\rangle = \Lambda^\mu{}_\rho p^\rho U(\Lambda)\left|p, \sigma\right\rangle \tag{1.47}$$

Hence the effect of operating P^μ on $U(\Lambda)\left|p, \sigma\right\rangle$ is equivalently to produce an eigenvector with eigenvalue Λp. Therefore the state $U(\Lambda)\left|p, \sigma\right\rangle$ can be written as a linear combination of $\left|\Lambda p, \sigma\right\rangle$

$$U(\Lambda)\left|p, a\right\rangle = \sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda, p)\left|\Lambda p, \sigma\right\rangle \tag{1.48}$$

We can use suitable linear combination to ensure that $C_{\sigma,\sigma'}$ is a block-diagonal matrix, which means that $C_{\sigma,\sigma'}$ furnishes a representation of the inhomogeneous Lorentz group. Furthermore, it is natural to connect a state with the component of a irreducible representation. Hence, we have to clarify the structure of $C_{\sigma,\sigma'}$ in irreducible representation of the inhomogeneous Lorentz group.

For each p^μ with different p^2 and p^0, we define a standard four-momentum k^μ and some standard Lorentz transformations $L^\mu{}_\nu(p)$ corresponding to each p^μ, so that

$$p^\mu = L^\mu{}_\nu(p)k^\mu \tag{1.49}$$

and define

$$|p,\sigma\rangle = N(p)U(L(p))\,|k,\sigma\rangle \tag{1.50}$$

where $N(p)$ is a normalization factor. Operate an inhomogeneous Lorentz transformation on $|p,\sigma\rangle$ we have

$$U(\Lambda)\,|p,\sigma\rangle = N(p)U(L(\Lambda p))U(L^{-1}(\Lambda p)\Lambda L(p))\,|k,\sigma\rangle \tag{1.51}$$

Here we notice that the transformation

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p) \tag{1.52}$$

takes makes k^μ invariant. Thus all such W form a subgroup, which is called **little group**. For any such W, eq.(1.48) goes to

$$U(W)\,|k,\sigma\rangle = \sum_{\sigma'} D_{\sigma,\sigma'}(W)\,|k,\sigma'\rangle \tag{1.53}$$

Obviously $D_{\sigma,\sigma'}$ furnish a representation of little group. Thus eq.(1.51) takes the form

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= N(p)\sum_{\sigma'} D_{\sigma,\sigma'}(W)U(L(\Lambda p))\,|k,\sigma'\rangle \\ &= \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} D_{\sigma,\sigma'}(W)\,|\Lambda p,\sigma'\rangle \end{aligned} \tag{1.54}$$

Thus the problem of determining $C_{\sigma,\sigma'}$ becomes the problem of finding the representation of little group and finding the normalization factors. Conventionally we adopt the normalization factors by

$$N(p) = \sqrt{\frac{k^0}{p^0}} \tag{1.55}$$

Table 1.4 shows the choice for k^μ and corresponding little group with respect to different kinds of p.

Mass Positive

According to table 1.4, for mass $M > 0$, the corresponding little group is three-dimension rotation group SO(3) whose unitary representation can be broken up to the direct sum of irreducible representation $D^j_{\sigma,\sigma'}(R)$ with dimensionality $2j+1$. Here $R \in$SO(3). We can prove that if Λ is a simple three-dimensional rotation, denoted by R, then

$$W(R,p) = R \tag{1.56}$$

Thus the little group for W has the same representation as SO(3). Hence the states for a massive particle have the same transformation under the rotations as in non-relativistic quantum mechanics.

p types	k^μ	little group
a. $p^2 = -M^2 < 0, p^0 > 0$	$(M, 0, 0, 0)$	SO(3)
b. $p^2 = -M^2 < 0, p^0 < 0$	$(-M, 0, 0, 0)$	SO(3)
c. $p^2 = 0, p^0 > 0$	$(\kappa, \kappa, 0, 0)$	ISO(2)
d. $p^2 = 0, p^0 < 0$	$(-\kappa, \kappa, 0, 0)$	ISO(2)
e. $p^2 = N^2 > 0$	$(0, N, 0, 0)$	SO(2,1)
f. $p^\mu = 0$	$(0, 0, 0, 0)$	SO(3,1)

Table 1.1: Standard k^μ and corresponding little group with respect to different kinds of p. Case (f) here shows the vacuum and cases (a), (c) respectively describe the particles with positive mass and zero mass. SO(2,1) and SO(3,1) are respectively Lorentz group in (2+1)- and (3+1)- dimensions. ISO(2) is Euclidean group in 2 dimensions.

Mass Zero

Select a time-like four-momenta $t^\mu = (1, 0, 0, 0)$. Then we must have

$$(Wt)^\mu (Wt)_\mu = -1 \tag{1.57}$$

$$(Wt)^\mu k_\mu = -1 \tag{1.58}$$

Thus Wt takes the form of

$$(Wt)^\mu = (1 + \zeta, \alpha, \beta, \zeta),\ \zeta = \frac{\alpha^2 + \beta^2}{2} \tag{1.59}$$

We find that the effect of W on both t and k is the same of that of the Lorentz transformation following

$$S^\mu{}_\nu(\alpha, \beta) = \begin{pmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{pmatrix} \tag{1.60}$$

So the effect of $S^{-1}W$ must be equivalent to a three-dimensional rotation along z-axis. Thus

$$S^{-1}(\alpha, \beta) W = R(\hat{\boldsymbol{z}}, \theta) \tag{1.61}$$

Therefore the element of the little group can be written as

$$W(\theta, \alpha, \beta) = S(\alpha, \beta) R(\hat{\boldsymbol{z}}, \theta) \tag{1.62}$$

We now work out the Lie algebra of the little group ISO(2). Let θ, α, β be infinitesimal, then according to eq.(1.11) we have

$$U(W) = \mathbb{1} + \mathrm{i}\alpha A + \mathrm{i}\beta B + \mathrm{i}\theta J_3 \tag{1.63}$$

where the operators A and B

$$A = J_2 + K_1 \tag{1.64}$$

$$B = -J_1 + K_2 \tag{1.65}$$

and their commutation relations

$$[J_3, A] = \mathrm{i}B \tag{1.66}$$

$$[J_3, B] = -\mathrm{i}A \tag{1.67}$$

$$[A, B] = 0 \tag{1.68}$$

Since A and B are commutative, they can correspond to a simultaneous eigenstate

$$\begin{aligned} A\,|k, a, b\rangle &= a\,|k, a, b\rangle \\ B\,|k, a, b\rangle &= b\,|k, a, b\rangle \end{aligned} \tag{1.69}$$

In fact, for massless particles we require $a = b = 0$. Such that

$$A\,|k, a, b\rangle = B\,|k, a, b\rangle = 0 \tag{1.70}$$

So only left the eigenstate of J_3

$$J_3\,|k, \sigma\rangle = \sigma\,|k, \sigma\rangle \tag{1.71}$$

where σ is the **helicity**. The helicity gives the component of angular momentum in the direction of motion.

From eq.(1.63) we have

$$U(W)\,|k, \sigma\rangle = \exp(\mathrm{i}\theta J_3)\,|k, \sigma\rangle = \exp(\mathrm{i}\theta\sigma)\,|k, \sigma\rangle \tag{1.72}$$

and therefore the representation matrix element

$$D_{\sigma,\sigma'}(W) = \exp(\mathrm{i}\theta\sigma)\delta_{\sigma,\sigma'} \tag{1.73}$$

We can prove from topological consideration that the helicity must take integers or half integers. Commonly, the massless particles of helicity ± 1 are called **photons** and the massless particles with helicity $\pm 1/2$ are called **neutrinos**-respectively:**neutrinos** for helicity $-1/2$ and **antineutrinos** for helicity $1/2$.

1.5 Space Inversion and Time Reversal

s Any member in a homogeneous Lorentz group can be generated from a member from its proper orthochronous subgroup by acting the following transformations

$$\mathscr{P} = \mathrm{diag}(-1, -1, -1, 1), \quad \mathscr{T} = \mathrm{diag}(1, 1, 1, -1) \tag{1.74}$$

The operators induced by those transformations are

$$\mathsf{P} = U(\mathscr{P}, 0), \quad \mathsf{T} = U(\mathscr{T}, 0) \tag{1.75}$$

such that

$$\mathsf{P}U(\Lambda, a)\mathsf{P}^{-1} = U(\mathscr{P}\Lambda\mathscr{P}^{-1}, \mathscr{P}a) \tag{1.76}$$

$$\mathsf{T}U(\Lambda, a)\mathsf{T}^{-1} = U(\mathscr{T}\Lambda\mathscr{T}^{-1}, \mathscr{T}a) \tag{1.77}$$

for an operator induced by an arbitrary transformation $U(\Lambda, a)$. If we consider about infinitesimal case that

$$\Lambda = \delta + \omega, a = \epsilon \tag{1.78}$$

then we can obtain the properties of them by acting on Poincaré generators

$$\mathsf{P}\mathrm{i}J^{\rho\sigma}\mathsf{P}^{-1} = \mathrm{i}\mathscr{P}_{\mu}{}^{\rho}\mathscr{P}_{\nu}{}^{\sigma}J^{\mu\nu} \tag{1.79}$$

$$\mathsf{P}\mathrm{i}P^{\rho}\mathsf{P}^{-1} = \mathrm{i}\mathscr{P}_{\mu}{}^{\rho}P^{\mu} \tag{1.80}$$

$$\mathsf{T}\mathrm{i}J^{\rho\sigma}\mathsf{T}^{-1} = \mathrm{i}\mathscr{T}_{\mu}{}^{\rho}\mathscr{T}_{\nu}{}^{\sigma}J^{\mu\nu} \tag{1.81}$$

$$\mathsf{T}\mathrm{i}P^{\rho}\mathsf{T}^{-1} = \mathrm{i}\mathscr{T}_{\mu}{}^{\rho}P^{\mu} \tag{1.82}$$

Thus we have

$$\mathsf{P}H\mathsf{P}^{-1} = H, \quad \mathsf{T}H\mathsf{T}^{-1} = H \tag{1.83}$$

That the energy operator is commutative with P and T. [4]

[4] Here we enforce both P and T to be unitary and linear. This is because $\mathscr{P}^2 = \mathscr{T}^2 = \mathbb{1}$.

Chapter 2

Scattering Theory

2.1 'In' and 'Out' States

We now consider the state for a system consisting of many non-interacting particles. The state is described by four momentum p^μ, spin z-component σ and another quantum number n. We define an index set $I = \{1, 2, 3 \cdots\}$ to indicate the labels for all particles, thus we have

$$\begin{aligned} &U(\Lambda, a)\,|\Psi(p_i, \sigma_i, n_i; i \in I)\rangle \\ &= \prod_{i\in I} \exp(-\mathrm{i}a_\mu(\Lambda p_i)^\mu)\sqrt{\frac{(\Lambda p_i)^0}{p_i^0}} \sum_{\sigma_i'; i\in I}(\prod_{i\in I} D^{j_i}_{\sigma_i'\sigma_i}(W(\Lambda, p_i)))\,|\Psi(p_i, \sigma_i, n_i; i \in I)\rangle \end{aligned} \tag{2.1}$$

and their normalization

$$\langle\Psi(p_i, \sigma_i, n_i; i \in I)|\Psi(p_i', \sigma_i', n_i'; i \in I)\rangle = \prod_{i\in I} \delta^3(\boldsymbol{p}_i - \boldsymbol{p}_i')\delta_{\sigma_i'\sigma_i}\delta_{n_i'n_i} \tag{2.2}$$

For simplicity, we let α to denote the whole collection of quantum numbers. Thus the normalization rule becomes

$$\langle\Psi_\alpha|\Psi_{\alpha'}\rangle = \delta(\alpha - \alpha') \tag{2.3}$$

We now consider the 'in' and 'out' states $|\Psi_\alpha^\pm\rangle$ which respectively describe the states of particles at $t \to \pm\infty$-long before and long after the interaction (conventionally we use superscript '+' to denote 'in' state and '−' for 'out' state, even if they seem backward.). We assume that the Hamiltonian can be splited into two parts: the free particle Hamiltonian H_0 and interaction V

$$H = H_0 + V \tag{2.4}$$

Here we assume that the free particle Hamiltonian has eigenstates $|\Phi_\alpha\rangle$ with same energy spectrum as the total Hamiltonian

$$H_0\,|\Phi_\alpha\rangle = E_\alpha\,|\Phi_\alpha\rangle \tag{2.5}$$

and we define the 'in' and 'out' states by

$$H \left|\Psi_\alpha^\pm\right\rangle = E_\alpha \left|\Psi_\alpha^\pm\right\rangle \tag{2.6}$$

which satisfy

$$\int \mathrm{d}\alpha \mathrm{e}^{-\mathrm{i}E_\alpha\tau} g(\alpha) \left|\Psi_\alpha^\pm\right\rangle \to \int \mathrm{d}\alpha \mathrm{e}^{-\mathrm{i}E_\alpha\tau} a(\alpha) \left|\Phi_\alpha^\pm\right\rangle \tag{2.7}$$

for $\tau \to \pm\infty$ respectively. That is to say, the states yield to the eigenstates of free particles long before or long after the interaction. Notice that from eq.(2.7) we have

$$\left|\Psi_\alpha^\pm\right\rangle = \lim_{\tau\to\mp\infty} \exp(\mathrm{i}H\tau)\exp(-\mathrm{i}H\tau)\left|\Phi_\alpha\right\rangle = \Omega(\infty)\Omega(-\infty)\left|\Phi_\alpha\right\rangle \tag{2.8}$$

Therefore the 'in' and 'out' states are free particle states acting with an unitary operator. Thus they are obviously normalized.

2.2 Lippmann-Schwinger Equation

Now we decide to find specific solution to the scattering states $\left|\Psi_\alpha^\pm\right\rangle$. Since we have mentioned the split of Hamiltonian in eq.(2.4), then we must have

$$(E_\alpha - H_0)\left|\Psi_\alpha^\pm\right\rangle = V\left|\Psi_\alpha^\pm\right\rangle \tag{2.9}$$

The operator $E_\alpha - H_0$ is not inversible if it equals zero. Thus the equation cannot be simply solved. However, we can plus it by a infinitesimal operator $\mathrm{i}\epsilon$ and we can take a limit $\epsilon \to 0$ at the end of the calculation. Notice that the free particle state $\left|\Phi_\alpha\right\rangle$ is the eigenstate of H_0, i.e. $(E_\alpha - H_0)\left|\Phi_\alpha\right\rangle$. Therefore, the solution to eq.(2.9) must satisfy

$$\left|\Psi_\alpha^\pm\right\rangle = \left|\Phi_\alpha\right\rangle + (E_\alpha - H_0 \pm \mathrm{i}\epsilon)^{-1} V \left|\Psi_\alpha^\pm\right\rangle \tag{2.10}$$

or by inserting a complete set of $\left|\Phi_\beta\right\rangle$ in the second term

$$\begin{aligned} \left|\Psi_\alpha^\pm\right\rangle &= \left|\Phi_\alpha\right\rangle + \int \mathrm{d}\beta (E_\alpha - H_0 \pm \mathrm{i}\epsilon)^{-1}\left|\Phi_\beta\right\rangle \left\langle\Psi_\beta\right| V \left|\Psi_\alpha^\pm\right\rangle \\ &= \left|\Phi_\alpha\right\rangle + \int \mathrm{d}\beta (E_\alpha - E_\beta \pm \mathrm{i}\epsilon)^{-1} T_{\beta\alpha}^\pm \left|\Phi_\beta\right\rangle \end{aligned} \tag{2.11}$$

where

$$T_{\beta\alpha}^\pm = \left\langle\Psi_\beta\right| V \left|\Psi_\alpha^\pm\right\rangle \tag{2.12}$$

These equations are known as the **Lippmann-Schwinger equations**. In the future sections we will widely discuss the Lippmann-Schwinger equation.

2.3 The S-Matrix

In classical case, the scattering matrix, or S-matrix, of a system is defined by the ratio between the scattering state and the incident state. Likewise, in quantum mechanics, we define the S-matrix by the inner product of the 'in' and 'out' states

$$S_{\beta\alpha} = \left\langle \Psi_\beta^- \middle| \Psi_\alpha^+ \right\rangle \tag{2.13}$$

That is to say, the S-matrix gives the universal effect for the interaction of transitting quantum state α to β. Thus if the 'in' and 'out' states are same, which means there is no interaction, then $S = \delta(\beta - \alpha)$. Obviously S-matrix is unitary

$$S^\dagger S = SS^\dagger = 1 \tag{2.14}$$

Besides, from eq.(2.8) we can rewrite eq.(2.13)

$$\left\langle \Psi_\beta^- \middle| \Psi_\alpha^+ \right\rangle = \langle \Phi_\beta | \Omega(\infty)^\dagger \Omega(-\infty) | \Phi_\alpha \rangle = \langle \Phi_\beta | S | \Phi_\alpha \rangle \tag{2.15}$$

where

$$S = \Omega(\infty)^\dagger \Omega(-\infty) \tag{2.16}$$

If we consider the wave packet, i.e. the scattering states and the free particle states are superpositions

$$\left|\Psi_g^\pm(t)\right\rangle = \int \mathrm{d}\alpha \mathrm{e}^{-\mathrm{i}E_\alpha t} g(\alpha) \left|\Psi_\alpha^\pm\right\rangle \tag{2.17}$$

$$|\Phi_g(t)\rangle = \int \mathrm{d}\alpha \mathrm{e}^{-\mathrm{i}E_\alpha t} g(\alpha) |\Phi_\alpha\rangle \tag{2.18}$$

such that eq.(2.11) goes to

$$\left|\Psi_g^\pm(t)\right\rangle = |\Phi_g(t)\rangle + \int \mathrm{d}\alpha \int \mathrm{d}\beta (E_\alpha - E_\beta \pm \mathrm{i}\epsilon)^{-1} \mathrm{e}^{-\mathrm{i}E_\alpha t} T_{\beta\alpha}^\pm |\Phi_\beta\rangle \tag{2.19}$$

Let $\epsilon \to 0$, then the 'out' state writes

$$\left|\Psi_g^\pm(t)\right\rangle = \int \mathrm{d}\beta \mathrm{e}^{-\mathrm{i}E_\beta t} |\Phi_\beta\rangle [g(\beta) - 2\pi\mathrm{i} \int \mathrm{d}\alpha \delta(\beta - \alpha) g(\alpha) T_{\beta\alpha}^\pm] \tag{2.20}$$

From eq.(2.13) and (2.17) we have

$$\left|\Psi_g^\pm(t)\right\rangle = \int \mathrm{d}\beta \mathrm{e}^{-\mathrm{i}E_\beta t} |\Phi_\beta\rangle \int \mathrm{d}\alpha g(\alpha) S_{\beta\alpha} \tag{2.21}$$

Compare eq.(2.20) and (2.21) we have

$$\int \mathrm{d}\alpha g(\alpha) S_{\beta\alpha} = g(\beta) - 2\pi\mathrm{i} \int \mathrm{d}\alpha \delta(\beta - \alpha) g(\alpha) T_{\beta\alpha}^\pm \tag{2.22}$$

Hence

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2\pi\mathrm{i}\delta(E_\beta - E_\alpha) T_{\beta\alpha}^\pm \tag{2.23}$$

Eq.(2.23) gives another formalism for S-matrix. If, the interaction V is weak enought to ignore, then the interaction between the 'in' states and the free particle states can be also neglected. Thus eq.(2.23) goes to

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2\pi\mathrm{i}\delta(E_\beta - E_\alpha) \langle \Phi_\beta | V | \Phi_\alpha \rangle \tag{2.24}$$

2.4 Symmetries of the S-Matrix