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Summary of J.D.Jackson's Classical Electrodynamics

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Introduction to Electrostatics

1.1 Coulomb's Law

The force between two small charged bodies separated in air a distance large compared to their dimensions:

varies directly as the magnitude of each charge;

varies inversely as the square of the distance between them;

is directed along the line joining the charges and is attractive if the bodies are oppositely charged and repulsive if the bodies are similarly charged.

1.2 Electric Field

The force $\boldsymbol{F}$ between one point charge q_1, located at $\boldsymbol{x_1}$, and another point charge q_2, located at $\boldsymbol{x_2}$, can be written as:

$$\boldsymbol{F} = kq_1q_2\frac{\boldsymbol{x_1}-\boldsymbol{x_2}}{|\boldsymbol{x_1}-\boldsymbol{x_2}|^3}$$

The electric field at $\boldsymbol{x}$ due to q_1 is:

$$\boldsymbol{E} = kq_1\frac{\boldsymbol{x}-\boldsymbol{x_1}}{|\boldsymbol{x}-\boldsymbol{x_1}|^3}$$

where $k = 1$ in esu, $k = \dfrac{1}{4\pi\epsilon_0}$ in SI.

Linear superposition of electric field at $\boldsymbol{x}$ due to a system of point charges q_i, located at $\boldsymbol{mmx}$, can be written as:

$$\boldsymbol{E(x)} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i\frac{\boldsymbol{x}-\boldsymbol{x_i}}{|\boldsymbol{x}-\boldsymbol{x_i}|^3}$$

for continuous charge distribution, which can be described as charge density $\rho(\boldsymbol{x'})$,

$$\boldsymbol{E(x)} = \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{x'})\frac{\boldsymbol{x}-\boldsymbol{x'}}{|\boldsymbol{x}-\boldsymbol{x'}|^3}\mathrm{d}^3x'$$

for a system of point charges q_i, located at $\boldsymbol{x}_i$,

$$\rho(\boldsymbol{x}) = \sum_{i=1}^{n} q_i \delta(\boldsymbol{x} - \boldsymbol{x_i})$$

1.3 Gauss's Law

For a closed surface S and a point charge q, then electronic field $\boldsymbol{E}$ due to q satisfy:

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \begin{cases} q/\epsilon_0 & q \text{ lies inside } S \\ 0 & q \text{ lies outside } S \end{cases}$$

For a discrete set of point charges enclosed in surface S,

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\epsilon_0} \sum_{i=1} q_i$$

For continuous distribution,

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\epsilon_0} \int_V \rho(\boldsymbol{x}) \mathrm{d}^3 x$$

1.4 Differential Form of Gauss's Law

Electric field $\boldsymbol{E}$ defined in a volume V surrounded by a closed surface S satisfy,

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{E} \mathrm{d}^3 x$$

So Gauss's Law can be written as:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0$$

1.5 Scalar Potential

Scalar potential $\Phi(\boldsymbol{x})$ is defined by the equation:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi$$

Since expression of electric field has been given, scalar potential is given in terms of charge density by:

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\mathrm{d}^3x'$$

The work done in moving a point charge q from A to B is:

$$W = -\int_A^B \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l} = -q\int_A^B \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = q\int_A^B \boldsymbol{\nabla}\Phi\cdot\mathrm{d}\boldsymbol{l} = q(\Phi_B - \Phi_A)$$

This follows that the line integral of $\boldsymbol{E}$ is zero, thus resulting curl $\boldsymbol{E}$ is zero.

$$\oint \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{\nabla}\times\boldsymbol{E})\cdot\mathrm{d}\boldsymbol{S} = 0$$
$$\boldsymbol{\nabla}\times\boldsymbol{E} = 0$$

1.6 Surface Distribution of Charges and Dipoles and Discontinuities in the Electric Field and Potential

If a surface S, with a unit normal $\boldsymbol{n}$ direct from side 1 to side 2 of the surface, has a surface-charge density of $\sigma(\boldsymbol{x})$ and electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ on the either side of the surface. According to Gauss's Law,

$$(\boldsymbol{E}_1 - \boldsymbol{E}_2)\cdot\boldsymbol{n} = \sigma/\epsilon_0$$

The scalar potential at any position can be written as

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x'}|}\mathrm{d}S$$

Another problem is the potential due to dipole-layer distribution on a surface S, which can be imagined as letting a surface S with surface-charge density $\sigma(\boldsymbol{x})$ on it, and another surface S' with equal and opposite surface-charge density $\sigma(\boldsymbol{x'})$ on it at neighboring points. The dipole-layer distribution of strength $D(\boldsymbol{x})$ is:

$$D(\boldsymbol{x}) = \lim_{d(\boldsymbol{x})\to 0}\sigma(\boldsymbol{x})d(\boldsymbol{x})$$

where $d(\boldsymbol{x})$ indicates the local separation of the two surfaces.

With $\boldsymbol{n}$, the unit normal to the surface S, directed away from S', the potential can be written as:

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\mathrm{d}S - \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}+\boldsymbol{n}d|}\mathrm{d}S$$

Give the Taylor expansion of the second term:

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}+\boldsymbol{n}d|}=\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|}+\boldsymbol{n}d\cdot\boldsymbol{\nabla}(\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|})$$

Then the potential becomes:

$$\Phi(\boldsymbol{x})=\frac{1}{4\pi\epsilon_0}\int_S D(\boldsymbol{x'})\boldsymbol{n}\cdot\boldsymbol{\nabla}'(\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|})\mathrm{d}S'$$

Define dipole moment of a point dipole locally at the two surfaces $\boldsymbol{p}=D\mathrm{d}\boldsymbol{S}$, then:

$$\Phi(\boldsymbol{x})=\frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|^3}$$

We note that

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}'(\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|})\mathrm{d}S=-\mathrm{d}\Omega$$

Then the potential can be written as:

$$\Phi(\boldsymbol{x})=-\frac{1}{4\pi\epsilon_0}\int_S D(\boldsymbol{x'})\mathrm{d}\Omega$$

For volume or surface distribution, the potential is everywhere continuous, even within the charge distribution. With point or line charges, or dipole layers, the potential is not continuous. From the expression above, it is clear that the potential has a discontinuity of D/ϵ_0 from the inner side to the outer side.

1.7 Electrostatic Potential Energy and Energy Density; Capacitance

For a system of point charges $q_i(i=1,2\cdots n)$, located at x_i respectively, the potential energy of the charge q_i is given by:

$$W_i=q_i\Phi(\boldsymbol{x}_i)=\frac{q_i}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{|\boldsymbol{x}_i-\boldsymbol{x}_j|}$$

So the total potential energy of all the point charges due to all the forces acting between them is:

$$W=\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_iq_j}{|\boldsymbol{x}_i-\boldsymbol{x}_j|}=\frac{1}{8\pi\epsilon_0}\sum_{j\neq i}\frac{q_iq_j}{|\boldsymbol{x}_i-\boldsymbol{x}_j|}$$

For continuous charge distribution, the potential energy is

$$W = \frac{1}{8\pi\epsilon_0}\int\int\frac{\rho(\boldsymbol{x})\rho(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\mathrm{d}^3x\mathrm{d}^3x' = \frac{1}{2}\int\rho(\boldsymbol{x})\Phi(\boldsymbol{x})\mathrm{d}^3x = -\frac{\epsilon_0}{2}\int\Phi\boldsymbol{\nabla}^2\Phi\mathrm{d}^3x$$
$$= \frac{\epsilon_0}{2}\int|\boldsymbol{\nabla}^2\Phi|^2\mathrm{d}^3x = \frac{\epsilon_0}{2}\int|\boldsymbol{E}|^2\mathrm{d}^3x$$

Define energy density w as:

$$w = \frac{\epsilon_0}{2}|\boldsymbol{E}|^2$$

For a system of n conductors, each with potential V_i and total charge Q_i in otherwise empty space, the potential energy is:

$$W = \frac{1}{2}\sum_{i=1}^{n}Q_iV_i$$

Since the charge of each conductor can be written as:

$$Q_i = \sum_{j=1}^{n}C_{ij}V_j$$

The potential energy:

$$W = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}V_iV_j$$

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Boundary-Value Problems in Electrodynamics: I

2.1 Method of Images

Image charges are charges external to the region of interest which can simulate the required boundary conditions. The replacement of the actual problem with boundaries by an enlarged region with image charges but not boundaries is called the method of images.

2.2 Point Charge in the Presence of a Grounded Conducting Sphere

A point charge q located at y relative to the origin, around which is centered a grounded conducting sphere of radius a. With method of images, we seek a image charge q', located at y', satisfied the boundary condition $\Phi(|\boldsymbol{x}| = a) = 0$. By symmetry it is easy to see that the image charge lies on the ray from the origin to the charge q. The potential at $|\boldsymbol{x}| = a$ is:

$$\Phi(|\boldsymbol{x}| = a) = \frac{q}{4\pi\epsilon_0}\frac{1}{a|\boldsymbol{n} - \frac{y}{a}\boldsymbol{n'}|} + \frac{q'}{4\pi\epsilon_0}\frac{1}{y'|\boldsymbol{n'} - \frac{a}{y'}\boldsymbol{n}|}$$

where n and n' are separately the unit vector of x and y'. To make $\Phi(|\boldsymbol{x}| = a) = 0$,

$$q' = -\frac{a}{y}q, y' = \frac{a^2}{y}$$

Since the potential satisfy Poisson equation

$$\boldsymbol{\nabla}^2\Phi = -\sigma\delta(|\boldsymbol{x}| - a)/\epsilon_0$$

So the actual charge density on the sphere surface is

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial x}|_{x=a} = -\frac{q}{4\pi a^2}(a/y)\frac{1-\frac{a^2}{y^2}}{(1-2\frac{a}{y}\cos\theta+\frac{a^2}{y^2})^{3/2}}$$

where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.

We note that the expression of the potential can be expanded by Legendre polynomials. So the potential can also be written as:

$$\begin{aligned}\Phi(\boldsymbol{x}) &= \frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}+\frac{q'}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{x}-\boldsymbol{y'}|} \\ &= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{y^l}{x^{l+1}}\mathrm{P}_l(\cos\theta)+\frac{q'}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{y'^l}{x^{l+1}}\mathrm{P}_l(\cos\theta)\end{aligned}$$

So the surface-charge density is

$$\begin{aligned}\sigma &= -\epsilon_0\frac{\partial\Phi}{\partial x}|_{x=a} = \frac{q}{4\pi}\sum_{l=0}^{\infty}(l+1)\frac{y^l}{a^{l+2}}\mathrm{P}_l(\cos\theta)+\frac{q'}{4\pi}\sum_{l=0}^{\infty}(l+1)\frac{y'^l}{a^{l+2}}\mathrm{P}_l(\cos\theta) \\ &= \frac{q}{4\pi}\sum_{l=0}^{\infty}(l+1)\frac{y^l}{a^{l+2}}\mathrm{P}_l(\cos\theta)-\frac{a}{y}\frac{q}{4\pi}\sum_{l=0}^{\infty}(l+1)\frac{y'^l}{a^{l+2}}\mathrm{P}_l(\cos\theta) \\ &= \frac{q}{4\pi a^2}\sum_{l=0}^{\infty}(l+1)(\frac{y^l}{a^2}-\frac{a^{l+1}}{y^{l+1}})\mathrm{P}_l(\cos\theta)\end{aligned}$$

2.3 Point Charge in the presence of a Charged, Insulated, Conducting Sphere

The potential is the superposition of the potential mentioned in (2.2) and the potential of a point charge $(Q-q')$ at the origin

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}[\frac{q}{|\boldsymbol{x}-\boldsymbol{y}|}-\frac{aq}{y|\boldsymbol{x}-\frac{a^2}{y^2}\boldsymbol{y}|}+\frac{Q+\frac{a}{y}q}{|\boldsymbol{x}|}]$$

2.4 Point Charge Near a Conducting Sphere at Fixed Potential

The problem is of a point charge near a conducting sphere held at a fixed potential V. The potential is the same as the potential mentioned in (2.3), except that the charge at the origin replaced by $4\pi\epsilon_0 Va$.

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}[\frac{q}{|\boldsymbol{x}-\boldsymbol{y}|} - \frac{aq}{y|\boldsymbol{x}-\frac{a^2}{y^2}\boldsymbol{y}|}] + \frac{Va}{|\boldsymbol{x}|}$$

2.5 Conducting Sphere in a Uniform Electric Field by Method of Images

A uniform field can be thought of as being produced by appropriate positive and negative charges at infinity. If the two charges $\pm Q$, located at position $z = \mp R$. Let $R, Q \to \infty$, then the electric field is $E_0 = 2\frac{Q}{4\pi\epsilon_0 R^2}$. For a sphere of a radius a is placed at the origin, the potential will be due to the charge $\pm Q$ and the image charged $\mp Q' = \mp\frac{a}{R}$ at $z = \mp\frac{a^2}{R}$.

$$\begin{aligned}\Phi = &- \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+R^2+2rRcos\theta}} + \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+R^2-2rRcos\theta}} \\ &+ \frac{aQ}{4\pi\epsilon_0 R}\frac{1}{\sqrt{r^2+\frac{a^4}{R^2}-2\frac{a^2r}{R}cos\theta}} + \frac{aQ}{4\pi\epsilon_0 R}\frac{1}{\sqrt{r^2+\frac{a^4}{R^2}-2\frac{a^2r}{R}cos\theta}}\end{aligned}$$

Expand the potential by Legendre polynomials

$$\begin{aligned}\Phi = &- \frac{Q}{4\pi\epsilon_0 R}\sum_{l=0}^{\infty}(r/R)^l \mathrm{P}_l(\cos\theta) + \frac{Q}{4\pi\epsilon_0 R}\sum_{l=0}^{\infty}(-r/R)^l \mathrm{P}_l(\cos\theta) \\ &+ \frac{aQ}{4\pi\epsilon_0 Rr}\sum_{l=0}^{\infty}(a^2/rR)^l \mathrm{P}_l(\cos\theta) - \frac{aQ}{4\pi\epsilon_0 Rr}\sum_{l=0}^{\infty}(-a^2/rR)^l \mathrm{P}_l(\cos\theta)\end{aligned}$$

It is easy to see the potential include all the terms only when l is odd number. Furthermore, the terms with l greater than 1 will vanish in the limit $R \to \infty$. So the potential is

$$\Phi = -\frac{Q}{4\pi\epsilon_0}\frac{2Q}{R^2}r\cos\theta + \frac{Q}{4\pi\epsilon_0}\frac{2Q}{R^2}\frac{a^3}{r^2}\cos\theta = E_0(-r + \frac{a^3}{r^2})\cos\theta$$

The induced surface-charge density is

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial r}|_{r=a} = 3\epsilon_0 E_0\cos\theta$$

Another solution is to let $\Phi = \Phi_1 + \Phi_2$, where Φ_1 and Φ_2 are respectively the electric potential produced by the uniform electric field and induced charges. Φ_1 satisfy:

$$\Phi_1 = -E_0 r\cos\theta + \Phi_0$$

where Φ_0 is the potential at $r = 0$. Φ_2 satisfy Laplace equation and is nonrelative to ϕ.

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi_2}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi_2}{\partial\theta}) = 0,$$

$$\Phi_2|_{r=a} = -u_1|_{r=a}, \Phi_2|_{r\to\infty} \to \infty,$$

$$\Phi|_{\theta=0}, \Phi|_{\theta=\pi}., \text{ bounded}$$

solution to the equation:

$$\Phi_2 = \sum_{l=0}^{\infty}(A_l r^l + B_L r^{-l-1})\mathrm{P}_l(\cos\theta)$$

according to the boundary conditions we can get:

$$\Phi_2 = -\Phi_0\frac{a}{r} + \frac{E_0 a^3}{r^2}\cos\theta.$$

then

$$\Phi = \Phi_1 + \Phi_2 = \Phi_0(1 - \frac{a}{r}) - E_0(1 - \frac{a^3}{r^3})r\cos\theta$$

Let $\Phi_0 = 0$ we get the same result as the method of images get

$$\Phi = E_0(-r + \frac{a^3}{r^2})\cos\theta$$

3

Boundary-Value Problems in Electrodynamics: II

Reference: Methods of Mathematic Physics.

4

Multipoles, Electrostatics of Macroscopic Media, Dielectrics

4.1 Multiple Expansion

A localized distribution of charge is described by the charge density $\rho(\boldsymbol{x'})$, which is non vanishing only inside a sphere of radius R around some origin. The potential outside the sphere can be written as an expansion in spherical harmonics

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{\mathrm{Y}_{lm}(\theta,\phi)}{r^{l+1}}$$

Since the basic expression of potential is

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\mathrm{d}^3x'$$

To expand $1/|\boldsymbol{x}-\boldsymbol{x'}|$ in spherical harmonics

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}}\mathrm{Y^*}_{lm}(\theta',\phi')\mathrm{Y}_{lm}(\theta,\phi)$$

Then the coefficients q_{lm} are

$$q_{lm} = \int\mathrm{Y^*}_{lm}(\theta',\phi')r'^l\rho(\boldsymbol{x'})\mathrm{d}^3x'$$

These coefficients are called multipole moment. So the potential can be expanded as

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r}+\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{r^3}+\frac{1}{2}\sum_{i,j}Q_{ij}\frac{x_ix_j}{r^5}+\cdots\right)$$

where q is the total charge, $\boldsymbol{p}$ is the electric dipole moment

$$\boldsymbol{p} = \int\boldsymbol{x'}\cdot\rho(x')\mathrm{d}^3x'$$

Q_{ij} is the traceless quadrupole moment tensor

$$Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\boldsymbol{x'}) \mathrm{d}^3 x'$$

4.2 Multipole Expansion of the Energy of a Charge Distribution in an External Field

If a localized charge distribution described by $\rho(\boldsymbol{x})$ is placed in an external potential $\Phi(\boldsymbol{x})$, the electrostatic energy of the system is

$$W = \int \rho(\boldsymbol{x}) \Phi(\boldsymbol{x}) \mathrm{d}^3 x$$

Expand the potential in a Taylor series around a chosen origin

$$\Phi(\boldsymbol{x}) = \Phi(0) + \boldsymbol{x} \cdot \boldsymbol{\nabla} \Phi(0) + \frac{1}{2} \sum_i \sum_j x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} \Phi(0) + \cdots$$

So the energy takes the form

$$\begin{aligned} W &= \int \rho(\boldsymbol{x}) \Phi(\boldsymbol{x}) \mathrm{d}^3 x \\ &= \int \rho(\boldsymbol{x}) [\Phi(0) + \boldsymbol{x} \cdot \boldsymbol{\nabla} \Phi(0) + \frac{1}{2} \sum_i \sum_j x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} \Phi(0) + \cdots] \mathrm{d}^3 x \\ &= q\Phi(0) + \boldsymbol{p} \cdot \boldsymbol{\nabla} \Phi(0) + \frac{1}{6} Q : \boldsymbol{\nabla\nabla} \Phi(0) + \cdots \\ &= q\Phi(0) - \boldsymbol{p} \cdot \boldsymbol{E}(0) - \frac{1}{6} Q : \boldsymbol{\nabla} \cdot E(0) \end{aligned}$$

4.3 Elementary Treatment of Electrostatics with Ponderable Media

Electric polarization is given by

$$\boldsymbol{P} = \sum_i N_i \langle \boldsymbol{p_i} \rangle$$

where $\boldsymbol{p_i}$ is the dipole moment of the ith type of molecule in the medium and

N_i is the average number per unit volume of the ith type of molecule at the point $\boldsymbol{x}$.

The definition of electric displacement $\boldsymbol{D}$

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

Then the Maxwell equation reads

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$$

where rho indicates the free charge density.

As a simplification we suppose that the medium is isotropic. Then the induced polarization $\boldsymbol{P}$ is parallel to $\boldsymbol{E}$

$$\boldsymbol{P} = \epsilon_0 \chi \boldsymbol{E}$$

Therefore the electric displacement $\boldsymbol{D}$ is parallel to $\boldsymbol{E}$

$$\boldsymbol{D} = \epsilon \boldsymbol{E}$$

where

$$\epsilon = \epsilon_0(1 + \chi)$$

Furthermore the divergence equation will become

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon$$

For an interface between region 1 and 2, the normal components of $\boldsymbol{D}$ and tangental components of $\boldsymbol{E}$ satisfy the boundary conditions

$$\left\{ \begin{array}{l} (\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \boldsymbol{n}_{21} = \sigma \\ (\boldsymbol{E}_1 - \boldsymbol{E}_2) \times \boldsymbol{n}_{21} = 0 \end{array} \right.$$

4.4 Boundary-Value Problems with Dielectrics

4.4.1 A Point Charge Subjected in a Semi-infinite Dielectric Distance Away From the Interface

The plane interface takes the form $z = 0$. The point charge q, located at $z = d$, embedded in dielectrics ϵ_1. By using the method of images, we can give the form of the potential at any point $P(\rho, \theta, z)$ at either side of the interface

$$\Phi(z > 0) = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{\rho^2 + (d - z)^2}} + \frac{q'}{\sqrt{\rho^2 + (d + z)^2}}\right)$$

$$\Phi(z<0) = \frac{1}{4\pi\epsilon_2}\frac{q''}{\sqrt{\rho^2+(d-z)^2}}$$

where q', q'' are image charges. According to the boundary conditions at $z = 0$

$$-\epsilon_2\frac{\partial\Phi}{\partial z}|_{z<0} = -\epsilon_1\frac{\partial\Phi}{\partial z}|_{z>0}$$
$$-\epsilon_2\frac{\partial\Phi}{\partial\rho}|_{z<0} = -\epsilon_1\frac{\partial\Phi}{\partial\rho}|_{z>0}$$

These can be solved to yield the image charge

$$q' = (\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2})q$$
$$q'' = (\frac{2\epsilon_2}{\epsilon_1+\epsilon_2})q$$

Furthermore, we can give the polarization in each dielectric at $z = 0$

$$\boldsymbol{P}_i = (\epsilon_i - \epsilon_0)\boldsymbol{E}_i = (\epsilon_i-\epsilon_0)\boldsymbol{\nabla}\Phi_{z=0\pm}$$

Then the polarization charge density can be shown as

$$\sigma_p = -(\boldsymbol{P}_2 - \boldsymbol{P}_1)\cdot\boldsymbol{n}_{21} = -\frac{q}{2\pi}\frac{\epsilon_0(\epsilon_2-\epsilon_1)}{\epsilon_1(\epsilon_2+\epsilon_1)}\frac{d}{(\rho^2+d^2)^{3/2}}$$

4.4.2 Dielectric Sphere Placed in Uniform Electric Field

The sphere has a radius a and dielectric constant ϵ. The uniform electric field takes the direction of z axis and has magnitude E_0. It is easy to give the form of the potential inside and outside the sphere

$$\Phi_{in} = \sum_{l=0}^{\infty} A_l r^l \mathrm{P}_l(\cos\theta)$$

$$\Phi_{out} = \sum_{l=0}^{\infty}[B_l r^l + C_l r^{-(l+1)}]\mathrm{P}_l(\cos\theta)$$

From the boundary condition at infinity we find that the only non-vanishing B_l is $B_1 = -E_0$. The other coefficients are determined from the boundary condition

$$-\frac{1}{a}\frac{\partial\Phi_{in}}{\partial\theta}|_{r=a} = -\frac{1}{a}\frac{\partial\Phi_{out}}{\partial\theta}|_{r=a}$$
$$-\epsilon\frac{\partial\Phi_{in}}{\partial\theta}|_{r=a} = -\epsilon_0\frac{\partial\Phi_{out}}{\partial\theta}|_{r=a}$$

The first condition gives the relation

$$A_1 = B_1 + \frac{C_1}{a^3} = -E_0 + \frac{C_1}{a^3}$$

$$A_l = \frac{C_l}{a^{2l+1}}$$

The second condition gives the relation

$$\frac{\epsilon}{\epsilon_0} A_1 = -E_0 - 2\frac{C_1}{a^3}$$

$$\frac{\epsilon}{\epsilon_0} l A_l = -(l+1)\frac{C_1}{a^3}$$

It is easy to see that $A_l = C_l = 0$ for al the $l \neq 1$. Then according to the two equation left we can give the form of the remaining coefficients.

$$A_1 = -(\frac{3}{2 + \epsilon/\epsilon_0})E_0$$

$$A_1 = (\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2})a^3 E_0$$

Therefore the potential is

$$\Phi_{in} = -(\frac{3}{2 + \epsilon/\epsilon_0})E_0 r \cos\theta$$

$$\Phi_{out} = -E_0 r \cos\theta + (\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2})\frac{a^3}{r^3} E_0$$

So the polarization inside the sphere is

$$\boldsymbol{P} = (\epsilon - \epsilon_0)\boldsymbol{E}_{in} = -(\epsilon - \epsilon_0)\frac{\partial \Phi_{in}}{\partial z}\boldsymbol{e}_z = 3\epsilon_0(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2})\boldsymbol{E}_0$$

Then the polarization charge density at the interface takes the form of

$$\sigma_p = \boldsymbol{P} \cdot \boldsymbol{n}_r = 3\epsilon_0(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2})E_0 \cos\theta$$

5

Magnetostatics, Faraday's Law, Quasi-Static Fields

5.1 Introduction and Definitions

The definition of magnetic dipole

$$\boldsymbol{N} = \boldsymbol{\mu} \times \boldsymbol{B}$$

where $\boldsymbol{\mu}$ is the magnetic moment of the dipole and $\boldsymbol{B}$ is the magnetic-flux density.

Conservation of charge demands that the charge density at any point in space be related to the current density in that neighborhood by a continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$$

5.2 Biot and Savart Law

If $\mathrm{d}\boldsymbol{l}$ is an element of length of a filamentary wire that carries a current I and $\boldsymbol{x}$ is the coordinate vector from the wire to point P, then the elemental magnetic flux density $\mathrm{d}\boldsymbol{B}$ at P is given by

$$\mathrm{d}\boldsymbol{B} = kI\frac{\mathrm{d}\boldsymbol{l} \times \boldsymbol{x}}{|\boldsymbol{x}|^3}$$

where $k = 1$ in esu, $k = \dfrac{\mu_0}{4\pi}$ in SI.

To put a current element $I_1\mathrm{d}\boldsymbol{l}$ in the presence of a magnetic induction $\boldsymbol{B}$, the elementary force is

$$\mathrm{d}\boldsymbol{F} = I_1(\mathrm{d}\boldsymbol{l} \times \boldsymbol{B})$$

If the external field $\boldsymbol{B}$ is due to a closed current loop 2 with current I_2, then the total force which a closed current loop 1 with current I_1 experiences is

$$F_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\mathrm{d}\boldsymbol{l}_1 \times (\mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{x}_{12})}{|\boldsymbol{x}_{12}|^3} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{(\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)\boldsymbol{x}_{12}}{|\boldsymbol{x}_{12}|^3}$$

where $\boldsymbol{x}_{12}$ is the vector distance from line element $\mathrm{d}\boldsymbol{l}_2$ to $\mathrm{d}\boldsymbol{l}_1$.

If a current density $\boldsymbol{J}(\boldsymbol{x})$ is in an external magnetic flux density $\boldsymbol{B}(\boldsymbol{x})$, the elementary force law implies that the total force on the current distribution is

$$\boldsymbol{F} = \int \boldsymbol{J}(\boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x}) \mathrm{d}^3 x$$

Similarly the total torque is

$$\boldsymbol{N} = \int \boldsymbol{x} \times (\boldsymbol{J} \times \boldsymbol{B}) \mathrm{d}^3 x$$

5.3 Differential Equation of Magnetostatics and Ampere's Law

Basic form of magnetic flux density

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \boldsymbol{J}(\boldsymbol{x'}) \times \frac{\boldsymbol{x} - \boldsymbol{x'}}{|\boldsymbol{x} - \boldsymbol{x'}|^3} \mathrm{d}^3 x'$$

The divergence and rotation of magnetic flux density satisfy

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$
$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

Ampere's law indicates that the loop integral of the current density is the total current I passing through the closed curve C

$$\oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I$$

5.4 Vector Potential

The vector potential $\boldsymbol{A}$ is defined as

$$\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{x})$$

According to the rotation of magnetic flux density equals zero, the vector potential takes the form of

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\mathrm{d}^3x' + \boldsymbol{\nabla}\Psi(\boldsymbol{x})$$

where $\boldsymbol{\nabla}\Psi(\boldsymbol{x})$ is the gradient of an arbitrary scalar field. To exploit the freedom of choosing vector potential, we can make the convenient choice of gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$, which means Ψ =constant, so that

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\mathrm{d}^3x'$$

Then the divergence of $\boldsymbol{B}$ will become

$$\nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{J}$$

5.5 Vector Potential and Magnetic Induction for a Circular Current Loop

We consider the problem of a circular loop pf radius a, lying in the $x-y$ plane, centered at the origin, and carrying a current I. The current density $\boldsymbol{J}$ has only a component in the ϕ direction.

$$J_\phi = I\sin\theta'\delta(\theta-\pi/2)\frac{\delta(r'-a)}{a}$$

The vectorial current density $\boldsymbol{J}$ can be written as

$$\boldsymbol{J} = -J_\phi\sin\phi'\mathbf{i} + J_\phi\cos\phi'\mathbf{j}$$

According to the definition of vector potential

$$A_\phi = \frac{\mu_0 I}{4\pi a}\int \frac{\sin\theta'\delta(\theta-\pi/2)\delta(r'-a)}{|\boldsymbol{x}-\boldsymbol{x'}|}\cos\phi' r'^2\mathrm{d}r'\mathrm{d}\Omega$$

According to the result of elliptic integrals, the vector potential can be written in the expansion as

$$A_\phi = \frac{\mu_0 I a^2 r\sin\theta}{4(a^2+r^2)^{3/2}}[1+\frac{15a^2r^2\sin^2\theta}{8(a^2+r^2)^2}+\cdots]$$

Furthermore

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = \frac{\mu_0 I a^2\cos\theta}{2(a^2+r^2)^{3/2}}[1+\frac{15a^2r^2\sin^2\theta}{4(a^2+r^2)^2}+\cdots]$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}(rA_\phi) = -\frac{\mu_0 I a^2\sin\theta}{4(a^2+r^2)^{5/2}}[2a^2-r^2+\frac{15a^2r^2\sin^2\theta(4a^2-3r^2)}{8(a^2+r^2)^2}+\cdots]$$

$$B_\phi = 0$$

Substitute the spherical expansion for $1/|\boldsymbol{x}-\boldsymbol{x'}|$ in vector potential

$$A_\phi = \frac{\mu_0 I}{a}Re\sum_{l,m}\frac{\mathrm{Y}_{lm}(\theta,0)}{2l+1}\int \sin\theta'\delta(\theta-\pi/2)\delta(r'-a)\mathrm{e}^{\mathrm{i}\phi'}\frac{r_<^l}{r_>^{l+1}}\mathrm{Y}_{lm}^*(\theta',\phi')r'^2\mathrm{d}r'\mathrm{d}\Omega$$

The presence of $\mathrm{e}^{\mathrm{i}\phi}$ means only $m=\pm1$ will contribute to the sum. Hence

$$\begin{aligned}
A_\phi &= 2\pi\mu_0 I a\sum_{l=1}^{\infty}\frac{\mathrm{Y}_{l1}(\theta,0)}{2l+1}\frac{r_<^l}{r_>^{l+1}}\mathrm{Y}_{l1}(\pi/2,0)\\
&= 2\pi\mu_0 I a\sum_{l=1}^{\infty}\frac{\mathrm{Y}_{l1}(\theta,0)}{2l+1}\frac{r_<^l}{r_>^{l+1}}\sqrt{\frac{2l+1}{4\pi l(l+1)}}\mathrm{P}_l^1(0)\\
&= \begin{cases} 0 & l=2n\\ 2\pi\mu_0 I a\sum_{l=1}^{\infty}\frac{\mathrm{Y}_{l1}(\theta,0)}{2l+1}\frac{r_<^l}{r_>^{l+1}}\sqrt{\frac{2l+1}{4\pi l(l+1)}}\frac{(-1)^{n+1}\Gamma(n+3/2)}{\Gamma(n+1)\Gamma(3/2)} & l=2n+1\end{cases}
\end{aligned}$$

5.6 Magnetic Moment

The definition of magnetic moment density or magnetization is

$$M(\boldsymbol{x}) = \frac{1}{2}[\boldsymbol{x}\times\boldsymbol{J}(\boldsymbol{x})]$$

The magnetic moment $\boldsymbol{m}$

$$m = \frac{1}{2}\int \boldsymbol{x'}\times\boldsymbol{J}(\boldsymbol{x'})\mathrm{d}^3x'$$

Then the vector potential and magnetic induction becomes

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{x}}{|\boldsymbol{x}|^3}$$

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{m}-\boldsymbol{m})}{|\boldsymbol{x}|^3}$$

5.7 Force and Torque on and Energy of a Localized Current Distribution in an External Magnetic Induction

The force on a localized current distribution in an external magnetic field $\boldsymbol{B}$ id

$$\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{m} \cdot \boldsymbol{B})$$

The torque

$$\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B}$$

The potential energy

$$U = -\boldsymbol{m} \cdot \boldsymbol{B}$$

5.8 Macroscopic Equations, Boundary Conditions on B and H

The definition of magnetization is

$$\boldsymbol{M}(\boldsymbol{x}) = \sum_i N_i \langle \boldsymbol{m}_i \rangle$$

where $\boldsymbol{m}_i$ is the magnetic moment density of the ith molecule and N_i is the average number per unit volume of molecules of type i.

The magnetization contribute an effective current density

$$\boldsymbol{J}_M = \boldsymbol{\nabla} \times \boldsymbol{M}$$

The definition of magnetic field

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

Then the macroscopic equations become

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

For isotropic diamagnetic and paramagnetic substances the simple linear relation

$$\boldsymbol{M} = \chi_M \boldsymbol{H}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}$$

where

$$\mu = \mu_{)}(1 + \chi_M)$$

For an interface between region 1 and 3, the normal components of $\boldsymbol{B}$ and the tangential components of $\boldsymbol{H}$ satisfy the boundary conditions

$$\begin{cases} (\boldsymbol{B}_1 - \boldsymbol{B}_2) \cdot \boldsymbol{n} = 0 \\ \boldsymbol{n} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{\alpha} \end{cases}$$

where $\boldsymbol{\alpha}$ is the idealized surface current density.

5.9 Methods of Solving Boundary-Value Problems in Magnetostatics

Two important equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}$$

5.9.1 Generally Applicable Method of the Vector Potential

Since the divergence of magnetic induction equals zero, we can always introduce a vector potential

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

For linear media with $\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$, the second equation becomes

$$\boldsymbol{\nabla} \times (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{J}$$

With the choice of Coulomb's gauge that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, then

$$-\nabla^2 \boldsymbol{A} = \mu \boldsymbol{J}$$

5.9.2 J=0; Magnetic Scalar Potential

If the current density vanishes in some finite region of space, the second equation becomes $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}$. We can introduce magnetic scalar potential Φ_M

$$\boldsymbol{H} = -\boldsymbol{\nabla}\Phi_M$$

For linear media $\boldsymbol{B} = \mu \boldsymbol{H}$, the first equation becomes tg

$$\boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} \Phi_M) = 0$$

If μ is at least piecewise constant, in each region the magnetic scalar potential satisfies

$$\nabla^2 \Phi_M = 0$$

5.9.3 Hard Ferromagnets (M given and J=0)

5.9.3.1 Scalar Potential

Since $\boldsymbol{J} = 0$, the magnetic scalar potential satisfies

$$\nabla^2 \Phi_M = -\rho_M$$
$$\rho_M = -\boldsymbol{\nabla} \cdot \boldsymbol{M}$$

Therefore

$$\Phi_M(\boldsymbol{x}) = -\frac{1}{4\pi} \int \frac{\boldsymbol{\nabla}' \cdot \boldsymbol{M}(\boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} \mathrm{d}^3 x'$$

5.9.3.2 Vector Potential

Since the vector potential satisfies

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}_M$$
$$\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{M}$$

Therefore

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{\nabla}' \times \boldsymbol{M}(\boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} \mathrm{d}^3 x'$$

5.10 Uniformly Magnetized Sphere

A sphere of radius a, with a uniform permanent magnetization $\boldsymbol{M}$ of magnitude M_0 and parallel to the z axis. Since $\sigma_M = \boldsymbol{n} \cdot \boldsymbol{M} = M_0 \cos\theta$, then the potential is

$$\Phi_M = \frac{M_0 a^2}{4\pi} \int \frac{\cos\theta'}{|\boldsymbol{x} - \boldsymbol{x'}|} \mathrm{d}\Omega'$$

It is easy to get that the potential is

$$\Phi_M = \frac{1}{3}M_0 a^2 \frac{r_<}{r_>^2}\cos\theta = \begin{cases} \frac{1}{3}M_0 r\cos\theta & r < a \\ \frac{1}{3}M_0\frac{a^3}{r^2}\cos\theta & r > a \end{cases}$$

The magnetic field and magnetic induction inside the sphere are therefore

$$\boldsymbol{H}_{in} = -\frac{1}{3}\boldsymbol{M}$$
$$\boldsymbol{B}_{in} = \frac{2\mu_0}{3}\boldsymbol{M}$$

5.11 Magnetized Sphere in an External Field; Permanent Magnets

We have a problem of a uniformly magnetized sphere in an external field $\boldsymbol{B}_0 = \mu_0\boldsymbol{H}_0$. So

$$\boldsymbol{B}_{in} = \boldsymbol{B}_0 + \frac{2\mu_0}{3}\boldsymbol{M}$$
$$\boldsymbol{H}_{in} = \frac{1}{\mu_0}\boldsymbol{B}_0 - \frac{1}{3}\boldsymbol{M}$$

Therefore

$$\boldsymbol{M} = \frac{3}{\mu_0}(\frac{\mu-\mu_0}{\mu+2\mu_0})\boldsymbol{B}_0$$

5.12 Magnetic Shielding, Spherical Shell of Permeable Material in a Uniform Field

We consider a spherical shell of inner radius a and outer radius b, made of material of permeability μ, and placed in a formerly uniform constant magnetic induction $\boldsymbol{B} = \mu_0\boldsymbol{H}_0$. We decide to find the fields $\boldsymbol{B}$ and $\boldsymbol{H}$ everywhere. It is easy to find that the magnetic scalar potential Φ_M satisfy Laplace equation.

$$\nabla^2\Phi_M = 0$$

So

$$\Phi_M = \begin{cases} \sum_{l=0}^{\infty} A_l r^l \mathrm{P}_l(\cos\theta) & r < a \\ \sum_{l=0}^{\infty} (B_l r^l + C_l r^{-(l+1)}) \mathrm{P}_l(\cos\theta) & a < r < b \\ -H_0 r \cos\theta + \sum_{l=0}^{\infty} D_l r^{-(l+1)} \mathrm{P}_l(\cos\theta) & r > b \end{cases}$$

The boundary conditions are

$$\frac{\partial \Phi_M}{\partial \theta}|_{r=a,b+} = \frac{\partial \Phi_M}{\partial \theta}|_{r=a,b-}$$
$$\mu_0 \frac{\partial \Phi_M}{\partial r}|_{r=a,b+} = \mu_0 \frac{\partial \Phi_M}{\partial r}|_{r=a,b-}$$

All the coefficients with $l \neq 1$ vanishes. So the solutions are

$$A_1 = -\frac{9\mu_r}{(2\mu_r+1)(\mu_r+2) - 2\frac{a^3}{b^3}(\mu_r-1)^2} H_0$$

$$B_1 = -\frac{3}{(2\mu_r+1)(\mu_r+2) - 2\frac{a^3}{b^3}(\mu_r-1)^2} H_0$$

$$C_1 = -\frac{3(\mu_r-1)a^3}{(2\mu_r+1)(\mu_r+2) - 2\frac{a^3}{b^3}(\mu_r-1)^2} H_0$$

$$D_1 = \frac{(2\mu_r+1)(\mu_r-1)}{(2\mu_r+1)(\mu_r+2) - 2\frac{a^3}{b^3}(\mu_r-1)^2}(b^3-a^3) H_0$$

where $\mu_r = \mu/\mu_0$

5.13 Effect of a Circular Hole in a Perfectly Conducting Plane with an Asymptotically Uniform Tangential Magnetic Field on On Side

5.14 Faraday's Law of Induction

The magnetic flux linking the circuit is defined b

$$F = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$

The electromotive force around the circuit is

$$\mathscr{E} = \oint_C \boldsymbol{E}' \cdot \mathrm{d}\boldsymbol{l}$$

where $\boldsymbol{E}'$ is the electric field at the element $\mathrm{d}\boldsymbol{l}$ of the circuit C. Faraday's Law

$$\mathscr{E} = -k\frac{\mathrm{d}F}{\mathrm{d}t}$$

Thus the differential form of Faraday's Law is

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

5.15 Energy in the Magnetic Field

The magnetic energy at total space is

$$W = \frac{1}{2}\int \boldsymbol{H} \cdot \boldsymbol{B}\mathrm{d}^3x = \frac{1}{2}\int \boldsymbol{J} \cdot \boldsymbol{A}\mathrm{d}^3x$$

5.16 Energy and Self- and Mutual Inductances

5.16.1 Coefficients of Self- and Mutual Inductances

The concept of self- and mutual inductances are useful for system of current-carrying circuits. Imagine a system of N distinct circuits, the ith one with total current I_i, in otherwise empty space. Then the total magnetic energy

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij} I_i I_j$$

where L_i is the self inductances of the ith current circuit and M_{ij} is the mutual inductance between the ith and the jth current ciucuits. They are given by

$$L_i = \frac{\mu_o}{4\pi I_i^2}\int_{C_i} \mathrm{d}^3x_i \int_{C_i} \mathrm{d}^3x_i \frac{\boldsymbol{J}(\boldsymbol{x}_i) \cdot \boldsymbol{J}(\boldsymbol{x}_i')}{|\boldsymbol{x}_i - \boldsymbol{x}_i'|}$$

$$M_i = \frac{\mu_o}{4\pi I_i I_j}\int_{C_i} \mathrm{d}^3x_i \int_{C_j} \mathrm{d}^3x_j \frac{\boldsymbol{J}(\boldsymbol{x}_i) \cdot \boldsymbol{J}(\boldsymbol{x}_j')}{|\boldsymbol{x}_i - \boldsymbol{x}_j'|}$$

5.16.2