Mathematics Foundation of Computer Science Formula Booklet

Students of MLFCS 2021

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Contents

1 Ring Laws

1.1 Ring Laws

$a + 0 = a \quad a \times 1 = a \qquad \text{(Neutral elements)}$

$a + b = b + a \quad a \times b = b \times a \qquad \text{(Commutativity)}$

$a + (b + c) = (a + b) + c$
$a \times (b \times c) = (a \times b) \times c \qquad \text{(Distributivity)}$

$a \times 0 = 0 \qquad \text{(Annihilation)}$

1.2 Cancellation Laws

$a + c = b + c \quad \Rightarrow \quad a = b$

$a \times c = b \times c \quad \Rightarrow \quad a = b, \quad c \neq 0$

2 Sets

$[A \backslash B]$: Set A without elements shared with B

$A = \{a, b\}\,, \quad B = \{1, 2\}\,, \quad A \times B = \{(a, 1)\,, (a, 2)\,, (b, 1)\,, (b, 2)\}$

$|A|$: Cardinality of A

3 Relations

3.1 Relations

Graph: Binary relation $R \subseteq A^2$ on a single set

Vertices: Elements of A

Edges: Elements of R

$A^2 :\ A \times A$

Reflexivity: $(x, x) \in R$

Irreflexive: $\forall x \in A. (x, x) \notin R$

Reflexive-closure: $R \cup \{(x, y) \in A^2 \mid x = y\} \subseteq A^2$

Symmetry: $\forall (x, y) \in A^2.\ (x, y) \in R \Rightarrow (y, x) \in R$

Anti-symmetry: $\forall x \in A. (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$

Symmetric-enclosure: $\{(a, b), (b, a), (a, c), (c, a), ...\}$

Transitivity: $\forall x, y, z \in A.\ (x, y) \in R\ \wedge\ (y, z) \in R \Rightarrow (x, z) \in R$

Transitive-closure: $R \cup R;\ R \cup R;\ R;\ R \cup R;\ R;\ R \cup R...$

Transitive-closure: "All R Paths"

Order Relation: When $R \subseteq A^2$ is Reflexive, Anti-Symmetric and Transitive

Equivalence Relation: Reflexive, Symmetric, Transitive

Equivalence Class: Set of $\in A$ which are all $\in$ equivalence closure

4 Functions

4.1 Function Requirements

Definedness: $\forall a \in A \; \exists b \in B.\; (a, b) \in R$

Single-valuedness: $\forall a \in A \; \forall b, b' \in B.\; (a, b) \in B \;\wedge\; (a, b') \in R \Rightarrow b = b'$

4.2 Properties

Range / Image: $\{b \in B \mid \exists a \in A.\; (a, b) \in R\} \subseteq B$

Injectivity: $\forall a, a' \in A.\; a \neq a' \Rightarrow f(a) \neq f(b)$

Surjectivity: $\forall b \in B \; \exists a \in A.\; f(a) = b$

Bijectivity: $|F^{-1}[\{b\}]| = 1$

Bijectivity: Injective And Surjective Simultaneously

Forward Image: $F[X] = \{b \in B \mid \exists a \in X.\; f(a) = b\}$

Backward Image: $F^{-1}[Y] = \{a \in A \mid f(a) \in Y\}$

$F[\bar{X}]$: Compliment of pre-image

$\overline{F[X]}$: Compliment of forward-image

Everywhere defined: All of A is the pre-image

5 The Inner Product

5.1 Conversions

Linear Equation To Parametric:

$$x_1 = d+bx_2+cx_3... \Rightarrow \begin{pmatrix} d \\ 0 \\ 0 \\ ... \\ ... \end{pmatrix} +x_2\cdot \begin{pmatrix} b \\ 1 \\ 0 \\ ... \\ ... \end{pmatrix} +x_3\cdot \begin{pmatrix} c \\ 0 \\ 1 \\ ... \\ ... \end{pmatrix} +...+x_n\cdot \begin{pmatrix} n^{th}\ co-eff \\ ... \\ ... \\ ... \\ 1 \end{pmatrix}$$

Parametric to linear (line):
$ax_1 + bx_2 = d$
$a = -v_2$
$b = v_1$
$d = -v_2p_1 + v_1p_2$

Parametric to linear (plane):
$ax_1 + bx_2 + cx_3 = d$
$a = v_2w_3 - v_3w_2$
$b = v_3w_1 - v_1w_3$
$c = v_1w_2 - v_2w_1$
$d = ap_1 + bp_2 + cp_3$

5.2 Inner Product

$\langle \vec{v}, \vec{w} \rangle : \ v_1 \times w_1 + v_2 \times w_2 + ... + v_n \times w_n$

$\langle \vec{v} + \vec{w}, \vec{u} \rangle : \ \langle \vec{v}, \vec{u} \rangle + \langle \vec{w}, \vec{u} \rangle$

$\langle s \cdot \vec{v}, \vec{w} \rangle : \ s \cdot \langle \vec{v}, \vec{w} \rangle$

$\langle \vec{v}, \vec{w} \rangle : \ |\vec{v}| \times |\vec{w}| \times \cos a$

$Orthogonal\ Test : \ \langle \vec{v}, \vec{w} \rangle = 0$

$\langle \vec{v}, \vec{v} \rangle : \ |\vec{v}|^2$

$|\vec{v}| : \ \sqrt{\langle \vec{v}, \vec{v} \rangle}$

5.3 Geometry

$\vec{n} : \ \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\frac{d}{|\vec{n}|}$: Distance from origin

$\langle \vec{n}, X \rangle = d = \langle \vec{n}, P \rangle$ where X: Arbitrary Point, P: Point on the line

Projection of line $\vec{v}$ on line $\vec{n}$: $\frac{\langle \vec{n}, \vec{v} \rangle}{\langle \vec{n}, \vec{n} \rangle} \times \vec{n}$

Distance of Point Q to line P:
$\frac{\langle P, \vec{n} \rangle - \langle Q, \vec{n} \rangle}{|\vec{n}|} = \frac{d - \langle \vec{n}, Q \rangle}{|\vec{n}|} = \frac{\left\langle \vec{n}, \vec{QP} \right\rangle}{|\vec{n}|}$

$Q' : Q + \frac{d - \langle \vec{n}, Q \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n}$

$Q'' : Q + 2 \times \frac{d - \langle \vec{n}, Q \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n}$

6 Bases

6.1 Bases

Linear Combinations:

$(\sum_{i=1}^{n} a_i \cdot \vec{v_i}) + (\sum_{i=1}^{n} b_i \cdot \vec{v_i}) = \sum_{i=1}^{n} (a_i + b_i) \cdot \vec{v_i}$

$s \cdot (\sum_{i=1}^{n} a_i \cdot \vec{v_i}) = \sum_{i=1}^{n} (s \times a_i) \cdot \vec{v_i}$

Theorem 8 for linear independence:
$\sum_{i=1}^{n} a_i \cdot \vec{v_i} = \vec{0} \Rightarrow a_1 = a_2 = a_3 = ... = 0$

Value of particular co-efficient (coordinates):
$a_k = \frac{\langle \vec{w_k}, \vec{v_k} \rangle}{\langle \vec{v_k}, \vec{v_k} \rangle}$

Orthonormal: $\langle \vec{v}, \vec{v} \rangle = 1$

Positive definite:
$\langle \vec{v}, \vec{v} \rangle \geq 0$
$\langle \vec{v}, \vec{v} \rangle = 0 \Rightarrow \vec{v} = \vec{0}$

Computing Orthogonal bases from bases:
$\vec{w_1} = \vec{v_1}$
$\vec{w_2} = \vec{v_2} - \frac{\langle \vec{v_2}, \vec{w_1} \rangle}{\langle \vec{w_1}, \vec{w_1} \rangle} \cdot \vec{w_1}$
$\vec{w_3} = \vec{v_3} - \frac{\langle \vec{v_3}, \vec{w_1} \rangle}{\langle \vec{w_1}, \vec{w_1} \rangle} \cdot \vec{w_1} - \frac{\langle \vec{v_3}, \vec{w_2} \rangle}{\langle \vec{w_2}, \vec{w_2} \rangle} \cdot \vec{w_2}$

and so on...